Chapter II.2

Linear Systems

The linear systems theory of this chapter is far from exhaustive, and we will introduce only what is necessary to proceed with stability and invariant manifold theory. The reader is encouraged to consult the 1993 monograph of Bainov and Simeonov [9] for additional background, if desired.

The main object of interest in this chapter is the inhomogeneous linear equation

$$\dot{x} = A(t)x(t) + f(t), \qquad t \neq t_k \tag{II.2.1}$$

$$\Delta x = B_k x(t) + g_k, \qquad t = t_k, \tag{II.2.2}$$

and the associated homogeneous equation

$$\dot{z} = A(t)z(t), \qquad t \neq t_k \tag{II.2.3}$$

$$\Delta z = B_k z(t), \qquad t = t_k. \tag{II.2.4}$$

In what follows, we will always assume that $t \mapsto A(t)$, $t \mapsto f(t)$ are continuous from the right and possess limits on the left. This is sufficient to ensure local existence and uniqueness of solutions forward in time; see Theorem II.1.1.1 and the subsequent remark.

II.2.1 Cauchy Matrix

Let $X(t,s)$ denote the *Cauchy matrix* of the homogeneous ordinary differential equation (II.2.3). That is, $x(t; s, x_0) := X(t,s)x_0$ is the unique solution of (II.2.3) satisfying the initial condition $x(s; s, x_0) = x_0$. The Cauchy matrix has the following (defining) properties.

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- $X(t,t) = I$ for all $t \in \mathbb{R}$.
- $X(t,s)^{-1}$ exists for all $t,s \in \mathbb{R}$, and for $s < t$ we define $X(s,t) \equiv X(t,s)^{-1}$.
- $X(t,s) = X(t,0)X(0,s)$ for all $t,s \in \mathbb{R}$.
- $\frac{d}{dt}X(t,s) = A(t)X(t,s)$ at all arguments t, where A is continuous.
- $X(t,s) = I + \int_s^t A(u)X(u,s)du$ for all $t,s \in \mathbb{R}$.

Using the Cauchy matrix of the continuous part (II.2.3), we can construct the fundamental matrix solution of the impulsive system (II.2.3)–(II.2.4).

Theorem II.2.1.1. *Introduce the matrix-valued function $U(t,s)$ for $t \geq s$ by the equation*

$$U(t,s) = \begin{cases} X(t,s), & t_{k-1} \leq s \leq t < t_k \\ X(t,t_\ell)\left(\prod_{j=\ell}^{k+1}(I+B_j)X(t_j,t_{j-1})\right)(I+B_k)X(t_k,s) & t_{k-1} \leq s < t_k < t_\ell \leq t < t_{\ell+1}. \end{cases}$$

Then, $x(t) := U(t,s)x_0$ is defined on $[s,\infty)$ and is the unique solution of (II.2.3)–(II.2.4) satisfying the initial condition $x(s) = x_0$. If the matrices $I+B_k$ are invertible—that is, $\det(I+B_k) \neq 0$ for all $k \in \mathbb{Z}$—then, $U(s,t) := U(t,s)^{-1}$ is well-defined for all $s \leq t$. In this case, the solution $x(t) = U(t,s)x_0$ is defined on the entire real line. In the above equation, the product denotes multiplication from left to right: $\prod_{j=\ell}^{k+1} M_j = M_\ell M_{\ell-1}\cdots M_{k+2}M_{k+1}$.

Proof. This theorem can be proven by induction on the cardinality of $(s,t] \cap \{t_j : j \in \mathbb{Z}\}$. If this set is empty, then it is clear by definition of $X(t,s)$ that the $t \mapsto U(t,s)x_0 = X(t,s)x_0$ is the unique solution of (II.2.3)–(II.2.4) satisfying the initial condition $x(s) = x_0$, since there are no impulse times in $(s,t]$. Suppose now that the conclusion of the theorem is true for any interval $[s,t]$ such that $(s,t] \cap \{t_j : j \in \mathbb{Z}\}$ has cardinality at most $q \geq 0$. Let $[s,t]$ be any interval such that $|(s,t] \cap \{t_j : j \in \mathbb{Z}\}| = q+1$. Without loss of generality, we may assume

$$(s,t] \cap \{t_j : j \in \mathbb{Z}\} = \{t_1,\ldots,t_{q+1}\}.$$

From the induction hypothesis, the unique solution x of the initial condition $x(s) = x_0$ satisfies

$$x(t_q) = \left(\prod_{j=q}^{2}(I+B_j)X(t_j,t_{j-1})\right)(I+B_1)X(t_1,s)x_0.$$

Then, by definition of $X(t,s)$, the solution x satisfies $x(t_{q+1}^-) = X(t_{q+1}^-, t_q)$ $x(t_q)$. Since $v \mapsto X(v,s)$ is continuous, combining the previous calculation with the jump condition (II.2.4), we get

$$\begin{aligned} x(t_{q+1}) &= (I + B_{q+1})X(t_{q+1}, t_q)x(t_q) \\ &= \left(\prod_{j=q+1}^{2} (I + B_j)X(t_j, t_{j-1}) \right) (I + B_1)X(t_1, s)x_0. \end{aligned}$$

The conclusion follows since $x(t) = X(t, t_{q+1})x(t_{q+1})$. □

Definition II.2.1.1. *The matrix $U(t,s)$ introduced in Theorem II.2.1.1 is called the* Cauchy matrix *associated to the linear homogeneous impulsive differential equation* (II.2.3)*–*(II.2.4)*.*

Corollary II.2.1.1. *The Cauchy matrix enjoys the following properties.*

- $U(t,t) = I$ *for all* $t \in \mathbb{R}$.
- *If* $\det(I + B_k) \neq 0$ *for all* $k \in \mathbb{Z}$*, then* $U(t,s)^{-1}$ *exists for all* $t, s \in \mathbb{R}$*, and for* $s < t$ *we define* $U(s,t) \equiv U(t,s)^{-1}$.
- $U(t_3, t_1) = U(t_3, t_2)U(t_2, t_1)$ *whenever* $t_1 \leq t_2 \leq t_3$*. If the above condition on* $\{B_k : k \in \mathbb{Z}\}$ *holds, the conclusion holds for any* $t_1, t_2, t_3 \in \mathbb{R}$.
- $U(t,s) = I + \int_s^t A(u)U(u,s)du + \sum_{s<t_k\leq t} B_k U(t_k, s)$ *for all* $t \geq s$.
- $U(t_k, s) = (I + B_k)U(t_k^-, s)$ *for all* $s \in \mathbb{R}$*,* $t_k > s$.

