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Bifurcation Theory of Impulsive Dynamical Systems

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Bifurcation Theory of Impulsive Dynamical Systems

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Preface

Impulsive dynamical systems have become increasingly popular during the past decades because they provide a natural framework for mathematical modeling of many real-world phenomena. Applications of impulsive dynamical systems can be found in a variety of fields such as aeronautics, ecology, economics, epidemiology, finance, medicine and robotics, just to name a few. An impulsive dynamical system normally consists of three elements: a continuous system of differential equations, which governs the motion of the dynamical system between impulsive and resetting events; a discrete system of difference equations, which governs the way the system states are instantaneously changed when a setting event occurs; and a criterion for determining when the states of the system are to be reset. The solutions of impulsive dynamical systems are in general discontinuous, which often renders some of the standard analysis and control design methods ineffective. Nonetheless, significant progress has been made in theory and applications of impulsive dynamical systems in the past few decades, especially when the underlying continuous portions are described by ordinary differential equations. The latter are often referred to as impulsive (ordinary) differential equations. When time delays are present in the systems, they are also called impulsive retarded functional differential equations. There are added layers of challenges in studying such systems containing time delays because of a lack of some more ubiquitous properties in dynamical systems, such as continuity of the induced semiflow. However, much progress has been made in recent years, and many interesting results in stability, manifold theory and bifurcation analysis have been published for such systems. The purpose of this book is to present the recent progress in this direction and to demonstrate that the local stability and bifurcation analysis of these systems, while at times subtle, can be made rigorous and computationally viable. The scope has been expanded to address not only smooth local bifurcations but also some nonsmooth bifurcation phenomena that are unique to impulsive dynamical systems. Arguably, one of the most powerful techniques in the study of local bifurcations in finite-dimensional smooth dynamical systems is the combination of linearization, centre manifold reduction and normal form theory. This continues to hold true for retarded functional differential equations, of which delay differential equations are a subclass. There, the first two steps of linearization and centre manifold reduction are fundamentally different than in the finite-dimensional setting, but the normal form theory can function as usual because the centre dynamics are finite-dimensional. The primary objective of this book is to extend this programme to the case of impulsive retarded functional differential equations.

This book consists of four parts with twenty chapters in total. Part I is devoted to infinite-dimensional impulsive functional differential equations. Some preliminary background is provided in Chap. I.1, including the phase space of right-continuous regulated functions that we use throughout. A thorough treatment of the representation of solutions for linear systems and linear periodic systems is completed in Chaps. I.2 and I.3. Following this, nonlinear systems are considered in Chap. I.4 from the point of view of mild solutions, where we also discuss stability. Invariant manifold theory is the focus of Chaps. I.5, I.6 and I.7, with a discussion of the generic codimension-one smooth bifurcations appearing in Chap. I.8.

Finite-dimensional ordinary impulsive differential equations are considered in Part II. This part of the monograph can be read independently of Part I. The first two chapters contain a review of material that should be familiar to a reader who is versed in ordinary impulsive differential equations: existence and uniqueness of solutions, dependence on initial conditions and linear systems theory. Chapter II.3 contains finite-dimensional variants of some of the results from Chap. I.4, including linearized stability. Invariant manifold theory is covered in Chap. II.4, and methods of studying bifurcations of fixed points and periodic solutions are discussed in Chap. II.5.

Part III contains some special topics concerning singular phenomena and nonsmooth bifurcations. Chapter III.1 pertains to the robustness of bifurcations and hyperbolic orbits under continuous-time temporal smoothing of the impulse effect, as well as the sensitivity of bifurcation curves under such smoothing actions. In a sense, this chapter is an analysis of what might be considered the fundamental tenet of modeling with impulses: the transient dynamics that occur during a temporally short burst of activity can be ignored and replaced by a discrete jump in state. Chapter III.2 is a study of some nonsmooth bifurcations in impulsive systems, namely those caused by taking discrete delays and/or impulse times as system parameters.

The final part, Part IV, contains applications. Therein, we study stability and bifurcation in five mathematical models involving impulses and (discrete and distributed) delays. The subject areas include classical mechanics, infectious disease modeling, mathematical ecology and in-host viral replication dynamics.

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Reading Guide

The target audience of this book is applied mathematicians and scientists who want to understand more about their models, especially if these are based on impulsive dynamical with such functional dependence as delays. Part I is rather technical, as we build up the theory from scratch and require machinery from functional analysis and measure theory to do it consistently. To compare, Part II should be approachable by advanced undergraduates or early graduate students with sufficient exposure to ordinary differential equations and dynamical systems. The remaining Parts III and IV are special topics and applications.

We recommend that readers less familiar with impulsive dynamical systems (especially delay equations) first read only the opening pages of Chap. I.1, as these provide some background, are mostly nontechnical and illustrate the main theoretical issues of impulsive dynamical systems we aim to remedy with this book. The readers could then skim the applications in Part IV to see what the theory from Parts I and II could do for the analysis of their mathematical models. Following this, such readers have two options: read Part II if they are primarily interested in ordinary impulsive differential equations, or begin Part I in earnest if their models involve functional dependence like delays or integrals.

Readers who are most interested in the theoretical developments concerning invariant manifolds and bifurcations for impulsive functional differential equations are advised to begin with Part I. The finite-dimensional analogues of these results appear in Chaps. II.4 and II.5.

Part III should be of interest to those readers interested in the intersection of bifurcation theory and nonsmooth dynamics. The content of Chap. III.1 is accessible to any reader who has read Part II or is familiar with the basics of impulsive differential equations, while to fully understand Chap. III.2 it is advisable that the reader have read Chaps. I.3 and I.6.

Contents

I Impulsive Functional Differential Equations **1**

I.1 Introduction **3**

I.1.1 Nonautonomous Dynamical Systems 9
I.1.2 History Functions 11
I.1.3 The Space $\mathcal{RCR}$ of Right-Continuous Regulated. 12
I.1.4 Gelfand–Pettis Integration 17
I.1.5 Integral and Summation Inequalities 18
I.1.6 Comments 19

I.2 General Linear Systems **21**

I.2.1 Existence and Uniqueness of Solutions 22
I.2.2 Evolution Families 23
I.2.2.1 Phase Space Decomposition 25
I.2.2.2 Evolution Families are (Generally) Nowhere Continuous 25
I.2.2.3 Continuity under the L^2 Seminorm 26
I.2.3 Representation of Solutions of the 27
I.2.3.1 Pointwise Variation-of-Constants Formula 28
I.2.3.2 Variation-of-Constants Formula in the Space $\mathcal{RCR}$ 30
I.2.4 Stability 32
I.2.5 Comments 32

I.3 Linear Periodic Systems **35**

I.3.1 Monodromy Operator 36
I.3.2 Floquet Theorem 43
I.3.3 Floquet Multipliers, Floquet Exponents and. 44
I.3.4 Computational Aspects in Floquet Theory 45
I.3.4.1 Floquet Eigensolutions 46

I.3.4.2 Characteristic Equations for Finitely Reducible Linear Systems . . . 48
I.3.4.3 Characteristic Equations for Systems with Memoryless Continuous Part . . . 51
I.3.5 Comments . . . 52
I.4 Nonlinear Systems and Stability 55
I.4.1 Mild Solutions . . . 55
I.4.2 Dependence on Initial Conditions . . . 58
I.4.3 The Linear Variational Equation and Linearized. . . . 61
I.4.4 Comments . . . 65
I.5 Existence, Regularity and Invariance of Centre Manifolds 67
I.5.1 Preliminaries . . . 68
I.5.1.1 Spaces of Exponentially Weighted Functions . . . 68
I.5.1.2 η-Bounded Solutions from Inhomogeneities . . . 69
I.5.1.3 Substitution Operator and Modification of Nonlinearities . . . 74
I.5.2 Fixed-Point Equation and Existence... . . . 75
I.5.2.1 A Remark on Centre Manifold Representations: Graphs and Images . . . 77
I.5.3 Invariance and Smallness Properties . . . 77
I.5.4 Dynamics on the Centre Manifold . . . 79
I.5.4.1 Integral Equation . . . 79
I.5.4.2 Abstract Ordinary Impulsive Differential Equation . . . 80
I.5.4.3 A Remark on Coordinates and Terminology . . . 83
I.5.5 Reduction Principle . . . 83
I.5.5.1 Parameter Dependence . . . 88
I.5.6 Smoothness in the State Space . . . 89
I.5.6.1 Contractions on Scales of Banach Spaces . . . 90
I.5.6.2 Candidate Differentials of the Substitution Operators . . . 91
I.5.6.3 Smoothness of the Modified Nonlinearity . . . 92
I.5.6.4 Proof of Smoothness of the Centre Manifold . . . 94
I.5.6.5 Periodic Centre Manifold . . . 98
I.5.7 Regularity of Centre Manifolds... . . . 100
I.5.7.1 A Coordinate System and Pointwise $PC^{1,m}$-Regularity . . . 100
I.5.7.2 Reformulation of the Fixed-Point Equation . . . 102
I.5.7.3 A Technical Assumption on the Projections $P_c(t)$ and $P_u(t)$ . . . 103
I.5.7.4 Proof of $PC^{1,m}$-Regularity at Zero . . . 104
I.5.7.5 The Hyperbolic Part Is Pointwise $PC^{1,m}$-Regular at Zero . . . 105
I.5.7.6 Uniqueness of the Taylor Coefficients . . . 106

I.5.7.7 A Discussion on the Regularity of the Matrices $t \mapsto Y_j(t)$ . . . 107
I.5.8 Comments . . . 108

I.6 Computational Aspects of Centre Manifolds **111**
I.6.1 Euclidean Space Representation . . . 111
I.6.1.1 Definition and Taylor Expansion . . . 113
I.6.1.2 Dynamics on the Centre Manifold in Euclidean Space . . . 115
I.6.1.3 An Impulsive Evolution Equation and Boundary Conditions . . . 118
I.6.2 Approximation by the Taylor Expansion . . . 121
I.6.2.1 Evolution Equation and Boundary Conditions for Quadratic Terms . . . 121
I.6.2.2 Solution by the Method of Characteristics . . . 122
I.6.3 Visualization of Centre Manifolds . . . 125
I.6.3.1 An Explicit Scalar Example Without Delays . . . 126
I.6.3.2 Two-Dimensional Example with Quadratic Delayed Terms . . . 127
I.6.3.3 Detailed Calculations Associated with Example I.6.3.2 . . . 130
I.6.4 The Overlap Condition . . . 133
I.6.4.1 Distributed Delays . . . 134
I.6.4.2 Transformations that Enforce the Overlap Condition for Discrete Delays . . . 134
I.6.5 Comments . . . 137

I.7 Hyperbolicity and the Classical Hierarchy of Invariant Manifolds **139**
I.7.1 Preliminaries . . . 139
I.7.2 Unstable Manifold . . . 141
I.7.3 Stable Manifold . . . 143
I.7.4 Centre-Unstable Manifold . . . 144
I.7.5 Centre-Stable Manifold . . . 145
I.7.6 Dynamics on Finite-Dimensional... . . . 147
I.7.7 Linearized Stability and Instability, Revisited . . . 148
I.7.8 Hierarchy and Inclusions . . . 149

I.8 Smooth Bifurcations **151**
I.8.1 Centre Manifolds Depending Smoothly on Parameters . . . 151
I.8.2 Codimension-One Bifurcations for Systems with a Single Delay: Setup . . . 153
I.8.3 Fold Bifurcation . . . 154
I.8.3.1 Example: Fold Bifurcation in a Scalar System with Delayed Impulse . . . 161
I.8.3.2 Calculation of the Function $Y_{11}(t)$ for Example I.8.3.1 . . . 162

I.8.4 Hopf-Type Bifurcation and Invariant Cylinders 164
I.8.4.1 Example: Impulsive Perturbation from a Hopf Point 176
I.8.5 Calculations Associated to Example I.8.4.1 181
I.8.5.1 The Projection $P_c(t)$ and Matrix $\tilde{Y}(t)$ 182
I.8.5.2 Calculation of $\pi(t)$ and the Matrices $\mathcal{A}(t)$ and $\mathcal{B}$ 184
I.8.5.3 Calculation of $n^0(t)$: A Numerical Routine . . . 185
I.8.5.4 Calculation of h_2 187
I.8.6 A Recipe for the Analysis of Smooth Local Bifurcations 188
I.8.7 Comments 190

II Finite-Dimensional Ordinary Impulsive Differential Equations 191

II.1 Preliminaries 193
II.1.1 Existence and Uniqueness of Solutions 193
II.1.2 Dependence on Initial Conditions... 195
II.1.3 Continuity Conventions: Right- and Left-Continuity 196
II.1.4 Comments 197

II.2 Linear Systems 199
II.2.1 Cauchy Matrix 199
II.2.2 Variation-of-Constants Formula 201
II.2.3 Stability 203
II.2.4 Exponential Trichotomy 203
II.2.5 Floquet Theory 204
II.2.5.1 Homogeneous Systems 205
II.2.5.2 Periodic Solutions of Homogeneous Systems . . 208
II.2.5.3 Periodic solutions of Inhomogeneous Systems . . 209
II.2.5.4 Periodic Systems Are Exponentially Trichotomous 210
II.2.5.5 Stability 210
II.2.6 Generalized Periodic Changes of Variables 212
II.2.6.1 A Full State Transformation and Chain Matrices 212
II.2.6.2 Real Floquet Decompositions 214
II.2.6.3 A Real T-Periodic Kinematic Similarity 215
II.2.7 Comments 216

II.3 Stability for Nonlinear Systems 217
II.3.1 Stability 217
II.3.2 The Linear Variational Equation... 218
II.3.3 Comments 220

II.4 Invariant Manifold Theory 221
II.4.1 Existence and Smoothness . . . 221
II.4.2 Invariance Equation for Nonautonomous... . . . 223
II.4.3 Invariance Equation for Systems with... . . . 225
II.4.4 Dynamics on Invariant Manifolds . . . 227
II.4.5 Reduction Principle for the Centre Manifold . . . 228
II.4.6 Approximation by Taylor Expansion . . . 228
II.4.7 Parameter Dependence . . . 231
II.4.7.1 Centre Manifolds Depending on a Parameter . . 232
II.4.8 Comments . . . 233
II.5 Bifurcations 235
II.5.1 Reduction to an Iterated Map . . . 236
II.5.2 Codimension-one Bifurcations . . . 236
II.5.2.1 Fold Bifurcation . . . 237
II.5.2.2 Period-Doubling Bifurcation . . . 241
II.5.2.3 Cylinder Bifurcation . . . 243
II.5.3 Comments . . . 248
III Singular and Nonsmooth Phenomena 251
III.1 Continuous Approximation 253
III.1.1 Introduction . . . 254
III.1.1.1 Singular Unfolding of an Impulsive Differential Equation . . . 255
III.1.1.2 Preliminaries . . . 257
III.1.1.3 Time q Map . . . 257
III.1.1.4 The Realization Problem . . . 257
III.1.1.5 A Brief Discussion on the Continuity Convention . . . 258
III.1.2 Pointwise Convergence and the Candidate... . . . 259
III.1.3 Smoothness of the Time q Map . . . 260
III.1.4 Sensitivity and Realization . . . 269
III.1.5 An Important Comment (Or Warning)... . . . 273
III.1.6 Example: Continuous-Time Logistic... . . . 274
III.2 Non-smooth Bifurcations 279
III.2.1 Overview . . . 279
III.2.1.1 Bifurcations Involving Perturbations of Impulse Times . . . 279
III.2.1.2 Bifurcations Involving Crossings of Impulse Times and Delays . . . 281
III.2.2 Centre Manifolds Parameterized... . . . 283
III.2.2.1 Dummy Matrix System and Robustness of Spectral Separation . . . 284
III.2.2.2 Centre Manifold Construction . . . 287

III.2.3 Overlap Bifurcations . . . 289
III.2.3.1 Floquet Spectrum . . . 290
III.2.3.2 Symmetries of Periodic Solutions . . . 292
III.2.3.3 A State Transformation that Eliminates the Delay . . . 292
III.2.3.4 Bifurcations of Periodic Solutions . . . 294
III.2.3.5 The Introductory Example Revisited . . . 296
III.2.4 Comments . . . 300
IV Applications **301**
IV.1 Bifurcations in an Impulsively Damped or Driven Pendulum **303**
IV.1.1 Stability Analysis: The Model Without Delay . . . 304
IV.1.1.1 Downward Rest Position . . . 305
IV.1.1.2 Upward Rest Position . . . 306
IV.1.2 Stability Analysis: The Model with Delay . . . 308
IV.1.2.1 Downward Rest Position . . . 308
IV.1.2.2 Upward Rest Position . . . 309
IV.1.3 Cylinder Bifurcation at the Downward... . . . 309
IV.1.4 Cylinder Bifurcation at the Downward... . . . 313
IV.1.4.1 Floquet Multiplier Transversality Condition . . . 313
IV.1.4.2 Computation of the First Lyapunov Coefficient . . . 314
IV.2 The Hutchinson Equation with Pulse Harvesting **319**
IV.2.1 Dummy Matrix System: Setup for the Non-smooth Centre Manifold . . . 320
IV.2.2 Dynamics on the Centre Manifold . . . 321
IV.2.3 The Transcritical Bifurcation . . . 322
IV.3 Delayed SIR Model with Pulse Vaccination and Temporary Immunity **325**
IV.3.1 Introduction . . . 325
IV.3.2 Vaccinated Component Formalism . . . 326
IV.3.3 Existence of the Disease-free Periodic Solution . . . 327
IV.3.4 Stability of the Disease-free Periodic Solution . . . 329
IV.3.5 Existence of a Bifurcation Point . . . 330
IV.3.6 Transcritical Bifurcation in Terms of Vaccine Coverage at $R_0 = 1$ with One Vaccination Pulse Per Period . . . 330
IV.3.7 Numerical Bifurcation Analysis . . . 337
IV.4 Stage-Structured Predator–Prey System with Pulsed Birth **343**
IV.4.1 Model Derivation . . . 344
IV.4.2 Stability of the Extinction Equilibrium . . . 345

IV.4.3 Analysis of Predator-Free Periodic Solution 347
IV.4.3.1 Existence and Uniqueness of the Predator-Free Solution 347
IV.4.3.2 Stability 348
IV.4.4 Bifurcation at Extinction 350
IV.4.4.1 Calculation of the Matrix $Y_c(t)$ 351
IV.4.4.2 Centre Manifold Quadratic Dynamics and Bifurcation 353
IV.4.5 Discussion 353
IV.5 Dynamics of an In-host Viral Infection Model with Drug Treatment 355
IV.5.1 Derivation of the Delayed Terms 357
IV.5.2 Existence of a Disease-free Periodic Solution and a Disease-free Attractor 359
IV.5.3 Well-Posedness and Boundedness 360
IV.5.4 Numerical Bifurcation Analysis: Preamble 362
IV.5.4.1 Model Transformation 363
IV.5.4.2 Monodromy Operator Discretization 365
IV.5.4.3 Parameters 367
IV.5.5 Transcritical Bifurcation from the Disease-free Periodic Solution 368
IV.5.5.1 Results 369
Bibliography 373
Index 387

Part I

Impulsive Functional Differential Equations

Chapter I.1

Introduction

Many real-world processes exhibit continuous-time evolution with intermittent bursts of comparatively fast dynamics. In mathematical models of such processes, these bursts of activity are sometimes intrinsic to the dynamics. For example, the Hodgkin–Huxley model [68] is a nonlinear ordinary differential equation that describes the propagation of action potentials of neurons; here, the bursts of activity correspond to the action potentials and are an intrinsic feature of the model. In the Hodgkin–Huxley model, these bursts arise from slow–fast dynamics in the continuous-time model, but in other neuronal models such as integrate-and-fire [1], the bursts are introduced synthetically using a logic rule. In other situations, these bursts of activity or *impulses* enter into the model in the form of a control that is designed to (ideally) force or constrain the dynamics in a desired way. The applications of this idea are quite diverse, including control theory, multi-agent systems, epidemiology, population dynamics, medicine and robotics [108, 140, 163]. The mathematical formalism in which these ideas take concrete form is *impulsive dynamical systems*.

The theoretical foundations of ordinary impulsive differential equations are mostly contained in the monographs [9, 10, 85, 112, 125]. One important class of ordinary impulsive differential equations are those that have impulses at fixed times. These are systems of the form

$$\frac{dx}{dt} = f(t,x), \qquad t \neq t_k \tag{I.1.1}$$

$$\Delta x = g(k,x), \qquad t = t_k, \tag{I.1.2}$$

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where $\{t_k : k \in \mathbb{Z}\}$ is a monotone sequence of *impulse times* (typically unbounded for $k \to \infty$), f and g are the *vector field* and *jump function* that satisfy some appropriate regularity conditions, and the second equation is interpreted as

$$\Delta x = x(t_k^+) - x(t_k) = g(k, x(t_k)).$$

The solutions of (I.1.1)–(I.1.2) are continuous from the left and have limits on the right (although conventions differ; some authors use continuity from the right, although these notions are equivalent for $x \in \mathbb{R}^n$ finite-dimensional). In many practical problems, the right-hand side of (I.1.1)–(I.1.2) is *autonomous*—that is, $t \mapsto f(t,x)$ and $k \mapsto g(k,x)$ are constant for fixed x. When the sequence of impulses is of the form $t_k = kT$ for some $T > 0$, the result is a system

$$\frac{dx}{dt} = f(x), \qquad t \neq kT \tag{I.1.3}$$

$$\Delta x = g(x), \qquad t = kT. \tag{I.1.4}$$

The above system is periodically forced though the impulse effects, which occur every T units of time. System (I.1.3)–(I.1.4) falls under the category of *periodic* impulsive differential equations. Many dynamical aspects of these systems can be understood by transforming to discrete time through the use of the time T map. Using this formalism, bifurcations from fixed points and periodic solutions can be studied using either Lyapunov–Schmidt reduction or centre manifold reduction for maps [30]. One of the earliest applications of such an approach seems to be due to Lakmeche and Arino [84] in the context of chemotherapy modelling. Since then, numerous authors have studied bifurcations in specific impulsive differential equation models—the reader may consult [37, 48, 128, 145, 162, 164] for a few recent applications. In most of these papers, the impulse effect represents a control that is applied at fixed, regular times.

Impulsive dynamical systems may be classified into the incredibly broad class of *hybrid systems*. The latter can be specified by a differential inclusion on one subset of the phase space, and a set-valued map defined on another subset of the phase space. They include impulsive dynamical systems as a subclass but also can be used to describe systems with distinct continuous states and logical modes, hybrid automata, switched systems, Filippov systems and others. See [51] for background.

The model (I.1.1)–(I.1.2) is suitable for describing systems whose evolution law does not depend explicitly on the state of the system in the past. However, many processes do have explicit memory effects, or models that take these into account have improved fidelity to empirical observations. Consider the logistic growth (Verlhust) model

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

for a single species. The number K is the *carrying capacity* of the population. Every solution of this differential equation from a positive initial condition converges monotonically to the carrying capacity. However, any notion of carrying capacity must take into account either that such a quantity is constantly fluctuating [126] or that the carrying capacity represents some quantity which populations fluctuate about [118]. Since solutions of the logistic equation are monotone, such fluctuations cannot be realized. The Hutchinson equation [72]

$$\frac{dN}{dt} = rN\left(1 - \frac{N(t-\tau)}{K}\right)$$

includes a delayed term $N(t-\tau)$, which suggests that the density-dependent feedback takes τ units of time to affect the population dynamics. Under certain parameter configurations, Hutchinson's equation features sustained oscillations near the carrying capacity.

Many authors have considered theoretical questions related to including impulse effects in systems involving delays and other functional dependence. Broadly, impulsive retarded functional differential equations (impulsive RFDEs)

$$\frac{dx}{dt} = f(t, x_t), \qquad t \neq t_k \tag{I.1.5}$$

$$\Delta x = g(t, x_t), \qquad t = t_k \tag{I.1.6}$$

sometimes referred to simply as impulsive functional differential equations (impulsive FDEs) have been considered. $f(t,\cdot)$ and $g(t,\cdot)$ are functionals acting on some appropriate space of functions $\phi : [-r, 0] \to \mathbb{R}^n$, the history x_t is defined by $x_t(\theta) := x(t+\theta)$ for $\theta \in [-r, 0]$ and the jump condition (I.1.6) is understood as one of

$$\Delta x \in \{x(t_k) - x(t_k^-),\ x(t_k^+) - x(t_k)\}.$$

The choice corresponds to a *continuity convention*. The majority of literature on impulsive functional differential equations appears to take the convention of continuity from the right—that is, $\Delta x = x(t_k) - x(t_k^-)$—but there are some exceptions. Classical topics such as existence and uniqueness, continuability of solutions and stability are treated in [13, 14, 92, 93, 105]. In the typical case of a right-continuity convention, the jump condition (I.1.6) is usually taken to be one of the more explicit forms

$$\Delta x = g(t, x_{t^-}),$$

where $x_{t^-}(\theta) = x(t+\theta)$ for $\theta \in [-r, 0)$, and $x_{t^-}(0) = \lim_{s\to 0^-} x(t+s)$.

The transition from the finite-dimensional system (I.1.1)–(I.1.2) to the infinite-dimensional system (I.1.5)–(I.1.6) is far from smooth (pun absolutely intended), at least insofar as dynamical systems aspects are concerned. The difficulties mostly centre around two inherently connected observations.

1. The phase space of (I.1.5)–(I.1.6) must contain discontinuous functions.

2. The associated dynamical system is generally discontinuous everywhere (with respect to time).

To compare, the nonautonomous dynamical system $\phi : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}^n$ associated with the finite-dimensional system (I.1.1)–(I.1.2) has the property that $t \mapsto \phi(t, s, x)$ and $s \mapsto \phi(t, s, x)$ only have discontinuities in the set $\{t_k : k \in \mathbb{Z}\}$ of impulse times. The infinite-dimensional setting is far less well-behaved, and it is our view that this is precisely the reason that bifurcation theory techniques for impulsive functional differential equations have lagged behind in development.

The first of these observations, that some amount of discontinuity must be allowed in the initial condition of a Cauchy problem if continuability of solutions is to be considered, is responsible for the occasionally abstract presentation of existence and uniqueness results. These universally involve a condition on the composition

$$t \mapsto f(t, x_t)$$

whenever $x : I \to \mathbb{R}^n$ is an element of the class of solutions the author is considering. For example, Ballinger and Liu [14] take $x : I \to \mathbb{R}^n$ to be continuous at all but finitely many points in any compact set and assume that the composition $t \mapsto f(t, x_t)$ satisfies a Carathéodory condition, with the result being existence, uniqueness and continuability of solutions in the extended sense. If this composition is suitably continuous, then the same conclusions hold for classical solutions [13]. Regardless, under these assumptions the phase space is incomplete, which makes the situation less than ideal for considering dynamical systems aspects such as invariant manifold theory.

In order to more fully understand the difficulties in moving to the infinite-dimensional setting, we will study here a few simple examples. Our discussion will be a bit informal, as we have yet to properly define such concepts as solutions.

First, consider the following scalar initial-value problem:

$$\frac{dx}{dt} = 1, \qquad t \neq k \in \mathbb{Z} \tag{I.1.7}$$

$$\Delta x = x(t-1), \qquad t = k \in \mathbb{Z} \tag{I.1.8}$$

$$x_0 = 1, \tag{I.1.9}$$

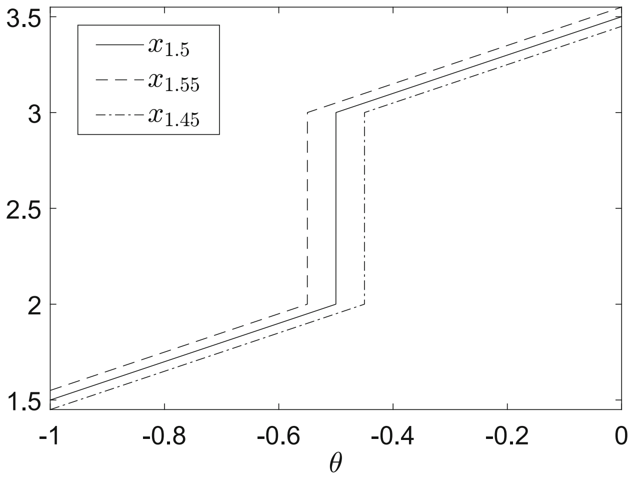

Figure I.1.1: Plots of $x_{1.5}$, $x_{1.55}$ and $x_{1.45}$ for the solution $x(t)$ from (I.1.10). These functions are discontinuous at $\theta = -0.5$, $\theta = -0.55$ and $\theta = -0.45$, respectively. The discontinuities are plotted with vertical lines for emphasis. We can now clearly see that for small $\epsilon \neq 0$, $||x_{1.5+\epsilon} - x_{1.5}|| \geq 1$

where $x_0(\theta) = 1$ is a constant initial function. Note that since the impulse effect requires the data at time $t-1$, we need to specify initial conditions on an interval $[-1, 0]$. By convention, the impulse at time $t = 0$ is ignored since our initial condition is specified at this same time. The solution can be computed directly: it is given by

$$x(t) = \begin{cases} 1, & t < 0 \\ 2^{\lfloor t \rfloor + 1} - 1 + t - \lfloor t \rfloor, & t \geq 0 \end{cases} \tag{I.1.10}$$

for $\lfloor \cdot \rfloor$ the floor (round down to the nearest integer) function. Indeed, one can verify that this function is differentiable for $t > 0$ non-integer with derivative 1, while for positive integers, $x(k) = 2^{k+1} - 1$ and $\lim_{s \to k^-} x(s) = 2^k$. Then, the impulse effect dictates that the solution must satisfy

$$x(k) = \lim_{s \to k^-} x(s) + \Delta x = 2^k + x(k-1) = 2^k + 2^k - 1 = 2^{k+1} - 1,$$

which is consistent with our claimed solution x above. However, if we consider the function $t \mapsto x_t$, for $x_t(\theta) = x(t + \theta)$ and $\theta \in [-1, 0]$, this function is discontinuous for *all* $t \geq 1$, in the sense that $\lim_{\epsilon \to 0} ||x_{t+\epsilon} - x_t|| \neq 0$, where the norm is the supremum norm; see Fig. I.1.1. In fact, one can verify that for $t > 1$ and $\epsilon \neq 0$ sufficiently small, $||x_{t+\epsilon} - x_t|| \geq x(\lfloor t \rfloor - 1) = 2^{\lfloor t \rfloor} - 1$. This example demonstrates one of the previously mentioned major technical differences between continuous time delay differential equations and those that involve impulses.

> The dynamical system generated by an impulsive functional differential equation is generally discontinuous everywhere (in the sense that $t \mapsto x_t$ is not continuous in the supremum norm, along solutions x).

It might seem natural to therefore endow the phase space with a different topology to avoid this issue. For example, one might consider what might happen if we instead equip the phase space with the topology induced by the L^2 norm. While this fixes the continuity problem of the dynamical system along *individual* solutions, it has the effect of breaking continuous dependence on initial conditions, as we show in Sect. I.2.2.3.

Observe also that the function $x_t(\theta) = x(t+\theta)$ for $\theta \in [-1, 0]$, with x the solution from (I.1.10), is discontinuous at $\theta = \lfloor t \rfloor - t$ for $t \geq 1$; see again Fig. I.1.1. The conventional wisdom when working with functional differential equations is to think of the solution *history segments* x_t as living in the phase space of the associated dynamical system. This is reasonable, since to specify an initial-value problem it is necessary to define the initial condition on an interval. For the present example, this interval is $[-1, 0]$ since the largest delay is 1. Finally, as we could have taken x_0 to be any continuous function (initial condition) in (I.1.9) and defined a solution $x(t)$ in a similar way, we come to the following observations concerning the phase space:

> The phase space of an impulsive functional differential equation must generally contain discontinuous functions. In particular, any "reasonable" phase space must contain the following subsets:
>
> - the continuous functions;
> - functions that are continuous except for at a single point.

The last of these two conditions makes it necessary to enlarge the phase space substantially from merely the continuous functions. Indeed, if we want the phase space to admit a vector space structure, we need to in fact allow for functions that are discontinuous at a countably infinite number of points.

Now that we have demonstrated the fundamental technical novelties of impulsive functional differential equations as compared to their counterparts without impulses, and our first task will be to put these systems into a rigorous nonautonomous dynamical systems framework. This will allow us to use classical techniques of nonlinear functional analysis to study the solution set of impulsive dynamical systems and determine the effects of parameter variation on these solutions. The first steps are to decide on a phase space and develop the theory of linear equations. Nonlinear equations will then be studied using the mild solution formalism, and from there we will be able to venture into the world of invariant manifolds and bifurcations.

I.1.1 Nonautonomous Dynamical Systems

Here we introduce some definitions related to nonautonomous dynamical systems. This is needed to provide a working theory for linear systems, but the terminology is very useful for nonlinear systems as well.

Definition I.1.1.1. *If X is a Banach space, a subset $\mathcal{M} \subseteq \mathbb{R} \times X$ is a* nonautonomous set over X*. For each $t \in \mathbb{R}$, the* fibre *is the set $\mathcal{M}(t) \subset C$ defined by*

$$\mathcal{M}(t) = \{x : (t, x) \in \mathcal{M}\}.$$

Definition I.1.1.2. *A* process on X *is a pair $(S, \mathcal{M})$, where $\mathcal{M}$ is a nonautonomous set over $\mathbb{R} \times X$ and $S : \mathcal{M} \to X$, whose action we denote by $S(t, (s, x)) = S(t, s)x$ satisfies the following:*

- *$\{t\} \times X \subset \mathcal{M}(t)$ and $S(t, t) = I_X$ for all $t \in \mathbb{R}$, where I_X is the identity operator on X.*
- *$S(t, s)x = S(t, v)S(v, s)x$ whenever $(s, x) \in \mathcal{M}(v)$ and $(v, S(v, s)x) \in \mathcal{M}(t)$.*

A process is a forward process *if for all $s \in \mathbb{R}$ and $x \in X$, $(t, S(t, s)x) \in \mathcal{M}(t)$ for all $t \geq s$. A process is an* all-time process *if for all $t, s \in \mathbb{R}$ and $x \in X$, $(t, S(t, s)x) \in \mathcal{M}(t)$.*

Processes will typically be identified in this monograph with solutions of impulsive differential equations, both finite- and infinite-dimensional. The definition of process stated above is rather different than the standard ones due to Dafermos [40] and Hale [57]. Regardless, the following definition can make some statements less verbose and more intuitive.

Definition I.1.1.3. *If $\mathcal{M}$ is a nonautonomous set over $\mathbb{R} \times X$ and $\mathcal{M}(t) \subset \mathbb{R} \times X$ is the associated t-fibre, define the* (t, s)-fibre *by*

$$\mathcal{M}(t, s) = \{x : (s, x) \in \mathcal{M}(t)\} \subset X.$$

The two-parameter semigroup associated with a process $(S, \mathcal{M})$ *is the family $S(t, s) : \mathcal{M}(t, s) \to X$ defined by $S(t, s)x = S(t, (s, x))$ for all $t, s \in \mathbb{R}$.*

The two-parameter semigroup is appropriately named. The following proposition follows easily from the definitions.

Proposition I.1.1.1. *The two-parameter semigroup $S(t, s) : \mathcal{M}(t, s) \to X$ associated with a process $(S, \mathcal{M})$ enjoys the following properties:*

1. *$S(t, t) = I_X$ for all $t \in \mathbb{R}$.*
2. *$S(t, s)x = S(t, v)S(v, s)x$ provided $x \in \mathcal{M}(v, s)$ and $S(v, s)x \in \mathcal{M}(t, v)$.*

Conversely, a two-parameter semigroup of operators $S(t,s) : \mathcal{M}(t,s) \to X$ satisfying properties 1 and 2 of Proposition I.1.1.1 determines a process.

Definition I.1.1.4. *A* two-parameter semigroup over *X is a family $S(t,s) : \mathcal{M}(t,s) \to X$ of functions, with $\mathcal{M}$ a nonautonomous set over $\mathbb{R} \times X$, satisfying properties 1 and 2 of Proposition I.1.1.1.*

Proposition I.1.1.2. *Let $S(t,s) : \mathcal{M}(t,s) \to X$ be a two-parameter semigroup over X. $S(t,s)$ is the two-parameter semigroup associated with $(\tilde{S}, \mathcal{M})$, with $\tilde{S}(t,(s,x)) := S(t,s)x$.*

Proof. We check that $(\tilde{S}, \mathcal{M})$ as defined is indeed a process. Since $S(t,t) = I_X$ has domain X and $S(t,t) : \mathcal{M}(t,t) \to X$, it follows that $\mathcal{M}(t,t) = X$. But this means that $\{t\} \times X \subset \mathcal{M}(t)$. Also, $\tilde{S}(t,t) = S(t,t) = I_X$. This verifies the first condition of a process. For the second condition, $(s,x) \in \mathcal{M}(v)$ and $(v, \tilde{S}(v,s)x) \in \mathcal{M}(t)$ imply $x \in \mathcal{M}(v,s)$ and $\tilde{S}(v,s)x \in \mathcal{M}(t,v)$. By definition of $\tilde{S}$ and the properties of the two-parameter family S, this gives us $S(t,s)x = S(t,v)S(v,s)x$, which implies $\tilde{S}(t,s)x = \tilde{S}(t,v)\tilde{S}(v,s)x$. □

In this way, a process uniquely determines its associated two-parameter semigroup and vice versa. This allows us to easily define linear processes, decompositions and invariant sets in terms of the two-parameter semigroup.

Definition I.1.1.5. *A process $(S, \mathcal{M})$ is* linear *if $\mathcal{M}(t,s)$ is a linear subspace of X for all $t,s \in \mathbb{R}$, $S(t,s) : \mathcal{M}(t,s) \to X$ is bounded and linear and $\mathcal{M}(t,s) = X$ whenever $t \geq s$.*

Definition I.1.1.6. *Let $(S, \mathcal{M})$ be a linear process, and consider the restriction of the associated two-parameter semigroup $S(t,s) : X \to X$ for $t \geq s$. We say that $(S, \mathcal{M})$ is* spectrally separated *if there exists a triple (P_s, P_c, P_u) of bounded projection-valued functions $P_i : \mathbb{R} \to \mathcal{L}(X)$ for $P_s + P_c + P_u = I_X$ such that the following hold for $i, j \in \{s, c, u\}$:*

1. *There exists a constant N such that $\sup_{t \in \mathbb{R}}(||P_s(t)|| + ||P_c(t)|| + ||P_u(t)||) = N < \infty$.*

2. *The projectors are mutually orthogonal: $P_i(t)P_j(t) = 0$ for $i \neq j$.*

3. *$S(t,s)P_i(s) = P_i(t)S(t,s)$ for all $t \geq s$.*

4. *Define $S_i(t,s)$ as the restriction of $S(t,s)$ to $X_i(s) = \mathcal{R}(P_i(s))$. The operators $S_c(t,s) : X_c(s) \to X_c(t)$ and $S_u(t,s) : X_u(s) \to X_u(t)$ are invertible, and we denote $S_c(s,t) = S_c(t,s)^{-1}$ and $S_u(s,t) = S_u(t,s)^{-1}$ for $s \leq t$.*

5. *For all $t, s, v \in \mathbb{R}$, we have*

$$S_c(t,s) = S_c(t,v)S_c(v,s), \qquad S_u(t,s) = S_u(t,v)S_u(v,s).$$

6. *There exist real numbers $a < 0 < b$ such that for all $\epsilon > 0$, there exists $K \geq 1$ such that*

$$||S_u(t,s)|| \leq Ke^{b(t-s)}, \qquad t \leq s \tag{I.1.11}$$
$$||S_c(t,s)|| \leq Ke^{\epsilon|t-s|}, \qquad t,s \in \mathbb{R} \tag{I.1.12}$$
$$||S_s(t,s)|| \leq Ke^{a(t-s)}, \qquad t \geq s. \tag{I.1.13}$$

Definition I.1.1.7. *Let $(S,\mathcal{M})$ be a spectrally separated linear process. Define the following nonautonomous sets over X for $i \in \{s,c,u\}$:*

$$X_i = \bigcup_{t\in\mathbb{R}} \{t\} \times X_i(t). \tag{I.1.14}$$

The nonautonomous sets X_s, X_c and X_u are the stable, centre and unstable fibre bundles[1]*, respectively.* $(S,\mathcal{M})$ *is* hyperbolic *if $X_c = \{0\}$, otherwise it is nonhyperbolic.*

Definition I.1.1.8. *Let $(S,\mathcal{M})$ be a process. A nonautonomous set $\mathcal{V}$ over X is*

- positively invariant *if $S(t,s)x \in \mathcal{V}(t)$ for all $t \geq s$, whenever $x \in \mathcal{V}(s)$;*
- negatively invariant *if $S(t,s)x \in \mathcal{V}(t)$ for all $t \leq s$, whenever $x \in \mathcal{V}(s)$;*
- invariant *if it is both positively and negatively invariant.*

From the definition of invariance and properties 4 and 5 of spectral separation, we directly get the following proposition.

Proposition I.1.1.3. *Let $(S,\mathcal{M})$ be linear spectrally separated. The stable fibre bundle is positively invariant, and the centre and unstable fibre bundles are (positively and negatively) invariant.*

The stable, centre and unstable fibre bundles of a linear process will eventually play the roles of the stable, centre and unstable subspaces from, for example, ordinary differential equations. A consequence of Propositions I.1.1.1 and I.1.1.2 is that any definition applicable to a process is also applicable to an arbitrary two-parameter semigroup. This includes linearity, invariance and spectral separation.

I.1.2 History Functions

It is necessary to define various history functions or window functions. Let $x : I \to X$ for some interval $I \subset \mathbb{R}$ and a Banach space X. Let $r > 0$ be finite, let $t \in I$ and assume $\inf I < t - r$.

[1]A nonautonomous set over X naturally has the structure of a topological fibre bundle over X with base space $\mathbb{R}$, hence the borrowing of this term here.

Definition I.1.2.1. *The* history of x at time $t \in I$ *is* $x_t : [-r, 0] \to X$ *defined by*

$$x_t(\theta) = x(t + \theta). \tag{I.1.15}$$

Definition I.1.2.2. *The* one-point left-limit history of x at $t \in I$ *is* $x_{t^-} : [-r, 0] \to X$ *defined by*

$$x_{t^-}(\theta) = \begin{cases} x(t+\theta), & \theta < 0 \\ \lim_{s \to 0^-} x(t+s), & \theta = 0, \end{cases} \tag{I.1.16}$$

provided the limit exists.

Definition I.1.2.3. *The* regulated left-limit history of x at $t \in I$ *is* $x_t^- : [-r, 0] \to X$ *defined by*

$$x_t^-(\theta) = \lim_{s \to 0^-} x(t + \theta + s),$$

provided the limits exist.

The various history functions are illustrated schematically in Fig. I.1.2. They all coincide when $x : I \to X$ is continuous, whereas if the set of discontinuities is reasonably well-behaved, then each of the history functions exists. Note that the regulated left-limit can be equivalently written in the more suggestive form

$$x_t^-(\theta) = \lim_{s \to 0^-} x_{t+s}(\theta),$$

so that in a pointwise (in θ) sense, we have the alternative definition

$$x_t^- = \lim_{s \to 0^-} x_{t+s}.$$

I.1.3 The Space $\mathcal{RCR}$ of Right-Continuous Regulated Functions

When working with impulsive functional differential equations, we will see that the natural phase space is that of the right-continuous regulated functions. Denote

$$\mathcal{RCR}(I, X) = \left\{ f : I \to X : \forall t \in I, \lim_{s \to t^+} f(s) = f(t) \text{ and } \lim_{s \to t^-} f(s) \text{ exists} \right\},$$

where $X \subseteq \mathbb{R}^n$ and $I \subseteq \mathbb{R}$. When X and I are closed,

$$\mathcal{RCR}_b(I, X) := \{ f \in \mathcal{RCR}(I, X) : ||f|| < \infty \}$$

is a Banach space with the norm $||f|| = \sup_{x \in I} |f(x)|$. We will also at times require the space $\mathcal{G}(I, X)$ of regulated functions from I into X; this is

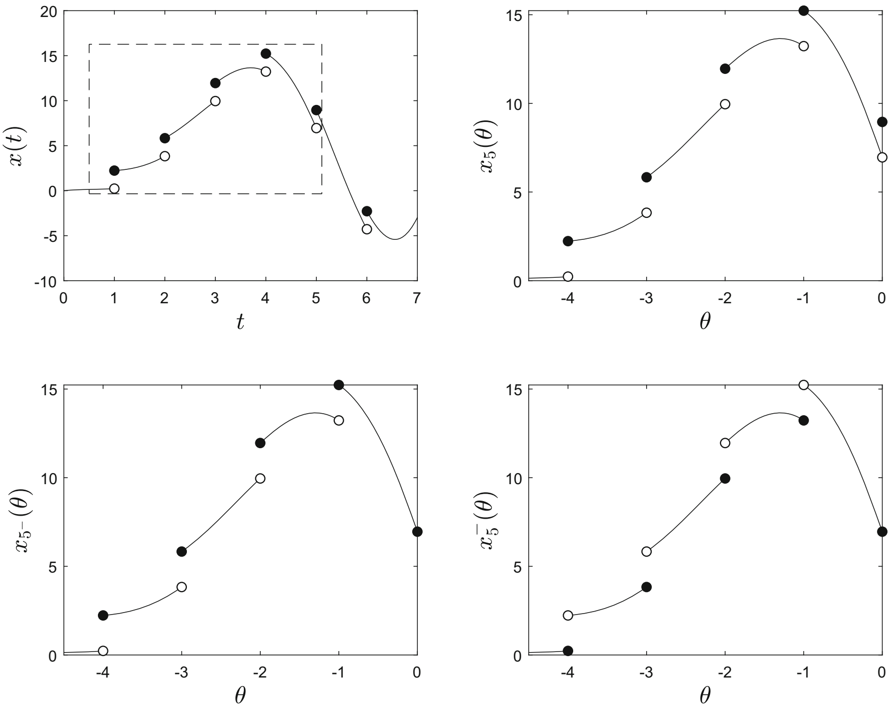

Figure I.1.2: History functions for an illustrative example of a right-continuous function (top left), with $r = 4.5$. The data contained within the dashed rectangle is (in a limiting sense) needed to define the various history functions. Filled-in circles indicate the value of the function at the relevant argument (t or θ), while empty circles denote appropriate limit points. Note that x_5 and the one-point left-limit x_{5^-} remain continuous from the right, but the regulated left-limit x_5^- is continuous from the left

merelythe set of functions $f : I \to X$ that possess left- and right-limits at each point, with no continuity sidedness restriction. One may consult Honig [69] for background on regulated functions, in particular, the claim that $\mathcal{G}(I, X)$ is complete. As $\mathcal{RCR}(I, X)$ is a closed subspace thereof, its completeness follows immediately. We will write $\mathcal{RCR} := \mathcal{RCR}([-r, 0], \mathbb{R}^n)$ when there is no ambiguity, and note that since $\mathcal{RCR}_b([-r, 0], \mathbb{R}^n) = \mathcal{RCR}([-r, 0], \mathbb{R}^n)$, we may identify $\mathcal{RCR}$ with its associated Banach space. The step functions are dense in $\mathcal{G}(I, X)$ and by extension, the subspace $\mathcal{RCR}(I, X)$. The proof of the following proposition appears in Honig [69].

Proposition I.1.3.1. *Let I be compact. For all $f \in \mathcal{G}(I, X)$, there exists a sequence of step functions $f_n : I \to X$ such that $||f_n - f|| \to 0$.*

Adapting the aforementioned proof to the explicitly right-continuous case, one obtains a specification to $\mathcal{RCR}(I, X)$.

Lemma I.1.3.1. *Let I be compact. For all $f \in \mathcal{RCR}(I, X)$, there exists a sequence of right-continuous step functions $f_n : I \to X$ such that $||f_n - f|| \to 0$.*

Regulated functions are integrable, as the following lemma guarantees. Its proof is simple and omitted.

Lemma I.1.3.2. *Let $f \in \mathcal{G}(I, \mathbb{R}^n)$ for some interval I. f is locally integrable —that is, $\int_S f(x)dx$ exists for all $S \subseteq I$ compact.*

In contrast to continuous functions, if $f \in \mathcal{RCR}(A, B)$ and $g \in \mathcal{RCR}(B, X)$ for A, B real intervals, it need not be true that $g \circ f \in \mathcal{RCR}(A, X)$. This is generally false even if f is continuous. As a simple example, take

$$f(x) = \begin{cases} 2x, & 0 \leq x < \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \leq x \leq 1, \end{cases}$$

and $g(x) = \lfloor x \rfloor$. Then, $f \in C([0, 1], \mathbb{R})$ and $g \in \mathcal{RCR}(\mathbb{R}, \mathbb{R})$, but the composition $g \circ f : [0, 1] \to \mathbb{R}$ has the form

$$g \circ f(x) = \begin{cases} 0, & x \neq \frac{1}{2} \\ 1, & x = \frac{1}{2} \end{cases}$$

and so is not continuous from the right. The following is a sufficient condition for the composition $g \circ f$ to be $\mathcal{RCR}(A, X)$ for *any* $g \in \mathcal{RCR}(B, X)$.

Lemma I.1.3.3. *If $f \in \mathcal{RCR}(A, B)$ for intervals $A, B \subset \mathbb{R}$, then $g \circ f \in \mathcal{RCR}(A, X)$ for any $g \in \mathcal{RCR}(B, X)$ provided the following conditions are satisfied:*

1. *For all $x \in A$, there exists $\epsilon > 0$ such that $f|_{[x,x+\epsilon]}$ is nondecreasing.*
2. *If $x \in A$ and $f(x) \neq f(x^-)$, there exists $\epsilon > 0$ such that $f|_{[x-\epsilon,x)}$ is nondecreasing.*

Proof. Let $x < \sup A$. Since $f|_{[x,x+\epsilon]}$ is nondecreasing for $\epsilon > 0$ sufficiently small, we have that for any $x_n \to x^+$, the sequence $f(x_n)$ is eventually nondecreasing. Since $f \in \mathcal{RCR}(A, B)$, we have $f(x_n) \to f(x)$. Then, as $g \in \mathcal{RCR}(B, X)$, we conclude $\lim_{n\to\infty} g \circ f(x_n) = g \circ f(x)$. Now let $x > \inf A$. There are two cases to consider. If $f(x) \neq f(x^-)$, then from the second condition $f|_{[x-\epsilon,x)}$ is nondecreasing for $\epsilon > 0$ small enough. A symmetric argument to how we proved the existence of right-limits can now be used to prove $\lim_{y\to x^-} f \circ g(y)$ exists. The more difficult case is if $f(x) = f(x^-)$—that is, when f is continuous at x.

For $\epsilon > 0$ sufficiently small, $\text{sgn}(f|_{[x-\epsilon,x)} - f(x))$ is constant. To see this, assume to the contrary that there exists $x_n \to x^-$ such that $\text{sgn}(f(x_n) - f(x))$

is oscillatory. This is impossible because $g(y) := \text{sgn}(f(y) - f(x))\nu$ is an element of $\mathcal{RCR}(B, X)$ for any $0 \neq \nu \in X$, and $g \circ f(x_n)$ has no limit as $n \to \infty$ even though $x_n \to x^-$. We can then without loss of generality assume $f(y) \geq f(x)$ for all $x \in [x - \epsilon, x)$ for some $\epsilon > 0$. Let $x_n \in [x - \epsilon, x)$ satisfy $x_n \to x$. Let x_{n_k} be any subsequence of x_n, and consider the sequence $k \mapsto f(x_{n_k})$. Since the limit exists and $f(x_{n_k}) \geq f(x)$, there exists a further subsequence such that $j \mapsto f(x_{n_{k_j}}) := u_j$ is nonincreasing. Since $g \in \mathcal{RCR}(B, X)$, $\lim_{j\to\infty} g(u_j)$ exists. As $u_j \to f(x)^-$, the limit does not depend on the choice of the subsequence. Thus, $\lim_{n\to\infty} g \circ f(x_n) = g(f(x)^-)$ for any sequence $x_n \to x^-$. □

Remark I.1.3.1. *Later on, one of the conditions needed to ensure the existence of solutions for an impulsive functional differential equation will concern the regularity of the composition $t \mapsto f(t, \phi_t)$ whenever $\phi \in \mathcal{RCR}(I, \mathbb{R}^n)$, where f is the functional defining the vector field and I is an interval. The motivation for Lemma I.1.3.3 is that functionals with time-dependent discrete delays involve terms of the form*

$$f_d(t, \phi) = \phi(d(t))$$

for state $\phi \in \mathcal{RCR}([-r, 0], \mathbb{R}^n)$ and some delay function $d : \mathbb{R} \to [-r, 0]$. In this case, $f_d(t, x_t) = x(t-d(t))$. Lemma I.1.3.3 provides a sufficient condition on $t - d(t)$ for the $t \mapsto x(t - d(t))$ to be an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ for any *$x \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$.*

We will eventually need spaces of function $f : I \to X$ that are differentiable from the right and whose right-hand derivatives are elements of $\mathcal{RCR}(I, X)$. Specifically, define the right-hand derivative by

$$d^+ f(t) = \lim_{\epsilon\to 0^+} \frac{f(t + \epsilon) - f(t)}{\epsilon},$$

and introduce the space

$$\mathcal{RCR}^1(I, X) = \{f \in \mathcal{RCR}(I, X) : d^+ f \in \mathcal{RCR}(I, X)\}.$$

This space is complete with respect to the norm $||f||_1 = ||f|| + ||d^+ f||$ when restricted to the subspace consisting of functions that are $|| \cdot |||_1$-bounded, although we will not make great use of this fact in this monograph.

We will need a few convergence and boundedness results for Perron–Stieltjes integrals involving right-continuous regulated functions and functions of bounded variation. These two results appear in [31] and are based on results by Tvrdý [142]. In what follows, $v^\intercal$ denotes the transpose of $v \in \mathbb{R}^n$. In the two lemmas below, we overload the notation and define $f^\intercal : [a, b] \to \mathbb{R}^{n*}$ by $f^\intercal(t) = [f(t)]^\intercal$.

Lemma I.1.3.4. *Let $f : [a,b] \to \mathbb{R}^n$ be of bounded variation and $g \in \mathcal{RCR}([a,b],\mathbb{R}^n)$. The integral $\int_a^b f^\intercal(t)dg(t)$ exists in the Perron–Stieltjes sense, and*

$$\left|\int_a^b f^\intercal(t)dg(t)\right| \leq (|f(a)| + |f(b)| + \mathrm{var}_a^b f)||g||, \tag{I.1.17}$$

where $\mathrm{var}_a^b f$ denotes the total variation of f on the interval $[a,b]$.

Lemma I.1.3.5. *Let $h_n \in \mathcal{RCR}([a,b],\mathbb{R}^n)$ and $h \in \mathcal{RCR}([a,b],R^n)$ be such that $||h_n - h|| \to 0$ as $n \to \infty$. For any $f : [a,b] \to \mathbb{R}^n$ of bounded variation, the Perron–Stieltjes integrals $\int_a^b f^\intercal(t)dh(t)$ and $\int_a^b f^\intercal(t)dh_n(t)$ exist and*

$$\lim_{n\to\infty} \int_a^b f^\intercal(t)dh_n(t) = \int_a^b f^\intercal(t)dh(t). \tag{I.1.18}$$

Next, we provide a generalization of a result by Ballinger and Liu [14], which can itself be seen as a weakened form of the result that if $x : \mathbb{R} \to \mathbb{R}^n$ is continuous, then $F : t \mapsto x_t \in C([-r,0],\mathbb{R}^n)$ is continuous as a function $F : \mathbb{R} \to C([-r,0],\mathbb{R}^n)$, where the codomain is the Banach space of continuous functions from $[-r,0]$ to $\mathbb{R}^n$ equipped with the supremum norm. The following lemma appears in [31], and we reproduce its proof here.

Lemma I.1.3.6. *Let $r > 0$ be finite, and let $\phi \in \mathcal{RCR}([a,b],\mathbb{R}^n)$ for some $b \geq a+r$. With $\phi_t : [-r,0] \to \mathbb{R}^n$ defined by (I.1.15), $t \mapsto ||\phi_t||$ is an element of $\mathcal{RCR}([a+r,b],\mathbb{R})$.*

Proof. Let $t \in [a+r,b]$ be fixed. We will only prove right-continuity, since the proof of the existence of left-limits is similar. It suffices to prove that for any decreasing sequence $s_n \downarrow 0$, we have $||\phi_{t+s_n}|| \to ||\phi_t||$. Let $\epsilon > 0$ be given. By right-continuity of ϕ, for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $0 < \mu < \delta$, then $|\phi(t+\mu) - \phi(t)| < \epsilon$. Therefore,

$$\begin{aligned} ||\phi_{t+s_n}|| &= \sup_{\mu\in[-r,0]} |\phi(t+s_n+\mu)| \leq \sup_{\mu\in[-r,s_n]} |\phi(t+\mu)| \\ &\leq \max\{||\phi_t||, \sup_{\mu\in[0,s_n]} |\phi(t+\mu)|\} \\ &\leq \max\{||\phi_t||, |\phi(t)| + \epsilon\} \leq ||\phi_t|| + \epsilon, \end{aligned}$$

provided $s_n < \delta$. On the other hand, since ϕ is bounded, there exists some sequence $x_n \in [-r,0]$ such that $|\phi_t(x_n)| \to ||\phi_t||$. By passing to a subsequence, we may assume $x_n \to \hat{x} \in [-r,0]$. If $\hat{x} > -r$, then we have

$$||\phi_{t+s_n}|| \geq \sup_{\mu\in[-r+s_n,0]} |\phi(t+\mu)| = |\phi(\hat{x})| = ||\phi_t||$$

provided $s_n < -\hat{x}$, while if $\hat{x} = -r$, we notice that the sequence $x'_n = t-r+s_n$ converges to $t+\hat{x}$, so that for all $\epsilon > 0$, there exists $N_3 > 0$ such that for $n \geq N$,

$$||\phi_{t+s_n}|| \geq |\phi(t+s_n) \geq ||\phi_t|| - \epsilon.$$

Therefore, if we let $s_{N_1} < \delta$ and $s_{N_2} < -\hat{x}$, then by setting $N = \max\{N_1, N_2, N_3\}$, it follows by the above three inequalities that for $n \geq N$,

$$-\epsilon \leq ||\phi_{t+s_n}|| - ||\phi_t|| \leq \epsilon.$$

We conclude $||\phi_{t+s_n}||$ converges to $||\phi_t||$. □

Using essentially the same argument, one can prove the following very slight generalization.

Lemma I.1.3.7. *Suppose* $\phi \in \mathcal{RCR}([a,b],\mathbb{R}^n)$, $X \in \mathcal{RCR}([a,b],\mathbb{R}^{n\times m})$ *and* $z \in \mathcal{RCR}([a,b],\mathbb{R}^m)$ *for some* $b \geq a+r$. *Then, the function* $t \mapsto ||\phi_t + X_t z(t)||$ *is an element of* $\mathcal{RCR}([a+r,b],\mathbb{R})$

The final element in our overview of right-continuous regulated functions is a characterization of the topological dual $\mathcal{RCR}^*$. A result from Tvrdy [142] provides such for the dual of the space of regulated left-continuous scalar-valued functions, and for our purposes, the obvious modification that is needed is the following. It can also be found in [31].

Lemma I.1.3.8. $F \in \mathcal{RCR}^*$ *if and only if there exists* $q \in \mathbb{R}^n$ *and* $p : [-r,0] \to \mathbb{R}^n$ *of bounded variation such that*

$$F(x) = q^\intercal x(0) + \int_{-r}^{0} p^\intercal(t)dx(t), \tag{I.1.19}$$

where the integral is a Perron–Stieltjes integral.

A final comment concerns the various left-limit histories introduced in Sect. I.1.2. When $x \in \mathcal{RCR}(I,\mathbb{R}^n)$, these histories are themselves regulated functions. The following proposition follows directly from the definitions of the history functions and the spaces $\mathcal{RCR}$ and $\mathcal{G}$.

Proposition I.1.3.2. *Let* $x \in \mathcal{RCR}(I,\mathbb{R}^n)$, *and assume* $\inf I < t - r$. *The left-limit histories* $x_{t^-} \in \mathcal{RCR}([-r,0],\mathbb{R}^n)$ *and* $x_t^- \in \mathcal{G}([-r,0],\mathbb{R}^n)$ *exist. Also,* $x_t^- : [-r,0] \to \mathbb{R}^n$ *is continuous from the left.*

I.1.4 Gelfand–Pettis Integration

In this monograph we will regularly need to integrate functions $f : \mathbb{R} \to X$ mapping into a Banach space X that have poor measurability properties. Such constraints make it difficult to establish the existence of a strong integral. Thankfully, for our purposes the following duality-based weak integration is sufficient.

Definition I.1.4.1. *Let* X *be a Banach space and* (S,Σ,μ) *a measure space. We say that* $f : S \to X$ *is* Pettis integrable *(or Gelfand–Pettis integrable) if* *there exists a set function* $I_f : \Sigma \to X$ *such that*

$$\varphi^* I_f(E) = \int_E \varphi^* f d\mu$$

for all $\varphi^* \in X^*$ *and* $E \in \Sigma$. I_f *is the* indefinite Pettis integral of f *and* $I_f(E)$ *the* Pettis integral of f on E.

In the above definition X^* is the topological dual of X. By abuse of notation, we will often write $I_f(E) = \int_E f d\mu$ when there is no ambiguity. The following proposition will be very useful; its proof is elementary and can be found in numerous textbooks on functional analysis and integration. For a brief introduction, one may consult Hille and Philips [65].

Proposition I.1.4.1. *The Pettis integral possesses the following properties:*

- *If* f *is Pettis integrable, then its indefinite Pettis integral is unique.*
- *If* $T : X \to X$ *is a bounded linear operator, then* $T\left(\int_E f d\mu\right) = \int_E (Tf)\, d\mu$ *whenever one of the integrals exists.*
- *If* $\mu(A \cap B) = 0$, *then* $\int_{A\cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$.
- $|| \int_E f d\mu|| \leq \int_E ||f|| d\mu$.

I.1.5 Integral and Summation Inequalities

We conclude this chapter with three inequalities. The first is an impulsive Gronwall–Bellman inequality for regulated functions. The result is similar to Lemma 2.3 from the 1993 monograph of Bainov and Simeonov [9], and the proof is omitted. The second one concerns an elementary estimation of sums of continuous functions at impulses, when the sequence of impulses satisfies a separation condition. The third allows for a coarse bound on sums of constant sequences.

Lemma I.1.5.1. *Suppose* $x \in \mathcal{RCR}([s,\alpha],\mathbb{R})$ *satisfies the inequality*

$$x(t) \leq C + \int_s^t (p(\mu)x(\mu) + h(\mu))d\mu + \sum_{s<t_i\leq t} (b_i x(t_i^-) + g_i) \tag{I.1.20}$$

for some nonnegative integrable function p, *integrable and bounded* h, *nonnegative constants* b_i, g_i *and* c, *and all* $t \in [s,\alpha]$. *For* $t \geq s$, *define*

$$z(t,s) = \exp\left(\int_s^t p(\mu)d\mu\right) \prod_{s<t_i\leq t} (1+b_i).$$

Then, $\mu \mapsto z(t,\mu)$ *is integrable, and the following inequality is satisfied:*

$$x(t) \leq Cz(t,s) + \int_s^t z(t,\mu)h(\mu)d\mu + \sum_{s<t_i\leq t} z(t,t_i)g_i. \tag{I.1.21}$$

Lemma I.1.5.2. *Let $f \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}_+)$, and suppose that $\{t_k\}$ satisfies $t_{k+1} - t_k \geq \xi$.*

1. *If f is nondecreasing, then $\sum_{s<t_i\leq t} f(t_i) \leq \frac{1}{\xi}\int_s^{t+\xi} f(\mu)d\mu$.*
2. *If f is nonincreasing, then $\sum_{s<t_i\leq t} f(t_i) \leq \frac{1}{\xi}\int_{s-\xi}^{t} f(\mu)d\mu$.*

Proof. Let $\{t_0, \dots, t_N\} = \{t_k : k \in \mathbb{Z}\} \cap (s, t]$. If f is nondecreasing, then

$$\sum_{s<t_i\leq t} f(t_i) = \sum_{i=0}^{N} f(t_i) = \frac{1}{\xi}\sum_{i=0}^{N} f(t_i)\xi \leq \frac{1}{\xi}\sum_{i=0}^{N} f(t_0 + i\xi)\xi \leq \frac{1}{\xi}\int_s^{t+\xi} f(\mu)d\mu.$$

The nonincreasing case is similar. □

Lemma I.1.5.3. *Suppose the sequence of impulses $\{t_k\}$ satisfies $\xi_1 \leq t_{k+1} - t_k \leq \xi_2$. Then,*

$$\frac{t-s}{\xi_1} - 1 \leq \#\{k \in \mathbb{Z} : s < t_k \leq t\} \leq \frac{t-s}{\xi_2} + 1.$$

I.1.6 Comments

Definition I.1.1.1 of a process uses the language of nonautonomous sets explicitly. The idea of these indexed families of sets has a long history of use in nonautonomous dynamical systems, and the term appears in this form in, for example, Koksch and Siegmund [81] and Rasmussen [120]. Our definition of process itself is certainly inspired by the one due to Dafermos [40] and Hale [57], but the motivation here is that we wish to allow for the process to not be defined for all (forward) time, regardless of the initial state $x \in X$. The reason here is that the solution maps of nonlinear impulsive functional differential equations will naturally define processes that are generally only well-defined for small increments forward in time. Instead of defining a process imprecisely as a partial function such as $U : \mathbb{R} \times \mathbb{R} \times X \to X$ for a final/initial time pair $(t, s) \in \mathbb{R} \times \mathbb{R}$ and initial state $x \in X$, we choose to bake this into the domain through the use of a nonautonomous set. The transition is made concrete by appealing to the two-parameter semigroup $S(t, s) : \mathcal{M}(t, s) \to X$.

The one-point left-limit history x_{t^-} is used extensively in the literature on impulsive functional differential equations, though the first appearance seems to be in the 1999 paper of Ballinger and Liu [13]. The term regulated left-limit history is introduced in Church and Liu [33], although the definition had been previously used in literature concerning stability.

The left-limit history functions x_{t^-} and x_t^- are distinct, and the reader may have some difficulty in navigating the literature as most authors use the symbol x_{t^-} exclusively in impulse conditions of the form

$$\Delta x = I(t_k, x_{t_k^-}),$$

although how the symbol x_{t^-} should be interpreted may vary. Zhang and Sun [160] consider stability of impulsive functional differential equations with fixed delays, where the impulse effect is of the form

$$\Delta x = I_k(x(t_k^-)) + J_k(x(t_k^- - r)),$$

which can be identified as the action of a functional on the regulated left-limit $x_{t_k}^-$. Inspired by this work, Lin, Li and O'Regan [91] consider stability of systems with impulse effect of the form

$$\Delta x = I_k(x(t_k^-)) + J_k(\mathbf{x}_{t_k^-}),$$

where we have placed the second x in boldface to emphasize that the notation used in the cited reference is what we refer to here as the one-point left-limit (x_{t^-}), but the definition used in that paper is the regulated left-limit (x_t^-). The regulated left-limit is also used in [26, 96, 150] and others, although we should point out that some of these papers suffer from minor technical errors mostly relating to the observation that if $x : I \to \mathbb{R}^n$ is continuous from the right, then $x_t^- : [-r, 0] \to \mathbb{R}^n$ is continuous from the *left* with finite jump discontinuities on the *right*—see Fig. I.1.2 for a visual aid. To contrast, the papers [13, 14, 93, 137] use the one-point left-limit.

In this monograph, we will primarily make use of the one-point left-limit. The reason for this is $m : \mathcal{RCR} \to \mathcal{RCR}$ defined by $m\phi = \phi_{0^-}$ is bounded and linear with norm 1. This allows for a rather elegant operator-theoretic definition of mild solutions that is amenable to the eventual construction of invariant manifolds.

The right-continuous regulated functions or *càdlàg*[2] functions are used extensively in probability and stochastic processes. The use of regulated functions in impulsive dynamical systems has one of its first appearances in the work of Bachar and Arijno [7] in 2004, where left-continuous regulated functions are used. The right-continuous regulated functions as they are used in impulsive dynamical systems were considered by Church and Liu [31, 33], where the integral inequalities from Sect. I.1.5 appear. Regulated functions $\mathcal{G}(I, \mathbb{R}^n)$ are extensively used by Federson and Schwabik [45] in their approach to solutions of impulsive functional differential equations through the lens of generalized ordinary differential equations. More recently, some authors have taken the regulated functions as the phase space in order to prove the existence of periodic solutions for some special classes of nonlinear impulsive FDE [6, 46] by fixed-point theory applied to Poincaré return maps.

[2]From the French: "continue à droite, limite à gauche", which translates to "continuous from the right with limit on the left".

Chapter I.2

General Linear Systems

In this chapter we will be primarily interested in the linear impulsive RFDE

$$\dot{x} = L(t)x_t + h(t), \qquad t \neq t_k \tag{I.2.1}$$

$$\Delta x = B(k)x_{t^-} + r_k, \qquad t = t_k. \tag{I.2.2}$$

The following assumptions will be needed throughout:

H.1 The representation

$$L(t)\phi = \int_{-r}^{0} [d_\theta \eta(t,\theta)]\phi(\theta)$$

holds, where the integral is taken in the Lebesgue–Stieltjes sense, the function $\eta : \mathbb{R} \times [-r, 0] \to \mathbb{R}^{n\times n}$ is jointly measurable and is of bounded variation and right-continuous on $[-r, 0]$ for each $t \in \mathbb{R}$, such that $|L(t)\phi| \leq \ell(t)||\phi||$ for some $\ell : \mathbb{R} \to \mathbb{R}$ locally integrable.

H.2 The sequence t_k is monotonically increasing with $|t_k| \to \infty$ as $|k| \to \infty$, and the representation

$$B(k)\phi = \int_{-r}^{0} [d_\theta \gamma_k(\theta)]\phi(\theta)$$

holds for $k \in \mathbb{Z}$ for functions $\gamma_k : [-r, 0] \to \mathbb{R}^{n\times n}$ of bounded variation and right-continuous, such that $|B(k)| \leq b(k)$.

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Remark I.2.0.1. *Assumption H.1 includes the case of discrete time-varying delays: for example, the linear differential-difference equation*

$$\dot{x} = \sum_{k=1}^{m} A_k(t)x(t - r_k(t))$$

with r_k continuous, is associated with a linear operator satisfying condition H.1 with $\eta(t,\theta) = \sum A_k(t)H_{-r_k(t)}(\theta)$, where $H_z(\theta) = 1$ if $\theta \geq z$ and zero otherwise. It also obviously includes a large class of distributed delays, such as those appearing in the differential equation

$$\dot{x} = \int_{-\tau}^{0} K(t,\theta)x(t+\theta)d\theta.$$

Similar results apply for the jump function $B(k)$ and assumption H.2. Moreover, each of $L(t)$ and $B(k)$ is well-defined on $\mathcal{RCR}$; see Theorem 2.23 from Chapter 3 of [66].

I.2.1 Existence and Uniqueness of Solutions

Definition I.2.1.1. *Let $(s,\phi) \in \mathbb{R} \times \mathcal{RCR}$. A function $x \in \mathcal{RCR}([s - r, \alpha), \mathbb{R}^n)$ for some $\alpha > s$ is an* integrated solution *of the linear impulsive RFDE* (I.2.1)–(I.2.2) satisfying the initial condition (s,ϕ) *if it satisfies $x_s = \phi$ and the integral equation*

$$x(t) = \begin{cases} \phi(0) + \int_s^t [L(\mu)x_\mu + h(\mu)]d\mu + \sum_{s<t_i\leq t}[B(i)x_{t_i^-} + r_i], & t > s \\ \phi(t-s), & s - r \leq t \leq s. \end{cases} \tag{I.2.3}$$

Lemma I.2.1.1. *Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$, let $\{r_k : k \in \mathbb{Z}\} \subset \mathbb{R}^n$ and let hypotheses H.1–H.2 hold. For all $(s,\phi) \in \mathcal{RCR}$, there exists a unique integrated solution $x \in \mathcal{RCR}([s - r, \infty), \mathbb{R}^n)$ of* (I.2.1)*–*(I.2.2) *satisfying the initial condition (s,ϕ).*

The above lemma follows by hypotheses H.1–2, the Banach fixed-point theorem, Lemma I.1.5.1 and typical continuation arguments. Note that h may be unbounded on the real line, but since it is regulated we are guaranteed its boundedness on every compact set—see Honig [69]. Any classical solution (in the sense of Ballinger and Liu [13]) is an integrated solution, so the definition is indeed appropriate. We will drop the adjective *integrated* from this point onwards, since this class of solutions will be used exclusively from this point on.

On the note of "classical" solutions, it will later be important that the impulsive RFDE (I.2.1)–(I.2.2) has a regularizing effect on initial conditions. Precisely, we have the following lemma.

Lemma I.2.1.2. *Under the conditions of Lemma I.2.1.1, the integrated solution $x : [s-r,\infty) \to \mathbb{R}^n$ is differentiable from the right on $[s,\infty)$. In particular, if $x : \mathbb{R} \to \mathbb{R}^n$ is a solution defined for all time, then $x \in \mathcal{RCR}^1(\mathbb{R},\mathbb{R}^n)$.*

Proof. The first conclusion follows by the integral representation of solutions with the remark that $\mu \mapsto L(\mu)x_\mu \in \mathcal{RCR}([s,\infty),\mathbb{R}^n)$. For the second part, one can show that the restriction of x to any interval of the form $[s,\infty)$ is differentiable from the right by applying the previous result to the restriction on $[s-r,\infty)$. Since s is arbitrary, the result is proven. □

I.2.2 Evolution Families

In this section we will specialize to the equation

$$\dot{x} = L(t)x_t, \qquad t \neq t_k \tag{I.2.4}$$

$$\Delta x = B(k)x_{t^-}, \qquad t = t_k. \tag{I.2.5}$$

Definition I.2.2.1. *Let hypotheses H.1–H.2 hold. For a given $(s,\phi) \in \mathbb{R} \times \mathcal{RCR}$, let $t \mapsto x(t;s,\phi)$ denote the unique solution of (I.2.4)–(I.2.5) satisfying $x_s(\cdot;s,\phi) = \phi$. The function $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ defined by $U(t,s)\phi = x_t(\cdot,s,\phi)$ for $t \geq s$ is the* evolution family *associated with the homogeneous equation (I.2.4)–(I.2.5).*

From here onwards, we will take the convention that if $L : \mathcal{RCR} \to \mathcal{RCR}$ is a linear operator, then $L\phi(\theta)$ for $\phi \in \mathcal{RCR}$ and $\theta \in [-r,0]$ should be understood as $[L(\phi)](\theta)$. Also, the symbol I_X will refer to the identity operator on X. When the context is clear, we will simply write it as I. Introduce the linear function $\chi_s : \mathbb{R}^n \to \mathcal{RCR}$ defined by

$$[\chi_s\xi](\theta) = \begin{cases} \xi, & \theta = s \\ 0, & \theta \neq s. \end{cases} \tag{I.2.6}$$

Lemma I.2.2.1. *The evolution family satisfies the following properties:*

1) $U(t,t) = I$ for all $t \in \mathbb{R}$.

2) For $s \leq t$, $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ is a bounded linear operator. In particular,

$$||U(t,s)|| \leq \exp\left(\int_s^t \ell(\mu)d\mu\right) \prod_{s<t_i\leq t}(1+b(i)). \tag{I.2.7}$$

3) For $s \leq v \leq t$, $U(t,s) = U(t,v)U(v,s)$.

4) For all $\theta \in [-r,0]$, $s \leq t+\theta$ and $\phi \in \mathcal{RCR}$, $U(t,s)\phi(\theta) = U(t+\theta,s)\phi(0)$.

5) For all $t_k > s$*, one has* $U(t_k, s) = (I + \chi_0 B(k))U(t_k^-, s)$.[1]

6) Let $C(t, s)$ *denote the evolution family on* $\mathcal{RCR}$ *associated with the "continuous" equation* $\dot{x} = L(t)x_t$*. The following factorization holds:*

$$U(t,s) = \begin{cases} C(t,s), & [s,t] \cap \{t_k\}_{k\in\mathbb{Z}} \in \{\{s\}, \emptyset\} \\ C(t,t_k) \circ (I + \chi_0 B(k)) \circ U(t_k^-, s), & t \geq t_k > s. \end{cases} \tag{I.2.8}$$

Proof. Properties (1), (3) and (4) are straightforward, given the uniqueness assertion of Lemma I.2.1.1 and the definition of the evolution family. Property (6) follows similarly once we can establish (5). To obtain boundedness, we use the integral equation (I.2.3) to get the estimate

$$\begin{aligned} |U(t,s)\phi(\theta)| &\leq ||\phi|| + \int_s^{t+\theta} |L(\mu)U(\mu,s)\phi| d\mu + \sum_{s<t_i\leq t+\theta} |B(i)U(t_i^-, s)\phi| \\ &\leq ||\phi|| + \int_s^t \ell(\mu)||U(\mu,s)\phi|| d\mu + \sum_{s<t_i\leq t} b(i)||U(t_i^-, s)\phi||. \end{aligned}$$

Since the upper bounds are independent of θ, denoting $X(t) = U(t,s)\phi$, we obtain

$$||X(t)|| \leq ||\phi|| + \int_s^t \ell(\mu)||X(\mu)|| d\mu + \sum_{s<t_i\leq t} b(i)||X(t_i^-)||.$$

By Lemma I.1.3.6, $t \mapsto ||X(t)||$ is an element of $\mathcal{RCR}([s-r,\infty),\mathbb{R})$. Using the Gronwall inequality of Lemma I.1.5.1, we obtain the desired boundedness (I.2.7). As for property (5),

$$\begin{aligned} U(t_k,s)\phi(0) &= \phi(0) + \int_s^{t_k} L(\mu)U(\mu,s)\phi d\mu + \sum_{s<t_i\leq t_k} B(i)U(t_i^-, s)\phi \\ &= U(t_k^-, s)\phi(0) + B(k)U(t_k^-, s)\phi \end{aligned}$$

and $U(t_k^-, s)\phi(\theta) = U(t_k, s)\phi(\theta)$ for $\theta < 0$. □

The connection between the evolution family and processes is given by the following lemma, whose proof now follows directly from Lemma I.2.2.1.

Lemma I.2.2.2. *Let* $\mathcal{M}$ *be the nonautonomous set over* $\mathbb{R} \times \mathcal{RCR}$ *with* t*-fibre*

$$\mathcal{M}(t) = \bigcup_{s\leq t} \{s\} \times \mathcal{RCR},$$

and define $S(t,(s,\phi)) = x(t; s, \phi)$*.* $(S, \mathcal{M})$ *is a forward, linear process, and* $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ *is its two-parameter semigroup.*

[1]Note here that the left limit is defined by $U(t_k^-, s)\phi(\theta) = U(t_k, s)\phi(\theta)$ for $\theta < 0$, while $U(t_k^-, s)\phi(0) = U(t_k, s)\phi(0^-)$.

I.2.2.1 Phase Space Decomposition

In the analysis of steady states of linear ordinary differential equations, the stable, centre and unstable subspaces play a key role. The appropriate generalization to impulsive functional differential equations is spectral separation of the evolution family. That is, the evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ is *spectrally separated* if it satisfies the properties of Definition I.1.1.6.

If the evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ is spectrally separated, the phase space admits a direct sum decomposition

$$\mathcal{RCR} = \mathcal{RCR}_s(t) \oplus \mathcal{RCR}_c(t) \oplus \mathcal{RCR}_u(t) \tag{I.2.9}$$

for each $t \in \mathbb{R}$. If $(s,\phi) \in \mathcal{RCR}_s$, Eq. (I.1.11) implies that $U(t,s)\phi$ decays to zero exponentially as $t \to \infty$. We say that in the stable fibre bundle, solutions decay exponentially in forward time. Similarly, in the unstable fibre bundle, solutions are defined for all time and decay exponentially in backward time. In the centre fibre bundle, solutions are defined for all time and grow slower than exponentially in forward and backward times. The difference between this decomposition and one more typical of autonomous or ordinary delay differential equations is that the factors of the decomposition are generally time-dependent; that is, they are determined by the t-fibres of the *invariant fibre bundles* $\mathcal{RCR}_s$, $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$.

I.2.2.2 Evolution Families are (Generally) Nowhere Continuous

The use of the phase space $\mathcal{RCR}$ causes the evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ to have some undesirable regularity properties. To illustrate this, consider the trivial impulsive functional differential equation

$$\begin{aligned} \dot{x} &= 0, & t &\neq t_k \\ \Delta x &= 0, & t &= t_k. \end{aligned}$$

The evolution family associated with the above homogeneous equation is equivalent to a one-parameter semigroup; $U(t,s) = V(t-s)$, where for $\xi \geq 0$,

$$V(\xi)\phi(\theta) = \begin{cases} \phi(\xi+\theta), & \xi+\theta \leq 0 \\ \phi(0), & \xi+\theta > 0. \end{cases}$$

Suppose $\phi \in \mathcal{RCR}$ has an internal discontinuity at some $d \in (-r, 0)$. Then, for $\epsilon > 0$ sufficiently small and any $0 \leq t < \min\{|d|, d+r\}$, one has

$$\begin{aligned} ||V(t-\epsilon)\phi - V(t)\phi|| &\geq |V(t-\epsilon)\phi(d-t) - V(t)\phi(d-t)| \\ &= |\phi(d-\epsilon) - \phi(d)|, \end{aligned}$$

which because of the discontinuity is guaranteed to be bounded away from zero for ϵ arbitrarily small. As such, $t \mapsto V(t)\phi$ is nowhere continuous from the left in $[0, \min\{|d|, d+r\})$. On the other hand, we also have

$$\begin{aligned}||V(t+\epsilon)\phi - V(t)\phi|| &\geq |V(t+\epsilon)\phi(d-t-\epsilon) - V(t)\phi(d-t-\epsilon)| \\ &= |\phi(d) - \phi(d-\epsilon)|,\end{aligned}$$

which is again bounded away from zero. We conclude that $t \mapsto V(t)\phi$ is nowhere continuous from the right on the interval $[0, \min\{|d|, d+r\})$. As a consequence, neither $t \mapsto U(t,s)\phi$ nor $s \mapsto U(t,s)\phi$ can generally be relied on to have any points of continuity from either side.

This lack of continuity continues to be a problem for arbitrary evolution families $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$. Indeed, suppose $U(t,s)\phi : [-r,0] \to \mathbb{R}^n$ has an internal discontinuity at some $d \in (-r,0)$—this could result from an impulse effect at some time $t_k \in (t-r,t)$. Because of the translation property (4) of Lemma I.2.2.1, we have for any $t' \geq t$ such that $0 \leq t'-t < \min\{|d|, d+r\}$

$$\begin{aligned}||U(t'+\epsilon,s)\phi - U(t',s)\phi|| &\geq |U(t'+\epsilon,s)\phi(d+t-t'-\epsilon) - U(t',s) \\ &\qquad \phi(d+t-t'-\epsilon)| \\ &= |U(t,s)\phi(d) - U(t,s)\phi(d-\epsilon)|,\end{aligned}$$

which is yet again bounded away from zero for ϵ arbitrarily small. In the same way as before, we conclude that $t' \mapsto U(t',s)\phi$ is nowhere continuous from the left or right, for $t' \in [t, t+\min\{|d|, d+r\})$.

