Chapter II.4

Invariant Manifold Theory

This chapter will be devoted to the invariant manifold theory of impulsive differential equations. At the theoretical level, we will assume only that the reference bounded solution has exponential trichotomy, but when we move into computational aspects we will assume that the dynamics are periodic. This will allow us to take advantage of the Floquet decomposition, with the result being that computation of invariant manifolds has much in common with the same procedure for ordinary differential equations without impulses.

In this chapter we will assume a semilinear decomposition

$$\dot{x} = A(t)x + f(t,x), \qquad t \neq t_k \tag{II.4.1}$$

$$\Delta x = B_k x + g_k(x), \qquad t = t_k, \tag{II.4.2}$$

where $f(t,0) = g_k(0) = 0$ and $D_2 f(t,0) = Dg_k(0) = 0$.

Definition II.4.0.1. *System* (II.4.1)–(II.4.2) *is* periodic *with period* $T > 0$ *and* c *impulses per period if* $A(t+T) = A(t)$*,* $f(t+T,\cdot) = f(t,\cdot)$*,* $B_{k+c} = B_k$*,* $g_{k+c} = g_k$ *and* $t_{k+c} = t_k + T$ *for all* $t \in \mathbb{R}$ *and* $k \in \mathbb{Z}$*.*

II.4.1 Existence and Smoothness

Definition II.4.1.1. *An* invariant manifold *for the trivial solution* $x = 0$ *is a subset* $W \subset \mathbb{R} \times \mathbb{R}^n$ *with the following properties:*

- $\mathbb{R} \times \{0\} \subset W$*;*
- *the sets* $W_t := \{x : (t,x) \in W\}$ *are submanifolds of* $\mathbb{R}^n$*;*
- *if* $x_s \in W_s$*, then* $x(t;s,x) \in W_t$ *as long as this solution is defined.*

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An invariant manifold is C^k if W_t is C^k for each t.

We will at this point drop the phrase "for the trivial solution $x = 0$", since we will always be referring to invariant manifolds at this solution. To define invariant manifolds at other solutions γ, one can simply perform a change of variables to translate γ to zero and get a system of the form (II.4.1)–(II.4.2).

Definition II.4.1.2. *Suppose the trivial solution $x = 0$ has exponential trichotomy. An invariant manifold W is a*

- stable manifold *if W_t is tangent to $X_s(t)$ at $0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$, and solutions $x(t) \in W_t$ converge exponentially to zero as $t \to \infty$;*
- centre manifold *if W_t is tangent to $X_c(t)$ at $0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$, and solutions $x(t) \in W_t$ have sub-exponential growth as $t \to \pm\infty$;*
- unstable manifold *if W_t is tangent to $X_u(t)$ at $0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$, and solutions $x(t) \in W_t$ converge exponentially to zero as $t \to -\infty$;*
- centre-stable manifold *if W_t is tangent to $X_{cs}(t)$ at $0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$, and solutions $x(t) \in W_t$ have sub-exponential growth as $t \to \infty$;*
- centre-unstable manifold *if W_t is tangent to $X_{cu}(t)$ at $0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$, and solutions $x(t) \in W_t$ have sub-exponential growth as $t \to -\infty$.*

Definition II.4.1.3. *Suppose $x = 0$ has exponential trichotomy. Let $P(t)$ denote the projection onto one of the stable, centre, unstable, centre-stable or centre-unstable fibre bundles associated to the linear part,*

$$\dot{x} = A(t)x, \qquad t \neq t_k \tag{II.4.3}$$
$$\Delta x = B_k x, \qquad t = t_k, \tag{II.4.4}$$

of (II.4.1)–(II.4.2). *A* local stable, centre, unstable, centre-stable or centre-unstable manifold *is a set of the form*

$$W^{loc} = \{(t, x + h(t, x)) : t \in \mathbb{R},\ x \in B_\delta(0) \cap \mathcal{R}(P(t)) \subset \mathbb{R}^n\},$$

for some $h : \mathbb{R} \times B_\delta(0) \to \mathbb{R}^n$, satisfying $h(t, 0) = 0$, $P(t)h(t, u) = 0$, with W^{loc} having the following properties:

- *If $(s, x_s) \in W^{loc}$, there exists $\epsilon > 0$ such that $(t, x(t; s, x_s)) \in W^{loc}$ for $|t - s| < \epsilon$.*
- *W_t^{loc} is tangent to $\mathcal{R}(P(t))$ at $0 \in \mathbb{R}^n$.*
- *Any solution that remains in W^{loc} for the asymptotic time ranges specified in Definition II.4.1.2 satisfies the same asymptotic growth or decay rates.*

A local invariant manifold is $PC^{1,m}$-regular at zero *if*

- $z \mapsto h(t,z)$ *is* C^m *in a neighbourhood of* $0 \in \mathbb{R}^p$*;*
- *for* $j = 0,\dots,m$ *and all* $z_1,\dots,z_j \in \mathbb{R}^p$*,* $t \mapsto D_2^j h(t,0)[z_1,\dots,z_j]$ *is continuous except at times* t_k *where it has limits on the left and, additionally, it is differentiable from the right everywhere.*

The $PC^{1,m}$-regular condition in addition to the tangency property implies that the function $h : \mathbb{R} \times B_\delta(0) \to \mathbb{R}^n$ has a Taylor expansion near $x = 0$ of the form

$$h(t,z) = \frac{1}{2!}h_2(t)z^2 + \frac{1}{3!}h_3(t)z^3 + \cdots + \frac{1}{m!}h_m(t)z^m + O(|z|^{m+1}), \quad \text{(II.4.5)}$$

for t fixed, and that the coefficients are differentiable from the right with discontinuities at impulse times t_k, where they have limits on the left.