II.2.2 Variation-of-Constants Formula

The Cauchy matrix can be used to analytically express the unique solution of the inhomogeneous equation (II.2.1)–(II.2.2) satisfying a given initial condition.

Theorem II.2.2.1. *The unique solution* $t \mapsto x(t; s, x_0)$ *of* (II.2.1)*–*(II.2.2) *satisfying the initial condition* $x(t; t, x_0) = x_0$ *can be expressed in the form*

$$x(t; s, x_0) = U(t,s)x_0 + \int_s^t U(t,\mu)f(\mu)d\mu + \sum_{s<t_k\leq t} U(t, t_k)g_k. \qquad \text{(II.2.5)}$$

Proof. Under the assumption that $A(t)$ and $f(t)$ are merely continuous from the right with limits on the left, we cannot prove that (II.2.5) is a solution of (II.2.1)–(II.2.2) by computing a derivative because a priori, this function is not differentiable. We can, however, easily check that it satisfies the jump condition. At times t_k, we have

$$\begin{aligned}
&x(t_k) - x(t_k^-) \\
&\quad = [U(t_k, s) - U(t_k^-, s)]x_0 + \int_s^{t_k} [U(t_k, \mu) - U(t_k^-, \mu)]f(\mu)d\mu \\
&\qquad + g_k + \sum_{s<t_j<t_k} [U(t_k, t_j) - U(t_k^-, t_j)]g_j \\
&\quad = B_k U(t_k^-, s)x_0 + \int_s^t B_k U(t_k^-, s)f(\mu)d\mu + g_k + \sum_{s<t_j<t_k} B_k U(t_k^-, t_j) \\
&\quad = B_k x(t_k^-) + g_k,
\end{aligned}$$

as required. Next, we prove that on each interval (t_j, t_{j+1}) for $s < t_j$, the variation-of-constants formula (II.2.5) is correct. Without loss of generality, assume $s = t_0$. For $\mu \in (t_0, t_1)$, we have $U(t, \mu) = X(t, \mu)$. Then, with $x_0 = x(t_0)$ and $t \in (t_0, t_1)$,

$$\begin{aligned}
&x_0 + \int_{t_0}^t (A(\mu)x(\mu) + f(\mu))d\mu \\
&\quad = x_0 + \int_{t_0}^t \left(A(\mu)\left[U(\mu, t_0)x_0 + \int_{t_0}^\mu U(\mu, v)f(v)dv\right] + f(\mu)\right) d\mu \\
&\quad = x_0 + \int_{t_0}^t A(\mu)X(\mu, t_0)d\mu x_0 + \int_{t_0}^t \int_{t_0}^\mu A(\mu)X(\mu, v)f(v)dv d\mu + \int_{t_0}^t f(\mu)d\mu \\
&\quad = x_0 + (X(t, t_0) - I)\, x_0 + \int_{t_0}^t \int_v^t A(\mu)X(\mu, v)d\mu f(v)dv + \int_{t_0}^t f(\mu)d\mu \\
&\quad = X(t, t_0)x_0 + \int_{t_0}^t (X(t, v) - I)f(v)dv + \int_{t_0}^t f(\mu)d\mu \\
&\quad = U(t, t_0)x_0 + \int_{t_0}^t U(t, v)f(v)dv = x(t),
\end{aligned}$$

as required by definition of solution. By the previous computation, we have

$$\begin{aligned}
x(t_1) &= (I + B_k)x(t_1^-) + g_1 \\
&= (I + B_1)X(t_1, t_0)x_0 + \int_{t_0}^{t_1} (I + B_1)X(t_1, \mu)f(\mu)d\mu + g_1 \\
&= U(t_1, t_0)x_0 + \int_{t_0}^{t_1} U(t_1, \mu)f(\mu)d\mu + U(t_1, t_1)g_1.
\end{aligned}$$

Equation (II.2.5) therefore holds on $[t_0, t_1]$. Assuming now that the variation-of-constants formula is correct for $t \in [t_0, t_k]$ for some $k \geq 1$, the same proof can be used to show that for $t \in (t_k, t_{k+1})$,

$$x(t) = x(t_k) + \int_{t_k}^{t} X(t,\mu) f(\mu) d\mu.$$

Substituting in the expression for $x(t_k)$ guaranteed by the induction hypothesis, one obtains (II.2.5) for $t \in (t_k, t_{k+1})$. At $t = t_{k+1}$, one uses the relation $x(t_{k+1}) = (I + B_k)x(t_{k+1}^-) + g_{k+1}$, and the result is after some simplification equivalent to (II.2.5). □

II.2.3 Stability

We recall now the definition of (Lyapunov) stability.

Definition II.2.3.1. *The inhomogeneous system* (II.2.1)–(II.2.2) *is*

- exponentially stable *if there exist* $K > 0$*,* $\alpha > 0$ *and* $\delta > 0$ *such that for all* $\phi, \psi \in \mathbb{R}^n$ *satisfying* $||\phi - \psi|| < \delta$*, one has* $||x(t; s, \phi) - x(t; s, \psi)|| \leq K||\phi - \psi||e^{-\alpha(t-s)}$ *for all* $t \geq s$*;*
- stable *if for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $\phi, \psi \in \mathbb{R}^n$ *satisfying* $||\phi - \psi|| < \delta$*, one has* $||x(t; s, \phi) - x(t; s, \psi)|| < \epsilon$ *for all* $t \geq s$*;*
- unstable *if it is not stable.*

Lemma II.2.3.1. *The inhomogeneous system* (II.2.1)–(II.2.2) *is stable (respectively, exponentially stable or unstable) if and only if the same is true for the associated homogeneous system* (II.2.3)–(II.2.4)*.*

Proof. One can easily verify from the variation-of-constants formula that $y(t) := x(t; s, \phi) - x(t; s, \psi)$ is a solution of the homogeneous system (II.2.3)–(II.2.4) satisfying $y(s) = \phi - \psi$. If the latter system is stable (respectively, exponentially stable), then in particular the difference between the trivial solution 0 and $y(t)$ can be bounded appropriately provided $||(\phi - \psi) - 0|| = ||\phi - \psi|| < \delta$ for some delta, which grants the stability assertion. The instability part follows similarly, as does the converse (that is, the stability of the inhomogeneous system implies the same for the homogeneous system). □

The above lemma states that, insofar as stability of linear systems is concerned, one needs to only consider homogeneous systems.

II.2.4 Exponential Trichotomy

Of use in later sections will be exponential trichotomy—referred to as spectral separation in Part I of this text.