I.2.2.3 Continuity under the L^2 Seminorm

While $t \mapsto U(t,s)\phi$ is generally discontinuous everywhere with respect to the uniform norm $||g|| = \sup_{\theta\in[-r,0]} |g(\theta)|$ that we have been using up until this point, the same is not true if one uses the L^2 norm. Indeed, for $0 < \epsilon < \epsilon_0 < r$ and a fixed ϵ_0, one can make the estimate

$$\int_{-r}^{0} |U(t+\epsilon,s)\phi(\theta) - U(t,s)\phi(\theta)|^2 d\theta \leq \int_{-r}^{-\epsilon} |U(t,s)\phi(\theta+\epsilon) - U(t,s)\phi(\theta)|^2 d\theta + \epsilon K$$

where K is some constant such that $||U(t+\epsilon,s)\phi - U(t,s)\phi||^2 \leq K$ for $\epsilon < \epsilon_0$. The integrand converges pointwise to zero almost everywhere and is uniformly bounded, so the dominated convergence theorem implies that $U(t+\epsilon,s)\phi \to U(t,s)\phi$ in the L^2 sense, with respect to the norm

$$||g||_2 = \left(\int_{-r}^{0} |g(\theta)|^2 d\theta\right)^{1/2}.$$

Consequently, $t \mapsto U(t,s)\phi$ is continuous for each $\phi \in \mathcal{RCR}$ with respect to the $||\cdot||_2$ norm. However,

$$U(t,s) : (\mathcal{RCR}, ||\cdot||_2) \to (\mathcal{RCR}, ||\cdot||_2)$$

is not bounded. To see this, let us again make use of our translation[2] semigroup $V(t) : \mathcal{RCR} \to \mathcal{RCR}$ from Sect. I.2.2.2. Assume there exists $K \geq 0$ such that $||U(t,s)\phi||_2 \leq K||\phi||_2$ for all $\phi \in \mathcal{RCR}$. If $t \geq s + r$, this implies the equation

$$r|\phi(0)| \leq K \left(\int_{-r}^{0} |\phi(\theta)|^2 d\theta \right)^{\frac{1}{2}},$$

which cannot hold for all $\phi \in \mathcal{RCR}$. As such, even though $\mathcal{RCR}$ is dense in $L^2([-r,0],\mathbb{R}^n)$, we cannot extend $U(t,s)$ to a bounded linear operator on L^2 and take advantage of the continuity of $t \mapsto U(t,s)\phi$ or the completeness of $\mathcal{L}^2$.

I.2.3 Representation of Solutions of the Inhomogeneous Equation

Given the evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ associated with the homogeneous equation (I.2.4)–(I.2.5), we now consider to what extent the solutions of the inhomogeneous equation (I.2.1)–(I.2.2) can be represented in the form of a variation-of-constants formula. It is worth revisiting the variation-of-constants formula of Hale [56] for the functional differential equation

$$\dot{x} = L(t)x_t + h(t).$$

The formula states that the solution $x : [s-r,\infty) \to \mathbb{R}^n$ satisfying the initial condition $x_s = \phi$ can be written in the form

$$x_t = X(t,s)\phi + \int_s^t X(t,\mu)\chi_0 h(\mu) d\mu \tag{I.2.10}$$

for all $t \geq s$, where $X(t,s) : C([-r,0],\mathbb{R}^n) \to C([-r,0],\mathbb{R}^n)$ is the evolution family associated with the homogeneous equation $\dot{x} = L(t)x_t$, and the integral is understood as one parameterized by the lag $\theta \in [-r,0]$—that is, for each $\theta \in [-r,0]$, one interprets the integral on the right-hand side to be the integral of $\mu \mapsto X(t,\mu)[\chi_0 h(\mu)](\theta) \in \mathbb{R}^n$ over $[s,t]$. As stated, the formula is technically incorrect because $\chi_0 h(\mu)$ is not in the domain $C([-r,0],\mathbb{R}^n)$ of $X(t,\mu)$. In this section we will prove an analogous formula for the inhomogeneous linear system (I.2.1)–(I.2.2), but this technical difficulty will be resolved by working with the phase space $\mathcal{RCR}$ at the outset. We will also interpret the integral in the weak sense. See the comments (Sect. I.2.5) for further discussion. The content of this section follows closely the presentation of Church and Liu [31].

[2] $u(t,\theta) = V(t)\phi(\theta)$ satisfies the partial differential equation $\frac{du}{dt} = \frac{du}{d\theta}$ on the half-line $t + \theta > 0$. This partial differential equation corresponds to a translation with speed one.

I.2.3.1 Pointwise Variation-of-Constants Formula

The first task is to decompose solutions of the inhomogeneous equation by means of superposition. Specifically, we write them as the sums of homogeneous solutions and a pair of inhomogeneous solutions with different inhomogeneities corresponding to the continuous forcing $h(t)$ and the impulsive forcing r_k. The result follows directly from Lemma I.2.1.1.

Lemma I.2.3.1. *Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$, and let hypotheses H.1–H.2 hold. Denote by $t \mapsto x(t; s, \phi; h, r)$ the solution of the linear inhomogeneous equation (I.2.1)–(I.2.2) for inhomogeneities $h = h(t)$ and $r = r_k$, satisfying the initial condition $x_s(\cdot; s, \phi; h, r) = \phi$. The following decomposition is valid:*

$$x(t; s, \phi; h, r) = x(t; s, \phi; 0, 0) + x(t; s, 0; h, 0) + x(t; s, 0; 0, r) \tag{I.2.11}$$

The following lemmas prove representations of the inhomogeneous impulsive and continuous terms $x_t(\cdot; s, 0; 0, r)$ and $x_t(\cdot; s, 0; h, 0)$, respectively.

Lemma I.2.3.2. *Under hypotheses H.1–H.2, one has*

$$x_t(\cdot; s, 0; 0, r) = \sum_{s<t_i\leq t} U(t, t_i)\chi_0 r_i \tag{I.2.12}$$

Proof. Denote $x(t) = x(t; s, 0; 0, r)$. Clearly, for $t \in [s, \min\{t_i : t_i > s\}$, one has $x_t = 0$. We may assume without loss of generality that$t_0 = \min\{t_i : t_i > s\}$. Then, $x_{t_0} = \chi_0 r_0$ due to (I.2.3). From Lemmas I.2.1.1 and I.2.2.1, we have $x_t = U(t, t_0)\chi_0 r_0$ for all $t \in [t_0, t_1)$, so (I.2.12) holds for all $t \in [s, t_1)$. Supposing by induction that $x_t = \sum_{s<t_i\leq t} U(t, t_i)\chi_0 r_i$ for all $t \in [s, t_k)$ for some $k \geq 1$, we have

$$\begin{aligned} x_{t_k} &= x_{t_k^-} + \chi_0 B(k) x_{t_k^-} + \chi_0 r_k \\ &= U(t_k, t_{k-1}) x_{t_{k-1}} + \chi_0 r_k \\ &= U(t_k, t_{k-1}) \textstyle\sum_{s<t_i\leq t_{k-1}} U(t_{k-1}, t_i)\chi_0 r_i + \chi_0 r_k \\ &= \textstyle\sum_{s<t_i\leq t_k} U(t, t_i)\chi_0 r_i. \end{aligned}$$

Equality (I.2.12) then holds for $t \in [t_k, t_{k+1})$ by applying Lemma I.2.2.1. □

Lemma I.2.3.3. *Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. Under hypotheses H.1–H.2, one has*

$$x_t(\theta; s, 0; h, 0) = \int_s^t U(t, \mu)[\chi_0 h(\mu)](\theta) d\mu, \tag{I.2.13}$$

where the integral is defined for each θ as the integral of the function $\mu \mapsto U(t, \mu)[\chi_0 h(\mu)](\theta)$ in $\mathbb{R}^n$.

Proof. We provide a brief sketch of the proof. The interested reader can consult Church and Liu [31] for details. Denote $x(t; s)h = x(t; s, 0; h, 0)$. The function $x(t,s) : \mathcal{RCR}([s,t],\mathbb{R}^n) \to \mathbb{R}^n$ is linear for each fixed $s \le t$ and extends uniquely to a linear functional $\tilde{x}(t,s) : \mathcal{L}_1([s,t],\mathbb{R}^n) \to \mathbb{R}^n$. One can show that it is also bounded, so there exists an integrable, essentially bounded $n \times n$ matrix-valued function $\mu \mapsto V(t,s,\mu)$ such that

$$\tilde{x}(t,s)h = \int_s^t V(t,s,\mu)h(\mu)d\mu. \tag{I.2.14}$$

One can then show that $V(t,s,\mu)$ is independent of s. Define $V(t,s) = V(t,\cdot,s)$ for any $t \ge s$ and $V(t,s) = 0$ for $s < t$. Let us denote $\tilde{x}(t) = \tilde{x}(t,s)h$ and $V_{t_i^-}(\theta,s) = V(t_i+\theta,s)$ when $\theta < 0$ and $V_{t_i^-}(0,s) = V(t_i^-,s)$. From the integral equation (I.2.3) and the representation (I.2.14), one can carefully show after a serious of changes of variables and applications of Fubini's theorem that

$$\int_s^t V(t,\mu)h(\mu)d\mu = \int_s^t \left[I + \int_\mu^t L(\mu)V_\nu(\cdot,\mu)d\nu + \sum_{s<t_i\le t} B(i)V_{t_i^-}(\cdot,\mu)\right] h(\mu)d\mu.$$

Since the above holds for all $h \in \mathcal{L}_1([s,t],\mathbb{R}^n)$, the *fundamental matrix* $V(t,s)$ satisfies

$$V(t,s) = \begin{cases} I + \int_s^t L(\mu)V_\mu(\cdot,s)d\mu + \sum_{s<t_i\le t} B(i)V_{t_i^-}(\cdot,s), & t \ge s \\ 0 & t < s, \end{cases} \tag{I.2.15}$$

almost everywhere. By uniqueness of solutions (Lemma I.2.1.1), $V(t,s)\xi = U(t,s)[\chi_0\xi](0)$ for all $\xi \in \mathbb{R}^n$. Since $\tilde{x}$ is an extension of x to $\mathcal{L}_1([s,t],\mathbb{R}^n) \supset \mathcal{RCR}([s,t],\mathbb{R}^n)$, representation (I.2.14) holds for $h \in \mathcal{RCR}([s,t],\mathbb{R}^n)$. Then, from the properties of V, one can verify that for all $t \ge s$,

$$\begin{aligned} x_t(\theta;s,0;h,0) &= \tilde{x}(t+\theta,s)h \\ &= \int_s^t U(t,\mu)[\chi_0 h(\mu)](\theta)d\mu, \end{aligned}$$

which is what was claimed by Eq. (I.2.13). □

With Lemma I.2.3.1 through Lemma I.2.3.3 at hand, we arrive at the variation-of-constants formula.

Lemma I.2.3.4. *Let $h \in \mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$. Under hypotheses H.1–H.2, one has the variation-of-constants formula*

$$x_t(\theta;s,\phi;h,r) = U(t,s)\phi(\theta) + \int_s^t U(t,\mu)[\chi_0 h(\mu)](\theta)d\mu + \sum_{s<t_i\le t} U(t,t_i)[\chi_0 r_i](\theta). \tag{I.2.16}$$

I.2.3.2 Variation-of-Constants Formula in the Space $\mathcal{RCR}$

The main result of the previous section—Lemma I.2.3.4—is a variation-of-constants formula in the Euclidean space. That is, for each $\theta \in [-r, 0]$, one can compute the right-hand side of (I.2.16), with the integral being that of a vector-valued function with codomain $\mathbb{R}^n$. The goal of this section will be to reinterpret the variation-of-constants formula in such a way that the integral appearing therein may be thought of as a weak integral in the space $\mathcal{RCR}$.

Lemma I.2.3.5. *Let $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$, and let hypotheses H.1–H.2 hold. The function $U(t, \cdot)[\chi_0 h(\cdot)] : [s, t] \to \mathcal{RCR}$ is Pettis integrable for all $t \geq s$ and*

$$\left[\int_s^t U(t, \mu)[\chi_0 h(\mu)]d\mu\right](\theta) = \int_s^t U(t, \mu)[\chi_0 h(\mu)](\theta)d\mu. \tag{I.2.17}$$

Proof. By Lemma I.1.3.8 and the uniqueness assertion of Proposition I.1.4.1, if we can show for all $p : [-r, 0] \to \mathbb{R}^n$ of bounded variation that the equality

$$\int_{-r}^0 p^\intercal(\theta)d\left[\int_s^t U(t, \mu)[\chi_0 h(\mu)](\theta)d\mu\right] = \int_s^t \left[\int_{-r}^0 p^\intercal(\theta)d\Big[U(t, \mu)[\chi_0 h(\mu)](\theta)\Big]\right]d\mu$$

holds, then Lemma I.2.3.5 will be proven. Note that the above is equivalent to

$$\int_{-r}^0 p^\intercal(\theta)d\left[\int_s^t V(t+\theta, \mu)h(\mu)d\mu\right] = \int_s^t \left[\int_{-r}^0 p^\intercal(\theta)d\Big[V(t+\theta, \mu)h(\mu)\Big]\right]d\mu. \tag{I.2.18}$$

Suppose first that h is a step function. In this case, a consequence of Eq. (I.2.15) is that $\theta \mapsto V(t+\theta, \mu)h(\mu)$ and $\mu \mapsto V(t+\theta, \mu)h(\mu)$ are piecewise-continuous, while Lemmas I.2.1.1 and I.2.3.3 imply that $\theta \mapsto \int_s^t V(t+\theta, \mu)h(\mu)d\mu$ is also piecewise-continuous, all with at most finitely many discontinuities on any given bounded set. Consequently, both integrals in (I.2.18) can be regarded as the Lebesgue–Stieltjes integrals, with Fubini's theorem granting the desired equality.

Given $h \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$, we approximate its restriction to the interval $[s, t]$ by a convergent sequence of right-continuous step functions h_n by Lemma I.1.3.1. Equation (I.2.18) is then satisfied with h replaced with h_n. Define the functions

$$J_n(\theta) = \int_s^t V(t+\theta, \mu)h_n(\mu)d\mu, \quad K_n(\mu) = \int_{-r}^0 p^\intercal(\theta)d\Big[V(t+\theta, \mu)h_n(\mu)\Big],$$
$$J(\theta) = \int_s^t V(t+\theta, \mu)h(\mu)d\mu, \qquad K(\mu) = \int_{-r}^0 p^\intercal(\theta)d\Big[V(t+\theta, \mu)h(\mu)\Big],$$

so that $\int_{-r}^{0} p^{\intercal}(\theta)dJ_n(\theta) = \int_s^t K_n(\mu)d\mu$. Using Lemma I.2.2.1, we can get the inequality

$$|J_n(\theta) - J(\theta)| \leq ||h_n - h|| \int_s^t \exp\left(\int_\mu^t \ell(\nu)d\nu\right) d\mu,$$

so $J_n \to J$ uniformly. The conditions of Lemma I.1.3.5 are satisfied, and we have the limit

$$\int_{-r}^{0} p^{\intercal}(\theta)dJ_n(\theta) \to \int_{-r}^{0} p^{\intercal}(\theta)dJ(\theta).$$

Conversely, for each $\mu \in [s,t]$, Lemma I.1.3.4 applied to the difference $K_n(\mu) - K(\mu)$ yields, together with Lemma I.2.2.1,

$$|K_n(\mu) - K(\mu)| \leq (|p(0)| + |p(-r)| + \mathrm{var}_{-r}^{0} p) \left(\int_s^t \exp\left(\int_y^t \ell(\nu)d\nu\right) dy\right) ||h_n - h||.$$

Thus, $K_n \to K$ uniformly, and the bounded convergence theorem implies $\int_s^t K_n(\mu)d\mu \to \int_s^t K(\mu)d\mu$. Combining these results, Eq. (I.2.18) holds and the lemma is proven. □

Lemmas I.2.3.4 and I.2.3.5 together grant the variation-of-constants formula in the Banach space $\mathcal{RCR}$.

Theorem I.2.3.1. *Let hypotheses H.1–H.2 hold, and let $h \in \mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$. The unique solution $t \mapsto x_t(\cdot;s,\phi;h,r) \in \mathcal{RCR}$ of the linear inhomogeneous impulsive system* (I.2.1)*–*(I.2.2) *with initial condition $x_s(\cdot;s,\phi;h,r) = \phi$ satisfies the variation-of-constants formula*

$$x_t(\cdot;s,\phi;h,r) = U(t,s)\phi + \int_s^t U(t,\mu)[\chi_0 h(\mu)]d\mu + \sum_{s<t_i\leq t} U(t,t_i)[\chi_0 r_i], \tag{I.2.19}$$

where the integral is interpreted in the Pettis sense and may be evaluated pointwise using (I.2.17).

As a simple corollary, if x is a solution defined on $[s-r,\infty)$, we can express $t \mapsto x_t$ defined on $[s,\infty)$ as the solution of an abstract integral equation.

Corollary I.2.3.1. *Let hypotheses H.1–H.2 hold, and let $h \in \mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$. Any solution $x : [s-r,\infty) \to \mathbb{R}^n$ of the linear inhomogeneous impulsive system* (I.2.1)*–*(I.2.2) *satisfies for all $t \geq s$ the abstract integral equation*

$$x_t = U(t,s)x_s + \int_s^t U(t,\mu)[\chi_0 h(\mu)]d\mu + \sum_{s<t_i\leq t} U(t,t_i)[\chi_0 r_i]. \tag{I.2.20}$$

Equation (I.2.20) will be the key to defining mild solutions in Chap. I.4 and, ultimately, will permit us to construct centre manifolds.

I.2.4 Stability

Stability (in the sense of Lyapunov) is a fundamental topic in dynamical systems. We remind the reader of its definition, which we will specify to the inhomogeneous linear system (I.2.1)–(I.2.2).

Definition I.2.4.1. *We say that the inhomogeneous impulsive RFDE* (I.2.1)–(I.2.2) *is*

- exponentially stable *if there exist* $K > 0$, $\alpha > 0$ *and* $\delta > 0$ *such that for all* $\phi, \psi \in \mathcal{RCR}$ *satisfying* $||\phi - \psi|| < \delta$, *one has* $||x_t(\cdot, s, \phi) - x_t(\cdot, s, \psi)|| \leq K||\phi - \psi||e^{-\alpha(t-s)}$ *for all* $t \geq s$;
- stable *if for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $\phi, \psi \in \mathcal{RCR}$ *satisfying* $||\phi - \psi|| < \delta$, *one has* $||x_t(\cdot, s, \phi) - x_t(\cdot, s, \psi)|| < \epsilon$ *for all* $t \geq s$;
- unstable *if it is not stable.*

A simple consequence of the superposition principle is that stability of the inhomogeneous equation can be directly inferred from the properties of the evolution family $U(t, s) : \mathcal{RCR} \to \mathcal{RCR}$ associated with its homogeneous part.

Proposition I.2.4.1. *The inhomogeneous impulsive RFDE* (I.2.1)–(I.2.2) *is exponentially stable if and only if there exist* $K > 0$ *and* $\alpha > 0$ *such that the evolution family* $U(t, s) : \mathcal{RCR} \to \mathcal{RCR}$ *associated with the homogeneous part* (I.2.4)–(I.2.5) *satisfies* $||U(t, s)|| \leq Ke^{-\alpha(t-s)}$ *for* $t \geq s$. *It is stable if and only if the evolution family is bounded: there exists* $K > 0$ *such that* $||U(t, s)|| \leq K$ *for* $t \geq s$.

There are several analytical criteria in the literature that guarantee exponential stability of linear impulsive RFDE. Some of these are based on analytical estimates and variation-of-constants formulas [4, 18, 98, 156], while others are proven using Lyapunov–Razumikhin methods [149, 166]. Of course, nonlinear stability criteria can be applied to the linear equation (I.2.4)–(I.2.5) as well. We will discuss nonlinear stability further in Chap. I.4.

I.2.5 Comments

This chapter contains results that appear in the paper *Smooth centre manifolds for impulsive delay differential equations* [31] by Church and Liu, published by Journal of Differential Equations in 2018. Most importantly, that publication contains the main results of Sect. I.2.3, as well as Lemmas I.2.1.1 and I.2.2.1.

The variation-of-constants formula (I.2.10) for functional differential equations due to Hale can be made rigorous in several ways, such as through adjoint semigroup theory and integrated semigroup theory. See the reference [62] for an overview on these ideas. In the autonomous setting, the textbook of Diekmann, Verduyn Lunel, van Gils and Walther [41] provides a very readable account based on adjoint semigroups. Here, we have proposed an arguably more elementary approach; use the phase space $\mathcal{RCR}$ and treat the integral in the weak sense. It is not possible (or at least quite nontrivial) to interpret the integral in (I.2.20) as a strong integral in $\mathcal{RCR}$ because $\mu \mapsto U(t,\mu)$ is generally nowhere continuous from the left or right—see the discussion of Sect. I.2.2.2.

Variation-of-constants formulas for impulsive delay differential equations have appeared in the literature at various earlier points, but only in the context of Euclidean space integrals and only when the impulses did not contain delays. See, for instance, Gopalsamy and Zhang [53], Anokhin, Berezansky and Braverman [4] and Berezansky and Braverman [18] for some early instances.

Chapter I.3

Linear Periodic Systems

We will once again be interested in the homogeneous linear system

$$\dot{x} = L(t)x_t, \qquad t \neq t_k \tag{I.3.1}$$

$$\Delta x = B(k)x_{t^-}, \qquad t = t_k \tag{I.3.2}$$

satisfying the conditions H.1 and H.2, but in this chapter we will assume that the system is also *periodic*. That is, there exists $T > 0$ and $q \in \mathbb{N} \setminus \{0\}$ such that $L(t + T) = L(t)$ for all $t \in \mathbb{R}$, $B(k + q) = B(k)$ and $t_{k+q} = t_k + T$ for all $k \in \mathbb{Z}$. One motivation for studying linear periodic systems comes from applications involving systems of the form

$$\begin{aligned} \dot{y} &= f(x_t), & t &\neq kT \\ \Delta y &= g(x_{t^-}), & t &= kT, \end{aligned}$$

for some $T > 0$. Such an impulsive functional differential has an *autonomous right-hand side*. The variational equation (also called the equation of perturbed motion) associated with a fixed point or periodic solution has the form of system (I.3.1)–(I.3.2). The analysis of this system can be used to inform the stability of the fixed point or periodic solution, construct invariant manifolds and study bifurcations. These topics will be covered in Sect. I.4.3, Chaps. I.5, I.7, and I.8, respectively.

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I.3.1 Monodromy Operator

In this section we will define a linear operator that plays the role of the Poincaré map from ordinary differential equations. We will demonstrate that this operator is compact and provide some definitions related to its spectrum. For reasons that will become apparent soon, it is sometimes convenient to work instead with the phase space $\mathcal{RCR}([-jT,0],\mathbb{R}^n)$ for some $j\geq 1$ such that $jT\geq r$. This can always be done, since each of $L(t)$ and $B(k)$ extends in an obvious, trivial way to $\mathcal{RCR}([-jT,0],\mathbb{R}^n)$. We then obtain the following proposition.

Proposition I.3.1.1. *There exists $j\in\mathbb{N}$ minimal such that $r\leq jT$, and the evolution family $U(t,s)$ on $\mathcal{RCR}$ associated with the periodic system (I.3.1)–(I.3.2) extends uniquely to an evolution family $\tilde{U}(t,s)$ on $\mathcal{RCR}([-jT,0],\mathbb{R}^n)$ satisfying the identity*

$$\tilde{U}(t,s)\phi(\theta)=U(t,s)\psi(\theta)$$

for all $\phi\in\mathcal{RCR}([-jT,0],\mathbb{R}^n)$ and $\theta\in[-r,0]$, where $\psi=\phi|_{[-r,0]}$. In particular, we have the representation

$$U(t,s)=\pi_{\rightarrow}\tilde{U}(t,s)\pi_{\leftarrow},$$

where the linear maps $\pi_{\leftarrow}:\mathcal{RCR}\to\mathcal{RCR}([-jT,0],\mathbb{R}^n)$ and $\pi_{\rightarrow}:\mathcal{RCR}([-jT,0],\mathbb{R}^n)\to\mathcal{RCR}$ are

$$\pi_{\leftarrow}\phi(\theta)=\begin{cases}\phi(\theta), & \theta\in[-r,0],\\ 0, & \theta\in[-jT,-r)\end{cases}\qquad\qquad \pi_{\rightarrow}\phi=\phi|_{[-r,0]}.$$

Following the above proposition, we denote $\mathcal{RCR}_j=\mathcal{RCR}([-jT,0],\mathbb{R}^n)$. For each $t\in\mathbb{R}$, define the *extended monodromy operators* $\tilde{Z}_t:\mathcal{RCR}_j\to\mathcal{RCR}_j$ and $Z_t:\mathcal{RCR}\to\mathcal{RCR}$ by

$$\tilde{Z}_t=\tilde{U}(t+jT,t),\qquad Z_t=U(t+jT,t). \tag{I.3.3}$$

Also define the *monodromy operators* $V_t:\mathcal{RCR}\to\mathcal{RCR}$ simply by

$$V_t=U(t+T,t).$$

Recall that a linear operator $L:X\to X$ on a Banach space X is *compact* if the image under L of any bounded subset of X is relatively compact.

Lemma I.3.1.1. *$\tilde{Z}_t$ is compact for each $t\in\mathbb{R}$.*

Proof. If $\phi\in\mathcal{RCR}_j$, then $\tilde{Z}_t\phi$ is continuous except at times $\theta_n\in[-jT,0]$ such that $t+jT+\theta_n\in\{t_k:k\in\mathbb{Z}\}$. At such times, $\tilde{Z}_t\phi$ is continuous from the right and has limits on the left. Let $\Theta=\{\theta_1,\dots,\theta_N\}$ denote the set of all such discontinuity points, guaranteed to be finite since $N = jq$. Thus $\mathcal{B} \subset \mathcal{RCR}_j$ is bounded, and then $Y := \tilde{V}_t(\mathcal{B})$ is a subset of PC_Θ, with

$$PC_\Theta = \{f \in \mathcal{RCR}_j : \text{the only points of discontinuity are in } \Theta\}.$$

Being closed in $\mathcal{RCR}_j$, PC_Θ is complete.

A subset of $Y \subset PC_\Theta$ is precompact if and only if it is uniformly bounded and quasi-equicontinuous—that is, for all $\epsilon > 0$, there exists $\delta > 0$ such that if $t_1, t_2 \in [\theta_{k-1}, \theta_k) \cap [-jT, 0]$ satisfy $|t_1 - t_2| < \delta$, then $||x(t_1) - x(t_2)|| < \epsilon$ for all $x \in Y$. One may consult Bainov and Simeonov [9] and associated references for a proof of this result. Uniform boundedness follows by Lemma I.2.2.1. As for quasi-equicontinuity, let $t_1 > t_2$ and $t = 0$ without loss of generality. We note that for all $\tilde{Z}_t x \in Y$,

$$\begin{aligned}
||\tilde{Z}_t x(t_1) - \tilde{Z}_t x(t_2)|| &= ||U(jT + t_1, jT + t_2)U(jT + t_2, 0)x(0) - U(jT + t_2, 0)x(0)|| \\
&= ||\chi_0 \circ [U(jT + t_1, jT + t_2) - I]U(jT + t_2, 0)x|| \\
&\leq \int_{jT+t_1}^{jT+t_2} \ell(\mu)d\mu \left(e^{\int_0^{jT} \ell(\mu)d\mu} \prod_{k=1}^{jq} (1 + b(k)) \right) C \\
&\equiv K \int_{jT+t_1}^{jT+t_2} \ell(\mu)d\mu,
\end{aligned}$$

where $||x|| \leq C$ for all $x \in \mathcal{B}$, and the inequality on the third line follows by Lemma I.2.2.1 and the integral form of solutions provided by Eq. (I.2.3). Choosing δ so that $\int_{jT+t_1}^{jT+t_2} \ell(\mu)d\mu < \epsilon/K$ for $|t_1 - t_2| < \delta$ whenever $t_1, t_2 \in [-jT, 0]$ we obtain the required quasi-equicontinuity of Y. We conclude that $\tilde{Z}_t$ is compact. □

Lemma I.3.1.2. *Z_t is compact for each $t \in \mathbb{R}$.*

Proof. This follows by Proposition I.3.1.1 and the compactness of $\tilde{Z}_t$ from Lemma I.3.1.1. □

Lemma I.3.1.3. *V_t^j is compact.*

Proof. This is clear, since $V_t^j = Z_t$. □

The eventual (i.e. jth power) compactness of V_t provides us with several useful results from the spectral theory of compact operators; one may consult the reference [79] for details. First, recall that if X is a real vector space, its *complexification* $X_\mathbb{C} = X \oplus X$ is a complex vector space with scalar multiplication defined by

$$(a + ib)(x_1, x_2) = (ax_1 - bx_2, bx_1 + ax_2).$$

We will regularly abuse notation and identify an element $(x_1, x_2) \in X_\mathbb{C}$ with the formal symbol $x_1 + ix_2$. If $L : X \to X$ is a linear map, its complexification $L_\mathbb{C} : X_\mathbb{C} \to X_\mathbb{C}$ is defined by

$$L_\mathbb{C}(x + iy) = L(x) + iL(y).$$

With this machinery, the spectral theory of compact operators grants the following characterization of the spectrum and generalized eigenspaces of the monodromy operator. One may consult Chapter IV.2 of [41] or Chapter III.6 of [79] for background and proofs.

Theorem I.3.1.1. *Let $t \in \mathbb{R}$, and let σ_t denote the spectrum of $V_t^{\mathbb{C}} := (V_t)_{\mathbb{C}}$, the complexification of V_t.*

1. *If $\lambda \in \sigma_t$ is nonzero, then λ and λ^* are eigenvalues of $V_t^{\mathbb{C}}$.*
2. *The generalized eigenspace $M_{\lambda,t} \subset \mathcal{RCR}_{\mathbb{C}}$ associated with the eigenvalue $\lambda \in \sigma_t$ is finite-dimensional and invariant under $V_t^{\mathbb{C}}$.*
3. *The Riesz projection*
$$P_{\lambda,t} = \frac{1}{2\pi i}\int_\gamma (\xi I - V_t^{\mathbb{C}})^{-1} d\xi \tag{I.3.4}$$
is a projection onto $M_{\lambda,t}$, where γ is a simple continuous closed contour in $\mathbb{C}$ such that λ is the only eigenvalue of $V_t^{\mathbb{C}}$ contained in its interior.
4. *If $\Lambda \subset \sigma_t$, then*
$$P_{\Lambda,t} = \sum_{\lambda\in\Lambda} P_{\lambda,t}$$
is a projection onto
$$M_{\Lambda,t} = \bigoplus_{\lambda\in\Lambda} M_{\lambda,t}.$$
5. *The projections $P_{\Lambda,t}$ commute with $V_t^{\mathbb{C}}$ and if Λ_1 and Λ_2 are disjoint, then $P_{\Lambda_1,t}P_{\Lambda_2,t} = 0$.*
6. *σ_t is bounded, and $0 \in \sigma_t$ is the only accumulation point.*

We also have the following theorem concerning eigenvalues of distinct monodromy operators and their generalized eigenspaces. The proof follows verbatim the proof of Theorem 3.3 from Chapter XIII.3 of [41].

Theorem I.3.1.2. *Let $t, s \in \mathbb{R}$ be given with $t \geq s$, and let $\lambda \in \mathbb{C} \setminus \{0\}$.*

- *$\lambda \in \sigma_t$ if and only if $\lambda \in \sigma_s$.*
- *The restriction of $U_{\mathbb{C}}(t, s)$ to $M_{\lambda,s}$ is a topological isomorphism onto $M_{\lambda,t}$.*

Due to the uniqueness of the eigenvalues across all of the monodromy operators, the following definition is appropriate.

Definition I.3.1.1. *The* Floquet multipliers *of the evolution family $U(t, s)$ are the eigenvalues $0 \neq \lambda \in \sigma_0$ of the (complexified) monodromy operator W_0. The* multiplier spectrum *of the evolution family $U(t, s)$ is denoted as $\sigma(U) := \sigma_0$.*

The projections of Theorem I.3.1.1 take values in the complexifications $M_{\Lambda,t} \subset \mathcal{RCR}_{\mathbb{C}}$. However, it will be helpful to know later that the eigenvectors of V_t come in complex conjugate pairs. This will be so if $P_{\Lambda,t}$ is the complexification of a real projection operator on $\mathcal{RCR}$. For this to be the case, it suffices to ensure that all conjugate Floquet multipliers are included in the set Λ. See Section IV, Theorem 2.18 and Corollary 2.19 of [41].

Definition I.3.1.2. *A subset $\Lambda \subset \mathbb{C}$ is* symmetric *if $\Lambda = \Lambda^*$—that is, it contains all of its complex conjugates.*

Corollary I.3.1.1. *Let $0 \notin \Lambda \subset \sigma(U)$. If Λ is symmetric, the projection $P_{\Lambda,t} : \mathcal{RCR}_{\mathbb{C}} \to \mathcal{RCR}_{\mathbb{C}}$ is the complexification of a projection operator on $\mathcal{RCR}$.*

By definition of complexification, if $x \in \mathcal{RCR}_{\mathbb{C}}$ is real (that is, $x = \xi + i0$ for some $\xi \in \mathcal{RCR}$), then $P_{\Lambda,t}x$ is also real. By abuse of notation, we will identify the complexification of said operator with its "real part" whenever no confusion should arise. That is, we say that

$$P_{\Lambda,t} : \mathcal{RCR} \to \mathcal{RCR}$$

is also a projection and is identified with its complexification. Similarly, we will sometimes blur the lines between a given operator $L : \mathcal{RCR} \to \mathcal{RCR}$ and its complexification $L_{\mathbb{C}} : \mathcal{RCR}_{\mathbb{C}} \to \mathcal{RCR}_{\mathbb{C}}$ whenever no confusion should result.

Define the time-varying projectors

$$P_u(t) = P_{\Lambda_c,t}, \qquad P_c(t) = P_{\Lambda_c,t}, \qquad P_s(t) = I - P_u(t) - P_c(t), \quad \text{(I.3.5)}$$

where $\Lambda_u = \{\lambda \in \sigma(U) : |\lambda| > 1\}$ and $\Lambda_c = \{\lambda \in \sigma(U) : |\lambda| = 1\}$. By Corollary I.3.1.1, the first two of these define projections on $\mathcal{RCR}$. The third one is a complementary projector.

The main result of this section is that the evolution family $U(t, s) : \mathcal{RCR} \to \mathcal{RCR}$ is spectrally separated. Specifically, the triples (P_s, P_c, P_u) are projectors onto the stable, centre and unstable fibre bundles. We need two preparatory lemmas.

Lemma I.3.1.4. *The projectors $P_i(t)$ for $i \in \{s, c, u\}$ are T-periodic.*

Proof. First, from periodicity and uniqueness of solutions, we have $U(t, s) = U(t + kT, s + kT)$ for any integer $k \geq 0$ and reals $t \geq s$. $P_i(t)$ is the projector through the spectral subset Λ_i associated with the complexified operator $U_1 := U_{\mathbb{C}}(t + T, t)$, and $P_i(t + kT)$ is the projector through the same subset, associated with $U_2 := U_{\mathbb{C}}(t + T + kT, t + kT)$, for all $k \in \mathbb{Z}$. Since $U_1 = U_2$, it follows that $P_i(t) = P_i(t + kT)$. □

Lemma I.3.1.5. *The restriction of $V_t^{\mathbb{C}}$ to the subspace $\mathcal{R}(P_s(t))$ has its spectrum contained in the ball $B_1(0) \subset \mathbb{C}$.*

Proof. Denote $P_{cu} = P_c + P_u$ and $\Lambda_{cu} = \Lambda_c \cup \Lambda_u$. Since the generalized eigenspaces $M_{\Lambda_{cu},t}$ are invariant under V_t, the same is true for the (closed) complement, $\mathcal{R}(P_s(t))$. Denote by $\hat{V}_t$ the restriction of V_t to said complement. Suppose by way of contradiction that ξ is a (generalized) eigenvector of $\hat{V}_t$ with eigenvalue λ with $|\lambda| \geq 1$. Then there exists $k \geq 1$ with $(V_t - \lambda I)^k \xi = (\hat{V}_t - \lambda I)\xi = 0$, so ξ is in fact a (generalized) eigenvector of V_t with eigenvalue λ and $|\lambda| \geq 1$. This means $\xi \in \mathcal{R}(P_{cu}(t))$, but we have assumed that $\xi \in \mathcal{R}(P_s(t))$ is a (generalized) eigenvector of $\hat{V}_t : \mathcal{R}(P_s(t)) \to \mathcal{R}(P_s(t))$. One can check that $P_s(t)P_{cu}(t) = 0$, which implies $\xi = 0$, a contradiction. □

Theorem I.3.1.3. *The evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ associated with the periodic system (I.3.1)–(I.3.2) is spectrally separated, with projectors (P_s, P_c, P_u) defined as in Eq. (I.3.5). Also, $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ are finite-dimensional.*

Proof. We prove the theorem by verifying properties 1–6 of Definition I.1.1.6 explicitly.

1. Since $||P_s|| \leq 1 + ||P_u|| + ||P_c||$, it suffices to prove that $||P_u(t)||$ and $||P_c(t)||$ are uniformly bounded. We will prove only uniform boundedness of $P_c(t)$, since the argument is similar for $P_u(t)$. Also, by periodicity (Lemma I.3.1.4), it suffices to prove uniform boundedness on $[0,T]$.

   Assume for the moment that property 3 and property 6 are satisfied (they will be proven later, independently of property 1). Suppose by way of contradiction that there exist $x_n \in \mathcal{RCR}$ and a sequence $t_n \in [0,T]$ with $||x_n|| = 1$ such that $||P_c(t_n)x_n|| = n$. We can then write

$$\begin{aligned} n = ||P_c(t_n)x_n|| &= ||U_c(t_n,T)U_c(T,t_n)P_c(t_n)x_n|| \\ &\leq ||U_c(t_n,T)|| \cdot ||P_c(T)U(T,t_n)x_n|| \\ &\leq C||U_c(t_n,T)|| \\ &\leq CKe^{\epsilon T} \end{aligned}$$

   for some constant $C \geq ||P_c(T)|| \cdot ||U(T,t_n)||$ (see Lemma I.2.2.1) and a constant K as in property 6 of spectral separation. This is a contradiction.

2. This follows by property 5 of Theorem I.3.1.1.

3. We first prove that for $i \in \{s,c,u\}$,

$$P_i(t)U(t+T, s+kT) = U(t+T, s+kT)P_i(s) \tag{I.3.6}$$

   for any $k \in \mathbb{N}$ chosen so that $s + (k-1)T \leq t < s + kT$. The argument follows that of Theorem 3.3, Section X.III of [41], and we reproduce it here. We can decompose V_t and V_s for $t \geq s$ as $V_t = A_t A_s$ and $V_s = A_s A_t$ with

$$A_t = U(t+T, s+kT), \quad A_s = U(s+kT, t)$$

whenever k satisfies the above inequality. This follows from the observation $V_{t+qT} = V_t$ and $V_{s+qT} = V_s$ for any $q \in \mathbb{Z}$. Since Λ_c and Λ_u are both finite, we can find contours Γ_c and Γ_u such that

$$\overline{\text{int}(\Gamma_c)} \cap (\sigma(V_t) \cup \sigma(V_s)) = \Lambda_c$$
$$\overline{\text{int}(\Gamma_u)} \cap (\sigma(V_t) \cup \sigma(V_s)) = \Lambda_u.$$

Observe that for any $z \in \mathbb{C}$,

$$A_t(zI - A_s A_t) = (zI - A_t A_s)A_t.$$

As a consequence, if $z \in \text{int}(\Gamma_c) \cup \text{int}(\Gamma_u)$, then we can multiply both sides of the above by $(zI - V_t)^{-1}$ on the left and $(zI - V_s)^{-1}$ on the right. The result is

$$(zI - V_t)^{-1} A_t = A_t (zI - V_s)^{-1}.$$

Multiplying by $\frac{1}{2\pi i}$ and integrating over either Γ_c or Γ_u result in the equation

$$P_i(t) A_t = A_t P_i(s), \quad i \in \{c, u\}.$$

This is equivalent to (I.3.6) for $i \in \{c, u\}$. The analogous result for P_s follows from the decomposition $P_s = I - P_c - P_u$.

Now, (I.3.6) implies $P(t)U(t, s+qT) = U(t, s+qT)P(s)$ for $q = k-1$. Thus,

$$\begin{aligned} P(t)U(t,s) &= P(t)U(t, s+qT)U(s+qT, s) \\ &= U(t, s+qT)P(s)U(s+T, s)^q \\ &= U(t, s+qT)P(s)^q U(s+T, s)^q \\ &= U(t, s+qT)U(s+T, s)^q P(s)^q \\ &= U(t,s)P(s), \end{aligned}$$

where we have used the fact that $P(s)$ is a projector and commutes with $U(s+T, s)$.

4. This follows from Theorem I.3.1.2.

5. When $t \geq v \geq s$, the identity $U_c(t,s) = U_c(t,v)U_c(v,s)$ holds by properties of the evolution family U. When $t \geq s \geq v$, we find $I = U_c(t,v)^{-1} U_c(t,s) U_c(s,v)$, which implies

$$U_c(v,s) = U_c(v,t)U_c(t,s). \tag{I.3.7}$$

Also,

$$U_c(t,s)=U_c(t,v)U_c(t,v)^{-1}U_c(t,s)=U_c(t,v)[U_c(v,t)U_c(t,s)]=U_c(t,v)U_c(v,s). \tag{I.3.8}$$

Equation (I.3.7) implies $U_c(t,s)=U_c(t,v)U_c(v,s)$ for $v\geq s\geq t$, while (I.3.8) grants it for $t\geq s\geq v$. If $v\geq t\geq s$, then

$$U_c(t,s)=U_c(t,v)U_c(t,v)^{-1}U_c(t,s)=U_c(t,v)U_c(v,t)U_c(t,s)=U_c(t,v)U_c(v,s).$$

If $s\geq t\geq v$, then

$$U_c(t,s)=U_c(s,t)^{-1}=[U_c(s,v)U(v,t)_c]^{-1}=U_c(t,v)U_c(v,s).$$

Similarly, the desired equality holds if $s\geq v\geq t$. We have proven that $U_c(t,s)=U_c(t,v)U_c(v,s)$ for all $t,v,s\in\mathbb{R}$. The proof is identical for U_u.

6. This section is split into two parts, where we prove the estimates for U_c and U_s separately. The proof for U_u is similar to the centre (U_c) case and is omitted.

Centre part: U_c. Let $\epsilon>0$ be given. Recall that $U_c(t,s)$ is the restriction of $U(t,s)$ to $\mathcal{R}(P_c(s))$, so by Lemma I.2.2.1 and periodicity, there exists $K>0$ such that for any $s\in\mathbb{R}$, we have $||U_c(t,s)||\leq K$ provided $t\in[s,s+T]$. As all eigenvalues of $U_c(s+T,s)$ satisfy $|\lambda|=1$, Gelfand's (spectral radius) inequality guarantees for any $\overline{\epsilon}>0$ the existence of an integer $k>0$ such that $||U_c(s+T,s)^k||<1+\overline{\epsilon}T$. If we let m_t be the greatest integer such that $s+m_tkT\leq t$ and $m_t^*\in\{0,\dots,k-1\}$ the greatest integer such that $s+m_tkT+m_t^*T\leq t$, then following Section XIII, Theorem 2.4 of [41], we can write

$$\begin{aligned}U(t,s)&=U(t,s+m_tkT+m_t^*T)U(s+m_tkT+m_t^*T,s+m_tkT)U(s+m_tkT,s)\\&=U(t-m_tkT-m_t^*T,s)U(s+m_t^*T,s)U(s+T,s)^{km_t}\\&=U(t-m_tkT-m_t^*T,s)U(s+T,s)^{m_t^*+km_t}.\end{aligned}$$

Set $\epsilon=\overline{\epsilon}/k$. We can then make the estimate

$$||U_c(t,s)||\leq K||U_c(s+T,s)^k||^{\frac{m_t^*}{k}+m_t}\leq K(1+\epsilon T)^{\frac{t-s}{kT}}\leq Ke^{\epsilon(t-s)}.$$

The proof is similar when $t\leq s$, and we obtain $||U_c(t,s)||\leq Ke^{\epsilon|t-s|}$.

Stable part: U_s. Let $t\geq s$. Since $U_s(s+T,s)$ has its spectrum contained in the complex unit ball by Lemma I.3.1.5, there exists $k>0$ such that $||U_s(s+T,s)^k||\leq(1+aT)$ for some $a<0$. The rest of the proof follows by the same reasoning as the proof for the centre part, and we obtain $||U(t,s)||\leq Ke^{a(t-s)}$ as required.

The finite-dimensionality of $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ follows easily from Theorem I.3.1.1 and Corollary I.3.1.1.

□

Remark I.3.1.1. *The centre and unstable fibre bundles $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ are subsets of $\mathcal{RCR}^1$. This is because $V_t : \mathcal{RCR} \to \mathcal{RCR}$ has range in $\mathcal{RCR}^1$ and the t-fibres of these bundles consist of eigenvectors of V_t. The same is not generally true for the stable fibre bundle, being the range of $I - P_u(t) - P_c(t)$.*

I.3.2 Floquet Theorem

The Floquet Theorem of ordinary differential equations allows the transformation from a time-periodic linear system into an autonomous one, by way of a time-periodic change of coordinates. For impulsive systems, a similar result holds—we will review it in Sect. II.2.5. For infinite-dimensional systems, the best we can generally hope for is the existence of a transformation on each invariant fibre bundle that reduces the associated dynamics to an autonomous flow.

Theorem I.3.2.1. *Let $\Lambda \in \sigma(U)$ be finite and symmetric, and denote by $U_\Lambda(t,s) : M_{\Lambda,s} \to M_{\Lambda,t}$ the restriction of $U(t,s)$ to $M_{\Lambda,s}$. There exists $W \in \mathcal{L}(M_{\Lambda,0})$ and $t \mapsto \alpha(t) \in \mathcal{L}(M_{\Lambda,0}, M_{\Lambda,t})$ with the following properties:*

- *α is T-periodic, $\alpha(t)$ is invertible and there exists $\beta \geq 1$ such that for all $\phi \in M_{\Lambda,0}$,*
$$\beta^{-1}||\phi|| \leq \sup_{t\in\mathbb{R}} ||\alpha(t)\phi|| \leq \beta||\phi||.$$
- *$U_\Lambda(t,0)\phi = \alpha(t)e^{tW}\phi$ for all $\phi \in M_{\Lambda,0}$.*

Proof. Define $W = \frac{1}{T}\log U_\Lambda(T,0)$, where we choose the logarithm to be a branch that includes the (finite set of nonzero) eigenvalues of $U_\Lambda(T,0)$. Defining $\alpha(t) = U_\Lambda(t,0)e^{-tW}$, one may verify (compare to Proposition 4.4 and Theorem 4.5 from Section XIII of [41]; the proofs in the present case are essentially identical) that α is periodic and $U_\Lambda(t,0)$ satisfies the claimed decomposition. Uniform boundedness of α above and below follows by its periodicity and boundedness of $U_\Lambda(t,0)$ on $[-T,T]$; see the related proof of Theorem I.3.1.3. $\alpha(t)$ is clearly invertible. □

Theorem I.3.2.1 grants the change of coordinates $x_t = \alpha_t z(t)$ for $x_t \in M_{\Lambda,t}$ and $z(t) \in M_{\Lambda,0}$ finite-dimensional. In the new coordinate system, z satisfies the ordinary differential equation

$$\dot{z} = Wz.$$

Through the use of an appropriate coordinate map, one can obtain a concrete ordinary differential equation in $\mathbb{R}^p$ for some appropriate $p \in \mathbb{N}$. We will discuss this in more detail in Sect. I.3.4. The decomposition $U_\Lambda(t,0) = \alpha(t)e^{tW}$ will serve as a convenient way to, eventually, make the dynamics on invariant manifolds more concrete.

I.3.3 Floquet Multipliers, Floquet Exponents and Stability

The Floquet multipliers of the evolution family $U(t,s)$ determine the stability of the periodic homogeneous system (I.3.1)–(I.3.2). We first have a definition.

Definition I.3.3.1. *Let X be a complex n-dimensional vector space. A linear operator $L : X \to X$ is* diagonalizable *if there exists a basis $B = \{x_1, \dots, x_n\}$ such that the $n \times n$ matrix L_B of L relative to the basis B is diagonalizable.*

It is simple to verify that the above notion of diagonalizability is well-defined, in that if L_B is diagonalizable and B_2 is another basis, then L_{B_2} is also diagonalizable.

Theorem I.3.3.1. *The periodic impulsive RFDE (I.3.1)–(I.3.2) is exponentially stable if and only if $\sigma(U) \subset B_1(0)$. It is stable if and only if $\sigma(U) \subset \overline{B_1(0)}$ and the linear operator $W : \mathcal{RCR}_c^{\mathbb{C}}(0) \to \mathcal{RCR}_c^{\mathbb{C}}(0)$ from the Floquet decomposition on the centre fibre bundle is diagonalizable.*

Proof. First, suppose $\sigma(U) \subset B_1(0)$. The sets $\Lambda_u = \{\lambda \in \sigma(U) : |\lambda| > 1\}$ and $\Lambda_c = \{\lambda \in \sigma(U) : |\lambda| = 1\}$ are empty, and it follows that $P_s(t) = I$. By spectral separation, there exist $K \geq 0$ and $\alpha < 0$ such that

$$||U(t,s)|| = ||U(t,s)P_s(s)|| \leq Ke^{a(t-s)},\ t \geq s,$$

which implies exponential stability with $\epsilon = -a$. Conversely, if $||U(t,s)|| \leq Ce^{-\epsilon(t-s)}$ for some $C > 0$ and $\epsilon > 0$, then Λ_u and Λ_c must be empty. Indeed, if Λ_c were nonempty, there would exist $\xi \in \mathcal{RCR}$ nonzero such that $||U(t+kT,t)\xi|| = ||\xi||$ for all $k \in \mathbb{N}$. By exponential stability, this would imply $||\xi|| \leq C||\xi||e^{-\epsilon kT}$ and, subsequently, $\xi = 0$, which is a contradiction. By the same argument, Λ_u is empty, and with Lemma I.3.1.5 we conclude $\sigma(U) \subset B_1(0)$.

Now suppose $\sigma(U) \subset \overline{B_1(0)}$ and W is diagonalizable. The former assumption implies $P_u(t) = 0$. Recall that $U_c(t,s) : \mathcal{RCR}_c(s) \to \mathcal{RCR}_c(t)$ defines an all-time process on $\mathcal{RCR}_c$ by definition of spectral separation, so we may write

$$U_c(t,s) = \alpha(t)e^{(t-s)W}\alpha^{-1}(s)$$

using Theorem I.3.2.1 with $\Lambda = \Lambda_c$. Moreover, by definition of W all of its eigenvalues have zero real part. From diagonalizability, it follows that $t \mapsto ||e^{tW}||$ is bounded. Then,

$$\begin{aligned}||U(t,s)\phi|| &\leq ||U(t,s)P_s(s)|| + ||U(t,s)P_c(s)||\\ &\leq Ke^{a(t-s)} + ||U_c(t,s)||\\ &\leq K + ||\alpha(t)|| \cdot ||\alpha^{-1}(s)|| \cdot ||e^{(t-s)W}||.\end{aligned}$$

The condition $\beta^{-1}||\phi|| \leq \sup_t ||\alpha(t)\phi|| \leq \beta||\phi||$ from the Floquet Theorem implies that $||\alpha^{-1}(s)||$ is bounded, and with the periodicity of α we conclude that $||U(t,s)||$ is bounded by a constant independent of $t \geq s$.

Suppose now that $||U(t,s)|| \leq K$ for all $t \geq s$. By similar arguments to the previous case, we must have $\sigma(U) \subset \overline{B_1(0)}$. If W were not diagonalizable, then the Jordan canonical form of W_B relative to some basis B would contain a block whose exponential grows at least linearly in t. In particular, there exist $\delta > 0$ and $D > 0$ such that $||e^{tW_B}|| \geq Dt$ for $|t| \geq \delta$. If $C : \mathcal{RCR}_c^{\mathbb{C}}(0) \to \mathbb{C}^c$ denotes the coordinate map satisfying $C(r_i) = e_i$ for $B = \{r_1, \ldots, r_n\}$, then we can write $W = C^{-1}W_BC$. From the convergent power series definition of the exponential, it follows that

$$||U_c(t,s)|| = ||\alpha(t)e^{(t-s)W}\alpha^{-1}(s)|| \geq \frac{1}{\beta}||C^{-1}e^{(t-s)W_B}C\alpha^{-1}(s)|| \geq \frac{D(t-s)}{||C|| \cdot ||C^{-1}||}$$

for $|t-s| \geq \delta$. If $\phi \in \mathcal{RCR}_c(s)$, the above implies $||U(t,s)\phi|| = ||U_c(t,s)\phi|| \to \infty$ as $t \to \infty$, which contradicts the uniform boundedness of $U(t,s)$. $\square$

The Floquet multipliers characterize the rates of expansion, contraction and rotation associated with the discrete-time maps $\phi \mapsto V_t\phi$ for any $t \in \mathbb{R}$, and through iteration they characterize the stability. To get a sense of the growth rate in continuous time, we have the following definition.

Definition I.3.3.2. *The* Floquet spectrum *of the evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ is the set $\lambda(U) = \{\frac{1}{T}\log(\mu) : \mu \in \sigma(U)\}$, where the principal branch of the logarithm is taken. Its elements are called* Floquet exponents.

The Floquet exponents truly are the (average) growth rates of solutions in the invariant fibre bundles associated with their respective Floquet multipliers, as made precise by Theorem I.3.2.1. As for stability, we can reformulate Theorem I.3.3.1 in terms of Floquet exponents as follows.

Corollary I.3.3.1. *The periodic impulsive RFDE* (I.3.1)–(I.3.2) *is exponentially stable if and only if all Floquet exponents have negative real parts. It is stable if and only if all Floquet exponents have zero real part and the linear operator $W : \mathcal{RCR}_c^{\mathbb{C}}(0) \to \mathcal{RCR}_c^{\mathbb{C}}(0)$ from the Floquet decomposition on the centre fibre bundle is diagonalizable.*

I.3.4 Computational Aspects in Floquet Theory

The Floquet exponents (equivalently, multipliers) determine the stability of a periodic impulsive RFDE. It is important in applications to be able not only to compute Floquet exponents but also to calculate the (generalized) eigenvectors of the monodromy operator V_t. Such computations are necessary for the analysis of local bifurcations in nonlinear systems, for example. In this section we will therefore take some time to discuss these more computational matters. We will assume for simplicity that $j = 1$, so that $U(T, 0)$ is a monodromy operator.

I.3.4.1 Floquet Eigensolutions

Suppose μ is a Floquet multiplier of $U(T, 0)$. Let $B = \{\xi_1, \ldots, \xi_m\}$ be a basis for the generalized eigenspace $M_{\mu,0}$. Then,

$$U(T, 0)\xi_j = \mu\xi_j$$

for $j = 1, \ldots, m$. From the Floquet Theorem I.3.2.1, there exists a linear operator $W : M_{\mu,0} \to M_{\mu,0}$ and $\alpha(t) : M_{\mu,0} \to M_{\mu,t}$ periodic in t such that $U_\mu(t, 0) = \alpha(t)e^{tW}$. As in the proof of Theorem I.3.3.1, let $C : M_{\mu,0} \to \mathbb{C}^m$ denote the coordinate map satisfying $C(\xi_j) = e_j$. Let $W_B = PJP^{-1}$ be the Jordan canonical form of the matrix W_B of the operator W relative to the basis B. If we compute the action of $U_\mu(t, 0)$ on the basis elements ξ_j, we get

$$U_\mu(t, 0)\xi_j = \alpha(t)C^{-1}Pe^{tJ}P^{-1}e_j. \tag{I.3.9}$$

Also, since $U_\mu(T, 0) = e^{TW}$, it follows that the eigenvectors satisfy the equation $e^{TW}\xi_j = \mu\xi_j$. In terms of coordinate maps and the Jordan form of W_B, this gives

$$C^{-1}Pe^{TJ}P^{-1}C\xi_j = \mu\xi_j \;\Rightarrow\; Pe^{TJ}P^{-1}(C\xi_j) = \mu(C\xi_j).$$

It follows that $\frac{1}{T}\log\mu$ is the only eigenvalue of J. That is, J is a Jordan matrix whose only eigenvalue is the Floquet exponent $\lambda = \frac{1}{T}\log\mu$. We can then express $e^{tJ}P^{-1}e_j$ as a sum of the form

$$e^{tJ}P^{-1}e_j = \sum_{i=1}^{m} t^{i-1}e^{\lambda t}v_i$$

for some vectors $v_i \in \mathbb{R}^m$. If we now define the function $z_j(t) = U(t, 0)\xi_j(0)$, we can use the above summation formula in (I.3.9) to obtain the representation

$$z_j(t) = \sum_{i=1}^{m} p_i(t)t^{i-1}e^{\lambda t},$$

where $p_i(t) = [\alpha(t)C^{-1}Pv_i](0)$. By definition, $p_i : \mathbb{R} \to \mathbb{C}^n$ is periodic and right-differentiable with its only discontinuities at times t_k where it is continuous from the right and has a finite jump discontinuity. Moreover, z_j is a solution of the (complexified) homogeneous system (I.3.1)–(I.3.2). This proves the following theorem.

Theorem I.3.4.1. *λ is a Floquet exponent of the evolution family $U(t,s)$ associated with the linear homogeneous system (I.3.1)–(I.3.2) if and only if the latter admits a solution of the form $x(t) = e^{\lambda t}p(t)$ for a nonzero T-periodic $p \in \mathcal{RCR}(\mathbb{R}, \mathbb{C}^m)$. Moreover, the m-dimensional generalized eigenspace $M_{e^{\lambda T},t}$ is spanned by elements x_t such that $x \in \mathcal{RCR}(\mathbb{R}, \mathbb{C}^n)$ is a solution of (I.3.1)–(I.3.2) admitting a decomposition of the form*

$$x(t) = e^{\lambda t} \sum_{i=1}^{m} p_i(t) t^{i-1}, \tag{I.3.10}$$

for T-periodic $p_i \in \mathcal{RCR}(\mathbb{R}, \mathbb{C}^m)$.

Definition I.3.4.1. *A solution of the linear homogeneous system (I.3.1)–(I.3.2) having a decomposition of the form (I.3.10) is a* Floquet eigensolution with exponent λ. *Its* rank *is* $d = \max\{i = 1, \ldots, m : p_i \neq 0\}$.

Any solution contained in a generalized eigenspace $M_{e^{\lambda T},t}$ for a given Floquet exponent λ is a linear combination of history functions consisting of complex vector polynomials with periodic coefficients, multiplied by the exponential growth factor $e^{\lambda t}$. This motivates the following definition.

Definition I.3.4.2. *The* generalized eigenspace of $U(t,s)$ with Floquet exponent λ *is the invariant fibre bundle E_λ with t-fibre $E_\lambda(t) = M_{e^{\lambda T},t}$.*

Corollary I.3.4.1. *Considered as a subset of $\mathcal{RCR}(\mathbb{R}, \mathbb{C}^n)$, for each Floquet exponent λ there exists a maximal linearly independent set $S = \{\phi^{(1)}, \ldots, \phi^{(m)}\}$ of Floquet eigensolutions with exponent λ for some natural number m. The generalized eigenspace with Floquet exponent λ is given by the linear span of the histories of the elements of S; that is, $E_\lambda(t) = \operatorname{span}\{\phi_t^{(1)}, \ldots, \phi_t^{(m)}\}$.*

Proof. The existence of S follows from the finite-dimensionality (Theorem I.3.1.1) of $M_{e^{\lambda T},s}$ for any s, and that these eigenspaces are isomorphic (Theorem I.3.1.2). That S can be chosen to be linearly independent in $\mathcal{RCR}(\mathbb{R}, \mathbb{C}^n)$ follows because one can impose that the histories $\{\phi_0^{(1)}, \ldots, \phi_0^{(m)}\}$ form a basis for $M_{e^{\lambda T},0} \subset \mathcal{RCR}([-T,0], \mathbb{C}^n)$, which then implies the independence of S in $\mathcal{RCR}(\mathbb{R}, \mathbb{C}^n)$. The characterization of the t-fibre $E_\lambda(t)$ then follows from Theorem I.3.4.1. ☐

The following corollary to Theorem I.3.4.1 provides a dynamical characterization of Floquet exponents. Its proof is straightforward—one merely substitutes the Floquet eigensolution ansatz $x(t) = e^{\lambda t}p(t)$ into the periodic homogeneous system (I.3.1)–(I.3.2).

Corollary I.3.4.2. *Define* $\exp_\lambda : [-T, 0] \to \mathbb{C}$ *by* $\exp_\lambda(\theta) = e^{\lambda\theta}$. *$\lambda$ is a Floquet exponent of the evolution family $U(t,s)$ associated with the linear homogeneous system (I.3.1)–(I.3.2) if and only if there exists a nonzero T-periodic*

$p \in \mathcal{RCR}(\mathbb{R}, \mathbb{C}^n)$ *satisfying the impulsive functional differential equation*

$$\dot{p} + \lambda p = L(t)[\exp_\lambda p_t], \qquad t \neq t_k \tag{I.3.11}$$

$$\Delta p = B(k)[\exp_\lambda p_{t^-}], \qquad t = t_k. \tag{I.3.12}$$

It is worth contrasting the result of Corollary I.3.4.2 with the notion of characteristic equation from autonomous ordinary differential equations or delay differential equations. A good reference on these topics for autonomous functional differential equations is Chapter 7 of the book by Jack Hale and Sjoerd Verduyn Lunel [58]. For an autonomous retarded functional differential equation

$$\dot{x} = Lx_t,$$

the function $v \exp_\lambda$ for some nonzero $v \in \mathbb{C}^n$ is always contained within the generalized eigenspace E_λ associated with the eigenvalue λ of the infinitesimal generator of the strongly continuous (solution) semigroup. In particular, $t \mapsto e^{\lambda t} v$ must be a solution. Substituting this into the above delay differential equation and simplifying produce the equation $\lambda v = L(\exp_\lambda v)$, which can be written equivalently as

$$(L(\exp_\lambda) - \lambda I)v = 0, \tag{I.3.13}$$

where we define $L(\exp_\lambda) = \begin{bmatrix} L(e_1 \exp_\lambda) & \cdots & L(e_n \exp_\lambda) \end{bmatrix} \in \mathbb{C}^{n \times n}$. The term in parentheses in (I.3.13) is a complex $n \times n$ matrix—the *characteristic matrix* (see Section I.3 of [41])—and $v \in \mathbb{C}^n$. It follows that λ is an eigenvalue if and only if

$$\det(L(\exp_\lambda) - \lambda I) = 0. \tag{I.3.14}$$

Equation (I.3.14) is the *characteristic equation* for $\dot{x} = Lx_t$. It is a scalar equation that is generally transcendental in λ. It is difficult to formulate a similar equation for impulsive RFDE because the equivalent dynamical characterization for Floquet exponents is precisely given by Corollary I.3.4.2, which contains functional terms that cannot be simplified further. See Sect. I.3.5 for related comments.

I.3.4.2 Characteristic Equations for Finitely Reducible Linear Systems

For some special classes of impulsive RFDE, one can define a characteristic equation in a straightforward way or at least reduce the problem of computing Floquet exponents to a finite-dimensional problem.

Definition I.3.4.3. *The linear periodic system* (I.3.1)–(I.3.2) *is* finitely reducible *if* $L(t)$ *and* $B(k)$ *can be written in the form*

$$\tilde{L}(t)\phi = \sum_{j=0}^{\ell} \tilde{L}_j(t)\phi(-iT), \tag{I.3.15}$$

$$\tilde{B}(k)\phi = \int_{-h_k}^{0} C_k(s)\phi(s)ds + \sum_{j=0}^{\ell} \tilde{B}_j(k)\phi(-iT), \tag{I.3.16}$$

for some $\ell \geq 0$ *and matrices* $\tilde{A}_i(t)$ *and* $\tilde{B}_i(k)$, *and continuous* $C_k : [-h_k, 0] \to \mathbb{R}^{n\times n}$, *with* $h_k \leq t_k - t_{k-1}$.

In the following, recall that $q > 0$ is a positive integer for which $\tilde{B}(k+q) = \tilde{B}(k)$ and $t_{k+q} = t_k + T$ for $k \in \mathbb{Z}$, where T is the period of $\tilde{L}$.

Theorem I.3.4.2. *Let the linear periodic system* (I.3.1)–(I.3.2) *be finitely reducible. Let* $X_\lambda(t,s)$ *denote the Cauchy matrix of the ordinary differential equation*

$$\dot{p} = \left(-\lambda I + \sum_{j=0}^{\ell} e^{-\lambda jT}\tilde{L}_j(t)\right)p. \tag{I.3.17}$$

λ *is a Floquet exponent if and only if there exist* $p_1, \dots, p_c \in \mathbb{C}^n$ *not all zero such that*

$$\left(I + \int_{-h_k}^{0} C_k(s)X_\lambda(t_k+s,t_k)e^{\lambda s}ds + \tilde{B}_0(k)\right)X_\lambda(t_k,t_{k-1})p_{k-1}$$
$$= \left(I - \sum_{j=1}^{\ell}\tilde{B}_j(k)e^{-\lambda jT}\right)p_k, \tag{I.3.18}$$

for $k = 1, \dots, q$, *where we define* $p_0 := p_q$. *If this is the case, then* $x(t) = p(t)e^{\lambda t}$ *is a Floquet eigensolution with the periodic function* $p : \mathbb{R} \to \mathbb{C}^n$ *defined by*

$$p(t) = X_\lambda(t,t_k)p_{[k]_c}, \qquad t \in [t_k, t_{k+1}).$$

Proof. If $L(t)$ can be written as $\tilde{L}(t)$ from (I.3.15), then the right-hand side of the functional differential equation (I.3.11) becomes

$$L(t)[\exp_\lambda p_t] = \left(\sum_{j=0}^{\ell}\exp(-\lambda jT)\tilde{L}_j(t)\right)p(t),$$

where we used the periodicity of p. As a consequence, between the impulse times the function p actually satisfies the ordinary differential equation

(I.3.17), parameterized by the parameter λ. At the impulse times, Eq. (I.3.12) becomes

$$p(t) - p(t^-) = \int_{-h_k}^{0} C_k(s)e^{\lambda s}p(t+s)ds + \tilde{B}_0(k)p(t^-) + \sum_{j=1}^{\ell} \tilde{B}_j(k)e^{-\lambda jT}p(t), \tag{I.3.19}$$

with $t = t_k$, after exploiting the periodicity of p. Since $h_k \leq t_k - t_{k-1}$, Eq. (I.3.17) implies $p(t_k + s) = X_\lambda(t_k + s, t_k)p(t_k^-)$ for all $s \in [-h_k, 0]$. Substituting into (I.3.19), it follows that

$$\Delta p(t_k) = \left(\sum_{j=1}^{\ell} \tilde{B}_j(k)e^{-\lambda jT}\right) p(t_k) + \left(\int_{-h_k}^{0} C_k(s)e^{\lambda s}X_\lambda(t_k+s, t_k)ds + \tilde{B}_0(k)\right) p(t_k^-). \tag{I.3.20}$$

Assuming $x(t) = p(t)e^{\lambda t}$ is a Floquet eigensolution, Eq. (I.3.20) relates the value of $p(t_k^-)$ with that at $p(t_k)$, while the ordinary differential equation (I.3.17) determines the evolution of the state $p(t_k)$ to $p(t_{k+1}^-)$. Applying these results to the interval $[t_0, t_q] = [t_0, t_0 + T]$, the theorem is proven. □

To summarize, one can check whether a given λ is a Floquet exponent of a finitely reducible periodic system by solving the cyclic system of finite-dimensional equations (I.3.18). An explicit result is provided by the following corollary.

Corollary I.3.4.3. *Suppose the linear periodic system* (I.3.1)–(I.3.2) *is finitely reducible with* $C_k = 0$ *and* $\det(I + \tilde{B}_0(k)) \neq 0$ *for* $k = 1, \ldots, q$. λ *is a Floquet exponent if and only if it satisfies the* characteristic equation

$$\det\left(I - \prod_{k=1}^{q} X_\lambda(t_{k-1}, t_k)(I + \tilde{B}_0(k))^{-1}\left(I - \sum_{j=1}^{\ell} \tilde{B}_j(k)e^{-\lambda jT}\right)\right) = 0, \tag{I.3.21}$$

where the product denotes composition from left to right: $\prod_{k=a}^{b} M_k = M_a \cdots M_b$.

Proof. From (I.3.18) of Theorem I.3.4.2, one can uniquely write

$$p_{k-1} = X_\lambda(t_{k-1}, t_k)(I + \tilde{B}_0(k))^{-1}M_k p_k,$$

where $M_k p_k$ is the term on the right-hand side of (I.3.18) for M_k the matrix term in parentheses. It follows that

$$p_0 = \left(\prod_{k=1}^{q} X_\lambda(t_{k-1}, t_k)(I + \tilde{B}_0(k))^{-1}M_k\right) p_c.$$

From the cyclic condition, we must have $p_q = p_0$. It follows that there must be a nonzero solution of the equation

$$\left(I - \prod_{k=1}^{c} X_\lambda(t_{k-1}, t_k)(I + \tilde{B}_0(k))^{-1}M_k\right) p_0 = 0,$$

from which we obtain Eq. (I.3.21). □

Equation (I.3.21) provides a generalization of the characteristic equation (I.3.14) from autonomous functional differential equations to impulsive RFDEs with discrete delays being multiples of the period. Its practical applications are limited, however, as delays are typically not multiples of the period.

There are some additional special cases that can be dealt with analytically, such as when only discrete delays are present and these are rational multiples of the period. For example, consider

$$\begin{aligned} \dot{x} &= ax(t) + bx\left(t - \frac{1}{2}\right), && t \notin \mathbb{Z} \\ \Delta x &= cx(t^-) + dx\left(t - \frac{1}{2}\right), && t \in \mathbb{Z}, \end{aligned}$$

for constants a, b, c and d. If one assumes an ansatz $x(t) = p(t)e^{\lambda t}$ Floquet eigensolution and defines $w(t) = p(t - 1/2)$, then $z(t) = (p(t), w(t))$ satisfies

$$\begin{aligned} \dot{u} &= \begin{bmatrix} a & be^{-\frac{1}{2}\lambda} \\ be^{-\frac{1}{2}\lambda} & a \end{bmatrix} z, && t \notin \mathbb{Z} \\ \Delta u &= cu(t^-) + \begin{bmatrix} 0 & de^{-\frac{1}{2}\lambda} \\ de^{-\frac{1}{2}\lambda} & 0 \end{bmatrix} u(t), && t \in \mathbb{Z}. \end{aligned}$$

If $c \neq -1$, the above system can be shown to have a periodic solution if and only if

$$\det\left(\exp\left(-\begin{bmatrix} a & be^{-\frac{1}{2}\lambda} \\ be^{-\frac{1}{2}\lambda} & a \end{bmatrix}\right)\frac{1}{1+c}\begin{bmatrix} 1 & -de^{-\frac{1}{2}\lambda} \\ -de^{-\frac{1}{2}\lambda} & 1 \end{bmatrix} - I\right) = 0.$$

Though difficult to solve analytically, the above is nonetheless an explicit equation satisfied by every Floquet exponent. This tactic of defining extra lagged states becomes infeasible very quickly, however. For example, if there is a single rational delay of the form $\frac{p}{q} < 1$ with impulses at the integers, then $q - 1$ additional lagged states (such as w from above) are needed to use the same method. In general, some form of numerical discretization is needed to approximate the Floquet multipliers directly from the monodromy operator; see the comments in Sect. I.3.5.

I.3.4.3 Characteristic Equations for Systems with Memoryless Continuous Part

A final class of systems for which an explicit characteristic equation is available is that where the continuous-time dynamics are memoryless—that is, have no delays (distributed, discrete or otherwise). A result of this type appears in [30], where an explicit change of variables is used to eliminate the delay entirely. See Sect. IV.1.2 for an application.

Theorem I.3.4.3. *For $\mathcal{RCR} = \mathcal{RCR}([-r,0],\mathbb{R}^n)$, $B(k) \in \mathcal{L}(\mathcal{RCR})$ and $\lambda \in \mathbb{C}$, let $\mathcal{B}_\lambda(k) \in \mathcal{RCR}$. Let $(t,s) \mapsto X(t,s)$ denote the Cauchy matrix of the ordinary differential equation $\dot{x} = A(t)x$ for $A \in \mathcal{RCR}(\mathbb{R},\mathbb{R}^{n\times n})$ being T-periodic. Assume $B_{k+q} = B_k$ for some $q > 0$, and $t_{k+q} = t_k + T$ for $k \in \mathbb{Z}$. If $t_{k+1} - t_k > r$ for all $k \in \mathbb{Z}$, then $\mu \neq 0$ is a Floquet multiplier of the periodic system*

$$\dot{x} = A(t)x, \qquad t \neq t_k \tag{I.3.22}$$

$$\Delta x = B(k)x_{t^-}, \qquad t = t_k \tag{I.3.23}$$

if and only if it satisfies the characteristic equation

$$\det\left(\mu I - \prod_{k=q}^{1}\left(X(t_k^-, t_{k-1}) + B(k)X_{t_k^-}(\cdot, t_{k-1})\right)\right) = 0. \tag{I.3.24}$$

Proof. Let $\phi \in \mathcal{RCR}$. Taking into account the spacing $t_{k+1} - t_k > r$ between subsequent impulses and solving (I.3.22)–(I.3.23) forward from the initial condition $(t_0, \phi) \in \mathbb{R} \times \mathcal{RCR}$, it follows that

$$\begin{aligned}\phi_{t_k}(\theta) &= \begin{cases} X(t_k + \theta, t_{k-1})\phi_{t_{k-1}}(0), & -r \leq \theta < 0 \\ \left[X(t_k^-, t_{k-1}) + B(k)X_{t_k^-}(\cdot, t_{k-1})\right]\phi_{t_{k-1}}(0), & \theta = 0 \end{cases} \\ &\equiv R_k(\theta)\phi_{t_{k-1}}(0).\end{aligned}$$

We can therefore represent the monodromy operator M_0 (from the initial time t_0) in the form

$$M_0\phi(\theta) = R_q(\theta)R_{q-1}(0)R_{q-2}(0)\cdots R_2(0)R_1\phi(0).$$

If $M_0\phi = \mu\phi$, then, in particular, we must have $M_0\phi(0) = \mu\phi(0)$, from which the necessary condition (I.3.24) for μ to be a Floquet multiplier is obtained. On the other hand, if $\mu \neq 0$ satisfies (I.3.24), then there is a nontrivial solution $\phi(0)$ of the equation $R_q(0)\cdots R_1(0)\phi(0) = \mu\phi(0)$. Define

$$\tilde{\phi}(\theta) = \frac{1}{\mu}R_q(\theta)R_{q-1}(0)\cdots R_2(0)R_1(0)\phi(0).$$

By construction, $\tilde{\phi}(0) = \phi(0)$ and $M_0\tilde{\phi} = \mu\tilde{\phi}$, so μ is indeed a Floquet multiplier. □

I.3.5 Comments

Sections 1–4 of this chapter contain results that appear in the paper *Smooth centre manifolds for impulsive delay differential equations* [31] by Church and Liu, published by Journal of Differential Equations in 2018. The presentation here has been streamlined, and some of the proofs have been improved.

Characteristic matrices (and, consequently, characteristic equations) have been derived for linear periodic differential-difference equations [129, 138] under minimal assumptions, and there is certainly a possibility of extending these results to impulsive functional differential equations. This being said, in implementing the cited characteristic matrix construction, one must ultimately perform some kind of discretization and collocation procedure to approximate an abstract linear operator. Church and Liu [32] introduced a discretization scheme for the monodromy operator of the linear homogeneous impulsive differential-difference equation with a single discrete delay:

$$\begin{aligned} \dot{x} &= A(t)x(t) + B(t)x(t-\tau), && t \neq k\tau \\ \Delta x &= Cx(t^-) + Dx(t-\tau), && t = k\tau, \end{aligned} \tag{I.3.25}$$

where the impulses occur at periodic times $k\tau$ for $k \in \mathbb{Z}$, and $A(t)$ and $B(t)$ are periodic with period τ, but no proof of convergence was supplied. More recently, a numerical method based on Chebyshev expansion was proposed in [29] for the discretization of the monodromy operator, and convergence was proven in an operator norm on the relevant sequence space. This approach has limitations in that the period and delay must be commensurate. Explicit characteristic equations when the period and delay are equal and the impulses contain no delays have been derived in the autonomous [73] and periodic [132] setting.

We mentioned briefly in Sect. I.3.4.2 that when there are only discrete delays that are rational multiples of the period, the technique of defining additional lagged states will ultimately produce an explicit (although likely not analytically tractable) characteristic equation. See the related discussion in the conclusion of Szalai, Stépán and Hogan [138].

Finally, we should mention that if the exact computation of Floquet multipliers (or equivalently, exponents) is not necessary and all that one wants is to verify stability or instability, Rouché's theorem can be used to derive counts of the number of Floquet multipliers outside of the unit disc in the complex plane. We mention [52, 77, 89, 122] for some applications to stability of delay differential equations and later Chap. IV.3 for an application to stability in an infectious disease model. On a related note is the argument principle, which was used extensively by Stépan [136] in the study of stability and characteristic equations (characteristic functions) for delay differential equations. Many results appearing in that reference are applicable to systems with impulses and delays. See also Hassard [61], Kaslik and Sivasundaram [78] and Shi and Wang [127] for other applications.

Chapter I.4

Nonlinear Systems and Stability

I.4.1 Mild Solutions

Our attention shifts now to the semilinear system

$$\dot{x} = L(t)x_t + f(t, x_t), \qquad t \neq t_k \tag{I.4.1}$$

$$\Delta x = B(k)x_{t^-} + g(k, x_{t^-}), \qquad t = t_k, \tag{I.4.2}$$

for nonlinearities $f : \mathbb{R} \times \mathcal{RCR} \to \mathbb{R}^n$ and $g : \mathbb{Z} \times \mathcal{RCR} \to \mathbb{R}^n$. Note that we could instead replace $\mathcal{RCR}$ in the domains of f and g with some open subset thereof, and the results of this and subsequent chapters would remain correct with trivial modifications. For notational simplicity, however, we will typically leave the domain as $\mathcal{RCR}$ and allow the reader to fill in any minor changes as needed. Additional assumptions on the nonlinearities, evolution family and sequence of impulses may include the following:

H.3 For $j = 0, \dots, m$, and any $\phi, \psi^{(1)}, \dots, \psi^{(j)} \in \mathcal{RCR}([\alpha - r, \beta], \mathbb{R}^n)$, the function $t \mapsto D^j f(t, \phi_t)[\psi_1^{(1)}, \dots, \psi_t^{(j)}]$ is an element of $\mathcal{RCR}([\alpha, \beta], \mathbb{R}^n)$.

H.4 The evolution family $U(t, s) : \mathcal{RCR} \to \mathcal{RCR}$ associated with the homogeneous equation (I.2.4)–(I.2.5) is spectrally separated.

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H.5 $\phi \mapsto f(t,\phi)$ and $\phi \mapsto g(k,\phi)$ are C^m for some $m \geq 1$ for each $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, and there exists $\delta > 0$ such that for each $j = 0, \ldots, m$, there exist $c_j : \mathbb{R} \to \mathbb{R}^+$ locally bounded and a positive sequence $\{d_j(k) : k \in \mathbb{Z}\}$ such that

$$||D^j f(t,\phi) - D^j f(t,\psi)|| \leq c_j(t)||\phi - \psi||,$$
$$||D^j g(k,\phi) - D^j g(k,\psi)|| \leq d_j(k)||\phi - \psi||,$$

for $\phi, \psi \in B_\delta(0) \subset \mathcal{RCR}$. Also, there exists $q > 0$ such that $||D^j f(t,\phi)$ $|| \leq qc_j(t)$ and $||D^j g_k(\phi)|| \leq qd_j(k)$ for $\phi \in B_\delta(0)$.

H.6 $f(t,0) = g(k,0) = 0$ and $Df(t,0) = Dg(k,0) = 0$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$.

H.7 There exists a constant $\xi > 0$ such that $t_{k+1} - t_k \geq \xi$ for all $k \in \mathbb{Z}$.

Remark I.4.1.1. *Consider the general nonlinear system*

$$\begin{aligned} \dot{y} &= F(t, y_t), && t \neq t_k \\ \Delta y &= G(t, y_{t^-}), && t = t_k, \end{aligned}$$

and suppose $\gamma(t)$ is a given reference solution—this could be an equilibrium, periodic solution or any other solution of interest. If the nonlinearities are differentiable, one can define

$$\begin{aligned} f(t,\phi) &= F(t,\phi+\gamma_t) - F(t,\gamma_t) - Df(t,\gamma_t)\phi, \\ g(k,\phi) &= G(k,\phi+\gamma_{t)k^-}) - G(k,\gamma_{t_k^-}) - DG(k,\gamma_{t_k^-})\phi, \\ L(t) &= DF(t,\gamma_t), \\ B(k) &= DG(k,\gamma_{t_k^-}). \end{aligned}$$

Then, after the change of coordinates $x = y+\gamma$, the above impulsive functional differential equation is equivalent to the semilinear system (I.4.1)–(I.4.2)*, and one has $f(t,0) = g(k,0) = 0$, $Df(t,0) = DG(k,0) = 0$.*

Definition I.4.1.1. *A* mild solution *of the semilinear equation* (I.4.1)–(I.4.2) *is a function $x : [s,T] \to \mathcal{RCR}$ such that for all $s \leq t < T$, the function $x(t) = x_t$ satisfies the integral equation*

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\mu)[\chi_0 f(\mu, x(\mu))]d\mu + \sum_{s<t_i\leq t} U(t,t_i)[\chi_0 g(i, x(t_i^-))], \tag{I.4.3}$$

and $x(t)(\theta) = x(t+\theta)(0)$ whenever $\theta \in [-r.0]$ satisfies $t + \theta \in [s,T]$, where U is the evolution family associated with the homogeneous equation (I.2.4)–(I.2.5)*, and the integral is interpreted in the Pettis sense.*

Remark I.4.1.2. *The right-hand side of Eq.* (I.4.3) *is well-posed under conditions H.1–H.3 in the sense that it naturally defines for* $s \leq t < T$*, a nonlinear operator from* $\mathcal{RCR}([s-r,t],\mathbb{R}^n)$ *into* $\mathcal{RCR}$*. Note also that for a function* $x : [s,T] \to \mathcal{RCR}$*, we denote* $x(t_i^-)(\theta) = x(t_i)(\theta)$ *for* $\theta < 0$ *and* $x(t_i^-)(0) = x(t_i)(0^-)$*.*

If $x : [s-r,T) \to \mathbb{R}^n$ is a *classical solution*—that is, x is differentiable from the right, continuous except at impulse times t_k, continuous from the right on $[s-r,T]$ and its derivative satisfies the differential equation (I.4.1)–(I.4.2)—then, $t \mapsto x_t$ is a mild solution. This can be seen by defining the inhomogeneities $h(t) \equiv f(t,x_t)$ and $r_k \equiv g(k, x_{t_k^-})$, solving the equivalent linear equation (I.2.1)–(I.2.2) with these inhomogeneities and initial condition $(s,x_s) \in \mathbb{R}\times\mathcal{RCR}$ in the integrated sense and applying Corollary I.2.3.1. For this reason, we will work with Eq. (I.4.3) exclusively from now on.

Additionally, one should note that the assumption H.5 implies that the nonlinearities are uniformly locally Lipschitz continuous. Together with the other assumptions, this implies the local existence and uniqueness of mild solutions through a given $(s,\phi) \in \mathbb{R}\times\mathcal{RCR}$. Namely, we have the following lemma, which may be seen as a local, nonlinear version of Lemma (I.2.1.1), with an extension of Lemma I.2.1.2. Its proof is an application of the Banach fixed-point theorem and is omitted. The idea is nearly identical to a portion of the proof of Proposition I.4.3.1 in the following section, and the interested reader may consult it for reference.