Proving the existence of local invariant manifolds and their $PC^{1,m}$ regularity is formally equivalent to all of the work done in Chaps. I.5, I.6, and I.7 and is in fact implied by the relevant theorems therein. Indeed, taking the delay range $r = 0$ directly recovers the case of impulsive differential equations. As such, the following theorem need not be proven.

Theorem II.4.1.1. *Suppose the trivial solution* $x = 0$ *has exponential trichotomy. There exist local stable, centre, unstable, centre-stable and centre-unstable manifolds. These manifolds are* $PC^{1,m}$ *regular provided* (II.4.1)–(II.4.2) *if* PC^m*. The Taylor coefficients* $h_j(t)$ *in* (II.4.5) *are bounded, and the asymptotic form of that equation holds uniformly for* $t \in \mathbb{R}$ *provided* (II.4.1)–(II.4.2) *if* PC^{m+1}*.*

II.4.2 Invariance Equation for Nonautonomous Systems

The dynamics on any invariant manifold can be characterized by the abstract results in Sect. I.7.6. However, the situation here is a fair bit simpler because the projection matrices $P_j(t)$ onto the stable, centre and unstable fibre bundles are much more regular than the associated operators in the infinite-dimensional case.

This section will be devoted to the derivation of the *invariance equation* associated to a given local invariant manifold. Throughout, $P(t)$ will denote a projection onto one of the stable, centre, unstable, centre-stable or centre-unstable fibre bundles. The invariant manifold in question will be assumed to be $PC^{1,m}$-regular at zero and is represented in the form

$$W^{loc} = \{(t, x + h(t,x)) : t \in \mathbb{R},\ x \in B_\delta(0) \cap \mathcal{R}(P(t)) \subset \mathbb{R}^n\} \quad \text{(II.4.6)}$$

for $h : \mathbb{R} \times B_\delta(0) \to \mathbb{R}^n$ and satisfying $h(t,0) = 0$ and $P(t)h(t,x) = 0$.

The following lemma characterizes the regularity of the projector $P(t)$. It will be needed in the subsequent sections.

Lemma II.4.2.1. *Let $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a matrix-valued function satisfying $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$. Then, P satisfies the matrix impulsive differential equation*

$$\dot{P}(t) = A(t)P(t) - P(t)A(t), \qquad t \neq t_k, \tag{II.4.7}$$

$$\Delta P(t_k) = B_k P(t_k-) - P(t_k)B_k, \qquad t = t_k. \tag{II.4.8}$$

More succinctly, at times $t = t_k$ we have $P(t_k)[I + B_k] = [I + B_k]P(t_k^-)$.

Proof. For ease of presentation, we will assume $A(t)$ is continuous on $[t_k, t_{k+1})$, so that $t \mapsto U(t,s)$ will be differentiable on each of $[t_k, t_{k+1})$, but the result remains true (in the sense of integrated solutions) under weaker PC^0 conditions. Let $t \in (t_k, t_{k+1})$. Then, we can write $U(t,t_k) = X(t,t_k)$ for X the Cauchy matrix of the continuous part, $\dot{x} = A(t)x$. This matrix is invertible, from which it follows that

$$P(t) = U(t,t_k)P(t_k)U^{-1}(t,t_k).$$

The right-hand side is differentiable, from which it follows that $P'(t)$ exists, with

$$\begin{aligned}\dot{P}(t) &= A(t)U(t,t_k)P(t_k)U(t,t_k)^{-1} + U(t,t_k)P(t_k)[-U(t,t_k)^{-1}A(t)]\\ &= A(t)P(t) - P(t)A(t),\end{aligned}$$

as claimed. As for the impulse times, since $U(t_k, t_k^-) = I + B_k$, the definition of P implies $P(t_k)[I + B_k] = [I + B_k]P(t_k^-)$. Rearranging gives (II.4.8). $\square$

Suppose $x(t)$ is a solution on the invariant manifold. Then, at each time t we can write $u(t) = z + h(t,z)$ for some $z \in \mathcal{R}(P(t))$. Substituting this ansatz into the impulsive differential equation (II.4.1)–(II.4.2), we get

$$\begin{aligned}A(t)(z+h) + f(t,z+h) &= \partial_t h + [I + \partial_z h]\dot{z}, &&t \neq t_k\\ B_k(z+h) + g_k(z+h) &= \Delta_t h(t, z+\Delta x) + \left[I + \int_0^1 \partial_z h(t_k^-, z + s\Delta z)ds\right]\Delta z, &&t = t_k.\end{aligned}$$