Definition II.2.4.1. *The homogeneous system* (II.2.3)–(II.2.4) *has exponential trichotomy if there exist projection-valued functions* $t \mapsto P_c(t)$ *and* $t \mapsto P_u(t)$ *on* $\mathbb{R}^n$ *such that its Cauchy matrix* $U(t,s)$ *satisfies the following:*

1. $\sup_{t\in\mathbb{R}} ||P_c(t)|| + ||P_u(t)|| = N < \infty$.

2. $P_c(t)P_u(t) = P_u(t)P_c(t) = 0$.

3. $U(t,s)P_j(s) = P_j(t)U(t,s)$ *for all* $t \geq s$ *and* $j \in \{c,u\}$.

4. *Define* $U_j(t,s)$ *as the restriction of* $U(t,s)$ *to* $X_j(s) = \mathcal{R}(P_j(s))$ *for* $j \in \{c,u,s\}$, *where we set* $P_s = I - P_c - P_u$. *The linear maps* $U_j(t,s) : X_j(s) \to X_j(t)$ *are invertible for* $j \in \{c,u\}$, *and we denote* $U_j(s,t) = U_j(t,s)^{-1}$ *for* $t \geq s$.

5. *For all* $t,s,v \in \mathbb{R}$, $U_j(t,s) = U_j(t,v)U_j(v,s)$ *for* $j \in \{c,u\}$.

6. *There exist real numbers* $a < 0 < b$ *such that for all* $\epsilon > 0$, *there exists* $K \geq 1$ *such that*

$$||U_u(t,s)|| \leq Ke^{b(t-s)}, \qquad t \leq s \tag{II.2.6}$$

$$||U_c(t,s)|| \leq Ke^{\epsilon|t-s|}, \qquad t,s \in \mathbb{R} \tag{II.2.7}$$

$$||U_s(t,s)|| \leq Ke^{a(t-s)}, \qquad t \geq s. \tag{II.2.8}$$

Definition II.2.4.2. *Let* (II.2.3)–(II.2.4) *have exponential trichotomy. Define the sets* $X_j = \{(t,x) : t \in \mathbb{R}, x \in \mathcal{R}(P_j(t))\}$ *for* $j \in \{s,c,u\}$. X_s, X_c *and* X_u *are, respectively, the* stable, centre and unstable fibre bundles.

$$X_{cs} = \{(t, x+y) : x \in X_c(t),\ y \in X_s(t)\} = \{(t,x) : x \in \mathcal{R}(P_c(t) + P_s(t))\}$$
$$X_{cu} = \{(t, x+y) : x \in X_c(t),\ y \in X_u(t)\} = \{(t,x) : x \in \mathcal{R}(P_c(t) + P_u(t))\}$$

are, respectively, the centre-stable *and* centre-unstable *fibre bundles. For each of these, the* t-fibre *is the set* $X_j(t) = \{x : (t,x) \in X_j\}$.

The fibre bundles introduced in the above definition play the role of the invariant subspaces from autonomous ordinary differential equations. There are simpler descriptions of these objects available—in particular, one can define an equivalent time-invariant description—if $\det(I + B_k) \neq 0$ for all $k \in \mathbb{Z}$, since then the dynamics are reversible. Since we do not assume this, we will stick with the definition above.

II.2.5 Floquet Theory

The Floquet theory allows for the transformation of a periodically driven homogeneous system into an autonomous ordinary differential equation. This will be helpful later when we consider invariant manifold theory. In this section we begin with the homogeneous equation before proceeding to inhomogeneous equations. First, two definitions are as follows.

Definition II.2.5.1. *The inhomogeneous system* (II.2.1)–(II.2.2) *is* periodic *if there exist real* $T > 0$ *and* $c \in \mathbb{N}$ *such that* $A(t+T) = A(t)$, $f(t+T) = f(t)$, $B_{k+c} = B_k$, $g_{k+c} = g_k$ *and* $t_{k+c} = t_k + T$ *for all* $t \in \mathbb{R}$ *and* $k \in \mathbb{Z}$*. The* period *is* T*, and the* number of impulses per period *is* c*.*

Definition II.2.5.2. *The homogeneous system* (II.2.1)–(II.2.2) *is* periodic *if there exist real* $T > 0$ *and* $c \in \mathbb{N}$ *such that* $A(t+T) = A(t)$, $B_{k+c} = B_k$ *and* $t_{k+c} = t_k + T$ *for all* $t \in \mathbb{R}$ *and* $k \in \mathbb{Z}$*. The* period *is* T*, and the* number of impulses per period *is* c*.*

II.2.5.1 Homogeneous Systems

Definition II.2.5.3. *Suppose the homogeneous system* (II.2.3)–(II.2.4) *is periodic (with period* T*). Each of* $M_t := U(t+T, t)$ *for* $t \in \mathbb{R}$ *is called a* monodromy matrix*. The eigenvalues are called* Floquet multipliers*.*

Proposition II.2.5.1. *If the homogeneous system* (II.2.3)–(II.2.4) *is periodic (with period* T*), then* $U(t+T, s+T) = U(t, s)$ *for all* $t \geq s$*.*

Proof. This follows from existence and uniqueness of solutions together with periodicity (period T). □

Lemma II.2.5.1. *For any* $t, s \in \mathbb{R}$*,* M_t *and* M_s *have the same eigenvalues.*

Proof. First suppose $t \geq s$. The monodromy matrices satisfy the equation

$$M_t U(t, s) = U(t, s) M_s.$$

Suppose v eigenvalue of M_s with eigenvalue $\mu \neq 0$. Then,

$$M_t(U(t, s)v) = U(t, s)M_s v = U(t, s)\mu v = \mu(U(t, s)v),$$

so $U(t, s)v$ is an eigenvector of M_t with the same eigenvalue, provided $w := U(t, s)v \neq 0$. If $w = 0$, then $M_s v = U(s+T, t)U(t, s)v = U(s+T, t)w = 0$, which would contradict v being an eigenvector of M_s. On the other hand, 0 is an eigenvalue of M_s if and only if there is at least one $k \in \{0, \ldots, c-1\}$ such that $\det(I + B_k) = 0$, which is then equivalent to 0 being an eigenvalue of M_t. From Lemma II.2.5.1, we have

$$M_{s+T} = U(s+2T, s+T) = U(s+T, s) = M_s.$$

Therefore, M_s and M_{s+T} have the same eigenvalues. If $\sigma(M)$ denotes the set of eigenvalues of M, then the previous results imply the inclusions $\sigma(M_t) \subseteq \sigma(M_s)$ for $t \geq s$ and $\sigma(M_s) = \sigma(M_{s+jT})$ for all $j \geq 0$. Together, these imply $\sigma(M_s) = \sigma(M_t)$. □

As a consequence of the previous lemma, the following definition is reasonable.