Lemma I.4.1.1. *Under assumptions H.1–H.5, for all* $(s,\phi) \in \mathbb{R}\times\mathcal{RCR}$*, there exists a unique mild solution* $x^{(s,\phi)} : [s,s+\alpha) \to \mathcal{RCR}$ *of* (I.4.3) *for some* $\alpha = \alpha(s,\phi) > 0$*, satisfying* $x(s) = \phi$*. Moreover, the function*

$$t \mapsto y(t) := \begin{cases} x^{(s,\phi)}(t)(0), & t \in [s,s+\alpha) \\ \phi(s-t), & t \in [s,s-r) \end{cases}$$

is an element of $\mathcal{RCR}([s-r,s+\alpha),\mathbb{R}^n)$*, the restriction to* $[s,\alpha)$ *is differentiable from the right, it is continuous except at impulse times* $\{t_k : k \in \mathbb{Z}\}$ *and* $x^{(s,\phi)}(t) = y_t$*. That is, it is a classical solution. If one defines the nonautonomous set*

$$\mathcal{M} = \bigcup_{\phi\in\mathcal{RCR}} \bigcup_{s\in\mathbb{R}} \bigcup_{t\in[s,s+\alpha)} \{t\}\times\{s\}\times\{\phi\},$$

then $S : \mathcal{M} \to \mathcal{RCR}$ *with* $S(t,s)x = x^{(s,\phi)}(t)$ *is a process on* $\mathcal{RCR}$ *and* $x \mapsto S(t,s)x$ *is continuous. Finally, if* $x : \mathbb{R} \to \mathcal{RCR}$ *is a mild solution defined for all time, then the function* $y(t) = x(t)(0)$ *is an element of* $\mathcal{RCR}^1(\mathbb{R},\mathbb{R}^n)$*, and its only discontinuities occur in* $\{t_k : k \in \mathbb{Z}\}$ *and* $x(t) = y_t$*.*

Combining the discussion following Definition I.4.1.1 with Lemma I.4.1.1, it follows that $S(t,s)$ satisfies the following abstract integral equation wherever it is defined.

$$S(t,s)\phi = U(t,s)\phi + \int_s^t U(t,\mu)\chi_0 f(\mu, S(\mu,s)\phi)d\mu + \sum_{s<t_i\le t} U(t,t_i)\chi_0 g(i, S(t_i^-,s)\phi). \tag{I.4.4}$$

Definition I.4.1.2. *We say that the process $S(t,s) : \mathcal{M} \to \mathcal{RCR}$ guaranteed by Lemma I.4.1.1 is* generated *by the impulsive RFDE* (I.4.1)–(I.4.2).

I.4.2 Dependence on Initial Conditions

Of use later will be a result concerning the smoothness of the process $S : \mathcal{M} \to \mathcal{RCR}$ with respect to arguments $\phi \in \mathcal{RCR}$. This result is interesting in its own right and will be useful later in proving the periodicity of invariant manifolds.

Theorem I.4.2.1. *Under hypotheses H.1–H.6, the process $S : \mathcal{M} \to \mathcal{RCR}$ is C^m. More precisely, $\phi \mapsto S(t,s)\phi$ is C^m in a neighbourhood of ϕ, provided $(t,s,\phi) \in \mathcal{M}$. Also, $DS(t,s)\phi \in \mathcal{L}(\mathcal{RCR})$ for given $\phi \in \mathcal{RCR}$ satisfies for $t \ge s$ the abstract integral equation*

$$DS(t,s)\phi = U(t,s) + \int_s^t U(t,\mu)\chi_0 Df(\mu, S(\mu,s)\phi)DS(\mu,s)d\mu$$
$$+ \sum_{s<t_i\le t} U(t,t_i)\chi_0 Dg(t_i, S(t_i^-,s)\phi)DS(t_i^-,s). \tag{I.4.5}$$

Proof. We will prove only that S is C^1, the proof of higher-order smoothness being an essentially identical albeit notationally cumbersome extension thereof. Let $s \in \mathbb{R}$ be fixed. Let $\psi \in \mathcal{RCR}$ be given. For given $\nu > 0$, denote by $B_\nu(\psi)$ the closed ball centred at ψ with radius ν in $\mathcal{RCR}$.

The proof is based on a formal differentiation procedure justified by the fibre contraction theorem. Introduce for given $\epsilon, \delta, \nu > 0$ the normed vector space $(X_{\epsilon,\delta,\nu}, ||\cdot||)$, where $X_{\epsilon,\delta,\nu}$ consists of the functions $\phi : [s-r, s+\epsilon] \times B_\delta(\psi) \to B_\nu(\psi)$ such that $x \mapsto \phi(t,x)$ is continuous for each t, $\phi(t,x)(\theta) = \phi(t+\theta,x)(0)$ whenever $\theta \in [-r,0]$ and $[t+\theta,t] \subset [s-r,s+\epsilon]$, and $|\phi|| < \infty$ for the norm given by

$$||\phi||_{\epsilon,\delta,\nu} = \sup_{\substack{t\in[s-r,s+\epsilon] \\ ||x-\psi||\le\delta}} ||\phi(t,x)||.$$

It can be easily verified that $(X_{\epsilon,\delta,\nu}, ||\cdot||)$ is a Banach space. With $\mathcal{L}(\mathcal{RCR})$ the bounded linear operators on $\mathcal{RCR}$, introduce also the space $(\mathbf{X}_{\epsilon,\delta}, ||\cdot||)$

consisting of functions $\Phi : [s-r, s+\epsilon] \times \mathcal{RCR} \to \mathcal{L}(\mathcal{RCR})$ such that $x \mapsto \Phi(t,x)$ is continuous for each t, $\Phi(t,x)h(\theta) = \Phi(t+\theta,x)h(0)$ for all $h \in \mathcal{RCR}$, and $||\Phi|| < \infty$, where the norm is $||\Phi(t,x)|| = \sup_{||h||=1} ||\Phi(t,x)h||_{\epsilon,\delta,\nu}$. Clearly, $(\mathbf{X}_{\epsilon,\delta}, ||\cdot||)$ is complete.

Define a pair of nonlinear operators

$$\Lambda_1 : X_{\epsilon,\delta,\nu} \to X_{\epsilon,\delta,\nu},$$

$$\Lambda_1(\phi)(t,x) = \chi_{[s-r,s)}(t)x(t-s) + \chi_{[s,s+\epsilon]}(t)\left[U(t,s)x(s) + \int_s^t U(t,s)\chi_0 f(\mu, \phi(\mu,x))d\mu \right.$$
$$\left. + \sum_{s<t_i\leq t} U(t,t_i)\chi_0 g(t_i, \phi(t_i^-, x))\right]$$

$$\Lambda_2 : X_{\epsilon,\delta} \times \mathbf{X}_{\epsilon,\delta} \to \mathbf{X}_{\epsilon,\delta}$$

$$\Lambda_2(\phi,\Phi)(t,x)h = \chi_{[s-r,s)}(t)I_{\mathcal{RCR}}h + \chi_{[s,s+\epsilon]}(t)\left[U(t,s)h + \int_s^t U(t,\mu)\chi_0 Df(\mu,\phi(\mu,x))\Phi(\mu,x)h d\mu + \right.$$
$$\left. + \sum_{s<t_i\leq t} U(t,\mu)\chi_0 Dg(i, \phi(t_i^-, x)))\Phi(t_i^-, x)h\right],$$

where in the definition of Λ_2 we have $h \in \mathcal{RCR}$. By choosing ϵ and δ small enough, Λ_1 can be shown to be a uniform contraction. Indeed, if we denote $\kappa = \sup_{||x-\psi||\leq 2\delta} ||x||$, the mean-value theorem grants the estimate

$$\begin{aligned} &||\Lambda_1(\phi) - \Lambda_1(\gamma)|| \\ &\quad \leq \kappa \sup_{t\in[s,s+\epsilon]} \left(\int_s^t ||U(t,\mu)||c_1(\mu)d\mu + \sum_{s<t_i\leq t} ||U(t,t_i)||d_1(i)\right) ||\phi - \gamma|| \\ &\quad \equiv \kappa L_\epsilon ||\phi - \gamma||. \end{aligned}$$

We can always obtain a uniform contraction by taking ϵ small enough. Also, note that $t \mapsto \Lambda_1(\phi)(t,x) \in \mathcal{RCR}$, $x \mapsto \Lambda_1(\phi, x)$ is continuous and $\Lambda_1(\phi)(t,x)(\theta) = \Lambda_1(\phi)(t+\theta,x)(0)$. To ensure the appropriate boundedness, if we denote $\overline{\kappa} = \sup_{||x-\psi||\leq\delta} k_0(x)$, the estimate

$$\begin{aligned} ||\Lambda_1(\phi) - \psi|| &\leq ||\phi - \psi|| + \overline{\kappa} \sup_{t\in[s,s+\epsilon]} \left(\int_s^t ||U(t,\mu)||c_0(\mu)d\mu + \sum_{s<t_i\leq t} ||U(t,t_i)||d_0(i)\right) \\ &\equiv \delta + \overline{\kappa} M_\epsilon \end{aligned}$$

implies it is sufficient to choose $\epsilon, \delta, \nu > 0$ small enough so that $\delta + \overline{\kappa}M_\epsilon < \nu$. This can always be done because $M_\epsilon \to 0$ as $\epsilon \to 0$ due to H.5 and Lemma I.2.2.1.

The continuity of $\phi \mapsto \Lambda_2(\phi, \Phi)$ for fixed $\Phi \in \mathbf{X}_{\epsilon,\delta}$ follows by the estimate

$$\begin{aligned} ||\Lambda_2(\phi,\Phi) - \Lambda_2(\gamma,\Phi)|| \leq &\left(\int_s^{s+\epsilon} ||U(s+\epsilon,\mu)||c_1(\mu)||(\phi(\mu,x) - \gamma(\mu,x)||d\mu \right. \\ &\left. + \sum_{s<t_i\leq s+\epsilon} ||U(s+\epsilon,t_i)||d_1(i)||\phi(t_i^-,x) - \gamma(t_i^-,x)||\right) ||\Phi||. \end{aligned}$$

Also, for each $\phi \in B_\delta(\psi)$, it is readily verified that $||\Lambda_2(\phi,\Phi) - \Lambda_2(\phi,\Gamma)|| \leq \kappa L_\epsilon ||\Phi - \Gamma||$, which by previous choices of $\epsilon, \delta, \nu > 0$ indicates that $\Phi \mapsto \Lambda_2(\phi,\Phi)$ is a uniform contraction.

We are ready to prove the statement of the theorem. Denote by (x_n, x_n') the iterates of the map $\Lambda : X_{\epsilon,\delta,\nu} \times \mathbf{X}_{\epsilon,\delta,\nu} \to X_{\epsilon,\delta,\nu} \times \mathbf{X}_{\epsilon,\delta,\nu}$ defined by $\Lambda(x,x') = (\Lambda_1(x), \Lambda_2(x,x'))$ and initialized at (x_0, x_0') with $x_0(t,x) = x$ and $x_0'(t,x) = I_{\mathcal{RCR}}$. The fibre contraction theorem—see [67] for the original, more abstract result or Theorem 1.176 of [27] for a more concrete formalism—implies convergence $(x_n, x_n') \to (x, x')$. Note also that $Dx_0 = x_0'$. If we suppose $Dx_n = x_n'$ for some $n \geq 0$, then for $t \geq s$, Lemma I.2.3.5 implies that for each $\theta \in [-r, 0]$,

$$\begin{aligned} Dx_{n+1}(t,\phi)(\theta) = D\Bigg[&U(t,s)x_n(s,\phi)(\theta) + \int_s^t U(t,\mu)\chi_0 f(\mu, x_n(\mu,\phi))(\theta)d\mu \\ &+ \sum_{s<t_i\leq t} U(t,t_i)\chi_0 g(i, x_n(t_i^-,\phi))(\theta)\Bigg] \\ = D\Bigg[&U(t,s)x_n(s,\phi)(\theta) + \int_s^t V(t+\theta,\mu)f(\mu, x_{n+1}(\mu,\phi))d\mu \\ &+ \sum_{s<t_i\leq t} V(t+\theta,t_i)g(i, x_{n+1}(t_i^-,\phi))\Bigg] \\ = &U(t,s)x_n'(s,\phi)(\theta) + \int_s^t V(t+\theta,\mu)Df(\mu, x_n(\mu,\phi))x_n'(\mu,\phi)d\mu \\ &+ \sum_{s<t_i\leq t} V(t+\theta,t_i)Dg(i, x_n(t_i^-,\phi))x_n'(t_i^-,\phi), \end{aligned}$$

which is precisely $\Lambda_2(x_n, x_n')(t,\phi)(\theta) = x_{n+1}'(t,\phi)(\theta)$. For $t < s$, it is easily checked that $Dx_{n+1}(t,\phi) = x_{n+1}'(t,\phi)$. This proves that $Dx_{n+1}(\theta) = x_{n+1}'(\theta)$ pointwise in θ. To prove the result uniformly, we note that the difference quotient can be written for $t \geq s$ as

$$\begin{aligned} &\frac{1}{||h||}\big(x_{n+1}(t,\phi+h) - x_{n+1}(t,\phi) - x_{n+1}'(t,\phi)h\big) \\ &= \int_s^t U(t,\mu)\chi_0 \frac{1}{||h||}\big(f(\mu, x_n(\mu,\phi+h)) - f(\mu, x_n(\mu,\phi)) - Df(\mu, x_n(\mu,\phi))Dx_n(\mu,\phi)h\big)d\mu \\ &\quad + \sum_{s<t_i\leq t} U(t,t_i)\chi_0 \frac{1}{||h|||}\big(g(i, x_n(t_i^-,\phi+h)) - g(i, x_n(t_i^-,\phi)) - Dg(i, x_n(t_i^-,\phi))Dx_n(t_i^-,\phi)h\big). \end{aligned}$$

Since x_n is differentiable by the induction hypothesis, the integrand and summand converge uniformly to zero as $h \to 0$. Thus, x_{n+1} is differentiable and $Dx_{n+1} = x_{n+1}'$, so by induction $Dx_n = x_n'$ for each n. Also, by construction, x_n' is continuous for each n and, being the uniform limit of continuous functions, $x' = \lim_{n\to\infty} x'$ is continuous. By the fundamental theorem of calculus,

$$\begin{aligned}\frac{x(\phi+h)-x(\phi)-x'(\phi)h}{||h||} &= \lim_{n\to\infty}\frac{x_n(\phi+h)-x_n(\phi)-Dx_n(\phi)h}{||h||}\\ &= \lim_{n\to\infty}\int_0^1\frac{1}{||h||}\left[x_n'(\phi+(\lambda-1)h)-x_n'(\phi)\right]hd\lambda\\ &= \int_0^1\frac{1}{||h||}\left[x'(\phi+(\lambda-1)h)-x'(\phi)\right]hd\lambda\to 0\end{aligned}$$

as $h\to 0$. By definition, x is differentiable and $Dx=x'$.

If we define $y(t)\phi=x(t,\phi)$ for the fixed point $x:[s-r,s+\epsilon]\times B_\delta(\psi)\to B_\nu(\psi)$, then y satisfies $y(t)\phi=S(t,s)\phi$ for $(t,\phi)\in[s,s+\epsilon]\times B_\delta(\psi)$. This can be seen by comparing the fixed-point equation $y(t)=\Lambda_1(y)(t,\phi)$ with the abstract integral equation (I.4.4). We conclude that S is C^1 (fibrewise). The correctness of Eq. (I.4.5) follows by comparing to the fixed-point equation associated with Λ_2. □

We should remind the reader that although $\phi\mapsto S(t,s)\phi$ is smooth for fixed $t\geq s$, the same is decidedly not true for $s\mapsto S(t,s)\phi$ and $t\mapsto S(t,s)\phi$ for other arguments fixed. The lack of regularity of the latter two functions is implied by the discussion of Sect. I.2.2.2.

I.4.3 The Linear Variational Equation and Linearized Stability

The Fréchet derivative $DS(t,s,\phi)\in\mathcal{L}(\mathcal{RCR})$ of the process associated with the semilinear equation (I.4.1)–(I.4.2) satisfies the abstract integral equation (I.4.5). Evaluating at $\phi=0$, we see that $DS(t,s,0)=U(t,s)$, the evolution family associated with the homogeneous equation. We are therefore fully justified in referring to

$$\begin{aligned}\dot z &= L(t)z_t, && t\neq t_k\\ \Delta z &= B(k)z_{t^-}, && t=t_k\end{aligned}$$

as the *linearization* of the semilinear equation (I.4.1)–(I.4.2). In general, if $\gamma:\mathbb{R}\to\mathbb{R}^n$ is a solution of the nonlinear system

$$\dot y = F(t,y_t),\qquad t\neq t_k \tag{I.4.6}$$

$$\Delta y = G(t,y_{t^-}),\qquad t=t_k, \tag{I.4.7}$$

and then we refer to the linear system

$$\dot z = DF(t,\gamma_t)z_t,\qquad t\neq t_k \tag{I.4.8}$$

$$\Delta z = DG(k,\gamma_{t_k^-})z_{t^-},\qquad t=t_k, \tag{I.4.9}$$

as the *linearization at* γ or the *variational equation* associated with γ. When γ is a periodic solution, it is sometimes called the *equation of perturbed motion*.

The *linearized stability principle* is the statement that the stability of the linearization (I.4.8)–(I.4.9) to a certain extent informs the stability of γ in the nonlinear system (I.4.6)–(I.4.7). To make this statement precise, we should of course define the notation of stability we will need.

Definition I.4.3.1. *Suppose* $F(t,0) = G(t,0) = 0$. *The trivial solution* $\gamma = 0$ *of* (I.4.6)–(I.4.7) *is*

- stable *if for all* $\epsilon > 0$ *and* $s \in \mathbb{R}$, *there exists* $\delta = \delta(\epsilon, s) > 0$ *such that if* $||\phi|| < \delta$, *then* $||S(t,s,\phi)|| < \epsilon$ *for all* $t \geq s$;
- uniformly stable *if it is stable and* δ *can be chosen independent of* s;
- attracting *if for all* $s \in \mathbb{R}$, *there exists* $\delta = \delta(s) > 0$ *such that if* $||\phi|| < \delta$, *then* $||S(t,s,\phi)|| \to 0$ *as* $t \to \infty$;
- uniformly attracting *if it is attracting and* δ *can be chosen independent of* s;
- asymptotically stable *if it stable and attracting;*
- uniformly asymptotically stable *if it is uniformly stable and uniformly attracting;*
- exponentially stable *if there exist* δ, α *and* $K > 0$ *such that* $||S(t,s,\phi)|| \leq Ke^{-\alpha(t-s)}$ *for all* $t \geq s$, *whenever* $||\phi|| < \delta$.

By Remark I.4.1.1, no generality is lost by defining stability with respect to a constant (zero) solution. The following proposition, which concerns linearized stability of the zero solution, therefore applies also to nonconstant solutions.

Proposition I.4.3.1 (Linearized Stability)**.** *Let assumptions H.1–H.7 hold. Assume that for all* $\delta > 0$ *sufficiently small, there exists* $c(\delta) \geq 0$ *satisfying* $\lim_{\delta \to 0^+} c(\delta) = 0$ *and such that*

$$||f(t,\phi) - f(t,\psi)|| \leq c||\phi - \psi|| \tag{I.4.10}$$

$$||g(k,\phi) - g(k,\psi)|| \leq c||\phi - \psi||, \tag{I.4.11}$$

for all $t \in \mathbb{R}$, $k \in \mathbb{Z}$ *and* $\phi, \psi \in B_\delta(0)$. *If the evolution family* $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ *associated with the linearization of* (I.4.1)–(I.4.2) *is hyperbolic and* $\mathcal{RCR}_u(t) = \{0\}$, *the fixed point* 0 *is exponentially stable.*

Proof. Since $\mathcal{RCR}_u(t) = \{0\}$ and the linearization is hyperbolic, the evolution family satisfies $||U(t,s)|| \leq Ke^{a(t-s)}$ for some $a < 0$, for all $t \geq s$. We first prove that the fixed point is stable. Let $\epsilon > 0$ be arbitrary, and choose some $\epsilon' \leq \epsilon$ small enough so that

$$\left(1 + \frac{1}{\xi}\right) c(\epsilon') < \frac{1}{2}, \tag{I.4.12}$$

where c is the constant from the statement of the theorem. Next, choose $\delta > 0$ satisfying

$$\delta < \frac{\epsilon'}{2K}. \tag{I.4.13}$$

Let $||\phi|| < \delta$, and introduce the space of history-valued functions X_ϕ, defined by

$$X_\phi = \left\{ \begin{array}{l} z : [s, \infty) \to \mathcal{RCR} \ : \ \exists y \in \mathcal{RCR}([s-r, \infty), \mathbb{R}^n), \\ z(t) = y_t, ||z|| < \epsilon', \ z(s) = \phi \end{array} \right\},$$

on which we introduce the norm $||z|| = \sup_{t \geq s} ||z(t)||$, where the latter is the typical supremum norm. X_ϕ is clearly equivalent up to isometry as a normed space to the subspace

$$\{w \in \mathcal{RCR}([s-r, \infty) : ||w|| < \epsilon, w_s = \phi\} \subset \mathcal{RCR}([s-r, \infty), \mathbb{R}^n).$$

Since the latter is complete, the same is true of X_ϕ. Consider the formal expression

$$F(z)[t] = U(t,s)\phi + \int_s^t U(t,\mu)[\chi_0 f(\mu, z(\mu))]d\mu + \sum_{s<t_i\leq t} U(t,t_i)[\chi_0 g(i, z(t_i^-))]$$

for $z \in X_\phi$. By assumption H.3 and Lemma I.2.3.5, the above defines a nonlinear map $F : X_\phi \to X$, with

$$X = \{z : [s,\infty) \to \mathcal{RCR} \ : \ \exists y \in \mathcal{RCR}([s-r,\infty), \mathbb{R}^n), z(t) = y_t, ||z|| < \infty\} \supset X_\phi.$$

We claim that $\mathrm{im}(F) \subset X_\phi$. We can estimate the nom $||F(z)[t]||$ using $||U(t,s)|| \leq Ke^{a(t-s)}$, the fundamental theorem of calculus for Banach space-valued C^1 functions and Lemma I.1.5.2. The result is

$$||F(z)[t]|| \leq Ke^{a(t-s)}\delta + \frac{K(1-e^{a(t-s)})}{-a} c(\epsilon')\epsilon' + \frac{K(1-e^{a(t+\xi-s)})}{-a\xi} c(\epsilon')\epsilon',$$

and from inequalities (I.4.12) and (I.4.13) together with $a < 0$, it follows that $||F(z)[t]|| < \epsilon'$ for all $t \geq s$, independent of s. Since mild solutions are precisely fixed points of F, it follows that whenever $||\phi|| < \delta$, the process satisfies $||S(t,s)\phi|| < \epsilon' \leq \epsilon$ for all $t \geq s$. That is, the fixed point is (uniformly) stable.

To get exponential stability, repeat the above argument but with the stronger condition that $\epsilon' \leq \epsilon$ is small enough to guarantee in addition to (I.4.12), the inequality

$$\rho := a + Kc(\epsilon')\left(1 + \frac{1}{\xi}\right) < 0. \tag{I.4.14}$$

We begin with the integral equation (I.4.4) for the mild solution. We have the estimate

$$\begin{aligned}||S(t,s)\phi|| \leq Ke^{a(t-s)}||\phi|| + \int_s^t Ke^{a(t-\mu)}||S(\mu,s)\phi||c(\epsilon')d\mu \\ + \sum_{s<t_i\leq t} Ke^{a(t-t_i)}||S(t_i^-,s)\phi||c(\epsilon')\end{aligned}$$

for all $t \geq s$, provided $||\phi|| < \delta$, where $\delta > 0$ is again chosen according to (I.4.13). Multiplying both sides by e^{-at}, this is equivalent to

$$\begin{aligned}e^{-at}||S(t,s)\phi|| \leq Ke^{-as}||\phi|| + \int_s^t Kc(\epsilon')e^{-a\mu}||S(\mu,s)\phi||d\mu \\ + \sum_{s<t_i\leq t} Kc(\epsilon')e^{-at_i}||S(t_i^-,s)\phi||.\end{aligned}$$

Applying the Gronwall lemma I.1.5.1 to $t \mapsto e^{-at}||S(t,s)\phi||$, we eventually obtain

$$e^{-at}||S(t,s)\phi|| \leq Ke^{-as}||\phi|| \exp\left((t-s)Kc(\epsilon') + \frac{(t+\xi-s)}{\xi}\log(1+Kc(\epsilon'))\right).$$

Multiplying by e^{at} and exploiting $\log(1+x) \leq x$ for $x > 0$, we obtain

$$||S(t,s)\phi|| \leq K(1+Kc(\epsilon'))||\phi||e^{\rho(t-s)},$$

and since $\rho < 0$ from the assumption (I.4.14) that ϵ' is chosen small enough, we obtain exponential stability. □

Remark I.4.3.1. *If γ is a periodic orbit of the* autonomous *delay differential equations* without impulses, *its centre fibre bundle is always at least one-dimensional [41] because $t \mapsto \gamma'(t)$ is a solution of the variational equation. Nevertheless, this orbit will be asymptotically stable modulo phase shifts provided the centre fibre bundle is one-dimensional and the unstable bundle is trivial. In this sense, the statement of Proposition I.4.3.1 is not optimized for this very special case. In this monograph, however, we will typically assume that our systems are temporally forced by impulses, so this consideration need not be taken into account.*

We will generally describe a solution as being hyperbolic or nonhyperbolic depending on whether the centre fibre bundle of its linearization is trivial or not.

Definition I.4.3.2. *Suppose the linearization associated with a solution γ is spectrally separated. γ is* hyperbolic *if the centre fibre bundle is trivial. Otherwise, it is said to be* nonhyperbolic*.*

From Remark I.4.1.1, statements concerning the fixed point at zero for the semilinear equation (I.4.1)–(I.4.2) can be translated into statements concerning the reference solution $\gamma : \mathbb{R} \to \mathbb{R}^n$ of the fully nonlinear equation (I.4.6)–(I.4.7).

I.4.4 Comments

Theorem I.4.2.1 and its proof appear in [31]. Under some weaker assumptions that we have made in this chapter, Federson and Schwabik [45] have proven existence and uniqueness of solutions of nonlinear impulsive RFDE with regulated initial conditions using a generalized ordinary differential equations approach, under the assumption that the impulse effects involve no delays. Ballinger and Liu [13, 14] previously investigated existence and uniqueness of solutions with initial conditions that are more regular, being piecewise-continuous. Continuous dependence on initial conditions has previously been considered by Liu and Ballinger [94], Federson and Schwabik [45] and Church and Liu [31], with Section I.4.2 being based on the work of the latter. Existence and uniqueness of solutions for impulsive functional differential equations in general Banach spaces have been studied by many authors, and we refer the reader to the short list [59, 64, 74, 113, 154, 155, 168] for some recent work.

We have not discussed here some of the more "nonlinear" methods of proving stability, which include Lyapunov functions and Lyapunov functionals. The body of literature on these methods is vast. More recently, there has been a great deal of research done on developing sufficient condition under which a given nonlinear time-delayed system can be stabilized by impulses. When Lyapunov functions and functionals are involved, these sufficient conditions often involve linear matrix inequalities. For a very short list of results along these lines, see [47, 88, 90, 100, 104, 148, 159, 167]. An alternative approach is invariance stabilization [34], which exploits the dynamics on invariant manifolds in the design of a controller to stabilize a time-delayed system. Related to stability is synchronization, for which the Lyapunov method continues to be an indispensable tool; see [25, 42, 63, 99, 141, 148, 152, 167].

An alternative approach to proving stability of nonlinear dynamical systems involves the use of fixed-point theory. The fixed-point theory in stability of functional differential equations arose with the aim of handling equations for which Lyapunov methods are too strict to grant asymptotic stability. For example [20], fixed-point methods can handle equations where the vector field is unbounded in time, where time-varying delays might be unbounded and nondifferentiable, and when the lag term $t - r(t)$ might very well approach a finite limit as $t \to \infty$. The use of fixed-point methods for stability seems to have been initiated by Burton and Furomochi in the year 2001 [21]. More recently, these ideas have been applied to impulsive functional differential equations [95, 119].

Chapter I.5

Existence, Regularity and Invariance of Centre Manifolds

In Sect. I.2.2.1 we introduced spectral separation for the evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ associated to a linear impulsive functional differential equation. This results in a decomposition of the phase space as the internal direct sum $\mathcal{RCR} = \mathcal{RCR}_s(t) \oplus \mathcal{RCR}_c(t) \oplus \mathcal{RCR}_u(t)$ of three fibre bundles, respectively: the stable, centre and unstable fibre bundles. These can be thought of as time-varying vector spaces, and the evolution family restricted to these fibre bundles exhibits growth characteristics that are distinct. In the stable fibre bundle, solutions decay exponentially to zero in forward time. Solutions in the centre fibre bundle are defined for all time and exhibit at most subexponential growth in forward and backward time, and solutions in the unstable fibre bundle are also defined for all time and decay to zero exponentially in reverse time.

For a nonlinear impulsive functional differential equation, if the evolution family of the linearization of some equilibrium point is spectrally separated, the invariant fibre bundles are in some sense nonlinearly distorted by the nonlinearities in the vector field and jump map, and the result is a local stable, centre and unstable manifold. The centre manifold in particular contains useful information pertaining to small solutions near the equilibrium and can for this reason be used for the detection of bifurcations. The present chapter is devoted to several aspects of the centre manifold, including its existence, smoothness (in both the phase space and with respect to time), invariance, reduction principle, restricted dynamics and its approximation by Taylor expansion. In Chap. I.7 we touch on aspects of the other classical

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invariant manifolds (although in less detail) and, in general, the dynamics of the nonautonomous process on restriction to them.

I.5.1 Preliminaries

The local centre manifold (and, indeed, every invariant manifold we consider in this monograph) is defined through the solution of a particular fixed-point equation. In order to formulate this equation correctly, we need to take into account the expected growth rates of solutions on the centre manifold, which, as we know from linear systems theory, may exhibit subexponential growth in both forward and reverse time. This section is devoted to the introduction of Banach spaces that satisfy these growth rate conditions, as well as some useful results from linear systems and substitution operators that we will later need to define the Lyapunov–Perron operator that will define the fixed-point equation.

I.5.1.1 Spaces of Exponentially Weighted Functions

Denote $PC(\mathbb{R},\mathbb{R}^n)$ the set of functions $f:\mathbb{R}\to\mathbb{R}^n$ that are continuous everywhere except for at times $t\in\{t_k:k\in\mathbb{Z}\}$ where they are continuous from the right and have limits on the left. Define a weighted norm $||f||_\eta = \sup_{t\in\mathbb{R}} e^{-\eta|t|}||\phi(t)||$ for functions $f:\mathbb{R}\to X$ for Banach space X. We define an analogous norm for sequences indexed by $\mathbb{Z}$.

$$\begin{aligned}
\mathcal{PC}^\eta &= \{\phi:\mathbb{R}\to\mathcal{RCR}:\phi(t)=f_t, f\in PC(\mathbb{R},\mathbb{R}^n), ||\phi||_\eta<\infty\}\\
B^\eta(\mathbb{R},\mathcal{RCR}) &= \{f:\mathbb{R}\to\mathcal{RCR}:||f||_\eta<\infty\}\\
PC^\eta(\mathbb{R},\mathbb{R}^n) &= \{f\in PC(\mathbb{R},\mathbb{R}^n):||f||_\eta<\infty\}\\
B^\eta_{t_k}(\mathbb{Z},\mathbb{R}^n) &= \{f:\mathbb{Z}\to\mathbb{R}^n:||f||_\eta<\infty\}.
\end{aligned}$$

Also, if $\mathcal{M}\subset\mathbb{R}\times\mathcal{RCR}$ is a nonautonomous set over $\mathcal{RCR}$, we define the space $\mathcal{PC}^\eta(\mathbb{R},\mathcal{M})$ of piecewise-continuous functions taking values in $\mathcal{M}$ by

$$\mathcal{PC}^\eta(\mathbb{R},\mathcal{M}) = \{f\in\mathcal{PC}^\eta : f(t)\in\mathcal{M}(t)\}.$$

If X^η is one of the above spaces, then the normed space $X^{\eta,s}=(X^\eta,||\cdot||_{\eta,s})$ with norm

$$||F||_{\eta,s} = \begin{cases} \sup_{t\in\mathbb{R}} e^{-\eta|t-s|}||F(t)||, & \mathrm{dom}(F)=\mathbb{R}\\ \sup_{k\in\mathbb{Z}} e^{-\eta|t_k-s|}||F(k)||, & \mathrm{dom}(F)=\mathbb{Z},\end{cases}$$

is complete. Broadly speaking, elements of these spaces will be referred to as η-bounded.

I.5.1.2 η-Bounded Solutions from Inhomogeneities

In this section we will characterize the η-bounded solutions of the inhomogeneous linear equation

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\mu)[\chi_0 F(\mu)]d\mu + \sum_{s<t_i\le t} U(t,t_i)[\chi_0 G_i], \qquad -\infty < s \le t < \infty, \tag{I.5.1}$$

for inhomogeneous terms F and G. As defined in Definition I.1.1.6, we recall now that $\mathcal{RCR}_c(t) = \mathcal{R}(P_c(t))$, where P_c is the projection onto the centre bundle of the linear part of (I.4.1)–(I.4.2).

Lemma I.5.1.1. *Let $\eta \in (0, \min\{-a, b\})$ and let H.1, H.2 and H.5 hold. Then,*

$$\mathcal{RCR}_c(\nu) = \{\varphi \in \mathcal{RCR} : \exists x \in \mathcal{PC}^\eta,\ x(t) = U(t,s)x(s),\ x(\nu) = \varphi\}. \tag{I.5.2}$$

Proof. If $\varphi \in \mathcal{RCR}_c(\nu)$, then $P_c(\nu)\varphi = \varphi$ and the function $x(t) = U(t,\nu) P_c(\nu)\varphi = U_c(t,\nu)\varphi$ is defined for all $t \in \mathbb{R}$, satisfies $x(t) = U(t,s)x(s)$, $x(\nu) = \varphi$, $x(t)(\theta) = x(t+\theta)(0)$, and by choosing $\epsilon < \eta$, there exists $K > 0$ such that

$$e^{-\eta|t|}||x(t)|| \le Ke^{\epsilon|\nu|}e^{-(\eta-\epsilon)|t|}||\varphi|| \le Ke^{\epsilon|\nu|}||\varphi||.$$

Finally, as $x(t) = [U(t,s)x(s)(0)]_t$ for all $t \in \mathbb{R}$, we conclude $x \in \mathcal{PC}^\eta$.

Conversely, suppose $\varphi \in \mathcal{RCR}$ admits some $x \in \mathcal{PC}^\eta$ such that $x(t) = U(t,s)x(s)$ and $x(\nu) = \varphi$. Let $||x||_\eta = \overline{K}$. We will show that $P_s(\nu)\varphi = P_u(\nu)\varphi = 0$, from which we will conclude $\varphi \in \mathcal{RCR}_c(\nu)$.

By spectral separation, we have for all $\rho < \nu$,

$$\begin{aligned} e^{-\eta|\rho|}||P_s(\nu)\varphi|| &= e^{-\eta|\rho|}||U_s(\nu,\rho)P_s(\rho)x(\rho)|| \\ &\le e^{-\eta|\rho|}Ke^{a(\nu-\rho)}||P_s(\rho)|| \cdot ||x(\rho)|| \\ &\le K\overline{K}e^{a(\nu-\rho)}||P_s(\rho)||, \end{aligned}$$

which implies $||P_s(\nu)\varphi|| \le K\overline{K}e^{a\nu}||P_s(\rho)||\exp(\eta|\rho| - a\rho)$. Since $\eta < -a$ and $\rho \mapsto ||P_s(\rho)||$ is bounded, taking the limit as $\rho \to -\infty$ we obtain $||P_s(\nu)\varphi||| \le 0$. Similarly, for $\rho > \nu$, we have

$$\begin{aligned} e^{-\eta|\rho|}||P_u(\nu)\varphi|| &= e^{-\eta|\rho|}||U_u(\nu,\rho)P_u(\rho)x(\rho)|| \\ &\le e^{-\eta|\rho|}Ke^{b(\nu-\rho)}||P_u(\rho)|| \cdot ||x(\rho)|| \\ &\le K\overline{K}e^{b(\nu-\rho)}||P_u(\rho)||, \end{aligned}$$

which implies $||P_u(\nu)\varphi|| \le K\overline{K}e^{b\nu}||P_u(\rho)||\exp(\eta|\rho| - b\rho)$. Since $\eta <$b and $\rho \mapsto ||P_u(\rho)||$ is bounded, taking the limit $\rho \to \infty$ we obtain $||P_u(\nu)\varphi|| \le 0$. Therefore, $P_s(\nu)\varphi = P_u(\nu)\varphi = 0$, and we conclude that $P_c(\nu)\varphi = \varphi$ and $\varphi \in \mathcal{RCR}_c(\nu)$. ∎

Lemma I.5.1.2. *Let conditions H.1, H.2 and H.5 be satisfied. Let* $h \in \mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$. *The integrals*

$$\int_s^t U(t,\mu)P_c(\mu)[\chi_0 h(\mu)]d\mu, \qquad \int_v^t U(t,\mu)P_u(\mu)[\chi_0 h(\mu)]d\mu$$

are well-defined as Pettis integrals for all $s,t,v \in \mathbb{R}$, *where we define* $\int_b^a f d\mu = -\int_a^b f d\mu$ *when* $a < b$.

Proof. The nontrivial cases are where $t \leq s$ and $t \leq v$. For the former, defining $H(\mu) = \chi_0 h(\mu)$ we have the string of equalities

$$\begin{aligned} U_c(t,s)P_c(s)\int_t^s U(s,\mu)H(\mu)d\mu &= U_c(t,s)\int_t^s U_c(s,\mu)P_c(\mu)H(\mu)d\mu \\ &= \int_t^s U_c(t,\mu)P_c(\mu)H(\mu)d\mu \\ &= \int_t^s U(t,\mu)P_c(\mu)H(\mu)d\mu \\ &= -\int_s^t U(t,\mu)P_c(\mu)H(\mu)d\mu. \end{aligned}$$

The first integral on the left exists due to Lemma I.2.3.5 and Proposition I.1.4.1. The subsequent equalities follow by Proposition I.2.3.5 and the definition of spectral separation. The case $t \leq v$ for the other integral is proven similarly. □

Define the (formal) linear operators $\mathcal{K}_s^\eta : \mathcal{PC}^{\eta,s} \oplus B_{t_k}^\eta(\mathbb{Z},\mathbb{R}^n) \rightarrow B^\eta(\mathbb{R},\mathcal{RCR})$ by the equation

$$\begin{aligned} \mathcal{K}_s^\eta(F,G)(t) = &\int_s^t U(t,\mu)P_c(\mu)[\chi_0 F(\mu)]d\mu - \int_t^\infty U(t,\mu)P_u(\mu)[\chi_0 F(\mu)]d\mu \\ &+ \int_{-\infty}^t U(t,\mu)P_s(\mu)[\chi_0 F(\mu)]d\mu + \sum_s^t U(t,t_i)P_c(t_i)[\chi_0 G_i]dt_i \\ &- \sum_t^\infty U(t,t_i)P_u(t_i)[\chi_0 G_i]dt_i + \sum_{-\infty}^t U(t,t_i)P_s(t_i)[\chi_0 G_i]dt_i, \end{aligned} \tag{I.5.3}$$

indexed by $s \in \mathbb{R}$, where the external direct sum $\mathcal{PC}^{\eta,s} \oplus B_{t_k}^{\eta,s}(\mathbb{Z},\mathbb{R}^n)$ is identified as a Banach space with norm $||(f,g)||_{\eta,s} = ||f||_{\eta,s} + ||g||_{\eta,s}$, and the summations are defined as follows:

$$\sum_a^b F(t_i)dt_i = \begin{cases} \displaystyle\sum_{a<t_i\leq b} F(t_i), & a \leq b \\ \displaystyle -\sum_b^a F(t_i)dt_i, & b < a. \end{cases}$$

Lemma I.5.1.3. *Let H.1, H.2, H.5 and H.7 hold, and let* $\eta \in (0, \min\{-a,b\})$.

1. *The function* $\mathcal{K}_s^\eta : \mathcal{PC}^{\eta,s} \oplus B_{t_k}^{\eta,s}(\mathbb{Z},\mathbb{R}^n) \to B^{\eta,s}(\mathbb{R},\mathcal{RCR})$ *with* $\eta \in (0,\min\{-a,b\})$ *and defined by formula* (I.5.3) *is linear and bounded. In particular, the norm satisfies*

$$||\mathcal{K}_s^\eta|| \leq C\left[\frac{1}{\eta-\epsilon}\left(1+\frac{e^{(\eta-\epsilon)\xi}}{\xi}\right)+\frac{1}{-a-\eta}\left(1+\frac{2e^{(\eta-a)\xi}}{\xi}\right)+\frac{1}{b-\eta}\left(1+\frac{2e^{(b+\eta)\xi}}{\xi}\right)\right] \tag{I.5.4}$$

for some constants C *and* ϵ *independent of* s.

2. $\mathcal{K}_s^\eta$ *has range in* $\mathcal{PC}^{\eta,s}$, *and* $v = \mathcal{K}_s^\eta(F,G)$ *is the unique solution of* (I.5.1) *in* $\mathcal{PC}^{\eta,s}$ *satisfying* $P_c(s)v(s) = 0$.

3. *The expression* $\mathcal{K}_*(F,G)(t) = (I - P_c(t))K_s^0(F,G)(t)$ *uniquely defines, independent of* s, *a bounded linear map*

$$\mathcal{K}_* : \mathcal{PC}^0 \oplus B_{t_k}^0(\mathbb{Z},\mathbb{R}^n) \to \mathcal{PC}^0.$$

Proof. Let $\epsilon < \min\{\min\{-a,b\}-\eta,\eta\}$. To show that $\mathcal{K}_s^\eta$ is well-defined, we start by mentioning that all improper integrals and infinite sums appearing on the right-hand side of (I.5.3) can be interpreted as limits of well-defined finite integrals and sums, due to Lemma I.2.3.5, Lemma I.5.1.2 and Proposition I.1.4.1. For brevity, write

$$\mathcal{K}_s^\eta(F,G) = \left(K_1^{u,f} - K_1^{c,F} + K_1^{u,F}\right) + \left(K_2^{u,G} - K_2^{c,G} + K_2^{s,G}\right),$$

where each term corresponds to the one in (I.5.3) in order of appearance.

We start by proving the convergence of the improper integrals. Denote

$$I(v) = \int_t^v U(v,\mu)P_u(\mu)[\chi_0 F(\mu)]d\mu,$$

and let $v_k \nearrow \infty$. We have, for $m > n$ and n sufficiently large so that $v_m > 0$,

$$\begin{aligned}
||I(v_m) - I(v_n)|| &\leq \int_{v_n}^{v_m} KNe^{b(t-\mu)}|F(\mu)|d\mu \\
&\leq \int_{v_n}^{v_m} KNe^{b(t-\mu)}e^{\eta\mu}||F||_\eta d\mu \\
&= KN||F||_\eta e^{bt}\int_{v_n}^{v_m} e^{\mu(\eta-b)}d\mu \\
&= \frac{KN||F||_\eta}{b-\eta}e^{bt}\left(e^{-v_n(b-\eta)} - e^{-v_m(b-\eta)}\right) \\
&\leq \frac{KN||F||_\eta}{b-\eta}e^{bt}e^{-v_n(b-\eta)}.
\end{aligned}$$

Therefore, $I(v_k) \in \mathcal{RCR}$ is Cauchy and thus converges; namely, it converges to the improper integral $K^{u,F}(t)$. One can similarly prove that $K^{s,F}(t)$

converges. For the infinite sums, we employ similar estimates; if we denote $S = \sum_{t<t_i<\infty} ||U_u(t,t_i)[\chi_0 G_i]||$ and assume without loss of generality that $t_0 = 0$, a fairly crude estimate (that we will later improve) yields

$$\begin{aligned} S &\leq \sum_{t<t_i<\infty} KNe^{b(t-t_i)}e^{\eta|t_i|}||G||_\eta \\ &= \sum_{-|t|<t_i\leq 0} KN||G||_\eta e^{bt}e^{|t_i|(b+\eta)} + \sum_{0<t_k<\infty} KN||G||_\eta e^{bt}e^{-(b-\eta)t_i} \\ &\leq KNe^{bt}\left(\frac{|t|}{\xi}e^{|t|(b+\eta)} + \frac{1}{1-e^{-(b-\eta)\xi}}\right)||G||_\eta. \end{aligned}$$

Thus, $K^{u,G}(t)$ converges uniformly. One can show by similar means that $K^{s,F}(t)$ and $K^{s,G}(t)$ both converge. Therefore, $\mathcal{K}_s^\eta(F,G)(t) \in \mathcal{RCR}$ exists. We can now unambiguously state that $\mathcal{K}_s^\eta$ is clearly linear.

Our next task is to prove that $||K_s^\eta(F,G)]||_{\eta,s} \leq Q||(F,G)||_{\eta,s}$ for constant Q satisfying the estimate of equation (I.5.4). We will prove the bounds only for $||K^{u,F}||_{\eta,s}$, $||K^{u,G}||_{\eta,s}$, $||K^{c,F}||_{\eta,s}$ and $||K^{c,G}||_{\eta,s}$; the others follow by similar calculations. For $t<s$, we have

$$\begin{aligned} &e^{-\eta|t-s|}||K^{u,F}(t)|| \\ &\leq e^{-\eta|t-s|}\int_t^\infty KNe^{b(t-\mu)}|F(\mu)|d\mu \\ &\quad\leq e^{\eta(t-s)}KN\left[\int_t^s e^{b(t-\mu)}e^{\eta|\mu-s|}||F||_{\eta,s}d\mu + \int_s^\infty e^{b(t-\mu)}e^{\eta|\mu-s|}||F||_{\eta,s}d\mu\right] \\ &\quad= e^{\eta(t-s)}KN||F||_{\eta,s}\left[\int_t^s e^{b(t-\mu)}e^{\eta(s-\mu)}d\mu + \int_s^\infty e^{b(t-\mu)}e^{\eta(\mu-s)}d\mu\right] \\ &\quad= e^{\eta(t-s)}KN||F||_{\eta,s}\left[e^{bt+\eta s}\frac{e^{-(b+\eta)t}-e^{-(b+\eta)s}}{b+\eta} + e^{bt-\eta s}\frac{e^{-(b-\eta)s}}{b-\eta}\right] \\ &\quad\leq KN||F||_{\eta,s}\frac{1}{b-\eta}. \end{aligned}$$

The above inequality is also satisfied for $t \geq s$, and we conclude $||K^{u,F}||_{\eta,s} \leq KN(b-\eta)^{-1}||(F,G)||_{\eta,s}$. Next, for $t<s$,

$$\begin{aligned} &e^{-\eta|t-s|}||K^{u,G}(t)|| \\ &\quad\leq e^{-\eta|t-s|}\sum_{t<t_i<\infty} KNe^{b(t-t_i)}|G_i| \\ &\quad\leq e^{\eta(t-s)}KN\left[\sum_{t<t_i<s} e^{b(t-t_i)}e^{\eta|t_i-s|}||G||_{\eta,s} + \sum_{s\leq t_i<\infty} e^{b(t-t_i)}e^{\eta|t_i-s|}||G||_{\eta,s}\right] \\ &\quad\leq e^{\eta(t-s)}KN||G||_{\eta,s}\frac{1}{\xi}\left[\int_{t-\xi}^s e^{b(t-\mu)}e^{\eta(s-\mu)}d\mu + \int_{s-\xi}^\infty e^{b(t-\mu)}e^{\eta(\mu-s)}d\mu\right] \end{aligned}$$

$$\leq e^{\eta(t-s)}\frac{KN||G||_{\eta,s}}{\xi}\left[e^{bt+\eta s}\frac{e^{-(b+\eta)(t-\xi)}-e^{-(b+\eta)x}}{b+\eta}+e^{bt-\eta s}\frac{e^{-(b-\eta)(s-\xi)}}{b-\eta}\right]$$
$$\leq \frac{2KN||G||_{\eta,s}}{\xi(b-\eta)}\cdot e^{(b+\eta)\xi},$$

where we have made use of Lemma I.1.5.2 to estimate the sums. The same conclusion is valid for $t\geq s$, and it follows that $||K^{u,G}||_{\eta,s}\leq 2KNe^{(b+\eta)\xi}(\xi(b-\eta))^{-1}||(F,G)||_{\eta,s}$. Next, for $t\leq s$,

$$\begin{aligned}e^{-\eta|t-s|}||K^{c,G}(t)|| &\leq e^{\eta(t-s)}KN||G||_{\eta,s}\sum_{t<t_i\leq s}e^{\epsilon(t_i-t)}e^{\eta(s-t_i)}\\ &\leq e^{\eta(t-s)}\frac{KN||G||_{\eta,s}}{\xi}\int_{s-\xi}^{t}e^{\epsilon(\mu-t)}e^{\eta(s-\mu)}d\mu\\ &= e^{\eta(t-s)}\frac{KN||G||_{\eta,s}}{\xi(\eta-\epsilon)}\left(e^{\epsilon(s-\xi-t)}e^{\eta\xi}-e^{-\eta(t-s)}\right)\\ &\leq \frac{KN||G||_{\eta,s}}{\xi(\eta-\epsilon)}e^{(\eta-\epsilon)\xi}.\end{aligned}$$

This estimate continues to hold for all $t,s\in\mathbb{R}$. To compare to the integral term, for $s\leq t$, we have

$$\begin{aligned}e^{-\eta|t-s|}||K^{c,F}(t)|| &\leq e^{-\eta(t-s)}KN||F||_{\eta,s}\int_{s}^{t}e^{\epsilon(t-\mu)}e^{\eta(\mu-s)}d\mu\\ &= e^{-\eta(t-s)}KN||F||_{\eta,s}\frac{1}{\eta-\epsilon}\left(e^{\eta(t-s)}-e^{\epsilon(t-s)}\right)\\ &\leq \frac{KN||F||_{\eta,s}}{\eta-\epsilon},\end{aligned}$$

and this estimate persists for all $t,s\in\mathbb{R}$. Similar estimates for the other integrals and sums appearing in (I.5.3) ultimately result in the bound appearing in (I.5.4). This proves part 1.

To prove part 2, denote $v=\mathcal{K}_s^{\eta}(F,G)$. It is clear from the definition of v, the orthogonality of the projection operators and Proposition I.1.4.1 that $P_c(s)v(s)=0$. Also, for all $-\infty<z\leq t<\infty$, denoting $\overline{F}=\chi_0F$ and $\overline{G}_i=\chi_0G$, we have

$$\begin{aligned}&U(t,z)v(z)+\int_z^t U(t,\mu)\overline{F}(\mu)d\mu+\sum_z^t U(t,t_i)\overline{G}_i dt_i\\ &=U(t,z)v(z)+\int_z^t U(t,\mu)P_c(\mu)\overline{F}(\mu)d\mu-\int_t^z U(t,\mu)P_u(\mu)\overline{F}(\mu)d\mu\\ &\quad+\int_z^t U(t,\mu)P_u(\mu)\overline{F}(\mu)d\mu+\sum_z^t U(t,t_i)P_c(t_i)\overline{G}_i dt_i\\ &\quad-\sum_t^z U(t,t_i)P_u(t_i)\overline{G}_i dt_i+\sum_z^t U(t,t_i)P_u(t_i)\overline{G}_i dt_i\end{aligned}$$

$$
\begin{aligned}
&= \int_s^t U(t,\mu)P_c(\mu)\overline{F}(\mu)d\mu - \int_t^\infty U(t,\mu)P_u(\mu)\overline{F}(\mu)d\mu + \int_{-\infty}^t U(t,\mu)P_s(\mu)\overline{F}(\mu)d\mu \\
&\quad + \sum_s^t U(t,t_i)P_c(t_i)\overline{G}_i dt_i - \sum_t^\infty U(t,t_i)P_u(t_i)\overline{G}_i dt_i + \sum_{-\infty}^t U(t,t_i)P_s(t_i)\overline{G}_i dt_i \\
&= v(t),
\end{aligned}
$$

so that $t \mapsto v(t)$ solves the integral equation (I.5.1). This also demonstrates that $v \in \mathcal{PC}^\eta$. To show that it is the only solution in $\mathcal{PC}^\eta$ satisfying $P_c(s)v(s) = 0$, suppose there is another $r \in PC^\eta$ that satisfies $P_c(s)r(s) = 0$. Then the function $w := v - r$ is an element of $\mathcal{PC}^\eta$ that satisfies $w(t) = U(t,z)w(z)$ for $-\infty < z \leq t < \infty$. By Lemma I.5.1.1, we have $w(s) \in \mathcal{RCR}_c(s)$. But since $P_c(s)w(s) = 0$ and $P_c(s)$ is the identity on $\mathcal{RCR}_c(s)$, we obtain $w(s) = 0$. Therefore, $w(t) = U(t,s)0 = U_c(t,s)0 = 0$ for all $t \in \mathbb{R}$, and we conclude $v = r$, proving the uniqueness assertion.

For assertion 3, we compute first

$$
\begin{aligned}
\mathcal{K}_*(F,G)(t) = &\int_{-\infty}^t U(t,\mu)P_s(\mu)[\chi_0 F(\mu)]d\mu - \int_t^\infty U(t,\mu)P_u(\mu)[\chi_0 F(\mu)]d\mu \\
&\sum_{-\infty}^t U(t,t_i)P_s(t_i)[\chi_0 G_i]dt_i - \sum_t^\infty U(t,t_i)P_u(t_i)[\chi_0 G_i]dt_i.
\end{aligned}
$$

Routine estimation using inequalities (I.1.11)–(I.1.13) together with Lemma I.1.5.2 produces the bound

$$
||\mathcal{K}_*(F,G)(t)|| \leq KN\left(\frac{-1}{a} + \frac{1}{b} - \frac{e^{-a\xi}}{a\xi} + \frac{e^{b\xi}}{b\xi}\right)||(F,G)||,
$$

and as the bound is independent of t, s, the result is proven. □

I.5.1.3 Substitution Operator and Modification of Nonlinearities

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a C^∞ bump function satisfying

i) $\xi(y) = 1$ for $0 \leq y \leq 1$,

ii) $0 \leq \xi(y) \leq 1$ for $1 \leq y \leq 2$,

iii) $\xi(y) = 0$ for $y \geq 2$.

We modify the nonlinearities of (I.4.1)–(I.4.2) in the centre and hyperbolic directions separately. For $\delta > 0$ and $s \in \mathbb{R}$, we let

$$
F_{\delta,s}(t,x) = f(t,x)\xi\left(\frac{||P_c(s)x||}{N\delta}\right)\xi\left(\frac{||(P_s(s)+P_u(s))x||}{N\delta}\right) \tag{I.5.5}
$$

$$
G_{\delta,s}(k,x) = g(k,x_{0^-})\xi\left(\frac{||P_c(s)x_{0^-}||}{N\delta}\right)\xi\left(\frac{||(P_s(s)+P_u(s))x_{0^-}||}{N\delta}\right). \tag{I.5.6}
$$

Notice that $G_{\delta,s}(k,x)$ takes the pointwise left-limit in the evaluation (I.5.6). The proof of the following lemma and corollary will be omitted. They can be proven by emulating the proof of Lemma 6.1 from [70] and taking into account the uniform boundedness of the projectors P_i; see property 1 of Definition I.1.1.6.

Lemma I.5.1.4. *Let $f(t,\cdot)$ and $g(k,\cdot)$ be uniformly (in $t \in \mathbb{R}$ and $k \in \mathbb{Z}$) Lipschitz continuous on the ball $B_{\mathcal{RCR}}(\delta, 0)$ in $\mathcal{RCR}$ with mutual Lipschitz constant $L(\delta)$, and let $f(t,0) = g_k(0) = 0$. The functions*

$$F_{\delta,s} : \mathbb{R} \times \mathcal{RCR} \to \mathbb{R}^n, \qquad G_{\delta,s} : \mathbb{Z} \times \mathcal{RCR} \to \mathbb{R}^n$$

are globally, uniformly (in $t \in \mathbb{R}$ and $k \in \mathbb{Z}$) Lipschitz continuous with mutual Lipschitz constant L_δ that satisfies $L_\delta \to 0$ as $\delta \to 0$, independent of s.

Corollary I.5.1.1. *The substitution operator*

$$R_{\delta,s} : \mathcal{PC}^{\eta,s} \to PC^{\eta,s}(\mathbb{R}, \mathbb{R}^n) \oplus B^{\eta,s}_{t_k}(\mathbb{Z}, \mathbb{R}^n)$$

defined by $R_{\delta,s}(x)(t,k) = (F_{\delta,s}(t,x(t)), G_{\delta,s}(k,x(t_k)))$ is globally Lipschitz continuous with Lipschitz constant $\tilde{L}_\delta$ that satisfies $\tilde{L}_\delta \to 0$ as $\delta \to 0$. Moreover, the Lipschitz constant is independent of η, s.

Corollary I.5.1.2. $||(F_{\delta,s}(t,x), G_{\delta,s}(k,x))|| \leq 4\delta L_\delta$ *for all $x \in \mathcal{RCR}$ and $(t,k) \in \mathbb{R} \times \mathbb{Z}$.*

Remark I.5.1.1. *The explicit connection between $L(\delta)$ (the Lipschitz constant for f and g) and L_δ and $\tilde{L}_\delta$ is complicated and depends in part on the choice of cutoff function ξ and the constant N.*

I.5.2 Fixed-Point Equation and Existence of a Lipschitz Centre Manifold

Let $\eta \in (\epsilon, \min\{-a, b\})$ and define a mapping $\mathcal{F}_s : \mathcal{PC}^{\eta,s} \times \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ by

$$\mathcal{F}_s(u,\varphi) = U(\cdot,s)\varphi + \mathcal{K}^\eta_s(R_{\delta,s}(u)). \tag{I.5.7}$$

Note that by Lemma I.5.1.3 and Corollary I.5.1.1, the operator is well-defined, $\mathcal{K}^\eta_s$ is bounded and R_δ is globally Lipschitz continuous for each $\delta > 0$, provided H.1–H.7 hold. Choose δ small enough so that

$$\tilde{L}_\delta ||K^\eta_s||_\eta < \frac{1}{2}. \tag{I.5.8}$$

Notice that δ can be chosen so that (I.5.8) can be satisfied independent of s, due to Lemma I.5.1.3. If $||\varphi|| < r/(2K)$, then $\mathcal{F}_s(\cdot,\varphi)$ leaves $\overline{B(r,0)} \subset \mathcal{PC}^{\eta,s}$ invariant. Moreover, $\mathcal{F}_s(\cdot,\varphi)$ is Lipschitz continuous with Lipschitz constant $\frac{1}{2}$. One may notice that r is arbitrary. We can now prove the following:

Theorem I.5.2.1. *Let conditions H.1–H.7 hold. If δ is chosen as in (I.5.8), then there exists a globally Lipschitz continuous mapping $u_s^* : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ such that $u_s = u_s^*(\varphi)$ is the unique solution in $\mathcal{PC}^{\eta,s}$ of the equation $u_s = \mathcal{F}_s(u_s, \varphi)$.*

Proof. The discussion preceding the statement of Theorem I.5.2.1 indicates that $\mathcal{F}_s(\cdot, \varphi)$ is a contraction mapping on $\overline{B(r,0)} \subset \mathcal{PC}^{\eta,s}$ for every $r > ||\varphi||2K$. Since the latter is a closed subspace of the Banach space $\mathcal{PC}^{\eta,s}$, the contraction mapping principle implies the existence of the function u_s^*. To show that it is a Lipschitz continuous, we note

$$\begin{aligned}||u_s^*(\varphi) - u_s^*(\psi)||_{\eta,s} &= ||\mathcal{F}_s(u_s^*(\varphi), \varphi) - \mathcal{F}_s(u_s^*(\psi), \psi))||_{\eta,s} \\ &\leq K||\varphi - \psi|| + \frac{1}{2}||u_s^*(\varphi) - u_s^*(\psi)||_{\eta,s}.\end{aligned}$$

Therefore, u_s^* is Lipschitz continuous with Lipschitz constant $2K$. □

Definition I.5.2.1 (Lipschitz Centre Manifold)**.** *The centre manifold, $\mathcal{W}_c \subset \mathbb{R} \times \mathcal{RCR}$, is the nonautonomous set whose t-fibres for $t \in \mathbb{R}$ are given by*

$$\mathcal{W}_c(t) = \text{Im}\{\mathcal{C}(t, \cdot)\}, \tag{I.5.9}$$

where $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ is the (fibrewise) Lipschitz map defined by $\mathcal{C}(t, \phi) = u_t^*(\phi)(t)$. Its dimension *is equal to* $\dim(\mathcal{RCR}_c)$*.*

Remark I.5.2.1. *The centre manifold depends non-canonically on the choice of cutoff function from Sect. I.5.1.3. That is, the centre manifold is not unique, so we are committing an abuse of syntax by referring to such a construct generally as "the" centre manifold. One must always understand that the definition of the centre manifold is with respect to a particular cutoff function. Also, since $C(t, \cdot) : \mathcal{RCR}_c(t) \to \mathcal{RCR}$ has a $\dim(\mathcal{RCR}_c(t))$-dimensional domain, it is appropriate to say that the centre manifold also has this dimension.*

The construction above implies the centre manifold is fibrewise Lipschitz. We can prove a stronger result, namely that the Lipschitz constant can be chosen independent of the given fibre.

Corollary I.5.2.1. *There exists a constant $L > 0$ such that $||\mathcal{C}(t, \phi) - \mathcal{C}(t, \psi)|| \leq L||\phi - \psi||$ for all $t \in \mathbb{R}$ and $\phi, \psi \in \mathcal{RCR}_c(t)$.*

Proof. Denote $u^\phi = u_t(\phi)$ and $u^\psi = u_t(\psi)$. A preliminary estimation appealing to the fixed-point equation (I.5.7) yields

$$||\mathcal{C}(t, \phi) - \mathcal{C}(t, \psi)|| \leq ||\phi - \psi|| + ||(\mathcal{K}_t^\eta(R_\delta u^\phi) - \mathcal{K}_t^\eta(R_\delta u^\psi))(t)||.$$

By Corollary I.5.1.2, each of $R_\delta u^\phi$ and $R_\delta u^\psi$ is uniformly bounded, so Lemma I.5.1.3 implies the existence of a constant $c > 0$ such that

$$\begin{aligned}||(\mathcal{K}_t^\eta(R_{\delta,t}u^\phi) - \mathcal{K}_t^\eta(R_{\delta,t}u^\psi))(t)|| &\leq c||(R_{\delta,t}u^\phi - R_{\delta,t}u^\psi)(t)|| \\ &\leq c \sup_{s\in\mathbb{R}} ||(R_{\delta,t}u^\phi - R_{\delta,t}u^\psi)(s)||e^{-\eta|t-s|} \\ &\leq c\tilde{L}_\delta ||u^\phi - u^\psi||_{\eta,t} \\ &\leq c\tilde{L}_\delta 2K||\phi - \psi||,\end{aligned}$$

and in the last line, we used the Lipschitz constant from Theorem I.5.2.1. Combining this result with the previous estimate for $||\mathcal{C}(t,\phi) - \mathcal{C}(t,\psi)||$ yields the uniform Lipschitz constant. By Corollary I.5.1.1, the Lipschitz constant has the claimed property. □

I.5.2.1 A Remark on Centre Manifold Representations: Graphs and Images

Our initial definition of the centre manifold was as the fibre bundle whose t-fibres are the images of $\mathcal{C}(t,\cdot)$. However, sometimes one likes to think of the centre manifold as being the graph of a function. To accomplish this, one can use the hyperbolic part. Let us define the function $\mathcal{H} : \mathcal{RCR}_c \to \mathcal{RCR}$ by $\mathcal{H}(t,\phi) = (I - P_c(t))\mathcal{C}(t,\phi)$. In this way, the centre manifold can be identified with the graph of the hyperbolic part of the centre manifold. Indeed, by part 2 of Lemma I.5.1.3, we have the decomposition $\mathcal{C}(t,\phi) = \phi + (I - P_c(t))\mathcal{C}(t,\phi)$, so that

$$\begin{aligned}\mathcal{W}_c(t) &= \{\phi + \mathcal{H}(t,\phi) : \phi \in \mathcal{RCR}_c(t)\} \\ &\sim \{(\phi, \mathcal{H}(t,\phi)) : \phi \in \mathcal{RCR}_c(t)\} = \mathrm{Graph}(\mathcal{H}(t,\cdot)).\end{aligned}$$

Since $\mathcal{RCR}_c(t)$ and its complement $\mathcal{R}(I - P_c(t)) = \mathcal{RCR}_s(t) \oplus \mathcal{RCR}_u(t)$ have only 0 in their intersection, this identification makes sense. When one reduces down to ordinary differential equations, one usually thinks of precisely the function $\mathcal{H}$ as *being* the centre manifold. This ambiguity between the function $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$, the fibre bundle $\mathcal{W}_c$, the hyperbolic part $\mathcal{H} : \mathcal{RCR}_c \to \mathcal{RCR}$ and its graph can sometimes make statements about centre manifolds imprecise. In this thesis, the term *centre manifold* without any additional qualifiers will always mean the fibre bundle $\mathcal{W}_c$.

I.5.3 Invariance and Smallness Properties

Recall that by Lemma I.4.1.1, there is a process $(S, \mathcal{M})$ on $\mathcal{RCR}$ such that $t \mapsto S(t,s)\phi$ is the unique mild solution of (I.4.3) through the initial condition (s,ϕ) defined on an interval $[s, s+\alpha)$. With this in mind, the centre manifold is locally positively invariant with respect to $\mathcal{S}$.

Theorem I.5.3.1 (Centre Manifold: Invariance and Inclusion of Bounded Orbits)**.** *Let conditions H.1–H.7 hold. The centre manifold $\mathcal{W}_c$ enjoys the following properties.*

1. *$\mathcal{W}_c$ is locally positively invariant: if $(s,\phi)\in\mathcal{W}_c$ and $||S(t,s)\phi||<\delta$ for $t\in[s,T]$, then $(t,S(t,s)\phi)\in\mathcal{W}_c$ for $t\in[s,T]$.*
2. *If $(s,\phi)\in\mathcal{W}_c$, then $S(t,s)\phi=u_t^*(P_c(t)S(t,s)\phi)(t)=\mathcal{C}(t,P_c(t)S(t,s)\phi)$.*
3. *If $(s,\phi)\in\mathcal{W}_c$, there exists a unique mild solution $u\in\mathcal{PC}^{\eta,s}$ of the semilinear system*
$$\begin{aligned}\dot{x}&=L(t)x_t+F_{\delta,s}(t,x_t), && t\neq t_k\\ \Delta x&=B(k)x_{t^-}+G_{\delta,s}(k,x_{t^-}), && t=t_k\end{aligned}$$
with the property that $u(t)\in\mathcal{W}_c(t)$ for $t\in\mathbb{R}$, $||u||_{\eta,s}\le\delta$ and $u(s)=\phi$.
4. *If $x:\mathbb{R}\to\mathcal{RCR}$ is a mild solution of (I.4.3) satisfying $||x||<\delta$, then $(t,x(t))\in\mathcal{W}_c$ for all $t\in\mathbb{R}$.*
5. *$\mathbb{R}\times\{0\}\subset\mathcal{W}_c$ and $\mathcal{C}(t,0)=0$ for all $t\in\mathbb{R}$.*

Proof. Let $(s,\phi)\in\mathcal{W}_c$ and denote $x(t)=S(t,s)\phi$, with $||x||<\delta$. Since $(s,\phi)\in\mathcal{W}_c$, there exists $\varphi\in\mathcal{RCR}_c(s)$ such that $\phi=u_s^*(\varphi)(s)$. Define $\hat{x}=u_s^*(\varphi)$. Then, it follows that $\varphi=P_c(s)\phi$, $\hat{x}(s)=\phi=P_c(s)\phi+K_s^\eta(R(\hat{x}))(s)$, and
$$\begin{aligned}\hat{x}(t)&=U(t,s)\varphi+\mathcal{K}_s^\eta(R_\delta(\hat{x}))(t)\\ &=U(t,s)\varphi+\Bigg[U(t,s)K_s^\eta(R_{\delta,s}(\hat{x}))(s)+\int_s^t U(t,\mu)\chi_0F_{\delta,s}(\mu,\hat{x}(\mu))d\mu\\ &\quad+\sum_{s<t_i\le t}U(t,t_i)\chi_0G_{\delta,s}(i,\hat{x}(t_i))\Bigg]\\ &=U(t,s)\hat{x}(s)+\int_s^t U(t,\mu)\chi_0F_{\delta,s}(\mu,\hat{x}(\mu))d\mu+\sum_{s<t_i\le t}U(t,t_i)\chi_0G_{\delta,s}(i,\hat{x}(t_i))\end{aligned}$$
for all $t\in[s,T]$. But since $||x(t)||<\delta$ on $[s,T]$, uniqueness of mild solutions (Lemma I.2.1.1 with Theorem I.2.3.1) implies that $x=\hat{x}|_{[s,T]}$.

Let $v\in[s,T]$ and define $z:\mathbb{R}\to\mathcal{RCR}$ by $z=\hat{x}-U(\cdot,v)P_c(v)\hat{x}(v)$. One can easily verify that
$$\begin{aligned}z(t)&=U(t,v)z(v)+\int_v^t U(t,\mu)U(t,\mu)\chi_0F_{\delta,s}(\mu,\hat{x}(\mu))d\mu\\ &\quad+\sum_{v<t_i\le t}U(t,t_i)\chi_0G_{\delta,s}(i,\hat{x}(t_i))\end{aligned}$$
for all $t\in[v,\infty)$ and that $P_c(v)z(v)=0$. On the other hand, since $||\hat{x}||<\delta$, we have $R_{\delta,s}(\hat{x})=R_{\delta,v}(\hat{x})$. From these two observations and Lemma I.5.1.3, $z=\mathcal{K}_v^\eta(R_{\delta,v}(\hat{x}))|_{[v,\infty)}$, so that we may write
$$\hat{x}=U(\cdot,v)P_c(v)\hat{x}(v)+\mathcal{K}_v^\eta(R_{\delta,v}(\hat{x}))=u_v^*(P_c(v)\hat{x}(v)).$$

Therefore, $\hat{x}(v) = u_v^*(P_c(v)\hat{x}(v))(v)$, and since $x(v) = \hat{x}(v)$, this proves that $(v, x(v)) \in \mathcal{W}_c$ and, through essentially the same proof, that

$$x(v) = u_v^*(P_c(v)x(v))(v) = \mathcal{C}(v, x(v))(v).$$

The proofs of the other three assertions of the theorem follow by similar arguments and are omitted. □

The modification of the nonlinearity R_δ results in the function u_s^* that defines the centre manifold having a uniformly small hyperbolic part. Namely, we have the following lemma.

Lemma I.5.3.1. *Define $\widehat{P}_c : \mathcal{PC}^\eta \to \mathcal{PC}^\eta(\mathbb{R}, \mathcal{RCR}_c)$ by $\widehat{P}_c\phi(t) = P_c(t)\phi(t)$. If $\delta > 0$ is sufficiently small, then $||(I - \widehat{P}_c)u_s^*||_0 < \delta$. More precisely, it is sufficient to chose $\delta > 0$ small enough so that $NL_\delta||\mathcal{K}_s^\eta|| < \frac{1}{4}$.*

Proof. Recall that u_s^* satisfies the fixed-point equation $u_s^* = U(\cdot, s)\varphi + \mathcal{K}_s^\eta(R_{\delta,s}(u_s^*))$. Thus, with $\widehat{P}_h = I - \widehat{P}_c$,

$$\widehat{P}_h u_s^* = \widehat{P}_h \circ \mathcal{K}_s^\eta(R_{\delta,s}(u_s^*))$$

because $U(t, s)$ is an isomorphism of $\mathcal{RCR}_c(s)$ onto $\mathcal{RCR}_c(t)$ and $\varphi \in \mathcal{RCR}_c(s)$. By Corollary I.5.1.2, we have for all $t \in \mathbb{R}$ that $||R_{\delta,s}(u_s^*(t))|| \leq 4\delta L_\delta$, which implies $R_{\delta,s}(u_s^*) \in B^0(\mathbb{R}, \mathbb{R}^n) \oplus B^0_{t_k}(\mathbb{Z}, \mathbb{R}^n)$. We obtain the claimed result by applying the second conclusion of Lemma I.5.1.3 and taking δ sufficiently small, recalling from Corollary I.5.1.1 that $L_\delta \to 0$ as $\delta \to 0$. The explicit estimate for δ comes from the bound $||\widehat{P}_h\mathcal{K}_s^\eta(R_{\delta,s}(u_s^*)||_0 \leq N||K_s^\eta||4\delta L_\delta$. □

I.5.4 Dynamics on the Centre Manifold

The centre manifold can be identified with a $\dim(\mathcal{RCR}_c(t))$-dimensional invariant fibre bundle over $\mathbb{R} \times \mathcal{RCR}$. A natural question to ask is how the process $(S, \mathcal{M})$ behaves when restricted to the centre manifold. We address this now.

I.5.4.1 Integral Equation

On the centre manifold, components of mild solutions on the centre fibre bundle are decoupled. The following lemma states how the components in the centre fibre bundle evolve. The proof follows from Theorem I.5.3.1.

Lemma I.5.4.1 (Dynamics on the Centre Manifold: Integral Equation)**.** *Let $y : \mathbb{R} \to \mathcal{RCR}$ satisfy $y(t) \in \mathcal{W}_c(t)$ with $||y|| < \delta$. Consider the projection of y onto the centre fibre bundle: $w(t) = P_c(t)y(t)$. The projection satisfies the integral equation*

$$\begin{aligned} w(t) = U(t,s)w(s) &+ \int_s^t U(t,\mu)P_c(\mu)\chi_0 F_{\delta,\mu}(\mu, \mathcal{C}(\mu, w(\mu)))d\mu \\ &+ \sum_{s<t_i\leq t} U(t,t_i)P_c(t_i)\chi_0 G_{\delta,t_i}(i, \mathcal{C}(t_i, w(t_i))). \end{aligned} \tag{I.5.10}$$

I.5.4.2 Abstract Ordinary Impulsive Differential Equation

Lemma I.5.4.1 describes the dynamics of the centre fibre bundle component of the centre manifold in terms of an integral equation. With an additional assumption on the jump map, we can extend this result to an ordinary impulsive differential equation on a Banach space.

Definition I.5.4.1. *A sequence of functionals $J(k,\cdot):\mathcal{RCR}\to\mathbb{R}^n$ for $k\in\mathbb{Z}$ satisfies the* overlap condition *(with respect to the sequence $\{t_k:k\in\mathbb{Z}\}$ of impulse times) if*

$$\lim_{\epsilon\to 0^+} J(k,\phi+\chi_{[\theta,\theta+\epsilon)}h)=J(k,\phi)$$

for all $\phi\in\mathcal{RCR}$ and $h\in\mathcal{RCR}$, whenever $\theta=t_j-t_k\in[-r,0)$.

The overlap condition roughly states that the jump functional does not have observable "memory" at times in the past that happens to correspond to impulse times. As the definition is somewhat abstract, we will give an example.

Example I.5.4.1. *Consider the scalar impulse effect defined according to*

$$\Delta x = x(t-r), \qquad t=t_k,$$

where $t_k=k\in\mathbb{Z}$ are the integers. The functional associated to the above is simply $J(\phi)=\phi(-r)$. If r is a positive integer, the overlap condition will not *be satisfied because with $\theta=t_k-t_{k+r}=-r\in[-r,0)$, we have*

$$J(\phi+\chi_{[\theta,\theta+\epsilon)}h)=\phi(-r)+h(-r)\neq\phi(-r)=J(\phi)$$

for all $\epsilon>0$ and $h\in\mathcal{RCR}$ with $h(-r)\neq 0$. However, if r is not *an integer, then since any $\theta=t_j-t_k\in[-r,0)$ must be an integer, it follows that for $\epsilon>0$ small enough, $-r\notin[\theta,\theta+\epsilon)$. From here, we can conclude that J satisfies the overlap condition.*

Remark I.5.4.1. *The overlap condition is equivalent to the statement that $J(k,\cdot)$ admits a continuous extension to a particular closed subspace of $\mathcal{G}$ $([-r,0],\mathbb{R}^n)$; see later Lemma I.6.1.1.*

The overlap condition is mostly important for functionals that define discrete delays, since the regularization incurred from distributed delays generally forces the overlap condition to be satisfied. See Sect. I.6.4 for a more thorough discussion. We make use of the overlap condition in proof of the following theorem. The details are somewhat subtle, and we will spend a fair bit more time on them in Sect. I.5.7.

Theorem I.5.4.1 (Dynamics on the Centre Manifold: Abstract Impulsive Differential Equation)**.** *Let $y\in\mathcal{RCR}^1(\mathbb{R},\mathbb{R}^n)$ satisfy $y_t\in\mathcal{W}_c(t)$ with $||y||<$*

δ. Consider the projection $w(t) = P_c(t)y_t$ and define the linear operators $\mathcal{L}(t) : \mathcal{RCR}^1 \to \mathcal{RCR}$ and $\mathcal{J}(k) : \mathcal{RCR} \to \mathcal{G}([-r, 0], \mathbb{R}^n)$ by

$$\mathcal{L}(t)\phi(\theta) = \begin{cases} L(t)\phi, & \theta = 0 \\ d^+\phi(\theta), & \theta < 0 \end{cases}, \qquad \mathcal{J}(k)\phi(\theta) = \begin{cases} B(k)\phi, & \theta = 0 \\ \phi(\theta^+) - \phi(\theta), & \theta < 0 \end{cases} \tag{I.5.11}$$

If the jump functionals $B(k)$ and $g(k, \cdot)$ satisfy the overlap condition, then $w : \mathbb{R} \to \mathcal{RCR}^1$ satisfies, pointwise, the abstract impulsive differential equation

$$d^+w(t) = \mathcal{L}(t)w(t) + P_c(t)\chi_0 F_\delta(t, \mathcal{C}(t, w(t))), \qquad t \neq t_k \tag{I.5.12}$$

$$\Delta w(t_k) = \mathcal{J}(k)w(t_k^-) + P_c(t_k)\chi_0 G_\delta(k, \mathcal{C}(t_k, w(t_k))), \qquad t = t_k, \tag{I.5.13}$$

where $w(t_k^-)(\theta) := \lim_{\epsilon\to 0^-} w(t_k - \epsilon)(\theta)$ and $\Delta w(t_k)(\theta) := w(t_k)(\theta) - w(t_k^-)(\theta)$ for $\theta \in [-r, 0]$.

Proof. For brevity, denote $F(\mu) = F_{\delta,\mu}(\mu, \mathcal{C}(\mu, w(\mu)))$, $\overline{F}(\mu) = \chi_0 F(\mu)$, $\mathbf{F}(\mu) = P_c(\mu)\chi_0 F(\mu)$, and analogously for G_δ. We begin by noting that equation (I.5.10) allows us to write the finite difference $w_\epsilon(t) = w(t + \epsilon) - w(t)$ as

$$\begin{aligned} w_\epsilon(t) = [U(t+\epsilon, s) - U(t, s)]w(s) + (U(t+\epsilon, t) - I)\int_s^t U(t, \mu)\mathbf{F}(\mu)d\mu \\ + U(t+\epsilon, t)P_c(t)\int_t^{t+\epsilon} U(t, \mu)\overline{F}(\mu)d\mu + (U(t+\epsilon, t) - I) \\ \times \sum_{s<t_i\leq t} U(t, t_i)\mathbf{G}(i) + U(t+\epsilon, t)\sum_{t<t_i\leq t+\epsilon} U(t, t_i)\mathbf{G}(i). \end{aligned} \tag{I.5.14}$$

First, we show that $d^+U(t, s)\phi = \mathcal{L}(t)U(t, s)\phi$ pointwise for $\phi \in \mathcal{RCR}$. For $\theta = 0$, we have

$$\frac{1}{\epsilon}(U(t+\epsilon, s)\phi(0) - U(t, s)\phi(0)) = \frac{1}{\epsilon}\int_t^{t+\epsilon} L(\mu)U(\mu, s)\phi d\mu,$$

which converges to $L(t)U(t, s)\phi$ as $\epsilon \to 0^+$. For $\theta < 0$ and $\epsilon > 0$ sufficiently small,

$$\begin{aligned} &\frac{1}{\epsilon}(U(t+\epsilon, s)\phi(\theta) - U(t, s)\phi(\theta)) \\ &= \frac{1}{\epsilon}(\phi(t+\epsilon+\theta-s) - \phi(t+\theta-s)) \longrightarrow d^+\phi(t+\theta-s) = d^+U(t, s)\phi(\theta). \end{aligned}$$

Therefore, $d^+U(t, s)\phi = \mathcal{L}(t)U(t, s)\phi$ pointwise, as claimed. Since $U(t, t) = I$, this also proves the pointwise convergence

$$\frac{1}{\epsilon}(U(t+\epsilon, t) - I)\phi \to \mathcal{L}(t)\phi.$$

Next, we show that

$$\frac{1}{\epsilon}U(t+\epsilon,t)P_c(t)\int_t^{t+\epsilon}U(t,\mu)\overline{F}(\mu)d\mu \to P_c(t)\overline{F}(t)=\mathbf{F}(t) \tag{I.5.15}$$

pointwise as $\epsilon\to 0^+$. We do this by first proving that the sequence

$$x_n := \frac{1}{\epsilon_n}U(t+\epsilon_n,t)P_c(t)\int_t^{t+\epsilon_n}U(t,\mu)\overline{F}(\mu)d\mu$$

is pointwise Cauchy for each sequence $\epsilon_n\to 0^+$. Assuming without loss of generality that ϵ_n is strictly decreasing, we have for all $n\geq m$,

$$\begin{aligned}x_n-x_m &= \left[\frac{1}{\epsilon_n}U(t+\epsilon_n,t)-\frac{1}{\epsilon_m}U(t+\epsilon_m,t)\right]P_c(t)\int_t^{t+\epsilon_n}U(t,\mu)\chi_0F(\mu)d\mu\\ &\quad+\frac{1}{\epsilon_m}U(t+\epsilon,t)\int_{t+\epsilon_m}^{t+\epsilon_n}U_c(t,\mu)P_c(\mu)\chi_0F(\mu)d\mu.\end{aligned}$$

Both integrals can be made arbitrarily small in norm by taking $n,m\geq N$ and N large enough. Since $\frac{1}{\epsilon}U(t+\epsilon,t)$ is pointwise convergent as $\epsilon\to 0^+$, we obtain that the sequence x_n is pointwise Cauchy and is hence pointwise convergent. Direct calculation of the limit in the pointwise sense yields (I.5.15). Combining all of the above results with Eq. (I.5.14) gives the pointwise equality

$$\begin{aligned}d^+w(t) &= \mathcal{L}(t)U(t,s)w(s)+\mathcal{L}(t)\int_s^t U(t,\mu)\mathbf{F}(\mu)d\mu\\ &\quad+\mathbf{F}(t)+\mathcal{L}(t)\sum_{s<t_i\leq t}U(t,t_i)\mathbf{G}(i)\\ &= \mathcal{L}(t)w(t)+\mathbf{F}(t),\end{aligned}$$

which is equivalent to (I.5.12).