Since $P(t)h(t,u) = 0$, we get $z = P(t)u(t)$. Applying Lemma II.4.2.1, one can check that

$$\dot{z} = A(t)z + P(t)f(t,z+h), \qquad t \neq t_k \tag{II.4.9}$$

$$\Delta z = B_k z + P(t_k)g_k(z+h), \qquad t = t_k. \tag{II.4.10}$$

Combining these two results, we arrive at the *invariance equation* for the invariant manifold:

$$A(t)h + (I - P(t))f(t, z + h) = \partial_z h[Az + Pf(t, z + h)] + \partial_t h, \qquad t \neq t_k, \tag{II.4.11}$$

$$B_k h + (I - P(t_k))g_k(z + h) = \left[\int_0^1 \partial_z h(t_k^-, z + sr_k)ds\right] r_k + \Delta_t h(t_k, z + r_k), \quad t = t_k, \tag{II.4.12}$$

where in the above $r_k = r_k(z, h) := B_k z + P(t_k)g_k(z+h)$, all unspecified time evaluations are at $t = t_k^-$ in (II.4.12), and we define $\Delta_t h(t_k, y) = h(t_k, y) - h(t_k^-, y)$.

The pair of Eqs. (II.4.11)–(II.4.12) defines an impulsive partial differential equation satisfied by the function $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ defining the invariant manifold.

II.4.3 Invariance Equation for Systems with Periodic Linear Part

When the linear part (II.4.3)–(II.4.4) is periodic, we can take advantage of Floquet theory to simplify the form of the invariance equation (II.4.11)–(II.4.12). More generally, we will assume a kinematic similarity as introduced in Sect. II.2.6.3. Let

$$x = Q_s(t)y_s + Q_c(t)y_c + Q_u(t)y_u, \tag{II.4.13}$$

be a real T-periodic change of variables of the form introduced in Corollary II.2.6.4. Each Q_j could be a chain matrix, a Floquet periodic matrix or some combination thereof, and the optimal choice will depend on the situation at hand. After completing the change of variables, (II.4.1)–(II.4.2) become

$$\dot{y}_s = \Lambda_s y_s + \tilde{f}_s(t, y_s, y_c, y_u), \qquad t \neq t_k \tag{II.4.14}$$

$$\dot{y}_c = \Lambda_c y_c + \tilde{f}_c(t, y_s, y_c, y_u), \qquad t \neq t_k \tag{II.4.15}$$

$$\dot{y}_u = \Lambda_u y_u + \tilde{f}_u(t, y_s, y_c, y_u), \qquad t \neq t_k \tag{II.4.16}$$

$$\Delta y_s = \Omega_s y_s + \tilde{g}_s(k, y_s, y_c, y_u), \qquad t = t_k \tag{II.4.17}$$

$$\Delta y_c = \Omega_c y_c + \tilde{g}_c(k, y_s, y_c, y_u), \qquad t = t_k \tag{II.4.18}$$

$$\Delta y_u = \Omega_u y_u + \tilde{g}_u(k, y_s, y_c, y_u), \qquad t = t_k, \tag{II.4.19}$$

with the nonlinearities

$$\tilde{f}_j(t, y, z, w) = Q_j^+(t)f(t, Q_s(t)y + Q_c(t)z + Q_u(t)w),$$
$$\tilde{g}_j(k, y, z, w) = Q_j^+(t_k)g_k(Q_s(t_k^-)y + Q_c(t_k^-)z + Q_u(t_k^-)w).$$

Recall that for a matrix M with linearly independent columns, the symbol M^+ denotes its left-inverse. The dynamics have been decoupled into stable (y_s), centre (y_c) and unstable (y_u) directions.

Denote σ_s, σ_c and σ_u the sets of Floquet multipliers of the decoupled linear parts, so that $\sigma(M_0) = \sigma_s \cup \sigma_c \cup \sigma_u$, with M_0 being the monodromy matrix of the original linear part (II.4.3)–(II.4.3). To derive the invariance equation for the invariant manifold $\mathcal{W}_f$, we will partition (II.4.14)–(II.4.19) as

$$\dot{y} = Uy + \tilde{f}_1(t, y, z), \qquad t \neq t_k, \tag{II.4.20}$$

$$\dot{z} = Vz + \tilde{f}_2(t, y, z), \qquad t \neq t_k, \tag{II.4.21}$$

$$\Delta y = R_k y + \tilde{g}_1(k, y, z), \qquad t = t_k, \tag{II.4.22}$$

$$\Delta z = S_k z + \tilde{g}_2(k, y, z), \qquad t = t_k, \tag{II.4.23}$$

where the linear part of the y equations has only the Floquet exponents σ_f, and the linear part of the z equations has only the Floquet exponents $\sigma(M_0) \setminus \sigma_f$. This partitioning is always attainable. The nonlinearities $\tilde{f}_i$ and $\tilde{g}_i$ will be some vectors involving those of (II.4.14)–(II.4.19).