Definition II.2.5.4. *The* Floquet multipliers of the linear system (II.2.3)–(II.2.4) *are the eigenvalues of the monodromy matrix M_0. The latter is given by*

$$M_0 = \prod_{k=c}^{1}(I + B_k)X(t_k, t_{k-1}). \tag{II.2.9}$$

Theorem II.2.5.1 (Floquet Decomposition)**.** *Let $\mathcal{B} := \{\xi_1, \ldots, \xi_p\}$ be a union of canonical bases for the direct sum of generalized eigenspaces of M_0 with nonzero eigenvalues. The restriction of $U(t,0)$ to span$\{\xi_1, \ldots, \xi_p\}$ is invertible for $t \geq 0$. Let $\Phi(t)$ be the $n \times p$ matrix whose jth column is $\Phi_j(t) = U(t,0)\xi_j$, defined for $t \in \mathbb{R}$. There exist a $p \times p$ complex matrix Λ and a T-periodic $n \times p$ complex matrix $Q(t)$ such that $\Phi(t) = Q(t)e^{t\Lambda}$ for all $t \geq 0$. The eigenvalues of Λ are*

$$\sigma(\Lambda) = \left\{\frac{1}{T}\log(\mu) : \mu \textit{ is a nonzero Floquet multiplier}\right\}.$$

Proof. Note, since $U(T,0)$ is invertible on $\mathcal{B}$ (generalized eigenspaces of a matrix are invariant under its action), it suffices to prove first that $U(t,0)$ is invertible for $t \in (0,T)$. Suppose not, then there exist $t \in (0,T)$ and ξ_j such that $U(t,0)\xi_j = 0$. But this implies $U(T,0)\xi_j = 0$, and since $M_0 = U(T,0)$, we conclude ξ_j is an eigenvector with eigenvalue zero. As the generalized eigenspaces are disjoint, we have obtained a contradiction.

Since $M_0 = U(T,0)$ is invertible on span$\{\xi_1, \ldots, \xi_p\}$, there exists an invertible $p \times p$ matrix V such that $\Phi(T) = \Phi(0)V$. Define $\Lambda = \frac{1}{T}\log V$, where the logarithm is any branch that defined on the spectrum of V. Define $Q(t) = \Phi(t)e^{-t\Lambda}$. By definition, we have $\Phi(t) = Q(t)e^{t\Lambda}$. For periodicity, we observe

$$\begin{aligned} Q(t+T) &= \Phi(t+T)e^{-(t+T)\Lambda} = U(t+T,T)\Phi(T)V^{-1}e^{-t\Lambda} \\ &= U(t,0)\Phi(0)VV^{-1}e^{-t\Lambda} = \Phi(t)e^{-t\Lambda} = Q(t), \end{aligned}$$

as required. The last thing to prove is the characterization of the spectrum of Λ. Let $\nu \in \mathcal{B}$ be a generalized eigenvector of rank m with eigenvector μ for M_0. Since $\mathcal{B}$ is a union of canonical bases, there is a Jordan chain $\{\nu_1, \ldots, \nu_m\} \subseteq \mathcal{B}$ such that $\nu_j = (M_0 - \mu I)\nu_{j+1}$ and $\nu_m = \nu$. Relative to the basis $\mathcal{B}$, we can write $\nu_j = \xi_{r_j}$ for some new index r_j so that the previous equation becomes $\xi_{r_j} = (M_0 - \mu I)\xi_{r_{j+1}}$. The right-hand side can be written as

$$\begin{aligned} (M_0 - \mu I)\xi_{r_{j+1}} &= U(T,0)\xi_{r_{j+1}} - \mu\xi_{r_{j+1}} = \Phi(T)e_{r_{j+1}} - \mu\xi_{r_{j+1}} \\ &= \Phi(0)Ve_{j+1} - \mu\xi_{r_{j+1}}. \end{aligned}$$

Since $\Phi(0)$ has linearly independent columns, the left-inverse $\Phi^+(0)$ exists. Then, since $\xi_j = \Phi(0)e_j$, multiplying $\Phi^+(0)$ on the left on both sides of $\xi_{r_j} = (M_0 - \mu I)\xi_{r_{j+1}}$, it follows that

$$e_{r_j} = (V - \mu I)e_{r_{j+1}}.$$

We conclude that $e_{r_m}, \ldots, e_{r_1}$ is a Jordan chain for eigenvalue μ of V. Since $\Lambda = \frac{1}{T}\log(V)$, the result follows. □

Remark II.2.5.1. *One can replace $\mathcal{B}$ with any basis for the direct sum of generalized eigenspaces of M_0 with nonzero eigenvalues. Indeed, the only place we used the previous description of $\mathcal{B}$ was in determining the spectrum of Λ. If one writes $\tilde{\Phi}(0) = \tilde{\Phi}(0)Z$ for invertible Z, where the columns of $\tilde{\Phi}(0)$ are a canonical basis of Jordan chains, and defines $\tilde{\Phi}(t) = U(t,0)\tilde{\Phi}(0)$, then one can apply the theorem directly to $\tilde{\Phi}(t) = \tilde{Q}(t)e^{t\tilde{\Lambda}}$, where $\tilde{\Phi}(T) = \tilde{\Phi}(0)\tilde{V}$ and $\tilde{\Lambda} = \frac{1}{T}\log\tilde{V}$. However, if $\Phi(T) = \Phi(0)V$, then $\tilde{\Phi}(T) = \tilde{\Phi}(0)ZVZ^{-1}$, so $\tilde{V}$ and V are similar and thus have the same eigenvalues. The same therefore holds for $\tilde{\Lambda}$ and $\Lambda = \frac{1}{T}\log V$.*

Corollary II.2.5.1. *With the notation of Theorem II.2.5.1, introduce a family of subspaces $\mathcal{X}_t$ of $\mathbb{R}^n$ indexed by $t \in \mathbb{R}$ as follows:*

$$\mathcal{X}_t = \{U(t,0)x : x \in \operatorname{span}(\mathcal{B})\}.$$

The nonautonomous dynamical system $U(t,s) : \mathcal{X}_s \to \mathcal{X}_t$ is equivalent to the ordinary differential equation

$$\dot{y} = \Lambda y \tag{II.2.10}$$

under the time-periodic change of variables $x(t) = Q(t)y(t)$. More precisely, any solution $x : \mathbb{R} \to \mathbb{R}^n$ of (II.2.3)–(II.2.4) such that $x(t) \in \mathcal{X}_t$ for all $t \in \mathbb{R}$ can be written in the form $x(t) = Q(t)y(t)$, where y is a solution of (II.2.10).