To obtain the difference equation (I.5.13), we similarly identify $w_\epsilon(t_k)(\theta) := w(t_k)(\theta)-w(t_k-\epsilon)(\theta)$ with the decomposition

$$\begin{aligned}w_\epsilon(t_k) &= [U(t_k,s)-U(t_k-\epsilon,s)]w(s)+\int_{t_k-\epsilon}^{t_k}U(t,\mu)\mathbf{F}(\mu)d\mu\\ &\quad+\int_s^{t_k-\epsilon}[U(t_k,\mu)-U(t_k-\epsilon,\mu)]\mathbf{F}(\mu)d\mu\\ &\quad+\sum_{t_k-\epsilon<t_i\leq t_k}U(t_k,t_i)\mathbf{G}(i)+\sum_{s<t_i\leq t_k-\epsilon}[U(t_k,t_i)-U(t_k-\epsilon,t_i)]\mathbf{G}(i).\end{aligned}$$

Using Lemmas I.2.2.1 and I.2.3.5, the above is seen to converge pointwise as $\epsilon\to 0^+$, with limit

$$\begin{aligned}\Delta w(t_k) &= \tilde{\mathcal{J}}(k)U(t_k^-,s)w(s)+\tilde{\mathcal{J}}(k)\int_s^{t_k}U(t_k^-,\mu)\mathbf{F}(\mu)d\mu\\ &\quad+\mathbf{G}(k)+\tilde{\mathcal{J}}(k)\sum_{s<t_i<t_k}U(t_k^-,t_i)\mathbf{G}(i),\end{aligned} \tag{I.5.16}$$

where $\tilde{\mathcal{J}}(k)\phi(\theta) = \chi_0(\theta)B(k)\phi + \chi_{(-r,0)}(\theta)[\phi(\theta) - \phi(\theta^-)]$, and we assume without loss of generality that $r > 0$ is large enough so that $t_k - r \neq t_j$ for all $j < k$ and all $k \in \mathbb{Z}$. Let us denote

$$U^-(t,s)\phi(\theta) = \lim_{\epsilon\to 0^+} U(t-\epsilon,s)\phi(\theta)$$

the strong left-limit of the evolution family at t. This limit is well-defined pointwise, and due to the overlap condition, we have

$$\tilde{\mathcal{J}}(k)U(t_k^-,\xi)\phi = \mathcal{J}(k)U^-(t_k,\xi)\phi \tag{I.5.17}$$

pointwise for all $\xi < t_k$. Moreover, since

$$w(t_k^-) = U^-(t_k,s)w(s) + \int_s^{t_k} U^-(t_k,\mu)\mathbf{F}(\mu)d\mu + \sum_{s<t_i<t_k} U^-(t_k,t_i)\mathbf{G}(i), \tag{I.5.18}$$

we can obtain equation (I.5.13) by substituting (I.5.17) and (I.5.18) into (I.5.16). □

We will not make much use of the abstract differential equation (I.5.12)–(I.5.13) and have included it mostly for the purpose of comparison with analogous results for delay differential equations. As we will see, the integral equation (I.5.10) will be more than sufficient.

I.5.4.3 A Remark on Coordinates and Terminology

It is a slight abuse of terminology to describe (I.5.12)–(I.5.13) as an impulsive differential equation on the centre manifold. More precisely, it is the dynamical system associated to the projection onto the centre fibre bundle associated to a given solution in the centre manifold. This precise description is, however, quite verbose, and for this reason we will usually call (I.5.12)–(I.5.13) the impulsive differential equations on the centre manifold, even if this is not exactly what it is.

The evolution equations (I.5.12)–(I.5.13) are quite abstract. It is an evolution equation in the centre fibre bundle that, despite being (in many situations) finite-dimensional, is still rather difficult to use in practice because the fibres $\mathcal{RCR}_c(t)$ are not themselves constant in time. What is needed is an appropriate coordinate system. This would in principle allow for the derivation of an impulsive differential equation in $\mathbb{R}^p$ for $p = \dim \mathcal{RCR}_c$. We expand on precisely this idea in Sects. I.5.7 and I.6.1.

I.5.5 Reduction Principle

Given a nonhyperbolic equilibrium, one may want to study the orbit structure near this equilibrium under parameter perturbation in the vector field or jump map defining the impulsive functional differential equation (I.4.1)–(I.4.2). Assuming the sufficient conditions for the existence of a centre manifold are satisfied, part 2 of Theorem I.5.3.1 implies that on the centre manifold, the dynamics are completely determined by those of the component in the centre fibre bundle. Part 3 of the same theorem guarantees that the small bounded solutions are all contained on the centre manifold. Lemma I.5.4.1 completely characterizes these dynamics in terms of an integral equation (I.5.10). As a consequence, bifurcations can be detected by analysing this integral equation instead, and no loss of generality occurs by looking only on the centre manifold (at least for small perturbations of the parameter).

The next natural question is the following. If we detect a bifurcation on the centre manifold and the branch of solutions (or union of solutions, e.g. a torus) is stable when restricted to the centre manifold, are we guaranteed that this solution is stable in the infinite-dimensional system provided $\mathcal{RCR}_u$ is trivial? The answer is yes, and the following results make this precise. This is sometimes referred to as the centre manifold reduction. They are inspired by similar results for ordinary differential equations in both finite- and infinite-dimensional systems; see for instance Theorem 2.2 from Chapter 10 of Hale and Verduyn Lunel's introductory text [58] for functional differential equations, Theorem 3.22 from Chapter 2 of [60] for ordinary differential equations in Banach spaces and the classic text of Jack Carr [22] for finite-dimensional ordinary differential equations, as well as some extensions to infinite-dimensional problems. However, we will require the vector field to be slightly more regular than previously.

Definition I.5.5.1. *The functional $f:\mathbb{R}\times\mathcal{RCR}\to\mathbb{R}^n$ is* additive composite regulated (ACR) *if for all $x\in\mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$, $Y\in\mathcal{RCR}(\mathbb{R},\mathbb{R}^{n\times m})$ and $z\in\mathcal{RCR}(\mathbb{R},\mathbb{R}^m)$, the function $t\mapsto f(t,x_t+Y_tz(t))$ is an element of $\mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$.*

Remark I.5.5.1. *ACR functionals are quite common in applications. For example, suppose $f:\mathbb{R}\times\mathcal{RCR}\to\mathbb{R}^n$ can be written in the form*

$$f(t,\phi)=F\left(t,A(t)\phi(-d(t)),\int_{-r}^{0}K(t,\theta)\phi(\theta)d\theta\right)$$

for $d\in\mathcal{RCR}(\mathbb{R},[-r,0])$, $A\in\mathcal{RCR}(\mathbb{R},\mathbb{R}^{n\times n})$, $K:\mathbb{R}\times[-r,0]\to\mathbb{R}^{n\times n}$ integrable in its second variable, continuous from the right in its first variable and uniformly bounded and $F:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ jointly continuous from the right in its first variable and continuous in its other variables. It is clear that

$$t\mapsto A(t)[x_t(-d(t))+Y_t(-d(t))z(t)]=A(t)[x(t-d(t))+Y(t-d(t))z(t)]$$

is an element of $\mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$. As for the integral term, the function

$$t\mapsto\int_{-r}^{0}K(t,\theta)[x(t+\theta)+Y(t+\theta)z(t)]d\theta]$$

can be seen to be an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ *by applying the dominated convergence theorem. From the assumptions on* F*, we conclude that* f *is ACR. The same holds true for vector fields with multiple time-varying delays and distributed delays.*

Lemma I.5.5.1. *Assume* $\mathcal{RCR}_u = \{0\}$*. Let* $\Phi_t = [\ \phi_t^{(1)} \ \ \cdots \ \ \phi_t^{(p)}\ ]$ *be a row array whose elements form a basis for* $\mathcal{RCR}_c(t)$*, the latter being assumed* p*-dimensional for* p *finite, such that* $\Phi_t = U_c(t,0)\Phi_0$*. Given a mild solution* $x_{(\cdot)} : I \to \mathcal{RCR}$*, write* $P_c(t)x_t = \Phi_t z(t)$ *for some* $z \in \mathbb{R}^p$*, so that*

$$x_t = \Phi_t z(t) + h(t, z(t)) + y_t^S$$

with $z \in \mathbb{R}^p$*,* $h(t,z) := (I - P_c(t)\mathcal{C}(t, \Phi_t z)$*, and* $y_t^S \in \mathcal{RCR}_s(t)$ *is a remainder term. Assume the matrix-valued function* $Y_c(t)$ *defined by the equation* $P_c(t)\chi_0 = \Phi_t Y_c(t)$ *is continuous from the right and possesses limits on the left. There exist positive constants* ρ, C *and* α *such that for all* $t \geq s$*, the remainder term satisfies*

$$||y_t^S|| \leq C||y_s^S - h(s, z(s))||e^{-\alpha(t-s)},$$

provided $||x_t|| \leq \rho$ *for all* $t \geq s$*.*

Proof. One can carefully verify that $z(t)$ and y_t^S, respectively, satisfy the following integral equations for all $t \geq s$:

$$z(t) = z(s) + \int_s^t Y_c(\mu)\mathcal{F}(\mu, z(\mu), y_\mu^S)d\mu + \sum_{s<t_i\leq t} U_c(t_i)\mathcal{G}(i, z(t_i), y_{t_i}^S), \tag{I.5.19}$$

$$\begin{aligned} y_t^S = U(t,s)[y_s^S - h(s, z(s))] + \int_s^t U(t,\mu)P_s(\mu)\chi_0[\mathcal{F}(\mu, z(\mu), y_\mu^S) \\ - \mathcal{F}(\mu, z(\mu), 0)]d\mu + \sum_{s<t_i\leq t} U(t,t_i)P_s(t_i)\chi_0[\mathcal{G}(i, z(t_i), y_{t_i}^S) \\ - \mathcal{G}(i, z(t_i), 0)], \end{aligned} \tag{I.5.20}$$

provided $\rho < \delta/N$, where $\mathcal{F}(t,z,y) = F_{\delta,0}(t, \Phi_t z + h(t,z) + y)$, $\mathcal{G}(k,z,y) = G_{\delta,0}(k, \Phi_{t_k} z + h(t_k, z) + y)$, and $Y_c(\mu)$ is defined by the equation $P_c(\mu)\chi_0 = \Phi_\mu Y_c(\mu)$. Because of our assumption on $Y_c(\mu)$, it follows (from the integral equation (I.5.19)) that z is continuous from the right and possesses limits on the left. If we remark that

$$y_t^S = (I - P_c(t))x_t = x_t - \Phi_t z(t),$$

we can use Lemma I.1.3.7 to conclude that $t \mapsto ||y_t^S||$ is an element of $\mathcal{RCR}(I, \mathbb{R})$. Using spectral separation and the Lipschitz condition on the substitution operator, we can use (I.5.20) to get the estimate

$$||y_t^S||e^{-at} \leq Ke^{-as}||y_s^S - h(s,z(s))|| + \int_s^t KL_\delta ||y_\mu^S||e^{-a\mu}d\mu$$
$$+ \sum_{s<t_i\leq t} KL_\delta ||y_{t_i}^S||e^{-at_i},$$

provided $||x_t|| \leq \rho$ for ρ sufficiently small. Next, we apply the Gronwall Inequality (Lemma I.1.5.1) to the function $t \mapsto ||y_t^S||e^{-at}$. After some simplifications, we get

$$||y_t^S|| \leq K(1+KL_\delta)||y_s^S - h(s,z(s))|| \exp\left(\left(a + KL_\delta\left(1+\frac{1}{\xi}\right)\right)(t-s)\right).$$

We can always guarantee that the exponential convergence rate is in the form $e^{-\alpha(t-s)}$ for $\alpha > 0$ by taking δ sufficiently small, since $a < 0$ and we have $L_\delta \to 0$ as $\delta \to 0$ by Corollary I.5.1.1. The result follows. □

The continuity condition on the matrix $t \mapsto Y_c(t)$ comes up in a few places in this monograph. Most noteworthy, it is used in Sect. I.5.7 to guarantee temporal regularity properties of the centre manifold.

Theorem I.5.5.1 (Local Attractivity of the Centre Manifold)**.** *Let the assumptions of Lemma I.5.5.1 be satisfied and let f be an ACR functional. There exists a neighbourhood V of $0 \in \mathcal{RCR}$ and positive constants K_1, α_1 such that if $t \mapsto x_t$ is a mild solution satisfying $x_t \in V$ for all $t \geq s$, then there exists $u_t \in \mathcal{W}_c(t)$ with the property that*

$$||x_t - u_t|| \leq K_1 e^{-\alpha_1(t-s)}$$

for all $t \geq s$. That is, every solution that remains close to the centre manifold in forward time is exponentially attracted to a particular solution on the centre manifold. More precisely, there exists $u \in \mathcal{RCR}([s,\infty),\mathbb{R}^n)$ such that $t \mapsto \Phi_t u(t)$ satisfies the abstract integral equation (I.5.10) for the coordinate on the centre manifold, and we have the estimates

$$||P_c(t)x_t - \Phi_t u(t)|| \leq Ke^{-a_1(t-s)},$$
$$||P_s(t)x_t - h(t,u(t))|| \leq Ke^{-a_1(t-s)}.$$

Proof. With the same setup as in the previous proof, let $u(t;u_s)$ for $t \geq s$ denote the solution of the integral equation

$$u(t) = u_s + \int_s^t Y_c(\mu)\mathcal{F}(\mu,u(\mu),0)d\mu + \sum_{s<t_i\leq t} Y_c(t_i)\mathcal{G}(i,u(t_i),0),$$

for given $u_s \in \mathbb{R}^p$. Define $w(t) = z(t) - u(t; u_s)$. With $x_t = \Phi_t z(t) + h(t, z(t)) + y_t^S$, we have the following integral equations for w and y_t^S:

$$y_t^S = U(t,s)[y_s^S - h(s, w(s) + u_s)] + \int_s^t U(t,\mu)P_s(\mu)\chi_0 M_1(\mu, w(\mu) \tag{I.5.21}$$

$$+ u(\mu; u_s), y_\mu^S)d\mu + \sum_{s<t_i\le t} U(t,t_i)P_s(t_i)\chi_0 M_2(i, w(t_i) + u(t_i; u_s), y_{t_i}^S),$$

$$w(t) = w(s) + \int_s^t Y_c(\mu)N_1(\mu, w(\mu), y_\mu^S)d\mu + \sum_{s<t_i\le t} Y_c(t_i)N_2(i, w(t_i), y_{t_i}^S), \tag{I.5.22}$$

with M_1, M_2, N_1 and N_2 defined by

$$\begin{aligned} M_1(\mu, a, b) &= \mathcal{F}(\mu, a, b) - \mathcal{F}(\mu, a, 0), \\ M_2(i, a, b) &= \mathcal{G}(i, a, b) - \mathcal{G}(i, a), \\ N_1(\mu, a, b) &= \mathcal{F}(\mu, a + u(\mu; u_s), b) - \mathcal{F}(\mu, u(\mu; u_s), 0), \\ N_2(i, a, b) &= \mathcal{G}(i, a + u(t_i; u_s), b) - \mathcal{G}(i, u(t_i; s), 0). \end{aligned}$$

The idea now is to reinterpret the integral equation for w as a fixed-point equation parameterized by $y_{(\cdot)}^S$ and $u(\cdot; u_s)$. Introduce the space

$$X = \{\phi \in \mathcal{RCR}([s,\infty), \mathbb{R}^p) : ||\phi(t)||e^{a(t-s)} \le K\}$$

equipped with the norm $||\phi|| = \sup_{t\ge s} ||\phi(t)||e^{a(t-s)}$. Define Tw by

$$(Tw)(t) = -\int_t^\infty Y_c(\mu)N_1(\mu, w(\mu), y_\mu^S)d\mu - \sum_{t<t_i<\infty} Y_c(t_i)N_2(i, w(t_i), y_{t_i}^S). \tag{I.5.23}$$

If $w \in X$, then from the assumption that f is an ACR functional we can conclude that $Tw \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. So we consider the nonlinear function $T : X \to \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. Notice that if w is a fixed point of T, then w satisfies the integral equation (I.5.22). Working backwards, it would then follow by Lemma I.5.5.1 that

$$v_t := \Phi_t[w(t) + u(t; u_s)] + h(t, w(t) + u(t; u_s)) + y_t^S \tag{I.5.24}$$

is a solution with the property that

$$\begin{aligned} ||P_c(t)v_t - \Phi_t u(t; u_s)]|| &= O(e^{-\gamma t}), \\ ||P_s(t)v_t - h(t, u(t; u_s))|| &= O(e^{-\gamma t}), \end{aligned}$$

as $t \to \infty$ (recall that if $w \in X$, then $w \to 0$ exponentially as $t \to \infty$, while $h(t, \cdot)$ is uniformly Lipschitz with respect to t). It is at this stage that we refer the reader to the proof of Theorem 2 of Carr's book [22]. The setup having been completed, the proof that T can be made a contraction on X provided δ is sufficiently small is the same as Carr's argument and is omitted. Specifically, we have the following conclusion: for $s \in \mathbb{R}$ and any (u_s, y_s^S) is sufficiently small, $T : X \to X$ is a contraction. In particular, by making this dependence on the fixed point explicit and writing $T : (\mathbb{R} \times \mathbb{R}^p \times \mathcal{RCR}_s(s)) \times X \to X$, one can ensure that T is a uniform contraction. In the same way we proved that the centre manifold is (uniformly in t) Lipschitz continuous, one can show that the fixed point $S^*(s, u, y^S)$ of $T(s, u, y^S)$ is uniformly (with respect to s) Lipschitz continuous in $\mathbb{R}^p \times \mathcal{RCR}_s(s)$, and the Lipschitz constant can be made as small as needed by taking δ sufficiently small.

Now, for a given $\phi \in \mathcal{RCR}$, define $u_s(\phi)$ and $y_s^S(\phi)$ according to

$$P_c(s)\phi = \Phi_s u_s(\phi), \quad y_s^S(\phi) = \phi - \Phi_s u_s(\phi) - h(s, u_s(\phi)).$$

Next, define $Q(s, \cdot, \cdot) : \mathbb{R}^p \times \mathcal{RCR}_s(s) \to \mathbb{R}^p \times \mathcal{RCR}_s(s)$ by

$$Q(s, u, \phi) = (u, \phi) + (\Phi_s S^*(s, u_s(\phi), y_s^S(\phi)), 0).$$

That is, $Q(s, \cdot, \cdot)$ is a nonlinear perturbation from the identity. If we let $\psi \in \mathcal{RCR}$, then the function $Q_\psi(s, \cdot, \cdot) : \mathbb{R}^p \times \mathcal{RCR}_s(s) \to \mathbb{R}^p \times \mathcal{RCR}_s(s)$ defined by

$$Q_\psi(s, u, \phi) = (u_s(\psi), \psi - \Phi_s u_s(\psi)) - (\Phi_s S^*(s, u_s(\phi), y_s^S(\phi)), 0)$$

satisfies the property that $Q(s, u, \phi) = (\psi_1, \psi_2)$ if and only if

$$Q_{\Phi_s \psi_1 + \psi_2}(s, u, \phi) = (u, \phi).$$

$S^*(s, \cdot, \cdot)$ is (uniformly in s) Lipschitz continuous with a Lipschitz constant that goes to zero as $\delta \to 0$. Since ψ does not factor into the nonlinear term, Q_ψ can be made a uniform (with respect to s and ψ) contraction by taking δ sufficiently small. As a consequence, every $(\psi_1, \psi_2) \in \mathbb{R}^p \times \mathcal{RCR}_s(s)$ is in the range of $Q(s, \cdot, \cdot)$ (in fact, $Q(s, \cdot, \cdot$ is a bijection).

Now, let x_t defined for $t \geq s$ be a mild solution with $||x_t||$ for $t \geq s$ sufficiently small. Write $x_s = \Phi_s x_s^c + x_s^S$ for $x_s^c \in \mathbb{R}^p$ and $x_s^S \in \mathcal{RCR}_s(s)$. Denote $(v_s^c, v_s^S) = Q^{-1}(s, \cdot, \cdot)(x_s^c, x_s^S)$. Take note that $v_s^S = x_s^S$. From the above discussion, it follows that with $u(t) = u(t; v_s^c)$, the asymptotic of the theorem is satisfied. By restricting to a sufficiently small neighbourhood of the origin, we can ignore the cutoffs on the vector field and jump map, thereby obtaining results that are applicable to mild solutions of the system without the cutoff nonlinearity. This proves the theorem. □

I.5.5.1 Parameter Dependence

The following heuristic discussion of parameter-dependent centre manifolds will be a bit imprecise. See Sect. I.8.1 for a more concrete presentation.

Suppose the (parameter-dependent) process $S(t,s;\epsilon) : \mathcal{RCR} \to \mathcal{RCR}$ is generated by a parameter-dependent impulsive RFDE with parameter $\epsilon \in \mathbb{R}^m$, and at $\epsilon = 0$, the equilibrium 0 is nonhyperbolic with a p-dimensional centre manifold. One then considers the spatially extended process on $\mathcal{RCR} \times \mathbb{R}^m$ defined by

$$(\phi, \epsilon) \mapsto (S(t,s;\epsilon)\phi, \epsilon)$$

$0 \in \mathcal{RCR} \times \mathbb{R}^m$ is now nonhyperbolic with a $(p+m)$-dimensional centre fibre bundle, so that the function $(x,\epsilon) \mapsto \mathcal{C}(t,x,\epsilon)$ defines a $(p+m)$-dimensional centre manifold. The dynamics on this centre manifold are trivial in the ϵ component, while those in the x component depend for each ϵ fixed on $x \mapsto \mathcal{C}(t,x,\epsilon)$.

For small parameters $\epsilon \neq 0$, there may be small solutions in the *parameter-dependent centre manifold* $\mathcal{W}_c^\epsilon$ defined by

$$\mathcal{W}_c^\epsilon(t) = \{\mathcal{C}(t,x,\epsilon) : x \in \mathcal{RCR}_c(t)\}$$

that are locally asymptotically stable when restricted to $\mathcal{W}_c^\epsilon$. There could also be stable attractors therein—in particular (by Theorem I.5.3.1), any small bounded solutions are contained in the centre manifold. The stability condition in addition to continuity with respect to initial conditions (Theorem I.4.2.1) and attractivity of the centre manifold (Theorem I.5.5.1) then grants the analogous stability of such small solutions or attractors when considered in the scope of the original infinite-dimensional system (I.4.1)–(I.4.2), provided ϵ is small enough and $\mathcal{RCR}_u$ is trivial.

To summarize, when the unstable fibre bundle is trivial, the dynamics on the centre manifold completely determine all nearby dynamics. Local stability assertions associated to small solutions and attractors on the parameter-dependent centre manifold carry over to those of the original infinite-dimensional system. The parameter-dependent centre manifold contains all such small solutions and attractors.

I.5.6 Smoothness in the State Space

In Sect. I.5.2, we proved the existence of invariant centre manifolds associated to the abstract integral equation (I.4.3). These invariant manifolds are images of a uniformly Lipschitz continuous function $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$. Our next task is to prove that the function $\mathcal{C}$ inherits the smoothness from the generating impulsive functional differential equation. To accomplish this, we will need to introduce an additional regularity assumption on the nonlinear parts of the vector field and jump map.

H.8 The functions c_j and sequences $\{d_j(k) : k \in \mathbb{Z}\}$ introduced in H.5 are bounded.

Note that H.8 is a purely nonautonomous property and is trivially satisfied if the vector field and jump functions are autonomous. Also, we will need to assume in this section that the centre fibre bundle is finite-dimensional.

H.9 $\mathcal{RCR}_c$ is finite-dimensional.

I.5.6.1 Contractions on Scales of Banach Spaces

The rest of this section will utilize several techniques from the theory of contraction mappings on scales of Banach spaces. In particular, many of the proofs that follow are inspired by those relating to smoothness of centre manifolds appearing in [41, 71, 144], albeit adapted somewhat so as to manage the explicitly nonautonomous and impulsive properties of the problem. The following lemma will be very helpful. It is taken from Section IX, Lemma 6.7 of [41], but also appears as Theorem 3 in [144].

Lemma I.5.6.1. *Let Y_0, Y, Y_1 be Banach spaces with continuous embeddings $J_0 : Y_0 \hookrightarrow Y$ and $J : Y \hookrightarrow Y_1$, and let Λ be another Banach space. Consider the fixed-point equation $y = f(y, \lambda)$ for $f : Y \times \Lambda \to Y$. Suppose the following conditions hold.*

b1) The function $g : Y_0 \times \Lambda \to Y_1$ defined by $(y_0, \lambda) \mapsto g(y_0, \lambda) = Jf(J_0 y_0, \lambda)$ is of class C^1, and there exist mappings

$$f^{(1)} : J_0 Y_0 \times \Lambda \to \mathcal{L}(Y),$$
$$f_1^{(1)} : J_0 Y_0 \times \Lambda \to \mathcal{L}(Y_1).$$

such that $D_1 g(y_0, \lambda)\xi = Jf^{(1)}(J_0 y_0, \lambda) J_0 \xi$ for all $(y_0, \lambda, \xi) \in Y_0 \times \Lambda \times Y_0$ and $Jf^{(1)}(J_0 y_0, \lambda) y = f_1^{(1)}(J_0 y_0, \lambda) J y$ for all $(y_0, \lambda, y) \in Y_0 \times \Lambda \times Y$.

b2) There exists $\kappa \in [0, 1)$ such that $f(\cdot, \lambda) : Y \to Y$ is Lipschitz continuous with Lipschitz constant κ, and each of $f^{(1)}(\cdot, \lambda)$ and $f_1^{(1)}(\cdot, \lambda)$ is uniformly bounded by κ.

b3) Under the previous condition, the unique fixed point $\Psi : \Lambda \to Y$ satisfying the equation $\Psi(\lambda) = f(\Psi(\lambda), \lambda)$ itself satisfies $\Psi = J_0 \circ \Phi$ for some continuous $\Phi : \Lambda \to Y_0$.

b4) $f_0 : Y_0 \times \Lambda \to Y$ defined by $(y_0, \lambda) \mapsto f_0(y_0, \lambda) = f(J_0 y_0, \lambda)$ has a continuous partial derivative

$$D_2 f : Y_0 \times \Lambda \to \mathcal{L}(\Lambda, Y).$$

b5) The mapping $(y, \lambda) \mapsto J \circ f^{(1)}(J_0 y, \lambda)$ from $Y_0 \times \Lambda$ into $\mathcal{L}(Y, Y_1)$ is continuous.

Then, the mapping $J \circ \Psi$ *is of class* C^1 *and* $D(J \circ \Psi)(\lambda) = J \circ \mathcal{A}(\lambda)$ *for all* $\lambda \in \Lambda$*, where* $\mathcal{A} = \mathcal{A}(\lambda)$ *is the unique solution of the fixed-point equation* $A = f^{(1)}(\Psi(\lambda), \lambda)A + D_2 f_0(\Phi(\lambda), \lambda)$.

The reason we will need this lemma is because substitution operators such as $R_\delta : \mathcal{PC}^{\eta,s} \to PC^{\eta,s}(\mathbb{R}, \mathbb{R}^n) \oplus B^{\eta,s}_{t_k}(\mathbb{Z}, \mathbb{R}^n)$ defined in Corollary I.5.1.1, though Lipschitz continuous, are generally not differentiable. The surprising result is that if one instead considers the codomain to be $PC^{\zeta,s}(\mathbb{R}, \mathbb{R}^n) \oplus B^{\zeta,s}_{t_k}(\mathbb{Z}, \mathbb{R}^n)$ for some $\zeta > \eta$, then the substitution operator becomes differentiable. Since X^η-type spaces admit continuous embeddings $J : X^{\eta_1} \hookrightarrow X^{\eta_2}$ whenever $\eta_1 \leq \eta_2$, the centre manifold itself can be considered to be embedded in any appropriate weighted Banach space with high enough exponent η. An appropriate application of Lemma I.5.6.1 applied to the defining fixed-point equation (I.5.7) of the centre manifold will allow us to prove that a composition of the embedding operator with the fixed point is a C^1 function. An inductive argument will ultimately get us to C^m smoothness.

I.5.6.2 Candidate Differentials of the Substitution Operators

Recall the definition of the modified nonlinearities

$$F_{\delta,s}(t,x) = f(t,x)\xi\left(\frac{||P_c(s)x||}{N\delta}\right)\xi\left(\frac{||(P_s(s)+P_u(s))x||}{N\delta}\right)$$
$$G_{\delta,s}(k,x) = g(k,x_{0-})\xi\left(\frac{||P_c(s)x_{0-}||}{N\delta}\right)\xi\left(\frac{||(P_s(s)+P_u(s))x_{0-}||}{N\delta}\right).$$

Since s is fixed, we may assume without loss of generality that $||\cdot||$ is smooth on $\mathcal{RCR}_c(0) \setminus \{0\}$. We introduce a symbolic modification of the fixed-point operator;

$$\mathcal{G}^{\eta,s}_\delta : \mathcal{PC}^{\eta,s} \times \times \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$$

defined in the same way as equation (I.5.7). The only difference here is that wish to make the dependence on η, s and δ explicit. We denote the associated fixed point by $\tilde{u}_{\eta,s}$, provided $\delta > 0$ is sufficiently small.

From this point on, our attention shifts to proving the smoothness of $\tilde{u}_{\eta,s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ as defined by the fixed point of (I.5.7). We begin with some notation. Define $\mathcal{PC}^\infty = \cup_{\eta>0}\mathcal{PC}^\eta$. Let

$$V^\eta = \{u \in \mathcal{PC}^\eta : ||(I - \widehat{P}_c)u||_0 < \infty\},$$

where $\widehat{P}_c$ is the projection operator from Lemma I.5.3.1. Equipped with the norm

$$||u||_{V^{\eta,s}} = ||\mathcal{P}_c u||_{\eta,s} + ||(I - \mathcal{P}_c)u||_0,$$

the space $V^{\eta,s}$ is complete, where the s-shifted definitions are as outlined at the beginning of Sect. I.5.1.1.

Let $\delta > 0$ be chosen as in Lemma I.5.3.1, define

$$V_\delta^\eta = \{u \in V^\eta : ||(I - \widehat{P}_c)u||_0 < \delta\}$$

and define $V_\delta^\eta(t) \subset \mathcal{RCR}$ by $V_\delta^\eta(t) = \{u(t) : u \in V_\delta^\eta\}$. Also, define the set $V_\delta^\infty = \cup_{\eta>0} V_\delta^\eta$. Set $B^\eta = PC^\eta(\mathbb{R}, \mathbb{R}^n) \oplus B_{t_k}^\eta(\mathbb{Z}, \mathbb{R}^n)$ and $B^\infty = \cup_{\eta>0} B^\eta$. Finally, the bounded p-linear maps from $X_1 \times \cdots \times X_p$ to Y for Banach spaces X_i and Y will be denoted as $\mathcal{L}^p(X_1 \times \cdots X_p, Y)$. We will denote it as $\mathcal{L}^p$ if there is no confusion.

By construction of the modified nonlinearity $R_{\delta,s}$ and the choice of δ from Lemma I.5.3.1, the functions $u \mapsto F_{\delta,s}(t,u)$ and $u \mapsto G_{\delta,s}(k,u)$ are C^m on $V_\delta^\eta(t)$ and $V_\delta^\eta(t_k)$, respectively, for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. We are therefore free to define

$$\tilde{F}_{\delta,s}^{(p)} u(t) = D^p \tilde{F}_{\delta,s}(t, u(t)), \qquad \tilde{G}_{\delta,s}^{(p)} u(k) = D^p G_{\delta,s}(t_k, u(t_k)),$$

for $1 \leq p \leq m$, where D^p denotes the pth Fréchet derivative with respect to the second variable. For each $u \in V_\delta^\infty$, we can define a p-linear map $\tilde{R}_{\delta,s}^{(p)}(u) : \mathcal{PC}^\infty \times \cdots \times \mathcal{PC}^\infty \to B^\infty$ by the equation

$$\tilde{R}_{\delta,s}^{(p)}(u)(v_1, \ldots, v_p)(t,k) = (F_{\delta,s}^{(p)} u(t)(v_1(t), \ldots, v_p(t)), G_{\delta,s}^{(p)} u(k)(v_1(t_k), \ldots, v_p(t_k))). \tag{I.5.25}$$

For $p = 0$, we define $\tilde{R}_{\delta,s}^{(0)} = R_{\delta,s}$.

I.5.6.3 Smoothness of the Modified Nonlinearity

In this section we elaborate on various properties of the substitution operator $R_{\delta,s}$ and its formal derivative $\tilde{R}_{\delta,s}^{(p)}$ introduced in equation (I.5.25). The first thing we need to do is extend condition H.5 to the modified nonlinearities.

Lemma I.5.6.2. *For $j = 1, \ldots, m$, there exist constants $\tilde{c}_j, \tilde{d}_j, \tilde{q} > 0$ such that*

$$\begin{aligned} &||D^j \tilde{F}_{\delta,s}(t,\phi) - D^j \tilde{F}_{\delta,s}(t,\psi)|| \leq \tilde{c}_j ||\phi - \psi||, \quad ||D^j \tilde{F}_{\delta,s}(t,\phi)|| \leq \tilde{q}\tilde{c}_j \quad \phi, \psi \in V_\delta^\infty(t) \\ &||D^j \tilde{G}_{\delta,s}(k,\phi) - D^j \tilde{G}_{\delta,s}(k,\psi)|| \leq \tilde{d}_j ||\phi - \psi||, \quad ||D^j \tilde{G}(k,\phi)|| \leq \tilde{q}\tilde{d}_j \quad \phi, \psi \in V_{\delta,s}^\infty(t_k). \end{aligned}$$

Proof. We prove only the Lipschitzian property for $D^j F_{\delta,s}$, since the boundedness and corresponding results for $D^j G_{\delta,s}$ are proven similarly. Denote

$$X(s,\phi) = \xi\left(\frac{||P_c(s)\phi||}{N\delta}\right)\xi\left(\frac{||(I - P_c(s))\phi||}{N\delta}\right).$$

When $\phi, \psi \in V_\delta^\infty(t)$, X is m-times continuously differentiable and its derivative is globally Lipschitz continuous. Moreover, the Lipschitz constant can be chosen independent of s because of the uniform boundedness (property

1) of the projection operators. Let Lip_X^k denote the Lipschitz constant for $D^kX(s,\cdot)$. Then,

$$\begin{aligned}
&D^j\tilde{F}_{\delta,s}(t,\phi) - D^j\tilde{F}_{\delta,s}(t,\psi) = D^j\left[f(t,\phi)X(s,\phi) - f(t,\psi)X(s,\phi)\right]\\
&= \sum_{N_1,N_2\in P_2(j)} D^{\#N_1}f(t,\phi)D^{\#N_2}X(s,\phi) - D^{\#N_1}f(t,\psi)D^{\#N_2}X(s,\psi)\\
&= \sum_{N_1,N_2\in P_2(j)} D^{\#N_1}[f(t,\phi)-f(t,\psi)]D^{\#N_2}X(s,\phi)\\
&+ D^{\#N_1}f(t,\psi)D^{\#N_2}[X(s,\phi)-X(s,\psi)],
\end{aligned}$$

where $P_2(j)$ denotes the set of partitions of length two from the set $\{1,\dots,j\}$ and $\#Y$ is the cardinality of Y. Restricted to the ball $B_{2\delta}(0)$, the Lipschitz constants for $D^jf(t,\cdot)$ and the boundedness estimates from H.5 then imply the estimate

$$||D^j\tilde{F}_{\delta,s}(t,\phi) - D^j\tilde{F}_{\delta,s}(t,\psi)|| \le \left(\sum_{N_1,N_2\in P_2(j)} (1+q)c_{\#N_1}(t)\mathrm{Lip}_X^{\#N_2}\right)||\phi-\psi||.$$

As each of c_j and d_j are bounded, the Lipschitz constant admits an upper bound. Outside of $B_{2\delta}(0)$, X and all of its derivatives are identically zero. □

Lemma I.5.6.3. *Let $1\le p\le m$, $\mu_i>0$ for $i=1,\dots,p$, $\mu=\mu_1+\cdots+\mu_p$ and $\eta\ge\mu$. Then we have $\tilde{R}^{(p)}_{\delta,s}(u)\in\mathcal{L}^p(\mathcal{PC}^{\mu_1}\times\cdots\times\mathcal{PC}^{\mu_p},B^\eta)$ for all $u\in V_\delta^\infty$, with*

$$\begin{aligned}
||\tilde{R}^{(p)}_{\delta,s}(u)||_{\mathcal{L}^p} &\le \sup_{t\in\mathbb{R}}||\tilde{F}^{(p)}_{\delta,s}u(t)||e^{-(\eta-\mu)|t|} + \sup_{k\in\mathbb{Z}}||\tilde{G}^{(p)}_{\delta,s}u(k)||e^{-(\eta-\mu)|t_k|}\\
&= ||\tilde{R}^{(p)}_{\delta,s}(u)||_{\eta-\mu}.
\end{aligned}$$

Also, $u\mapsto\tilde{R}^{(p)}_{\delta,s}(u)$ is continuous as a mapping $\tilde{R}^{(p)}_{\delta,s}:V_\delta^\sigma\to\mathcal{L}^p(\mathcal{PC}^{\mu_1}\times\cdots\times\mathcal{PC}^{\mu_p},B^\eta)$ if $\eta>\mu$, for all $\sigma>0$.

Proof. For brevity, denote $\tilde{R}_\delta=\tilde{R}_{\delta,s}$, and similarly for $\tilde{F}$ and $\tilde{G}$. It is easy to verify that $\tilde{R}^{(p)}_\delta(u)$ is p-linear. For boundedness,

$$\begin{aligned}
||\tilde{R}^{(p)}_\delta(u)||_{\mathcal{L}^p} &= \sup_{\substack{t\in\mathbb{R},k\in\mathbb{Z}\\ ||v||_{\vec{\mu}}=1}} ||\tilde{F}^{(p)}_\delta u(t)(v(t))||e^{-\eta|t|} + ||\tilde{G}^{(p)}_\delta u(k)(v(t_k))||e^{-\eta|t_k|}\\
&\le \sup_{\substack{t\in\mathbb{R}\\ ||v||_{\vec{\mu}}=1}} ||\tilde{F}^{(p)}_\delta u(t)(v(t))||e^{-\eta|t|} + \sup_{\substack{k\in\mathbb{Z}\\ ||w||_{\vec{\mu}}=1}} ||\tilde{G}^{(p)}_\delta u(k)(w(t_k))||e^{-\eta|t_k|}\\
&\le \sup_{\substack{t\in\mathbb{R}\\ ||v||_{\vec{\mu}}=1}} ||\tilde{F}^{(p)}_\delta u(t)||\cdot\left(\prod_j||v_j(t)||\right)e^{-\eta|t|} + \sup_{\substack{k\in\mathbb{Z}\\ ||w||_{\vec{\mu}}=1}} ||\tilde{G}^{(p)}_\delta u(k)||\cdot\left(\prod_j||w_j(t_k)||\right)e^{-\eta|t_k|}\\
&= \sup_{t\in\mathbb{R}}||\tilde{F}^{(p)}_\delta u(t)||e^{-(\eta-\mu)|t|} + \sup_{k\in\mathbb{Z}}||\tilde{G}^{(p)}_\delta u(k)||e^{-(\eta-\mu)|t_k|},
\end{aligned}$$

where $||v||_{\vec{\mu}=1}$ is the set of all $v = (v_1, \ldots, v_p) \in \mathcal{PC}^{\mu_1} \times \cdots \times \mathcal{PC}^{\mu_p}$ such that $||v_i||_{\mu_i} = 1$ for $i = 1, \ldots, p$. The latter term in the inequality is finite by Lemma I.5.6.2 whenever $\eta \geq \mu$. In particular, the latter lemma implies that for all $\phi \in V_\delta^\infty$, one has $\sup_{t\in\mathbb{R}} ||D^j \tilde{F}_\delta(t, \phi(t))|| \leq \tilde{q}\tilde{c}_j$, and similar for $\tilde{G}_k$. This uniform boundedness can then be used to prove the continuity of $u \mapsto \tilde{R}_\delta^{(p)}(u)$ when $\eta > \mu$; the proof follows that of [Lemma 7.3 [71]] and is omitted here. ☐

The proofs of the following lemmas are essentially identical to the proofs of [Corollary 7.5, Corollary 7.6, Lemma 7.7 [71]] and are omitted.

Lemma I.5.6.4. *Let $\eta_2 > k\eta_1 > 0$, $1 \leq p \leq k$. Then, $\tilde{R}_{\delta,s} : V_\delta^{\eta_1} \to \mathcal{L}^p(\mathcal{PC}^{\eta_1} \times \cdots \times \mathcal{PC}^{\eta_1}, B^{\eta_2})$ is C^k and $D^p \tilde{R}_{\delta,s} = \tilde{R}_{\delta,s}^{(p)}$.*

Lemma I.5.6.5. *Let $1 \leq p \leq m$, $\mu_i > 0$ for $i = 1, \ldots, p$, $\mu = \mu_1 + \cdots + \mu_p$ and $\eta \geq \mu$. Then, $\tilde{R}_{\delta,s}^{(p)} : V_\delta^\sigma \to \mathcal{L}^p(\mathcal{PC}^{\mu_1} \times \cdots \times \Pi^{\mu_p}, B^\eta)$ is C^{k-p} provided $\eta > \mu + (k-p)\sigma$.*

Lemma I.5.6.6. *Let $1 \leq p \leq k$, $\mu_i > 0$ for $i = 1, \ldots, p$, $\mu = \mu_1 + \cdots \mu_p$ and $\eta > \mu + \sigma$ for some $\sigma > 0$. Let $X : \mathcal{RCR}_c(s) \to V_\delta^\sigma$ be C^1. Then, $\tilde{R}_{\delta,s}^{(p)} \circ X : \mathcal{RCR}_c(s) \to \mathcal{L}^p(\mathcal{PC}^{\mu_1} \times \cdots \times \Pi^{\mu_p}, B^\eta)$ is C^1 and*

$$D\left(\tilde{R}_{\delta,s}^{(p)} \circ X\right)(\phi)(v_1, \ldots, v_p, \psi) = \tilde{R}_{\delta,s}^{(p+1)}(X(\phi))(v_1, \ldots, v_p, X'(\phi)\psi).$$

I.5.6.4 Proof of Smoothness of the Centre Manifold

With our preparations complete, we can formulate and prove the statement concerning the smoothness of the centre manifold.

Theorem I.5.6.1. *Let $\mathcal{J}_s^{\eta_2,\eta_1} : \mathcal{PC}^{\eta_1,s} \to \mathcal{PC}^{\eta_2,s}$ denote the (continuous) embedding operator for $\eta_1 \leq \eta_2$. Let $[\tilde{\eta}, \overline{\eta}] \subset (0, \min\{-a, b\})$ be such that $m\tilde{\eta} < \overline{\eta}$. Then, for each $p \in \{1, \ldots, m\}$ and $\eta \in (p\tilde{\eta}, \overline{\eta}]$, the mapping $J_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$ is of class C^p provided $\delta > 0$ is sufficiently small.*

Proof. The proof here follows the same lines as Theorem 7.7 from Section IX of [41]. To begin, we choose $\delta > 0$ small enough so that Lemma I.5.3.1 is satisfied in addition to having $NL_\delta ||\mathcal{K}_s^\eta|| < \frac{1}{4}$ for all $\eta \in [\tilde{\eta}, \overline{\eta}]$. Remark that this condition ensures that the centre manifold has range in V^η. By Lemma I.5.1.3 and Corollary I.5.1.1, this can always be done in such a way that the inequality holds for all $s \in \mathbb{R}$.

We proceed by induction on k. For $p = 1 = k$, we let $\eta \in (\tilde{\eta}, \overline{\eta}]$ and show that Lemma I.5.6.1 applies with

$$Y_0 = V_\delta^{\tilde{\eta},s}, \qquad Y = \mathcal{PC}^{\tilde{\eta},s}, \qquad Y_1 = \mathcal{PC}^{\eta,s}, \qquad \Lambda = \mathcal{RCR}_c(s)$$
$$f(u,\varphi) = \tilde{\mathcal{G}}_\delta^{\tilde{\eta},s}(u,\varphi), \quad f^{(1)}(u,\varphi) = \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_{\delta,s}^{(1)}(u), \quad f_1^{(1)}(u,\varphi) = \mathcal{K}_s^\eta \circ \tilde{R}_{\delta,s}^{(1)}(u),$$

with embeddings $J = \mathcal{J}_s^{\eta\tilde{\eta}}$ and $J_0 : V_\delta^{\tilde{\eta},s} \hookrightarrow \mathcal{PC}^{\tilde{\eta},s}$. To check condition b1, we must first verify the C^1 smoothness of

$$V_\delta^{\tilde{\eta},s} \times \mathcal{RCR}_c(s) \ni (u,\varphi) \mapsto g(u,\varphi) = \mathcal{J}_s^{\eta\tilde{\eta}} \left(U(\cdot,s)\varphi + \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_{\delta,s}(J_0 u) \right).$$

The embedding operator $\mathcal{J}_s^{\eta\tilde{\eta}}$ is itself C^1, as is $\varphi \mapsto U(\cdot,s)\varphi$ and $J_0 u \mapsto \tilde{R}_{\delta,s}(J_0 u)$, the latter due to Lemma I.5.6.4. C^1 smoothness of g then follows by continuity of the linear embedding J_0. Verification of the equalities $D_1 g(u,\varphi)\xi = Jf^{(1)}(J_0 u,\varphi)J_0\xi$ and $Jf^{(1)}(J_0 u,\varphi)\xi = f_1^{(1)}(J_0 u,\varphi)J\xi$ is straightforward. Condition b2 follows by boundedness of the embedding operators and the small Lipschitz constant for $\tilde{\mathcal{G}}_{\delta,s}^{\tilde{\eta},s}$. For condition b3, the fixed point is $\tilde{u}_{\tilde{\eta},s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\tilde{\eta},s}$, and we may factor it as $\tilde{u}_{\tilde{\eta},s} = J_0 \circ \Phi$ with $\Phi : \mathcal{RCR}_c(s) \to V_\delta^{\tilde{\eta},s}$ defined by $\Phi(\varphi) = \tilde{u}_{\tilde{\eta},s}(\varphi)$; the latter is continuous by Theorem I.5.2.1, and the factorization is justified by Lemma I.5.3.1. To check condition b4, we must verify that

$$V_\delta^{\tilde{\eta},s} \times \mathcal{RCR}_c(s) \ni (u,\varphi) \mapsto f_0(u,\varphi) = \tilde{\mathcal{G}}_{\delta,s}^{\tilde{\eta},s}(J_0 u,\varphi)$$

has a continuous partial derivative in its second variable—this is clear since f_0 is linear in φ. Finally, condition b5 requires us to verify that the map $(u,\varphi) \mapsto \mathcal{J}_s^{\eta\tilde{\eta}} \circ \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_{\delta,s}^{(1)}(J_0 u)$ is continuous from $V_\delta^{\tilde{\eta},s} \times \mathcal{RCR}_c(s)$ into $\mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$, but this once again follows by the continuity of the embedding operators and the smoothness of $\tilde{R}_{\delta,s}$ from Lemma I.5.6.4.

The conditions of Lemma I.5.6.1 are satisfied, and we conclude that $\mathcal{J}^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}$ is of class C^1 and that the derivative $D(\mathcal{J}^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}(\varphi)) \in \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is the unique solution $w^{(1)}$ of the equation

$$w^{(1)} = \mathcal{K}_s^{\tilde{\eta}} \circ \tilde{R}_{\delta,s}^{(1)}(\tilde{u}_s^{\tilde{\eta}}(\varphi))w^{(1)} + U(\cdot,s) := F_1(w^{(1)},\varphi). \tag{I.5.26}$$

The mapping $F_1 : \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s}) \times \mathcal{RCR}_c(s) \to \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is a uniform contraction for $\eta \in [\tilde{\eta}, \overline{\eta}]$—indeed, $F_1(\cdot,\varphi)$ is Lipschitz continuous with Lipschitz constant $\tilde{L}_\delta \cdot ||\mathcal{K}_s^{\eta}|| < \frac{1}{4}$; this follows from Lemma I.5.6.3 and is independent of s. Thus, $\tilde{u}_s^{(1)}(\varphi) \in \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\tilde{\eta},s}) \hookrightarrow \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ for $\eta \geq \tilde{\eta}$. Moreover, $\tilde{u}_s^{(1)} : \mathcal{RCR}_c(s) \to \mathcal{L}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is continuous if $\eta \in (\tilde{\eta}, \overline{\eta}]$.

Now, let $1 \leq p \leq k$ for $k \geq 1$ and suppose that for all $q \in \{1,\ldots,p\}$ and all $\eta \in (q\tilde{\eta}, \overline{\eta}]$, the mapping

$$\mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s} : \mathcal{RCR}_c(s) \to \mathcal{PC}^{\eta,s}$$

is of class C^q with $D^q(\mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_s^{\tilde{\eta}}) = \mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}^{(q)}$ and $\tilde{u}_{\tilde{\eta},s}^{(q)}(\varphi) \in \mathcal{L}^q(\mathcal{RCR}_c(s), \mathcal{PC}^{q\tilde{\eta},s})$ for each $\varphi \in \mathcal{RCR}_c(s)$, such that the mapping

$$\mathcal{J}_s^{\eta\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}^{(q)} : \mathcal{RCR}_c(s) \to \mathcal{L}^q(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$$

is continuous for $\eta \in (q\tilde{\eta}, \overline{\eta}]$. Suppose additionally that $\tilde{u}^{(q)}_{\tilde{\eta},s}(\varphi)$ is the unique solution $w^{(p)}$ of an equation

$$w^{(p)} = \mathcal{K}_s^{\tilde{\eta}p} \circ \tilde{R}^{(1)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi))w^{(p)} + H^{(p)}_{\tilde{\eta}}(\varphi) := F^{(p)}_{\tilde{\eta}}(w^{(p)}, \varphi), \tag{I.5.27}$$

with $H^1 = U(\cdot, s)$, and $H^{(p)}_x(\varphi)$ for $p \geq 2$ is a finite sum of terms of the form

$$\mathcal{K}_s^{px} \circ \tilde{R}^{(q)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi))(\tilde{u}^{(r_1)}_{\tilde{\eta},s}(\varphi), \cdots, \tilde{u}^{(r_q)}_{\tilde{\eta},s}(\varphi))$$

with $2 \leq q \leq p$, $1 \leq r_i < p$ for $i = 1, \dots, q$, such that $r_1 + \cdots + r_q = p$. Under such assumptions, the mapping $F^{(p)}_{\tilde{\eta}} : \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s}) \times \mathcal{RCR}_c(s) \to \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is a uniform contraction for all $\eta \in [p\tilde{\eta}, \overline{\eta}]$; see Lemma I.5.6.3.

Next, choose some $\eta \in ((p+1)\tilde{\eta}, \overline{\eta}]$, $\sigma \in (\tilde{\eta}, \eta/(p+1)]$ and $\mu \in ((p+1)\sigma, \eta)$. We will verify the conditions of Lemma I.5.6.1 with the spaces and functions

$$\begin{aligned}
Y_0 &= \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{p\sigma,s}), \qquad\qquad Y = \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\mu,s}),\\
[3pt]Y_1 &= \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})\\
f(u,\varphi) &= \mathcal{K}_s^{\mu} \circ \tilde{R}^{(1)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi))u + H^{(p)}_{\mu/p}(\varphi), \quad \Lambda = \mathcal{RCR}_c(s),\\
f^{(1)}(u,\varphi) &= \mathcal{K}_s^{\mu} \circ \tilde{R}^{(1)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi)) \in \mathcal{L}(Y),\\
f_1^{(1)}(u,\varphi) &= \mathcal{K}_s^{\eta} \circ \tilde{R}^{(1)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi)) \in \mathcal{L}(Y_1).
\end{aligned}$$

We begin with the verification of condition b1. We must check that

$$\mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{p\sigma,s}) \times \mathcal{RCR}_c(s) \ni (u,\varphi) \mapsto \mathcal{J}^{\eta\mu} \circ \mathcal{K}_s^{\mu} \circ \tilde{R}^{(1)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi))u + \mathcal{J}^{\eta\mu} \circ H^{(p)}_{\mu/p}(\varphi)$$

is of class C^1, where now $\mathcal{J}^{\eta_2\eta_1} : \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta_1,s}) \hookrightarrow \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta_2,s})$. The above mapping is C^1 with respect to $u \in \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{p\sigma,s})$ since it is linear in this variable. With respect to $\varphi \in \mathcal{RCR}_c(s)$, we have that $\varphi \mapsto \mathcal{J}^{\eta\mu}\mathcal{K}_s^{\mu} \circ \tilde{R}^{(1)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi))$ is C^1: this follows by Lemma I.5.6.6 with $\mu > (p+1)\sigma$ and the C^1 smoothness of $\varphi \mapsto \mathcal{J}^{\sigma\tilde{\eta}} \circ \tilde{u}_{\tilde{\eta},s}(\varphi)$ with $\sigma > \tilde{\eta}$. For the C^1 smoothness of the portion $\varphi \mapsto \mathcal{J}^{\eta\mu}H^{(p)}_{\mu/p}(\varphi)$, we get differentiability from Lemma I.5.6.6; we have that the derivative of $\varphi \mapsto H^{(p)}_{\mu/p}(\varphi)$ is a sum of terms of the form

$$\begin{aligned}
&\mathcal{K}_s^{\mu} \circ \tilde{R}^{(q+1)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi))(\tilde{u}^{(r_1)}_{\tilde{\eta},s}(\varphi), \dots, \tilde{u}^{(r_q)}_{\tilde{\eta},s}(\varphi))\\
&\qquad + \sum_{j=1}^{q} \mathcal{K}_s^{\mu} \circ \tilde{R}^{(q)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi))(\tilde{u}^{(r_1)}_{\tilde{\eta},s}(\varphi), \dots, \tilde{u}^{(r_j+1)}_{\tilde{\eta},s}(\varphi), \dots, \tilde{u}^{(r_q)}_{\tilde{\eta},s}(\varphi)),
\end{aligned}$$

and each $\tilde{u}^{(j)}_{\tilde{\eta},s}$ is understood as a map into $\mathcal{PC}^{j\sigma,s}$. Applying Lemma I.5.6.3 with $\mu > (p+1)\sigma$ grants continuity of $DH^{(p)}_{\mu/p}(\varphi)$ and, subsequently, to

$\mathcal{J}^{\eta\mu} DH^{(p)}_{\mu/p}(\varphi)$. The other embedding properties of condition b1 are easily checked. Condition b4 can be proven similarly.

The Lipschitz condition and boundedness of b2 follow by the choice of $\delta > 0$ at the beginning and the uniform contractivity of H_p described above. Condition b3 is proven by writing

$$\mathcal{J}^{\eta\mu} \circ \mathcal{K}^{\mu}_{s} \circ \tilde{R}^{(1)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s})(\varphi)) = \mathcal{K}^{\eta}_{s} \circ \tilde{R}^{(1)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi))$$

and applying Lemma I.5.6.3 together with the C^1 smoothness of $\tilde{u}_{\tilde{\eta},s}$ to obtain the continuity of $\varphi \mapsto \tilde{R}^{(1)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}) \in \mathcal{L}(Y, Y_1)$. This also proves the final condition b5 of Lemma I.5.6.1, and we conclude that $\tilde{u}^{(p)}_{\tilde{\eta},s} : \mathcal{RCR}_c(s) \to \mathcal{L}^p(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,s})$ is of class C^1 with $\tilde{u}^{(p+1)}_{\tilde{\eta},\mu} = D\tilde{u}^{(p)}_{\tilde{\eta},s} \in \mathcal{L}^{(p+1)}(\mathcal{RCR}_c(s), \mathcal{PC}^{\eta,\mu})$ given by the unique solution $w^{(p+1)}$ of the equation

$$w^{(p+1)} = \mathcal{K}^{\mu}_{s} \circ \tilde{R}^{(1)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi))w^{(p+1)} + H^{(p+1)}_{\mu/(p+1)}(\varphi), \qquad \text{(I.5.28)}$$

where $H^{(p+1)}_{\mu/(p+1)}(\varphi) = \mathcal{K}^{\mu}_{s} \circ \tilde{R}^{(2)}_{\delta,s}(\tilde{u}_{\tilde{\eta},s}(\varphi))(\tilde{u}^{(p)}_{\tilde{\eta},s}(\varphi), \tilde{u}^{(1)}_{\tilde{\eta},s}(\varphi)) + DH^{(p)}_{\mu/p}(\varphi)$. Similar arguments to the proof of the case $k = 1$ show that the fixed point $w^{(p+1)}$ is also contained in $\mathcal{L}^{(p+1)}(\mathcal{RCR}_c(s), \mathcal{PC}^{\tilde{\eta}(p+1),s})$, and the proof is complete. □

Corollary I.5.6.1. *$\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ is C^m and tangent at the origin to the centre bundle $\mathcal{RCR}_c$. More precisely, $\mathcal{C}(t, \cdot) : \mathcal{RCR}_c(t) \to \mathcal{RCR}$ is C^m and $D\mathcal{C}(t, 0)\phi = \phi$ for all $\phi \in \mathcal{RCR}_c(t)$.*

Proof. Let $\tilde{\eta}, \eta$ be as in the proof of Theorem I.5.6.1. Define the evaluation map $ev_t : \mathcal{PC}^{\eta} \to \mathcal{RCR}$ by $ev_t(f) = f(t)$. Since we can decompose the centre manifold as

$$\mathcal{C}(t, \phi) = ev_t(\tilde{u}_t(\phi)) = ev_t(\mathcal{J}^{\eta\tilde{\eta}}_t \tilde{u}_t(\phi)),$$

boundedness of the linear evaluation map on the space $\mathcal{PC}^{\eta,t}$ then implies the C^k smoothness of $\mathcal{C}(t, \cdot)$. To obtain the tangent property, we remark that

$$D\mathcal{C}(t, 0)\phi = ev_t\left(D\left(\mathcal{J}^{\eta\tilde{\eta}}_t \circ \tilde{u}_t(0)\right)\phi\right) = ev_t\left(\tilde{u}^{(1)}_{\eta,t}(0)\phi\right).$$

From equation (I.5.26) and Theorem I.5.3.1, we obtain $\tilde{u}^{(1)}_{\eta,t}(0) = U(\cdot, t)$, from which it follows that $D\mathcal{C}(t, 0)\phi = \phi$, as claimed. □

As a secondary corollary, we can prove that each derivative of the centre manifold is uniformly Lipschitz continuous. The proof is similar to that of Corollary I.5.2.1 if one takes into account the representation of the derivatives $\tilde{u}^{(p)}_{\tilde{\eta},s}$ as solutions of the fixed-point equations (I.5.28), whose right-hand side is a contraction with Lipschitz constant independent of s.

Corollary I.5.6.2. *For each $p \in \{1, \ldots, k\}$, there exists a constant $L(p) > 0$ such that the centre manifold satisfies $||D^p C(t,\phi) - D^p C(t,\psi)|| \leq L(p)||\phi - \psi||$ for all $t \in \mathbb{R}$ and $\phi, \psi \in \mathcal{RCR}_c(t)$.*

Additionally, each of the Taylor coefficients of the centre manifold is in fact bounded. This observation will be important in later chapters.

Corollary I.5.6.3. *There exist constants $\gamma_0, \ldots, \gamma_m$ such that the C^m centre manifold satisfies $||D^j\mathcal{C}(t,0)|| \leq \gamma_j$ for all $t \in \mathbb{R}$. If the centre manifold is C^{m+1}, the Taylor remainder*

$$R_m(t,\phi) = \mathcal{C}(t,\phi) - \sum_{j=1}^{m} D^j\mathcal{C}(t,0)\phi^j$$

admits an estimate of the form $||R_m(t,\phi)|| \leq \gamma^{(m)}||\phi||^{m+1}$ for $\phi \in B_\delta(0) \cap \mathcal{RCR}_c(t)$. The constants $\gamma^{(m)}$ and δ can be chosen independent of t.

Proof. From equation (I.5.28), the jth Taylor coefficient is given by

$$D^j\mathcal{C}(t,0) = ev_t\left(H^{(j+1)}_{\mu/(j+1)}(0)\right).$$

The first two coefficients ($j = 0$ and $j = 1$) are zero and the identity, respectively. These are bounded. A straightforward inductive argument on the form of the maps H then grants the uniform boundedness of $D^j\mathcal{C}(t,0)$. The claimed bound on the remainder term then follows from the uniform boundedness of $D^{m+1}\mathcal{C}(t,0)$ and the Lipschitz constant $L(j+1)$ from Corollary I.5.6.2. □

We readily obtain the smoothness of the centre manifold in the case where the semilinear equation is periodic. In particular, in such a situation some of the assumptions H.1–H.8 are satisfied automatically and can be ignored.

Corollary I.5.6.4. *Suppose the semilinear equation (I.4.1)–(I.4.2) satisfies the following conditions.*

- P.1 *The equation is periodic with period T and c impulses per period. That is, $L(t+T) = L(t)$ and $f(t+T,\cdot) = f(t)$ for all $t \in \mathbb{R}$, and $B(k+c) = B(k)$, $g(k+c,\cdot) = g(k,\cdot)$ and $t_{k+c} = t_k + T$ for all $k \in \mathbb{Z}$.*
- P.2 *Conditions H.1–H.3 and H.5–H.6 are satisfied.*

Then, the conclusions of Corollaries I.5.6.1 and I.5.6.2 hold.

I.5.6.5 Periodic Centre Manifold

In this section we will prove that the centre manifold is itself a periodic function, provided the conditions P.1–P.2 of Corollary I.5.6.4 are satisfied. We begin with a preparatory lemma.

Lemma I.5.6.7. *Define the operator* $N_s : \mathcal{RCR}_c(s) \to \mathcal{RCR}_c(s)$ *by*

$$N_s(\phi) = P_c(s)S(s+T,s)\mathcal{C}(s,\phi).$$

This operator is well-defined and invertible in a neighbourhood of $0 \in \mathcal{RCR}_c(s)$. *Moreover, the neighbourhood can be written* $U \cap \mathcal{RCR}_c(s)$ *for some open neighbourhood* $U \subset \mathcal{RCR}$ *of* $0 \in \mathcal{RCR}$, *independent of* s.

Proof. To show that N_s is invertible in a neighbourhood of the origin, we will use the inverse function theorem. The Fréchet derivative of N_s at 0 is given by

$$\begin{aligned} DN_s(0)\phi &= P_c(s) \circ DS(s+T,s)(0) \circ D\mathcal{C}(s,0)\phi \\ &= P_c(s+T) \circ U(s+T,s)\phi \\ &= U_c(s+T,s)\phi, \end{aligned}$$

where we used Corollary I.5.6.1 to calculate $D\mathcal{C}(s,0)$ and Theorem I.4.2.1 to calculate $DS(s+T,s)(0)$. Since $U(s+T,s)$ is an isomorphism (Theorem I.3.1.3) of $\mathcal{RCR}_c(s)$ with $\mathcal{RCR}_c(s+T) = \mathcal{RCR}_c(s)$, we obtain the claimed local invertibility.

To show that the neighbourhood may be written as claimed, we remark that $DN_s(x)$ is uniformly convergent (in the variable s) as $x \to 0$ to $DN_s(0)$. Indeed, we have the estimate

$$||DN_s(x) - DN_s(0)|| \leq ||U_c(s+T,s)P_c(s)|| \cdot ||D\mathcal{C}(s,x) - D\mathcal{C}(s,0)||,$$

and the Lipschitz property of Corollary I.5.6.2 together with uniform boundedness of the projector $P_c(s)$ and centre monodromy operator $U_c(s+T,s)$ grants the uniform convergence as $x \to 0$. As a consequence, the implicit function may be defined on a neighbourhood that does not depend on s. □

Theorem I.5.6.2. *There exists* $\delta > 0$ *such that* $\mathcal{C}(s+T,\phi) = \mathcal{C}(s,\phi)$ *for all* $s \in \mathbb{R}$ *whenever* $||\phi|| \leq \delta$.

Proof. By Lemma I.5.6.7, there exists $\delta > 0$ such that if $||\phi|| \leq \delta$, we can write $\phi = N_s(\psi)$ for some $\psi \in \mathcal{RCR}_c(s)$. By Theorem I.5.3.1 and the periodicity condition P.1,

$$\begin{aligned} \mathcal{C}(s+T,\phi) &= \mathcal{C}(s+T, N_s(\psi)) \\ &= \mathcal{C}(s+T, P_c(s+T)S(s+T,s)\mathcal{C}(s,\psi)) \\ &= S(s+T,s)\mathcal{C}(s,\psi) \\ &= S(s,s-T)\mathcal{C}(s,\psi) \\ &= \mathcal{C}(s, P_c(s)S(s,s-T)\mathcal{C}(s,\psi)) \\ &= \mathcal{C}(s, P_c(s)S(s+T,s)\mathcal{C}(s,\psi)) \\ &= \mathcal{C}(s, N_s(\psi)) = \mathcal{C}(s,\phi), \end{aligned}$$

where the identity $S(s+T,s) = S(s,s-T)$ follows due to periodicity and Lemma I.4.1.1. □

I.5.7 Regularity of Centre Manifolds with Respect to Time

In the previous section we were concerned with the smoothness of $\phi \mapsto \mathcal{C}(t,\phi)$. To contrast, in this section we are interested in to what degree the function $t \mapsto D^k\mathcal{C}(t,\phi)$ is differentiable, for each $k = 1,\ldots,m$. We should generally *not* expect this function to be differentiable; indeed, it would be very surprising if this were true given that the process $U(t,s)$ associated to the linearization is generally discontinuous everywhere (recall the discussion of Sect. I.2.2.2).

Perhaps it is better to motivate our ideas on regularity in time by explaining how we will be using the centre manifold in applications. From Taylor's theorem, $\mathcal{C}(t,\phi)$ admits an expansion of the form

$$\mathcal{C}(t,\phi) = D\mathcal{C}(t,0)\phi + \frac{1}{2}D^2\mathcal{C}(t,0)[\phi]^2 + \cdots + \frac{1}{m!}D^m\mathcal{C}(t,0)[\phi]^m + O(||\phi||^{m+1}),$$

where $[\phi]^k = [\phi,\ldots,\phi]$ with k factors, and the $O(||\phi||^{m+1})$ terms generally depend on t. By Theorem I.5.6.2, under periodicity assumptions these terms will be uniformly bounded in t for $||\phi||$ sufficiently small. This expansion can in principle be used in the dynamics equation (I.5.12)–(I.5.13) on the centre manifold or its integral version (I.5.10), which will permit us to classify bifurcations in impulsive RFDE. In later sections we will want to make these dynamics equations concrete—that is, to pose them in a concrete vector space such as $\mathbb{R}^p$ for some $p \in \mathbb{N}$. By analogy with ordinary and delay differential equations, this should also allow us to obtain a partial differential equation for the Taylor coefficients $D^j\mathcal{C}(t,0)$. As these coefficients are time-varying, we should suspect this PDE to contain derivatives in time as well.

In summary, we need to consider the differentiability of the function $t \mapsto D^j\mathcal{C}(t,0)$ for $j = 1,\ldots,m$. Since we suspect that this function will not actually be differentiable, we might consider instead the differentiability of

$$t \mapsto D^j\mathcal{C}(t,0)[\phi_1,\ldots,\phi_j](\theta)$$

for each $\theta \in [-r,0]$ and j-tuples $\phi_1,\ldots,\phi_j$. While a more realistic goal, even this is too strong a condition. The first differential $D\mathcal{C}(t,0) : \mathcal{RCR}_c(t) \to \mathcal{RCR}$ of the centre manifold has a different domain for each t. As a consequence, we cannot even define the derivative of $t \mapsto D\mathcal{C}(t,0)\phi(\theta)$, since we must have $\phi \in \mathcal{RCR}_c(t)$ for the right-hand side to be well-defined. This problem is apparent for all higher differentials.

I.5.7.1 A Coordinate System and Pointwise $PC^{1,m}$-Regularity

To address the issue, the centre manifold having a "time-varying domain", let us first assume that $\mathcal{RCR}_c$ is finite-dimensional—that is, H.9 is satisfied. Note if we fix a sufficiently well-behaved coordinate system in $\mathcal{RCR}_c(t)$—for example, let $\phi_1, \ldots, \phi_p$ be a basis for $\mathcal{RCR}_c(0)$ and define $\phi_i(t) = U_c(t,0)\phi_i$ for $i = 1, \ldots, p$ to be a basis for $\mathcal{RCR}_c(t)$—then the function $w(t)$ of Lemma I.5.4.1 and Theorem I.5.4.1 can be written as $w(t) = \Phi_t z(t)$ for $z \in \mathbb{R}^p$, where $\Phi_t = [\ \phi_1(t) \quad \cdots \quad \phi_p(t)\ ]$. This motivates us to consider instead a centre manifold in these coordinates.

Definition I.5.7.1. *The function $C : \mathbb{R} \times \mathbb{R}^p \to \mathcal{RCR}$ defined by*

$$C(t,z) = \mathcal{C}(t, \Phi_t z) \tag{I.5.29}$$

is the centre manifold in terms of the basis array Φ.

If $\mathcal{C}(t,\cdot)$ is C^m-smooth, the chain rule implies the same is true for $C(t,\cdot)$. It follows that

$$\begin{aligned}\mathcal{C}(t, w(t)) = DC(t,0)z(t) + \frac{1}{2}D^2C(t,0)[z(t), z(t)] + \cdots \\ + \frac{1}{m!}D^mC(t,0)[z(t)]^m + O(||w(t)||^{m+1}),\end{aligned}$$

so insofar as dynamics on the centre manifold are concerned, it is enough to study the differentiability of $t \mapsto D^jC(t,0)[z_1, \ldots, z_p](\theta)$ for p-tuples $z_1, \ldots, z_p \in \mathbb{R}^p$. Specifically, the temporal regularity we will attempt to prove is given in the following definition.

Definition I.5.7.2. *A function $F : \mathbb{R} \times \mathbb{R}^p \to \mathcal{RCR}$ is* pointwise $PC^{1,m}$-regular at zero *if it satisfies the following conditions:*

- *$x \mapsto F(t,x)$ is C^m in a neighbourhood of $0 \in \mathbb{R}^p$, uniformly in t;*
- *for $j = 0, \ldots, m$, $D^jF(t,0)[z_1, \ldots, z_j](\theta)$ is differentiable from the right with limits on the left separately with respect to t and θ, for all $z_1, \ldots, z_j \in \mathbb{R}^p$.*

With this in mind, the result we will prove is as follows.

Theorem I.5.7.1. *Let $\phi_1, \ldots, \phi_p$ be a basis for $\mathcal{RCR}_c(0)$, and define*

$$\Phi_t = [\ U_c(t,0)\phi_1 \quad \cdots \quad U_c(t,0)\phi_p\ ].$$

If the centre manifold $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ is (fibrewise) C^m, then the centre manifold in terms of the basis array Φ is pointwise $PC^{1,m}$-regular at zero provided certain technical conditions are met (assumption H.10). Moreover, if $\theta \in \mathcal{RCR}(\mathbb{R}, [-r, 0])$, then $t \mapsto C(t,z)(\theta(t))$ is continuous from the right with limits on the left for all $z \in \mathbb{R}^p$, and $z \mapsto C(t,z)$ is Lipschitz continuous, uniformly for $t \in \mathbb{R}$.

The technical condition will be introduced in Sect. I.5.7.3.

I.5.7.2 Reformulation of the Fixed-Point Equation

Given that $C(t,z) = \mathcal{C}(t,\Phi_t z)$, we can equivalently write $C(t,z) = v_t(z)(t)$ with $v_t : \mathbb{R}^p \to \mathcal{PC}^{\eta,t}$ the unique fixed point of the equation

$$v_t(z) = \Phi_{(\cdot)} z + \mathcal{K}_t^\eta(R_{\delta,t}(v_t(z))) \tag{I.5.30}$$

for each $|z|$ small enough, where $\mathcal{K}_t^\eta$ is as defined in Eq. (I.5.3) and $R_{\delta,t}$ is the substitution operator from Sect. I.5.1.3. Notice also that the nonlinear operator defining the right-hand side of the equation admits the same Lipschitz constant as original fixed-point operator $\mathcal{G}$ from Eq. (I.5.7). Up to an appropriate embedding, the jth differential $v_t^{(j)}$ satisfies for $j \geq 2$ a fixed-point equation of the form

$$v_t^{(j)} = \mathcal{K}_t^\eta \circ R_{\delta,t}^{(1)}(v_t) v_t^{(j)} + \mathcal{K}_t^\eta \circ H^{(j)}(v_t, v_t^{(1)}, \ldots, v_t^{(j-1)}), \tag{I.5.31}$$

with the right-hand side defining a uniform contraction in $v_t^{(j)}$. $H^{(j)}$ can be written as a finite linear combination of terms of the form

$$R_{\delta,t}^{(q)}(v_t)[v_t^{(r_1)}, \ldots, v_t^{(r_q)}],$$

for $q \in \{2,\ldots,j\}$, such that $r_1 + \cdots + r_q = j$. All of this follows from (the proof of) Theorem I.5.6.1. Explicitly,

$$H^{(j)} = -R_{\delta,t}^{(1)}(v_t)v_t^{(j)} + D_z^j[R_{\delta,t}(v_t(z))],$$

and one can verify by induction on j that $H^{(j)}$ contains no term of the form $R^{(1)}(v_t)v_t^{(j)}$ and that the coefficients in the aforementioned linear combination are independent of t. To compare, for $j=0$ and $j=1$, we can compute directly from the definition of the fixed point and by using Corollary I.5.6.1 and the chain rule that

$$v_t(0)(\cdot) = 0, \tag{I.5.32}$$

$$v_t^{(1)}(0)(\cdot) = \Phi_{(\cdot)}. \tag{I.5.33}$$

The assumption $Df(t,0) = Dg(k,0) = 0$ implies $R_\delta^{(1)}(0) = 0$, so the fixed-point equation (I.5.31) implies

$$v_t^{(j)}(0)(\mu) = \left[\mathcal{K}_t^\eta \circ H^{(j)}(0, \Phi_{(\cdot)}, v_t^{(2)}(0)(\cdot), \ldots, v_t^{(j-1)}(0)(\cdot))\right](\mu) \tag{I.5.34}$$

for $j \geq 2$. By definition of the basis array Φ, the following lemma is proven.

Lemma I.5.7.1. *If the centre manifold is C^1, then the centre manifold in terms of the basis array Φ is pointwise $PC^{1,1}$-regular at zero. If the centre* *manifold is C^m, then the centre manifold in terms of the basis array Φ is pointwise $PC^{1,m}$-regular at zero provided*

$$t \mapsto \left[\mathcal{K}_t^\eta \circ H^{(j)}(0, \Phi_{(\cdot)}, v_t^{(2)}(0)(\cdot), \dots, v_t^{(j-1)}(0)(t))\right](t)[z_1, \dots, z_j](\theta),$$
$$\theta \mapsto \left[\mathcal{K}_t^\eta \circ H^{(j)}(0, \Phi_{(\cdot)}, v_t^{(2)}(0)(\cdot), \dots, v_t^{(j-1)}(0)(t))\right](t)[z_1, \dots, z_j](\theta)$$

are each, for $j = 2, \dots, m$ differentiable from the right with limits on the left, for all $z_1, \dots, z_j \in \mathbb{R}^p$.

I.5.7.3 A Technical Assumption on the Projections $P_c(t)$ and $P_u(t)$

By definition of the bounded linear map $\mathcal{K}_t^\eta$ from (I.5.3), it will be necessary to differentiate (in t) integrals involving terms of the form $\mu \mapsto U(t,\mu)P_s(\mu)\chi_0$ and $\mu \mapsto U(t,\mu)P_u(\mu)\chi_0$. Generally, if we assume $\mathcal{RCR}_u(0)$ to be q-dimensional (guaranteed by Theorem I.3.1.3 if the linearization is periodic, for example), then we can fix a basis $\psi_1, \dots, \psi_q$ for $\mathcal{RCR}_u(0)$ and construct a basis array

$$\Psi_t = [\ U_u(t,0)\psi_1 \quad \cdots \quad U_u(t,0)\psi_q\ ]$$

for $\mathcal{RCR}_u(t)$ that is formally analogous to the basis array Φ_t for the centre fibre bundle. Under spectral separation assumptions, $U_u(t,s) : \mathcal{RCR}_u(s) \to \mathcal{RCR}_u(t)$ and $U_c(t,s) : \mathcal{RCR}_c(s) \to \mathcal{RCR}_c(t)$ are topological isomorphisms, from which it follows that there exist unique $Y_c(t) \in \mathbb{R}^{p\times n}$ and $Y_u(t) \in \mathbb{R}^{q\times n}$ such that

$$\begin{aligned} P_c(t)\chi_0 &= \Phi_t Y_c(t), \\ P_u(t)\chi_0 &= \Psi_t Y_u(t). \end{aligned} \tag{I.5.35}$$

Recall $p = \dim(\mathcal{RCR}_c)$. Even under periodicity conditions, computing the action of these projections on the functional $\chi_0 \in \mathcal{RCR}([-r,0], \mathbb{R}^{n\times n})$ is quite nontrivial and requires computing the abstract contour integral (I.3.4). Though this can in principle be done numerically by discretizing the monodromy operator, there is little in the way of theoretical results guaranteeing, for example, that the matrix functions $t \mapsto Y_c(t)$ and $t \mapsto Y_u(t)$ are, respectively, elements of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^{p\times n})$ and $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^{q\times n})$. Such a result would make the differentiation of the integrals appearing in the definition of $\mathcal{K}_t^\eta$ much more reasonable. We therefore introduce another hypothesis. We will discuss it in a bit more detail in Sect. I.5.7.7.

****

H.10 There are (finite) basis arrays Φ and Ψ for $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$, respectively, for which the matrix functions $t \mapsto Y_c(t)$ and $t \mapsto Y_u(t)$ from equation (I.5.35) are continuous from the right and possess limits on the left.

I.5.7.4 Proof of $PC^{1,m}$-Regularity at Zero

We deal first with the continuity of $t \mapsto C(t,z)(\theta(t))$ from the right and the existence of its left-limits. Since $C(\cdot, z) = v_t(z)(\cdot) \in \mathcal{PC}^{\eta,t}$, it can be identified with a history function $t \mapsto c_t$ for some $c \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$. But this implies $C(t,z)(\theta(t)) = c_t(\theta(t)) = c(t + \theta(t))$. The conclusion follows because $c \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ and $\theta \in \mathcal{RCR}(\mathbb{R}, [-r, 0])$, and right-continuity and limits respect composition. As for the Lipschitzian claim, it follows by similar arguments to the proof of the original centre manifold Theorem I.5.2.1 and Corollary I.5.2.1.

Using the definition of the linear map $\mathcal{K}_t^\eta$ in (I.5.3) and equation (I.5.34), we can explicitly write $v_t^{(j)}(0)(t)$ as

$$
\begin{aligned}
v_t^{(j)}(0)(t) = &\int_{-\infty}^{t} U(t,\mu)[I - P_c(\mu) - P_u(\mu)]\chi_0 \hat{H}_1^{(j)}(\mu) d\mu \\
&- \int_t^\infty U(t,\mu) P_u(\mu) \chi_0 \hat{H}_1^{(j)}(\mu) d\mu \\
&+ \sum_{-\infty}^{t} U(t,t_i)[I - P_c(t_i) - P_u(t_i)]\chi_0 \hat{H}_2^{(j)}(t_i) dt_i \\
&- \sum_{t}^{\infty} U(t,t_i) P_u(t_i) \chi_0 \hat{H}_2^{(j)}(t_i) dt_i,
\end{aligned}
$$

where each of $\hat{H}_1^{(j)}(\mu)$ and $\hat{H}_2^{(j)}(t_i)$ and $H^{(j)}$ are related by the equations

$$
\begin{aligned}
H^{(j)} &= \sum_i c_i R_{\delta,t}^{(r_i)}(0)[\Phi_{(\cdot)}^{d_{i,1}}, [v_t^{(2)}(0)(t)]^{d_{i,2}}, \dots, [v_t^{(j-1)}(0)(t)]^{d_{i,j-1}}] \\
\hat{H}_1^{(j)}(\mu) &= \sum_i c_i D^{r_i} f(\mu, 0)[\Phi_\mu^{d_{i,1}}, [v_t^{(2)}(0)(\mu)]^{d_{i,2}}, \dots, [v_t^{(j-1)}(0)(\mu)]^{d_{i,j-1}}] \\
\hat{H}_2^{(j)}(t_k) &= \sum_i c_i D^{r_i} g(k, 0)[\Phi_{t_k}^{d_{i,1}}, [v_t^{(2)}(0)(t_k)]^{d_{i,2}}, \dots, [v_t^{(j-1)}(0)(t_k)]^{d_{i,j-1}}].
\end{aligned}
$$

The first line follows from the definition of $H^{(j)}$, while the other two come from the definition of the substitution operator. Note also that we have suppressed the inputs $z_1, \dots, z_j$; technically, each of $\hat{H}_1^{(j)}(\mu)$ and $\hat{H}_2^{(j)}(\mu)$ are j-linear maps from $\mathbb{R}^p$ to $\mathcal{RCR}$. Using assumption H.10, we can equivalently write $v_t^{(j)}(0)(t)$ as

$$
\begin{aligned}
v_t^{(j)}(0)(t) = &\int_{-\infty}^{t} [U(t,\mu)\chi_0 - \Phi_t Y_c(\mu) - \Psi_t Y_u(\mu)] \hat{H}_1^{(j)}(\mu) d\mu \\
&- \int_t^\infty \Psi_t Y_u(\mu) \hat{H}_1^{(j)}(\mu) d\mu
\end{aligned}
$$

$$+\sum_{-\infty}^{t}[U(t,t_i)\chi_0-\Phi_tY_c(t_i)-\Psi_tY_u(t_i)]\hat{H}_2^{(j)}(t_i)dt_i$$
$$-\sum_{t}^{\infty}\Psi_tY_u(t_i)\hat{H}_2^{(j)}(t_i)dt_i. \tag{I.5.36}$$

At this stage, we remark that Theorem I.5.6.1 implies $v_t^{(i)}(0)(\cdot)[z_1,\dots,z_i]\in\mathcal{PC}^\infty$ for $i=1,\dots,j-1$, while Φ_t is pointwise differentiable from the right by its very definition. With these details and assumption H.3, $\mu\mapsto\hat{H}_1^{(j)}(\mu)[z_1,\dots,z_j]$ is an element of $\mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$ for every tuple $z_1,\dots,z_j\in\mathbb{R}^p$. From assumption H.10, $v_t^{(j)}(0)(t)$ is pointwise differentiable from the right if and only if the limit

$$\lim_{\epsilon\to0^+}\frac{1}{\epsilon}\int_t^{t+\epsilon}U(t+\epsilon,\mu)\chi_0\hat{H}_1^{(j)}(\mu)d\mu$$

exists pointwise. From Eq. (I.2.15) and Lemma I.2.3.5, we can equivalently write the integral above in terms of the fundamental matrix solution:

$$\int_t^{t+\epsilon}U(t+\epsilon,\mu)\chi_0\hat{H}_1^{(j)}(\mu)d\mu$$
$$=\int_t^{t+\epsilon}\chi_{(-\infty,t+\epsilon+\theta]}(\mu)\left(I+\int_\mu^{t+\epsilon+\theta}L(\zeta)V_\zeta(\cdot,\mu)d\zeta\right)\hat{H}_1^{(j)}(\mu)d\mu.$$

If $\theta<0$, then the integrand vanishes when $\epsilon<-\theta$. Since $\mu\mapsto\hat{H}_1^{(j)}(\mu)$ is continuous from the right, we conclude that

$$\lim_{\epsilon\to0^+}\frac{1}{\epsilon}\int_t^{t+\epsilon}U(t,\mu)\chi_0\hat{H}_1^{(j)}(\mu)d\mu=\chi_0\hat{H}_1^{(j)}(t),$$

so that $t\mapsto v_t^{(j)}(0)(t)$ is differentiable from the right (for θ fixed), as claimed. The proof of existence of limits on the left is similar and omitted.

To get the analogous result for θ, it is worth recalling that from the fixed-point formulation, $v_t^{(j)}(0)$ is a j-linear map from $\mathbb{R}^p$ to $\mathcal{PC}^{\eta,t}$. As a consequence, for all $t\in\mathbb{R}$, $\theta\in[-r,0]$ and $z_1,\dots,z_j\in\mathbb{R}^p$ the equation

$$v_t^{(j)}(0)(t)[z_1,\dots,z_j](\theta)=v_t^{(j)}(0)(t+\theta)[z_1,\dots,z_j](0)$$

is satisfied. The analogous differentiability and limit results for θ therefore follow from those of t, completing the proof.