In the (y, z) coordinates, the t-fibre $\mathcal{W}_f(t)$ of the invariant manifold is the solution set of the equation

$$z = \tilde{h}(t, y), \tag{II.4.24}$$

with $\tilde{h} : \mathbb{R} \times \mathbb{R}^{\dim X_f} \to \mathbb{R}^{n - \dim X_f}$ defined explicitly in terms of the function h in (II.4.6) by

$$\tilde{Q}(t)\tilde{h}(t, y) = h(t, Q_f(t)y),$$

where $\tilde{Q}(t)$ is the matrix $Q = [\ Q_s \quad Q_c \quad Q_u\ ]$ without the Q_f part. For example, if $Q_f = Q_c$, then $\tilde{Q} = [\ Q_s \quad Q_u\ ]$. These details are unimportant, since we can work directly with (II.4.24). Also, from this point on we will drop the tildes in (II.4.24) and simply write $z = h(t, y)$.

To derive the invariance equation, we substitute (II.4.24) into (II.4.20)–(II.4.23). Working first with the differential equation (II.4.21), we get

$$Vh(t, y) + \tilde{f}_2(t, y, h(t, y)) = \partial_t h(t, y) + \partial_y h(t, y)\dot{y}. \tag{II.4.25}$$

The next step would be to substitute (II.4.20) into (II.4.25) and replace all instances of z with $h(t, y)$. As for the jumps, substituting $z = h(t, y)$ into (II.4.23) gives the equation

$$S_k h(t_k^-, y) + \tilde{g}_2(k, y, h(t_k^-, y)) = h(t_k, y + \Delta y) - h(t_k^-, y).$$

We can write the right-hand side equivalently as

$$h(t_k, y + \Delta y) - h(t_k^-, y) = \Delta_t h(t_k, y + \Delta y) + \int_0^1 \partial_y h(t_k^-, y + s\Delta y)\Delta y ds,$$

where $\Delta_t h(t_k, v) = h(t_k, v) - h(t_k^-, v)$. Every instance of Δy can now be replaced with (II.4.22), and all appearances of z therein are replaced with $h(t_k^-, y)$. This entire discussion then leads to the complete invariance equation.

Theorem II.4.3.1. *The invariant manifold $\mathcal{W}_f$ in the (y, z) coordinates of system (II.4.20)–(II.4.23) can be expressed as the solution set of $z = h(t, y)$, where the function $h : \mathbb{R} \times \mathbb{R}^{\dim X_f} \to \mathbb{R}^{n - \dim X_f}$ is periodic in its first variable and satisfies the impulsive partial differential equation*

$$Vh + \tilde{f}_2(t, y, h) = \partial_t h + (\partial_y h)[Uy + \tilde{f}_1(t, y, h)], \qquad t \neq t_k \tag{II.4.26}$$

$$S_k h + \tilde{g}_2(k, y, h) = \Delta_t h(t_k, y + r_k) + \int_0^1 \partial_y h(t_k^-, y + s r_k) r_k ds, \quad t = t_k, \tag{II.4.27}$$

where $h = h(t, y)$ in the first equation, $h = h(t_k^-, y)$ in the second equation (unless otherwise specified), $\Delta_t h(t_k, v) = h(t_k, v) - h(t_k^-, v)$, and $r_k = r_k(y, h) := R_k y + \tilde{g}_1(k, y, h(t_k^-, y))$.

II.4.4 Dynamics on Invariant Manifolds

In the most general (nonautonomous) setting, the dynamics on a given invariant manifold can be derived from (II.4.9)–(II.4.10). Set $z(t) = \Phi(t) w(t)$ for $\Phi(t)$ a basis matrix for $\mathcal{R}(P(t))$ and some $w(t) \in \mathbb{R}^{\dim X_c}$. Then, the function w satisfies the impulsive differential equation

$$\dot{w} = \Phi^+(t) P(t) f(t, \Phi(t) w + h(t, \Phi(t_k^-) w)), \qquad t \neq t_k$$
$$\Delta w = \Phi^+(t_k) P(t_k) g_k(\Phi(t_k^-) w + h(t_k^-, \Phi(t_k^-) w)), \qquad t = t_k.$$

The above system essentially describes the nonlinear part of the dynamics on the centre manifold. Indeed, the transformation $z = \Phi(t) w$ quotients out the linear part. However, this transformation is not generally uniformly bounded, so it is difficult to compare growth rates of solutions of the above equation with those on the invariant manifold.

The drawbacks described in the previous paragraph are remedied if the linear part (II.4.3)–(II.4.4) is periodic. In this case, the dynamics on the invariant manifold are topologically equivalent near the origin to

$$\dot{y} = Uy + \tilde{f}_1(t, y, h(t, y)), \qquad t \neq t_k \tag{II.4.28}$$
$$\Delta y = R_k y + \tilde{g}_1(k, y, h(t_k^-, y)), \qquad t = t_k. \tag{II.4.29}$$

Solutions of (II.4.28)–(II.4.29) near the origin are in one-to-one correspondence with those on the invariant manifold. For more information on notions of topological equivalence for impulsive systems, we refer the reader to [28] and the references cited therein.