Proof. Let x be a solution of (II.2.3)–(II.2.4) such that $x(t) \in \mathcal{X}_t$. Write $x(0) = \Phi(0)h$ for some $h \in \mathbb{R}^p$. By uniqueness of solutions, $x(t) = U(t,0)\Phi(0)h = \Phi(t)h$, so by Theorem II.2.5.1, we can write $x(t) = Q(t)e^{t\Lambda}h$. With $y = e^{t\Lambda}h$, the claim is proven. □

Corollary II.2.5.2. *If $p = n$—that is, $\mathcal{B}$ is a basis for $\mathbb{R}^n$—the time-periodic change of coordinates $x = Q(t)y$ transforms the ordinary impulsive differential equation (II.2.3)–(II.2.4) into the autonomous ordinary differential equation (II.2.10). In this case, $|Q(t)|$ and $|Q^{-1}(t)|$ are both bounded. $\mathcal{B}$ is a basis for $\mathbb{R}^n$ if and only if $\det(I + B_k) \neq 0$ for $k = 0, \ldots, c-1$.*

Proof. The first part follows by Corollary II.2.5.1. As for the second part, since $\mathcal{B}$ is a basis for the direct sum of generalized eigenspaces of M_0 with nonzero eigenvalue, the assertion that $\mathcal{B}$ is a basis for $\mathbb{R}^n$ is equivalent to 0 not being an eigenvalue of M_0. Since $M_0 = \prod_{k=c}^{1}(I + B_k)X(t_k, t_{k-1})$ and each of $X(t_k, t_{k-1})$ has full rank, zero can only be an eigenvalue of $\det(I + B_k) = 0$ for at least one $k \in \{1, \ldots, c\}$. Since $B_{k+c} = B_k$, this proves the claim. □

Corollary II.2.5.2 is an analogue of the Floquet theorem from ordinary differential equations. It appears in the 1993 monograph of Bainov and Simeonov [9]. Theorem II.2.5.1 is the generalization of the Floquet decomposition to the case where the jump maps $x \mapsto x + B_k x$ to not be one-to-one, while Corollary II.2.5.1 gives the change of variables on the "non-singular fibre bundle" $\mathcal{X}_t$ that renders the dynamics autonomous.

In essentially the same way Theorem II.2.5.1 is proven, we can establish a more general version that roughly states that for a homogeneous linear periodic system, the dynamics on any of its invariant fibre bundles—except for the "singular" portion of the stable fibre bundle or centre-stable fibre bundle—are driven by an autonomous ordinary differential equation. The proof is omitted. First, a quick definition is as follows.

Definition II.2.5.5. *Let $X_f(t)$ be one of the following:*

- *one of the centre, unstable or centre-unstable fibre bundles;*
- *the* reversible stable fibre bundle*, $X_s^\infty(t) = \{\xi \in X_s(t) : \forall t' > t,\ U(t',t)\ \xi \neq 0\}$;*
- *the* reversible centre-stable fibre bundle*, $X_{cs}^\infty(t) = X_c(t) \oplus X_s^\infty(t)$.*

Let $\{\xi_1, \ldots, \xi_p\}$ be a basis for $X_f(0)$. The matrix-valued function $\Phi_f(t) = U(t,0)[\xi_1, \cdots, \xi_p]$ is a basis matrix for *X_f.*

Theorem II.2.5.2. *Let X_f be one of the following fibre bundles:*

- *one of the centre, unstable or centre-unstable fibre bundles;*
- *the reversible stable fibre bundle or reversible centre-stable fibre bundle.*

The restriction of $U(t,0)$ to any basis for $X_f(0)$ is invertible for $t \geq 0$, so any basis matrix $\Phi_f(t)$ for X_f can be uniquely extended to the entire real line. There exist a $p \times p$ complex matrix Λ_f and a T-periodic $n \times p$ complex matrix $Q_f(t)$ such that $\Phi(t) = Q_f(t)e^{t\Lambda_f}$ for all $t \geq 0$. The eigenvalues of Λ_f are

$$\sigma(\Lambda) = \left\{ \frac{1}{T} \log(\mu) : M_0 \xi = \mu \xi,\ \xi \in X_f(0) \right\}.$$

As defined in the above theorem, $X_s^\infty(t)$ is spanned by the generalized eigenvectors of M_t having with Floquet multipliers μ satisfying $0 < |\mu| < 1$. As a consequence, if $\det(I + B_k) \neq 0$ for $k = 0, \ldots, c-1$, then $X_s^\infty = X_s$.

II.2.5.2 Periodic Solutions of Homogeneous Systems

The Floquet multipliers allow us to identify periodic solutions of the homogeneous system.

Proposition II.2.5.2. *The homogeneous system* (II.2.3)–(II.2.4) *has a nontrivial jT-periodic solution for $j \in \mathbb{N}$ if and only if there exists a Floquet multiplier μ satisfying $\mu^j = 1$. In this case, the jT-periodic solutions are precisely $x(t) = U(t,0)\xi$, where $\xi \in \mathbb{R}^n$ satisfies $M_0^j \xi = \xi$.*

Proof. $x(t)$ is a nontrivial periodic solution of period jT if and only if $x(jT) = x(0) \neq 0$, which is equivalent to the equation $U(jT,0)x(0) = x(0)$. Since $U(jT,0) = M_0^j$, this implies that $x(0)$ satisfies the equation $M_0^j x(0) = x(0)$, so that 1 is an eigenvalue of M_0^j. Since the eigenvalues of M_0^j are the jth powers of the eigenvalues of M_0, there must be a Floquet multiplier μ satisfying $\mu^j = 1$. □

II.2.5.3 Periodic solutions of Inhomogeneous Systems

From the variation-of-constants formula, a periodic solution $x(t)$ is uniquely determined by its value at time $t = 0$. Indeed, starting from the variation-of-constants formula (II.2.5), setting $s = 0$ and assuming the periodic ansatz $x(0) = x(T)$, the necessary and sufficient condition for the existence of a periodic solution is that there exists a solution x_0 of the equation

$$(U(T,0) - I)x_0 = \int_0^T U(T,\mu)f(\mu)d\mu + \sum_{0<t_k\leq T} U(T,t_k)g_k. \qquad \text{(II.2.11)}$$

The matrix on the left-hand side will be invertible precisely if 1 is not an eigenvalue of $U(T,0)$. Since the latter is precisely the monodromy matrix M_0, we obtain the following lemma.