I.5.7.5 The Hyperbolic Part Is Pointwise $PC^{1,m}$-Regular at Zero

Later we will need to also consider the Taylor expansions of the *hyperbolic part* $H:\mathbb{R}\times\mathbb{R}^p\to\mathcal{RCR}$ of the centre manifold in terms of a basis array Φ, defined by

$$H(t,z) = (I - P_c(t))C(t,z). \tag{I.5.37}$$

The hyperbolic part is guaranteed to be C^m-smooth in z, since $(I - P_c(t))$ is linear. To show that it is pointwise $PC^{1,m}$-regular at zero, we notice that $H(t,z) = h_t(z)(t)$, where $h_t(z)$ can be written as

$$h_t(z) = (I - P_c(t))\Phi_{(\cdot)}z + \mathcal{K}_* \circ R_{\delta,t}(v_t(z))$$

in $\mathcal{PC}^0$. However, since $(I - P_c(t))$ is uniformly bounded, $\mathcal{K}_* = (I - P_c(t))\mathcal{K}_t^\eta$ is well-defined as a map from η-bounded inhomogeneities into $\mathcal{PC}^{\eta,t}$. Setting $z = 0$, it follows that

$$\begin{aligned} h_t(0) &= 0, \\ h_t^{(1)}(0)(t) &= 0 \\ h_t^{(j)}(0)(t) &= \mathcal{K}_* \circ H^{(j)}(0, \Phi_{(\cdot)}, v_t^{(2)}(0)(\cdot), \dots, v_t^{(j-1)}(0)(\cdot)). \end{aligned}$$

On the other hand, for $z \neq 0$ we have

$$h_t(z)(t) = \mathcal{K}_* \circ R_{\delta,t}(v_t(z))(t).$$

By the same argument as in the proof of Theorem I.5.7.1, we can make the following conclusion.

Corollary I.5.7.1. *The hyperbolic part $H(t,z) = (I - P_c(t))C(t,z)$ of the centre manifold in terms of the basis array Φ is pointwise $PC^{1,m}$-regular at zero. Moreover, if $\theta \in \mathcal{RCR}(\mathbb{R}, [-r, 0])$, then $t \mapsto H(t,z)(\theta(t))$ is continuous from the right and has limits on the left for all $z \in \mathbb{R}^p$, and $z \mapsto H(t,z)$ is Lipschitz continuous uniformly for $t \in \mathbb{R}$.*

I.5.7.6 Uniqueness of the Taylor Coefficients

Theorem I.5.7.1 guarantees that the coefficients in the Taylor expansion

$$C(t,z) = DC(t,0)z + \frac{1}{2}D^2C(t,0)[z,z] + \cdots + \frac{1}{m!}C^mC(t,0)[z,\dots,z] + O(||z||^{m+1})$$

are pointwise differentiable from the right and have limits on the left. However, the centre manifold $\mathcal{C} : \mathcal{RCR}_c \to \mathcal{RCR}$ used to define the representation in terms of the basis array Φ depends non-canonically on the choice of cutoff function used to define the substitution operator $R_{\delta,t}$. However, this cutoff function does not actually factor into the coefficients $D^jC(t,0)$. Indeed, each of $\mu \mapsto v_t^{(j)}(0)(\mu)$ is a sum of improper integrals and convergent series that depend only the lower-order terms $v_t^{(i)}(0)(\cdot)$ for $i < j$—see equation (I.5.36)—and is independent of the cutoff function. By induction, we can see from (I.5.32)–(I.5.34) that, in fact, none of these lower-order terms depend on the cutoff function. The same arguments apply to the hyperbolic part. Since this is the only non-canonical element in the definition of the centre manifold (indeed, the renorming is only relevant outside of a small neighbourhood of $0 \in \mathcal{RCR}$ and so does not affect Taylor expansions), the following corollary is proven.

Corollary I.5.7.2. *Let Φ be a basis array for $\mathcal{RCR}_c$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct centre manifolds, and let C_1 and C_2, respectively, be the centre manifolds with respect to the basis array Φ. Also, let H_1 and H_2 be the respective hyperbolic parts. Then, for $j = 1, \dots, m$, we have $D^j C_1(t,0) = D^j C_2(t,0)$ and $D^j H_1(t,0) = D^j H_2(t,0)$. That is, the Maclaurin series expansion of the centre manifold in terms of the basis array Φ is unique, as is that of the hyperbolic part.*

I.5.7.7 A Discussion on the Regularity of the Matrices $t \mapsto Y_j(t)$

Hypothesis H.10 introduces a technical assumption on the matrices appearing in the decomposition (I.5.35). It is our goal in this section to formally demonstrate that there is reason to suspect that this hypothesis holds generally, although proving this result would likely be difficult. We will consider only $t \mapsto Y_c(t)$, since the discussion for $t \mapsto Y_u(t)$ is the same.

When the linearization "has no delayed terms" and is spectrally separated as a finite-dimensional system, $t \mapsto Y_c(t)$ is automatically continuous from the right with limits on the left. Abstractly, having no delayed terms means that the functionals defining the linearization have support in the subspace $\mathcal{RCR}^0 = \{\chi_0 \xi : \xi \in \mathbb{R}^n\}$. Let us prove the claim. Let $X(t,s)$ denote the Cauchy matrix associated to the linearization

$$\dot{x} = L(t)x(t), \qquad t \neq t_k \tag{I.5.38}$$

$$\Delta x = B(k)x(t^-), \qquad t = t_k. \tag{I.5.39}$$

The projection $P_c(t)$ onto the associated centre fibre bundle satisfies the equation

$$X(t,s)P_c(s) = P_c(t)X(t,s)$$

for all $t \geq s$. However, since $X^{-1}(t,s)$ exists for all $t,s \in \mathbb{R}$—see Chap. II.2 or the monograph [9] for the relevant background on linear impulsive differential equations in finite-dimensional spaces—we have $P_c(t) = X(t,0)P_c(0)X^{-1}(t,0)$ for all $t \in \mathbb{R}$. Moreover, $t \mapsto X(t,0)$ is continuous from the right and has limits on the left, from which it follows that the same is true for $P_c(t)$. Similarly, each of $P_s(t)$ and $P_u(t)$ is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^{n\times n})$. If we write $P_c(t) = \Phi(t)Y_c(t)$ for $\Phi(t) = X(t,0)\Phi(0)$ a matrix whose columns form a basis for $\mathcal{RCR}_c(t)$, then the observation that the columns of $\Phi(t)$ are linearly independent implies we can write

$$Y_c(t) = \Phi^+(t)P_c(t),$$

where $\Phi^+(t)$ is the left-inverse of $\Phi(t)$. Since the rank of $t \mapsto \Phi(t)$ is constant, $t \mapsto \Phi^+(t)$ is continuous from the right and has limits on the left. It follows that $t \mapsto Y_c(t)$ is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^{p\times n})$. If (I.5.38)–(I.5.39) is now considered as an impulsive RFDE with phase space $\mathcal{RCR}([-r,0],\mathbb{R}^n)$ for some $r > 0$, then we can write

$$U(t,s)\phi(\theta) = \begin{cases} X(t+\theta,s)\phi(0), & t+\theta \geq s \\ \phi(t+\theta-s), & t+\theta < s. \end{cases}$$

If one defines $\mathcal{P}_j(t) : \mathcal{RCR} \to \mathcal{RCR}$ by

$$\mathcal{P}_j(t)\phi(\theta) = X(t+\theta,t)P_j(t)\phi(0),$$

one can verify directly $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ is spectrally separated with the triple of projectors $(\mathcal{P}_s, \mathcal{P}_c, \mathcal{P}_u)$. But then,

$$\mathcal{P}_c(t)\chi_0(\theta) = X(t+\theta,t)P_c(t) = X(t+\theta,t)\Phi(t)Y_c(t) = \Phi(t+\theta)Y_c(t) = \Phi_t(\theta)Y_c(t).$$

We already know that $t \mapsto Y_c(t)$ is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^{p\times n})$, and since this same matrix satisfies the decomposition $\mathcal{P}_c(t)\chi_0 = \Phi_t Y_c(t)$, we are done.

In the general case, the situation is far more subtle since the projector $t \mapsto P_c(t)$ is not even pointwise continuous. Consider the periodic case. $\mathcal{RCR}_c(t)$ is the invariant subspace of the monodromy operator V_t that contains, in particular, nontrivial elements $\phi \in \mathcal{RCR}(t)$ with the property that $||V_t\phi|| = ||\phi||$. However, since $V_t = U(t+T,t)$, any such element of $\mathcal{RCR}_c(t)$ will have discontinuities on the set $D_t = \{\theta \in [-r,0] : t+\theta \in \{t_k : k \in \mathbb{Z}\}\}$. Generally, D_t is nonempty and nonconstant; the discontinuities move by translation to the left as t increases. Consequently, the discontinuities of $P_c(t)\phi$ for fixed $\phi \in \mathcal{RCR}$ are nonconstant in t, so $t \mapsto ||P_c(t)\phi||$ is generally discontinuous (from the right and left) at any $t \in \mathbb{R}$ such that D_t is nonempty. As such, one cannot take advantage of any regularity properties of $t \mapsto P_c(t)$ even in the pointwise sense.

I.5.8 Comments

Some of the content of this chapter appears in the two papers *Smooth centre manifolds for impulsive delay differential equations* and *Computation of centre manifolds and some codimension-one bifurcations for impulsive delay differential equations* by Church and Liu, published in Journal of Differential Equations [31, 33] in 2018 and 2019, respectively. Some improvements have been made in the present monograph, however. For example, in the first of the two publications, smoothness of the centre manifold was only proven in the periodic case. The second of the two publications considers only discrete delays.

Some early results on the existence of invariant manifolds for impulsive differential equations in the infinite-dimensional context are due to Bainov et al. [11, 12], where they prove the existence of integral manifolds (subsets of the phase space consisting of entire solutions) identified as perturbations of linear invariant subspaces. Centre manifolds are not considered, however, and in this context, the linear dynamics on a given Banach space are assumed to be reversible, so in particular the restriction to the stable subspace defines an all-time process. Exponential trichotomy is assumed on the dynamics of the linear part, which is similar to what we have assumed in this chapter. Aside from these and related investigations into stable manifolds under weaker notions of hyperbolicity than exponential dichotomy—see [16] and the references cited therein—and some recent results on Lipschitz-smooth stable manifolds for impulsive delay differential equations [8], there has not been much investigation in this area.

Our proof of smoothness of the centre manifold uses formal differentiation in conjunction with Lemma I.5.6.1 on fixed points of contractions on a scale of Banach spaces. The latter technique as it applies to proving the smoothness of centre manifolds was introduced in 1987 by Vanderbauwhede and Van Gils [144]. See [44, 70, 71, 107] for a few other applications. Regularity in time of the coefficients in the Taylor expansion of the centre manifold for nonautonomous systems seems to not be as well-studied. See Theorem A.1 of [116] by Potzschë and Rasmussen for a regularity result for invariant manifolds for nonautonomous ordinary differential equations, and the references cited therein for relevant proof methodologies.

Chapter I.6

Computational Aspects of Centre Manifolds

In this chapter we will once again be studying the semilinear equation

$$\dot{x} = L(t)x_t + f(t, x_t), \qquad t \neq t_k \tag{I.6.1}$$

$$\Delta x = B(k)x_{t^-} + g(k, x_{t^-}), \qquad t = t_k. \tag{I.6.2}$$

We introduce some important conditions:

C.1 The linearization is periodic with period T and c impulses per period. That is, $L(t+T) = L(t)$ for all $t \in \mathbb{R}$, $B(k+c) = B(k)$ and $t_{k+c} = t_k + T$ for all $k \in \mathbb{Z}$.

C.2 Conditions H.1–H.3, H.5–H.6, and H.10 are satisfied.

C.3 The sequences of functionals $B(k)$ and $g(k, \cdot)$ satisfy the overlap condition.

Conditions C.1 and C.2 are strong enough to guarantee that the (local) centre manifold exists and is smooth, and in terms of the basis array Φ, it is pointwise $PC^{1,m}$-regular at zero. Condition C.3 will be needed to ensure that the dynamics on the centre manifold are well-defined. We will assume them throughout this chapter.

I.6.1 Euclidean Space Representation

The centre manifold

$$C : \mathbb{R} \times \mathbb{R}^p \to \mathcal{RCR}$$

K. E. M. Church, X. Liu, *Bifurcation Theory of Impulsive Dynamical Systems*, IFSR International Series in Systems Science and Systems Engineering 34, https://doi.org/10.1007/978-3-030-64533-5_6

in terms of Φ has range in $\mathcal{RCR}_c \oplus \mathcal{RCR}_s$. If $y : \mathbb{R} \to \mathcal{RCR}$ is a solution in the centre manifold—that is, $y(t) = S(t,s)y(s)$ for all $t \geq s$ with $y(t) \in \mathcal{W}_c(t)$—then we can use property 2 of Theorem I.5.3.1 and part 2 of Lemma I.5.1.3 to write it as

$$y(t) = \Phi_t z(t) + H(t, z(t)), \tag{I.6.3}$$

where $\Phi_t z(t) = P_c(t)y(t)$ for some $z : \mathbb{R} \to \mathbb{R}^p$. Recall $H(t,z) = (I - P_c(t))\mathcal{C}(t, \Phi_t z)$. By definition of the basis array Φ and the observation that any mild solution such as y defined for all time must in fact be the history function of some element of $\mathcal{RCR}^1(\mathbb{R},\mathbb{R}^n)$—see Lemma I.4.1.1—we must have $z \in \mathcal{RCR}^1(\mathbb{R},\mathbb{R}^p)$. The present chapter is essentially an investigation into how the above representation of solutions in the centre manifold can help us obtain a concrete version of the dynamics restricted to the centre manifold.

We need to introduce some extra notation. The set $M^{n\times m}(\mathbb{R}^k)$ denotes the set of $n \times m$ matrices with entries in the vector space $\mathbb{R}^k$. If $A \in M^{n\times m}(\mathbb{R}^k)$, $A_{i,j}$ denotes the entry in its ith row and jth column. The notation $[A]_{a:b}$ denotes the $(b-a+1)\times m$ matrix whose rows coincide with rows a through b of A.

For a j-dimensional multi-index $\xi = (\xi_1, \ldots, \xi_j)$, where $\xi_i \in \mathbb{N}$, we define $|\xi| = \sum_i \xi_i$. For $u \in \mathbb{R}^j$ and a j-dimensional multi-index ξ with $|\xi| = m$, the ξ power of u is $u^\xi = u_1^{\xi_1} \cdots u_j^{\xi_j}$. If X is a vector space and $U \in X^j$, we similarly define $U^\xi \in X^{|\xi|}$ by

$$U^\xi = (U_1, \ldots, U_1, U_2, \ldots, U_2, \ldots, U_j \ldots, U_j),$$

where the factor U_i appears ξ_i times. If $u \in X$ and $m \in \mathbb{N}$, we define $u^m \in X^m$ by $u^m = (u, \ldots, u)$.

For a vector multi-index $\xi = (\xi_1, \ldots, \xi_j)$, where each $\xi_i \in \{e_1', \ldots, e_k'\}$ for $\{e_i' : i = 1, \ldots, k\}$ the standard ordered basis of $\mathbb{R}^{k*}$, we write $|\xi| = j$ and define $(u_1 \cdots u_j)^\xi$ for $u_i \in \mathbb{R}^k$ as follows:

$$(u_1 \cdots u_j)^\xi = (\xi_1 u_1) \cdots (\xi_j u_j).$$

For vectors in $\mathbb{R}^n$ written in component form, $(u_1, \ldots, u_n) \cdot (v_1, \ldots, v_n) = \sum_i u_i v_i$ denotes the standard inner product.

If $A \in \mathbb{R}^{m\times n}$ and $B \in M^{n\times k}(\mathbb{R}^\ell)$, we define the overloaded product $A * B \in M^{m\times k}(\mathbb{R}^\ell)$ by the equation

$$[A * B]_{i,j} = \sum_{u=1}^{n} A_{i,u} B_{u,j}. \tag{I.6.4}$$

It is readily verified that if $A \in \mathbb{R}^{m\times m}$ is invertible, then $A * B = C$ if and only if $B = A^{-1} * C$. Moreover, $*$ satisfies the Leibniz's law

$$\frac{d}{dt} A(t) * B(t) = \left(\frac{d}{dt} A(t)\right) * B(t) + A(t) * \left(\frac{d}{dt} B(t)\right)$$

whenever $t \mapsto A(t)$ and $t \mapsto B(t)$ are differentiable. Clearly, when $\ell = 1$, the overloaded product reduces to the standard matrix product.

I.6.1.1 Definition and Taylor Expansion

The first task is to replace the hyperbolic part of the centre manifold with something even more concrete. First, we recall from the Floquet Theorem I.3.2.1 that we can write the basis array equivalently as

$$\Phi_t = \alpha(t)e^{tW}\Phi_0,$$

for $\alpha(t) : \mathcal{RCR}_c(0) \to \mathcal{RCR}_c(t)$ T-periodic and pointwise differentiable from the right with limits on the left, and $W \in \mathcal{L}(\mathcal{RCR}_c(0))$. Let $\Lambda \in \mathbb{C}^{p\times p}$ denote the matrix associated with W with respect to the ordered basis consisting of the columns of the array Φ_0, so that

$$e^{tW}\Phi_0 = Re^{t\Lambda}$$

for some $R \in \mathcal{L}(\mathbb{R}^p, \mathcal{RCR}_c(0))$. Then, the basis array satisfies the Floquet decomposition

$$\Phi_t = Q_t e^{t\Lambda}, \tag{I.6.5}$$

where $Q_t = \alpha(t)R$. By construction, we can identify $Q_t(\theta) = Q(t + \theta)$ for some $Q \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^{n\times p})$ that is T-periodic.

Remark I.6.1.1. *We can guarantee that Λ is real by instead taking $\alpha(t) : \mathcal{RCR}_c(0) \to \mathcal{RCR}_c(t)$ to be $2T$-periodic. The reason this can be done is because we know that the spaces $\mathcal{RCR}_c(kT)$ are isomorphic for $k \in \mathbb{Z}$ via the monodromy operator V_0, by Theorem I.3.1.2. This then implies that there is an invertible matrix M such that $\Phi_T = \Phi_0 M$. But then,*

$$\Phi_{2T} = V_0\Phi_0 M = \Phi_0 M^2.$$

Using (I.6.5), it follows that

$$\Lambda = \frac{1}{2T}\log(M^2),$$

which is guaranteed to have a real logarithm. Moreover, if $\log(M)$ is real, then $\frac{1}{T}\log(M)$ still coincides with Λ as defined above. If the basis matrix Φ_0 is taken to be real, this will result in Q_t being real as well.

Let us introduce a change of variables. Starting from Eq. (I.6.3), we let $u(t) = e^{t\Lambda}z(t)$ so that $\Phi_t z(t) = Q_t u(t)$. We then define $h : \mathbb{R}\times\mathbb{R}^n \times [-r, 0] \to \mathbb{R}^n$ by the equation

$$h(t, u, \theta) = \mathcal{H}(t, Q_t u)(\theta). \tag{I.6.6}$$

With this transformation completed, Eq. (I.6.3) becomes

$$y(t) = Q_t u(t) + h(t, u(t), \cdot). \tag{I.6.7}$$

The function $h : \mathbb{R} \times \mathbb{R}^p \times [-r, 0] \to \mathbb{R}^n$ will be referred to as the *Euclidean space representation of the centre manifold*. Introduce the left-limit in the first variable

$$h(t^-, u, \theta) = \lim_{\epsilon \to 0^-} h(t + \epsilon, u, \theta).$$

We have the following approximation theorem.

Theorem I.6.1.1. *The Euclidean representation $h : \mathbb{R} \times \mathbb{R}^p \times [-r, 0] \to \mathbb{R}^n$ of any centre manifold enjoys the following properties:*

1. *h admits a Taylor expansion near $u = 0$:*

$$h(t, u, \theta) = \frac{1}{2!} h_2(t, \theta) u^2 + \frac{1}{3!} h_3(t, \theta) u^3 + \cdots + \frac{1}{m!} h_m(t, \theta) u^m + O(u^{m+1}),$$

   with $h_i(t, \theta) = D_2^i h(t, 0, \theta)$, and this Taylor expansion is unique and does not depend on the choice of cutoff function.

2. *$t \mapsto h_i(t, \cdot)$ is periodic for $i = 2, \ldots, m$, and each of $t \mapsto h_i(t, \theta)$ and $\theta \mapsto h_i(t, \theta)$ is differentiable from the right with limits on the left.*

3. *$P_c(t) h_i(t, \cdot) = 0$ for $i = 2, \ldots, m$.*

4. *If $u \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ and $\theta \in [-r, 0]$, then we have $\lim_{s \to 0^-} h(t + s, u(t + s), \theta) = h(t^-, u(t^-), \theta)$. Also, $t \mapsto h(t, u(t), \theta(t))$ is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ whenever $\theta \in \mathcal{RCR}(\mathbb{R}, [-r, 0])$.*

Proof. The Taylor expansion is a consequence of Theorem I.5.2.1 and the definition of the Euclidean space representation of the centre manifold. Next, $t \mapsto \mathcal{C}(t, \phi)$ is periodic, from which it follows that the same is true of the differentials $D_2^j \mathcal{C}(t, 0)$. Since $h(t, u, \cdot) = (I - P_c(t)) \mathcal{C}(t, Q_t u)$, and each of $t \mapsto Q_t$ and $t \mapsto P_c(t)$ is also periodic, the same is true for $t \mapsto h(t, \cdot, \cdot)$ and its differentials $t \mapsto D_2^j h(t, \cdot, \cdot)$. For the projection, linearity of the differential implies that $D_2^j H(t, 0) = (I - P_c(t)) D_2^j C(t, 0)$, so that $P_c(t) D_2^j H(t, 0) = 0$. The same is true for h by its definition. As for the limit relation, the fundamental theorem of calculus and the triangle inequality can be used to obtain

$$\begin{aligned} ||h(t+s, &u(t+s), \theta(t+s)) - h(t^-, u(t^-), \theta(t^-))|| \\ &\leq L||u(t+s) - u(t^-)|| + ||h(t+s, u(t^-), \theta(t+s)) - h(t^-, u(t^-), \theta(t+s))|| \\ &\quad + ||h(t^-, u(t^-), \theta(t+s)) - h(t^-, u(t^-), \theta(t^-))||, \end{aligned}$$

where L is a Lipschitz constant for $x \mapsto D_2 h(t, x, \cdot)$ valid uniformly for all t. Note that this Lipschitz constant is guaranteed to exist by Theorem I.5.2.1, the uniform boundedness of the projectors and the periodicity of $t \mapsto Q_t$. Since $u(t+s) \to u(t^-)$, the first of the two terms converges to zero as $s \to 0^-$. As for the second, since $t \mapsto H(t, z)(\theta(t))$ has limits on the left, the same is true of $t \mapsto h(t, u(t^-), \theta(t))$ for $u(t^-)$ fixed. For the third term, $\theta \mapsto h(t^-, u(t^-), \theta) \in \mathcal{RCR}$ gives the limit. This proves all assertions concerning limits from the left. Limits from the right are proven analogously. □

I.6.1.2 Dynamics on the Centre Manifold in Euclidean Space

Recall that the centre fibre bundle component $w(t) = P_c(t)y(t)$ satisfies the integral equation (I.5.10) whenever $y : \mathbb{R} \to \mathcal{RCR}$ is a mild solution with $y(t) \in \mathcal{W}_c(t)$. From (I.6.7), it follows that $w(t) = \Phi_t z(t)$. But also, from Lemma I.4.1.1, we can formally identify $y : \mathbb{R} \to \mathcal{RCR}$ with a right-continuous regulated function $\tilde{y} : \mathbb{R} \to \mathbb{R}^n$ by way of $y(t) = \tilde{y}_t$. Substituting into the integral equation from Lemma I.5.4.1, using assumption H.10 to write $P_c(s)\chi_0 = \Phi_s Y_c(s)$ and recalling that y is a *small* solution, so we can ignore the impact of the cutoff function on the nonlinearity, this gives after some simplification

$$\Phi_t z(t) = \Phi_t z(s) + \Phi_t \int_s^t Y_c(\mu) f(\mu, y_\mu) d\mu + \Phi_t \sum_{s<t_i\leq t} Y_c(t_i) g(i, y_{t_i^-}),$$

where we recall by Theorem I.5.3.1 that $\mathcal{C}(\mu, w(\mu)) = \mathcal{C}(\mu, P_c(\mu)y(\mu)) = y(\mu)$. Since the columns of Φ_t form a basis for $\mathcal{RCR}_c(t)$, we can apply the coordinate map defined by $\phi_i(t) \mapsto e_i$ to eliminate the basis array Φ_t from each side. The result is the following integral equation in $\mathbb{R}^p$:

$$z(t) = z(s) + \int_s^t Y_c(\mu) f(\mu, y_\mu) d\mu + \sum_{s<t_i\leq t} Y_c(t_i) g(i, y_{t_i^-}). \qquad \text{(I.6.8)}$$

It is here that our derivation becomes a bit subtle. To motivate the next step, we prove a result concerning the overlap condition, mild solutions and regulated left-limit histories.

Lemma I.6.1.1. *Suppose $F(k, \cdot) : \mathcal{RCR} \to \mathbb{R}^n$ satisfies the overlap condition. If $x \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ defines a mild solution $t \mapsto x_t$, then $F(k, x_{t_k^-})$ is well-defined in the sense that there exists a continuous extension $F(k, \cdot) : \mathcal{RCR} \oplus D(k) \to \mathbb{R}^n$, where*

$$D(k) = \{\phi \in \mathcal{G}([-r, 0], \mathbb{R}^n) : \phi(\theta) = 0 \textit{ for } \theta \neq t_j - t_k \in [-r, 0),\ t < k\},$$

we have $x_{t_k}^- \in \mathcal{RCR} \oplus D(k)$, the subspace $\mathcal{RCR} \oplus D(k)$ is closed in $\mathcal{G}([-r, 0], \mathbb{R}^n)$ and $F(k, x_{t_k^-}) = F(k, x_{t_k}^-)$.

Proof. First, observe that $D(k)$ is finite-dimensional because the sequence of impulses is unbounded both forward and backward in time, so the set $\{t_j - t_k \in [-r, 0) : j < k\}$ is finite. If the cardinality of this set is $m(k)$, then $\dim D(k) = n \cdot m(k)$. It follows that $\mathcal{RCR} \oplus D(k)$ is closed. We define the extension of $F(k, \cdot)$ to $\mathcal{RCR} \oplus D(k)$ by $F(k, \phi) = F(k, \phi_{\mathcal{RCR}})$, where $\phi = \phi_{\mathcal{RCR}} + \phi_{D(k)}$ according to the direct sum decomposition. Moreover, one can verify (using the overlap condition and continuity of F on $\mathcal{RCR}$) that if $\phi_n \in \mathcal{RCR} \oplus D(k)$ satisfies $\lim_{n\to\infty} \phi_n = \phi \in \mathcal{RCR} \oplus D(k)$, then $\lim_{n\to\infty} F(k, \phi_n) = F(k, \phi)$. That is, the extension is indeed continuous.

By Lemma I.4.1.1, a solution $x \in \mathcal{RCR}(\mathbb{R}, \mathbb{R}^n)$ is classical, so its discontinuities are a subset of $\{t_k : k \in \mathbb{Z}\}$. As a consequence, we can write

$$x_{t_k^-} - \sum_{t_k - t_j \in [-r,0]} \chi_{t_k - t_j} \Delta x(t_k - t_j) = x_{t_k}^-.$$

We can now apply $F(k,\cdot)$ to both sides, and the overlap condition together with the extension property to $\mathcal{RCR} \oplus D(k)$ implies $F(k, x_{t_k^-}) = F(k, x_{t_k}^-)$. In particular, we have $x_{t_k}^- \in \mathcal{RCR} \oplus D(k)$. □

To see why the above lemma is so helpful, let us take right-derivatives of both sides of the integral equation (I.6.8) and check finite differences across the jump times t_k. The result is the impulsive differential equation

$$\dot{z} = Y_c(t) f(t, y_t), \qquad t \neq t_k \tag{I.6.9}$$

$$\Delta z = Y_c(t_k) g(k, y_{t_k^-}), \qquad t = t_k, \tag{I.6.10}$$

where the derivative operator is understood as the right-hand derivative $\dot{x} = \frac{d^+}{dt} x(t)$. In (I.6.9), we can replace y_t with $\mathcal{C}(t, w(t)) = \Phi_t z(t) + h(t, z(t), \cdot)$ and get a well-behaved ordinary differential equation. In the second equation we have to take a pointwise left-limit. This is a problem, however, because the pointwise left-limit is a limit in the θ variable at a single point. In particular,

$$y_{t^-}(\theta) = \begin{cases} \Phi_t(\theta) z(t) + H(t, z(t), \theta), & \theta < 0 \\ \Phi_{t^-}(0) z(t^-) + \lim_{s \to 0^-} H(t+s, z(t+s), 0), & \theta = 0. \end{cases}$$

Substituting this into the jump condition (I.6.10) would result in an implicit equation for $z(t_k)$, which makes the impulsive differential equation (I.6.9)–(I.6.10) somewhat difficult to work with. Lemma I.6.1.1 solves this problem; if $g(k,\cdot) : \mathcal{RCR} \to \mathbb{R}^n$ satisfies the overlap condition, we can replace $y_{t_k^-}$ inside $g(k,\cdot)$ with the regulated left-limit $y_{t_k}^-$. In view of Theorem I.6.1.1, y_t^- can be written as

$$y_t^- = Q_t^- u(t^-) + h(t^-, u(t^-), \cdot). \tag{I.6.11}$$

Substituting (I.6.11) into (I.6.10) and completing the change of variables $u(t) = e^{t\Lambda} z(t)$ described in Sect. I.6.1.1, the following theorem is proven.

Theorem I.6.1.2. *Suppose that $g(k,\cdot) : \mathcal{RCR} \to \mathbb{R}^n$ satisfies the overlap condition and conditions C.1 and C.2 are satisfied. Then, the abstract dynamics on the centre manifold described by the integral equation (I.5.10) in the variable $w(t) \in \mathcal{RCR}$ are equivalent under the time-periodic change of variables $w(t) = Q_t u(t)$ for $u \in \mathbb{R}^p$ to the ordinary impulsive differential equation*

$$\frac{d^+}{dt} u(t) = \Lambda u(t) + e^{t\Lambda} Y_c(t) f(t, Q_t u(t) + h(t, u(t), \cdot)), \qquad t \neq t_k \tag{I.6.12}$$

$$\Delta u = e^{t\Lambda} Y_c(t) g(k, Q_t^- u(t^-) + h(t^-, u(t^-), \cdot)), \qquad t = t_k. \tag{I.6.13}$$

Remark I.6.1.2. *$t \mapsto e^{t\Lambda}Y_c(t)$ is periodic. To see this, remark that from the definition of Y_c and the periodicity of the projectors P_c,*

$$\begin{aligned}\Phi_t Y_c(t) &= P_c(t)\chi_0 = P_c(t+T)\chi_0 = \Phi_{t+T}Y_c(t+T) \\ &= Q_{t+T}e^{(t+T)\Lambda}Y_c(t+T) = \Phi_t e^{T\Lambda}Y_c(t+T),\end{aligned}$$

from which Φ_t being a basis for $\mathcal{RCR}_c(t)$ implies the equality $Y_c(t) = e^{T\Lambda}Y_c(t+T)$. As a consequence, $t \mapsto e^{t\Lambda}Y_c(t) := \overline{Y}_c(t)$ satisfies

$$\overline{Y}_c(t+T) = e^{(t+T)\Lambda}Y_c(t+T) = e^{(t+T)\Lambda)}e^{-T\Lambda}Y_c(t) = e^{t\Lambda}Y_c(t) = \overline{Y}_c(t),$$

so it is periodic as claimed. If the nonlinear terms are also periodic (with the same period T and $g(k+c,\cdot) = g(k,\cdot)$ for all $k \in \mathbb{Z}$), then the same is true of the impulsive differential equation (I.6.12)–(I.6.12).

By Theorem I.5.3.1, the ordinary impulsive differential equation (I.6.12)–(I.6.13) completely characterizes the dynamics of all small solutions. From the perspective of bifurcations, this is quite useful because we can study a concrete impulsive differential equation in $\mathbb{R}^p$ to detect the birth or destruction of periodic solutions or other invariant structures. When $\mathcal{RCR}_u(t)$ is empty, stability transitions can be analyzed. If one needs only terms of order two (e.g. saddle-node bifurcation), then the center manifold does not need to be calculated. In this case, the dynamics on the center manifold are characterized by the following corollary.

Corollary I.6.1.1. *Under the hypotheses of Theorem I.6.1.2, the dynamics on the centre manifold to quadratic order are equivalent by a time-periodic change of variables to those of the impulsive differential equation*

$$\dot{u} = \Lambda u + e^{t\Lambda}Y_c(t)\left[\frac{1}{2}D^2 f(t,0)[Q_t u]^2\right] + O(u^3), \qquad t \neq t_k \tag{I.6.14}$$

$$\Delta u = e^{t\Lambda}Y_c(t)\left[\frac{1}{2}D^2 g(k,0)[Q_t^- u(t^-)]^2\right] + O(u^3), \qquad t = t_k. \tag{I.6.15}$$

For Hopf bifurcation conditions, for example, we require the reduced dynamics equations to be explicit to cubic order. Recall from Theorem I.6.1.1 that we can write

$$h(t,u,\theta) = \frac{1}{2!}h_2(t,\theta)u^2 + \frac{1}{3!}h_3(t,\theta)u^3 + \cdots$$

for symmetric multilinear mappings $h_i(t,\theta) : (\mathbb{R}^p)^i \to \mathbb{R}^n$ defined by $h_i(t,\theta) = D_2^i h(t,0,\theta)$. It is then easily verified that to cubic order, the reduced dynamics are

$$\begin{aligned}&\dot{u} = \Lambda u + e^{\Lambda t}Y_c(t) \\ &\left[\frac{1}{2!}D^2 f(t,0)[Q_t u]^2 + \frac{1}{3!}\left(D^3 f(t,0)[Q_t u]^3 + 3D^2 f(t,0)[Q_t u, h_2(t,\theta)u^2]\right)\right], \quad t \neq t_k\end{aligned} \tag{I.6.16}$$

$$\Delta u = e^{\Lambda t}Y_c(t)$$
$$\left[\frac{1}{2!}D^2g(k,0)[Q_t^- u]^2 + \frac{1}{3!}\left(D^3g(k,0)[Q_t^- u]^3 + 3D^2g(k,0)[Q_t^- u, h_2(t^-,\theta)u^2]\right)\right], ct = t_k. \tag{I.6.17}$$

I.6.1.3 An Impulsive Evolution Equation and Boundary Conditions

In the same way that the centre manifold associated with a nonhyperbolic equilibrium of an ordinary differential equation satisfies a nonlinear partial differential equation, the centre manifold of an impulsive RFDE satisfies a nonlinear impulsive evolution equation. This is what we prove in this section.

At this stage, we should define a pair of linear operators that are in a certain sense "generators" of the evolution family $U(t,s) : \mathcal{RCR} \to \mathcal{RCR}$. They are

$$\mathcal{L}(t)\phi = \begin{cases} B(t)\phi, & \theta = 0 \\ d^+\phi(\theta), & \theta < 0 \end{cases}, \qquad \mathcal{J}(k)\phi(\theta) = \begin{cases} B(k)\phi, & \theta = 0 \\ \phi(\theta^+) - \phi(\theta), & \theta < 0. \end{cases} \tag{I.6.18}$$

We introduced these generators in Theorem I.5.4.1, but it is worth recalling them now. Also, we define $\Delta_\theta^+ : \mathcal{RCR} \to \mathcal{G}([-r,0),\mathbb{R}^n)$ by $\Delta_\theta^+\phi(\theta) = \phi(\theta^+) - \phi(\theta)$. This operator permits a decomposition of $\mathcal{J}(k)$ into

$$\mathcal{J}(k) = \chi_0 B(k) + \chi_{[-r,0)}\Delta_\theta^+.$$

Next, we introduce yet another jump operator. $\Delta_t : \mathcal{RCR}(\mathbb{R},\mathbb{R}^n) \to \mathcal{G}([-r,0],\mathbb{R}^n)$ is defined by

$$\Delta_t\phi(\theta) = \phi_t(\theta) - \lim_{s\to t^-}\phi_s(\theta).$$

We will also need the notion of the regulated left-limit of an $\mathcal{RCR}$-valued function.

Definition I.6.1.1. *For a function $f : \mathbb{R} \to \mathcal{RCR}$, we define the* regulated left-limit *$f^- : \mathbb{R} \to F([-r,0],\mathbb{R}^n)$ by the formal expression*

$$f^-(t)(\theta) = \lim_{s\to 0^-} f(t+s)(\theta).$$

Note that if $x \in \mathcal{RCR}(\mathbb{R},\mathbb{R}^n)$, then for $t \mapsto x_t$, the regulated left-limit x_t^- is an element of $\mathcal{G}([-r,0],\mathbb{R}^n)$ and, in particular, it is continuous from the left. Moreover, $\Delta_t x_t = x_t - x_t^-$. The following proposition is clear, given Lemma I.6.1.1.

Proposition I.6.1.1. *Let $x : \mathbb{R} \to \mathbb{R}^n$ be continuous except at times t_k, where it is right-continuous and has limits on the left. Then, $\Delta_\theta^+ x_t^-(\theta) = \Delta_t x_t(\theta)$ for $\theta < 0$ and all $t \in \mathbb{R}$. If the functionals $B(k)$ and $g(k,\cdot)$ satisfy the overlap condition and x is a solution of the impulsive RFDE* (I.4.1)–(I.4.2)*, then*

$$B(k)x_{t_k^-} = B(k)x_{t_k}^-, \quad g(k, x_{t_k^-}) = g(k, x_{t_k}^-). \tag{I.6.19}$$

Finally, if $x : \mathbb{R} \to \mathbb{R}^n$ is differentiable from the right, we define $\frac{d^+}{dt}x_t$ by the equation

$$\left[\frac{d^+}{dt}x_t\right](\theta) = \frac{d^+}{dt}x_t(\theta).$$

Let $x : \mathbb{R} \to \mathbb{R}^n$ be a complete solution such that $t \mapsto x_t \in \mathcal{W}_t$. If assumption C.3 is satisfied, this solution satisfies the abstract evolution equation

$$\frac{d^+}{dt}x_t = \mathcal{L}(t)x_t + \chi_0 f(t, x_t), \qquad t \neq t_k \tag{I.6.20}$$

$$\Delta_t x_t = \mathcal{J}(k)x_t^- + \chi_0 g(k, x_t^-), \qquad t = t_k, \tag{I.6.21}$$

with $\mathcal{L}$ and $\mathcal{J}$ as defined in (I.6.18). In addition, the following boundary condition must be satisfied across the jump interfaces $t + \theta = t_k$ for all $k \in \mathbb{Z}$ and $\theta < 0$ because of Proposition I.6.1.1:

$$\Delta_t x_t(\theta) = \Delta_\theta^+ x_t^-(\theta), \qquad t + \theta = t_k,\ \theta < 0. \tag{I.6.22}$$

Along the lines $t + \theta = s$ for $s \notin \{t_k : k \in\}$, the condition $\Delta_t x_t(\theta) = \Delta_\theta^+ x_t^-(\theta)$ is uninformative because x is continuous at $s = t+\theta$. Note that all left-limits are now regulated left-limits because we have used Eq. (I.6.19) of Proposition I.6.1.1. It is at this stage that we make the substitution (I.6.7) to write x_t in terms of the Euclidean space representation of the centre manifold. The following theorem characterizes the Euclidean space representation of the centre manifold in terms of an impulsive evolution equation.

Theorem I.6.1.3. *For any solution u of the finite-dimensional ordinary impulsive differential equation (I.6.12)–(I.6.13), the Euclidean space representation of the centre manifold is a solution of the following boundary-value problem (invariance equation):*

$$\left.\begin{aligned}
&Q_t(\theta)[\dot{u} - \Lambda u] + \partial_t h(t, u, \theta) + \partial_u h(t, u, \theta)\dot{u} = \partial_\theta h(t, u, \theta),\\
&\theta < 0, t \neq t_k\\
&Q_t(\theta)\Delta u + \Delta_t h(t, u + \Delta u, \theta) + \Omega(t, h, u, \theta)\Delta u = \Delta_\theta^+ h(t^-, u, \theta),\\
&\theta < 0,\ (t = t_k \vee t + \theta = t_k)
\end{aligned}\right\} \tag{I.6.23}$$

$$\left.\begin{aligned}
&Q_t(0)[\dot{u} - \Lambda u] + \partial_t h(t, u, 0) + \partial_u h(t, u, 0)\dot{u} = L(t)h(t, u, \cdot) + f(t, Q_t u + h(t, u, \cdot)),\\
&\theta = 0, t \neq t_k\\
&Q_t(0)\Delta u + \Delta_t h(t, u + \Delta u, 0) + \Omega(t, h, u, 0)\Delta u = B(k)h(t^-, u, \cdot) + g(k, Q_t^- u + h(t^-, u, \cdot)),\\
&\theta = 0, t = t_k,
\end{aligned}\right\} \tag{I.6.24}$$

where we denote $u = u(t)$ when $t \neq t_k$ and $u = u(t^-)$ when $t = t_k$, $\Delta u = u(t) - u(t^-)$, we define Ω by

$$\Omega(t, h, u, \theta) = \int_0^1 \partial_u h(t^-, u + s\Delta u, \theta)ds,$$

and all derivatives in t and θ are the right-derivatives $\frac{\partial^+}{\partial t}$ and $\frac{\partial^+}{\partial \theta}$.

Proof. First, we remark that Q_t satisfies the following abstract impulsive differential equation:

$$\begin{aligned} \frac{d^+}{dt}Q_t(\theta) + Q_t\Lambda &= \chi_0 L(t)[Q_t \exp_\Lambda] + \chi_{[-r,0)}\frac{d^+}{d\theta}Q_t(\theta), && t \neq t_k \\ \Delta_t Q_t(\theta) &= \chi_0 B(k)[Q_t^- \exp_\Lambda] + \chi_{[-r,0)}\Delta_\theta^+ Q_t^-(\theta), && t = t_k, \end{aligned} \tag{I.6.25}$$

where $\exp_\Lambda(\theta) = e^{\Lambda\theta}$. It can be derived from the equality $\Phi_t = Q_t e^{\Lambda t}$ and Proposition I.6.1.1. Substituting the ansatz $x_t = Q_t u(t) + h(t, u(t), \cdot)$ into Eq. (I.6.20), we obtain when $\theta < 0$ the equality

$$\frac{d^+}{dt}[Q_t]u + Q_t\dot{u} + \partial_t h + \partial_u h\dot{u} = \frac{d^+}{d\theta}[Q_t]u + \partial_\theta h,$$

which is equivalent to the first equation from (I.6.23) if one takes into account (I.6.25). When $\theta = 0$, the same approach results in the first equation from (I.6.24).

Next, we substitute the ansatz into (I.6.21). If one denotes $u = u(t^-)$, when $\theta < 0$ and $t = t_k$, the result reduces to[1]

$$-Q_t\Delta u + \Delta_\theta^+ h(t^-, u, \theta) = h(t, u + \Delta u, \theta) - h(t^-, u, \theta)$$

after cancelling several duplicate terms. The above is equivalent to

$$Q_t\Delta u + \Delta_t h(t, u + \Delta u, \theta) + h(t^-, u + \Delta u, \theta) - h(t^-, u, \theta) = \Delta_\theta^+ h(t^-, u, \theta).$$

The fundamental theorem of calculus implies $\Omega(t, h, u, \theta) = h(t^-, u+\Delta u, \theta) - h(t^-, u, \theta)$, and from this, we obtain the second equation of (I.6.23). The equation for $t + \theta = t_k$ is obtained by checking the boundary condition (I.6.22), while the equation for $\theta = 0$ is obtained by the same methods. □

Remark I.6.1.3. *Note that $h(t, \theta, u)$ must possess discontinuities along the lines $t + \theta = t_k$ for u fixed. These discontinuities are captured by the second equation of* (I.6.23) *when $\theta < 0$ and in the second equation of* (I.6.24) *when $\theta = 0$. When $t = t_k - \theta \notin \{t_j : j < k\}$, we have $\Delta u = \Delta u(t) = 0$ in the second equation of* (I.6.23)*, and the result is the constraint $h(t, u, \theta) = h(t^-, u, \theta^+)$. In particular, even though we know that $\theta \mapsto h(t, u, \theta)$ is continuous from the right, the same is* not *true of $\theta \mapsto h(t^-, u, \theta)$; the latter is continuous from the left.*

The boundary-value problem (I.6.23)–(I.6.24) is implicit in terms of the variable $\dot{u}$ and Δu. To obtain an explicit boundary-value problem for $(t, u, \theta) \mapsto h(t, u, \theta)$, one would replace every instance of $\dot{u}$ and Δu with the Eqs. (I.6.12) and (I.6.13). The resulting equations take up a lot of space, so we do not write them out explicitly.

[1] Note that $Q_t^-(\theta^+) = Q_t(\theta)$ for $\theta < 0$.

I.6.2 Approximation by the Taylor Expansion

Equations (I.6.23)–(I.6.24) and (I.6.12)–(I.6.13) of Theorems I.6.1.3 and I.6.1.2 yield a system of impulsive partial delay differential equations and boundary conditions for the Euclidean space representation of the centre manifold.

In the u coordinates, the dynamics on the centre manifold are given by (I.6.12)–(I.6.13). If one seeks to obtain the $O(||u||^k)$ dynamics on the center manifold, it is necessary to compute the terms of order $O(||u||^{k-1})$ of the center manifold h. Quadratic terms are needed to analyze Hopf-like bifurcations, for instance. The quadratic coefficient $h_2(t,\theta)$ of the centre manifold can be represented in the form

$$h_2(t,\theta)[u,v]$$
$$= \begin{bmatrix} c^1_{11}(t,\theta)u_1v_1 + \cdots c^1_{1p}(t,\theta)u_1v_p + c^1_{21}(t,\theta)u_2v_1 + c^1_{22}(t,\theta)u_2v_2 + \cdots + c^1_{pp}(t,\theta)u_pv_p \\ \vdots \\ c^n_{11}(t,\theta)u_1v_1 + \cdots c^n_{1p}(t,\theta)u_1v_p + c^n_{21}(t,\theta)u_1v_1 + c^n_{22}(t,\theta)u_2v_2 + \cdots + c^n_{pp}(t,\theta)u_pv_p \end{bmatrix},$$

and similarly for the higher-order terms, where symmetrically, $c_{ij} = c_{ji}$. In terms of vector multi-indices, we can write

$$h_m(t,\theta)[u_1,\ldots,u_m] = \sum_{|\xi|=m} c_\xi(t,\theta)(u_1\cdots u_m)^\xi \tag{I.6.26}$$

for multi-index $\xi = (\xi_1,\ldots,\xi_m)$ and $\xi_i \in \{\emptyset, e'_1,\ldots,e'_p\}$.

As a consequence of the above observations, one can substitute an appropriate order $O(||u||^k)$ expansion of the impulsive differential equation (I.6.12)–(I.6.13) into the evolution equation and boundary conditions (I.6.23)–(I.6.24) to obtain a $O(||u||^k)$ impulsive evolution equation for the center manifold.

I.6.2.1 Evolution Equation and Boundary Conditions for Quadratic Terms

For the calculation of cubic order dynamics (e.g. Hopf bifurcation), one needs to calculate h_2 before the dynamics on the center manifold (I.6.16)–(I.6.17) can be studied. Substituting the aforementioned equation into the evolution equation and boundary conditions (I.6.23)–(I.6.24) and keeping only the order two terms in u, we obtain the following rather large equation:

$$\frac{1}{2}Q_t(\theta)e^{\Lambda t}Y_c(t)D_2^2 f(t,0)[Q_tu]^2 + \frac{1}{2}\partial_t h_2(t,\theta)u^2 + h_2(t,\theta)[\Lambda u,u] = \frac{1}{2}\partial_\theta h_2(t,\theta)u^2,$$
$$\theta < 0, t \neq t_k \tag{I.6.27}$$

$$\frac{1}{2}Q_t(\theta)e^{\Lambda t}Y_c(t)D^2g(k,0)[Q_t^-u]^2 + \frac{1}{2}\Delta_t h_2(t,\theta)u^2 = \frac{1}{2}\Delta_\theta^+ h_2(t^-,\theta)u^2 \quad \theta < 0, t \in \{t_k, t_k - \theta\} \tag{I.6.28}$$

$$\frac{1}{2}Q_t(0)e^{\Lambda t}Y_c(t)D_2^2 f(t,0)[Q_t u]^2 + \frac{1}{2}\partial_t h_2(t,0)u^2 + h_2(t,\theta)[\Lambda u, u] = \frac{1}{2}L(t)h_2(t,\cdot)u^2 + \frac{1}{2}D_2^2 f(t,0)[Q_t u]^2, \theta = 0, t \neq t_k \tag{I.6.29}$$

$$\frac{1}{2}Q_t(0)e^{\Lambda t}Y_c(t)D^2g(k,0)[Q_t^-u]^2 + \frac{1}{2}\Delta_t h_2(t,0)u^2 = \frac{1}{2}J(k)h_2(t^-,\cdot)u^2 + \frac{1}{2}D^2g(k,0)[Q_t^-u]^2, \theta = 0, t = t_k. \tag{I.6.30}$$

In this equation, all partial derivatives are right-hand derivatives. Notice that upon expansion, the coefficients of each binomial $u^\xi = u_{\xi_1}u_{\xi_2}$ generate a system of coupled linear impulsive partial differential equations for the associated coefficients c_ξ of the quadratic expansion of the center manifold. This system can be solved by a variation of the method of characteristics; this is done in Sect. I.6.2.2.

The pattern established here continues to mth-order expansions. In particular, each multinomial $u^\xi = u_1^{\xi_1}\cdots u_m^{\xi_m}$ with $\sum_m \xi_k = m$ generates a system of coupled impulsive PDEs for the u^ξ coefficient of $h_m(t,\theta)$. The order $i < m$ expansions $h_i(t,\theta)$ are generally needed to compute the order m terms, so the procedure must be done iteratively. The calculations quickly become taxing, and the use of computer algebra software is highly recommended to keep track of all of the differentials.

I.6.2.2 Solution by the Method of Characteristics

The system of impulsive partial differential equations (I.6.27)–(I.6.30) must be solved in order to obtain the quadratic-order term of the center manifold. A similar equation can be derived for the pth-order terms, and this equation will typically depend on the lower-order terms. For notational simplicity, we will only present the method as it applies to computing the quadratic term h_2.

First, some preparations. Given the representation (I.6.26), we can write

$$h_2(t,\theta)[u,u] = \sum_{\xi\in\Xi} h_2^\xi(t,\theta)(uu)^\xi,$$

for $u \in \mathbb{R}^p$, where Ξ is a set of p-dimensional multi-indices in two variables that is both *permutation-free* (i.e. $(e_i,e_j) \in \Xi$ implies $(e_j,e_i) \in \Xi$ if and only if $i = j$) and *complete* (i.e. for every p-dimensional multi-index in two variables ζ, either $\zeta \in \Xi$ or $\zeta = (e_i,e_j)$ and $(e_j,e_i) \in \Xi$). In this setting we have

$$h_2^\xi(t,\theta) = \begin{cases} 2c_\xi, & \xi = (e_i,e_j), i \neq j \\ c_\xi, & \xi = (e_i,e_i). \end{cases}$$

Writing everything in terms of scalar products, there exists a $\beta \times \beta$ matrix Λ_2 with $\beta = \binom{p+1}{2}$ such that

$$h_2(t,\theta)[u,u] = [\ (uu)^{\zeta_1} \quad \cdots \quad (uu)^{\zeta_\beta}\ ] * h_2^{\Xi}, \tag{I.6.31}$$

$$h_2(t,\theta)[\Lambda u,u] = [\ (uu)^{\zeta_1} \quad \cdots \quad (uu)^{\zeta_\beta}\ ] * \Lambda_2 * h_2^{\Xi}, \tag{I.6.32}$$

where $h_2^{\Xi} = (h_2^{\zeta_1}, \ldots, h_2^{\zeta_\beta}) \in (\mathbb{R}^n)^\beta$ is interpreted as a $(\beta \times 1)$ array whose ith entry is $h_2^{\zeta_i}$, and $\Xi = \{\zeta_1, \ldots, \zeta_\beta\}$. As such, the matrix multiplication needs to be interpreted in an overloaded sense as in Eq. (I.6.4). For example, with $p = 3$, $\Xi = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$ and the data

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{array}{lll} \zeta_1 = (e_1,e_1), & \zeta_2 = (e_1,e_2), & \zeta_3 = (e_1,e_3), \\ \zeta_4 = (e_2,e_2), & \zeta_5 = (e_2,e_3), & \zeta_6 = (e_3,e_3), \end{array}$$

we first calculate $h_2[\Lambda u, u]$. Written in terms of the coefficients $h_2^{\zeta_i}$, it is

$$h_2[\Lambda u,u] = h_2\left[\begin{bmatrix} u_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right] = h_2^{11}u_1u_2 + \frac{1}{2}h_2^{12}u_2^2 + \frac{1}{2}h_2^{13}u_2u_3.$$

We can then readily extract the matrix Λ_2 satisfying the expression (I.6.32), and we find it is

$$\Lambda_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Next, we write

$$Q_t(\theta)e^{\Lambda t}Y(t)D_2^2 f(t,0)[Q_t u]^2 = [\ (uu)^{\zeta_1} \quad \cdots \quad (uu)^{\zeta_\beta}\ ] * \mathcal{F}(t,\theta) \tag{I.6.33}$$

$$Q_{t_k}(\theta)e^{\Lambda t_k}Y(t_k)D^2 g_k(0)[Q_{t_k^-}u]^2 = [\ (uu)^{\zeta_1} \quad \cdots \quad (uu)^{\zeta_\beta}\ ] * \mathcal{G}_k(\theta) \tag{I.6.34}$$

$$D_2^2 f(t,0)[Q_t u]^2 = [\ (uu)^{\zeta_1} \quad \cdots \quad (uu)^{\zeta_\beta}\ ] * a(t) \tag{I.6.35}$$

$$D^2 g_k(0)[Q_{t_k^-}u]^2 = [\ (uu)^{\zeta_1} \quad \cdots \quad (uu)^{\zeta_\beta}\ ] * b_k, \tag{I.6.36}$$

where $\mathcal{F}(t,\theta)$, $\mathcal{G}_k(\theta)$, $a(t)$ and b_k are $\beta \times 1$ arrays with entries in $\mathbb{R}^n$. Note that as $Q_t e^{\Lambda t} = \Phi_t$, Eqs. (I.6.33) and (I.6.34) could be simplified further. Substituting equations (I.6.31)–(I.6.36) into (I.6.27)–(I.6.30) and cancelling fractions, it follows upon comparing powers $(uu)^{\zeta_i}$ that h_2^{Ξ} must satisfy the impulsive functional differential equation

$$\mathcal{F}(t,\theta) + \partial_t h_2^{\Xi}(t,\theta) + 2\Lambda_2 * h_2^{\Xi}(t,\theta) = \partial_\theta h_2^{\Xi}(t,\theta), \quad \theta < 0, t \neq t_k \tag{I.6.37}$$

$$\mathcal{G}_k(\theta) + \Delta_t h_2^{\Xi}(t,\theta) = \Delta_\theta^- h_2^{\Xi}(t,\theta), \quad \theta < 0, t \in \{t_k, t_k - \theta\} \tag{I.6.38}$$

$$\mathcal{F}(t,0) + \partial_t h_2^{\Xi}(t,\theta) + 2\Lambda_2 * h_2^{\Xi}(t,0) = L(t) \odot h_2^{\Xi}(t,\cdot) + a(t), \quad \theta = 0, t \neq t_k \tag{I.6.39}$$

$$\mathcal{G}_k(0) + \Delta_t h_2^{\Xi}(t,0) = J(k) \odot h_2^{\Xi}(t^-,\cdot) + b_k, \quad \theta = 0, t = t_k, \tag{I.6.40}$$

and we define the overloaded operator $L(t)\odot$ by

$$L(t) \odot h_2^{\Xi}(t,\cdot) = \begin{bmatrix} L(t)h^{\zeta_1}(t,\cdot) \\ \vdots \\ L(t)h^{\zeta_\beta}(t,\cdot) \end{bmatrix},$$

and similarly for $J(k)$. Note also that all derivatives are taken from the right: $\partial_t = \frac{\partial^+}{\partial t}$ and $\partial_\theta = \frac{\partial^+}{\partial \theta}$. The following proposition characterizes the solutions of the above inhomogeneous linear impulsive system.

Proposition I.6.2.1. *Every solution $z = z(t,\theta)$ of the inhomogeneous system (I.6.37)–(I.6.40) can be expressed in the form*

$$z(t,\theta) = e^{2\Lambda_2\theta} * \left[n(t+\theta) - \int_\theta^0 e^{-2\Lambda_2 s} * \mathcal{F}(t-s+\theta, s)ds - \sum_{\theta < t_k - t \leq 0} e^{-2\Lambda_2 t_k} * \mathcal{G}_k(t - t_k + \theta) \right], \tag{I.6.41}$$

where $t \mapsto n(t)$ is a solution of the inhomogeneous linear impulsive delay differential equation

$$\begin{aligned} \mathcal{F}(t,0) + \dot{n}(t) + 2\Lambda_2 * n(t) &= L(t) \odot [e^{2\Lambda_2(\cdot)} * n_t] + m(t), \quad t \neq t_k \\ \mathcal{G}_k(0) + \Delta n(t) &= J(k) \odot [e^{2\Lambda_2(\cdot)} * n_{t^-}] + n_k, \quad t = t_k \end{aligned} \tag{I.6.42}$$

$$\begin{aligned} m(t) &= a(t) - L(t) \odot \left[\int_{(\cdot)}^0 e^{-2\Lambda_2 s} * \mathcal{F}(t-s+\cdot, s)ds + \sum_{(\cdot) < t_k - t \leq 0} e^{-2\Lambda_2 t_k} * G_k(t - t_k + \cdot) \right] \\ n_k &= b_k - J(k) \odot \left[\int_{(\cdot)}^0 e^{-2\Lambda_2 s} * \mathcal{F}(t_k^- - s + \cdot, s)ds + \sum_{(\cdot) < t_j - t_k \leq 0} e^{-2\Lambda_2 t_j} * G_j(t_k^- - t_j + \cdot) \right]. \end{aligned} \tag{I.6.43}$$

Proof. Solving the Eqs. (I.6.37)–(I.6.38) along the characteristic lines $t + \theta =$ constant, one can show that every solution has the form of (I.6.41) for some function n. To show that such a function n satisfies the impulsive delay differential equation (I.6.42)–(I.6.43), one substitutes the ansatz into Eqs. (I.6.39)–(I.6.40), taking note that $\partial_t z(t,0) = \dot{n}(t)$ and $\Delta_t z(t,0) = \Delta n(t)$. □

Solving the inhomogeneous system (I.6.42) is a nontrivial matter. Moreover, there are infinitely many solutions of the form prescribed by the above proposition, since the inhomogeneous equation (I.6.42) can have many bounded solutions. We must recall some additional properties of the centre manifold to identify the unique solution $z = h_2^{\Xi}$ corresponding to the true coefficient vector of $h_2(t,\theta)$ in the expansion (I.6.31). We state the result in the form of a corollary. It is essentially a consequence of Theorem I.6.1.1.

Corollary I.6.2.1. *Let the centre manifold be $PC^{1,2}$-regular at zero, and let the assumptions of Theorem I.5.2.1 hold. The $(\beta \times 1)$ array h_2^{Ξ} with $[h_2^{\Xi}]_i = h_2^{\zeta_i}$ in the expansion*

$$h_2(t,\theta)[u,u] = \sum_{i=1}^{\beta} h_2^{\zeta_i}(t,\theta)[uu]^{\zeta_i}$$

is the unique solution of the inhomogeneous linear impulsive PDE (I.6.37)–(I.6.40) *satisfying the following constraints:*

1. *Projection constraint: $P_c(t)h_2^{\zeta_i}(t,\cdot) = 0$ for all $t \in [0,T)$ and $i = 1,\ldots,\beta$.*

2. *Periodicity constraint: $t \mapsto h_2^{\Xi}(t,\cdot)$ is periodic.*

I.6.3 Visualization of Centre Manifolds

The discontinuous invariant manifolds we have presented here seem to be relatively new in the dynamical systems literature. To aid with visualization, here we consider two examples of centre manifold computation. The first (Sect. I.6.3.1) is a finite-dimensional toy example where the centre manifolds can all be explicitly calculated. The second one (Sect. I.6.3.2) is two-dimensional with quadratic delays and requires Taylor expansions. Both examples contain a parameter ϵ that controls the “size” of the impulse effect and, for all ϵ small enough, the centre manifold is one-dimensional. This allows us to visualize the centre manifolds as depending on the parameter ϵ and study how the introduction of impulses affects their geometry. Both these examples are intentionally simple; in particular, the linear parts contain no delays.

I.6.3.1 An Explicit Scalar Example Without Delays

Consider the finite-dimensional impulsive system

$$\dot{x} = x^2, \qquad t \neq k, \qquad \Delta x = 0, \qquad t = k \tag{I.6.44}$$

$$\dot{y} = -y, \qquad t \neq k, \qquad \Delta y = \epsilon y, \qquad t = k. \tag{I.6.45}$$

This system has several useful properties. First, for all $\epsilon \in (-1, e-1)$, the unique equilibrium at the origin is nonhyperbolic with a one-dimensional centre fiber bundle $\mathcal{RCR}_c = \text{span}(e_1)$ that is constant in time. Moreover, when $\epsilon = 0$, this system is a classical example of a system with infinitely many centre manifolds, the only analytic one being the x-axis. The centre manifolds are all graphs of

$$y = c \exp\left(\frac{1}{x}\right) \chi_{(-\infty,0)}(x), \qquad c \in \mathbb{R}. \tag{I.6.46}$$

Our first step will be to introduce a time-dependent change of variables. Define $y = (1+\epsilon)^{\lfloor t \rfloor - t} w$. This change of variables eliminates the impulse effect from (I.6.44)–(I.6.45) entirely. The result is the autonomous system

$$\dot{x} = x^2, \tag{I.6.47}$$

$$\dot{w} = (\log(1+\epsilon) - 1)w. \tag{I.6.48}$$

The centre manifolds of the above autonomous system are all orbits. Explicitly solving the differential equations and rearranging show that they can be represented in the form

$$w = c \exp\left(\frac{1}{x}(1 - \log(1+\epsilon))\right) \chi_{(-\infty,0)}(x),$$

where $c \in \mathbb{R}$ is a constant. Inverting the change of variables and simplifying the expression somewhat, the centre manifolds of the original system (I.6.44)–(I.6.45) can be written as the graphs of

$$y = c(1+\epsilon)^{\lfloor t \rfloor - t} \left(\frac{e}{1+\epsilon}\right)^{1/x} \chi_{(-\infty,0)}(x) := h_\epsilon(t,x) \tag{I.6.49}$$

See Fig. I.6.1 for a visualization.

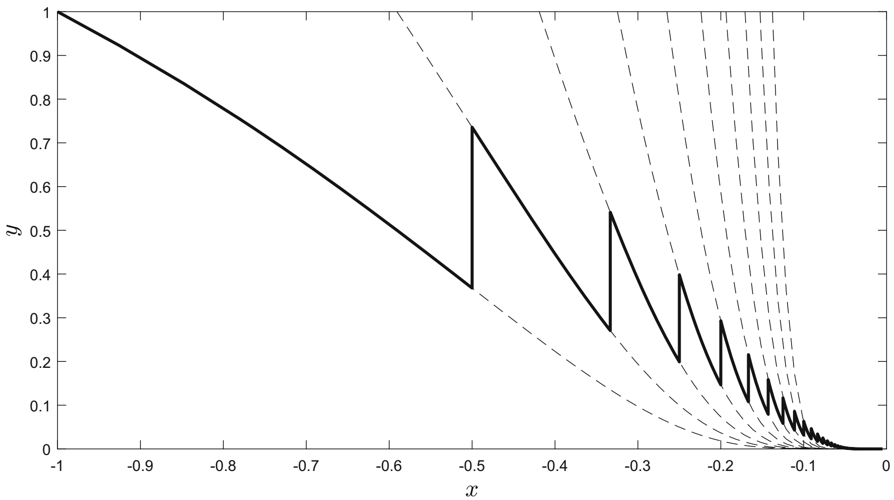

Figure I.6.1: The forward orbit through $(-1,1)$ of system (I.6.44)–(I.6.45) plotted for time $t \in [0,200]$ and parameter $\epsilon = 1$, illustrated by the thick solid line. On the intervals $[k,k+1)$ for $k = 0,\dots,9$, the graphs of $y = c\exp\left(\frac{1}{x}\right)$ (Eq. (I.6.46)) on which the solution travels are indicated by dashed lines and plotted for $x < 0$

I.6.3.2 Two-Dimensional Example with Quadratic Delayed Terms

We consider in this section the equation with a single discrete delay

$$\dot{x} = -x + y^2, \qquad t \neq 2k\pi, \qquad \Delta x = \epsilon y(t^-), \qquad t = 2k\pi \tag{I.6.50}$$

$$\dot{y} = x - x^2(t-\pi) - y^2(t-\pi), \qquad t \neq 2k\pi, \qquad \Delta y = 0, \qquad t = 2k\pi \tag{I.6.51}$$

$$\dot{\epsilon} = 0, \qquad t \neq 2k\pi, \qquad \Delta\epsilon = 0, \qquad t = 2k\pi. \tag{I.6.52}$$

Considered in isolation, the planar system (I.6.50)–(I.6.51) has, with ϵ treated as a parameter, a single zero Floquet exponent for all $\epsilon \in \mathbb{R}$. Thus, for each ϵ, the centre manifold at the origin is one-dimensional. Taking ϵ as a state variable, we obtain (I.6.50)–(I.6.52), and it is for this system that we will calculate (approximate) the two-dimensional centre manifold at the origin. Taking one-dimensional slices for fixed ϵ small will produce the centre manifolds for the parameterized system (I.6.50)–(I.6.51).

The linearization of (I.6.50)–(I.6.52) at $(0,0,0)$ admits the monodromy operator V_t and associated resolvent $R(z;V_t)$

$$V_t\xi(\theta) = \begin{bmatrix} e^{-(2\pi+\theta)} & 0 & 0 \\ 1-e^{-(2\pi+\theta)} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xi(0) := V(\theta)\xi(0), \tag{I.6.53}$$

$$R(z;V_t) = z^{-1}(I_{\mathcal{RCR}} + V[I - z^{-1}V(0)]^{-1}\text{ev}_0), \qquad z \neq 0,1. \tag{I.6.54}$$

One can similarly calculate a basis matrix Φ_t for the centre fiber bundle, the projection $P_c(t) : \mathcal{RCR} \to \mathcal{RCR}_c(t)$ and the matrix $Y(t)$. We find

$$\Phi_t = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_c(t)\phi(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\phi(0), \quad Y(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{I.6.55}$$

It follows that we can take $Q_t = \Phi_t$ and $\Lambda = 0$ in the Floquet decomposition. The nonlinearity f of the vector field contains only the second-order term, and we have

$$Q_t(\theta)u = \begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix}, \qquad \frac{1}{2}D^2 f(0)[Q_t u]^2 = \begin{bmatrix} u_1^2 \\ -u_1^2 \\ 0 \end{bmatrix}. \tag{I.6.56}$$

Similarly, the nonlinearity of the jump map g also contains only the second-order term, and we have at $t = 2k\pi$,

$$\frac{1}{2}D^2 g(0)[Q_{2k\pi^-}u]^2 = \begin{bmatrix} u_1u_2 \\ 0 \\ 0 \end{bmatrix}. \tag{I.6.57}$$

Using (I.6.55), (I.6.56), and (I.6.57), we can read off the impulsive delay differential equations and boundary conditions (I.6.27)–(I.6.30) for the second-order term h_2. These are listed in I.6.3.3, where the rest of the calculations are completed. In particular, the coefficients c_ξ of the quadratic-order expansion $h_2u^2 = h_2^{11}u_1^2 + h_2^{12}u_1u_2 + h_2^{22}u_2^2$ are computed therein. Given that $h = \frac{1}{2}h_2u^2 + O(u^3)$, the quadratic-order expansion of the centre manifold is found to be

$$h(t,u,\theta) = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u_1^2 + \frac{1}{2}h_2^{12}(t,\theta)u_1u_2 + O(u^3), \tag{I.6.58}$$

where the function $t \mapsto h_2^{12}(t,\theta)$ is 2π-periodic, has discontinuities along the lines $t+\theta = 2k\pi$ and is given, for $t \geq 2\pi$, by (I.6.66).

As ϵ is stationary in (I.6.50)–(I.6.52), we actually have $u_2 = \epsilon$. Therefore, the parameter-dependent centre manifold for the planar system (I.6.50)–(I.6.51) is obtained by replacing u_2 with ϵ and u_1 with u in (I.6.58) and dropping the third row, as this last row corresponds to the dynamics in ϵ. The result is

$$h_\epsilon(t,\theta,u) = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} u^2 + \frac{1}{2}\epsilon \tilde{h}_2^{12}(t,\theta)u + O(u^3) := h_{\epsilon,2}(t,\theta,u) + O(u^3), \tag{I.6.59}$$

where $\tilde{h}_2^{12}$ denotes the first two rows of h_2^{12} (the third row is identically zero). When $\epsilon = 0$, the centre manifold is identical to the one that would be obtained by the usual adjoint-based method for autonomous delay differential equations. This can be verified by direct calculation. A static portrait of $h_{\epsilon,2}$ at the fixed time argument $t = \pi$ with $\epsilon = -0.5$ is provided in Fig. I.6.2, while a contour plot of the quadratic coefficient $\tilde{h}_2^{12}$ appears in Fig. I.6.3.

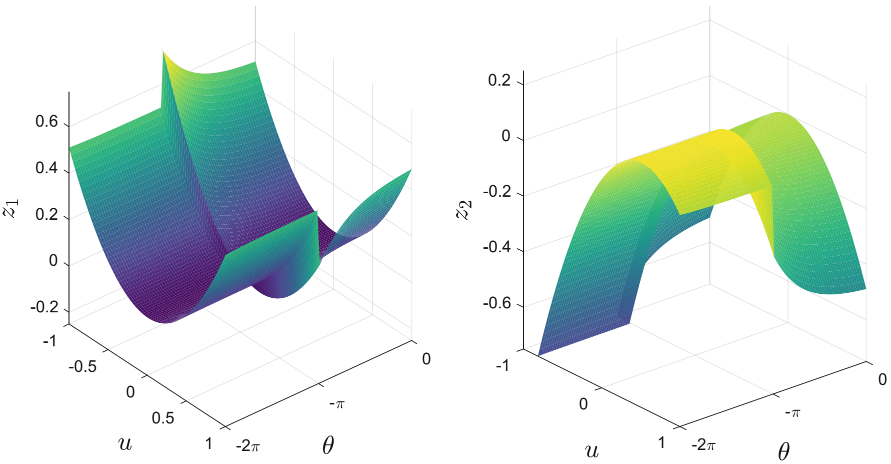

Figure I.6.2: The two components $(z_1, z_2) = h_{\epsilon,2}(t,\theta,u)$ of the quadratic-order truncation of the parameter-dependent centre manifold for Example I.6.3.2, plotted at the time snapshot $t = \pi$ on the grid $(u,\theta) \in [-1,1] \times [-2\pi, 0]$, with parameter $\epsilon = -0.5$. Notice the discontinuity along the plane $\theta = -\pi$; this occurs because $t + \theta = 0$ is an integer multiple of 2π. The phase space was taken to be $\mathcal{RCR}([-2\pi,0],\mathbb{R}^2)$ mainly for visualization; it allows us to visualize a wider range of θ arguments than if we were to restrict to $[-\pi,0]$, the latter being implied by the range of the delay in (I.6.50)–(I.6.52)

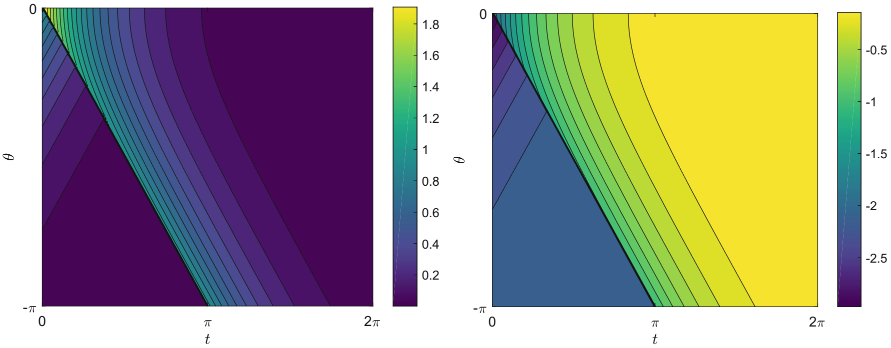

Figure I.6.3: Contour plot of $(t,\theta) \mapsto \tilde{h}_2^{12}(t,\theta)$ for $(t,\theta) \in [0,2\pi) \times [-\pi,0]$. Left: the first component $e_1^\intercal \tilde{h}_2^{12}$. Right: the second component $e_2^\intercal \tilde{h}_2^{12}$. We wish to emphasize two discontinuities: the localized (in space, θ) jump across the line $t+\theta=0$ and the global discontinuity across the "periodic boundary" $t=0$ and $t \to 2\pi^-$. These contour plots provide a complete description of the quadratic term since $t \mapsto \tilde{h}_2^{12}(t,\theta)$ is periodic with period 2π

I.6.3.3 Detailed Calculations Associated with Example I.6.3.2

Substituting (I.6.55), (I.6.56), and (I.6.57) into Eqs. (I.6.27)–(I.6.30), we obtain

$$\begin{aligned}
\partial_t h_2 u^2 - \partial_\theta h_2 u^2 &= 0, && \theta < 0, t \neq 2\pi k \\
\begin{bmatrix} 0 \\ u_1 u_2 \\ 0 \end{bmatrix} + \Delta_t h_2 u^2 - \Delta_\theta^+ h_2(t^-,\theta) &= 0, && \theta < 0, t = 2\pi k \\
\partial_t h_2 u^2 - \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} h_2(t,0) u^2 - \begin{bmatrix} u_1^2 \\ -u_1^2 \\ 0 \end{bmatrix} &= 0, && \theta = 0, t \neq 2\pi k \\
\begin{bmatrix} 0 \\ u_1 u_2 \\ 0 \end{bmatrix} + \Delta_t h_2 u^2 - \begin{bmatrix} u_1 u_2 \\ 0 \\ 0 \end{bmatrix} &= 0, && \theta = 0, t = 2\pi k.
\end{aligned}$$

The second-order term $h_2(t,\theta)u^2$ is given by

$$h_2 u^2 = c_{11} u_1^2 + 2c_{12} u_1 u_2 + c_{22} u_2^2 = h_2^{11} u_1^2 + h_2^{12} u_1 u_2 + h_2^{22} u_2^2$$

for $h_2^\zeta \in \mathbb{R}^3$, so there is a system of three 3-dimensional systems to solve. In this example, the impulsive PDEs for each of the coefficients $h_2^{\zeta_i}$ of (I.6.37)–

(I.6.40) decouple because $\Lambda = 0$ implies $\Lambda_2 = 0$. With respect to the multi-index ordering $\zeta^1 = (e_1, e_1)$, $\zeta^2 = (e_1, e_2)$, $\zeta^3 = (e_2, e_2)$, we have

$$\partial_t h_2^{\Xi} = \partial_\theta h_2^{\Xi}, \qquad \theta < 0, t \neq 2\pi k \tag{I.6.60}$$

$$\begin{bmatrix} 0 \\ e_2 \\ 0 \end{bmatrix} + \Delta_t h_2^{\Xi} = \Delta_\theta^+ h_2^{\Xi}(t^-, \theta), \qquad \theta < 0, t = 2\pi k \tag{I.6.61}$$

$$\partial_t h_2^{\Xi} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \odot h_2^{\Xi}(t,0) + \begin{bmatrix} e_1 - e_2 \\ 0 \\ 0 \end{bmatrix}, \qquad \theta = 0, t \neq 2k\pi \tag{I.6.62}$$

$$\begin{bmatrix} 0 \\ e_2 \\ 0 \end{bmatrix} + \Delta_t h_2^{\Xi} = \begin{bmatrix} 0 \\ e_1 \\ 0 \end{bmatrix}, \qquad \theta = 0, t = 2k\pi. \tag{I.6.63}$$

We will solve the equations for $h_2^{\zeta_i}$ individually.

The u_1^2 Coefficient

The partial differential equation (I.6.60)–(I.6.61) becomes the trivial transport equation:

$$\partial_t h_2^{\zeta} = \partial_\theta h_2^{\zeta}, \qquad t \neq 2k\pi \tag{I.6.64}$$

$$\Delta_t h_2^{\zeta} = \Delta_\theta^+ h_2^{\zeta}(t^-, \theta), \qquad t = 2k\pi, \tag{I.6.65}$$

for $\zeta = (e_1, e_1)$. Therefore, both the functions $\mathcal{F}$ and $\mathcal{G}$ of Proposition I.6.2.1 are zero, and it follows that $h_2^{11}(t,\theta) = h_2^{11}(t+\theta, 0)$. The latter is determined solely by the boundary conditions (I.6.62)–(I.6.63). Namely, $t \mapsto h_2^{11}(t,0)$ satisfies the impulsive differential equation

$$\begin{aligned} \partial_t h_2^{11}(t,0) &= \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} h_2^{11}(t,0) + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, && t \neq 2\pi k \\ \Delta_t h_2^{11}(t,0) &= 0, && t = 2\pi k. \end{aligned}$$

It follows that $h_2^{11}(t,\theta)$ is given by

$$h_2^{11}(t,\theta) = h_2^{11}(t+\theta, 0) = \begin{bmatrix} e^{-(t+\theta)} & 0 & 0 \\ 1 - e^{-(t+\theta)} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} h_2^{11}(0,0) + \begin{bmatrix} 1 - e^{-(t+\theta)} \\ e^{-(t+\theta)} - 1 \\ 0 \end{bmatrix}.$$

Finally, we apply the constraint of Corollary I.6.2.1 to identify the unknown constant $h_2^{11}(0,0) = (\alpha, \beta, \gamma)$. We therefore require both $P_c(t)h_2^{11}(t,\cdot) = 0$ and $h_2^{11}(t,\theta) = h_2^{11}(t+2\pi,\theta)$, where the period is 2π. Evaluating these two constraints and simplifying produce the systems of equations

$$\begin{bmatrix} \alpha+\beta \\ \gamma \end{bmatrix} = 0,$$

$$\begin{bmatrix} (\alpha-1)(e^{-(t+\theta)} - e^{-(t+\theta+2\pi)}) \\ (1-\alpha)(e^{-(t+\theta)} - e^{-(t+\theta+2\pi)}) \\ \gamma \end{bmatrix} = 0.$$

It follows that $\alpha = 1$, $\beta = -1$ and $\gamma = 0$, so that the coefficient $h_2^{11}(t,\theta)$ is the constant

$$h_2^{11}(t,\theta) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The u_2^2 Coefficient

With $\zeta = (e_2, e_2)$, the partial differential equation (I.6.60)–(I.6.61) for h_2^{22} similarly reduces to the transport equation (I.6.64)–(I.6.65), so that we have $h_2^{22}(t,\theta) = h_2^{22}(t+\theta,0)$. The boundary condition (I.6.62)–(I.6.63) contains no inhomogeneous terms, and it follows that

$$h_2^{22}(t,\theta) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 1-e^{-t} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} c_{22}(0,0).$$

The periodicity constraint $h_2^{22}(t+2\pi,\theta) = h_2^{22}(t,\theta)$ implies that $h_2^{22}(0,0) = (\alpha,\beta,\gamma)$ satisfies $\alpha = 0$. The projection constraint $P_c(t)h_2^{22}(t,\cdot) = 0$ then yields $\beta = 0$ and $\gamma = 0$, from which we conclude that $h_2^{22} \equiv 0$.

The u_1u_2 Coefficient

Contrary to the previous two coefficients, there is an inhomogeneity in the impulsive partial differential equation (I.6.60)–(I.6.61) for the final index $\zeta = (e_1, e_2)$. Specifically, in the notation of Proposition I.6.2.1,

$$\mathcal{G}_k = [\ 0 \quad 1 \quad 0\ ]', \qquad \mathcal{F} = 0, \qquad b_k = [\ 1 \quad 0 \quad 0\ ]', \qquad a = 0,$$

which means that the coefficient h_2^{12} is of the form

$$h_2^{12}(t,\theta) = n(t+\theta) - \sum_{\theta < 2k\pi - t \leq 0} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

whereas n is a solution of the impulsive differential equation

$$\dot{n} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} n, \qquad t \neq 2k\pi$$

$$\Delta n = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad t = 2k\pi.$$

The general solution of the above system is given by

$$n(t) = X(t,0)\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{cases} \sum_{0<2k\pi\leq t} X(t,2k\pi)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, & t > 0 \\ -\sum_{t<2k\pi\leq 0} X(t,2k\pi)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, & t \leq 0. \end{cases}$$

$$X(t,s) = \begin{bmatrix} e^{-(t-s)} & 0 & 0 \\ 1-e^{-(t-s)} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Checking the projection condition $P_c(t)h_2^{12}(t,\cdot) = 0$, we find that $\gamma = 0$ and $\alpha + \beta = 0$. Verifying the periodicity condition $h_2^{12}(t + 2\pi, \theta) = h_2^{12}(t, \theta)$ at $\theta = 0$ and $t = 0$, we see that α must satisfy the equation $e^{-2\pi}\alpha + 1 = \alpha$, which implies

$$\alpha = \frac{1}{1-e^{-2\pi}}, \qquad \beta = -\frac{1}{1-e^{-2\pi}}, \qquad \gamma = 0.$$

It is not necessary to check at other arguments t and θ because Corollary I.6.2.1 guarantees that the constants α, β and γ are uniquely specified. Therefore, the u_1u_2 coefficient vector is given, for $t \geq 2\pi$, by

$$\begin{aligned} h_2^{12}(t,\theta) = \frac{1}{1-e^{-2\pi}} X(t+\theta,0)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \sum_{\theta<2k\pi-t\leq 0}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ + \sum_{0<2k\pi\leq t+\theta} X(t+\theta,2k\pi)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}. \end{aligned} \qquad \text{(I.6.66)}$$

For $t < 2\pi$, one may extend backwards by periodicity.

I.6.4 The Overlap Condition

Throughout this chapter, we have assumed that the overlap condition (Definition I.5.4.1) is satisfied. We will study this condition in a bit more detail now, verifying for what classes of impulse effect it is satisfied and when it might be violated. We will later (Chap. IV.3) study a mathematical model in which the overlap condition is explicitly violated. In these instances, it is sometimes possible to circumvent the overlap condition by defining additional state variables. This is the technique used in [30], for example. For ease of presentation, we will focus our attention on impulse effects that involve only distributed delays or discrete delays.

I.6.4.1 Distributed Delays

Suppose that $g_k : [-r, 0] \times \mathbb{R}^n \to \mathbb{R}^n$ for $k \in \mathbb{Z}$ is continuous. We claim that the sequence of functionals $J(k, \cdot) : \mathcal{RCR} \to \mathbb{R}^n$ defined by

$$J(k, \phi) = \int_{-r}^{0} g_k(s, \phi(s))ds$$

satisfies the overlap condition, regardless of the sequence of impulse effects. To see why, first observe that since g_k is continuous, $s \mapsto g_k(s, \phi(s))$ is integrable on $[-r, 0]$ whenever $\phi \in \mathcal{G}([-r, 0], \mathbb{R}^n)$. This is a consequence of the fact that regulated functions (i.e. elements of $\mathcal{G}([-r, 0], \mathbb{R}^n)$) are actually Riemann integrable since they are uniform limits of step functions. Next, it is straightforward to verify that we have $\lim_{\epsilon \to 0^+} \phi + \chi_{[t,t+\epsilon)}h = \phi + \chi_t h$ pointwise in $\mathcal{G}([-r, 0], \mathbb{R}^n)$ for any $h \in \mathcal{RCR}$ and $t \in [-r, 0)$. This together with the Lebesgue dominated convergence theorem implies $\lim_{\epsilon \to 0^+} J(k, \phi + \chi_{[t,t+\epsilon)}h) = J(k, \phi)$, as required.

The continuity of the sequence of functionals g_k can be weakened somewhat and the conclusion that $J(k, \cdot)$ satisfies the overlap condition will remain valid. For example, it is enough to require $s \mapsto g_k(s, \phi(s))$ integrable on $[-r, 0]$ for all $\phi \in \mathcal{G}([-r, 0], \mathbb{R}^n)$.