II.4.5 Reduction Principle for the Centre Manifold

The centre manifold (at zero) contains several important classes of solutions, namely:

- all sufficiently small bounded solutions;
- all sufficiently small periodic solutions.

As a consequence, any small solution or attractor that is formed at a bifurcation point must necessarily be contained within the (parameter-dependent) centre manifold. The following theorem provides more detail.

Theorem II.4.5.1. *Suppose X_u is trivial. There exists a neighbourhood V of $0 \in \mathbb{R}^n$ such that any solution $x : [s, \infty) \to \mathbb{R}^n$ for which $x(t) \in V$ for $t \geq s$ converges exponentially towards $\mathcal{W}_c$. That is, there exists a solution $u(t) \in \mathcal{W}_c(t)$ such that $||x(t) - u(t)|| \leq K_1 e^{-\alpha_1(t-s)}$ for some positive constants K_1 and α_1.*

II.4.6 Approximation by Taylor Expansion

We have discussed a few ways to represent invariant manifolds in this chapter. In the periodic case, we can always express $\mathcal{W}_f$ as the (time-varying) graph of a function $h : \mathbb{R} \times \mathbb{R}^{\dim X_f} \to \mathbb{R}^{n-\dim X_f}$, where $t \mapsto h(t, v)$ is periodic and the Taylor expansion

$$h(t, v) = \frac{1}{2}h_2(t)v^2 + \cdots + \frac{1}{m!}h_m(t)v^m + O(||v||^{m+1})$$

holds uniformly in t near $v = 0$. Each of the coefficients h_j is periodic and differentiable from the right everywhere, with discontinuities only at the impulse times. The idea is to substitute the above Taylor expansion ansatz into the invariance equation (whichever is appropriate to the given situation) and compare powers of v, starting at degree two and proceeding higher until the desired expansion is computed. Since the Taylor coefficients of the invariant manifold are unique, this process yields a unique solution at each order of the expansion. Rather than develop this procedure abstractly, we will consider an example.

Example II.4.6.1. *Consider the following two-dimensional impulsive differential equation:*

$$\begin{aligned}
\dot{u} &= -u + v^2, \quad t \notin \mathbb{Z} & \Delta u &= 0.5u^3, \quad t \in \mathbb{Z} \\
\dot{v} &= v - w^2, \quad t \notin \mathbb{Z} & \Delta v &= -v, \quad t \in \mathbb{Z} \\
\dot{w} &= \alpha uw, \quad t \notin \mathbb{Z} & \Delta w &= 0, \quad t \in \mathbb{Z},
\end{aligned}$$

where $\alpha \in \mathbb{R}$ is a parameter. We will determine the invariance equation for the centre manifold and obtain its Taylor approximation. The first thing to do is to transform the above system into the form (II.4.20)–(II.4.23). This is very nearly complete; the w component corresponds to the centre component for all values of α, while (u, v) corresponds to "leftover" components, z. However, the dynamics are not as simple as they could be, since the $z = (u, v)$ dynamics involve a singular stable direction (the v component) but the continuous-time portion in this direction is nonzero. To fix this, we can use a chain matrix for X_0. This is easily computed: $Q_0(t) = e^{[t]_1}$.

If we set $z = (u, Q_0(t)v)$ and $y = w$, then the above system becomes

$$\begin{aligned}
\dot{y} &= \alpha y z_1, & t &\notin \mathbb{Z}\\
\dot{z} &= \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} e^{2[t]_1} z_2^2 \\ -e^{-[t]_1} y^2 \end{bmatrix}, & t &\notin \mathbb{Z}\\
\Delta y &= 0, & t &\in \mathbb{Z}\\
\Delta z &= \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} z + \begin{bmatrix} 0.5 z_1^3 \\ 0 \end{bmatrix}, & t &\in \mathbb{Z}.
\end{aligned}$$

Compare to (II.4.14)–(II.4.19) for details. The centre manifold can be represented in the form

$$z_1 = h_1(t, y), \quad z_2 = h_2(t, y)$$

for a pair h_1, h_2 of scalar-valued functions that are periodic in their first variable. Writing $h = [\ h^1 \quad h^2\ ]^\intercal$, the invariance equation is

$$\begin{aligned}
\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} h + \begin{bmatrix} e^{2[t]_1}(\pi_2 h)^2 \\ -e^{-[t]_1} y^2 \end{bmatrix} &= \partial_t h + (\partial_y h)\alpha y \pi_1 h, & t &\notin \mathbb{Z}\\
\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} h + \begin{bmatrix} 0.5(\pi_1 h)^3 \\ 0 \end{bmatrix} &= \Delta_t h(t_k, y), & t &\in \mathbb{Z},
\end{aligned} \tag{II.4.30}$$

where $\pi_1 h = h_1$ and $\pi_2 h = h_2$. Note: the function r_k from Theorem II.4.3.1 is identically zero, hence why the partial derivative $\partial_y h$ does not appear in the jump condition of the invariance equation.