Lemma II.2.5.2. *The inhomogeneous equation* (II.2.1)–(II.2.2) *has a unique T-periodic solution if and only if* 1 *is not a Floquet multiplier of the associated homogeneous equation; that is,* $\det(M_0 - I) \neq 0$.

More generally, one might want to know under what conditions there is a jT-periodic solution for natural number j. The ansatz $x(jT) = 0$ leads to the equation

$$(U(jT,0) - I)x_0 = \int_0^{jT} U(jT,\mu)f(\mu)d\mu + \sum_{0<t_k\leq jT} U(jT,t_k)g_k.$$

Since $U(jT,0) = U(T,0)^j = M_0^j$, the previous lemma has the following simple generalization.

Theorem II.2.5.3. *The inhomogeneous equation* (II.2.1)–(II.2.2) *has a unique jT-periodic solution if and only if no Floquet multiplier μ of the associated homogeneous equation is a jth root of unity; that is, $\mu^j \neq 1$ for all $\mu \in \sigma(M_0)$.*

If $1 \in \sigma(M_0)$, there will be either infinitely many periodic solutions or none, depending on whether the right-hand side of (II.2.11) is in the range of $M_0 - I$. Similar conclusions hold for jT-periodic solutions. Existence of periodic solutions in the critical case where $\det(M_0 - I) = 0$ is discussed in Bainov and Simeonov [9], and we refer the interested reader to this resource.

II.2.5.4 Periodic Systems Are Exponentially Trichotomous

The invariant fibre bundles of a periodic system induce an exponential trichotomy.

Theorem II.2.5.4. *The periodic system* (II.2.3)–(II.2.4) *has exponential trichotomy. The projectors* P_c*,* P_u *and* $P_s = I - (P_c + P_u)$ *are projections onto the centre, unstable and stable fibre bundles* X_c*,* X_u *and* X_s*, respectively. These projectors are also periodic with period* T*.*

Proof Outline. Define $P_j(t)$ by the integral

$$P_j(t) = \frac{1}{2\pi i}\int_{\Gamma_j}(zI - M_t)^{-1}dz,$$

where Γ_j is a simple closed contour in $\mathbb{C}$ such that the only eigenvalues μ of M_0 contained in its closure are, respectively, those with $|\mu| > 0$ for $j = c$, $|\mu| = 1$ for $j = u$ and $|\mu| < 1$ for $j = s$, oriented counterclockwise relative to its interior. One can show that with this choice of projections, all properties of exponential trichotomy are satisfied. The proof is quite long; see Theorem I.3.1.3 for details. □

Sometimes it is desirable to have an explicit formula for one of the projections $P(t)$ onto an invariant fibre bundle. When there are $c = 1$ impulses per period, we have a fairly nice formula. Let $M_0 = VJV^{-1}$ be the Jordan canonical form of the monodromy matrix M_0, and let $X(t, s)$ be the Cauchy matrix of the continuous part $\dot{x} = A(t)x$. Let $t \in [t_0, t_0 + T)$. Then, we have

$$\begin{aligned} M_t &= X(t+T, t_0+T)[I + B_1]X(t_0+T, t) = X(t, t_0)[I + B]X(t_0+T, t_0)X^{-1}(t, t_0) \\ &= X(t, t_0)M_0X^{-1}(t, t_0) = V(t)JV^{-1}(t), \end{aligned}$$

where we set $V(t) = X(t, t_0)V$. The projection $P_j(t)$ can then be equivalently written in the form

$$P_j(t) = V(t)\left[\frac{1}{2\pi i}\int_{\Gamma_j}(zI - J)^{-1}dz\right]V^{-1}(t), \qquad \text{(II.2.12)}$$

where the contour Γ_j is as stated in Theorem II.2.5.4. The contour integral in (II.2.12) is easy to evaluate because J is a Jordan matrix and the integrand no longer depends on t. Since P_j is periodic, it is enough to compute it for $t \in [t_0, t_0 + T)$.

II.2.5.5 Stability

The following theorem characterizes the stability of the homogeneous system in terms of the Floquet multipliers. It follows directly from the associated infinite-dimensional version, Theorem I.3.3.1, from Part I of this text. The proof is identical apart from symbolic changes and slight changes to presentation, so it will be omitted.

Theorem II.2.5.5. *The homogeneous system* (II.2.3)–(II.2.4) *is exponentially stable if and only if all Floquet multipliers* μ *satisfy* $|\mu| < 1$. *It is stable if and only if all Floquet multipliers satisfy* $|\mu| \leq 1$, *and to those Floquet multipliers satisfying* $|\mu| = 1$, *the generalized eigenspaces contain only rank 1 eigenvectors—equivalently, each block in the complex Jordan form of* M_0 *corresponding to one of the Floquet multipliers satisfying* $|\mu| = 1$ *is one-dimensional.*

Stability for periodic linear systems is therefore completely determined by the Floquet multipliers—that is, the eigenvalues μ of M_0. These satisfy the *characteristic equation*

$$\det(M_0 - \mu I) = 0. \tag{II.2.13}$$

Recall that M_0 is given explicitly by (II.2.9). There is, however, another way to compute the Floquet multipliers. The following proposition is a direct consequence of Theorem II.2.5.1.

Proposition II.2.5.3. *If* $M_0\xi = \mu\xi$ *and* $\mu \neq 0$, *the function* $x(t) = U(t,0)\xi$ *can be written in the form* $x(t) = q(t)e^{\lambda t}$, *where* $\lambda = \frac{1}{T}\log\mu$ *and* q *is (generally) complex-valued and* T*-periodic. Conversely, if* $x(t) = q(t)e^{\lambda t}$ *is a solution of* (II.2.3)–(II.2.4) *with* q *a complex-valued* T*-periodic function, then* $\mu = e^{T\lambda}$ *is a Floquet multiplier and* $M_0 q(0) = \mu q(0)$.

Let us substitute the ansatz $x(t) = q(t)e^{\lambda t}$ into (II.2.3)–(II.2.4). After some cancellation, one arrives at the following impulsive differential equation for q:

$$\lambda q + \dot{q} = A(t)q, \qquad t \neq t_k \tag{II.2.14}$$

$$\Delta q = B_k q, \qquad t = t_k. \tag{II.2.15}$$

Let X_λ denote the Cauchy matrix of the continuous part of Eq. (II.2.14). That is, $X_\lambda(t,s)$ satisfies $X_\lambda(t,t) = I$ for $t \in \mathbb{R}$ and

$$\frac{d}{dt}X_\lambda(t,s) = (A(t) - \lambda I)X_\lambda(t,s).$$

By Proposition II.2.5.2, system (II.2.14)–(II.2.15) has a T-periodic solution if and only if

$$\det\left(\prod_{k=c}^{1}(I + B_k)X_\lambda(t_k, t_{k-1}) - I\right) = 0. \tag{II.2.16}$$

Notice that the product term is precisely the monodromy matrix M_0 for (II.2.14)–(II.2.15). If one can compute all solution λ of the equation (II.2.16), then one can compute the Floquet multipliers $\mu = e^{T\lambda}$. The numbers λ have a special name.