I.6.4.2 Transformations that Enforce the Overlap Condition for Discrete Delays

For (discrete delay) functionals of the form

$$\phi \mapsto J(k, \phi) = g_k(\phi(-r_1), \ldots, \phi(-r_\ell))$$

for $0 \leq r_1 < \cdots < r_\ell \leq r$ and continuous $g_k : (\mathbb{R}^n)^\ell \to \mathbb{R}^n$, the overlap condition fails if there exists an impulse time t_j such that $t_j - r_n = t_m$ for some $n \in \{1, \ldots, \ell\}$ and $m < j$.

We will demonstrate how one can introduce additional state variables to transform an impulsive functional differential equation that fails the overlap condition into one for which the condition is satisfied. Since the procedure can quickly become complicated, it is best explained for a simple class of systems. Consider

$$\dot{x} = f(x_t), \qquad t \neq kT \tag{I.6.67}$$

$$\Delta x = g(x(t^-), x(t-T)), \qquad t = kT, \tag{I.6.68}$$

for $k \in \mathbb{Z}$, where $g : (\mathbb{R}^n)^2 \to \mathbb{R}^n$ is continuous and f satisfies assumption H.3. This system clearly fails the overlap condition. Consider the following modification of (I.6.67)–(I.6.67), which includes an additional state variable y:

$$\dot{x} = f(x_t), \qquad t \neq kT \tag{I.6.69}$$
$$\dot{y} = 0, \qquad t \neq kT \tag{I.6.70}$$
$$\Delta x = g(x(t^-), y(t^-)), \qquad t = kT \tag{I.6.71}$$
$$\Delta y = -y(t^-) + x(t^-) + g(x(t^-), y(t^-)), \qquad t = kT. \tag{I.6.72}$$

For brevity, we will call (I.6.69)–(I.6.72) the *transformed system*. Using the jump condition for x, we get

$$x((k-1)T) = x((k-1)T^-) + \Delta x = x((k-1)T^-) + g(x((k-1)T^-), y((k-1)T^-)).$$

On the other hand, since y is right-continuous, the differential equation and jump condition for y imply

$$\begin{aligned} y(kT^-) = y((k-1)T) &= x((k-1)T^-) + g(x((k-1)T^-), y((k-1)T^-)) \\ &= x((k-1)T). \end{aligned}$$

In other words, if $t = kT$, then $y(t^-) = x(t-T)$, so the jump condition for Δx reduces to the one from (I.6.68). The new impulsive functional differential equation satisfies the overlap condition because it has no delays in the impulse effects at all. The explicit correspondence between solutions of the new system and the original one is provided by the following proposition, whose proof is a fairly direct consequence of the above discussion.

Proposition I.6.4.1. *Let $x : \mathbb{R} \to \mathbb{R}^n$ be a solution of (I.6.67)–(I.6.68). Then, the function $(x(t), y(t))$ with y defined by $y(t) = x(kT)$ for $t \in [kT, (k+1)T)$ is a solution of the transformed system. Conversely, if (x, y) is a solution of the transformed system defined for $t \in \mathbb{R}$, then x is a solution of (I.6.67)–(I.6.68).*

Analogous statements can be made for solutions defined on a semiaxis $[a, \infty)$, and assertions for stability, asymptotic stability and instability carry over between each of these systems.

Proposition I.6.4.2. *Suppose $f(0) = 0$ and $g(0, 0) = 0$. The trivial solution is asymptotically stable in (I.6.67)–(I.6.68) if and only if the trivial solution is asymptotically stable in the transformed system. Analogous statements hold for stability and instability.*

Proof. Suppose zero is asymptotically stable in the original system. Then for all $\epsilon > 0$, there exists $\delta > 0$ such that if $||\phi|| < \delta$, then $||x_t|| < \epsilon$ for $t \geq T$

and $\lim_{t\to\infty} x(t) = 0$. Since g is continuous and f satisfies H.3, there exists $\delta_2 > 0$ such that the solution $(x(t), y(t))$ of the transformed system with initial condition $(x_0, y_0) = \Phi_0$ satisfies $||(x_T, y_T)|| < \delta$ provided $||\Phi_0|| < \delta_2$. Consider now the function $\tilde{x}(t) = x(t+T)$. Then $\tilde{x}_0 = x_T$ satisfies $||\tilde{x}_0|| < \delta$. It is straightforward to verify that $\tilde{x}$ is a solution of (I.6.67)–(I.6.68), from which it follows that $||\tilde{x}_t|| < \epsilon$ for $t \geq 0$ and $\lim_{t\to\infty} \tilde{x}(t) \to 0$. Thus, $||x_t|| < \epsilon$ for $t \geq T$ and $\lim_{t\to\infty} x(t) = 0$. From (I.6.72) and the previous discussion on the dynamics of the y component, we conclude $||y_t|| < \epsilon$ for $t \geq T$ and $\lim_{t\to\infty} y(t) = 0$. It follows that zero is asymptotically stable in the transformed system, since if we take $||(x_0, y_0)|| < \min\{\epsilon, \delta_2\}$, then we are guaranteed $||(x_t, y_t)|| < \epsilon$ for $t \geq 0$ and $\lim_{t\to\infty}(x(t), y(t)) = 0$. The converse is proven in a similar way, as are the analogous statements for stability and instability. □

For systems with several impulses per period, the construction is similar to the transformed system. One defines new state variables y_j with trivial continuous-time dynamics, with impulse effects that ensure that these new state variables correspond to lagged x variables. It is possible to set up the impulses for the y_j equations in such a way that they contain no delays. These are then transplanted into the original impulse effect for the x equations. The result is that the new system has no discrete-delayed impulses, so the overlap condition is satisfied. The general procedure is rather involved if one wants to keep the dimension as small as possible, so we do not develop it fully.

More generally, if both discrete and distributed delays are present, the same kind of trick can be used. For example, the scalar impulse effect

$$\Delta x = x(t-1)\int_{-1}^{0} g(x(t+s))ds, \qquad t \in \mathbb{Z} \tag{I.6.73}$$

fails the overlap condition with the sequence of impulses corresponding to the integers. If one introduces an additional variable y with continuous dynamics $\dot{y} = 0$ and considers the "augmented" impulse effect

$$\Delta x = y(t^-)\int_{-1}^{0} g(x(t+s))ds, \qquad t \in \mathbb{Z}$$

$$\Delta y = -y(t^-) + x(t^-) + y(t^-)\int_{-1}^{0} g(x(t+s))ds, \qquad t \in \mathbb{Z},$$

then from the jump conditions in x, we get

$$x(k-1) = x((k-1)^-) + y((k-1)^-)\int_{-1}^{0} g(x(k-1+s))ds.$$

From the dynamics $\dot{y} = 0$ and the jump condition for y, we get

$$y(k^-) = y(k-1) = x((k-1)^-) + y((k-1)^-)\int_{-1}^{0} g(x(k-1+s))ds$$

$$= x(k-1).$$

This is exactly the property we want, since then

$$\Delta x(k) = y(k^-)\int_{-1}^{0} g(x(k+s))ds = x(k-1)\int_{-1}^{0} g(x(k+s))ds,$$

which coincides with (I.6.73).

I.6.5 Comments

The content of this chapter features extended, more general versions of results appearing in *Computation of centre manifolds and some codimension-one bifurcations for impulsive delay differential equations* [33] by Church and Liu, published by Journal of Differential Equations in 2019. The approximation scheme for centre manifolds for delay differential equations (without impulses) seems to date back to 1997 with the work of Ait Babram, Hbid and Arino [5]. For ordinary differential equations, it is of course well-known.

Chapter I.7

Hyperbolicity and the Classical Hierarchy of Invariant Manifolds

In Chap. I.5 we studied the existence and smoothness of centre manifolds and their invariance properties. Chapter I.6 was devoted to computational aspects. Now we expand the scope. We discuss the existence and smoothness of unstable, stable and centre-stable manifolds, thereby establishing the classical hierarchy of invariant manifolds for impulsive functional differential equations. The presentation in this chapter will, however, be less thorough than the analogous ones for the centre manifold, since many of the proofs are similar.

I.7.1 Preliminaries

Let us introduce a few more spaces of exponentially weighted functions. The notation will be similar to what we have in Sect. I.5.1.1. First, we define

$$PC(-s,\mathbb{R}^n)=\{f|_{(-\infty,s]}:f\in PC(\mathbb{R},\mathbb{R}^n)\},\quad PC(+s,\mathbb{R}^n)=\{f|_{[s,\infty)}:f\in PC(\mathbb{R},\mathbb{R}^n)\}.$$

Next, for $s\in\mathbb{R}$, define $\mathbb{Z}_{-s}=\{k\in\mathbb{Z}:t_k\leq s\}$ and $\mathbb{Z}_{+s}=\{k\in\mathbb{Z}:t_k\geq s\}$. We introduce some exponentially weighted function spaces

$$\begin{aligned}\mathcal{PC}^{\eta,-s}&=\{\phi:(-\infty,s]\to\mathcal{RCR}:\phi(t)=f_t,f\in PC(-s,\mathbb{R}^n),||\phi||_{\eta,-s}<\infty\}\\ \mathcal{PC}^{\eta,+s}&=\{\phi:[s,\infty)\to\mathcal{RCR}:\phi(t)=f_f,f\in PC(+(s-r),\mathbb{R}^n),||\phi||_{\eta,+s}<\infty\}\\ B^{\eta,-s}(-s,\mathcal{RCR})&=\{f:(-\infty,s]\to\mathcal{RCR}:||f||_{\eta,-s}<\infty\}\\ B^{\eta,+s}(+s,\mathcal{RCR})&=\{f:[s,\infty)\to\mathcal{RCR}:||f||_{\eta,+s}<\infty\}\end{aligned}$$

K. E. M. Church, X. Liu, *Bifurcation Theory of Impulsive Dynamical Systems*, IFSR International Series in Systems Science and Systems Engineering 34, https://doi.org/10.1007/978-3-030-64533-5_7

$$
\begin{aligned}
&PC^{\eta,-s}(-s,\mathbb{R}^n) = \{f \in PC(-s,\mathbb{R}^n) : ||f||_{\eta,-s} < \infty\}\\
&PC^{\eta,+s}(+s,\mathbb{R}^n) = \{f \in PC(+s,\mathbb{R}^n) : ||f||_{\eta,+s} < \infty\}\\
&B^{\eta,-s}_{t_k}(\mathbb{Z}_{-s},\mathbb{R}^n) = \{f : \mathbb{Z}_{-s} \to \mathbb{R}^n : ||f||_{\eta,-s} < \infty\}\\
&B^{\eta,+s}_{t_k}(\mathbb{Z}_{-s},\mathbb{R}^n) = \{f : \mathbb{Z}_{+s} \to \mathbb{R}^n : ||f||_{\eta,+s} < \infty\},
\end{aligned}
$$

with the norms

$$
||f||_{\eta,-s} = \begin{cases} \sup_{t\leq s} ||f(t)||e^{-\eta(t-s)}, & \mathrm{dom}(f) = (-\infty, s] \\ \sup_{k\in\mathbb{Z}_{-s}} ||f(k)||e^{-\eta(t_k-s)}, & \mathrm{dom}(f) = \mathbb{Z}_{-s} \end{cases}
$$

$$
||f||_{\eta,+s} = \begin{cases} \sup_{t\geq s} ||f(t)||e^{-\eta(t-s)}, & \mathrm{dom}(f) = [s, \infty) \\ \sup_{k\in\mathbb{Z}_{+s}} ||f(k)||e^{-\eta(t_k-s)}, & \mathrm{dom}(f) = \mathbb{Z}_{+s}. \end{cases}
$$

Next, we introduce analogues of the linear operator $\mathcal{K}^\eta_s$ from Sect. I.5.1.2. Formally,

$$
\begin{aligned}
\mathcal{K}^\eta_{-s} &: \mathcal{PC}^{\eta,-s} \oplus B^{\eta,-s}_{t_k}(\mathbb{Z}_{-s},\mathbb{R}^n) \to B^{\eta,-s}(-s,\mathcal{RCR})\\
\mathcal{K}^\eta_{+s} &: \mathcal{PC}^{\eta,+s} \oplus B^{\eta,+s}_{t_k}(\mathbb{Z}_{+s},\mathbb{R}^n) \to B^{\eta,+s}(+s,\mathcal{RCR})\\
\mathcal{K}^\eta_{0-s} &: \mathcal{PC}^{\eta,-s} \oplus B^{\eta,-s}_{t_k}(\mathbb{Z}_{-s},\mathbb{R}^n) \to B^{\eta,-s}(-s,\mathcal{RCR})\\
\mathcal{K}^\eta_{0+s} &: \mathcal{PC}^{\eta,+s} \oplus B^{\eta,+s}_{t_k}(\mathbb{Z}_{+s},\mathbb{R}^n) \to B^{\eta,+s}(+s,\mathcal{RCR})
\end{aligned}
$$

defined as follows:

$$
\begin{aligned}
\mathcal{K}^\eta_{-s}(F,G)(t) = \int_s^t U(t,\mu)P_u(\mu)[\chi_0 F(\mu)]d\mu + \int_{-\infty}^t U(t,\mu)[I - P_u(\mu)][\chi_0 F(\mu)]d\mu \qquad \text{(I.7.1)}\\
+ \sum_s^t U(t,t_i)P_u(t_i)[\chi_0 G_i]dt_i + \sum_{-\infty}^t U(t,t_i)[I - P_u(t_i)][\chi_0 G_i]dt_i,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}^\eta_{+s}(F,G)(t) = \int_s^t U(t,\mu)P_s(\mu)[\chi_0 F(\mu)]d\mu - \int_t^\infty U(t,\mu)[I - P_s(\mu)][\chi_0 F(\mu)]d\mu \qquad \text{(I.7.2)}\\
+ \sum_s^t U(t,t_i)P_s(t_i)[\chi_0 G_i]dt_i - \sum_t^\infty U(t,t_i)[I - P_s(t_i)][\chi_0 G_i]dt_i.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}^\eta_{0-s}(F,G)(t) = \int_s^t U(t,\mu)[I - P_s(\mu)][\chi_0 F(\mu)]d\mu + \int_{-\infty}^t U(t,\mu)P_s(\mu)[\chi_0 F(\mu)]d\mu \qquad \text{(I.7.3)}\\
+ \sum_s^t U(t,t_i)[I - P_s(t_i)][\chi_0 G_i]dt_i + \sum_{-\infty}^t U(t,t_i)P_s(t_i)[\chi_0 G_i]dt_i,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}^\eta_{0+s}(F,G)(t) = \int_s^t U(t,\mu)P_s(\mu)[\chi_0 F(\mu)]d\mu - \int_t^\infty U(t,\mu)[I - P_s(\mu)][\chi_0 F(\mu)]d\mu \qquad \text{(I.7.4)}\\
+ \sum_s^t U(t,t_i)P_s(t_i)[\chi_0 G_i]dt_i - \sum_t^\infty U(t,t_i)[I - P_s(t_i)][\chi_0 G_i]dt_i.
\end{aligned}
$$

The following result is the appropriate analogue of Lemma I.5.1.3. Its proof is similar to that of the aforementioned result and is omitted.

Lemma I.7.1.1. *Let H.1, H.2, H.5 and H.7 hold.*

1. *For $\eta \in (0, \min\{-a, b\})$, $\mathcal{K}^{\eta}_{-s}$ and $\mathcal{K}^{\eta}_{0-s}$ are bounded linear maps with norms that can be chosen independent of s. For any compact interval $J \subset (0, \min\{-a, b\})$, the norms are bounded uniformly for $\eta \in J$.*

2. *For $-\eta \in (0, \min\{-a, b\})$, $\mathcal{K}^{\eta}_{+s}$ and $\mathcal{K}^{\eta}_{0+s}$ are bounded linear maps with norms that can be chosen independent of s. For any compact interval $J \subset (-\min\{-a, b\}, 0)$, the norms are bounded uniformly for $\eta \in J$.*

3. *With η satisfying the above inequality, $\mathcal{K}^{\eta}_{-s}$ has range in $\mathcal{PC}^{\eta,-s}$ and $v = \mathcal{K}^{\eta}_{-s}(F, G)$ is the unique solution of (I.5.1) in $\mathcal{PC}^{\eta,-s}$ such that $P_s(s)v(s) = 0$.*

4. *With η satisfying the above inequality, $\mathcal{K}^{\eta}_{+s}$ has range in $\mathcal{PC}^{\eta,+s}$ and $v = \mathcal{K}^{\eta}_{+s}(F, G)$ is the unique solution of (I.5.1) in $\mathcal{PC}^{\eta,+s}$ such that $P_u(s)v(s) = 0$.*

5. *With η satisfying the above inequality, $\mathcal{K}^{\eta}_{0-s}$ has range in $\mathcal{PC}^{\eta,-s}$ and $v = \mathcal{K}^{\eta}_{0-s}(F, G)$ is the unique solution of (I.5.1) in $\mathcal{PC}^{\eta,-s}$ such that $(P_c(s) + P_s(s))v(s) = 0$.*

6. *With η satisfying the above inequality, $\mathcal{K}^{\eta}_{0+s}$ has range in $\mathcal{PC}^{\eta,-s}$ and $v = \mathcal{K}^{\eta}_{0+s}(F, G)$ is the unique solution of (I.5.1) in $\mathcal{PC}^{\eta,+s}$ such that $(P_c(s) + P_u(s))v(s) = 0$.*

I.7.2 Unstable Manifold

Let $\eta \in (0, \min\{-a, b\})$. At this stage, we reintroduce the substitution operators

$$R_{-s} : \mathcal{PC}^{\eta,-s} \to B^{\eta,-s}(-s, \mathbb{R}^n) \oplus B^{\eta,-s}_{t_k}(\mathbb{Z}_{-s}, \mathbb{R}^n),$$

defined by $R_{-s}(x)(t, k) = (f(t, x(t)), g(k, x(t_k)_{0^-}))$. One can then prove the following lemma.

Lemma I.7.2.1. *Let H.4 and H.7 hold. The substitution operator defined above is m-times continuously differentiable. Moreover, on the ball $B_\delta(0)$ in $\mathcal{PC}^{\eta,-s}$, the substitution operator is Lipschitz continuous with Lipschitz constant L_δ that satisfies $L_\delta \to 0$ as $\delta \to 0$ (and is independent of s).*

This lemma is the reason we do not need to cut off the nonlinearity. It is a consequence of the fact that if $u \in \mathcal{PC}^{\eta,-s} \cap B_\delta(0)$ for $\eta > 0$, then $||u(t)|| \le \delta$ for all $t \le s$. Let $\eta \in (0, \min\{-a, b\})$, and introduce a map $\mathcal{F}_{-s} : \mathcal{PC}^{\eta,-s} \times \mathcal{RCR}_u(s) \to \mathcal{PC}^{\eta,-s}$ defined by

$$\mathcal{F}_{-s}(u, \varphi) = U(\cdot, s)\varphi + \mathcal{K}^{\eta}_{-s}(R_{-s}(u)).$$

In the same way that we proved Theorem I.5.2.1, one can show that if $||\varphi|| < \delta_1$ is small enough, then $\mathcal{F}_{-s}(\cdot,\varphi)$ has a unique fixed point in some ball $B_{\delta_2}(0) \cap \mathcal{PC}^{\eta,-s}$. Moreover, δ_1 and δ_2 can be chosen independent of s, and the fixed point is (uniformly in s) Lipschitz continuous with respect to φ.

Theorem I.7.2.1. *Let assumptions H.1–H.7 hold. There exist δ_1 and $\delta_2 > 0$ such that for all $\varphi \in B_{\delta_1}(0) \cap \mathcal{RCR}_u(s)$, there is a unique $u^*_{-s} = u^*_{-s}(\varphi) \in B_{\delta_2}(0) \cap \mathcal{PC}^{\eta,-s}$ such that $u^*_{-s} = \mathcal{F}^{\eta}_{-s}(u^*_{-s},\varphi)$.*

Definition I.7.2.1. *The* local unstable manifold*, $\mathcal{W}_u \subset \mathbb{R} \times \mathcal{RCR}$, is the nonautonomous set whose t-fibres for $t \in \mathbb{R}$ are given by*

$$\mathcal{W}_u(t) = \mathrm{Im}\{\mathcal{U}(t,\cdot)\}, \tag{I.7.5}$$

where $\mathcal{U} : \mathcal{RCR}_u \cap B_{\delta_1}(0) \to \mathcal{RCR}$ is the (fibrewise) Lipschitz map defined by $\mathcal{U}(t,\phi) = u^*_{-t}(\phi)(t)$.

Corollary I.7.2.1. *There exists a constant $L > 0$ such that $||\mathcal{U}(t,\phi) - \mathcal{U}(t,\psi)|| \leq L||\phi - \psi||$ for all $t \in \mathbb{R}$ and $\phi,\psi \in \mathcal{RCR}_u(t)$.*

The local unstable manifold is both positively and negatively invariant under the nonautonomous process $S(t,s) : \mathcal{RCR} \to \mathcal{RCR}$, in the following sense. The proof is similar to the proof of part 1 of Theorem I.5.3.1.

Theorem I.7.2.2. *Let conditions H.1–H.7 hold.*

- *For φ sufficiently small and $t \leq s$, we have $u^*_{-s}(\varphi)(t) \in \mathcal{W}_u(t)$. In particular, if $(s,\phi) \in \mathcal{W}_u$, then there exists a unique mild solution $u \in \mathcal{PC}^{\eta,-s}$ of (I.4.1)–(I.4.2) with the property that $u(t) \in \mathcal{W}_u(t)$, $||u||_{\eta,-s} \leq \delta_2$, and $u(s) = \phi$.*
- *If $(s,\phi) \in \mathcal{W}_u$, there exists $T > s$ such that $(t,S(t,s)\phi) \in \mathcal{W}_u$ for $t \in [s,T]$.*

To prove smoothness of the unstable manifold (in the state space), we will apply the implicit function theorem to the solutions of the equation $\mathcal{F}_{-s} = 0$, with

$$\mathcal{F}_{-s}(u,\varphi) = u - \mathcal{F}_{-s}(u,\varphi).$$

Because of Lemma I.7.2.1, $\mathcal{F}_s$ is m-times continuously differentiable, $\mathcal{F}_{-s}(0,0) = 0$ and the differential at zero is $D_u\mathcal{F}_{-s}(0,0) = I$. One can then directly apply the implicit function theorem to guarantee the existence of an m-times continuously differentiable $\varphi \mapsto \tilde{u}^*_{-s}(\varphi)$ defined on some neighbourhood $B_\rho(0) \subset \mathcal{RCR}_u(s)$, such that $\mathcal{F}_{-s}(\tilde{u}^*_{-s}(\varphi),\varphi) = 0$. By restricting to $B_{\delta_1}(0)$, we get the equality $\tilde{u}^*_{-s} = u^*_{-s}$. Since this operation allows formal differentiation of the fixed-point equation, we immediately get $Du^*_{-s}(0) = U(\cdot,s)$. Finally, as the evaluation functional $ev_s : \mathcal{PC}^{\eta,-s} \to \mathcal{RCR}$ defined by $ev_s(f) = f(s)$ is linear and bounded, the following theorem is proven.

Theorem I.7.2.3. *$\mathcal{U}(t,\cdot) : \mathcal{RCR}_u(t) \to \mathcal{RCR}$ is m-times continuously differentiable, and $D\mathcal{U}(t,0)\phi = \phi$ for all $\phi \in \mathcal{RCR}_u(t)$. Each of $t \mapsto D^j\mathcal{U}(t,0)$ is uniformly bounded.*

I.7.3 Stable Manifold

The construction here is essentially symmetric to the one for the unstable manifold. With $-\eta \in (0, \min\{-a, b\})$, we can define the substitution operator

$$R_{+s} : \mathcal{PC}^{\eta,+s} \to B^{\eta,+s}(+s, \mathbb{R}^n) \oplus B^{\eta,+s}_{t_k}(\mathbb{Z}_{+s}, \mathbb{R}^n),$$

with the same formula as previously. In the same way as before, the following lemma is applicable. It is a consequence of the fact that, if $||u||_{\eta,+s} \le \delta$ and $\eta < 0$, then $||u(t)|| \le \delta$ for all $t \ge s$.

Lemma I.7.3.1. *Let H.4 and H.7 hold. The substitution operator defined above is m-times continuously differentiable. Moreover, on the ball $B_\delta(0)$ in $\mathcal{PC}^{\eta,+s}$, the substitution operator is Lipschitz continuous with Lipschitz constant L_δ that satisfies $L_\delta \to 0$ as $\delta \to 0$.*

We can then proceed to define the fixed-point operator $\mathcal{F}_{+s} : \mathcal{PC}^{\eta,+s} \times \mathcal{RCR}_s(s) \to \mathcal{PC}^{\eta,+s}$ by

$$\mathcal{F}_{+s}(u, \varphi) = U(\cdot, s)\varphi + \mathcal{K}^{\eta}_{+s}(R_{+s}(u))$$

and ultimately obtain the following results. They are proven similarly to the analogous results in Sect. I.5.2.

Theorem I.7.3.1. *Let assumptions H.1–H.7 hold. There exist δ_1 and $\delta_2 > 0$ such that for all $\varphi \in B_{\delta_1}(0) \cap \mathcal{RCR}_u(s)$, there is a unique $u^*_{+s} = u^*_{+s}(\varphi) \in B_{\delta_2}(0) \cap \mathcal{PC}^{\eta,+s}$ such that $u^*_{+s} = \mathcal{F}^{\eta}_{+s}(u^*_{+s}, \varphi)$.*

Definition I.7.3.1. *The* local stable manifold*, $\mathcal{W}_s \subset \mathbb{R}\times\mathcal{RCR}$, is the nonautonomous set whose t-fibres for $t \in \mathbb{R}$ are given by*

$$\mathcal{W}_s(t) = \text{Im}\{\mathcal{T}(t,\cdot)\},$$

where $\mathcal{T} : \mathcal{RCR}_s \cap B_{\delta_1}(0) \to \mathcal{RCR}$ is the (fibrewise) Lipschitz map defined by $\mathcal{T}(t,\phi) = u^*_{+t}(\phi)(t)$.

Corollary I.7.3.1. *There exists a constant $L > 0$ such that $||\mathcal{T}(t,\phi) - \mathcal{T}(t,\psi)|| \le S||\phi - \psi||$ for all $t \in \mathbb{R}$ and $\phi, \psi \in \mathcal{RCR}_s(t)$.*

Theorem I.7.3.2. *Let conditions H.1–H.7 hold. If $(s_1, \phi) \in \mathcal{W}_s$ and ϕ is sufficiently small, then $(t, S(t,s_1)\phi) \in \mathcal{W}_s$ for all $t \ge s_1$. Additionally, for each $\gamma > 0$, there exist $\delta > 0$ and $C > 0$ such that for all $\varphi \in \mathcal{RCR}_s(s_1)$ with $||\varphi|| \le \delta$, we have the estimate*

$$||S(t,s_1)\mathcal{T}(s_1,\varphi)|| \le Ce^{(a+\gamma)(t-s_1)}.$$

Also, $0 \in \mathcal{W}_s(t)$ for all $t \in \mathbb{R}$.

Theorem I.7.3.3. *$\mathcal{T}(t,\cdot) : \mathcal{RCR}_s(t) \to \mathcal{RCR}$ is m-times continuously differentiable, and $D\mathcal{T}(t,0)\phi = \phi$ for all $\phi \in \mathcal{RCR}_s(t)$. Each of $t \mapsto D^j\mathcal{T}(t,0)$ is uniformly bounded.*

I.7.4 Centre-Unstable Manifold

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a C^∞ cutoff function as introduced in Sect. I.5.1.3. For $\delta > 0$ and $s \in \mathbb{R}$, we define the cutoff nonlinearities

$$F_{\delta,-s}(t,x) = f(t,x)\xi\left(\frac{||(P_c(s)+P_u(s))x||}{N\delta}\right)\xi\left(\frac{||P_s(s)x||}{N\delta}\right)$$
$$G_{\delta,-s}(k,x) = g(k,x_{0-})\xi\left(\frac{||(P_c(s)+P_u(s))x_{0-}||}{N\delta}\right)\xi\left(\frac{||P_s(s)x_{0-}||}{N\delta}\right).$$

Next, let $\eta \in (0,\min\{-a,b\})$, and define $R_{0-s} : \mathcal{PC}^{\eta,-s} \to B^{\eta,-s}(-s,\mathbb{R}^n) \oplus B^{\eta,-s}_{t_k}(\mathbb{Z}_{-s},\mathbb{R}^n)$ by

$$R_{\delta,0-s}(x)(t,k) = (F_{\delta,-s}(t,x(t)), G_{\delta,-s}(k,x(t_k)_{0-})).$$

One can then derive an analogue of Lemma I.5.1.4 and Corollary I.5.1.1. If one then introduces a nonlinear map $\mathcal{F}_{0-s} : PC^{\eta,-s} \times \mathcal{RCR}_c(s) \oplus \mathcal{RCR}_u(s) \to PC^{\eta,-s}$ by

$$\mathcal{F}_{0-s}(u,\varphi) = U(\cdot,s)\varphi + K^\eta_{0-s}(R_{\delta,0-s}(u)),$$

then by essentially the same proof as Theorem I.5.2.1, one obtains the following.

Theorem I.7.4.1. *If $\delta > 0$ is chosen sufficiently small, there exists a globally Lipschitz continuous mapping $u^*_{0-s} : \mathcal{RCR}_c(s) \oplus \mathcal{RCR}_u(s) \to PC^{\eta,-s}$ such that $u_{0-s} = u_{0-s}(\varphi)$ is the unique solution in $\mathcal{PC}^{\eta,-s}$ of the equation $u_{0-s} = \mathcal{F}_{0-s}(\varphi, u_{0-s})$.*

Definition I.7.4.1 (Lipschitz Centre-Unstable Manifold)**.** *The centre-unstable manifold, $\mathcal{W}_{cu} \subset \mathbb{R}\times\mathcal{RCR}$, is the nonautonomous set whose t-fibres for $t \in \mathbb{R}$ are given by*

$$\mathcal{W}_{cu}(t) = \mathrm{Im}\{\mathcal{CU}(t,\cdot)\},$$

where $\mathcal{CU}(t,\cdot) : \mathcal{RCR}_c(t) \oplus \mathcal{RCR}_u(t) \to \mathcal{RCR}$ is the fibrewise Lipschitz map defined by $\mathcal{CU}(t,\varphi) = u^*_{0-t}(\varphi)(t)$.

Corollary I.7.4.1. *There exists a constant $L > 0$ such that $||\mathcal{CU}(t,\phi) - \mathcal{CU}(t,\psi)|| \le S||\phi-\psi||$ for all $t \in \mathbb{R}$ and $\phi,\psi \in \mathcal{RCR}_c(t) \oplus \mathcal{RCR}_u(t)$.*

The centre-unstable manifold has much the same invariance properties as the centre manifold.

Theorem I.7.4.2. *Let conditions H.1–H.7 hold. The centre-unstable manifold $\mathcal{W}_{cu}$ enjoys the following properties:*

1. *$\mathcal{W}_{cu}$ is locally positively invariant: if $(s, \phi) \in \mathcal{W}_{cu}$ and $||\phi|| < \delta$, there exists $T > s$ such that $(t, S(t, s)\phi) \in \mathcal{W}_{cu}$ for $t \in [s, T]$.*

2. *If $(s, \phi) \in \mathcal{W}_{cu}$, there exists a unique mild solution $u \in \mathcal{PC}^{\eta,-s}$ of the semilinear system*

$$\begin{aligned} \dot{x} &= L(t)x_t + F_{\delta,-s}(t, x_t), & t \neq t_k \\ \Delta x &= B(k)x_{t^-} + G_{\delta,-s}(k, x_{t^-}), & t = t_k \end{aligned}$$

   with the property that $u(t) \in \mathcal{W}_{cu}(t)$, for $t \leq s$, $||u||_{\eta,-s} \leq \delta$, and $u(s) = \phi$.

3. *If u is a mild solution satisfying $||u(t)|| < \delta$ on $(-\infty, s]$, then $(t, u(t)) \in \mathcal{W}_{cu}$ for $t \leq s$.*

4. *$\mathbb{R} \times \{0\} \subset \mathcal{W}_{cu}$ and $\mathcal{CU}(t, 0) = 0$ for all $t \in \mathbb{R}$.*

To obtain smoothness of the centre manifold, we need to assume H.1–H.7 in addition to H.8, H.9 and the finite-dimensionality of $\mathcal{RCR}_u$. The proof follows the same lines as the one of Theorem I.5.6.1 and the relevant Corollarys I.5.6.1 and I.5.6.3. The proof is omitted.

Theorem I.7.4.3. *Assume that conditions H.1–H.8 are satisfied, and both $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ are finite-dimensional. $\mathcal{CU} : \mathcal{RCR}_c \oplus \mathcal{RCR}_u \to \mathcal{RCR}$ is C^m and tangent at the origin to the centre-unstable fibre bundle $\mathcal{RCR}_c \oplus \mathcal{RCR}_u$. More precisely, $\mathcal{CU}(t, \cdot) : \mathcal{RCR}_c(t) \oplus \mathcal{RCR}_u(t) \to \mathcal{RCR}$ is C^m and $D\mathcal{CU}(t, 0)\phi = \phi$ for all $\phi \in \mathcal{RCR}_c(t) \oplus \mathcal{RCR}_u(t)$. Each of $t \mapsto D^j\mathcal{CU}(t, 0)$ is uniformly bounded.*

I.7.5 Centre-Stable Manifold

The construction here is essentially symmetric to the one for the centre-unstable manifold. Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a C^∞ cutoff function as introduced in Sect. I.5.1.3. For $\delta > 0$ and $s \in \mathbb{R}$, we define the cutoff nonlinearities as

$$\begin{aligned} F_{\delta,+s}(t, x) &= f(t, x)\xi\left(\frac{||(P_c(s) + P_s(s))x||}{N\delta}\right)\xi\left(\frac{||P_u(s)x||}{N\delta}\right) \\ G_{\delta,+s}(k, x) &= g(k, x_{0^-})\xi\left(\frac{||(P_c(s) + P_s(s))x_{0^-}||}{N\delta}\right)\xi\left(\frac{||P_u(s)x_{0^-}||}{N\delta}\right). \end{aligned}$$

Let $-\eta \in (0, \min\{-a, b\})$, and define $R_{0+s} : \mathcal{PC}^{\eta,+s} \to B^{\eta,+s}(-s, \mathbb{R}^n) \oplus B^{\eta,+s}_{t_k}(\mathbb{Z}_{+s}, \mathbb{R}^n)$ by

$$R_{\delta,0+s}(x)(t, k) = (F_{\delta,+s}(t, x(t)), G_{\delta,+s}(k, x(t_k)_{0^-})).$$

One can once again derive an analogue of Lemma I.5.1.4 and Corollary I.5.1.1. If one then introduces a nonlinear map $\mathcal{F}_{0+s} : PC^{\eta,+s} \times \mathcal{RCR}_c(s) \oplus \mathcal{RCR}_s(s) \to PC^{\eta,+s}$ by

$$\mathcal{F}_{0+s}(u, \varphi) = U(\cdot, s)\varphi + K^{\eta}_{0+s}(R_{\delta,0+s}(u)),$$

then by essentially the same proof as Theorem I.5.2.1, one obtains the following.

Theorem I.7.5.1. *If $\delta > 0$ is chosen sufficiently small, there exists a globally Lipschitz continuous mapping $u^*_{0+s} : \mathcal{RCR}_c(s) \oplus \mathcal{RCR}_u(s) \to PC^{\eta,+s}$ such that $u_{0+s} = u_{0-s}(\varphi)$ is the unique solution in $\mathcal{PC}^{\eta,+s}$ of the equation $u_{0+s} = \mathcal{F}_{0+s}(\varphi, u_{0+s})$.*

Definition I.7.5.1 (Lipschitz Centre-Stable Manifold)**.** *The centre-stable manifold, $\mathcal{W}_{cs} \subset \mathbb{R} \times \mathcal{RCR}$, is the nonautonomous set whose t-fibres for $t \in \mathbb{R}$ are given by*

$$\mathcal{W}_{cu}(t) = \text{Im}\{\mathcal{CT}(t, \cdot)\},$$

where $\mathcal{CT}(t, \cdot) : \mathcal{RCR}_c(t) \oplus \mathcal{RCR}_s(t) \to \mathcal{RCR}$ is the fibrewise Lipschitz map defined by $\mathcal{CT}(t, \varphi) = u^*_{0+t}(\varphi)(t)$.

Corollary I.7.5.1. *There exists a constant $L > 0$ such that $||\mathcal{CT}(t, \phi) - \mathcal{CT}(t, \psi)|| \leq S||\phi - \psi||$ for all $t \in \mathbb{R}$ and $\phi, \psi \in \mathcal{RCR}_c(t) \oplus \mathcal{RCR}_s(t)$.*

Theorem I.7.5.2. *The centre-stable manifold enjoys the following properties:*

- *$\mathcal{W}_{cu}$ is locally positively invariant with respect to $S(t, s)$; if $(s, \phi) \in \mathcal{W}_{cs}$ and $||\phi|| < \delta$, there exists $T > 0$ such that $(t, S(t, s)\phi) \in \mathcal{W}_{cs}$ for $t \in [s, T]$.*

- *If $(s, \phi) \in \mathcal{W}_{cu}$, there exists a unique mild solution $u \in \mathcal{PC}^{\eta,+s}$ of the semilinear system*

$$\begin{aligned} \dot{x} &= L(t)x_t + F_{\delta,+s}(t, x_t), && t \neq t_k \\ \Delta x &= B(k)x_{t^-} + G_{\delta,+s}(k, x_{t^-}), && t = t_k \end{aligned}$$

  with the property that $u(t) \in \mathcal{W}_{cs}(t)$, for $t \geq s$, $||u||_{\eta,+s} \leq \delta$, and $u(s) = \phi$.

- *If u is a mild solution satisfying $||u(t)|| < \delta$ on $[s, \infty)$, then $(t, u(t)) \in \mathcal{W}_{cs}$ for $t \geq s$.*

- *$\mathbb{R} \times \{0\} \subset \mathcal{W}_{cs}$ and $\mathcal{CT}(t, 0) = 0$ for all $t \in \mathbb{R}$.*

The same proof of smoothness we used for the centre manifold does not work here because $\mathcal{RCR}_{cs}(t) := \mathcal{RCR}_c(t) \oplus \mathcal{RCR}_s(t)$ is always infinite- dimensional, so we cannot assume by a suitable renorming that $x \mapsto ||x||$ is smooth on $\mathcal{RCR}_{cs}(t) \setminus \{0\}$.

I.7.6 Dynamics on Finite-Dimensional Invariant Manifolds

Suppose $\mathcal{RCR}_j$ is one of the finite-dimensional invariant fibre bundles $\mathcal{RCR}_c$, $\mathcal{RCR}_u$ or $\mathcal{RCR}_{cu} = \mathcal{RCR}_c \oplus \mathcal{RCR}_u$. Then, analogously to Lemma I.5.4.1, for any sufficiently small $y : \mathbb{R} \to \mathcal{RCR}$ satisfying $y(t) \in \mathcal{W}_j(t)$, the j-fibre bundle component $w(t) = P_j(t)y(t)$ satisfies the integral equation

$$w(t) = U(t,s)w(s) + \int_s^t U(t,\mu)P_j(\mu)\chi_0 f(\mu, y_\mu)d\mu + \sum_{s<t_i\leq t} U(t,t_i)P_j(t_i)\chi_0 g(k, y_{t_i^-}).$$

If one poses that $w(t) = X_t z(t)$ for X_t an array whose columns form a basis for $\mathcal{RCR}_j(t)$ and satisfy $X_t = U(t,0)X_0$ for all $t \in \mathbb{R}$, and we let $Y_j(t)$ be a matrix-valued function satisfying $P_j(t)\chi_0 = X_t Y_j(t)$, then the function $z : \mathbb{R} \to \mathbb{R}^p$ for $p = \dim(\mathcal{RCR}_j)$ satisfies

$$w(t) = w(s) + \int_s^t Y_j(\mu) f(\mu, y_\mu)d\mu + \sum_{s<t_i\leq t} Y_j(t_i)g(i, y_{t_i^-}).$$

y_t can be written in the equivalent form as

$$\begin{aligned} y_t &= X_t z(t) + M(t, z(t)) \\ M(t,z) &= (I - P_j(t))\mathcal{M}(t, X_t z), \end{aligned}$$

for $z \in \mathbb{R}^p$ and $\mathcal{M}(t,\cdot) : \mathcal{RCR}_j(t) \to \mathcal{RCR}$ defining the fibres of the relevant invariant manifold $\mathcal{W}_j$. The dynamics on said invariant manifold are therefore determined by those of the $p = \dim(\mathcal{RCR}_j)$-dimensional ordinary impulsive differential equation

$$\begin{aligned} \dot{z} &= Y_j(t) f(t, X_t z(t) + M(t, z(t))), && t \neq t_k \\ \Delta z &= Y_j(t) g(k, X_t^- z(t^-) + [M(t, z(t))]^-), && t = t_k, \end{aligned}$$

assuming $t \mapsto Y_j(t)$ is an element of $\mathcal{RCR}(\mathbb{R}, \mathbb{R}^{n\times p})$. In this sense we have a generalized, non-periodic version of Theorem I.6.1.2. However, since $\xi \mapsto X_t\xi$ is not necessarily bounded (it is guaranteed to be unbounded if $\mathcal{W}_j$ is, for example, the unstable manifold and $\mathcal{RCR}_u$ is nontrivial), the topological conjugacy between the above system and the dynamics on the invariant manifold is only local in time.

Under periodicity assumptions, we can write $X_t = Q_t e^{t\Lambda}$ in the Floquet decomposition. If one then defines $m(t,u,\theta) = M(t, Q_t u)(\theta)$ for $u \in \mathbb{R}^p$, then under this same assumption of regularity on $t \mapsto Y_j(t)$ and the overlap condition, the function $u(t) = e^{t\Lambda}z(t)$ satisfies

$$\begin{aligned} \dot{u} &= \Lambda u(t) + e^{t\Lambda} Y_j(t) f(t, Q_t u(t) + m(t, u(t), \cdot)), && t \neq t_k \\ \Delta u &= e^{t\Lambda} Y_j(t) g(k, Q_t^- u(t^-) + m(t^-, u(t^-), \cdot)), && t = t_k. \end{aligned}$$

Since this transformation is equivalent to $w(t) = Q_t u(t)$ and $t \mapsto Q_t$ is uniformly bounded, the conjugacy between the above impulsive differential equation and the dynamics on the invariant manifold is all-time. In this way we obtain a generalization of Theorem I.6.1.2 to two other finite-dimensional invariant manifolds. Under conditions that are fully analogous to those of Theorem I.5.7.1, the function $M : \mathbb{R}\times\mathbb{R}^p \to \mathcal{RCR}$ is pointwise $PC^{1,m}$-regular at zero. The Euclidean space representation $m : \mathbb{R} \times \mathbb{R}^p \times [-r, 0] \to \mathbb{R}^n$ then has all properties outlined in Theorem I.6.1.1, except of course that its Taylor coefficients m_i satisfy $P_j(t)m_i(t, \cdot) = 0$ for the correct projector.

I.7.7 Linearized Stability and Instability, Revisited

Proposition I.4.3.1 states that when the unstable and centre fibre bundles are trivial, the fixed point 0 of the nonlinear equation (I.4.1)–(I.4.2) is exponentially stable. With the help of the unstable manifold, we can prove a converse, completing our extension of the classical linearized stability/instability theorem. The following lemma will be helpful in proving the linearized instability theorem; its proof is an elementary consequence of the definition of instability and is omitted.

Lemma I.7.7.1. *Suppose there exists $\epsilon > 0$ such that for all $s \in \mathbb{R}$, there exist sequences $x_n \in \mathcal{RCR}$ and $t_n \in \mathbb{R}$ with $x_n \to 0$ and $t_n > s$ satisfying $t_n \to \infty$, such that $||S(t_n, s)x_n|| \geq \epsilon$. Then, the fixed point 0 is unstable.*

Theorem I.7.7.1. *Let assumptions H.1–H.7 hold. The fixed point $0 \in \mathcal{RCR}$ of the nonlinear equation (I.4.1)–(I.4.2) is unstable if $\mathcal{RCR}_u$ is nontrivial. If for all $\delta > 0$ sufficiently small, there exists $c(\delta) \geq 0$ satisfying $\lim_{\delta\to 0^+} c(\delta) = 0$ and such that*

$$||f(t, \phi) - f(t, \psi)|| \leq c||\phi - \psi||$$
$$||g(k, \phi) - g(k, \psi)|| \leq c||\phi - \psi||,$$

for all $t \in \mathbb{R}$, $k \in \mathbb{Z}$ and $\phi, \psi \in B_\delta(0)$, then the fixed point is exponentially stable provided both $\mathcal{RCR}_c$ and $\mathcal{RCR}_u$ are trivial.

Proof. The exponential stability result is precisely Proposition I.4.3.1. For the instability result, let $s \in \mathbb{R}$ be given. Let $s < t_n \to \infty$, and let $\varphi_n \in \mathcal{RCR}_u(t_n)$ be a sequence such that $||\varphi_n|| = \delta_1$, with $\delta_1 > 0$ the constant from Theorem I.7.2.1. Consider the sequence $\xi_n = \mathcal{U}(t_n, \varphi_n)$. From the fixed-point equation (for the unstable manifold), it follows that $P_u(t_n)\xi_n = \varphi_n$ for all n. Also, we have

$$\delta_1 = ||\varphi_n|| = ||P_u(t_n)\xi_n|| \leq N||\xi_n||,$$

from which we get the lower bound $||\xi_n|| \geq \frac{1}{N}\delta_1$ for all $n \in \mathbb{N}$. From Theorem I.7.2.2, there exists a mild solution $u_n \in \mathcal{PC}^{\eta,-t_n}$ satisfying $u_n(t_n) = \xi_n$.

From the bound $||u_n||_{\eta,-t_n} \leq \delta_2$, we have the exponential estimate $||u_n(t)|| \leq \delta_2 e^{\eta(t-t_n)}$ for all $t \leq t_n$. In particular, we have $||u_n(s)|| \leq \delta_2 e^{\eta(s-t_n)}$. Since $t_n \to \infty$, we get $u_n(s) \to 0$ as $n \to \infty$. Applying Lemma I.7.7.1 with the sequence $x_n = u_n(s)$ and $\epsilon = \frac{1}{N}\delta_1$, we get the claimed result. $\square$

I.7.8 Hierarchy and Inclusions

The classical hierarchy of invariant manifolds is so named because of the chain of inclusions

$$\mathcal{W}_s \quad \subseteq \quad \mathcal{W}_{cs} \quad \supseteq \quad \mathcal{W}_c \quad \subseteq \quad \mathcal{W}_{cu} \quad \supseteq \quad \mathcal{W}_u,$$

for which at least two inclusions must always be strict. These inclusions can easily be checked using the definitions of the relevant invariant manifolds based on the fixed-point operators. One can form an *extended hierarchy* of invariant manifolds if there is a finer decomposition of the phase space by the evolution family into invariant fibre bundles—see Pötzsche [114] for the idea as it applies to dynamic equations on measure chains. This is the case for periodic systems, since there is generally an infinite family of invariant fibre bundles parameterized by the Floquet multipliers.

Chapter I.8

Smooth Bifurcations

The centre manifold reduction provides a framework in which bifurcations of fixed points and periodic solutions can be studied. In this section we will explain how the centre manifold reduction can be adapted to take into account parameters, prove two generic bifurcation patterns and present a general recipe for how one might study smooth local bifurcations in impulsive functional differential equations.

I.8.1 Centre Manifolds Depending Smoothly on Parameters

Suppose we have a system depending on a parameter $\epsilon \in \mathbb{R}^p$ for some $p > 0$:

$$\dot{x} = f(t, x_t, \epsilon), \qquad t \neq t_k \tag{I.8.1}$$

$$\Delta x = g(k, x_{t^-}, \epsilon), \qquad t = t_k. \tag{I.8.2}$$

Suppose $f(t, 0, 0) = g(k, 0, 0)$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, so that $x = 0$ is an equilibrium when $\epsilon = 0$. If the linearization has a c-dimensional centre fibre bundle when $\epsilon = 0$, we can construct a *centre manifold depending on the parameter* ϵ as follows.

We consider the following extension of (I.8.1)–(I.8.2) to the phase space $\mathcal{RCR}([-r, 0], \mathbb{R}^{n+p})$:

$$\dot{x} = f(t, x_t, \epsilon), \qquad t \neq t_k \tag{I.8.3}$$

$$\dot{\epsilon} = 0, \qquad t \neq t_k \tag{I.8.4}$$

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$$\Delta x = g(k, x_{t^-}, \epsilon), \qquad t = t_k \tag{I.8.5}$$
$$\Delta \epsilon = 0, \qquad t = t_k. \tag{I.8.6}$$

The above system has no parameter dependence, and if one fixes the initial condition for the ϵ component—that is, some function $\epsilon \in \mathcal{RCR}([-r,0],\mathbb{R}^p)$—then the solution of the x component through some initial condition pair $(s,\phi) \in \mathbb{R} \times \mathcal{RCR}$ will coincide with the solution of (I.8.1)–(I.8.2) with parameter $\epsilon(0)$ through the same initial condition. Thus, the above system in the extended phase space contains, after projection to the first component, every solution of the original parameter-dependent system.

One can identify the functional on the right-hand side of the first two Eqs. (I.8.3)–(I.8.4) with $F(t,\cdot) : \mathcal{RCR}([-r,0],\mathbb{R}^{n+p}) \to \mathbb{R}^{n+p}$ defined by

$$F(t,\phi) = [\begin{array}{cc} f(t,\phi_1,\phi_2(0)) & 0 \end{array}]^\intercal,$$

for $\phi = (\phi_1,\phi_2) \in \mathcal{RCR}([-r,0],\mathbb{R}^n) \times \mathcal{RCR}([-r,0],\mathbb{R}^p)$. One can similarly identify the jump functional in (I.8.5)–(I.8.6) with the analogous $G(k,\cdot)$. Abstractly, the result is the impulsive system

$$\dot{X} = F(t,X_t), \qquad t \neq t_k$$
$$\Delta X = G(k,X_{t^-}), \qquad t = t_k$$

without parameters. Assuming this system satisfies conditions H.1–H.7, the centre fibre bundle of the linearization at $(0,0)$ will be $(c+p)-$dimensional, and there will be a $(c+p)$-dimensional centre manifold. The centre fibre bundle will always take the form

$$\mathcal{RCR}_c(t) = (\mathcal{RCR}_c^0(t) \times \{0\}) \oplus \text{span}\{(\phi_t^1, e_1)\} \oplus \cdots \oplus \text{span}\{(\phi_t^p, e_p)\},$$

where $\mathcal{RCR}_c^0$ is the c-dimensional centre fibre bundle associated to the linearization of (I.8.1)–(I.8.2) at $\epsilon = 0$, each $\phi_t^i \in \mathcal{RCR}$ is nonzero and e_i is the ith standard basis vector in $\mathbb{R}^p$. We can therefore represent $\mathcal{RCR}_c(t)$ in the equivalent way

$$\mathcal{RCR}_c(t) = (\mathcal{RCR}_c^0(t) \times \{0\}) \oplus \{[\begin{array}{ccc} \phi_t^1 & \cdots & \phi_t^p \end{array}]\epsilon : \epsilon \in \mathbb{R}^p\} \sim \mathcal{RCR}_c^0(t) \oplus \mathbb{R}^p,$$

where the latter identification is up to isomorphism (for each t fixed). As a result, the centre manifold can be expressed in the form

$$\mathcal{W}_c(t) = \{\mathcal{C}(t,\phi,\epsilon) : \phi \in \mathcal{RCR}_c^0(t), \epsilon \in \mathbb{R}^p\},$$

for some $\mathcal{C}(t,\cdot) : \mathcal{RCR}_c^0(t) \times \mathbb{R}^p \to \mathcal{RCR}$ smooth (under assumptions H.1–H.7). Since the centre manifold consists of solutions of (I.8.3)–(I.8.6) and the dynamics in the ϵ component are trivial, it follows that $\pi_2[\mathcal{C}(t,\phi,\epsilon)] = \epsilon$, where $\pi_2 : \mathcal{RCR} \times \mathbb{R}^p \to \mathbb{R}^p$ is the projection onto the second component.

The *centre manifold depending on the parameter* ϵ is the nonautonomous set $\mathcal{W}_c^\epsilon$ over $\mathcal{RCR}$ with t-fibre

$$\mathcal{W}_c^\epsilon(t) = \{\pi_1[\mathcal{C}(t,\phi,\epsilon)] : \phi \in \mathcal{RCR}_c^0(t)\},$$

where $\pi_1 : \mathcal{RCR} \times \mathbb{R}^p \to \mathcal{RCR}$ is the projection onto the first component. For $|\epsilon|$ sufficiently small, this nonautonomous set will be locally (near zero) invariant under the dynamics of (I.8.1)–(I.8.2) and contain all small solutions, so bifurcations can then be studied by examining the dynamics on this invariant manifold. The next section will demonstrate how this is done in practice.

I.8.2 Codimension-One Bifurcations for Systems with a Single Delay: Setup

Consider the n-dimensional system

$$\dot{x} = A(t)x(t) + B(t)x(t-r) + q_1(t)\epsilon + f(t, x(t), x(t-r), \epsilon), \quad t \neq k \in \mathbb{Z} \tag{I.8.7}$$

$$\Delta x = Cx(t^-) + Ex(t-r) + q_2\epsilon + g(x(t^-), x(t-r), \epsilon), \qquad t = k \in \mathbb{Z}, \tag{I.8.8}$$

where q_1 and q_2 are column vectors, and $t \mapsto f(t,\cdot,\cdot,\cdot)$, $t \mapsto A(t)$, $t \mapsto B(t)$ and $t \mapsto q_1(t)$ are periodic with period 1. We will assume that f and g are sufficiently smooth and $f(t,0,0,0) = g(0,0,0) = 0$, so that 0 is an equilibrium point when $\epsilon = 0$. Also, we assume $Df(t,0,0,0) = 0$ and $Dg(0,0,0) = 0$, so that f and g contain all terms of order 2 and above in $x(t)$, $x(t-r)$ and ϵ. We will also assume that the discrete delay satisfies $r < 1$. The linearization at $x = 0$ with parameter $\epsilon = 0$ is

$$\dot{x} = A(t)x(t) + B(t)x(t-r), \qquad t \neq k \tag{I.8.9}$$

$$\Delta x = Cx(t^-) + Ex(t-r), \qquad t = k. \tag{I.8.10}$$

We will assume throughout that the unstable fibre bundle of the above linear system is q-dimensional. We will also need the formally adjoint equation

$$\frac{d}{ds}z_1 = -z_1(s)A(s) - z_1(s+r)E(s+r), \qquad s \notin \{k, k-r\} \tag{I.8.11}$$

$$\Delta z_1 = \begin{cases} -z_1(k)C(I+C)^{-1}, & s = k \\ -z_1(k)E, & s = k-r, \end{cases} \tag{I.8.12}$$

and the augmented linear homogeneous system

$$\dot{\pi} = A(t)\pi(t) + B(t)\pi(t-r) + q_1(t)\epsilon, \qquad t \neq k \tag{I.8.13}$$

$$\Delta\pi = C\pi(t^-) + E\pi(t-r) + q_2\epsilon, \qquad t = k \tag{I.8.14}$$

$$\dot{\epsilon} = 0, \qquad t \neq k, \tag{I.8.15}$$

$$\Delta\epsilon = 0, \qquad t = k. \tag{I.8.16}$$

Finally, we denote

$$F(t, x_t, \epsilon) = f(t, x_t(0), x_t(-r), \epsilon)$$
$$G(x_t, \epsilon) = g(x_t(0), x_t(-r), \epsilon)$$

I.8.3 Fold Bifurcation

Suppose the linearized system (I.8.9)–(I.8.10) at $\epsilon = 0$ has one-dimensional centre fibre bundle: $\mathcal{RCR}_c^0(t) = \mathrm{span}\{\phi_t\}$. This means that zero is a Floquet exponent, and there is a single rank 1 Floquet eigensolution ϕ_t.

To study bifurcations at the origin for (I.8.7)–(I.8.8) at $\epsilon = 0$, we must expand the state space by taking ϵ as an additional state variable as in Sect. I.8.1. This results in the *augmented* system

$$\frac{d}{dt}\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \mathcal{L}(t)\begin{bmatrix} x_t \\ \epsilon_t \end{bmatrix} + \begin{bmatrix} f(t, x(t), x(t-r), \epsilon) \\ 0 \end{bmatrix}, \quad t \neq k \tag{I.8.17}$$

$$\Delta\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \mathcal{J}\begin{bmatrix} x_{t^-} \\ \epsilon_{t^-} \end{bmatrix} + \begin{bmatrix} g(x(t^-), x(t-r), \epsilon) \\ 0 \end{bmatrix}, \quad t = k, \tag{I.8.18}$$

and the linear functionals $\mathcal{L}$ and $\mathcal{J}$ are defined by

$$\mathcal{L}(t)\begin{bmatrix} w \\ y \end{bmatrix} = \begin{bmatrix} A(t) & q_1(t) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w(0) \\ y(0) \end{bmatrix} + \begin{bmatrix} B(t) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w(-r) \\ y(-r) \end{bmatrix},$$
$$\mathcal{J}\begin{bmatrix} w \\ y \end{bmatrix} = \begin{bmatrix} C & q_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w(0) \\ y(0) \end{bmatrix} + \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w(-r) \\ y(-r) \end{bmatrix}.$$

Observe that the linearization of the augmented system (I.8.17)–(I.8.18) is (I.8.13)–(I.8.16). The centre fibre bundle, in particular, has become two-dimensional, but the dimension of the unstable fibre bundle is unchanged.

Lemma I.8.3.1. *The centre fibre bundle $\mathcal{RCR}_c(t)$ associated to the linearization of the augmented system* (I.8.17)–(I.8.18) *is two-dimensional. A basis matrix is*

$$\Phi_t = \begin{bmatrix} \phi_t & \pi_t \\ 0 & 1 \end{bmatrix},$$

where ϕ_t spans the centre fibre bundle of the original linearization (I.8.9)–(I.8.10) *at $\epsilon = 0$, and $t \mapsto (\pi_t(0), 1)$ is a Floquet eigensolution of rank ≤ 2 with exponent zero of the augmented homogeneous system* (I.8.13)–(I.8.16)*. Also, the unstable fibre bundle $\mathcal{RCR}_u(t)$ of the augmented system remains q-dimensional.*

Proof. $(\phi_t, 0)$ is a solution in the centre fibre bundle, so $\dim \mathcal{RCR}_c(t) \geq 1$. On the other hand, any solution (x, ϵ) of the linearization (I.8.13)–(I.8.16) must satisfy $\epsilon =$ constant, so in searching for other solutions in $\mathcal{RCR}_c(t)$ with such a nonzero constant, we may without loss of generality assume a solution of the form $(\pi_t, 1)$, where $t \mapsto \pi_t(0)$ solves

$$\dot{\pi} = A(t)\pi(t) + B(t)\pi(t-r) + q_1(t), \qquad t \neq k \tag{I.8.19}$$

$$\Delta\pi = C\pi(t^-) + E\pi(t-r) + q_2, \qquad t = k. \tag{I.8.20}$$

Observe, however, that for any pair of solutions π_t and ω_t of the above system, the difference $h_t = \pi_t - \omega_t$ satisfies the homogeneous equation (I.8.9)–(I.8.10) and, consequently, if $(\pi_t, 1) \in \mathcal{RCR}_c(t)$, then π_t is unique up to addition by a multiple of ϕ_t. Thus, $\dim \mathcal{RCR}_c(t) \leq 2$. Moreover, because of the inhomogeneity, any solution of the form $(\pi_t, 1)$ cannot satisfy $||\pi_t|| \to 0$ as $t \to \infty$ or $t \to -\infty$. By spectral separation, if follows that if $t \mapsto (\pi_t, 1)$ is an eigensolution, then $(\pi_t, 1) \in \mathcal{RCR}_c(t)$.

Next, we prove that $\mathcal{RCR}_u(t) = \mathcal{RCR}_u^0(t) \times \{0\}$, thereby proving that $\mathcal{RCR}_u(t)$ is q-dimensional. Let $(\omega_t, \epsilon) \in \mathcal{RCR}_u(t)$, and assume by way of contradiction that $\epsilon = 1$, because if $\epsilon = 0$ then we must have $\omega_t \in \mathcal{RCR}_c^0(t)$, and if $\epsilon \neq 1$ we can rescale. But this means $(\omega_t, 1) \in \mathcal{RCR}_u(t)$, which is a contradiction to the above result that all such eigensolutions must be in $\mathcal{RCR}_c(t)$. Thus, $\mathcal{RCR}_u(t)$ contains only eigensolutions of the form $(\omega_t, 0)$, and so $\mathcal{RCR}_u(t) = \mathcal{RCR}_u^0(t) \times \{0\}$ is c-dimensional.

Finally, by way of contradiction, assume that $\dim \mathcal{RCR}_c(t) = 1$ and $\mathcal{RCR}_c(t)$ is spanned by $(\phi_t, 0)$. Consider the element $(0, 1) \in \mathcal{RCR} = \mathcal{RCR}([-r, 0], \mathbb{R}^n) \times \mathcal{RCR}([-r, 0], \mathbb{R})$. Due to the decomposition $\mathcal{RCR} = \mathcal{RCR}_c(0) \oplus \mathcal{RCR}_s(0)$, there exists a unique $c \in \mathbb{R}$ such that $(0, 1) = c(\phi_0, 0) + (a_0, b_0)$ for some $(a_0, b_0) \in \mathcal{RCR}_s(0)$. Consequently, $b_0 = 1$, and it follows that $(a_0, 1) \in \mathcal{RCR}_s(0)$. The solution $t \mapsto (a_t, 1)$ through $(a_0, 1)$ is defined on $[0, \infty)$, and by forward invariance of $\mathcal{RCR}_s$, we have $(a_t, 1) \in \mathcal{RCR}_s(t)$, which is a contradiction because as has been proven above, such a solution can only be in $\mathcal{RCR}_c(t)$. That π_t is a Floquet eigensolution with exponent zero and of rank ≤ 2 is a consequence the Floquet decomposition (I.6.5). ☐

Lemma I.8.3.2. *Let $\rho(t) \in \mathbb{R}^{n^*}$ be a nontrivial periodic solution of the formally adjoint equation (I.8.11)–(I.8.12), normalized with respect to ϕ_t such that*

$$N(\rho, \phi) := \int_0^1 \rho(t)[\phi(t) + rB(t)\phi(t-r)]dt + \rho(0)rE\phi(-r) = 1. \tag{I.8.21}$$

This normalization is always attainable. Define the quantity

$$a_{01} = \int_0^1 \rho(t)q_1(t)dt + \rho(0)q_2. \tag{I.8.22}$$

The Floquet eigensolution $t \mapsto (\pi_t(0), 1)$ from Lemma I.8.3.1 is rank 1 if and only if $a_{01} = 0$. Under the above normalization, π_t satisfies the equality

$$\pi_1 = \pi_0 + a_{01}\phi_0, \tag{I.8.23}$$

and the matrix Λ *of the Floquet decomposition* $\Phi_t = Q_t e^{\Lambda t}$ *for* $t \mapsto Q_t$ *of period one is*

$$\Lambda = \begin{bmatrix} 0 & a_{01} \\ 0 & 0 \end{bmatrix}.$$

Proof. The proof of this result makes use of a modification and slight generalization of [[3], Theorem 1 & Lemma 5] adapted to the right-continuous solution formalism. Specialized to our class of systems, it says the following. The proof is omitted.

Proposition I.8.3.1. *The inhomogeneous linear system*

$$\begin{aligned} \dot{x} &= A(t)x + B(t)x(t-r) + f(t), & t &\neq k \\ \Delta x &= Cx(t^-) + Ex(t-r) + g, & t &= k \end{aligned}$$

has a periodic solution if and only if

$$\int_0^1 \rho_i(s)f(s)ds + \rho_i(0)g = 0 \tag{I.8.24}$$

for every nontrivial periodic solution ρ_i *of the formally adjoint homogeneous system* (I.8.11)–(I.8.12)*. Also, the number of linearly independent periodic solutions of the homogeneous system* (I.8.9)–(I.8.10) *and its formal adjoint* (I.8.11)–(I.8.12) *is the same.*

Using Proposition I.8.3.1, it is clear that the condition $a_{01} = 0$ is equivalent to the periodicity of π_t. Next we show that Λ has the claimed form. To begin, we remark that $(\pi(t), 1)$ being a Floquet eigensolution of rank ≤ 2 and exponent zero implies it can be written in the form $\pi(t) = tv(t) + w(t)$ for periodic v and w. Substituting into (I.8.19)–(I.8.20) and factoring, we can write

$$t(\dot{v} - Av - Bv(t-r)) + v + \dot{w} + rBv(t-r) = Aw + Bw(t-r) + q_1, \quad t \neq k \tag{I.8.25}$$

$$t(\Delta v - Cv(t^-) - Ev(t-r)) + \Delta w + rEv(t-r) = Cw(t^-) + Bw(t-r) + q_2, \quad t = k. \tag{I.8.26}$$

Since w is periodic and hence bounded, it follows that the order t terms must vanish. Consequently, v must in fact be a periodic solution of (I.8.9)–(I.8.10), and it follows that $v_t = c\phi_t$ for a constant c. But this in turn means

$$\begin{aligned} \pi_1(\theta) &= (1+\theta)v_1(\theta) + w_1(\theta) \\ &= \theta v_0(\theta) + w_0(\theta) + v_0(\theta) \\ &= \pi_0(\theta) + c\phi_0(\theta). \end{aligned}$$

Thus, Φ_1 satisfies the decomposition

$$\Phi_1 = \Phi_0 \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} := \Phi_0 M.$$

If $a_{01} = 0$, we must chose $c = 0$, with the result being the matrix $\Lambda = \log M = 0_{2\times 2}$ as claimed. On the other hand, if $a_{01} \neq 0$, we substitute our ansatz $v(t) = c\phi(t)$ into (I.8.25)–(I.8.26) to obtain the following inhomogeneous system for w:

$$\begin{aligned} \dot{w} &= Aw + Bw(t-r) + q_1 - c[\phi(t) + rB\phi(t-r)], && t \neq k \\ \Delta w &= Cw(t^-) + Ew(t-r) + q_1 - crE\phi(t-r), && t = k. \end{aligned}$$

Applying Proposition I.8.3.1 to the above inhomogeneous system, we conclude that as w is periodic, c must satisfy the equation

$$a_{01} - cN(\rho, \phi) = 0.$$

It follows that $N(\rho, \phi) \neq 0$, and as it is linear in ρ, we can always attain the normalization condition $N(\phi, \rho) = 1$. Therefore, $c = a_{01}$, and we get

$$\Lambda = \log M = \log \begin{bmatrix} 1 & a_{01} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a_{01} \\ 0 & 0 \end{bmatrix}.$$

□

Lemma I.8.3.3. *Write the matrix $Y(t) \in \mathbb{R}^{2\times(n+1)}$ satisfying $P_c(t)\chi_0 = \Phi_t Y(t)$ in block form*

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

for $Y_{i1} \in \mathbb{R}^{1\times n}$ and $Y_{i2} \in \mathbb{R}^{1\times 1}$. We have $Y_{21} = 0$ and

$$\phi_t Y_{11}(t) = \frac{1}{2\pi i} \int_{\Gamma_1} (zI - V_t^0)^{-1} \chi_0 dz, \tag{I.8.27}$$

where $V_t^0 : \mathcal{RCR} \to \mathcal{RCR}$ is the monodromy operator associated to the linear system (I.8.9)–(I.8.10) and Γ_1 is a positively oriented contour whose interior is bounded away from zero, enclosing $1 \in \mathbb{C}$ and no other eigenvalue of V_t^0.

Proof. By definition, we have

$$\Phi_t \begin{bmatrix} Y_{11}(t) \\ Y_{21}(t) \end{bmatrix} = \frac{1}{2\pi i} \int_{\Gamma_1} (zI - V_t)^{-1} \text{diag}(1, 1, \dots, 1, 1, 0) \chi_0 dz, \tag{I.8.28}$$

where the diagonal matrix has n ones. For bounded linear operator X, denote by $R(z; X) = (zI - X)^{-1}$ its resolvent operator. We start by partially computing ψ, defined by

$$R(z; V_t)\text{diag}(1, \dots, 1, 0)\chi_0 = \psi \in \mathcal{RCR}([-r, 0], \mathbb{R}^{n+1}).$$

By linearity of the monodromy operator V_t, if we write

$$\psi = \begin{bmatrix} \psi_\pi \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \psi_\epsilon \end{bmatrix}$$

for $\psi_\pi \in \mathcal{RCR}([-r,0],\mathbb{R}^n)$ and $\psi_\epsilon \in \mathcal{RCR}([-r,0],\mathbb{R})$, the equation for ψ is equivalent to

$$\mathrm{diag}(1,\dots,1,0)\chi_0 = z\psi - V_t \begin{bmatrix} \psi_\pi \\ 0 \end{bmatrix} - V_t \begin{bmatrix} 0 \\ \psi_\epsilon \end{bmatrix}.$$

Note, however, that the dynamics of the augmented system (I.8.13)–(I.8.16) are trivial in ϵ, and the left-hand side of the above equation is zero in the ϵ component. Consequently, $\psi_\epsilon \equiv 0$. Similarly, $V_t[\ \psi_\pi \quad 0\ ]^\intercal \in \mathcal{RCR}([-r,0],\mathbb{R}^n) \times \{0\}$, so the equation for the component ψ_π takes the form

$$\chi_0 = z\psi_\pi - V_t^0\psi_\pi,$$

which implies ψ_π satisfies the equation $R(z;V_t^0)\chi_0 = \psi_\pi$. Taking all of the above into account together with the representation of Φ_t from Lemma I.8.3.1, we see that Eq. (I.8.28) is equivalent to

$$\begin{bmatrix} \phi_t Y_{11}(t) + \pi_t Y_{21}(t) \\ Y_{21}(t) \end{bmatrix} = \frac{1}{2\pi i}\int_{\Gamma_1} \begin{bmatrix} (zI - V_t^0)^{-1}\chi_0 \\ 0 \end{bmatrix} dz,$$

which readily implies $Y_{21} = 0$ and the characterization of Y_{11} specified in Eq. (I.8.27). □

With the above three lemmas at hand, we can compute the quadratic-order dynamics on the centre manifold (I.6.14)–(I.6.15). First, however, we note that because the dynamics of ϵ in (I.8.17)–(I.8.18) are trivial, we can abuse notation and write $u = (u_1, u_2) \in \mathbb{R}^2$ instead as $u = (u,\epsilon)$.

Lemma I.8.3.4. *The dynamics on the parameter-dependent centre manifold, to quadratic order, are given by*

$$\dot{u} = a_{01}\epsilon + \frac{1}{2}Y_{11}(t)D^2F(t,0)[(\phi_t u + (\pi_t - a_{01}t\phi_t)\epsilon, \epsilon)]^2, \quad t \neq k \quad \text{(I.8.29)}$$

$$\Delta u = \frac{1}{2}Y_{11}(0)D^2G(0)[(\phi_0^- u + \pi_0^-\epsilon, \epsilon)]^2, \quad t = k, \quad \text{(I.8.30)}$$

for $|\epsilon|$ small, and all differentials are in the $\mathcal{RCR}$ variable.

Proof. First, using Lemmas I.8.3.1 and I.8.3.2, we can calculate

$$Q_t = \begin{bmatrix} \phi_t & \pi_t - a_{01}t\phi_t \\ 0 & 1 \end{bmatrix}.$$

Next, the quadratic-order dynamics for (u, ϵ) are given by (I.6.14)–(I.6.15):

$$\frac{d}{dt}\begin{bmatrix} u \\ \epsilon \end{bmatrix} = \Lambda \begin{bmatrix} u \\ \epsilon \end{bmatrix} + \frac{1}{2}Y(t)\begin{bmatrix} D^2F(t,0)\left[Q_t\begin{bmatrix} u \\ \epsilon \end{bmatrix}\right]^2 \\ 0 \end{bmatrix}, \qquad t \neq k,$$

$$\Delta \begin{bmatrix} u \\ \epsilon \end{bmatrix} = \frac{1}{2}Y(t)\begin{bmatrix} D^2G(0)\left[Q_{t^-}\begin{bmatrix} u \\ \epsilon \end{bmatrix}\right]^2 \\ 0 \end{bmatrix}, \qquad t = k.$$

By periodicity of Q_t and $Y(t)$, we can replace Q_{t^-} with Q_{0^-} and $Y(t)$ with $Y(0)$ in the second equation, since the jumps occur at the integers. Also, because of the structure of $Y(t)$ supplied by Lemma I.8.3.3, the above reduces to Eqs. (I.8.29)–(I.8.30) together with the trivial equations $\dot{\epsilon} = 0$ and $\Delta\epsilon = 0$. ☐

Theorem I.8.3.1 (Fold Bifurcation)**.** *Let the centre fibre bundle associated to the linear system* (I.8.7)*–*(I.8.8) *be one-dimensional, with* $\lambda = 0$ *being the only Floquet exponent with zero real part. Let* a_{01} *be as stated in Lemma I.8.3.2, and let* π_t *satisfy Eq.* (I.8.23)*. Assume that the functions* $\rho(t)$ *and* $\phi(t)$ *satisfy the normalization condition* (I.8.21) *and introduce the quantities*

$$a_{20} = \int_0^1 Y_{11}(s)D^2F(s,0)[(\phi_s,0)]^2 ds + Y_{11}(0)D^2G(0)[(\phi_0^-,0)]^2,$$

$$a_{11} = \int_0^1 Y_{11}(s)D^2f(s,0)[(\phi_s,0),(\pi_s - a_{01}s\phi_s,1)]ds$$
$$+ Y_{11}(0)D^G(0)[(\phi_0^-,0),(\pi_0^-,1)]$$

The following are true.

1. *If* $a_{01} \neq 0$ *and* $a_{20} \neq 0$*, the nonlinear system* (I.8.7)*–*(I.8.8) *undergoes a fold (saddle-node) bifurcation of periodic orbits from the equilibrium* 0 *at parameter* $\epsilon = 0$*. More precisely, the iterated discrete-time dynamics on the parameter-dependent centre manifold are locally topologically equivalent near* $(u,\epsilon) = 0$ *to the quadratic-order truncated dynamics*

$$u \mapsto u + a_{01}\epsilon + \frac{1}{2}a_{20}u^2.$$

2. *Suppose* $a_{01} = 0$*. If* $a_{11} \neq 0$ *and* $a_{20} \neq 0$*, and the nonlinear system* (I.8.7)*–*(I.8.8) *has a stationary solution* $x = 0$ *for all* $|\epsilon|$ *sufficiently small, then this system undergoes a transcritical bifurcation of periodic orbits from the equilibrium* 0 *at parameter* $\epsilon = 0$*. More precisely, the iterated discrete-time dynamics on the parameter-dependent centre manifold are locally topologically equivalent near* $(u,\epsilon) = 0$ *to the*

quadratic-order truncated dynamics

$$u \mapsto u + a_{11}\epsilon u + \frac{1}{2}a_{20}u^2.$$

3. *If $\mathcal{RCR}_u^0(t)$ is trivial, the stability of fixed points of the iterated dynamics carries over to the analogous bifurcating periodic orbits in* (I.8.7)–(I.8.8)*.*

Proof. Starting from the quadratic-order dynamics (I.8.29)–(I.8.30) on the centre manifold, we define the stroboscopic (Poincaré) map $u \mapsto S(u,\epsilon)$ mapping the state u at time $t = 0$ to the state at time $t = 1$ for parameter ϵ. This function is smooth, and following [30], it admits a Taylor expansion of the form

$$S(u,\epsilon) = u + q_{01}\epsilon + \frac{1}{2}q_{20}u^2 + q_{11}u\epsilon + \frac{1}{2}q_{02}\epsilon^2 + O(||(u,\epsilon)||^3)$$

near $(u,\epsilon) = 0$. Each of the coefficients q_{ij} is a solution of a particular initial-value problem evaluated at $t = 1$. Namely, $q_{ij} = v_{ij}(1)$, where $v_{ij}(0) = 0$ and

$$\begin{aligned}
\dot{v}_{01} &= a_{01}, && t \in \mathbb{R}\\
\dot{v}_{20} &= Y_{11}(t)D^2F(t,0)[(\phi_t,0)]^2, && t \neq k\\
\dot{v}_{11} &= \frac{1}{2}Y_{11}(t)D^2F(t,0)[(\phi_t,0),(\pi_t - a_{01}t\phi_t,1)], && t \neq k\\
\Delta v_{20} &= Y_{11}(0)D^2G(0)[(\phi_0^-,0)]^2, && t = k\\
\Delta v_{11} &= \frac{1}{2}Y_{11}(0)D^2G(0)[(\phi_0^-,1),(\pi_0^-,1)], && t = k.
\end{aligned}$$

The differential equation for v_{02} is not shown because it will not be needed. Solving the above differential equations, it follows that the stroboscopic map admits the Taylor expansion

$$S(u,\epsilon) = u + a_{01}\epsilon + \frac{1}{2}a_{20}u^2 + a_{11}\epsilon u + O(\epsilon^2 + ||(u,\epsilon)||^3).$$

As the dynamics on the centre manifold are periodic, the orbit structure and bifurcations can be determined by analyzing the iterated map $u \mapsto S(u,\epsilon)$. The conclusions 1–2 of the theorem now follow directly from the saddle-node bifurcation theorem for maps [151]. The stability assertion in the presence of a trivial unstable fibre bundle follows by Theorem I.5.5.1. □

Remark I.8.3.1. *The above theorem could easily be generalized to the setting where there is more than one impulse per period. The period can also be any positive real number, since one can always rescale time so that the period is unity. Note also that there is no need to calculate π_t unless the nondegeneracy condition $a_{01} \neq 0$ of the saddle-node bifurcation fails.*

The following corollary is a statement of the generic fold bifurcation theorem for the system (I.8.7)–(I.8.8). The genericity conditions are the parametric nonstationarity condition $a_{01} \neq 0$ and the nonvanishing quadratic condition $a_{20} \neq 0$.

Corollary I.8.3.1 (Generic Fold Bifurcation). *For any generic impulsive delay differential equation* (I.8.7)–(I.8.8) *having at* $\epsilon = 0$ *the equilibrium* 0 *with a single Floquet exponent* $\lambda = 0$ *and one-dimensional centre fibre bundle, there is a neighbourhood* N *of* $0 \in \mathcal{RCR}$ *and a smooth invertible change of parameters* $\eta = \eta(\epsilon)$ *satisfying* $\eta(0) = 0$*, such that for* $\eta > 0$*, there are exactly two periodic orbits of period* 1 *in* N *that trivialize to the equilibrium as* $\eta \to 0^+$*, while for* $\eta < 0$ *there are no periodic orbits in* N*.*

I.8.3.1 Example: Fold Bifurcation in a Scalar System with Delayed Impulse

Consider the scalar equation with discrete delay

$$\dot{x} = \log(2)x - x^2(t - 1/2) + \epsilon\sigma(t), \qquad t \neq k \tag{I.8.31}$$

$$\Delta x = -\frac{1}{\sqrt{2}}x(t - 1/2) + \epsilon, \qquad t = k, \tag{I.8.32}$$

where $\sigma(t)$ is periodic with period 1. When $\epsilon = 0$, the linearization at the origin has $\phi(t) = 2^{t-\lfloor t \rfloor}$ as the unique periodic solution up to scaling. Moreover, every solution of the linearization at 0 of the above impulsive delay differential equation is eventually a solution of (i.e. (I.8.31)–(I.8.32) is FD-reducible [35] to) the system

$$\dot{z} = \log(2)z, \qquad t \neq k$$

$$\Delta z = -\frac{1}{2}z, \qquad t = k.$$

Since any solution of (I.8.31)–(I.8.32) defined for all time must satisfy the above finite-dimensional system, and since it has only the Floquet exponent 0 with multiplicity 1, we conclude that the linearization of (I.8.31)–(I.8.32) has $\mathcal{RCR}_c(t) = \text{span}\{\phi_t\}$ and $\mathcal{RCR}_u(t) = \{0\}$. We are therefore in a position to apply Theorem I.8.3.1.

The formal adjoint system to the linearization is

$$\dot{y} = -\log(2)y, \qquad t \neq k - 1/2$$

$$\Delta y = \frac{1}{\sqrt{2}}y(k), \qquad t = k - 1/2.$$

From this, we can calculate the nontrivial periodic solution

$$\rho(t) = \begin{cases} 2^{-t+\lfloor t \rfloor}, & 0 \leq t - \lfloor t \rfloor < \frac{1}{2} \\ 2^{1-t+\lfloor t \rfloor}, & \frac{1}{2} \leq t - \lfloor t \rfloor < 1, \end{cases}$$

by solving the equation in reverse time from $(t, y) = (1, 1)$. Next we verify the normalization condition (I.8.21). We have

$$\begin{aligned} N(\rho, \phi) &= \int_0^1 \rho(t)\phi(t)dt + \rho(0)\frac{1}{2} \cdot \frac{-1}{\sqrt{2}}\phi(-1/2) \\ &= \int_0^{\frac{1}{2}} 1dt + \int_{\frac{1}{2}}^1 2dt - \frac{1}{2\sqrt{2}}2^{\frac{1}{2}} \\ &= 1, \end{aligned}$$

so ρ is already normalized relative to ϕ. The calculation of the function $Y_{11}(t)$ is carried out in Sect. I.8.3.2; we find that $Y_{11}(t) = \rho(t)$. We have enough information to calculate the coefficients a_{01} and a_{20}:

$$\begin{aligned} a_{01} &= 1 + \int_0^{1/2} 2^{-s}\sigma(s)ds + \int_{1/2}^1 2^{1-s}\sigma(s)ds, \\ a_{20} &= \int_0^1 -2\phi(s-1/2)^2 ds = -\frac{3}{\log(2)}. \end{aligned}$$

Since a_{20} is nonzero, Theorem I.8.3.1 guarantees a saddle-node bifurcation occurs at $\epsilon = 0$ assuming $a_{01} \neq 0$.

For example, if we choose $\sigma(t) = \sin(2\pi t)$, then $a_{01} = 0.88881 \pm 10^{-5}$ is positive and a_{20} is negative. Reading off the discrete-time dynamics from the theorem, we predict that there should be a single, locally asymptotically stable periodic orbit when $\epsilon > 0$, the origin should be semistable when $\epsilon = 0$, and there should be no small periodic orbits when $\epsilon < 0$. These conclusions should all hold true provided $|\epsilon|$ is sufficiently small. Indeed, we can verify numerically that these conclusions are consistent for $0 \leq \epsilon \leq 1$, although the bifurcating periodic orbit appears to undergo a period doubling bifurcation between $\epsilon = 1$ and $\epsilon = 1.2$; see Fig. I.8.1. Solutions rapidly diverge in the regime $\epsilon < 0$, and we do not provide accompanying figures.