Let us compute the fourth-order approximation of the centre manifold. We write

$$h(t, y) = \begin{bmatrix} h_{1,2}(t)y^2 + h_{1,3}(t)y^3 + h_{1,4}(t)y^4 \\ h_{2,2}(t)y^2 + h_{2,3}(t)y^3 + h_{2,4}(t)y^4 \end{bmatrix} + O(|y|^5)$$

for periodic functions $h_{i,j}$ of period one. Substituting the above into (II.4.30) and comparing y^2 coefficients, we get

$$\begin{bmatrix} -h_{1,2} \\ -e^{-[t]_1} \end{bmatrix} = \partial_t \begin{bmatrix} h_{1,2} \\ h_{2,2} \end{bmatrix}, \quad t \notin \mathbb{Z} \qquad \begin{bmatrix} 0 \\ -h_{2,2} \end{bmatrix} = \Delta_t \begin{bmatrix} h_{1,2} \\ h_{2,2} \end{bmatrix}, \quad t \in \mathbb{Z}.$$

The unique periodic solution is $h_{1,2}(t) = 0$ and $h_{2,2}(t) = e^{-[t]_1} - 1$. We can now update our expression for h

$$h(t, y) = \begin{bmatrix} h_{1,3}(t)y^3 + h_{1,4}(t)y^4 \\ (e^{-[t]_1} - 1)y^2 + h_{2,3}(t)y^3 + h_{2,4}(t)y^4 \end{bmatrix} + O(|y|^5).$$

Substituting this into (II.4.30) *and equating cubic terms* y^3*, the result is*

$$\begin{bmatrix} -h_{1,3} \\ 0 \end{bmatrix} = \partial_t \begin{bmatrix} h_{1,3} \\ h_{2,3} \end{bmatrix}, \quad t \notin \mathbb{Z} \qquad \begin{bmatrix} 0 \\ -h_{2,3} \end{bmatrix} = \Delta_t \begin{bmatrix} h_{1,3} \\ h_{2,3} \end{bmatrix}, \quad t \in \mathbb{Z}.$$

The unique periodic solution is the trivial solution $h_{1,3} = h_{2,3} = 0$*. Updating our expression for* h *yet again,*

$$h(t, y) = \begin{bmatrix} h_{1,4}(t)y^4 \\ (e^{-[t]_1} - 1)y^2 + h_{2,4}(t)y^4 \end{bmatrix} + O(|y|^5).$$

Finally, substituting into (II.4.30) *and equating coefficients on* y^4 *terms, the result is*

$$\begin{bmatrix} -h_{1,4} + e^{2[t]_1}(e^{-[t]_1} - 1)^2 \\ 0 \end{bmatrix} = \partial_t \begin{bmatrix} h_{1,4} \\ h_{2,4} \end{bmatrix}, \quad t \notin \mathbb{Z}$$
$$\begin{bmatrix} 0 \\ -h_{2,4} \end{bmatrix} = \Delta_t \begin{bmatrix} h_{1,4} \\ h_{2,4} \end{bmatrix}, \quad t \in \mathbb{Z}.$$

There is a nontrivial periodic solution: $h_{2,4} = 0$ *and*

$$h_{1,4}(t) = \frac{e^{-[t]_1 - 1}}{3(1 - e^{-1})}(e - 1)^3 + e^{-[t]_1} \int_0^{[t]_1} e^s(e^{2s} - 2e^s + 1)ds. \qquad \text{(II.4.31)}$$

Note that $h_{1,4} > 0$*. The latter can be identified with the unique periodic solution of*

$$\dot{q} = -q + e^{2[t]_1}(e^{-[t]_1} - 1)^2.$$

To fourth order, the function h *representing the centre manifold is given by*

$$h(t, y) = \begin{bmatrix} h_{1,4}(t)y^4 \\ (e^{-[t]_1} - 1)y^2 \end{bmatrix} + O(|y|^5),$$

where $h_{1,4}$ *is the positive function from* (II.4.31)*. The dynamics on the centre manifold are topologically conjugate to those of* $\dot{y} = \alpha y h_1(t, y)$*. Substituting in the above expression for* h*, we get*

$$\dot{y} = \alpha h_{1,4}(t)y^5 + O(\alpha|y|^6).$$

Since $h_{1,4}$ *is positive, we conclude that the zero solution of the original impulsive system is* unstable *if* $\alpha > 0$*,* stable *if* $\alpha = 0$*, and* asymptotically stable *if* $\alpha < 0$*. These last two assertions follow by the reduction principle, Theorem II.4.5.1. See Fig. II.4.1 for a comparison.*