Definition II.2.5.6. *The complex numbers λ that solve* (II.2.16) *are the* Floquet exponents*. The set of all Floquet exponents is denoted $\lambda(U)$ and is called the* Floquet spectrum*.*

Equation (II.2.16) will have infinitely many solutions because $\lambda = \frac{1}{T}\log\mu$ and the logarithm has infinitely many branches. Namely, if λ is a Floquet exponent, then so is $\lambda + \frac{2\pi i}{T}$. As such, when solving (II.2.16), one should focus only on solutions in the strip

$$\left\{\lambda \in \mathbb{C} : \Im(\lambda) \in \left[0, \frac{2\pi}{T}\right)\right\}.$$

II.2.6 Generalized Periodic Changes of Variables

The changes of variables we introduced in Sect. II.2.5 transform some or all components of a periodic impulsive system into an autonomous ordinary differential equation. The downside is that the resulting ordinary differential equation might be complex-valued. In this section we consider other periodic changes of variables that will be useful in later applications.

II.2.6.1 A Full State Transformation and Chain Matrices

Corollary II.2.5.1 grants a transformation that very nearly renders the dynamics of (II.2.3)–(II.2.4) autonomous. The barrier is the *singular fibre bundle*, X_0, whose t-fibres are given by

$$X_0(t) = \{\xi \in X_s(t) : \exists t' > t : U(t', t)\xi = 0\}.$$

Denote $P_0(t)$ the projection onto $X_0(t)$. To any solution $x : \mathbb{R} \to \mathbb{R}^n$ such that $P_0(s)x(s) \neq 0$, there necessarily exist some $t' > s$ such that $P_0(t)x(t) = 0$ for all $t \geq t'$. This suggests we form a basis matrix of $X_0(t)$ not in the way that is done in Theorem II.2.5.2, but rather in a piecewise fashion.

Definition II.2.6.1. *Let $\Psi_0, \ldots, \Psi_{c-1}$ denote matrices whose columns are bases for the t_j-fibres $X(t_j)$ of a fibre bundle X. Define for $k \in \{0, \ldots, c-1\}$ and $t \in [t_k, t_{k+1})$ the matrix $Q(t) = U(t, t_k)\Psi_k$. Then, extend Q to $Q : \mathbb{R} \to \mathbb{R}^{n\times q}$ by periodicity, where $q = \dim X_0(0)$. We will call Q a* chain matrix *for X.*

We can now apply Theorem II.2.5.2 and pose a transformation that maps x into its components in each of X_s^∞, X_c, X_u and X_0. These components will be decoupled, and the dynamics for all components aside from X_0 will be autonomous.

Theorem II.2.6.1. *Let Q_0 be a chain matrix for X_0. Let Φ_s, Φ_c and Φ_u be basis matrices for X_s^∞, X_c and X_u, respectively. The change of variables $x = Q_s y + Q_c z + Q_u w + Q_0 q$ is invertible and transforms the homogeneous impulsive system* (II.2.3)–(II.2.4) *into the decoupled system*

$$\dot{y} = \Lambda_s y, \tag{II.2.17}$$

$$\dot{z} = \Lambda_c z, \tag{II.2.18}$$

$$\dot{w} = \Lambda_u w, \tag{II.2.19}$$

$$\dot{q} = 0, \qquad t \neq t_k \tag{II.2.20}$$

$$\Delta q = Q_0^+(t_k)[(I + B_k)Q_0(t_k^-) - \Delta Q_0(t_k)]q, \qquad t = t_k, \tag{II.2.21}$$

where $\Phi_j = Q_j e^{t\Lambda_j}$ are the respective Floquet decompositions, and for a matrix M with independent columns, the symbol M^+ denotes its left-inverse. The transformation and its inverse are uniformly bounded.

Proof. Since the columns of $Q_s(t)$, $Q_c(t)$, $Q_u(t)$ and $Q_0(t)$ are bases $X_s^\infty(t)$, $X_c(t)$, $X_u(t)$ and $X_0(t)$, respectively, and these subspaces have trivial intersection, the transformation is invertible. Substituting $x = Q_s y + Q_c z + Q_u w + Q_0 q$ into (II.2.3), we find

$$\begin{aligned} &A(Q_s y + Q_c z + Q_u w + Q_0 q) \\ &\quad = (AQ_s - Q_s\Lambda_s)y + Q_s\dot{y} + (AQ_c - Q_c\Lambda_c)z + Q_c\dot{z} \\ &\qquad + (AQ_u - Q_u\Lambda_u)w + Q_u\dot{w} + AQ_0 q + Q_0\dot{q}. \end{aligned}$$

After cancelling several terms, we get

$$0 = Q_s(\dot{y} - \Lambda_s y) + Q_c(\dot{z} - \Lambda_c z) + Q_u(\dot{w} - \Lambda_u w) + Q_0\dot{q}.$$

This implies the first four equations, (II.2.17)–(II.2.20). It is easy to check that $Q_j(t_k) = [I + B_k]Q_j(t_k^-)$ for $j = c, s, u$. Substituting $x = Q_s y + Q_c z + Q_u w + Q_0 q$ into (II.2.4), this implies

$$(I + B_k)Q_0(t_k^-)q(t_k^-) = Q_0(t_k)q(t_k) - Q_0(t_k^-)q(t_k^-).$$

Denoting $\Delta q = \Delta q(t_k)$, $q = q(t_k^-)$, $\Delta Q_0 = \Delta Q_0(t_k)$ and $Q_0^- = Q_0(t_k^-)$, we can expand the above as

$$(I + B_k)Q_0^- q = Q_0^-(q + \Delta q) + \Delta Q_0(q^- + \Delta q) - Q_0^- q.$$

Cancelling $Q_0^- q$ on either side, this is equivalent to

$$(I + B_k)Q_0(t_k^-)q(t_k^-) = Q_0(t_k)\Delta q + \Delta Q_0(t_k)q(t_k^-).$$

Rearranging and multiplying by $Q_0^+(t_k)$ on both sides give (II.2.21). The boundedness of the transformation and its inverse is clear from the periodicity of each of Q_s, Q_c, Q_u, together with the observation that Q_0 is periodic and the left-limits $Q_0(t_k^-)$ are full column rank. □

II.2.6.2 Real Floquet Decompositions

In some applications, the utility of the Floquet decomposition is less than the transformation of a periodic system to an autonomous one, but rather in the decoupling of the stable, centre and unstable parts. This is where the emphasis is placed in Theorem II.2.6.1. However, sometimes we also want the resulting dynamics of the transformed equation to be real. The following provides a sufficient condition for all matrices in the statement of Theorem II.2.6.1 to be real, or for there to exist real matrices such that the statement holds. The proof is a consequence of the existence of a real logarithm of a real matrix [39] and is omitted.