I.8.3.2 Calculation of the Function $Y_{11}(t)$ for Example I.8.3.1

First, we remark that because of the periodicity of the monodromy operator and the matrix $Y(t)$, it suffices to compute the restriction of $Y_{11}(t)$ to the interval $[0, 1)$ and extend periodically. We begin by computing the monodromy operator V_t^0 on this restriction. One can verify that this is given by

$$V_t^0\xi(\theta) = \begin{cases} 2^{\theta+1}\xi(0), & t + \theta < 0, t \leq \frac{1}{2} \\ 2^{\theta}\xi(0), & t + \theta \geq 0, t \leq \frac{1}{2} \\ 2^{\theta}(2\xi(0) - 2^{t-\frac{1}{2}}\xi(t-1/2)), & t > \frac{1}{2}. \end{cases}$$

Next, we solve the equation $(zI - V_t^0)^{-1}\chi_0 = \psi$. This is equivalent to

$$\chi_0 = z\psi - V_t^0\psi. \tag{I.8.33}$$

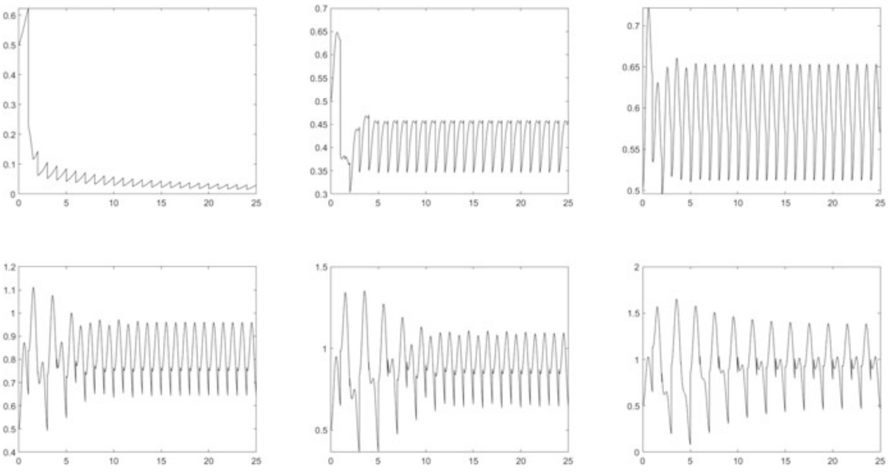

Figure I.8.1: Simulations of the scalar impulsive system (I.8.31)–(I.8.32) from Sect. I.8.3.1 for various parameters ϵ, with the forcing function $\sigma(t) = \sin(2\pi t)$, from the constant initial condition $x_0 = \frac{1}{2}$. Time t on the horizontal with $x(t)$ on the vertical axis. Top row left to right: solutions with $\epsilon = 0$, $\epsilon = 0.2$ and $\epsilon = 0.4$. Bottom row from left to right: $\epsilon = 0.8$, $\epsilon = 1$ and $\epsilon = 1.2$

We do two cases separately: first, with $t \le \frac{1}{2}$, and then with $t > \frac{1}{2}$.

If $t \le \frac{1}{2}$, Eq. (I.8.33) evaluated at $\theta = 0$ produces the following algebraic equation for $\psi(0)$:

$$1 = z\psi(0) - \psi(0).$$

Therefore, $\psi(0) = (z-1)^{-1}$. Evaluating (I.8.33) at $\theta < 0$ and substituting in the constraint $\psi(0) = (z-1)^{-1}$ produce the equation

$$0 = z\psi(\theta) - \frac{1}{z-1}\begin{cases} 2^{1+\theta}, & t+\theta < 0 \\ 2^{\theta}, & t+\theta \ge 0. \end{cases}$$

Solving the above equation for $\psi(\theta)$, combining the two results and simplifying, we have determined that for $t \le \frac{1}{2}$,

$$(zI - V_t^0)^{-1}\chi_0 = \frac{2^{\theta}}{z(z-1)}\begin{cases} z, & \theta = 0 \\ 2, & t+\theta < 0 \\ 1, & t+\theta \ge 0. \end{cases} \tag{I.8.34}$$

Next, we consider the case $t > \frac{1}{2}$. Evaluating Eq. (I.8.33) at $\theta = 0$ and $\theta = \frac{1}{2} - t < 0$, we obtain the following pair of linear equations for the unknowns $\psi(0)$ and $\psi(1/2 - t)$:

$$\begin{aligned} 1 &= (z-2)\psi(0) + 2^{t-\frac{1}{2}}\psi(1/2 - t) \\ 0 &= -2^{\frac{3}{2}-t} + (z+1)\psi(1/2 - t). \end{aligned}$$

Solving this equation, we find

$$\begin{bmatrix} \psi(0) \\ \psi(1/2-t) \end{bmatrix} = \frac{1}{z(z-1)} \begin{bmatrix} z+1 \\ 2^{\frac{3}{2}-t} \end{bmatrix}. \tag{I.8.35}$$

Next, evaluating (I.8.33) at $\theta < 0$ and expressing one of the terms as matrix product yield the equation

$$0 = z\psi(\theta) - 2^{\theta}\begin{bmatrix} 2 & -2^{t-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \psi(0) \\ \psi(1/2-t) \end{bmatrix}.$$

Solving the equation for $\psi(\theta)$ and combining it with (I.8.35), we conclude that for $t > \frac{1}{2}$,

$$(zI - V_t^0)^{-1}\chi_0 = \frac{2^{\theta}}{z(z-1)}\begin{cases} z+1, & \theta = 0 \\ 2, & \theta < 0. \end{cases} \tag{I.8.36}$$

Next we calculate the Dunford integral $(2\pi i)^{-1}\int_{\Gamma_1}(zI - V_t^0)^{-1}\chi_0 dz$. Using (I.8.34) and (I.8.36) together with residue theorem, we obtain after much simplification

$$\frac{1}{2\pi i}\int_{\Gamma_1}(zI - V_t^0)^{-1}\chi_0(\theta)dz = \begin{cases} 2^{1+\theta}, & t \leq \frac{1}{2}, t+\theta < 0 \text{ or } t > \frac{1}{2} \\ 2^{\theta}, & t \leq \frac{1}{2}, t+\theta \geq 0. \end{cases}$$

By Lemma I.8.3.3, we can calculate $Y_{11}(t)$ by multiplying the above by $1/\phi_t(\theta)$, and the result should be independent of θ. Initially, we obtain

$$\frac{1}{\phi_t(\theta)}\frac{1}{2\pi i}\int_{\Gamma_1}(zI - V_t^0)^{-1}\chi_0(\theta)dz = \begin{cases} 2^{1-t+\lfloor t+\theta \rfloor}, & t \leq \frac{1}{2}, t+\theta < 0 \text{ or } t > \frac{1}{2} \\ 2^{-t+\lfloor t+\theta \rfloor}, & t \leq \frac{1}{2}, t+\theta \geq 0. \end{cases} \tag{I.8.37}$$

When $t > \frac{1}{2}$, we have $1 - t + \lfloor t+\theta \rfloor = 1 - t$. Conversely, when $t \leq \frac{1}{2}$, we have

$$\begin{aligned} t+\theta < 0 \quad &\Rightarrow \quad 1 - t + \lfloor t+\theta \rfloor = -t \\ t+\theta \geq 0 \quad &\Rightarrow \quad -t + \lfloor t+\theta \rfloor = -t. \end{aligned}$$

Therefore, in both cases, we see that (I.8.37) can be written independent of θ, with the result being $Y_{11}(t) = \rho(t)$ on $[0, 1)$. Extending by periodicity, the claim is proven.

I.8.4 Hopf-Type Bifurcation and Invariant Cylinders

This time we will assume that (I.8.9)–(I.8.10) have a pair $\pm i\omega$ of complex-conjugate Floquet exponents, and there are no other Floquet exponents with zero real part. This means the centre fibre bundle has the real basis matrix $\Phi_t^0 = [\ \phi_{1,t} \quad \phi_{2,t}\ ]$, so that $\mathcal{RCR}_c^0(t) = \text{span}\{\phi_{1,t}(\theta), \phi_{2,t}(\theta)\}$. As the Floquet exponents are $\pm i\omega$, Eq. (I.6.5) implies the decomposition

$$\Phi_t^0 = Q_t^0 \exp\left(\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t\right), \tag{I.8.38}$$

where $t \mapsto Q_t$ is periodic with columns in $\mathcal{RCR}$. Finally, we let dim $\mathcal{RCR}_u^0(t) = c$.

The analysis of this section is similar to the previous one, with some modifications. To motivate our first result, recall that a Neimark–Sacker bifurcation occurs in the iterated map

$$z \mapsto A(\epsilon)z + \frac{1}{2}B(\epsilon)[z,z] + \frac{1}{6}C(\epsilon)[z,z,z] + O(||z||^4) \tag{I.8.39}$$

for a 2×2 matrix $A(\epsilon)$ and symmetric multilinear maps $B(\epsilon)$ and $C(\epsilon)$, at $z = 0$ with parameter $\epsilon = 0$, provided the following are satisfied [82]:

- the eigenvalues $\mu_1(\epsilon)$ and $\mu_2(\epsilon)$ of $A(\epsilon)$ satisfy $\mu_i(0) = e^{\pm i\omega}$, and $e^{ik\omega} \neq 1$ for $k = 1, 2, 3, 4$;
- the crossing condition[1] $r'(0) \neq 0$ is satisfied, where
$$r'(0) = \frac{1}{2}\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\mu_1(\epsilon)\mu_2(\epsilon) \neq 0;$$
- the first Lyapunov coefficient [82] $d(0)$,
$$\begin{aligned} d(0) = \Re\Bigg(e^{-i\omega}\frac{1}{2}\Big[&\langle p, C_0[q,q,\overline{q}]\rangle + 2\langle p, B_0[q,(I-A_0)^{-1}B_0[q,\overline{q}]]\rangle \\ &+ \langle p, B_0[\overline{q},(e^{2i\omega}I - A_0)^{-1}B_0[q,q]]\rangle\Big]\Bigg), \end{aligned} \tag{I.8.40}$$
satisfies $d(0) \neq 0$, where $\langle a, b\rangle = \overline{a}_1 b_1 + \overline{a}_2 b_2$ is the standard inner product on $\mathbb{C}^2$, $A_0 = A(0)$, $B_0 = B(0)$ and $C_0 = C(0)$, q satisfies $A_0 q = e^{i\omega}q$, p satisfies $A_0^{\intercal}p = e^{-i\omega}p$, and $\langle p, q\rangle = 1$.

Our approach in analyzing the Hopf bifurcation condition in (I.8.7)–(I.8.8) will be to first expand the state space as in equation (I.8.17)–(I.8.18) and determine the nontrivial dynamics on the parameter-dependent centre manifold near $(x_t, \epsilon) = (0, 0)$. This will be a two-dimensional impulsive differential equation. The iterated dynamics of the associated stroboscopic map at parameter $\epsilon = 0$ will be compared to (I.8.39), while the dynamics for $|\epsilon|$ small

[1] Precisely, this condition states the eigenvalues must cross the boundary of $|z| = 1$ in $\mathbb{C}$ transversally. This is equivalent to the modulus $|\mu_i(\epsilon)|$ being increasing or decreasing at $\epsilon = 0$, which, given that $|z| = z\overline{z}$, is equivalent to the condition we have supplied.

will provide a way to calculate $r'(0)$. This will allow us to effectively lift the Neimark–Sacker bifurcation into the nonlinear dynamics of (I.8.7)–(I.8.8).

To begin, we introduce some additional notation. The symbols D_x and D_ϵ will denote the partial derivative operators acting on functions $H : \mathcal{RCR} \times \mathbb{R} \to \mathbb{R}^n$. We then set $H_{xx} = D_x^2 H(0)$, $H_{xxx} = D_x^3 H(0)$ and $H_{x\epsilon} = D_\epsilon D_x H(0)$, where the first two are symmetric bilinear and trilinear maps on $\mathcal{RCR}$, respectively, and the latter is a linear operator on $\mathcal{RCR}$. For $H(t, \cdot, \cdot) : \mathcal{RCR} \times \mathbb{R} \to \mathbb{R}^n$, we overload the notation and write, for example, $H_{xx}(t) = H(t, \cdot, \cdot)_{xx}$.

The following three lemmas provide the foundation of our result. They are analogues of Lemmas I.8.3.1, I.8.3.2 and I.8.3.3, and the proofs follow from the same reasoning.

Lemma I.8.4.1. *The centre fibre bundle $\mathcal{RCR}_c(t)$ associated to the linearization of the parameter-augmented system (I.8.17)–(I.8.18) is three-dimensional. A basis matrix is*

$$\Phi_t = \begin{bmatrix} \phi_{1,t} & \phi_{2,t} & \pi_t \\ 0 & 0 & 1 \end{bmatrix},$$

where ϕ_t spans the centre fibre bundle of the original linearization (I.8.9)–(I.8.10) at $\epsilon = 0$, and $t \mapsto (\pi_t(0), 1) = (\pi(t), 1)$ is a Floquet eigensolution with exponent zero of the homogeneous impulsive delay differential equation (I.8.13)–(I.8.16). Also, the unstable fibre bundle $\mathcal{RCR}_u(t)$ of the parameter-augmented system remains c-dimensional.

Lemma I.8.4.2. *The Floquet eigensolution $t \mapsto (\pi_t(0), 1)$ is rank 1, so $\pi(t) = \pi_t(0)$ is the unique periodic solution of the system (I.8.13)–(I.8.14) with $\epsilon \equiv 1$. The matrices Λ and Q_t of the Floquet decomposition $\Phi_t = Q_t e^{\Lambda t}$ are*

$$\Lambda = \begin{bmatrix} \Lambda_\omega & 0_{2\times 1} \\ 0_{1\times 2} & 0 \end{bmatrix}, \qquad \Lambda_\omega = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \qquad Q_t = \begin{bmatrix} Q_t^0 & \pi_t \\ 0 & 1 \end{bmatrix}, \tag{I.8.41}$$

where Q_t^0 is the same periodic matrix appearing in (I.8.38).

Lemma I.8.4.3. *Write the matrix $Y(t) \in \mathbb{R}^{3\times(n+1)}$ defined by $P_c(t)\chi_0 = \Phi_t Y(t)$ in block form*

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \\ Y_{31} & Y_{32} \end{bmatrix}$$

for $Y_{i1} \in \mathbb{R}^{1\times n}$ and $Y_{i2} \in \mathbb{R}^{1\times 1}$. We have $Y_{31} = 0$ and

$$\phi_{1,t} Y_{11}(t) + \phi_{2,t} Y_{21}(t) = \frac{1}{2\pi i} \int_\Gamma (zI - V_t^0)^{-1} \chi_0 dz, \tag{I.8.42}$$

where $V_t^0 : \mathcal{RCR} \to \mathcal{RCR}$ is the monodromy operator associated to the linear system (I.8.9)–(I.8.10) and Γ is a positively oriented contour whose interior *is bounded away from zero, enclosing both of* $e^{\pm i\omega}$ *and no other eigenvalues of* V_t^0*.*

At this stage, we should point out that because of the trivial dynamics of the parameter ϵ and the form of Φ_t, the Euclidean space representation of the three-dimensional centre manifold of the augmented system (I.8.17)–(I.8.18) takes the form

$$\overline{h}(t,(u,\epsilon),\theta) = \begin{bmatrix} h(t,u,\epsilon,\theta) \\ \epsilon \end{bmatrix}$$

for $h(t,u,\epsilon,\theta) \in \mathbb{R}^n$ and $u \in \mathbb{R}^2$. Consequently, Lemma I.8.4.1 though Lemma I.8.4.3 together with Theorem I.6.1.2 imply that the dynamics on parameter-dependent centre manifold at parameter ϵ are

$$\dot{u} = \Lambda_\omega u + e^{\Lambda_\omega t}\begin{bmatrix} Y_{11}(t) \\ Y_{21}(t) \end{bmatrix} F(t, Q_t^0 u + \epsilon\tilde{\pi}_t + h(t,u,\epsilon,\cdot),\epsilon), \qquad t \neq k \tag{I.8.43}$$

$$\Delta u = \begin{bmatrix} Y_{11}(0) \\ Y_{21}(0) \end{bmatrix} G(Q_{0^-}^0 u + \epsilon\tilde{\pi}_{0^-} + h(0^-,u,\epsilon,\cdot),\epsilon), \qquad t = k. \tag{I.8.44}$$

Next we consider $\tilde{h}(t,u,\theta) := h(t,u,0,\theta)$, the centre manifold at parameter $\epsilon = 0$. As we will need the cubic order terms in the iterated map (I.8.39), it will be necessary to compute $\tilde{h}$ to quadratic order in u. As Theorem I.6.1.1 implies the expansion $\tilde{h} = \frac{1}{2}\tilde{h}_2 + O(||u||^3)$, the following lemma is of use and implies the fairly striking result that, for Hopf bifurcation conditions, the projection constraint of Proposition I.6.2.1 is not actually needed to determine the quadratic term of the centre manifold at $\epsilon = 0$.

Lemma I.8.4.4. *In terms of the expansion*

$$\tilde{h}_2(t,\theta)[u,u] = h_2^{11}u_1^2 + h_2^{12}u_1u_2 + h_2^{22}u_2^2,$$

the vector function $h_2^{\Xi} = (h_2^{11}, h_2^{12}, h_2^{22}) \in (\mathbb{R}^n)^3$ *is the unique periodic solution of period one of the impulsive partial delay differential equation* (I.6.37)–(I.6.40) *satisfying the constraint* $P_c(t)h_2^{ij}(t,\cdot) = 0$ *for all* $t \in [0,1)$ *and un-*

ordered pairs $i, j \in \{1, 2\}$, *with the data*

$$\Lambda_2 = \begin{bmatrix} 0 & -\frac{\omega}{2} & 0 \\ \omega & 0 & -\omega \\ 0 & \frac{\omega}{2} & 0 \end{bmatrix},$$

$$\mathcal{F}(t,\theta) = \big(\phi_{1,t}(\theta)Y_{11}(t) + \phi_{2,t}(\theta)Y_{21}(t)\big) \begin{bmatrix} F_{xx}(t)[Q^0_{t,1}]^2 \\ 2F_{xx}(t)[Q^0_{t,1}, Q^0_{t,2}] \\ F_{xx}(t)[Q^0_{t,2}]^2 \end{bmatrix},$$

$$\mathcal{G}(\theta) = (\phi_{1,0}(\theta)Y_{11}(0) + \phi_{2,0}(\theta)Y_{21}(0)) \begin{bmatrix} G_{xx}[Q^0_{0-,1}]^2 \\ 2G_{xx}[Q^0_{0-,1}, Q^0_{0-,2}] \\ G_{xx}[Q^0_{0-,2}]^2 \end{bmatrix}.$$

$$a(t) = \begin{bmatrix} F_{xx}(t)[Q^0_{t,1}]^2 \\ 2F_{xx}(t)[Q^0_{t,1}, Q^0_{t,2}] \\ F_{xx}(t)[Q^0_{t,2}]^2 \end{bmatrix}, \qquad b_0 = \begin{bmatrix} G_{xx}[Q^0_{0-,1}]^2 \\ 2G_{xx}[Q^0_{0-,1}, Q^0_{0-,2}] \\ G_{xx}[Q^0_{0-,2}]^2 \end{bmatrix}.$$

Moreover, the inhomogeneous linear system (I.6.42)–(I.6.43) *from Proposition I.6.2.1 has a unique periodic solution* $n(t)$ *of period* T, *so the vector* h_2^{Ξ} *of coefficients of the centre manifold at parameter* $\epsilon = 0$ *is given precisely by the right-hand side of Eq.* (I.6.41). *Also, if* $\mathcal{RCR}_u(t)$ *is trivial, the set*

$$n_t + \mathcal{RCR}_c^\dagger(t) \subset \mathcal{RCR}^3$$

is globally attracting, where $\mathcal{RCR}_c^\dagger(t)$ *is the centre fibre bundle associated to the homogeneous equation*

$$\dot n(t) + 2\Lambda_2 * n(t) = L(t) \odot [e^{2\Lambda_2(\cdot)} * n_t], \qquad t \neq t_k \tag{I.8.45}$$

$$\Delta n(t) = J(k) \odot [e^{2\Lambda_2(\cdot)} * n_{t^-}], \qquad t = t_k. \tag{I.8.46}$$

Proof. Since $\Lambda = \Lambda_\omega = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, one can readily compute

$$\begin{aligned} h_2[\Lambda u, u] &= h_2(t,\theta)\left[\begin{bmatrix} \omega u_2 \\ -\omega u_1 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right] \\ &= \omega\left(c_{11}u_2u_1 + c_{12}u_2^2 - c_{21}u_1^2 - c_{22}u_1u_2\right) \\ &= \omega\left(h_2^{11}u_2u_1 + \frac{1}{2}h_2^{12}u_2^2 - \frac{1}{2}h_2^{12}u_1^2 - h_2^{22}u_1u_2\right) \\ &= \begin{bmatrix} u_1^2 & u_1u_2 & u_2^2 \end{bmatrix} * \Lambda_2 * \begin{bmatrix} h_2^{11} \\ h_2^{12} \\ h_2^{22} \end{bmatrix}, \end{aligned}$$

so that Λ_2 does indeed have the claimed form. Verifying that $\mathcal{F}$, $\mathcal{G}$, a and b_0 are as stated in the lemma can be done in a similar manner, taking into account Lemmas I.8.4.2, I.8.4.3 and the symmetric bilinearity of the second differentials F_{xx} and G_{xx}.

Since (I.6.42)–(I.6.43) are inhomogeneous, we can write any given periodic solution in the form $n(t) = n_0(t) + M(t)y$, where $n_0(t)$ is a particular periodic solution, $M(t)$ is a matrix whose columns consist of a maximal linearly independent set (finite, due to the Floquet theory) of real periodic solutions of the homogeneous equation and y is a real vector of appropriate dimension. To characterize $M(t)$, we write down the homogeneous equation

$$\dot{n} + 2\Lambda_2 * n = L(t) \odot [e^{2\Lambda_2(\cdot)} * n_t], \qquad t \neq k \tag{I.8.47}$$

$$\Delta n = J(k) \odot [e^{2\Lambda_2(\cdot)} * n_{t^-}], \qquad t = k, \tag{I.8.48}$$

where we remember that multiplications involving Λ_2 and its exponentials are treated as array multiplications. Introducing a change of variables $w(t) = e^{2\Lambda_2 t} * n(t)$ for $w \in (\mathbb{R}^n)^{\beta\times 1}$, we find by applying the Leibniz's law that w satisfies the homogeneous equation

$$\dot{w} = L(t) \odot w_t, \qquad t \neq k \tag{I.8.49}$$

$$\Delta w = J(k) \odot w_{t^-}, \qquad t = k. \tag{I.8.50}$$

Thus, the dynamical system for w is merely the β-fold product of the homogeneous system (I.8.9)–(I.8.10) with itself. Recall that this system has, a priori, no nontrivial periodic solutions, and the only Floquet exponents on the imaginary axis are $\pm i\omega$. Since the eigenvalues of $2\Lambda_2$ are $\lambda = 0$ and $\lambda = \pm i2\omega$, the transformation $u \mapsto e^{2\Lambda_2 t} * u$ is uniformly bounded, so every periodic solution of (I.8.47)–(I.8.48) must be of the form $e^{-2\Lambda_2 t} * w(t)$ for a periodic solution $w(t)$ of (I.8.49)–(I.8.50). Since $\omega \neq 0$, the only periodic solution of this form is the trivial solution, thereby proving that the T-periodic solution of the inhomogeneous equation is unique. That $n_t + \mathcal{RCR}_c^\dagger(t)$ is attracting when $\mathcal{RCR}_u(t)$ is trivial follows by the uniform boundedness of the transformation and the spectral separation of $\mathcal{RCR}^3$ by the homogeneous system (I.8.49)–(I.8.50). □

We cannot hope to obtain an explicit, fully general formula for the solution $c(t, \theta)$ encoding the coefficients of $\tilde{h}_2(t, \theta)$. The difficulty arises in solving the inhomogeneous impulsive delay system (I.6.42)–(I.6.43) of Proposition I.6.2.1. See later Example I.8.4.1.

At this stage, we will assume that one has computed the second-order term $\tilde{h}_2$ of the centre manifold at parameter $\epsilon = 0$, or some sufficiently precise numerical approximation thereof. To prove the following lemma, we could apply the method of [30], as is done in the proof of Theorem I.8.3.1. However, the calculations are obviously a bit messier in this case. For the sake of transparency, we provide a self-contained proof.

Lemma I.8.4.5. *The iterated dynamics defined by the Stroboscopic (Poincaré) map associated to the impulsive differential equation* (I.8.43)–(I.8.44) *on the* *parameter-dependent centre manifold, at the fixed parameter $\epsilon = 0$, are given to cubic order by*

$$\begin{aligned}
u \mapsto \Omega(1)u &+ \frac{1}{2!}\left(\int_0^1 \Omega(1)\tilde{Y}(s)F_{xx}(s)[\Phi_s^0 u]^2 ds + \tilde{Y}(0)G_{xx}[\Phi_{1^-}^0 u]^2\right)\\
&+ \frac{1}{3!}\left(\int_0^1 \Omega(1)\tilde{Y}(s)\left[F_{xxx}(s)[\Phi_s^0 u]^3 + 3F_{xx}(s)[\Phi_s^0 u, \tilde{h}_2(s,\cdot)[\Omega(s)u]^2] + \cdots\right.\right.\\
&+ \left.3F_{xx}(s)\left[\Phi_s^0 u, \Phi_s^0 \int_0^s \tilde{Y}(t)F_{xx}(t)[\Phi_t^0 u]^2 dt\right] ds\right] + \cdots\\
&+ \tilde{Y}(0)\left[G_{xxx}[\Phi_{1^-}^0 u]^3 + 3G_{xx}[\Phi_{1^-}^0 u, \tilde{h}_2(1^-,\cdot)[\Omega(1)u]^2] + \cdots\right.\\
&+ \left.\left.3G_{xx}\left[\Phi_{1^-}^0 u, \Phi_{1^-}^0 \int_0^1 \tilde{Y}(t)F_{xx}(t)[\Phi_t^0 u]^2 dt\right]\right]\right) + O(||u||^4),
\end{aligned} \tag{I.8.51}$$

where $\Lambda_\omega = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, $\Omega(t) = e^{\Lambda_\omega t}$ *and* $\tilde{Y}(t) = \begin{bmatrix} Y_{11}(t) \\ Y_{21}(t) \end{bmatrix}$.

Proof. The cubic order dynamics (I.6.16)–(I.6.17) on the centre manifold at parameter $\epsilon = 0$ are

$$\begin{aligned}
\dot{u} &= \Lambda u + e^{\Lambda t}\tilde{Y}(t)\left[\frac{1}{2}D^2F(t)[Q_t^0 u]^2\right.\\
&\quad + \left.\frac{1}{3!}(D^3F(t)[Q_t^0 u]^3 + 3D^2F(t)[Q_t^0 u, \tilde{h}_2(t,\cdot)u^2])\right], t \neq k\\
\Delta u &= \tilde{Y}(0)\left[\frac{1}{2!}D^2G[Q_{k^-}u]^2\right.\\
&\quad \left. + \frac{1}{3!}(D^3G[Q_{k^-}^0 u]^3 + 3D^2G[Q_{k^-}^0 u, \tilde{h}_2(k^0,\cdot)u^2])\right], t = k,
\end{aligned}$$

where $D^2F = F_{xx}$ and analogously for the third derivatives and for G. Let $t \mapsto S(t,u)$ be the unique solution of the above ordinary impulsive differential equation, defined for time $t \geq 0$ and satisfying the initial condition $S(0,u) = u$. It follows that $t \mapsto S(t,u)$ satisfies the impulsive differential equation

$$\begin{aligned}
\dot{S} &= \Lambda S + e^{\Lambda t}\tilde{Y}(t)]\\
&\quad \left[\frac{1}{2}D^2F(t)[Q_t^0 S]^2 + \frac{1}{3!}(D^3F(t)[Q_t^0 S]^3 + 3D^2F(t)[Q_t^0 S, \tilde{h}_2(t,\cdot)S^2])\right],\\
&\quad t \neq k
\end{aligned} \tag{I.8.52}$$

$$\begin{aligned}
\Delta S &= \tilde{Y}(0)\left[\frac{1}{2!}D^2G[Q_{k^-}^0 S]^2 + \right.\\
&\quad \left.\frac{1}{3!}(D^3G[Q_{k^-}^0 S]^3 + 3D^2G[Q_{k^-}^0 S, \tilde{h}_2(k^-,\cdot)S^2])\right], t = k.
\end{aligned} \tag{I.8.53}$$

Our objective is to compute a degree three Taylor expansion of $u \mapsto S(1,u)$ near $u = 0$. The function $S : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is C^3 in its second variable and C^1 in its first variable except at times $t \in \mathbb{Z}$, where it is right-continuous [30]. Consequently, the multiple partial derivatives $t \mapsto \partial_{u^k} S := S_{u^k}$ for $k = 1, 2, 3$ themselves satisfy a set of impulsive differential equations that can be calculated by formally differentiating (I.8.52)–(I.8.53) with respect to u, keeping in mind that $S = S(t,u)$. Also, we have the initial conditions $S_u(0,0) = I$, $S_{uu}(0,0) = 0$ and $S_{uuu}(0,0) = 0$. The formal differentiation process produces

$$\begin{aligned}
\dot{S}_u &= \Lambda S_u + e^{\Lambda t}\tilde{Y}(t)\bigg(D^2F(t)[Q_t^0 S, Q_t^0 S_u] + \frac{1}{3!}\Big(3D^3F(t)[Q_t^0 S, Q_t^0 S, Q_t^0 S_u] + \cdots \\
&\qquad + 3D^2F(t)[Q_t^0 S_u, \tilde{h}_2(t,\cdot)S^2] + 6D^2F(t)[Q_t^0 S, \tilde{h}_2(t,\cdot)[S,S_u]]\Big)\bigg),\ t \neq k \\
\dot{S}_{uu} &= \Lambda S_{uu} + e^{\Lambda t}\tilde{Y}(t)\bigg(D^2F(t)[Q_t^0 S_u]^2 + D^2F(t)[Q_t^0 S, Q_t^0 S_{uu}] + \cdots \\
&\qquad + \frac{1}{3!}\Big(3D^3F(t)[Q_t^0 S, Q_t^0 S, Q_t^0 S_{uu}] + 6D^3F(t)[Q_t^0 S, Q_t^0 S_u, Q_t^0 S_u] + \cdots \\
&\qquad + 3D^2F(t)[Q_t^0 S_{uu}, \tilde{h}_2(t,\cdot)S^2] + 12D^2F(t)[Q_t^0 S_u, \tilde{h}_2(t,\cdot)[S,S_u] + \cdots \\
&\qquad + 6D^2F(t)[Q_t^0 S, \tilde{h}_2(t,\cdot)[S_u,S_u] + \tilde{h}_2(t,\cdot)[S,S_{uu}]]\Big),\ t \neq k \\
\Delta S_u &= \tilde{Y}(k)\bigg(D^2G[Q_{k^-}^0 S, Q_{k^-}^0 S_u] + \frac{1}{3!}\Big(3D^3G[Q_{0^-}^0 S, Q_{k^-}^0 S, Q_{k^-}^0 S_u] + \cdots \\
&\qquad + 3D^2G[Q_{k^-}^0 S_u, \tilde{h}_2(k^-,\cdot)S^2] + 6D^2G[Q_{k^-}^0 S, \tilde{h}_2(k^-,\cdot)[S,S_u]]\Big)\bigg),\ t = k \\
\Delta S_{uu} &= \tilde{Y}(k)\bigg(D^2G[Q_{k^-}^0 S_u]^2 + D^2G[Q_{k^-}^0 S, Q_{k^-}^0 S_u] + \cdots \\
&\qquad + \frac{1}{3!}\Big(3D^2G[Q_{k^-}^0 S, Q_{k^-}^0 S, Q_{k^-}^0 S_{uu}] + 6D^2G[Q_{k^-}^0 S, Q_{k^-}^0 S_u, Q_{k^-}^0 S_u] + \cdots \\
&\qquad + 3D^2G[Q_{k^-}^0 S_{uu}, \tilde{h}_2(k^-,\cdot)S^2] + 12D^2G[Q_{k^-}^0 S_u, \tilde{h}_2(k^-,\cdot)[S,S_u] + \cdots \\
&\qquad + 6D^2G[Q_{k^-}^0 S, \tilde{h}_2(k^-,\cdot)[S_u,S_u] + \tilde{h}_2(k^-,\cdot)[S,S_{uu}]]\Big),\ t = k,
\end{aligned}$$

and we refrain from calculating the impulsive differential equation for S_{uuu} for now. Take note that each of $S = S(t,u)$, $S_u = D_2S(t,u)$ and $S_{uu} = D_2^2 S(t,u)$ is evaluated at an arbitrary $u \in \mathbb{R}^2$. If one calculates the impulsive differential equation for S_{uuu} and evaluates each of S, S_u and S_{uu} at $u = 0$, many terms cancel because $S(t,0)=0$. The result is

$$
\begin{aligned}
\dot{S}_u &= \Lambda S_u, \\
\dot{S}_{uu} &= \Lambda S_{uu} + e^{\Lambda t}\tilde{Y}(t) D^2 F(t)[Q_t^0 S_u]^2, \\
\dot{S}_{uuu} &= \Lambda S_{uuu} + e^{\Lambda t}\tilde{Y}(t)\Big[D^3 F(t)[Q_t^0 S_u]^3 + 3D^2 F(t)[Q_t^0 S_u, \tilde{h}_2(t,\cdot)[S_u]^2] + \cdots \\
&\quad + 3D^2 F(t)[Q_t^0 S_u, Q_t^0 S_{uu}]\Big], \\
\Delta S_u &= 0, \\
\Delta S_{uu} &= \tilde{Y}(k) D^2 G[Q_{k^-}^0 S_u]^2, \\
\Delta S_{uuu} &= \tilde{Y}(k)\Big[D^3 G[Q_{k^-}^0 S_u]^3 + 3D^2 G[Q_{k^-}^0 S_u, \tilde{h}_2(k^-,\cdot)[S_u]^2] + \cdots \\
&\quad + 3D^2 G[Q_{k^-}^0 S_u, Q_{k^-}^0 S_{uu}]\Big].
\end{aligned}
$$

where we have suppressed the conditions $t \neq k$ and $t = k$.

It follows that $S_u(0,t) = e^{\Lambda t} = \Omega(t)$, so that $Q_t^0 S_u = \Phi_t^0$. Taking this into account, solving the above impulsive differential equations at the prescribed initial conditions and substituting into the Taylor expansion

$$
S(1,u) = S_u(1,0)u + \frac{1}{2!}S_{uu}(1,0)[u,u] + \frac{1}{3!}S_{uuu}(1,0)[u,u,u] + O(||u||^4)
$$

produce the right-hand side of (I.8.51). As $u \mapsto S(1,u)$ is precisely the stroboscopic (Poincaré) map, the lemma is proven. □

Remark I.8.4.1. *If the vector field and jump map have no quadratic terms in the state x_t—that is, if $F_{xx} = G_{xx} = 0$—one does not need to compute $\tilde{h}_2$ at all, since the evaluations of $\tilde{h}_2$ in the iterated dynamics* (I.8.51) *only appear in the action of the second differentials $D^2F(s)$ and D^2G.*

Remark I.8.4.2. *Technically, we have not computed the actual symmetric multilinear maps $S_{uu}(1,0)$ and $S_{uuu}(1,0)$. We have only obtained the action of these maps on the elements (u,u) and (u,u,u). It will be necessary in the computation of the Lyapunov coefficient $d(0)$ to evaluate these maps at non-identical tuples; see Eq.* (I.8.40). *To rectify this, we have three suggestions.*

- *One can use the expression* (I.8.51) *as a starting point for a numerical differentiation to approximate the action of the multilinear maps on the appropriate tuples in the expression for $d(0)$. We do this for the example of Sect. I.8.4.1.*

- *Compute the multilinear maps directly. This is done later in Sect. II.5.2.3, when we consider the cylinder bifurcation again in the finite-dimensional context.*

- *Avoid the formal expression of the stroboscopic map entirely, and approximate it numerically by time integration. Use numerical differentiation to approximate the multilinear maps.*

With these preparatory lemmas in place, we are ready to state and prove our bifurcation theorem at a Hopf point.

Theorem I.8.4.1 (Cylinder Bifurcation)**.** *With the notation and assumptions of Lemma I.8.4.1 through Lemma I.8.4.3, suppose the following nondegeneracy conditions are satisfied:*

G.1 $e^{im\omega} \neq 1$ *for* $m = 1, 2, 3, 4$.

G.2 $\gamma(0) \neq 0$, *where*

$$\gamma(0) = \frac{1}{2}\left(\mathrm{tr}\mathcal{B} + \int_0^1 \mathrm{tr}\mathcal{A}(s)ds\right), \tag{I.8.54}$$

where $\mathcal{B} \in \mathbb{R}^{2\times 2}$ *and* $\mathcal{A}(t) \in \mathbb{R}^{2\times 2}$ *are defined by*

$$\begin{aligned}\mathcal{B} &= \tilde{Y}(0)\big(G_{xx}[\pi_{0^-}, Q^0_{0^-}] + G_{\epsilon x}Q^0_{0^-}\big)\\ \mathcal{A}(t) &= \Omega(t)\tilde{Y}(t)\big(F_{xx}(t)[\pi_t, Q^0_t] + F_{\epsilon x}(t)Q^0_t\big).\end{aligned}$$

G.3 The first Lyapunov coefficient $d(0)$ *associated to the two-dimensional discrete-time map* (I.8.51) *of Lemma I.8.4.5 is nonzero.*

Then, the equilibrium point at the origin of the nonlinear impulsive delay differential equation (I.8.7)–(I.8.8) *undergoes a bifurcation to an invariant cylinder at the critical parameter* $\epsilon = 0$. *Specifically, for* $|\epsilon|$ *small, there is a unique periodic orbit* $t \mapsto y(t, \epsilon)$ *that satisfies* $y_t(\cdot, \epsilon) \to 0$ *as* $\epsilon \to 0$, *in addition to a two-dimensional parameter-dependent invariant fibre bundle* $\Sigma_\epsilon \subset \mathbb{S}^1 \times \mathcal{RCR}$ *that exists for* $d(0)\gamma(0)\epsilon < 0$ *and is periodic. The* t*-fibre* $\Sigma_\epsilon(t)$ *can be locally realized as*

$$\Sigma_\epsilon(t) = Q^0_t \sigma_\epsilon(t) + O(\epsilon),$$

where $t \mapsto \sigma_\epsilon(t) \subset \mathbb{R}^2$ *is periodic with its image a curve of diameter* $O(\sqrt{\epsilon})$, *and continuous in the Hausdorff metric except at integer times, where it is continuous from the right. Also, if in addition* $\mathcal{RCR}^0_u(t)$ *is trivial, then*

- y_ϵ *is asymptotically stable for* $\gamma(0)\epsilon < 0$, *stable for* $\epsilon = 0$ *and unstable for* $\gamma(0)\epsilon > 0$, *while* $\Sigma_\epsilon(t)$ *is attracting for* $\gamma(0)\epsilon > 0$ *provided* $d(0) < 0$;
- y_ϵ *is asymptotically stable for* $\gamma(0)\epsilon < 0$ *and unstable for* $\gamma(0)\epsilon \geq 0$, *while* $\Sigma_\epsilon(t)$ *is unstable for* $\gamma(0)\epsilon < 0$ *provided* $d(0) > 0$.

Moreover, the assertions concerning the stability and existence of the periodic orbit $y(t, \epsilon)$ *for* $\epsilon \neq 0$ *are true regardless of the nondegeneracy condition G.3.*

Proof. The persistence of the equilibrium to a periodic orbit for $|\epsilon|$ small follows by the remark that the iterated dynamics satisfy, to linear order, $u \mapsto \Omega(1)u$ at the parameter $\epsilon = 0$. As $\Omega(1)$ is invertible, the implicit function theorem guarantees the iterated dynamics posses a unique, small fixed point for $0 < |\epsilon| \ll 1$. Lifting this fixed point into the nonlinear impulsive delay differential equation, the result is a unique, small periodic orbit.

The periodic orbit $v_\epsilon(t)$ is C^1 in ϵ. Near $\epsilon = 0$, we can infer from (I.8.43)–(I.8.44) that $v_\epsilon(t) = O(\epsilon^2)$. If we perform a time- and parameter-dependent change of coordinates $y = z + v_\epsilon$, (I.8.43)–(I.8.44) become

$$\dot{z} = \Lambda_\omega z + e^{\Lambda_\omega t}\tilde{Y}(t)\left[F(t, Q_t^0(v_\epsilon + z) + \pi_t\epsilon, \epsilon) - F(t, Q_t^0 z + \pi_t\epsilon, \epsilon)\right], \quad t \neq k \tag{I.8.55}$$

$$\Delta z = e^{\Lambda_\omega k}\tilde{Y}(k)\left[G(Q_{k^-}^0(v_\epsilon + z) + \pi_{k^-}\epsilon, \epsilon) - G(Q_{k^-}^0 v_\epsilon + \pi_{k^-}\epsilon, \epsilon)\right], \qquad t = k. \tag{I.8.56}$$

Comparing to the iterated map (I.8.39), the Floquet multipliers of the trivial equilibrium $z = 0$ of (I.8.55)–(I.8.56) are precisely the eigenvalues of $A(\epsilon)$ for the iterated map obtained via the stroboscopic (Poincaré) map for the above dynamical system. The linearization of the above system is

$$\begin{aligned}
\dot{z} &= \Lambda_\omega z + e^{\Lambda_\omega t}\tilde{Y}(t)\left[D_x F(t, Q_t^0 v_\epsilon + \pi_t\epsilon, \epsilon)Q_t^0\right] z, && t \neq k \\
\Delta z &= e^{\Lambda_\omega k}\tilde{Y}(k)\left[D_x G(Q_{k^-}^0 v_\epsilon + \pi_{k^-}\epsilon, \epsilon)Q_{k^-}^0\right] z, && t = k,
\end{aligned}$$

and it follows from Liouville's formula for impulsive differential equations [9] that the product of the Floquet multipliers is

$$\begin{aligned}
\tilde{\gamma}(\epsilon)^2 := \mu_1(\epsilon)\mu_2(\epsilon) = \det\Big(I + \tilde{Y}(0)D_x G(Q_{0^-}^0 v_\epsilon(0^-) + \pi_{0^-}\epsilon, \epsilon)Q_{0^-}^0\Big) \cdots \\
\times \exp\left(\int_0^1 \operatorname{tr}\left[e^{\Lambda_\omega t}\tilde{Y}(t)D_x f(t, Q_t^0 v_\epsilon(t) + \pi_t\epsilon, \epsilon)Q_t^0\right] dt\right).
\end{aligned}$$

Using Jacobi's formula and the asymptotic $v_\epsilon(t) = O(\epsilon^2)$, we can readily calculate the derivative of $\mu_1(\epsilon)\mu_2(\epsilon)$ at $\epsilon = 0$. The formula from (I.8.54) and the sufficient condition $\gamma(0) \neq 0$ for the transversality condition follows by the observation that $r'(0) = \frac{1}{2}\tilde{\gamma}'(0) = \gamma(0)$.

It follows that under assumptions G.1 through G.3, the discrete-time dynamical system defined by the Stroboscopic map $S_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$ of (I.8.55)–(I.8.56) undergoes a Neimark–Sacker bifurcation at parameter $\epsilon = 0$. Inverting the change of variables, the same is true of the original system (I.8.43)–(I.8.44). There is a closed curve $\sigma_\epsilon(0) \subset \mathbb{R}^2$ that exists for $d(0)\gamma(0) < 0$, is invariant under S_ϵ and is attracting and stable for $\gamma(0)\epsilon > 0$ and unstable for $\gamma(0)\epsilon < 0$. The fixed point $v_\epsilon(0)$ of S_ϵ satisfies the stability and attraction properties of the theorem, and the same is true of $t \mapsto v_\epsilon(t)$. The stability and attraction persist when considered in the infinite-dimensional context of (I.8.7)–(I.8.8) provided $\mathcal{RCR}_u(t) = \{0\}$ because of Theorem I.5.5.1.

Let $t \mapsto X_\epsilon(t; 0, w)$ denote the unique solution of (I.8.55)–(I.8.56). Because $\sigma_\epsilon(0)$ is invariant under S_ϵ, the fibre bundle

$$\sigma_\epsilon = \{(t, x(t)) : \exists w \in \sigma_\epsilon(0), x(t) = X(t; 0, w)\}$$

is invariant under the process X_ϵ and the t-fibre $\sigma_\epsilon(t)$ is periodic with period 1. The continuity of $t \mapsto X(t;0,w)$ for $t \in [k,k+1)$ and $k \in \mathbb{Z}$ leads naturally to the continuity of fibre $t \mapsto \sigma_\epsilon(t)$. Also, the diameter [82] of $\sigma_\epsilon(0)$ is $O(\sqrt{\epsilon})$, and continuity implies the same of $\sigma_\epsilon(t)$. Moreover, $||\sigma_\epsilon(t)|| = O(\sqrt{\epsilon})$, and we can write $\sigma_\epsilon(t) = v_\epsilon(t) + \sigma_\epsilon^0(t)$ for another closed curve $\sigma_\epsilon^0(t)$ of diameter $O(\sqrt{\epsilon})$.

By invariance of the parameter-dependent centre manifold, it follows from the representation (I.6.3) of solutions on the centre manifold and the description of the matrix Q_t from (I.8.41) that

$$\Sigma_\epsilon(t) = Q_t^0\sigma_\epsilon(t) + \pi_t\epsilon + \tilde{h}(t,\sigma_\epsilon(t),\epsilon,\cdot) \subset \mathcal{RCR}$$

naturally defines an invariant fibre bundle $\Sigma \subset \mathbb{R} \times \mathcal{RCR}$ of the nonlinear impulsive delay differential equation (I.8.7)–(I.8.8), with

$$\Sigma_\epsilon = \bigcup_{t\in\mathbb{R}} \{t\} \times \Sigma_\epsilon(t).$$

The fibre bundle is periodic in the sense that $\Sigma_\epsilon(t+1) = \Sigma_\epsilon(t)$, and so can be identified as being a subset of $\mathbb{S}^1 \times \mathcal{RCR}$. Since $h = O(||(u,\epsilon)||^2)$, we can write

$$\begin{aligned}\Sigma_\epsilon(t) &= Q_t^0\sigma_\epsilon(t) + \pi_t\epsilon + O(||(\sigma_\epsilon(t),\epsilon)||^2)\\ &= Q_t^0\sigma_\epsilon(t) + O(\epsilon + ||(v_\epsilon(t) + \sigma_\epsilon^0(t),\epsilon)||^2)\\ &= Q_t^0\sigma_\epsilon(t) + O(\epsilon + |\sqrt{\epsilon}+\epsilon|^2)\\ &= Q_t^0\sigma_\epsilon(t) + O(\epsilon).\end{aligned}$$

Finally, the attractivity properties of Σ_ϵ follow by recognizing that it is a locally attracting invariant set within the centre manifold and applying Theorem I.5.5.1. □

If we wanted to be more verbose, we would describe this bifurcation pattern as a *Neimark–Sacker bifurcation from a discontinuous periodic solution*. As for the term *cylinder bifurcation*, we discuss this now. Because the system (I.8.7)–(I.8.8) is periodic, there is a nonautonomous dynamical system

$$S : \mathbb{R}\times(\mathbb{S}^1\times\mathcal{RCR}) \to \mathbb{S}^1\times\mathcal{RCR}, \qquad S(t,(s,\phi)) = (s+t \bmod 1, x^{(s,\phi)}(t)),\ t \geq s$$

associated to it, where $x^{(s,\phi)} : [s, s+\alpha) \to \mathcal{RCR}$ is the unique solution through the initial condition (s,ϕ) and defined on a maximal interval of existence. This dynamical system satisfies the semigroup properties $S(t,(t,\cdot)) = I_{\mathcal{RCR}}$ and

$$S(t,(s,S(s,(u,\phi)))) = S(t,(u,\phi))$$

whenever $t \geq s \geq u$. As such, $\mathbb{S}^1 \times \mathcal{RCR}$ is the state space of the nonautonomous dynamical system generated by (I.8.7)–(I.8.8).

Each of the t-fibres $\Sigma_\epsilon(t)$ of the bifurcating invariant fibre bundle are homeomorphic to the circle $\mathbb{S}^1$. The set

$$\Sigma_\epsilon = \bigcup_{t\in\mathbb{R}} \{t \bmod 1\} \times \Sigma_\epsilon(t) \subset \mathbb{S}^1 \times \mathcal{RCR}$$

naturally has the structure of a topological manifold with boundary $\{0\} \times \Sigma_\epsilon(0)$, while every interior slice $\{s\} \times \Sigma_\epsilon(s)$ for $s \neq 0$ has a neighbourhood homeomorphic to the open cylinder $(0,1)\times\mathbb{S}^1$. The nontrivial boundary is the result of $t \mapsto \sigma_\epsilon(t)$ being periodic but lacking continuity at the integers, being only right-continuous there and generically possessing a finite jump. The name *cylinder bifurcation* we propose stems from this fact. When the impulse effect is trivial, there are no discontinuities in the time evolution of $t \mapsto \sigma_\epsilon(t)$, and we obtain the classical bifurcation pattern to an invariant torus. Such torus bifurcations typically occur from periodic orbits in autonomous delay differential equations or from equilibrium points in periodically forced delay differential equations [121].

The generic cylinder bifurcation is as follows, with the genericity conditions being $e^{ik\omega} \neq 1$ for $k = 1, 2, 3, 4$, $\gamma(0) \neq 0$ and $d(0) \neq 0$.

Corollary I.8.4.1 (Generic Cylinder Bifurcation)**.** *For any generic impulsive delay differential equation* (I.8.7)–(I.8.8) *having at* $\epsilon = 0$ *the equilibrium* 0 *with a single pair of complex-conjugate Floquet exponent* $\lambda = \pm i\omega$ *for* $\omega \in (0, 2\pi)$ *and two-dimensional centre fibre bundle, there is a neighbourhood* N *of* $0 \in \mathcal{RCR}$ *and a smooth invertible change of parameters* $\eta = \eta(\epsilon)$ *satisfying* $\eta(0) = 0$ *such that for* $\eta > 0$*, there is an invariant cylinder in* $\mathbb{S}^1 \times N$ *that trivializes to* $\mathbb{S}^1 \times \{0\}$ *(i.e. to the equilibrium) as* $\eta \to 0^+$*, together with a unique periodic orbit in* N *that persists for all* $|\eta|$ *sufficiently small and trivializes to the equilibrium as* $\eta \to 0$*.*

I.8.4.1 Example: Impulsive Perturbation from a Hopf Point

We will study the effect of linear impulsive perturbations on a scalar delay differential equation at a Hopf point

$$\begin{aligned}
\dot{x} &= -\frac{\pi}{2}x(t-1) + s_2(t)x^2(t-1) + s_3(t)x^3(t), && t \neq kT \\
\Delta x &= \epsilon x(t^-), && t = kT,
\end{aligned}$$

where $T \geq 1$ is a fixed period not smaller than the delay and $s_2(t)$ and $s_3(t)$ are real periodic functions of period T. When $\epsilon = 0$, the impulse effect is trivial and the linearization of the associated delay differential equation has a pair of simple complex-conjugate eigenvalues $\pm i\frac{\pi}{2}$ on the imaginary axis, and all other have negative real parts. Thus, perturbing the vector field from this configuration would generically lead to a Hopf bifurcation if s_2 and s_3

were constant. Otherwise, if they were periodic and nonconstant, we would expect a bifurcation to an invariant torus [121].

One can easily verify that $[\ \cos\frac{\pi}{2}(t+\theta) \quad \sin\frac{\pi}{2}(t+\theta)\ ]$ is a basis matrix for $\mathcal{RCR}_c^0(t)$. Performing a rescaling of time $t \mapsto Tt$, the result is

$$\dot{x} = -\frac{T\pi}{2}x(t-1/T) + \sigma_2(t)x^2(t-1/T) + \sigma_3(t)x^3(t), \quad t \neq k \tag{I.8.57}$$

$$\Delta x = \epsilon x(t^-), \quad t = k, \tag{I.8.58}$$

where $\sigma_i(t) = Ts_i(t/T)$ is periodic with period 1. After this transformation,

$$\Phi_t^0 = [\ \cos(\tfrac{\pi}{2}T(t+\theta)) \quad \sin(\tfrac{\pi}{2}T(t+\theta))\ ]$$

is a the new basis for $\mathcal{RCR}_c^0(t)$, $\pm i\frac{T\pi}{2}$ are the Floquet exponents on the imaginary axis, and we have the representation $\Phi_t^0 = Q_t^0 \exp(t\Lambda_{\frac{T\pi}{2}})$. Therefore, ω and Q_t^0 are

$$\omega = \frac{T\pi}{2}, \qquad Q_t^0(\theta) = Q_0(\theta) = \left[\ \cos\left(\frac{T\pi}{2}\theta\right) \quad \sin\left(\frac{T\pi}{2}\theta\right)\ \right].$$

Using this information, we can verify the first nondegeneracy conditions G.1 of Theorem I.8.4.1. We find that G.1 is equivalent to the condition

$$T \text{ and } \frac{3T}{4} \text{ are not integers.}$$

To check the nondegeneracy condition G.2, we require $\tilde{Y}(t)$ and π_t. In I.8.5, we compute $\tilde{Y}(t)$ and find

$$\tilde{Y}(t) = e^{-\Lambda_\omega t}\frac{4}{4+\pi^2}\left[\begin{array}{c} 2 \\ \pi \end{array}\right].$$

The explicit calculation of π_t is a much more difficult problem: one must compute the unique periodic solution $\pi(t)$ of the linear inhomogeneous equation

$$\begin{aligned} \dot{y} &= -\frac{T\pi}{2}\pi(t-1/T), \quad && t \neq k \\ \Delta y &= 1, && t = k. \end{aligned} \tag{I.8.59}$$

When T is rational, the periodic solution can be computed explicitly, although the calculation can be very lengthy. Alternatively, although $\pi(t)$ is not asymptotically stable, every solution $y(t)$ satisfies $y_t \to \pi_t + e_t$ as $t \to \infty$ for some $e_t \in \mathcal{RCR}_c(t)$, and since the latter is completely characterized, one could numerically simulate an arbitrary solution and solve an appropriate linear equation to compute $\pi(t)$ to any desired precision. Still another way is to discretize the monodromy operator and compute an approximation using a discretized variation-of-constants formula. To demonstrate one of the (in principle) more analytically tractable cases however, we will focus our attention on a specific choice of T satisfying the nondegeneracy condition G.1. One of the simplest choices is $T = \frac{3}{2}$. As demonstrated in I.8.5, the periodic solution is $y(t) = e_1 \cdot S(t - \lfloor t \rfloor)$, where $S : [0, 1) \to \mathbb{R}^3$ is defined piecewise by the expression

$$S(t) = \begin{cases} e^{At}u_0, & t \in [0, 1/3) \\ e^{At}(e_3 + e^{A\frac{1}{3}}u_0), & t \in [1/3, 2/3) \\ e^{At}(e_2 + e^{A\frac{1}{3}}e_1 + e^{A\frac{2}{3}}u_0), & t \in [2/3, 1), \end{cases} \qquad A = -\frac{3\pi}{4}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \tag{I.8.60}$$

$$u_0 = (I - e^A)^{-1}(e_1 + e^{\frac{1}{3}A}e_2 + e^{\frac{2}{3}A}e_3), \tag{I.8.61}$$

where $\{e_1, e_2, e_3\}$ is the standard ordered basis of $\mathbb{R}^3$. The expression is rather cumbersome, so we do not display it explicitly in terms of elementary functions. A plot (Fig. I.8.6) of this periodic solution is provided in I.8.5. The nondegeneracy condition G.2 is then equivalent to

$$\gamma(0) = \frac{4}{4 + \pi^2}\left(1 - \pi \int_0^1 \sigma_2(u)e_2 \cdot S(u)du\right) \neq 0. \tag{I.8.62}$$

Being affine linear in the coefficient σ_2, the condition $\gamma(0) \neq 0$ is indeed generic.

At this stage, we will make a choice for the coefficient σ_2. Choosing $\sigma_2(t) = \frac{1}{2}\sin(2\pi t)$, numerical integration yields

$$\gamma(0) = \frac{4}{4 + \pi^2}\left(1 - \pi \int_0^1 \frac{1}{2}\sin(2\pi u)e_2 \cdot S(u)du\right) = 0.30854 \pm 10^{-5}, \tag{I.8.63}$$

so the nondegeneracy condition G.2 passes, and in particular, Theorem I.8.4.1 predicts the existence of a nontrivial periodic orbit $t \mapsto y(t, \epsilon)$ that is locally asymptotically stable for $\epsilon < 0$ and unstable for $\epsilon > 0$.

Next, we calculate the quadratic approximation h_2 of the centre manifold. We must solve the impulsive evolution equation (I.6.37)–(I.6.40) subject to the data and constraints from Lemma I.8.4.4. The first step is to calculate the unique periodic solution $n^0(t)$ satisfying (I.6.42)–(I.6.43). The second equation (I.6.43) is trivial for the present example, so $n^0(t)$ satisfies the inhomogeneous linear delay differential equation

$$\mathcal{F}(t, 0) + \dot{n} + 2\Lambda_2 n(t) = -\frac{3\pi}{4}e^{-\frac{2}{3}\Lambda_2}n(t - 2/3) + m(t), \tag{I.8.64}$$

with the functions $\mathcal{F}(t, \theta)$ and $m(t)$ being given in Sect. I.8.5.4. We devise a numerical routine in Sect. I.8.5.3 to approximate the periodic solution. With $\sigma_2(t) = \frac{1}{2}\sin(2\pi t)$, a plot of the result is provided in Fig. I.8.7. A numerical approximation of the coefficient vector h_2^{Ξ} is then obtained by substituting the approximation into (I.6.41), with the integral being computed by numerical quadrature.

Visualizing the quadratic term $\frac{1}{2}\tilde{h}_2$ of the two-dimensional centre manifold is slightly more complicated because even though the two independent parameters u_1, u_2 are real, $\tilde{h}_2(t,\cdot)[u_1,u_2]$ is an element of $\mathcal{RCR}$ for each $t \in \mathbb{R}$. However, since $(u_1,u_2) \mapsto \frac{1}{2}\tilde{h}_2(t,\theta)[u_1,u_2]$ is a scalar field with a zero at the origin, we can get a sense of the geometry in a neighbourhood of this point using the Hessian matrix. To provide a coarse but faithful depiction of the geometry, we will therefore generate a plot of the eigenvalues of

$$\theta \mapsto \texttt{eig}\left[\begin{array}{cc} h_2^{11}(t,\theta) & \frac{1}{2}h_2^{12}(t,\theta) \\ \frac{1}{2}h_2^{12}(t,\theta) & h_2^{22}(t,\theta) \end{array}\right] := \texttt{eig}(\mathrm{H}(t,\theta)),$$

for $\theta \in [-1,0]$, where $\texttt{eig}(\mathrm{H})$ denotes the eigenvalues of H, and the matrix above is indeed the Hessian of $(u_1,u_2) \mapsto h(t,\theta)(u_1,u_2)$ at the origin. See Fig. I.8.2. It is clear that the classification of the origin is generally nonconstant in (t,θ), as there are θ-intervals where the origin is a saddle, maximum or minimum for fixed t. These intervals themselves are nonconstant in t.

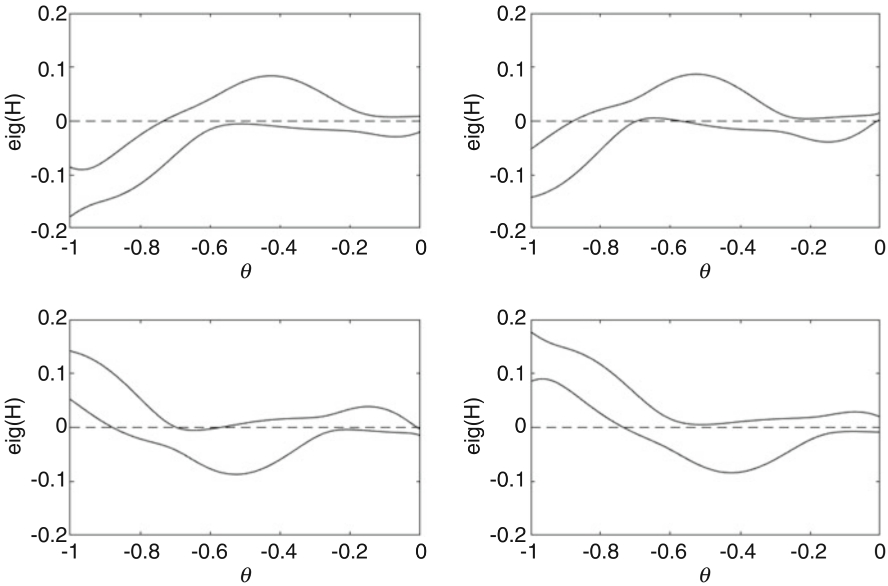

Figure I.8.2: Plots of the function $\theta \mapsto \texttt{eig}(\mathrm{H}(t,\theta))$ for fixed arguments of $t \in [-1,0]$. From top left counterclockwise, these times are $t=0$, $t=0,1$, $t=0.5$ and $t=0.6$. Notice the varying topological classification of the origin as θ and t are varied. For each argument θ, the origin is a local minimum when both curves are above the dashed line, a local maximum when both curves are below the dashed line, and a saddle point when the dashed line separates the curves

By fixing a particular $\theta \in [-1, 0]$, we can generate a contour plot of $(u_1, u_2) \mapsto \frac{1}{2}\tilde{h}_2(t,\theta)[u_1, u_2]$ for a few fixed arguments of t to see the transitions between different topological classifications of the origin. Fixed snapshots from one period of transitions of the contour plot are provided in Fig. I.8.3 for the same times used in the snapshots of the Hessian eigenvalue plot (Fig. I.8.2).

To compute the Lyapunov coefficient $d(0)$, we will need to fix a choice of σ_3. We choose $\sigma_3(t) = -2$. For our example, the Lyapunov coefficient was calculated by first defining a MATLAB function that computes the right-hand side of (I.8.51) using numerical quadrature (specifically, MATLAB's built-in trapezoidal method `trapz`) and our previously computed approximation of $\tilde{h}_2$. Then, we used numerical differentiation (with iterated use of `gradient`) to calculate the associated bilinear and trilinear maps in (I.8.40), the normalized right- and left- eigenvectors of A_0 and, finally, the Lyapunov coefficient $d(0)$. With our choices $T = \frac{3}{2}$, $\sigma_2(t) = \frac{1}{2}\sin(2\pi t)$ and $\sigma_3(t) = -2$, our approximation of the Lyapunov coefficient is $d(0) = -0.5604$. Since $\gamma(0) > 0$ and $d(0) < 0$, Theorem I.8.4.1 implies the existence of a locally attracting invariant cylinder when $\epsilon > 0$ is small.

A standard way to visualize bifurcations to invariant tori in autonomous scalar delay differential equations is to plot curves of the form $(x(t), x(t - r_1), x(t - r_2))$ for two delays r_1 and r_2 that can be chosen as desired. For periodically forced systems, the same thing can be done, or one can plot $(t \bmod T, x(t), x(t - r))$ and identify the hyperplanes $t = 0$ and $t = 1^-$ by "wrapping" the figure around a circle of fixed radius embedded in $\mathbb{R}^3$ to illustrate the torus as being a subspace of $\mathbb{S}^1 \times \mathbb{R}^2$. For impulsive delay differential equations of the form (I.8.7)–(I.8.8), one must choose the delays r_1 and r_2 to be positive integers, otherwise the curves $t \mapsto (x(t), x(t - r_1), x(t - r_2))$ will have discontinuities at times other than the integers.

For our example, we provide both with the illustrative parameter $\epsilon = \frac{1}{2}$. Figure I.8.4 is a plot of the attractor in the delayed variables $x(t)$, $x(t - 1)$ and $x(t-2)$, while Fig. I.8.5 is a plot of the curve $t \mapsto (t \bmod 1, x(t), x(t-1))$ wrapped around a cylinder of radius 7 (there is no deep significance to the choice of radius). That is, we plot the curve in the cylindrical coordinates (r, θ, z) with

$$r = |7 + x(t)|, \qquad \theta = 2\pi t, \qquad z = x(t - 1).$$

In rectangular coordinates, this corresponds to a plot of $(x_{rad,1}(t), x_{rad,2}(t), x(t - 1))$ with

$$\begin{bmatrix} x_{rad,1}(t) \\ x_{rad,2}(t) \end{bmatrix} = \begin{bmatrix} \cos(2\pi t)(7 + x(t)) \\ \sin(2\pi t)(7 + x(t)) \end{bmatrix}. \tag{I.8.65}$$

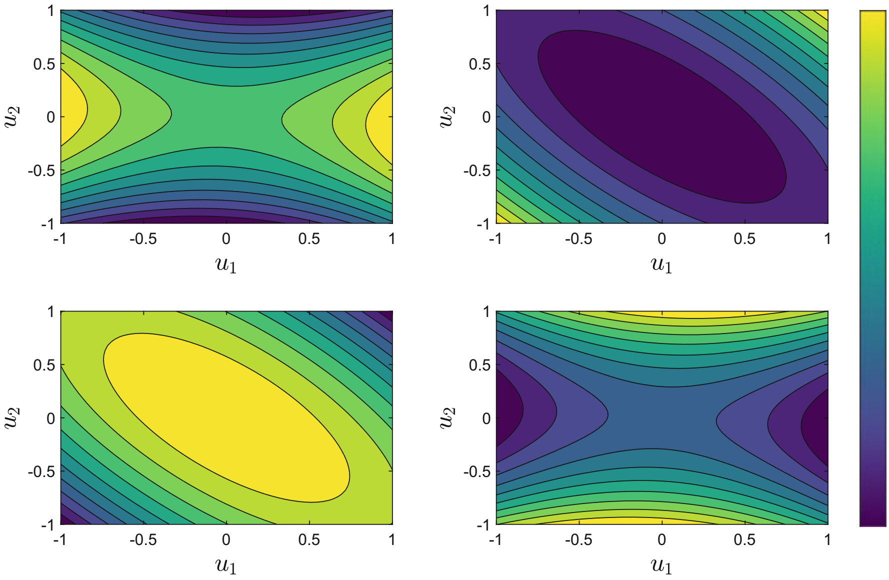

Figure I.8.3: Contour plots of $(u_1, u_2) \mapsto \frac{1}{2}\tilde{h}_2(t, 0)[u_1, u_2]$ for various arguments of $t \in [0, 1]$. In the colour map (displayed right), yellow (top) corresponds to more positive levels and purple (bottom) corresponds to more negative levels. From top left clockwise, the plot times are $t = 0$, $t = 0.1$, $t = 0.5$ and $t = 0.6$, corresponding to the origin being a saddle point, local minimum, saddle point and local maximum, respectively. The transition times between the different topological classifications in the interval $[0, 1]$ are $t = 0.0781$, $t = 0.262$, $t = 0.576$ and $t = 0.759$

The simulation is in agreement with Theorem I.8.4.1, as the cylinder is indeed present and attracting for $\epsilon > 0$.

I.8.5 Calculations Associated to Example I.8.4.1

This section is broken up into several parts. We begin with the calculations concerning the projection $P_c(t)$. Next, we calculate $\pi(t)$ and the matrices $\mathcal{A}(t)$ and $\mathcal{B}$ needed in the nondegeneracy condition $\gamma(0) \neq 0$. Following this, we calculate $\tilde{h}_2$. We conclude by checking the nondegeneracy condition $d(0) \neq 0$.

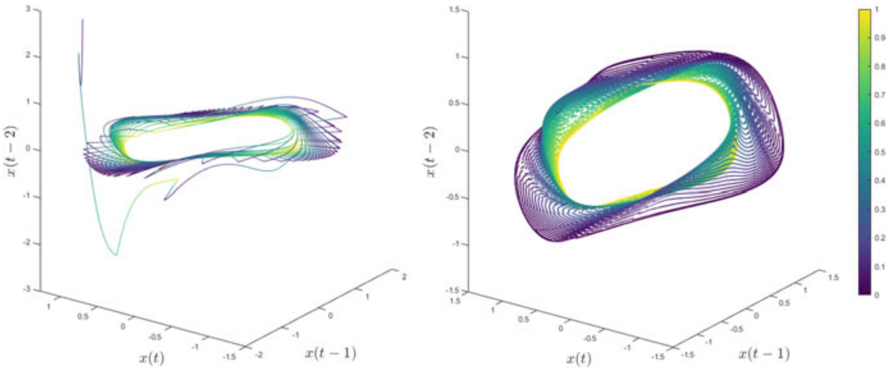

Figure I.8.4: Left: trajectory through the constant initial condition $x_0 = 8$ of the system (I.8.58)–(I.8.58) from Example I.8.4.1 for parameters $T = \frac{3}{2}$, $\sigma_2(t) = \frac{1}{2}\sin(2\pi t)$, $\sigma_3(t) = -2$, $\epsilon = \frac{1}{2}$ in the coordinates $x(t)$, $x(t-1)$ and $x(t-2)$, plotted for time $t \in [0, 80]$. Linear interpolation between left-limits $x(k^-)$ and points $x(k)$ for integer times $k \in \mathbb{Z}$ is shown. Right: attractor to which the solution in the left pane converges. In both panes, trajectories are coloured using the Viridis colourmap (displayed right) relative to the argument $t \bmod 1$, so that purple corresponds to integer arguments of $t = k \in \mathbb{Z}$, while yellow corresponds to the left-limits $t \to k^-$

I.8.5.1 The Projection $P_c(t)$ and Matrix $\tilde{Y}(t)$

Conveniently, since $P_c(t)$ is calculated with respect to the linearization at $\epsilon = 0$ and this system is autonomous, we have $V_t^0 = V_0^0$ for all $t \in \mathbb{R}$, and the projection

$$P_c(t) = \frac{1}{2\pi i}\int_\Gamma (zI - V_0^0)^{-1} dz$$

is constant. Precisely, V_0^0 is the monodromy operator associated to the autonomous system

$$\dot{x} = -\frac{T\pi}{2}x(t - 1/T). \tag{I.8.66}$$

To compute the vector $Y(t)$, we remark that because $P_c(t) = P_c$ is constant and $\Phi_t^0 = Q_0 e^{\Lambda_\omega t}$ for $\omega = \frac{T\pi}{2}$, we have the representation $Y(t) = e^{-\Lambda_\omega t}Y(0)$, so it suffices to compute $Y(0)$. Since $\Phi_0 = Q_0$, $Y(0)$ satisfies the equation $P_c\chi_0 = Q_0 Y(0)$. Q_0 is precisely the basis matrix for the centre eigenspace

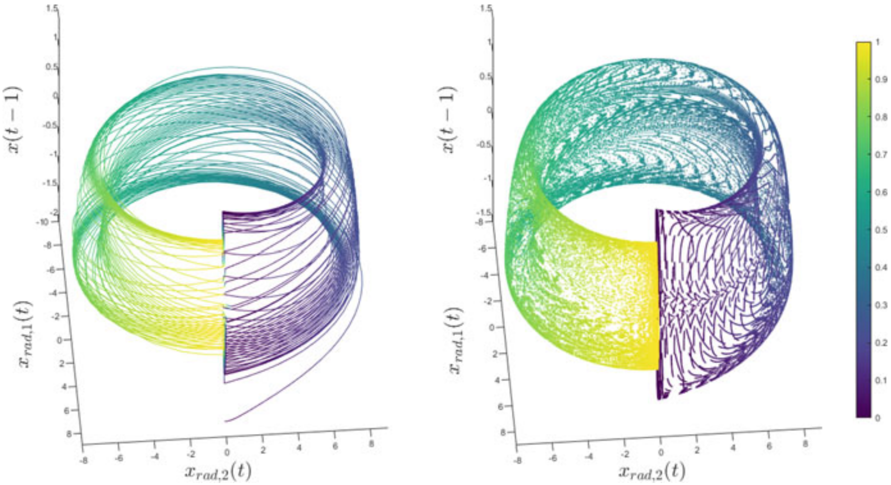

Figure I.8.5: The same trajectories and attractors from Fig. I.8.4 in the cylindrical coordinates defined in Eq. (I.8.65). The same colourmap is also used. In these coordinates it is much easier to visualize the cylindrical topology of the attractor as well as the discontinuity along the half-plane $\{x_{rad,2}(t) = 0, x_{rad,1}(t) \geq 0\}$ corresponding to the times $t = k \in \mathbb{Z}$. If the impulse effect were replaced by a parameter-dependent continuous-time-periodic linear forcing in the vector field, the structure above would generically be replaced by that of a torus, and the aforementioned half-plane discontinuity would not be present

of the infinitesimal generator associated to the semigroup of (I.8.66), so we can formally calculate $P_c\chi_0$ using adjoint-based methods; see [55, 58]. A basis matrix for the centre eigenspace of the infinitesimal generator of the formal adjoint of (I.8.66) is $\Psi(\theta) = \begin{bmatrix}\cos(\frac{T\pi}{2}\theta) & -\sin(\frac{T\pi}{2}\theta)\end{bmatrix}^\intercal$, so applying the usual bilinear form,

$$\begin{aligned}
&\langle \Psi, \Phi_0\rangle = \Psi(0)\Phi_0(0) - \frac{T\pi}{2}\int_{-1/T}^{0}\Psi(\theta+1/T)\Phi_0(\theta)d\theta \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \frac{\pi}{2}\int_{-1}^{0}\begin{bmatrix} \cos(\frac{\pi}{2}(\theta+1))\cos(\frac{\pi}{2}\theta) & \cos(\frac{\pi}{2}(\theta+1))\sin(\frac{\pi}{2}\theta) \\ -\sin(\frac{\pi}{2}(\theta+1))\cos(\frac{\pi}{2}\theta) & -\sin(\frac{\pi}{2}(\theta+1))\sin(\frac{\pi}{2}\theta) \end{bmatrix} d\theta \\
&= \begin{bmatrix} 1/2 & \pi/4 \\ \pi/4 & -1/2 \end{bmatrix}.
\end{aligned}$$

Normalizing Ψ with respect to Φ_0, it follows that the projection $P_c\chi_0$ is given

$$\begin{aligned} P_c\chi_0 &= \Phi_0\langle\Psi,\Phi_0\rangle^{-1}\langle\Psi,\chi_0\rangle \\ &= \left[\begin{array}{cc} \cos(\frac{T\pi}{2}\theta) & \sin(\frac{T\pi}{2}\theta) \end{array}\right] \left[\begin{array}{cc} \frac{8}{4+\pi^2} & \frac{4\pi}{4+\pi^2} \\ \frac{4\pi}{4+\pi^2} & -\frac{8}{4+\pi^2} \end{array}\right] \left[\begin{array}{c} 1 \\ 0 \end{array}\right] \\ &= \Phi_0(\theta)\frac{4}{4+\pi^2}\left[\begin{array}{c} 2 \\ \pi \end{array}\right]. \end{aligned}$$

Therefore, $\tilde{Y}(t) = e^{-\Lambda_\omega t}\frac{4}{4+\pi^2}[\begin{array}{cc} 2 & \pi \end{array}]^\intercal$, as claimed.

I.8.5.2 Calculation of $\pi(t)$ and the Matrices $\mathcal{A}(t)$ and $\mathcal{B}$

Now, we consider only the case $T = \frac{3}{2}$. To calculate $\pi(t)$, we define the shifts $y_0(t) = y(t)$, $y_1(t) = y(t-2/3)$ and $y_2(t) = y(t-4/3)$. Assuming y is periodic with period 1, it follows that the shifts satisfy the impulsive differential equation

$$\begin{aligned} \dot{y}_j &= -\frac{3\pi}{4}y_{i+1}, & t &\neq k + \frac{2}{3}j, \\ \Delta y_j &= 1, & t &= k + \frac{2}{3}j, \end{aligned}$$

for $j = 0, 1, 2$, and we define $y_3 = y_0$. The sequence of impulses is periodic with period 1, and the impulse times in the interval $(0, 1]$ are $\frac{1}{3}$, $\frac{2}{3}$ and 1. If u_0 is a given initial condition at time $t = 0$, it is easy to check that the solution at time $t = 1$ is given by

$$\vec{y}(1) = e_1 + e^{\frac{1}{3}A}(e_2 + e^{\frac{1}{3}A}(e_3 + e^{\frac{1}{3}A}u_0)).$$

Imposing the periodicity constraint $\vec{y}(1) = u_0$ and solving for u_0 yield the expression (I.8.61), which is well-defined because 1 is not an eigenvalue of A. Since $\pi(t)$ is the unique periodic solution of (I.8.59), there is exactly one periodic solution of the above inhomogeneous impulsive system. Thus, its first component must coincide with $\pi(t)$, thereby proving our claim. Figure I.8.6 is a plot of the periodic solution $\pi(t)$ together with the shifted components.

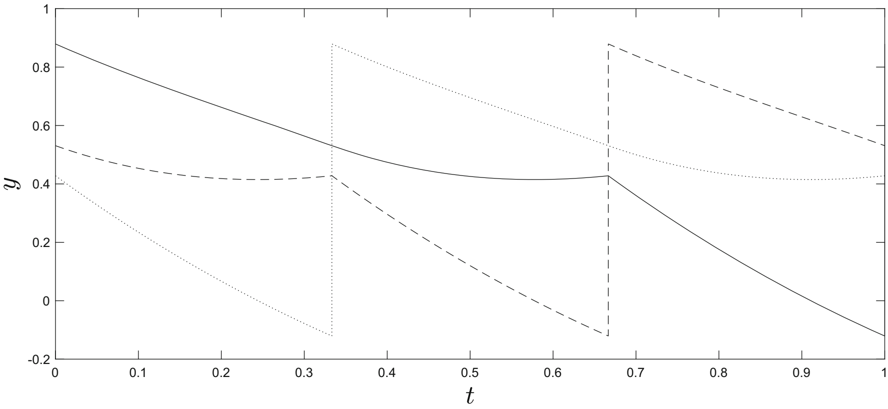

Figure I.8.6: The periodic solution $\pi(t) = y_0(t)$ (solid black line) and the two shifts $y_1(t)$ and $y_2(t)$ (black dashed line and dotted lines, respectively), plotted over one period $t \in [0, 1)$

To check the nondegeneracy condition G.2, we calculate each of $\mathcal{B}$ and $\mathcal{A}(t)$. Since $G_{xx} = 0$, $G_{\epsilon x}\phi = \phi(0)$ and $Q^0_{0^-} = [\ 1 \quad 0\ ]$, we readily compute

$$\begin{aligned}\mathcal{B} &= \tilde{Y}(0)(G_{xx}[\pi_{0^-}, Q^0_{0^-}] + G_{\epsilon x} Q^0_{0^-})\\ &= \frac{4\pi}{2}\begin{bmatrix} 2 \\ \pi \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\\ &= \frac{4}{4+\pi^2}\begin{bmatrix} 2 & 0 \\ \pi & 0 \end{bmatrix}.\end{aligned}$$

On the other hand, we have $F_{xx}(t)[\phi, \psi] = 2\sigma_2(t)\phi(-1/T)\psi(-1/T)$, $F_{x\epsilon} = 0$ and $Q^0_t(-1/T) = [\ 0 \quad -1\ ]$. Since $\Omega(t)\tilde{Y}(t) = \tilde{Y}(0)$ and we have chosen $T = \frac{3}{2}$, we have

$$\begin{aligned}\mathcal{A}(t) &= \Omega(t)\tilde{Y}(t)(F_{xx}(t)[\pi_t, Q^0_t] + F_{\epsilon\epsilon x} Q^0_t)\\ &= \frac{4}{4+\pi^2}\begin{bmatrix} 2 \\ \pi \end{bmatrix}\left(2\sigma_2(t)\pi(t-2/3)\begin{bmatrix} 0 & -1 \end{bmatrix}\right)\\ &= -\frac{8\sigma_2(t)\pi(t-2/3)}{4+\pi^2}\begin{bmatrix} 0 & 2 \\ 0 & \pi \end{bmatrix}.\end{aligned}$$

Taking into account that $\pi(t - 2/3) = y_1(t) = e_2 \cdot S(t)$, one obtains (I.8.62) after substituting $\mathcal{B}$ and $\mathcal{A}(t)$ into (I.8.54).

I.8.5.3 Calculation of $n^0(t)$: A Numerical Routine

The routine we propose here could certainly be adapted to more general settings, and our notation will at times suggest at a more general approach.

Our first step is to compute a periodic solution satisfying (I.8.64). To do this, we integrate the delay differential equation from the constant initial condition $0 \in \mathbb{R}^3$ until convergence is achieved to a superposition of some periodic solution $n^0(t)$ of period one, together with a linear combination of periodic solutions of period $\frac{8}{3}$ determined by the eigenvalues $\pm\frac{3\pi}{4}i$ of the homogeneous equation

$$\dot{n} + 2\Lambda_2 n(t) = -\frac{3\pi}{4}e^{-\frac{2}{3}\Lambda_2}n(t-2/3). \tag{I.8.67}$$

Symbolically, the solution $s(t)$ satisfying $s(0) = 0$ is simulated until the numerical convergence $s(t) \to \tilde{s}(t)$ is achieved with

$$\tilde{s}_t \in n_t^0 + \mathcal{RCR}_c^\dagger(t), \tag{I.8.68}$$

where $\mathcal{RCR}_c^\dagger(t)$ is the centre fibre bundle associated to the homogeneous equation for (I.8.67). That this decomposition can be realized in the limit is due to Lemma I.8.4.4.

Next, we construct an approximate basis for $\mathcal{RCR}_c^\dagger(t)$. This is done by integrating the homogeneous equation associated to (I.8.64) from a collection $\{x_0^1, \dots, x_0^K\}$ of arbitrary linearly independent initial conditions $x_0^i \in \mathcal{RCR}$ for $i = 1, \dots, K$, with the integration performed until the associated solutions $s^i(t)$ numerically converge to some $\tilde{s}^i(t)$ satisfying $\tilde{s}_t^i \in \mathcal{RCR}_c^\dagger(t)$. That this convergence is attainable essentially follows be Lemma I.8.4.4.

Having computed an approximate basis $\{\tilde{s}_t^1, \dots, \tilde{s}_t^K\}$ for $\mathcal{RCR}_c^\dagger(t)$, our goal is to extract n_t^0 from $\tilde{s}_t$ in the decomposition (I.8.68). To this end, we define the shifts

$$v = \tilde{s}_t - \tilde{s}_{t-1}, \qquad v_i = \tilde{s}_t^i - \tilde{s}_{t-1}^i, \qquad i = 1, \dots, K.$$

In an idealized sense, we have $v, v_i \in \mathcal{RCR}_c^\dagger(t)$, so that $v = \sum_{i=1}^K y_i v_i$ for some real constants y_i. In practice this equality is not attainable, so instead we search for a best approximation of v in the finite-dimensional subspace $W = \text{span}\{v_1, \dots, v_K\}$, where we now interpret v and W as being in $L^2 = L^2([-1,0], \mathbb{R}^3)$. The best approximation is $\sum_{i=1}^K y_i v_i$ with the vector $\vec{y} = (y_1, \dots, y_K) \in \mathbb{R}^K$ being the unique solution of

$$M\vec{y} = b, \qquad M_{ij} = \langle v_i, v_j\rangle, \qquad b_i = \langle v, v_i\rangle, \tag{I.8.69}$$

where $\langle f, g\rangle = \int_{-1}^0 f(t)\cdot g(t)dt$ is the standard inner product on L^2. It follows that the function

$$n^0(t) = \tilde{s}(t) - \sum_{i=1}^K y_i \tilde{s}_i(t) \tag{I.8.70}$$

is the best approximation to a periodic solution of (I.8.64), relative to the basis W, in the sense that the L^2 *periodicity error*

$$\mathsf{e}(n_t^0) = ||n_t^0 - n_{t-1}^0||_{L^2}$$

is minimized.