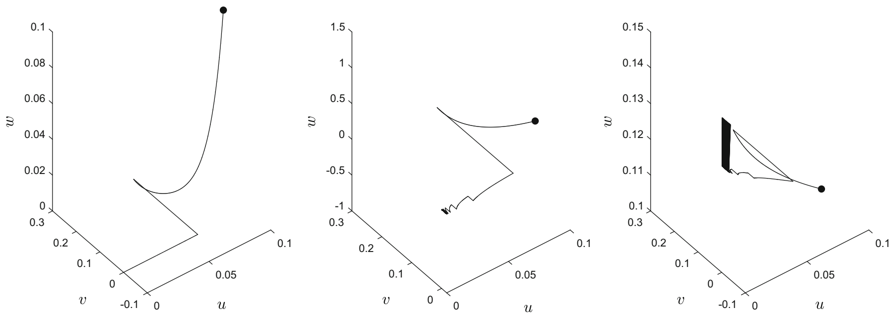

Figure II.4.1: Left to right: simulation from the initial condition $(u, v, w) = (0.1, 0.1, 0.1)$ at time $t = 0$ of the system from Example II.4.6.1 for parameters $\alpha = -100$, $\alpha = 0$ and $\alpha = 2$, respectively. The convergence rate for the case $\alpha < 0$ is incredibly small and numerically unstable, hence our decision to choose a large $\alpha = -100$. Time integration for the cases $\alpha < 0$ and $\alpha = 0$ was done for $t \in [0, 1000]$, and in the $\alpha > 0$ case for $t \in [0, 200]$. In all figures, the black dot denotes the initial condition

II.4.7 Parameter Dependence

In this section we will discuss how one can incorporate parameter-dependent systems into the invariant manifold framework. Suppose we have a system of the form

$$\begin{aligned} \dot{x} &= f(t, x, \epsilon), && t \neq t_k(\epsilon), \\ \Delta x &= g_k(x, \epsilon), && t = t_k(\epsilon), \end{aligned}$$

for a parameter $\epsilon \in \mathbb{R}^p$. We assume this system is periodic with period $T(\epsilon)$ with $q > 0$ impulses per period. Importantly, we assume the number of impulses per period *does not* change depending on the parameter. Suppose that $f(t, 0, 0) = g_k(0, 0) = 0$, so that 0 is an equilibrium point when $\epsilon = 0$. We will assume without loss of generality that $t_k(\epsilon) = 0$.

The first thing we will do is to perform a parameter-dependent rescaling of time so that the impulses occur on the integers. Specifically, set

$$t = t(\tau, \epsilon) = \left\{ \; t_k(\epsilon) + (\tau - k)(t_{k+1}(\epsilon) - t_k(\epsilon)), \quad \tau \in [k, k+1), \; k \in \mathbb{Z}. \right.$$

for rescaled time τ. Under this rescaling, $t = t_k(\epsilon)$ if and only if $\tau = k$. Moreover, $\tau \mapsto t$ is continuous, piecewise-linear and monotone increasing, so it has an inverse with the same properties. If we define $y(\tau) = x(t(\tau, \epsilon))$, then y satisfies the impulsive differential equation

$$\begin{aligned} \frac{dy}{d\tau} &= f(t(\tau, \epsilon), y, \epsilon)(t_{k+1}(\epsilon) - t_k(\epsilon)), && k < \tau < k+1 \\ \Delta y &= g_k(y, \epsilon), && \tau = k \in \mathbb{Z}. \end{aligned}$$

The above system is now periodic with period q, and q impulses per period. Moreover, it has the same level of regularity of the original system—if the original system is PC^m, so is the above. As such, we can always assume without loss of generality that a parameter-dependent system is in the form

$$\dot{x} = f(t, x, \epsilon), \qquad t \notin \mathbb{Z} \tag{II.4.32}$$
$$\Delta x = g_k(x, \epsilon), \qquad t \in \mathbb{Z}, \tag{II.4.33}$$

where the period is $q > 0$ and there are q impulses per period.

Next, we expand the state space of Eqs. (II.4.32)–(II.4.33) by taking ϵ as an additional state. The result is the system

$$\frac{d}{dt}\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} f(t, x, \epsilon) \\ 0 \end{bmatrix}, \quad t \notin \mathbb{Z} \qquad \Delta\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} g_k(x, \epsilon) \\ 0 \end{bmatrix}, \quad t \in \mathbb{Z}.$$

We can now apply the invariant manifold theory to the above system. Indeed, the above is equivalent to the semilinear form

$$\frac{d}{dt}\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} D_x f(t,0,0) & D_\epsilon f(t,0,0) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \epsilon \end{bmatrix} + \begin{bmatrix} F(t,x,\epsilon) \\ 0 \end{bmatrix}, \quad t \notin \mathbb{Z} \tag{II.4.34}$$

$$\Delta\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} D_x g_k(0,0) & D_\epsilon g_k(0,0) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \epsilon \end{bmatrix} + \begin{bmatrix} G_k(x,\epsilon) \\ 0 \end{bmatrix}, \qquad t \notin \mathbb{Z}, \tag{II.4.35}$$

with $F = f(t, x, \epsilon) - D_x f(t, 0, 0)x - D_\epsilon f(t, 0, 0)\epsilon$ and $G_k = g_k(x, \epsilon) - D_x g_k$ $(0, 0)x - D_\epsilon g_k(0, 0)\epsilon$. It follows that $DF(t, 0, 0) = 0$ and $DG_k(0, 0) = 0$ as required.