Proposition II.2.6.1. *Let $M_0 = VJV^{-1}$ denote the real Jordan canonical form of M_0. There exist real basis matrices Φ_s, Φ_c and Φ_u for X_s^∞, X_c and X_u, respectively, with real Floquet decompositions $\Phi_j(t) = Q_j(t)e^{t\Lambda_j}$ for $j \in \{s, c, u\}$, where Q_j are real and T-periodic and Λ_j are real, if and only if each Jordan block of J belonging to a negative real eigenvalue occurs an even number of times.*

Corollary II.2.6.1. *Let $\Phi(t)$ be a real basis matrix for one of X_s^∞, X_c or X_u. Let D be the unique non-singular (real) matrix such that $\Phi(T) = \Phi(0)D$, and let $D = VJV^{-1}$ be its real Jordan canonical form. There exists a real Floquet decomposition—that is, $Q(t)$ real and T-periodic and Λ real such that $\Phi(t) = Q(t)e^{t\Lambda}$—if and only if any Jordan block of J belonging to a negative real eigenvalue occurs an even number of times.*

Corollary II.2.6.2. *There exist real basis matrices Φ_s, Φ_c and Φ_u for X_s^∞, X_c and X_u, respectively, with real Floquet decompositions $\Phi_j(t) = Q_j(t)e^{t\Lambda_j}$ for $j \in \{s, c, u\}$, where Q_j are real and $2T$-periodic and Λ_j are real.*

Proof. Let $\Phi(t) \in \mathbb{R}^{m\times m}$ be a basis matrix for one of X_s^∞, X_c or X_u. Then,

$$\Phi(2T) = M_0(M_0\Phi(0)) = M_0(\Phi(0)D) = \Phi(0)D^2$$

for some invertible $D \in \mathbb{R}^{m\times m}$. Defining $\Lambda = \frac{1}{2T}\log(D)$, since D has no negative real eigenvalues, Λ is real. Then, $Q(t) := \Phi(t)e^{-t\Lambda}$ is $2T$-periodic. □

It is clear from the above proposition and corollary that, for example, the best real Floquet decomposition one can hope to obtain for the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x, && t \neq kT \\ \Delta x &= \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} x, && t = kT \end{aligned}$$

is one that is $2T$-periodic. For some applications, this is not enough, for example, in the analysis of period-doubling bifurcations, it is preferable to maintain the original period. We can accomplish this by way of chain matrices.

II.2.6.3 A Real T-Periodic Kinematic Similarity

We can use a system of chain matrices to transform any T-periodic impulsive system into a T-periodic impulsive system with a block structure. The proof is analogous to that of Theorem II.2.6.1 and is omitted.

Corollary II.2.6.3. *Let Q_0, Q_s, Q_c and Q_u be chain matrices for X_0, X_s^∞, X_c and X_u, respectively. Define $Q(t) = [\ Q_0(t) \quad Q_s(t) \quad Q_c(t) \quad Q_u(t)\ ]$. This matrix is bounded and periodic with a bounded inverse. The change of variables $x = Q(t)y$ transforms* (II.2.3)–(II.2.4) *into the piecewise-constant system*

$$\begin{aligned} \dot{y} &= 0, && t \neq t_k \\ \Delta y &= Q^{-1}(t_k)[(I + B_k)Q(t_k^-) - \Delta Q(t_k)]y, && t = t_k. \end{aligned}$$

One can also define a transformation $x = Q(t)y$, where $Q(t)$ is some combination of chain matrices and Floquet periodic matrices (i.e. coming from a Floquet decomposition), and the result will be some combination of the systems from Theorem II.2.6.1 and Corollary II.2.6.3. In some cases, it will be preferable to use the standard Floquet periodic matrices, and other times, it will be better to use chain matrices. Regardless, we have the following general corollary.

Corollary II.2.6.4. *There exists a real, T-periodic, linear change of variables $x = Q(t)y$ with $||Q(t)||$ and $||Q^{-1}(t)||$ uniformly bounded, such that* (II.2.3)–(II.2.4) *are transformed into a system of the form*

$$\begin{aligned} \dot{y} &= \Lambda_s y, \quad t \neq t_k & \Delta y &= \Omega_s y, \quad t = t_k \\ \dot{z} &= \Lambda_c w, \quad t \neq t_k & \Delta z &= \Omega_c z, \quad t = t_k \\ \dot{w} &= \Lambda_u z, \quad t \neq t_k & \Delta w &= \Omega_u w, \quad t = t_k, \end{aligned}$$

with real matrices Λ_j and Ω_j, $j \in \{s, c, u\}$. Let M_0 denote the monodromy matrix of (II.2.3)–(II.2.4)*, and write its spectrum (set of eigenvalues) as $\sigma(M_0) = \sigma_s \cup \sigma_c \cup \sigma_u$, with $|\sigma_s| < 1$, $|\sigma_c| = 1$ and $|\sigma_u| > 1$. Let $M_{0,y}$, $M_{0,z}$ and $M_{0,w}$ denote the monodromy matrices of the y, z and w subsystems. Then,*

$$\sigma(M_{0,y}) = \sigma_s, \quad \sigma(M_{0,z}) = \sigma_c, \quad \sigma(M_{0,w}) = \sigma_u.$$

In the above corollary, we used $|S| < 1$ as a shorthand for the sentence *all elements of S have absolute value less than one.* The symbols $|S| = 1$ and $|S| > 1$ are interpreted analogously.

II.2.7 Comments

The Floquet theory for impulsive differential equations is fully described in the monograph of Bainov and Simeonov [9], being partially developed in 1982 by Samoilenko and Perestyuk [124], although therein the assumption that matrices $I + B_k$ are invertible is assumed. We have intentionally dispensed with this requirement since it makes the theory far more flexible. The content of Sects. II.2.5.2 and II.2.5.3 appears in [9], as does Theorem II.2.5.5.