II.4.7.1 Centre Manifolds Depending on a Parameter

System (II.4.34)–(II.4.35) always has a centre manifold of dimension at least p. If $x = 0$ in (II.4.32)–(II.4.33) at parameter $\epsilon = 0$ is nonhyperbolic with a c-dimensional centre fibre bundle X_c, then the centre manifold of $(x, \epsilon) = (0, 0)$ in (II.4.34)–(II.4.35) will be $(c+p)$-dimensional. Applying the transformation from Sect. II.4.3 and partitioning the equations appropriately, the result will be a q-periodic system in the form

$$\begin{aligned}
\dot{y} &= U_1 y + U_2\epsilon + \tilde{F}_1(t, y, z, \epsilon), && t \notin \mathbb{Z} \\
\dot{z} &= V_1 z + V_2\epsilon + \tilde{F}_2(t, y, z, \epsilon), && t \notin \mathbb{Z} \\
\dot{\epsilon} &= 0, && t \notin \mathbb{Z} \\
\Delta y &= R_1(k)z + R_2(k)\epsilon + \tilde{G}_1(k, y, z, \epsilon), && t \in \mathbb{Z} \\
\Delta z &= S_1(k)z + S_2(k)\epsilon + \tilde{G}_2(k, y, z, \epsilon), && t \in \mathbb{Z} \\
\Delta\epsilon &= 0, && t \in \mathbb{Z},
\end{aligned}$$

where the linear part of the y equation with $\epsilon = 0$ has only Floquet multipliers with absolute value equal to unity, and the Floquet multipliers associated to the linear part of the z component at $\epsilon = 0$ are disjoint from the unit circle. A local centre manifold of the above system at $(0, 0, 0)$ is locally representable by the solution set of the equation

$$z = h(t, y, \epsilon)$$

for $h : \mathbb{R} \times \mathbb{R}^c \times \mathbb{R}^p \to \mathbb{R}^{n-c}$ periodic in its first variable. In the ϵ direction, the dynamics on the centre manifold are trivial since there are no linear or nonlinear terms. However, in the y (centre) direction they are

$$\dot{y} = U_1 y + U_2 \epsilon + \tilde{F}_1(t, y, h(t, y, \epsilon), \epsilon), \qquad t \notin \mathbb{Z} \tag{II.4.36}$$

$$\Delta y = R_1(k) y + R_2(k)\epsilon + \tilde{G}_1(k, y, h(t_k^-, y, \epsilon), \epsilon), \qquad t \in \mathbb{Z}, \tag{II.4.37}$$

for ϵ fixed and sufficiently small. The *local parameter-dependent centre manifold* is the set with t-fibres

$$\mathcal{W}_{c,\epsilon}^{loc}(t) = \{(y, h(t, y, \epsilon)) : ||(y, \epsilon)|| < \delta\}.$$

The dynamics on this invariant manifold are topologically conjugate near $y = 0$ to those of (II.4.36)–(II.4.37), provided $|\epsilon|$ is small enough. The reduction principle (Theorem II.4.5.1) also applies to the parameter-dependent centre manifold, allowing one to derive bifurcation results.

II.4.8 Comments

Taylor approximation of invariant manifolds for nonautonomous ordinary differential in Banach spaces was developed by Pötzsche and Rasmussen [116]. The same authors also developed these techniques for nonautonomous discrete-time systems in [117]. The construction for impulsive differential equations with delays was completed by Church and Liu [33]. The computational (e.g. invariance equation and Taylor expansion) aspects of this chapter can be considered as a specification of the latter results to finite-dimensional systems.

Theorem II.4.1.1 is apparently new. The existence of invariant manifolds in the reversible hyperbolic case—that is, where X_c and X_0 are empty—has been known for some time. See for instance Theorem 6.8 of [9]. In the non-hyperbolic case, there was perhaps a good reason to believe such manifolds existed. Indeed, they can be identified with the forward time evolution of the associated invariant manifold of the time T map. Still, the concrete result of Theorem II.4.1.1 and the representation furnished by Eq. (II.4.6) remained absent.

Linear periodic systems are examples of *reducible* systems. Such systems can be transformed into block form by way of a bounded linear transformation with a bounded inverse, where the blocks induce a natural spectral decomposition. The transformation is called a *kinematic similarity*. A theorem of Siegmund [130] implies that such kinematic similarities always exist for linear ordinary differential equations $\dot{x} = A(t)x$ provided $A(t)$ is locally integrable. A suitable generalization of such a result to nonautonomous impulsive differential equations would permit the derivation of a concrete dynamics equation on the centre manifold analogous to (II.4.26)–(II.4.27) for general nonautonomous differential equations, not necessarily under periodic conditions.