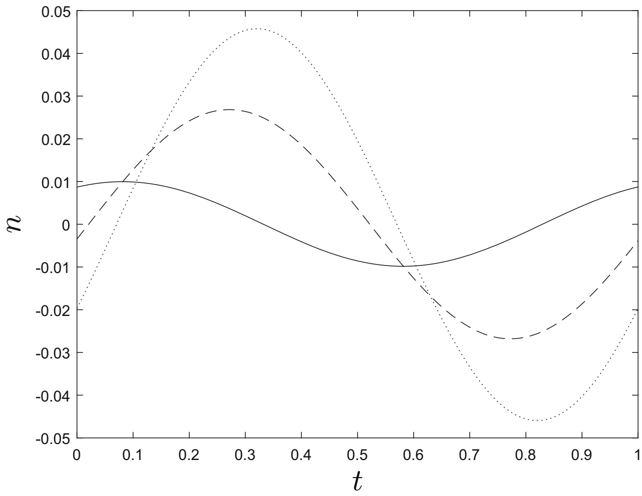

Figure I.8.7: The unique periodic solution $n^0 = (n_1^0, n_2^0, n_3^0) \in \mathbb{R}^3$ of the inhomogeneous linear system (I.8.64) computed using the numerical routine from I.8.5.3. The solid black curve is the plot of $n_1^0(t)$, while the dashed and dotted curves are those of $n_2^0(t)$ and $n_3^0(t)$, respectively

I.8.5.4 Calculation of h_2

Using Eq. (I.6.33), we find that $\mathcal{F}(t,\theta)$ and $m(t)$ are, in terms of an arbitrary $\sigma_2(t)$,

$$\mathcal{F}(t,\theta) = \frac{4}{4+\pi^2}\left(2\cos(\frac{3\pi}{4}\theta) + \pi\sin(\frac{3\pi}{4}\theta)\right)2\sigma_2(t)\begin{bmatrix}0\\0\\1\end{bmatrix} \tag{I.8.71}$$

$$m(t) = 2\sigma_2(t)\begin{bmatrix}0\\0\\1\end{bmatrix} + \frac{3\pi}{4}e^{-\frac{4}{3}\Lambda_2}\int_{-2/3} e^{-\frac{4}{3}\Lambda_2 s}\mathcal{F}\left(t-s-\frac{2}{3}, s\right)ds, \tag{I.8.72}$$

$$\Lambda_2 = \frac{3\pi}{4}\begin{bmatrix}0 & -\frac{1}{2} & 0\\ 1 & 0 & -1\\ 0 & \frac{1}{2} & 0\end{bmatrix}. \tag{I.8.73}$$

We implemented the numerical routine from Sect. I.8.5.3 with $K = 6$ in MATLAB R2018a. All numerical integration of the delay differential equations was done using `dde23` with default error tolerances. The random initial history functions x_0^i for $i = 1, \ldots, 6$ were produced using separate calls to `randn(3,1)` at each point $t \in [-2/3, 0]$ requested by the solver. With $\sigma_2(t) = \frac{1}{2}\sin(2\pi t)$, the result was the periodic solution plotted in Fig. I.8.7 with L^2 periodicity error $\mathtt{e}(n_t^0) < 3 \cdot 10^{-3}$.

I.8.6 A Recipe for the Analysis of Smooth Local Bifurcations

In the previous two sections we followed what might be described a "recipe" for the analysis of a smooth local bifurcation from a nonhyperbolic fixed point (or periodic solution, after change of variables) of impulsive functional differential equations (I.8.1)–(I.8.2) subject to periodicity conditions. In Part IV we will consider several applications of the centre manifold reduction in the analysis of local bifurcations. To aid in the structure of these chapters later, and also to make this recipe more explicit, we will list the steps now.

Check for Nonhyperbolicity

Suppose $x = 0$ is an equilibrium point of (I.8.1)–(I.8.2) for some subset $N \subset \mathbb{R}^p$ of parameter space. This will be the case if one has a known periodic solution $t \mapsto x(t,p)$ depending smoothly[2] on a parameter and performs a change of variables to place this periodic solution at zero. To check for nonhyperbolicity amounts to determining at which parameter(s) $\epsilon \in N$ the linearization

$$\begin{aligned} \dot{y} &= D_2 f(t,0,\epsilon), && t \neq t_k \\ \Delta y &= D_2 g(k,0,\epsilon), && t = t_k \end{aligned}$$

has Floquet exponents with zero real part (equivalently, Floquet multipliers on the unit circle). Here, $D_2 f(t,0,\epsilon)$ is the differential of $\phi \mapsto f(t,\phi,\epsilon)$ evaluated at $\phi = 0$, and similarly for $D_2 g(k,0,\epsilon)$.

Extend Phase Space and Write Equations in Semilinear Form

Assume without loss of generality that at $\epsilon = 0$, the fixed point $x = 0$ is nonhyperbolic with $c > 0$ Floquet exponents with zero real part. In the extended phase space, we can write the dynamics in semilinear form

$$\frac{d}{dt}\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} D_2 f(t,0,0) & D_3 f(t,0,0)\chi_0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_t \\ \epsilon_t \end{bmatrix} + r_f(t,x_t,\epsilon_t), \qquad t \neq t_k \tag{I.8.74}$$

$$\Delta\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} D_2 g(k,0,0) & D_3 g(k,0,0)\chi_0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{t^-} \\ \epsilon_{t^-} \end{bmatrix} + r_g(t,x_{t^-},\epsilon_{t^-}), \quad t = t_k, \tag{I.8.75}$$

[2]In the sense that $p \mapsto x_t(\cdot,p)$ is smooth for each t fixed.

where r_f and r_g are the quadratic (and above) nonlinearities (in x and ϵ) satisfying $r_f(t,0,0) = r_g(k,0,0) = 0$ with vanishing differentials at zero; $D_2 r_f(t,0,0) = D_2 g(k,0,0) = 0$, $D_3 r_f(t,0,0) = D_3 r_g(k,0,0) = 0$. Here, $D_3 f(t,0,0)$ denotes the derivative of $\epsilon \mapsto f(t,0,\epsilon)$ at $\epsilon = 0$, and analogously for g and the remainders. The interpretation of D_2 is the same as before.

Linear Data

Next, one must compute a basis matrix Φ_0 of $\mathcal{RCR}_c(0)$ for the linearization

$$\frac{d}{dt}z = \begin{bmatrix} D_2 f(t,0,0) & D_3 f(t,0,0)\chi_0 \\ 0 & 0 \end{bmatrix} z_t, \qquad t \neq t_k$$

$$\Delta z = \begin{bmatrix} D_2 g(k,0,0) & D_3 g(k,0,0)\chi_0 \\ 0 & 0 \end{bmatrix} z_{t^-}, \qquad t = t_k.$$

After doing this, one defines $\Phi_t = U_c(t,0)\Phi_0$ and computes the (real) Floquet decomposition $\Phi_t = Q_t e^{t\Lambda}$ and the matrix $Y_c(t) \in \mathbb{R}^{n\times(c+p)}$ satisfying $P_c(t)\chi_0 = \Phi_t Y_c(t)$. These steps will, in most situations, involve some form of numerical approximation. For example, one might use some form of discretization scheme on the monodromy operator V_0 to compute Φ_0. Then, to compute $Y(t)$ at some set of discrete points in the interval $[0,T]$ for T the period, one could discretize V_t and numerically integrate the action of the numerical resolvent $R(z;V_t)$ on the relevant representation (i.e. discretization) of χ_0 to compute the integral

$$P_c(t)\chi_0 = \frac{1}{2\pi i}\int_\Gamma R(z;V_t)\chi_0 dz.$$

Taylor Coefficients of the Centre Manifold

To approximate the Euclidean space representation of the centre manifold, one applies Theorem I.6.1.3 and substitutes the Taylor series ansatz

$$h(t,u,\theta) = \frac{1}{2}h_2(t,\theta)u^2 + \cdots + \frac{1}{m!}h_m(t,\theta)u^m + O(|u|^{m+1})$$

to obtain a sequence of linear evolution equations for the coefficients h_i. For $i = 2$, one use the method of characteristics from Sect. I.6.2.2. In most cases, this step will also require the help of a numerical solver.

Centre Manifold Reduction

Having determined all necessary data to write down the impulsive differential equation on the centre manifold, one applies Theorem I.6.1.2 to (I.8.74)–(I.8.75). After truncating to the order of the Taylor expansion (e.g. order 3 for a quadratic Taylor expansion), the result will be an impulsive differential equation in $c+p$ dimensions. The projection onto the first c components will give the dynamics on the parameter-dependent centre manifold for fixed parameter ϵ, up to prescribed order in $\mathbb{R}^c$. Local bifurcations can then be studied using time T maps (i.e. reduction to discrete time), now taking the dynamics as being dependent on the (small) parameter ϵ.

I.8.7 Comments

The content of Sect. I.8.2 through Sect. I.8.5 appears in *Computation of centre manifolds and some codimension-one bifurcations for impulsive delay differential equations* [33] by Church and Liu, published by Journal of Differential Equations in 2019. The existing literature on smooth bifurcations for impulsive functional differential equations has mostly consisted of numerical studies; see the references [146, 157, 158, 165] for some example. Church and Liu [32] have applied the results of this chapter to analytically study transcritical bifurcations and numerically verify cylinder bifurcations in a SIR model with pulse vaccination and temporary immunity; an abridged version of this publication appears later in Sect. IV.3.

The results of this section do not depend crucially on the assumption that discrete delay r satisfies $r<1$, but rather that the linear and nonlinear terms in the jump map satisfy the overlap condition and that $r\leq 1$. We make the assumption $r<1$ mostly for ease of presentation. The methodology can be extended in a straightforward way to accomodate the case $r>1$. In particular, the generic results of Corollaries I.8.3.1 and I.8.4.1 remain true for r arbitrary, provided the overlap condition is satisfied. They also hold if there is more than one delay and more than one impulse per period, under similar assumptions.

Part II

Finite-Dimensional Ordinary Impulsive Differential Equations

Chapter II.1

Preliminaries

II.1.1 Existence and Uniqueness of Solutions

An *impulsive differential equations with impulses at fixed times* is a type of nonautonomous dynamical system generated by

$$\frac{dx}{dt} = f(t,x), \qquad t \neq t_k \tag{II.1.1}$$

$$\Delta x = g_k(x), \qquad t = t_k, \tag{II.1.2}$$

for $x \in \mathbb{R}^n$. The symbol Δx should be understood as $\Delta x = x(t_k) - x(t_k^-)$, whereas $g_k(x)$ should be understood as $g_k(x(t_k^-))$, so that (II.1.2) more concretely states

$$x(t_k) - x(t_k^-) = g_k(x(t_k^-)).$$

The superscript minus sign denotes a left-limit; $x(t_k^-) = \lim_{s \to t_k^-} x(s)$. The sequence of impulses $\{t_k : k \in \mathbb{Z}\}$ is always assumed strictly increasing and unbounded as $k \to \pm\infty$.

For systems with impulses at fixed times, mild conditions on the impulse effect and sufficient continuity of the vector field are enough to guarantee local existence and uniqueness of solutions forward in time. The following results can be found in [9], except that in the reference continuity is assumed from the left—see Sect. II.1.3 for a discussion.

Theorem II.1.1.1 (Forward Existence and Uniqueness for Systems with Impulses at Fixed Times)**.** *Consider the impulsive differential equation*

$$\frac{dx}{dt} = f(t,x), \qquad t \neq t_k$$

$$\Delta x = g_k(x), \qquad t = t_k.$$

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Suppose the following conditions are valid:

(a) The function $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is continuous in the sets $[t_k, t_{k+1}) \times \Omega$ for $k \in \mathbb{Z}$, and, for each $k \in \mathbb{Z}$ and $x \in \Omega$, the limit of $f(t, y)$ as $(t, y) \to (t_k^-, x)$ exists and is finite.

(b) For any compact interval $K \subset \mathbb{R}$, $f(t, \cdot)$ is locally Lipschitz continuous, uniformly for $t \in K$.

(c) $x + g_k(x) \in \Omega$ for all $x \in \Omega$ and $k \in \mathbb{Z}$.

Then the initial-value problem

$$x(s) = x_0$$

has a unique solution for all $(s, x_0) \in \mathbb{R} \times \Omega$ defined in an interval of the form $[t_0, \omega)$ and is not continuable to the right of ω. That is, there is a unique function $x : [s, \omega) \to \Omega$ that satisfies $x(s) = x_0$, the differential equation $\dot{x} = f(t, x(t))$ except at times $t_k \in (s, \omega)$, where it is continuous from the right with limits on the left, and the jump condition $x(t_k) - x(t_k^-) = g_k(x(t_k^-))$ whenever $s \in (t_0, \omega)$.

Remark II.1.1.1. *The conclusions remain valid if condition (a) is replaced with the requirement that for any $x_n \to x \in \mathbb{R}^n$ and any $s \in \mathbb{R}$, if $s_n \to s$ is decreasing then $\lim_{n\to\infty} f(x_n, s_n) = f(x, s)$, while if $s_n \to s$ is increasing the limit $\lim_{n\to\infty} f(x_n, s_n)$ exists. However, in this case one must work with a weaker definition solution (an integrated solution) since the time-dependent discontinuities in the vector field will generally result in a lack of differentiability of the solution. Namely, a solution $x : [s, \omega) \to \Omega$ of the initial-value problem $x(s) = x_0$ is a function satisfying the integral equation*

$$x(t) = x_0 + \int_s^t f(u, x(u))du + \sum_{s<t_k\leq t} g_k(x(t_k^-)), \qquad s \leq t < \omega$$

and continuous except for at $t \in (s, \omega) \cap \{t_k : k \in \mathbb{Z}\}$.

By time reversal, one finds analogous conditions for existence of solutions defined on intervals that are not continuable to the left.

Corollary II.1.1.1 (Backward Existence and Uniqueness)**.** *As in Theorem II.1.1.1, the initial-value problem $x(s) = x_0$ has a unique solution for all $(s, x_0) \in \mathbb{R} \times \Omega$ defined on an interval of the form (γ, s) and is not continuable to the left of γ, if conditions (a) and (b) of Theorem II.1.1.1 are satisfied, and (c) is strengthened to the condition that the equation $x + g_k(x) = \eta$ has a unique solution $x \in \Omega$ for each $\eta \in \Omega$.*

Consequently, to establish existence and uniqueness of solutions defined on maximal intervals that are not continuable on either side, it suffices that the vector field f is continuous in t on $[t_k, t_{k+1})$, locally Lipschitz with respect to x, and for the map $J_k : \Omega \to \Omega$ defined by $J_k(x) = x + g_k(x)$ to exist and be invertible for all $k \in \mathbb{Z}$. This last condition makes it impossible for two distinct solutions to "merge" together by way of a discontinuity.

Theorem II.1.1.2 (Global Existence)**.** *Let the conditions of Theorem II.1.1.1 or Corollary II.1.1.1 hold, suppose $J^{\pm}(s, x_0)$ is the left $(-)$ or right $(+)$ maximal interval of existence of the initial-value problem $x(s) = x_0$, and let $\varphi(t)$ be the solution associated to this interval. If there exists a compact set $Q \subset \Omega$ such that $\varphi(t) \in Q$ for $t \in J^{\pm}(s, x_0)$, then $J^{\pm}(s, x_0)$ is unbounded; specifically,*

$$J^{\pm} \in \{(-\infty, s), (s, \infty)\}.$$

The converse of this result then states that a solution that is not continuable must approach the boundary of Ω at one of the endpoints of its interval of existence. In the case where Ω is unbounded or equal to $\mathbb{R}^n$, this means that there may be finite-time "blow-up" of solutions.

II.1.2 Dependence on Initial Conditions and Parameters

We introduce a parameter into the system (II.1.1)–(II.1.2):

$$\dot{x} = f(t, x, p), \quad t \neq t_k(p) \tag{P1}$$

$$\Delta x = g_k(x, p), \quad t = t_k(p), \tag{P2}$$

where $p \in \Pi \subset \mathbb{R}^d$. Since our main focus in this book is the analysis of bifurcations, it is natural for us to assume a fairly high level of regularity on the vector field. For these reasons, we introduce some assumptions.

F.1 $t_0(p) = 0$ for all $p \in \Pi$.

F.2 (P1)–(P2) is PC^k for some $k \geq 0$; that is,

- $D^m_{(2,3)} f(t, x, p)$ exist for $m = 0, \dots, k$, whenever $(s_n, x_n, p_n) \to (s, x, p)$, the limit $\lim_{n\to\infty} D^m_{(2,3)} f(s_n, x_n, p_n)$ exists and, if s_n is decreasing, the limit is precisely $Df^m_{(2,3)} f(s, x, p)$;
- g_j and t_j are C^k for all $j \in \mathbb{Z}$.

Remark II.1.2.1. *Condition F.2 for $m = 1$ is strong enough to guarantee local existence and uniqueness of solutions; see Remark II.1.1.1. Condition F.1 can always be assumed without loss of generality, by performing a change of variables.*

Under the assumptions F.1 and F.2, solutions of (P1)–(P2) have a good amount of regularity in terms of both initial conditions and parameters. The following is a generalization of Theorem 2.1 of the monograph of Bainov and Simeonov [9]. Its proof is omitted.

Theorem II.1.2.1. *Consider the solution map* $x(t;\cdot,\cdot,\cdot):\mathbb{R}\times\Omega\times\Pi\to\Omega$ *of the* PC^k *system* (P1)–(P2) *for some* $k\geq 1$*. Denote* $x(t)=x(t;s,x_0,p_0)$, $u=\frac{\partial x}{\partial x_0}(t;s,x_0,p_0)$, $v=\frac{\partial x}{\partial p}(t;s,x_0,p_0)$*. If* $s\notin\{t_k(p):k\in\mathbb{Z}\}$*, then* u *and* v *satisfy the following initial-value problems:*

$$\begin{aligned}
\dot{u} &= D_x f(t,x(t),p_0)u, && t\neq t_k(p_0)\\
\Delta u &= D_x g_k u, && t=t_k(p_0)\\
\dot{v} &= D_x f(t,x(t),p_0)v + D_p f(t,x(t),p_0), && t\neq t_k(p_0)\\
\Delta v &= D_x g_k v + D_p g_k - [f - f^- - (D_x g_k)f]D_p t_k(p_0), && t=t_k(p_0)\\
u(0) &= I_{n\times n},\\
v(0) &= 0,
\end{aligned}$$

where $D_x g_k$ *and* $D_p g_k$ *are evaluated at* $(x(t_k^-),p_0)$*,* $f=f(t_k,x(t_k),p_0)$*,* $f^-=f(t_k(p_0),x(t_k^-),p_0)$*.*

Remark II.1.2.2. *Technically, since the* PC^1 *assumption only guarantees existence and uniqueness of integrated solutions—see Remark II.1.1.1—the differential equations in the above theorem themselves will only generated integrated solutions. In practice, this will not be of great importance because the discontinuities in the vector field will be quite minute. In fact, it will usually be the case that* f *(and its differentials) are continuous on the sets* $[t_k,t_{k+1})\times\Omega\times\Pi$ *for all* $k\in\mathbb{Z}$*, so that the integrated solutions will be differentiable except at the impulse times.*

II.1.3 Continuity Conventions: Right- and Left-Continuity

In the study of impulsive dynamical systems, one must eventually make a choice as to how to define solutions at impulse times. Specifically, must decide whether left-continuity ($x(t_k)=x(t_k^-)$) or right-continuity ($x(t_k)=x(t_k^+)$) is imposed on the solution. This will affect the interpretation of the symbol Δx. For finite-dimensional systems such as we consider in this part of the present monograph, the definitions end up being equivalent in some sense.

For simplicity, suppose $f:\mathbb{R}^n\to\mathbb{R}^n$ is locally Lipschitz continuous so that the ordinary differential equation $\dot{x}=f(x)$ has local existence and uniqueness of solutions. Let $g_k:\mathbb{R}^n\to\mathbb{R}^n$ be continuous and $\{t_k:k\in\mathbb{Z}\}$ be a monotonically increasing unbounded (as $k\to\pm\infty$) sequence. Let $s\in\mathbb{R}\backslash\{t_k:k\in\mathbb{Z}\}$ and consider the solution $x(t;s,x_0)$ of $\dot{x}=f(x)$ satisfying $x(s;s,x_0)=x_0$. If we impose that x has a single jump at time $t_j=\min\{t_k:t_k>s\}$, then we have two reasonable choices assuming we only want a single jump discontinuity in a neighbourhood of t_j:

- define $x(t_j) = x(t_j^-) + I_j(x(t_j^-))$, and then continue the solution of the differential equation from $(t_j, x(t_j))$;
- let $x(t_j) = x(t_j^-)$, define $x(t_j^+) = x(t_j) + I_j(x(t_j))$, and continue onto the unique solution $y(t)$ that satisfies $y(t_j^+) = x(t_j^+)$.

In both cases, the value of $x(t_j^+)$ is the same, so both solutions will coincide in the interval (t_j, t_{j+1}). Obviously, this equivalence breaks down if $s = t_k$ for some k, but other structures such as periodic solutions are preserved. For instance, if $x(t)$ is a periodic solution that is continuous from the right and satisfies $x(t_k) = x(t_k^-) + g_k(x(t_k^-))$, then the function $y(t)$ defined by

$$y(t) = \begin{cases} x(t), & t \neq t_k \\ x(t^-), & t = t_k \end{cases}$$

is continuous from the left and satisfies $y(t_k^+) = y(t_k) + g_k(y(t_k))$.

To summarize, there is no loss of generality in taking solutions to be continuous from the right. Since this was the convention we took in Part 1 of this book, we will continue it here.

II.1.4 Comments

There are many monographs that cover the basics of ordinary impulsive differential equations with impulses at fixed times. These include the more classical works [9, 10, 85, 112, 125], in addition to some more modern treatments [2, 17]. This brief chapter serves only to provide enough detail so that in subsequent chapters, the equations we study will be on a rigorous footing. The interested reader certainly should consult any of these related references for additional background.

Chapter II.2

Linear Systems

The linear systems theory of this chapter is far from exhaustive, and we will introduce only what is necessary to proceed with stability and invariant manifold theory. The reader is encouraged to consult the 1993 monograph of Bainov and Simeonov [9] for additional background, if desired.

The main object of interest in this chapter is the inhomogeneous linear equation

$$\dot{x} = A(t)x(t) + f(t), \qquad t \neq t_k \tag{II.2.1}$$
$$\Delta x = B_k x(t) + g_k, \qquad t = t_k, \tag{II.2.2}$$

and the associated homogeneous equation

$$\dot{z} = A(t)z(t), \qquad t \neq t_k \tag{II.2.3}$$
$$\Delta z = B_k z(t), \qquad t = t_k. \tag{II.2.4}$$

In what follows, we will always assume that $t \mapsto A(t)$, $t \mapsto f(t)$ are continuous from the right and possess limits on the left. This is sufficient to ensure local existence and uniqueness of solutions forward in time; see Theorem II.1.1.1 and the subsequent remark.

II.2.1 Cauchy Matrix

Let $X(t,s)$ denote the *Cauchy matrix* of the homogeneous ordinary differential equation (II.2.3). That is, $x(t;s,x_0) := X(t,s)x_0$ is the unique solution of (II.2.3) satisfying the initial condition $x(s;s,x_0) = x_0$. The Cauchy matrix has the following (defining) properties.

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- $X(t,t) = I$ for all $t \in \mathbb{R}$.
- $X(t,s)^{-1}$ exists for all $t, s \in \mathbb{R}$, and for $s < t$ we define $X(s,t) \equiv X(t,s)^{-1}$.
- $X(t,s) = X(t,0)X(0,s)$ for all $t, s \in \mathbb{R}$.
- $\frac{d}{dt}X(t,s) = A(t)X(t,s)$ at all arguments t, where A is continuous.
- $X(t,s) = I + \int_s^t A(u)X(u,s)du$ for all $t, s \in \mathbb{R}$.

Using the Cauchy matrix of the continuous part (II.2.3), we can construct the fundamental matrix solution of the impulsive system (II.2.3)–(II.2.4).

Theorem II.2.1.1. *Introduce the matrix-valued function $U(t,s)$ for $t \geq s$ by the equation*

$$U(t,s) = \begin{cases} X(t,s), & t_{k-1} \leq s \leq t < t_k \\ X(t,t_\ell)\left(\prod_{j=\ell}^{k+1}(I+B_j)X(t_j,t_{j-1})\right)(I+B_k)X(t_k,s) & t_{k-1} \leq s < t_k < t_\ell \leq t < t_{\ell+1}. \end{cases}$$

Then, $x(t) := U(t,s)x_0$ is defined on $[s,\infty)$ and is the unique solution of (II.2.3)–(II.2.4) satisfying the initial condition $x(s) = x_0$. If the matrices $I + B_k$ are invertible—that is, $\det(I+B_k) \neq 0$ for all $k \in \mathbb{Z}$—then, $U(s,t) := U(t,s)^{-1}$ is well-defined for all $s \leq t$. In this case, the solution $x(t) = U(t,s)x_0$ is defined on the entire real line. In the above equation, the product denotes multiplication from left to right: $\prod_{j=\ell}^{k+1} M_j = M_\ell M_{\ell-1}\cdots M_{k+2}M_{k+1}$.

Proof. This theorem can be proven by induction on the cardinality of $(s,t] \cap \{t_j : j \in \mathbb{Z}\}$. If this set is empty, then it is clear by definition of $X(t,s)$ that the $t \mapsto U(t,s)x_0 = X(t,s)x_0$ is the unique solution of (II.2.3)–(II.2.4) satisfying the initial condition $x(s) = x_0$, since there are no impulse times in $(s,t]$. Suppose now that the conclusion of the theorem is true for any interval $[s,t]$ such that $(s,t] \cap \{t_j : j \in \mathbb{Z}\}$ has cardinality at most $q \geq 0$. Let $[s,t]$ be any interval such that $|(s,t] \cap \{t_j : j \in \mathbb{Z}\}| = q+1$. Without loss of generality, we may assume

$$(s,t] \cap \{t_j : j \in \mathbb{Z}\} = \{t_1, \ldots, t_{q+1}\}.$$

From the induction hypothesis, the unique solution x of the initial condition $x(s) = x_0$ satisfies

$$x(t_q) = \left(\prod_{j=q}^{2}(I+B_j)X(t_j,t_{j-1})\right)(I+B_1)X(t_1,s)x_0.$$

Then, by definition of $X(t,s)$, the solution x satisfies $x(t_{q+1}^-) = X(t_{q+1}^-, t_q)$ $x(t_q)$. Since $v \mapsto X(v,s)$ is continuous, combining the previous calculation with the jump condition (II.2.4), we get

$$\begin{aligned} x(t_{q+1}) &= (I + B_{q+1})X(t_{q+1}, t_q)x(t_q) \\ &= \left(\prod_{j=q+1}^{2} (I + B_j)X(t_j, t_{j-1}) \right) (I + B_1)X(t_1, s)x_0. \end{aligned}$$

The conclusion follows since $x(t) = X(t, t_{q+1})x(t_{q+1})$. □

Definition II.2.1.1. *The matrix $U(t,s)$ introduced in Theorem II.2.1.1 is called the* Cauchy matrix *associated to the linear homogeneous impulsive differential equation* (II.2.3)–(II.2.4).

Corollary II.2.1.1. *The Cauchy matrix enjoys the following properties.*

- $U(t,t) = I$ *for all* $t \in \mathbb{R}$.
- *If* $\det(I + B_k) \neq 0$ *for all* $k \in \mathbb{Z}$*, then* $U(t,s)^{-1}$ *exists for all* $t, s \in \mathbb{R}$*, and for* $s < t$ *we define* $U(s,t) \equiv U(t,s)^{-1}$.
- $U(t_3, t_1) = U(t_3, t_2)U(t_2, t_1)$ *whenever* $t_1 \leq t_2 \leq t_3$*. If the above condition on* $\{B_k : k \in \mathbb{Z}\}$ *holds, the conclusion holds for any* $t_1, t_2, t_3 \in \mathbb{R}$.
- $U(t,s) = I + \int_s^t A(u)U(u,s)du + \sum_{s<t_k\leq t} B_k U(t_k, s)$ *for all* $t \geq s$.
- $U(t_k, s) = (I + B_k)U(t_k^-, s)$ *for all* $s \in \mathbb{R}$*,* $t_k > s$.

II.2.2 Variation-of-Constants Formula

The Cauchy matrix can be used to analytically express the unique solution of the inhomogeneous equation (II.2.1)–(II.2.2) satisfying a given initial condition.

Theorem II.2.2.1. *The unique solution* $t \mapsto x(t; s, x_0)$ *of* (II.2.1)–(II.2.2) *satisfying the initial condition* $x(t; t, x_0) = x_0$ *can be expressed in the form*

$$x(t; s, x_0) = U(t,s)x_0 + \int_s^t U(t,\mu)f(\mu)d\mu + \sum_{s<t_k\leq t} U(t, t_k)g_k. \qquad \text{(II.2.5)}$$

Proof. Under the assumption that $A(t)$ and $f(t)$ are merely continuous from the right with limits on the left, we cannot prove that (II.2.5) is a solution of (II.2.1)–(II.2.2) by computing a derivative because a priori, this function is not differentiable. We can, however, easily check that it satisfies the jump condition. At times t_k, we have

$$
\begin{aligned}
&x(t_k) - x(t_k^-) \\
&\quad = [U(t_k, s) - U(t_k^-, s)]x_0 + \int_s^{t_k} [U(t_k, \mu) - U(t_k^-, \mu)]f(\mu)d\mu \\
&\qquad + g_k + \sum_{s<t_j<t_k} [U(t_k, t_j) - U(t_k^-, t_j)]g_j \\
&\quad = B_k U(t_k^-, s)x_0 + \int_s^t B_k U(t_k^-, s)f(\mu)d\mu + g_k + \sum_{s<t_j<t_k} B_k U(t_k^-, t_j) \\
&\quad = B_k x(t_k^-) + g_k,
\end{aligned}
$$

as required. Next, we prove that on each interval (t_j, t_{j+1}) for $s < t_j$, the variation-of-constants formula (II.2.5) is correct. Without loss of generality, assume $s = t_0$. For $\mu \in (t_0, t_1)$, we have $U(t, \mu) = X(t, \mu)$. Then, with $x_0 = x(t_0)$ and $t \in (t_0, t_1)$,

$$
\begin{aligned}
&x_0 + \int_{t_0}^t (A(\mu)x(\mu) + f(\mu))d\mu \\
&\quad = x_0 + \int_{t_0}^t \left(A(\mu) \left[U(\mu, t_0)x_0 + \int_{t_0}^{\mu} U(\mu, v)f(v)dv \right] + f(\mu) \right) d\mu \\
&\quad = x_0 + \int_{t_0}^t A(\mu)X(\mu, t_0)d\mu x_0 + \int_{t_0}^t \int_{t_0}^{\mu} A(\mu)X(\mu, v)f(v)dvd\mu + \int_{t_0}^t f(\mu)d\mu \\
&\quad = x_0 + (X(t, t_0) - I)\, x_0 + \int_{t_0}^t \int_v^t A(\mu)X(\mu, v)d\mu f(v)dv + \int_{t_0}^t f(\mu)d\mu \\
&\quad = X(t, t_0)x_0 + \int_{t_0}^t (X(t, v) - I)f(v)dv + \int_{t_0}^t f(\mu)d\mu \\
&\quad = U(t, t_0)x_0 + \int_{t_0}^t U(t, v)f(v)dv = x(t),
\end{aligned}
$$

as required by definition of solution. By the previous computation, we have

$$
\begin{aligned}
x(t_1) &= (I + B_k)x(t_1^-) + g_1 \\
&= (I + B_1)X(t_1, t_0)x_0 + \int_{t_0}^{t_1} (I + B_1)X(t_1, \mu)f(\mu)d\mu + g_1 \\
&= U(t_1, t_0)x_0 + \int_{t_0}^{t_1} U(t_1, \mu)f(\mu)d\mu + U(t_1, t_1)g_1.
\end{aligned}
$$

Equation (II.2.5) therefore holds on $[t_0, t_1]$. Assuming now that the variation-of-constants formula is correct for $t \in [t_0, t_k]$ for some $k \geq 1$, the same proof can be used to show that for $t \in (t_k, t_{k+1})$,

$$x(t) = x(t_k) + \int_{t_k}^{t} X(t, \mu) f(\mu) d\mu.$$

Substituting in the expression for $x(t_k)$ guaranteed by the induction hypothesis, one obtains (II.2.5) for $t \in (t_k, t_{k+1})$. At $t = t_{k+1}$, one uses the relation $x(t_{k+1}) = (I + B_k)x(t_{k+1}^-) + g_{k+1}$, and the result is after some simplification equivalent to (II.2.5). □

II.2.3 Stability

We recall now the definition of (Lyapunov) stability.

Definition II.2.3.1. *The inhomogeneous system* (II.2.1)–(II.2.2) *is*

- exponentially stable *if there exist* $K > 0$, $\alpha > 0$ *and* $\delta > 0$ *such that for all* $\phi, \psi \in \mathbb{R}^n$ *satisfying* $||\phi - \psi|| < \delta$, *one has* $||x(t; s, \phi) - x(t; s, \psi)|| \leq K||\phi - \psi||e^{-\alpha(t-s)}$ *for all* $t \geq s$;
- stable *if for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $\phi, \psi \in \mathbb{R}^n$ *satisfying* $||\phi - \psi|| < \delta$, *one has* $||x(t; s, \phi) - x(t; s, \psi)|| < \epsilon$ *for all* $t \geq s$;
- unstable *if it is not stable.*

Lemma II.2.3.1. *The inhomogeneous system* (II.2.1)–(II.2.2) *is stable (respectively, exponentially stable or unstable) if and only if the same is true for the associated homogeneous system* (II.2.3)–(II.2.4).

Proof. One can easily verify from the variation-of-constants formula that $y(t) := x(t; s, \phi) - x(t; s, \psi)$ is a solution of the homogeneous system (II.2.3)–(II.2.4) satisfying $y(s) = \phi - \psi$. If the latter system is stable (respectively, exponentially stable), then in particular the difference between the trivial solution 0 and $y(t)$ can be bounded appropriately provided $||(\phi - \psi) - 0|| = ||\phi - \psi|| < \delta$ for some delta, which grants the stability assertion. The instability part follows similarly, as does the converse (that is, the stability of the inhomogeneous system implies the same for the homogeneous system). □

The above lemma states that, insofar as stability of linear systems is concerned, one needs to only consider homogeneous systems.

II.2.4 Exponential Trichotomy

Of use in later sections will be exponential trichotomy—referred to as spectral separation in Part I of this text.

Definition II.2.4.1. *The homogeneous system* (II.2.3)–(II.2.4) *has exponential trichotomy if there exist projection-valued functions* $t \mapsto P_c(t)$ *and* $t \mapsto P_u(t)$ *on* $\mathbb{R}^n$ *such that its Cauchy matrix* $U(t, s)$ *satisfies the following:*

1. $\sup_{t\in\mathbb{R}} ||P_c(t)|| + ||P_u(t)|| = N < \infty$.

2. $P_c(t)P_u(t) = P_u(t)P_c(t) = 0$.

3. $U(t,s)P_j(s) = P_j(t)U(t,s)$ *for all* $t \geq s$ *and* $j \in \{c,u\}$.

4. *Define* $U_j(t,s)$ *as the restriction of* $U(t,s)$ *to* $X_j(s) = \mathcal{R}(P_j(s))$ *for* $j \in \{c,u,s\}$, *where we set* $P_s = I - P_c - P_u$. *The linear maps* $U_j(t,s) : X_j(s) \to X_j(t)$ *are invertible for* $j \in \{c,u\}$, *and we denote* $U_j(s,t) = U_j(t,s)^{-1}$ *for* $t \geq s$.

5. *For all* $t,s,v \in \mathbb{R}$, $U_j(t,s) = U_j(t,v)U_j(v,s)$ *for* $j \in \{c,u\}$.

6. *There exist real numbers* $a < 0 < b$ *such that for all* $\epsilon > 0$, *there exists* $K \geq 1$ *such that*

$$||U_u(t,s)|| \leq Ke^{b(t-s)}, \qquad t \leq s \tag{II.2.6}$$

$$||U_c(t,s)|| \leq Ke^{\epsilon|t-s|}, \qquad t,s \in \mathbb{R} \tag{II.2.7}$$

$$||U_s(t,s)|| \leq Ke^{a(t-s)}, \qquad t \geq s. \tag{II.2.8}$$

Definition II.2.4.2. *Let* (II.2.3)–(II.2.4) *have exponential trichotomy. Define the sets* $X_j = \{(t,x) : t \in \mathbb{R}, x \in \mathcal{R}(P_j(t))\}$ *for* $j \in \{s,c,u\}$. X_s, X_c *and* X_u *are, respectively, the* stable, centre and unstable fibre bundles.

$$X_{cs} = \{(t,x+y) : x \in X_c(t),\ y \in X_s(t)\} = \{(t,x) : x \in \mathcal{R}(P_c(t) + P_s(t))\}$$
$$X_{cu} = \{(t,x+y) : x \in X_c(t),\ y \in X_u(t)\} = \{(t,x) : x \in \mathcal{R}(P_c(t) + P_u(t))\}$$

are, respectively, the centre-stable *and* centre-unstable *fibre bundles. For each of these, the* t-fibre *is the set* $X_j(t) = \{x : (t,x) \in X_j\}$.

The fibre bundles introduced in the above definition play the role of the invariant subspaces from autonomous ordinary differential equations. There are simpler descriptions of these objects available—in particular, one can define an equivalent time-invariant description—if $\det(I + B_k) \neq 0$ for all $k \in \mathbb{Z}$, since then the dynamics are reversible. Since we do not assume this, we will stick with the definition above.

II.2.5 Floquet Theory

The Floquet theory allows for the transformation of a periodically driven homogeneous system into an autonomous ordinary differential equation. This will be helpful later when we consider invariant manifold theory. In this section we begin with the homogeneous equation before proceeding to inhomogeneous equations. First, two definitions are as follows.

Definition II.2.5.1. *The inhomogeneous system* (II.2.1)–(II.2.2) *is* periodic *if there exist real* $T > 0$ *and* $c \in \mathbb{N}$ *such that* $A(t+T) = A(t)$, $f(t+T) = f(t)$, $B_{k+c} = B_k$, $g_{k+c} = g_k$ *and* $t_{k+c} = t_k + T$ *for all* $t \in \mathbb{R}$ *and* $k \in \mathbb{Z}$*. The* period *is* T*, and the* number of impulses per period *is* c*.*

Definition II.2.5.2. *The homogeneous system* (II.2.1)–(II.2.2) *is* periodic *if there exist real* $T > 0$ *and* $c \in \mathbb{N}$ *such that* $A(t+T) = A(t)$, $B_{k+c} = B_k$ *and* $t_{k+c} = t_k + T$ *for all* $t \in \mathbb{R}$ *and* $k \in \mathbb{Z}$*. The* period *is* T*, and the* number of impulses per period *is* c*.*

II.2.5.1 Homogeneous Systems

Definition II.2.5.3. *Suppose the homogeneous system* (II.2.3)–(II.2.4) *is periodic (with period* T*). Each of* $M_t := U(t+T, t)$ *for* $t \in \mathbb{R}$ *is called a* monodromy matrix*. The eigenvalues are called* Floquet multipliers.

Proposition II.2.5.1. *If the homogeneous system* (II.2.3)–(II.2.4) *is periodic (with period* T*), then* $U(t+T, s+T) = U(t, s)$ *for all* $t \geq s$*.*

Proof. This follows from existence and uniqueness of solutions together with periodicity (period T). □

Lemma II.2.5.1. *For any* $t, s \in \mathbb{R}$*,* M_t *and* M_s *have the same eigenvalues.*

Proof. First suppose $t \geq s$. The monodromy matrices satisfy the equation

$$M_t U(t, s) = U(t, s) M_s.$$

Suppose v eigenvalue of M_s with eigenvalue $\mu \neq 0$. Then,

$$M_t(U(t, s)v) = U(t, s)M_s v = U(t, s)\mu v = \mu(U(t, s)v),$$

so $U(t, s)v$ is an eigenvector of M_t with the same eigenvalue, provided $w := U(t, s)v \neq 0$. If $w = 0$, then $M_s v = U(s+T, t)U(t, s)v = U(s+T, t)w = 0$, which would contradict v being an eigenvector of M_s. On the other hand, 0 is an eigenvalue of M_s if and only if there is at least one $k \in \{0, \ldots, c-1\}$ such that $\det(I + B_k) = 0$, which is then equivalent to 0 being an eigenvalue of M_t. From Lemma II.2.5.1, we have

$$M_{s+T} = U(s+2T, s+T) = U(s+T, s) = M_s.$$

Therefore, M_s and M_{s+T} have the same eigenvalues. If $\sigma(M)$ denotes the set of eigenvalues of M, then the previous results imply the inclusions $\sigma(M_t) \subseteq \sigma(M_s)$ for $t \geq s$ and $\sigma(M_s) = \sigma(M_{s+jT})$ for all $j \geq 0$. Together, these imply $\sigma(M_s) = \sigma(M_t)$. □

As a consequence of the previous lemma, the following definition is reasonable.

Definition II.2.5.4. *The* Floquet multipliers of the linear system (II.2.3)–(II.2.4) *are the eigenvalues of the monodromy matrix M_0. The latter is given by*

$$M_0 = \prod_{k=c}^{1}(I + B_k)X(t_k, t_{k-1}). \tag{II.2.9}$$

Theorem II.2.5.1 (Floquet Decomposition)**.** *Let $\mathcal{B} := \{\xi_1, \dots, \xi_p\}$ be a union of canonical bases for the direct sum of generalized eigenspaces of M_0 with nonzero eigenvalues. The restriction of $U(t, 0)$ to span$\{\xi_1, \dots, \xi_p\}$ is invertible for $t \geq 0$. Let $\Phi(t)$ be the $n \times p$ matrix whose jth column is $\Phi_j(t) = U(t, 0)\xi_j$, defined for $t \in \mathbb{R}$. There exist a $p \times p$ complex matrix Λ and a T-periodic $n \times p$ complex matrix $Q(t)$ such that $\Phi(t) = Q(t)e^{t\Lambda}$ for all $t \geq 0$. The eigenvalues of Λ are*

$$\sigma(\Lambda) = \left\{\frac{1}{T}\log(\mu) : \mu \textit{ is a nonzero Floquet multiplier}\right\}.$$

Proof. Note, since $U(T, 0)$ is invertible on $\mathcal{B}$ (generalized eigenspaces of a matrix are invariant under its action), it suffices to prove first that $U(t, 0)$ is invertible for $t \in (0, T)$. Suppose not, then there exist $t \in (0, T)$ and ξ_j such that $U(t, 0)\xi_j = 0$. But this implies $U(T, 0)\xi_j = 0$, and since $M_0 = U(T, 0)$, we conclude ξ_j is an eigenvector with eigenvalue zero. As the generalized eigenspaces are disjoint, we have obtained a contradiction.

Since $M_0 = U(T, 0)$ is invertible on span$\{\xi_1, \dots, \xi_p\}$, there exists an invertible $p \times p$ matrix V such that $\Phi(T) = \Phi(0)V$. Define $\Lambda = \frac{1}{T}\log V$, where the logarithm is any branch that defined on the spectrum of V. Define $Q(t) = \Phi(t)e^{-t\Lambda}$. By definition, we have $\Phi(t) = Q(t)e^{t\Lambda}$. For periodicity, we observe

$$\begin{aligned} Q(t + T) &= \Phi(t + T)e^{-(t+T)\Lambda} = U(t + T, T)\Phi(T)V^{-1}e^{-t\Lambda} \\ &= U(t, 0)\Phi(0)VV^{-1}e^{-t\Lambda} = \Phi(t)e^{-t\Lambda} = Q(t), \end{aligned}$$

as required. The last thing to prove is the characterization of the spectrum of Λ. Let $\nu \in \mathcal{B}$ be a generalized eigenvector of rank m with eigenvector μ for M_0. Since $\mathcal{B}$ is a union of canonical bases, there is a Jordan chain $\{\nu_1, \dots, \nu_m\} \subseteq \mathcal{B}$ such that $\nu_j = (M_0 - \mu I)\nu_{j+1}$ and $\nu_m = \nu$. Relative to the basis $\mathcal{B}$, we can write $\nu_j = \xi_{r_j}$ for some new index r_j so that the previous equation becomes $\xi_{r_j} = (M_0 - \mu I)\xi_{r_{j+1}}$. The right-hand side can be written as

$$\begin{aligned} (M_0 - \mu I)\xi_{r_{j+1}} &= U(T, 0)\xi_{r_{j+1}} - \mu\xi_{r_{j+1}} = \Phi(T)e_{r_{j+1}} - \mu\xi_{r_{j+1}} \\ &= \Phi(0)Ve_{j+1} - \mu\xi_{r_{j+1}}. \end{aligned}$$

Since $\Phi(0)$ has linearly independent columns, the left-inverse $\Phi^+(0)$ exists. Then, since $\xi_j = \Phi(0)e_j$, multiplying $\Phi^+(0)$ on the left on both sides of $\xi_{r_j} = (M_0 - \mu I)\xi_{r_{j+1}}$, it follows that

$$e_{r_j} = (V - \mu I)e_{r_{j+1}}.$$

We conclude that $e_{r_m}, \ldots, e_{r_1}$ is a Jordan chain for eigenvalue μ of V. Since $\Lambda = \frac{1}{T}\log(V)$, the result follows. □

Remark II.2.5.1. *One can replace $\mathcal{B}$ with any basis for the direct sum of generalized eigenspaces of M_0 with nonzero eigenvalues. Indeed, the only place we used the previous description of $\mathcal{B}$ was in determining the spectrum of Λ. If one writes $\tilde{\Phi}(0) = \tilde{\Phi}(0)Z$ for invertible Z, where the columns of $\tilde{\Phi}(0)$ are a canonical basis of Jordan chains, and defines $\tilde{\Phi}(t) = U(t,0)\tilde{\Phi}(0)$, then one can apply the theorem directly to $\tilde{\Phi}(t) = \tilde{Q}(t)e^{t\tilde{\Lambda}}$, where $\tilde{\Phi}(T) = \tilde{\Phi}(0)\tilde{V}$ and $\tilde{\Lambda} = \frac{1}{T}\log\tilde{V}$. However, if $\Phi(T) = \Phi(0)V$, then $\tilde{\Phi}(T) = \tilde{\Phi}(0)ZVZ^{-1}$, so $\tilde{V}$ and V are similar and thus have the same eigenvalues. The same therefore holds for $\tilde{\Lambda}$ and $\Lambda = \frac{1}{T}\log V$.*

Corollary II.2.5.1. *With the notation of Theorem II.2.5.1, introduce a family of subspaces $\mathcal{X}_t$ of $\mathbb{R}^n$ indexed by $t \in \mathbb{R}$ as follows:*

$$\mathcal{X}_t = \{U(t,0)x : x \in \operatorname{span}(\mathcal{B})\}.$$

The nonautonomous dynamical system $U(t,s) : \mathcal{X}_s \to \mathcal{X}_t$ is equivalent to the ordinary differential equation

$$\dot{y} = \Lambda y \tag{II.2.10}$$

under the time-periodic change of variables $x(t) = Q(t)y(t)$. More precisely, any solution $x : \mathbb{R} \to \mathbb{R}^n$ of (II.2.3)–(II.2.4) such that $x(t) \in \mathcal{X}_t$ for all $t \in \mathbb{R}$ can be written in the form $x(t) = Q(t)y(t)$, where y is a solution of (II.2.10).

Proof. Let x be a solution of (II.2.3)–(II.2.4) such that $x(t) \in \mathcal{X}_t$. Write $x(0) = \Phi(0)h$ for some $h \in \mathbb{R}^p$. By uniqueness of solutions, $x(t) = U(t,0)\Phi(0)h = \Phi(t)h$, so by Theorem II.2.5.1, we can write $x(t) = Q(t)e^{t\Lambda}h$. With $y = e^{t\Lambda}h$, the claim is proven. □

Corollary II.2.5.2. *If $p = n$—that is, $\mathcal{B}$ is a basis for $\mathbb{R}^n$—the time-periodic change of coordinates $x = Q(t)y$ transforms the ordinary impulsive differential equation (II.2.3)–(II.2.4) into the autonomous ordinary differential equation (II.2.10). In this case, $|Q(t)|$ and $|Q^{-1}(t)|$ are both bounded. $\mathcal{B}$ is a basis for $\mathbb{R}^n$ if and only if $\det(I + B_k) \neq 0$ for $k = 0, \ldots, c-1$.*

Proof. The first part follows by Corollary II.2.5.1. As for the second part, since $\mathcal{B}$ is a basis for the direct sum of generalized eigenspaces of M_0 with nonzero eigenvalue, the assertion that $\mathcal{B}$ is a basis for $\mathbb{R}^n$ is equivalent to 0 not being an eigenvalue of M_0. Since $M_0 = \prod_{k=c}^{1}(I + B_k)X(t_k, t_{k-1})$ and each of $X(t_k, t_{k-1})$ has full rank, zero can only be an eigenvalue of $\det(I + B_k) = 0$ for at least one $k \in \{1, \ldots, c\}$. Since $B_{k+c} = B_k$, this proves the claim. □

Corollary II.2.5.2 is an analogue of the Floquet theorem from ordinary differential equations. It appears in the 1993 monograph of Bainov and Simeonov [9]. Theorem II.2.5.1 is the generalization of the Floquet decomposition to the case where the jump maps $x \mapsto x + B_k x$ to not be one-to-one, while Corollary II.2.5.1 gives the change of variables on the "non-singular fibre bundle" $\mathcal{X}_t$ that renders the dynamics autonomous.

In essentially the same way Theorem II.2.5.1 is proven, we can establish a more general version that roughly states that for a homogeneous linear periodic system, the dynamics on any of its invariant fibre bundles—except for the "singular" portion of the stable fibre bundle or centre-stable fibre bundle—are driven by an autonomous ordinary differential equation. The proof is omitted. First, a quick definition is as follows.

Definition II.2.5.5. *Let $X_f(t)$ be one of the following:*

- *one of the centre, unstable or centre-unstable fibre bundles;*
- *the* reversible stable fibre bundle*, $X_s^\infty(t) = \{\xi \in X_s(t) : \forall t' > t,\ U(t', t)\ \xi \neq 0\}$;*
- *the* reversible centre-stable fibre bundle*, $X_{cs}^\infty(t) = X_c(t) \oplus X_s^\infty(t)$.*

Let $\{\xi_1, \ldots, \xi_p\}$ be a basis for $X_f(0)$. The matrix-valued function $\Phi_f(t) = U(t, 0)[\xi_1, \cdots, \xi_p]$ is a basis matrix for X_f*.*

Theorem II.2.5.2. *Let X_f be one of the following fibre bundles:*

- *one of the centre, unstable or centre-unstable fibre bundles;*
- *the reversible stable fibre bundle or reversible centre-stable fibre bundle.*

The restriction of $U(t, 0)$ to any basis for $X_f(0)$ is invertible for $t \geq 0$, so any basis matrix $\Phi_f(t)$ for X_f can be uniquely extended to the entire real line. There exist a $p \times p$ complex matrix Λ_f and a T-periodic $n \times p$ complex matrix $Q_f(t)$ such that $\Phi(t) = Q_f(t)e^{t\Lambda_f}$ for all $t \geq 0$. The eigenvalues of Λ_f are

$$\sigma(\Lambda) = \left\{ \frac{1}{T} \log(\mu) : M_0\xi = \mu\xi,\ \xi \in X_f(0) \right\}.$$

As defined in the above theorem, $X_s^\infty(t)$ is spanned by the generalized eigenvectors of M_t having with Floquet multipliers μ satisfying $0 < |\mu| < 1$. As a consequence, if $\det(I + B_k) \neq 0$ for $k = 0, \ldots, c-1$, then $X_s^\infty = X_s$.

II.2.5.2 Periodic Solutions of Homogeneous Systems

The Floquet multipliers allow us to identify periodic solutions of the homogeneous system.

Proposition II.2.5.2. *The homogeneous system (II.2.3)–(II.2.4) has a non-trivial jT-periodic solution for $j \in \mathbb{N}$ if and only if there exists a Floquet multiplier μ satisfying $\mu^j = 1$. In this case, the jT-periodic solutions are precisely $x(t) = U(t, 0)\xi$, where $\xi \in \mathbb{R}^n$ satisfies $M_0^j\xi = \xi$.*

Proof. $x(t)$ is a nontrivial periodic solution of period jT if and only if $x(jT) = x(0) \neq 0$, which is equivalent to the equation $U(jT,0)x(0) = x(0)$. Since $U(jT,0) = M_0^j$, this implies that $x(0)$ satisfies the equation $M_0^j x(0) = x(0)$, so that 1 is an eigenvalue of M_0^j. Since the eigenvalues of M_0^j are the jth powers of the eigenvalues of M_0, there must be a Floquet multiplier μ satisfying $\mu^j = 1$. □

II.2.5.3 Periodic solutions of Inhomogeneous Systems

From the variation-of-constants formula, a periodic solution $x(t)$ is uniquely determined by its value at time $t = 0$. Indeed, starting from the variation-of-constants formula (II.2.5), setting $s = 0$ and assuming the periodic ansatz $x(0) = x(T)$, the necessary and sufficient condition for the existence of a periodic solution is that there exists a solution x_0 of the equation

$$(U(T,0) - I)x_0 = \int_0^T U(T,\mu)f(\mu)d\mu + \sum_{0<t_k\leq T} U(T,t_k)g_k. \qquad \text{(II.2.11)}$$

The matrix on the left-hand side will be invertible precisely if 1 is not an eigenvalue of $U(T,0)$. Since the latter is precisely the monodromy matrix M_0, we obtain the following lemma.

Lemma II.2.5.2. *The inhomogeneous equation* (II.2.1)–(II.2.2) *has a unique T-periodic solution if and only if* 1 *is not a Floquet multiplier of the associated homogeneous equation; that is,* $\det(M_0 - I) \neq 0$.

More generally, one might want to know under what conditions there is a jT-periodic solution for natural number j. The ansatz $x(jT) = 0$ leads to the equation

$$(U(jT,0) - I)x_0 = \int_0^{jT} U(jT,\mu)f(\mu)d\mu + \sum_{0<t_k\leq jT} U(jT,t_k)g_k.$$

Since $U(jT,0) = U(T,0)^j = M_0^j$, the previous lemma has the following simple generalization.

Theorem II.2.5.3. *The inhomogeneous equation* (II.2.1)–(II.2.2) *has a unique jT-periodic solution if and only if no Floquet multiplier μ of the associated homogeneous equation is a jth root of unity; that is, $\mu^j \neq 1$ for all $\mu \in \sigma(M_0)$.*

If $1 \in \sigma(M_0)$, there will be either infinitely many periodic solutions or none, depending on whether the right-hand side of (II.2.11) is in the range of $M_0 - I$. Similar conclusions hold for jT-periodic solutions. Existence of periodic solutions in the critical case where $\det(M_0 - I) = 0$ is discussed in Bainov and Simeonov [9], and we refer the interested reader to this resource.

II.2.5.4 Periodic Systems Are Exponentially Trichotomous

The invariant fibre bundles of a periodic system induce an exponential trichotomy.

Theorem II.2.5.4. *The periodic system* (II.2.3)–(II.2.4) *has exponential trichotomy. The projectors* P_c*,* P_u *and* $P_s = I - (P_c + P_u)$ *are projections onto the centre, unstable and stable fibre bundles* X_c*,* X_u *and* X_s*, respectively. These projectors are also periodic with period* T*.*

Proof Outline. Define $P_j(t)$ by the integral

$$P_j(t) = \frac{1}{2\pi i}\int_{\Gamma_j}(zI - M_t)^{-1}dz,$$

where Γ_j is a simple closed contour in $\mathbb{C}$ such that the only eigenvalues μ of M_0 contained in its closure are, respectively, those with $|\mu| > 0$ for $j = c$, $|\mu| = 1$ for $j = u$ and $|\mu| < 1$ for $j = s$, oriented counterclockwise relative to its interior. One can show that with this choice of projections, all properties of exponential trichotomy are satisfied. The proof is quite long; see Theorem I.3.1.3 for details. □

Sometimes it is desirable to have an explicit formula for one of the projections $P(t)$ onto an invariant fibre bundle. When there are $c = 1$ impulses per period, we have a fairly nice formula. Let $M_0 = VJV^{-1}$ be the Jordan canonical form of the monodromy matrix M_0, and let $X(t,s)$ be the Cauchy matrix of the continuous part $\dot{x} = A(t)x$. Let $t \in [t_0, t_0 + T)$. Then, we have

$$\begin{aligned} M_t &= X(t+T, t_0+T)[I + B_1]X(t_0+T, t) = X(t,t_0)[I+B]X(t_0+T,t_0)X^{-1}(t,t_0) \\ &= X(t,t_0)M_0X^{-1}(t,t_0) = V(t)JV^{-1}(t), \end{aligned}$$

where we set $V(t) = X(t,t_0)V$. The projection $P_j(t)$ can then be equivalently written in the form

$$P_j(t) = V(t)\left[\frac{1}{2\pi i}\int_{\Gamma_j}(zI - J)^{-1}dz\right]V^{-1}(t), \tag{II.2.12}$$

where the contour Γ_j is as stated in Theorem II.2.5.4. The contour integral in (II.2.12) is easy to evaluate because J is a Jordan matrix and the integrand no longer depends on t. Since P_j is periodic, it is enough to compute it for $t \in [t_0, t_0 + T)$.

II.2.5.5 Stability

The following theorem characterizes the stability of the homogeneous system in terms of the Floquet multipliers. It follows directly from the associated infinite-dimensional version, Theorem I.3.3.1, from Part I of this text. The proof is identical apart from symbolic changes and slight changes to presentation, so it will be omitted.

Theorem II.2.5.5. *The homogeneous system* (II.2.3)–(II.2.4) *is exponentially stable if and only if all Floquet multipliers* μ *satisfy* $|\mu| < 1$. *It is stable if and only if all Floquet multipliers satisfy* $|\mu| \leq 1$, *and to those Floquet multipliers satisfying* $|\mu| = 1$, *the generalized eigenspaces contain only rank 1 eigenvectors—equivalently, each block in the complex Jordan form of* M_0 *corresponding to one of the Floquet multipliers satisfying* $|\mu| = 1$ *is one-dimensional.*

Stability for periodic linear systems is therefore completely determined by the Floquet multipliers—that is, the eigenvalues μ of M_0. These satisfy the *characteristic equation*

$$\det(M_0 - \mu I) = 0. \tag{II.2.13}$$

Recall that M_0 is given explicitly by (II.2.9). There is, however, another way to compute the Floquet multipliers. The following proposition is a direct consequence of Theorem II.2.5.1.

Proposition II.2.5.3. *If* $M_0\xi = \mu\xi$ *and* $\mu \neq 0$, *the function* $x(t) = U(t,0)\xi$ *can be written in the form* $x(t) = q(t)e^{\lambda t}$, *where* $\lambda = \frac{1}{T}\log\mu$ *and* q *is (generally) complex-valued and* T*-periodic. Conversely, if* $x(t) = q(t)e^{\lambda t}$ *is a solution of* (II.2.3)–(II.2.4) *with* q *a complex-valued* T*-periodic function, then* $\mu = e^{T\lambda}$ *is a Floquet multiplier and* $M_0 q(0) = \mu q(0)$.

Let us substitute the ansatz $x(t) = q(t)e^{\lambda t}$ into (II.2.3)–(II.2.4). After some cancellation, one arrives at the following impulsive differential equation for q:

$$\lambda q + \dot{q} = A(t)q, \qquad t \neq t_k \tag{II.2.14}$$

$$\Delta q = B_k q, \qquad t = t_k. \tag{II.2.15}$$

Let X_λ denote the Cauchy matrix of the continuous part of Eq. (II.2.14). That is, $X_\lambda(t,s)$ satisfies $X_\lambda(t,t) = I$ for $t \in \mathbb{R}$ and

$$\frac{d}{dt}X_\lambda(t,s) = (A(t) - \lambda I)X_\lambda(t,s).$$

By Proposition II.2.5.2, system (II.2.14)–(II.2.15) has a T-periodic solution if and only if

$$\det\left(\prod_{k=c}^{1}(I + B_k)X_\lambda(t_k, t_{k-1}) - I\right) = 0. \tag{II.2.16}$$

Notice that the product term is precisely the monodromy matrix M_0 for (II.2.14)–(II.2.15). If one can compute all solution λ of the equation (II.2.16), then one can compute the Floquet multipliers $\mu = e^{T\lambda}$. The numbers λ have a special name.

Definition II.2.5.6. *The complex numbers λ that solve (II.2.16) are the* Floquet exponents*. The set of all Floquet exponents is denoted $\lambda(U)$ and is called the* Floquet spectrum*.*

Equation (II.2.16) will have infinitely many solutions because $\lambda = \frac{1}{T}\log\mu$ and the logarithm has infinitely many branches. Namely, if λ is a Floquet exponent, then so is $\lambda + \frac{2\pi i}{T}$. As such, when solving (II.2.16), one should focus only on solutions in the strip

$$\left\{\lambda \in \mathbb{C} : \Im(\lambda) \in \left[0, \frac{2\pi}{T}\right)\right\}.$$

II.2.6 Generalized Periodic Changes of Variables

The changes of variables we introduced in Sect. II.2.5 transform some or all components of a periodic impulsive system into an autonomous ordinary differential equation. The downside is that the resulting ordinary differential equation might be complex-valued. In this section we consider other periodic changes of variables that will be useful in later applications.

II.2.6.1 A Full State Transformation and Chain Matrices

Corollary II.2.5.1 grants a transformation that very nearly renders the dynamics of (II.2.3)–(II.2.4) autonomous. The barrier is the *singular fibre bundle*, X_0, whose t-fibres are given by

$$X_0(t) = \{\xi \in X_s(t) : \exists t' > t : U(t', t)\xi = 0\}.$$

Denote $P_0(t)$ the projection onto $X_0(t)$. To any solution $x : \mathbb{R} \to \mathbb{R}^n$ such that $P_0(s)x(s) \neq 0$, there necessarily exist some $t' > s$ such that $P_0(t)x(t) = 0$ for all $t \geq t'$. This suggests we form a basis matrix of $X_0(t)$ not in the way that is done in Theorem II.2.5.2, but rather in a piecewise fashion.

Definition II.2.6.1. *Let $\Psi_0, \dots, \Psi_{c-1}$ denote matrices whose columns are bases for the t_j-fibres $X(t_j)$ of a fibre bundle X. Define for $k \in \{0, \dots, c-1\}$ and $t \in [t_k, t_{k+1})$ the matrix $Q(t) = U(t, t_k)\Psi_k$. Then, extend Q to $Q : \mathbb{R} \to \mathbb{R}^{n\times q}$ by periodicity, where $q = \dim X_0(0)$. We will call Q a* chain matrix *for X.*

We can now apply Theorem II.2.5.2 and pose a transformation that maps x into its components in each of X_s^∞, X_c, X_u and X_0. These components will be decoupled, and the dynamics for all components aside from X_0 will be autonomous.

Theorem II.2.6.1. *Let Q_0 be a chain matrix for X_0. Let Φ_s, Φ_c and Φ_u be basis matrices for X_s^∞, X_c and X_u, respectively. The change of variables $x = Q_s y + Q_c z + Q_u w + Q_0 q$ is invertible and transforms the homogeneous impulsive system (II.2.3)–(II.2.4) into the decoupled system*

$$\dot{y} = \Lambda_s y, \tag{II.2.17}$$

$$\dot{z} = \Lambda_c z, \tag{II.2.18}$$

$$\dot{w} = \Lambda_u w, \tag{II.2.19}$$

$$\dot{q} = 0, \qquad t \neq t_k \tag{II.2.20}$$

$$\Delta q = Q_0^+(t_k)[(I + B_k)Q_0(t_k^-) - \Delta Q_0(t_k)]q, \qquad t = t_k, \tag{II.2.21}$$

where $\Phi_j = Q_j e^{t\Lambda_j}$ are the respective Floquet decompositions, and for a matrix M with independent columns, the symbol M^+ denotes its left-inverse. The transformation and its inverse are uniformly bounded.

Proof. Since the columns of $Q_s(t)$, $Q_c(t)$, $Q_u(t)$ and $Q_0(t)$ are bases $X_s^\infty(t)$, $X_c(t)$, $X_u(t)$ and $X_0(t)$, respectively, and these subspaces have trivial intersection, the transformation is invertible. Substituting $x = Q_s y + Q_c z + Q_u w + Q_0 q$ into (II.2.3), we find

$$\begin{aligned} &A(Q_s y + Q_c z + Q_u w + Q_0 q) \\ &\quad = (AQ_s - Q_s\Lambda_s)y + Q_s\dot{y} + (AQ_c - Q_c\Lambda_c)z + Q_c\dot{z} \\ &\qquad + (AQ_u - Q_u\Lambda_u)w + Q_u\dot{w} + AQ_0 q + Q_0\dot{q}. \end{aligned}$$

After cancelling several terms, we get

$$0 = Q_s(\dot{y} - \Lambda_s y) + Q_c(\dot{z} - \Lambda_c z) + Q_u(\dot{w} - \Lambda_u w) + Q_0\dot{q}.$$

This implies the first four equations, (II.2.17)–(II.2.20). It is easy to check that $Q_j(t_k) = [I + B_k]Q_j(t_k^-)$ for $j = c, s, u$. Substituting $x = Q_s y + Q_c z + Q_u w + Q_0 q$ into (II.2.4), this implies

$$(I + B_k)Q_0(t_k^-)q(t_k^-) = Q_0(t_k)q(t_k) - Q_0(t_k^-)q(t_k^-).$$

Denoting $\Delta q = \Delta q(t_k)$, $q = q(t_k^-)$, $\Delta Q_0 = \Delta Q_0(t_k)$ and $Q_0^- = Q_0(t_k^-)$, we can expand the above as

$$(I + B_k)Q_0^- q = Q_0^-(q + \Delta q) + \Delta Q_0(q^- + \Delta q) - Q_0^- q.$$

Cancelling $Q_0^- q$ on either side, this is equivalent to

$$(I + B_k)Q_0(t_k^-)q(t_k^-) = Q_0(t_k)\Delta q + \Delta Q_0(t_k)q(t_k^-).$$

Rearranging and multiplying by $Q_0^+(t_k)$ on both sides give (II.2.21). The boundedness of the transformation and its inverse is clear from the periodicity of each of Q_s, Q_c, Q_u, together with the observation that Q_0 is periodic and the left-limits $Q_0(t_k^-)$ are full column rank. □

II.2.6.2 Real Floquet Decompositions

In some applications, the utility of the Floquet decomposition is less than the transformation of a periodic system to an autonomous one, but rather in the decoupling of the stable, centre and unstable parts. This is where the emphasis is placed in Theorem II.2.6.1. However, sometimes we also want the resulting dynamics of the transformed equation to be real. The following provides a sufficient condition for all matrices in the statement of Theorem II.2.6.1 to be real, or for there to exist real matrices such that the statement holds. The proof is a consequence of the existence of a real logarithm of a real matrix [39] and is omitted.

Proposition II.2.6.1. *Let $M_0 = VJV^{-1}$ denote the real Jordan canonical form of M_0. There exist real basis matrices Φ_s, Φ_c and Φ_u for X_s^∞, X_c and X_u, respectively, with real Floquet decompositions $\Phi_j(t) = Q_j(t)e^{t\Lambda_j}$ for $j \in \{s, c, u\}$, where Q_j are real and T-periodic and Λ_j are real, if and only if each Jordan block of J belonging to a negative real eigenvalue occurs an even number of times.*

Corollary II.2.6.1. *Let $\Phi(t)$ be a real basis matrix for one of X_s^∞, X_c or X_u. Let D be the unique non-singular (real) matrix such that $\Phi(T) = \Phi(0)D$, and let $D = VJV^{-1}$ be its real Jordan canonical form. There exists a real Floquet decomposition—that is, $Q(t)$ real and T-periodic and Λ real such that $\Phi(t) = Q(t)e^{t\Lambda}$—if and only if any Jordan block of J belonging to a negative real eigenvalue occurs an even number of times.*

Corollary II.2.6.2. *There exist real basis matrices Φ_s, Φ_c and Φ_u for X_s^∞, X_c and X_u, respectively, with real Floquet decompositions $\Phi_j(t) = Q_j(t)e^{t\Lambda_j}$ for $j \in \{s, c, u\}$, where Q_j are real and $2T$-periodic and Λ_j are real.*

Proof. Let $\Phi(t) \in \mathbb{R}^{m\times m}$ be a basis matrix for one of X_s^∞, X_c or X_u. Then,

$$\Phi(2T) = M_0(M_0\Phi(0)) = M_0(\Phi(0)D) = \Phi(0)D^2$$

for some invertible $D \in \mathbb{R}^{m\times m}$. Defining $\Lambda = \frac{1}{2T}\log(D)$, since D has no negative real eigenvalues, Λ is real. Then, $Q(t) := \Phi(t)e^{-t\Lambda}$ is $2T$-periodic. □

It is clear from the above proposition and corollary that, for example, the best real Floquet decomposition one can hope to obtain for the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x, && t \neq kT \\ \Delta x &= \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} x, && t = kT \end{aligned}$$

is one that is $2T$-periodic. For some applications, this is not enough, for example, in the analysis of period-doubling bifurcations, it is preferable to maintain the original period. We can accomplish this by way of chain matrices.

II.2.6.3 A Real T-Periodic Kinematic Similarity

We can use a system of chain matrices to transform any T-periodic impulsive system into a T-periodic impulsive system with a block structure. The proof is analogous to that of Theorem II.2.6.1 and is omitted.

Corollary II.2.6.3. *Let Q_0, Q_s, Q_c and Q_u be chain matrices for X_0, X_s^∞, X_c and X_u, respectively. Define $Q(t) = [\ Q_0(t) \quad Q_s(t) \quad Q_c(t) \quad Q_u(t)\ ]$. This matrix is bounded and periodic with a bounded inverse. The change of variables $x = Q(t)y$ transforms (II.2.3)–(II.2.4) into the piecewise-constant system*

$$\begin{aligned} \dot{y} &= 0, && t \neq t_k \\ \Delta y &= Q^{-1}(t_k)[(I + B_k)Q(t_k^-) - \Delta Q(t_k)]y, && t = t_k. \end{aligned}$$

One can also define a transformation $x = Q(t)y$, where $Q(t)$ is some combination of chain matrices and Floquet periodic matrices (i.e. coming from a Floquet decomposition), and the result will be some combination of the systems from Theorem II.2.6.1 and Corollary II.2.6.3. In some cases, it will be preferable to use the standard Floquet periodic matrices, and other times, it will be better to use chain matrices. Regardless, we have the following general corollary.

Corollary II.2.6.4. *There exists a real, T-periodic, linear change of variables $x = Q(t)y$ with $||Q(t)||$ and $||Q^{-1}(t)||$ uniformly bounded, such that (II.2.3)–(II.2.4) are transformed into a system of the form*

$$\begin{aligned} \dot{y} &= \Lambda_s y, \quad t \neq t_k & \Delta y &= \Omega_s y, \quad t = t_k \\ \dot{z} &= \Lambda_c w, \quad t \neq t_k & \Delta z &= \Omega_c z, \quad t = t_k \\ \dot{w} &= \Lambda_u z, \quad t \neq t_k & \Delta w &= \Omega_u w, \quad t = t_k, \end{aligned}$$

with real matrices Λ_j and Ω_j, $j \in \{s, c, u\}$. Let M_0 denote the monodromy matrix of (II.2.3)–(II.2.4), and write its spectrum (set of eigenvalues) as $\sigma(M_0) = \sigma_s \cup \sigma_c \cup \sigma_u$, with $|\sigma_s| < 1$, $|\sigma_c| = 1$ and $|\sigma_u| > 1$. Let $M_{0,y}$, $M_{0,z}$ and $M_{0,w}$ denote the monodromy matrices of the y, z and w subsystems. Then,

$$\sigma(M_{0,y}) = \sigma_s, \quad \sigma(M_{0,z}) = \sigma_c, \quad \sigma(M_{0,w}) = \sigma_u.$$

In the above corollary, we used $|S| < 1$ as a shorthand for the sentence *all elements of S have absolute value less than one.* The symbols $|S| = 1$ and $|S| > 1$ are interpreted analogously.

II.2.7 Comments

The Floquet theory for impulsive differential equations is fully described in the monograph of Bainov and Simeonov [9], being partially developed in 1982 by Samoilenko and Perestyuk [124], although therein the assumption that matrices $I + B_k$ are invertible is assumed. We have intentionally dispensed with this requirement since it makes the theory far more flexible. The content of Sects. II.2.5.2 and II.2.5.3 appears in [9], as does Theorem II.2.5.5.

Chapter II.3

Stability for Nonlinear Systems

In this chapter we will discuss some methods of proving stability for nonlinear systems. There are parallels here to Chap. I.4 from part I, but we will not go into as much detail. Here we will be interested in the general nonlinear system

$$\dot{x} = f(t, x), \qquad t \neq t_k \tag{II.3.1}$$
$$\Delta x = g_k(x), \qquad t = t_k, \tag{II.3.2}$$

under PC^1 regularity assumptions.

II.3.1 Stability

Before proceeding, we must of course define stability for the nonlinear system (II.3.1)–(II.3.2). Let $t \mapsto x(t; s, x_0)$ denote the solution of the above system satisfying the initial condition $x(s; s, x_0) = x_0$.

Definition II.3.1.1. *Suppose $f(t, 0) = g_k(0) = 0$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. The trivial solution* (II.3.1)–(II.3.2) *is*

- stable *if for all $\epsilon > 0$ and $s \in \mathbb{R}$, there exists $\delta = \delta(\epsilon, s) > 0$ such that if $||\phi|| < \delta$, then $||x(t; s, \phi)|| < \epsilon$ for all $t \geq s$;*
- uniformly stable *if it is stable and δ can be chosen independent of s;*
- attracting *if for all $s \in \mathbb{R}$ there exists $\delta = \delta(s) > 0$ such that if $||\phi|| < \delta$, then $||x(t; s, \phi)|| \to 0$ as $t \to \infty$;*

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- uniformly attracting *if it is attracting and δ can be chosen independent of s;*
- asymptotically stable *if it stable and attracting;*
- uniformly asymptotically stable *if it is uniformly stable and uniformly attracting;*
- exponentially stable *if there exists δ, α and $K > 0$ such that* $||x(t; s, \phi)|| \leq Ke^{-\alpha(t-s)}$ *for all $t \geq s$, whenever $||\phi|| < \delta$.*

Remark II.3.1.1. *To define stability of a nontrivial solution $\gamma : \mathbb{R} \to \mathbb{R}^n$, one may translate it into zero by a time-varying change of coordinates and consider the stability of the trivial solution in the new systems.*

II.3.2 The Linear Variational Equation and Linearized Stability

Definition II.3.2.1. *Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a solution of* (II.3.1)–(II.3.2)*. The* linear variational equation about γ *or the* linearization at γ *is the linear system*

$$\dot{z} = Df(t, \gamma(t))z, \qquad t \neq t_k \tag{II.3.3}$$

$$\Delta z = Dg_k(\gamma(t_k^-))z, \qquad t = t_k. \tag{II.3.4}$$

Definition II.3.2.2. *The solution γ has* exponential trichotomy *if the linearization at γ has exponential trichotomy.*

Definition II.3.2.3. *The solution γ is* hyperbolic *if it has exponential trichotomy and the centre fibre bundle is trivial. It is* nonhyperbolic *if the centre fibre bundle is nontrivial.*

The following proposition is a direct consequence of Theorem II.2.5.4.

Proposition II.3.2.1. *If the variational equation about γ is periodic (in the sense of Definition II.2.5.2), then γ has exponential trichotomy.*

The linearization contains some information about how solutions near γ evolve with time. It can be derived in several ways, but the most heuristic one is to assume that $x(t) = \gamma(t) + z(t)$ is some perturbed solution of γ. Substituting this ansatz into (II.3.1)–(II.3.2) results in

$$\dot{z} + f(t, \gamma(t)) = f(t, z + \gamma(t)), \qquad t \neq t_k$$

$$\Delta z + g_k(\gamma(t_k^-)) = g_k(z + \gamma(t_k^-)), \qquad t = t_k.$$

Expanding $f(t, \cdot)$ and $g_k(\cdot)$ in Taylor series in z at $\gamma(t)$ and $\gamma(t_k^-)$, respectively, and neglecting terms of order higher than linear, the result is an impulsive differential equation for z, given by (II.3.3)–(II.3.4). That this equation actually grants stability information is harder to prove. The main result is the following theorem, whose proof we omit. The interested reader can gain some insight from the analogous Theorem I.7.7.1.

The following is a linearized stability theorem stated relative to a given solution $\gamma : \mathbb{R} \to \mathbb{R}^n$. Its proof is a consequence of the more general Theorem I.7.7.1.

Theorem II.3.2.1. *Suppose* (II.3.1)–(II.3.2) *is* PC^1 *and, additionally, the following conditions hold.*

- $x \mapsto [f(t, \gamma(t) + x) - f(t, \gamma(t)) - Df(t, \gamma(t))x]$ *is uniformly (in* t*) Lipschitz continuous in the ball* $B_\delta(0) \subset \mathbb{R}^n$ *with Lipschitz constant* $c_1(\delta)$.
- $x \mapsto [g_k(\gamma(t_k^-) + x) - g_k(\gamma(t_k^-)) - Dg_k(\gamma(t_k^-))x]$ *is uniformly (in* k*) Lipschitz continuous in ball* $B_\delta(0) \subset \mathbb{R}^n$ *with Lipschitz constant* $c_2(\delta)$.
- *There exists* $\xi > 0$ *such that* $t_{k+1} - t_k \geq \xi$ *for all* $k \in \mathbb{Z}$.

If γ *has exponential trichotomy and the variational equation about* γ *is unstable, then* γ *is unstable. If the linear variational equation about* γ *is exponentially stable, then* γ *is exponentially stable provided the constants* $c_1(\delta)$ *and* $c_2(\delta)$ *satisfy* $c_1, c_2 \to 0$ *as* $\delta \to 0$.

Corollary II.3.2.1. *Let* $\gamma : \mathbb{R} \to \mathbb{R}^n$ *be a bounded solution of* (II.3.1)–(II.3.2). *Suppose* $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ *is* C^2 *on* $[t_k, t_{k+1}) \times \Omega$ *for all* $k \in \mathbb{Z}$ *and* $g_k : \Omega \to \mathbb{R}^n$ *is* C^2, *where* Ω *is some open set containing the image of* γ. *Suppose* $x \mapsto D_2 f(t, x)$ *and* $x \mapsto Dg_k(x)$ *are uniformly (in* t *and* k*) locally Lipschitz continuous.*

- *If the linear variational equation about* γ *is exponentially stable, then* γ *is exponentially stable.*
- *If* γ *has exponential trichotomy and the variational equation about* γ *is unstable, then* γ *is unstable.*

Proof. The assumption and the C^2 assumption imply that $Df(t, \cdot)$ and $g_k(\cdot)$ are uniformly (in t and k) Lipschitz continuous on any compact set containing γ. To see that the Lipschitz constant vanishes as this compact set shrinks uniformly to γ, we use the mean-value theorem on $\tilde{f}(t, z) = f(t, \gamma(t) + z) - f(t, \gamma(t)) - Df(t, \gamma(t))z$ to write

$$
\begin{aligned}
&||\tilde{f}(t, \phi) - \tilde{f}(t, \psi)|| \\
&\quad = \left|\left| \int_0^1 \tilde{D}_2 f(t, \phi + h(\psi - \phi))(\psi - \phi) \right|\right| \\
&\quad \leq \int_0^1 ||D_2 f(t, \gamma(t) + \phi + h(\psi - \phi)) - D_2 f(t, \gamma(t))|| \cdot ||\phi - \psi|| dh \\
&\quad \leq L 2\delta ||\phi - \psi||,
\end{aligned}
$$

for $\phi, \psi \in B_\delta(0)$ for $\delta < \delta'$, where L is a uniform (in t) Lipschitz constant for $x \mapsto D_2 f(t, x)$ on the closure of K,

$$K = \bigcup_{t \in \mathbb{R}} \{x \in \mathbb{R}^n : ||x - \gamma(t)|| \leq 2\delta'\},$$

and $\delta' > 0$ is some fixed constant. The Lipschitz constant on $\tilde{f}$ is $L2\delta$, which vanishes as $\delta \to 0$. A similar Lipschitz constant can be derived for $\tilde{g}_k(z) = g_k(\gamma(t_k^-) + z) - g_k(\gamma(t_k^-)) - Dg(\gamma(t_k^-))z$. The conclusion then follows from Theorem II.3.2.1. □

Corollary II.3.2.2. *Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a periodic solution of* (II.3.1)–(II.3.2) *with period T. Suppose $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is C^2 on $[t_k, t_{k+1}) \times \Omega$ for all $k \in \mathbb{Z}$, and $g_k : \Omega \to \mathbb{R}^n$ is C^2, where Ω is some open set containing the image of γ. Additionally, assume $f(t + T, \cdot) = f(t, \cdot)$, $g_{k+c} = g_k$ and $t_{k+c} = t_k + T$ for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$.*

- *If the linear variational equation about γ is exponentially stable, then γ is exponentially stable.*
- *If the linear variational equation about γ is unstable, then γ is unstable.*

Proof. The periodicity assumptions imply the Lipschitzian estimates on $D_2 f$ and Dg_k required by Corollary II.3.2.1. Proposition II.3.2.1 grants the required exponential trichotomy of γ. □

II.3.3 Comments

The exponential stability part of Corollary II.3.2.2 on periodic solutions and periodic systems is originally due to Simeonov and Bainov [131], with an instability result under exponential dichotomy appearing in the same authors' 1993 monograph [9].

Chapter II.4

Invariant Manifold Theory

This chapter will be devoted to the invariant manifold theory of impulsive differential equations. At the theoretical level, we will assume only that the reference bounded solution has exponential trichotomy, but when we move into computational aspects we will assume that the dynamics are periodic. This will allow us to take advantage of the Floquet decomposition, with the result being that computation of invariant manifolds has much in common with the same procedure for ordinary differential equations without impulses.

In this chapter we will assume a semilinear decomposition

$$\dot{x} = A(t)x + f(t,x), \qquad t \neq t_k \tag{II.4.1}$$

$$\Delta x = B_k x + g_k(x), \qquad t = t_k, \tag{II.4.2}$$

where $f(t,0) = g_k(0) = 0$ and $D_2 f(t,0) = Dg_k(0) = 0$.

Definition II.4.0.1. *System* (II.4.1)–(II.4.2) *is* periodic *with period* $T > 0$ *and* c *impulses per period if* $A(t+T) = A(t)$, $f(t+T,\cdot) = f(t,\cdot)$, $B_{k+c} = B_k$, $g_{k+c} = g_k$ *and* $t_{k+c} = t_k + T$ *for all* $t \in \mathbb{R}$ *and* $k \in \mathbb{Z}$.

II.4.1 Existence and Smoothness

Definition II.4.1.1. *An* invariant manifold *for the trivial solution* $x = 0$ *is a subset* $W \subset \mathbb{R} \times \mathbb{R}^n$ *with the following properties:*

- $\mathbb{R} \times \{0\} \subset W$;
- *the sets* $W_t := \{x : (t,x) \in W\}$ *are submanifolds of* $\mathbb{R}^n$;
- *if* $x_s \in W_s$, *then* $x(t;s,x) \in W_t$ *as long as this solution is defined.*

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An invariant manifold is C^k if W_t is C^k for each t.

We will at this point drop the phrase "for the trivial solution $x = 0$", since we will always be referring to invariant manifolds at this solution. To define invariant manifolds at other solutions γ, one can simply perform a change of variables to translate γ to zero and get a system of the form (II.4.1)–(II.4.2).

Definition II.4.1.2. *Suppose the trivial solution $x = 0$ has exponential trichotomy. An invariant manifold W is a*

- stable manifold *if W_t is tangent to $X_s(t)$ at $0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$, and solutions $x(t) \in W_t$ converge exponentially to zero as $t \to \infty$;*
- centre manifold *if W_t is tangent to $X_c(t)$ at $0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$, and solutions $x(t) \in W_t$ have sub-exponential growth as $t \to \pm\infty$;*
- unstable manifold *if W_t is tangent to $X_u(t)$ at $0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$, and solutions $x(t) \in W_t$ converge exponentially to zero as $t \to -\infty$;*
- centre-stable manifold *if W_t is tangent to $X_{cs}(t)$ at $0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$, and solutions $x(t) \in W_t$ have sub-exponential growth as $t \to \infty$;*
- centre-unstable manifold *if W_t is tangent to $X_{cu}(t)$ at $0 \in \mathbb{R}^n$ for all $t \in \mathbb{R}$, and solutions $x(t) \in W_t$ have sub-exponential growth as $t \to -\infty$.*

Definition II.4.1.3. *Suppose $x = 0$ has exponential trichotomy. Let $P(t)$ denote the projection onto one of the stable, centre, unstable, centre-stable or centre-unstable fibre bundles associated to the linear part,*

$$\dot{x} = A(t)x, \qquad t \neq t_k \tag{II.4.3}$$

$$\Delta x = B_k x, \qquad t = t_k, \tag{II.4.4}$$

of (II.4.1)–(II.4.2). A local stable, centre, unstable, centre-stable or centre-unstable manifold *is a set of the form*

$$W^{loc} = \{(t, x + h(t, x)) : t \in \mathbb{R},\ x \in B_\delta(0) \cap \mathcal{R}(P(t)) \subset \mathbb{R}^n\},$$

for some $h : \mathbb{R} \times B_\delta(0) \to \mathbb{R}^n$, satisfying $h(t, 0) = 0$, $P(t)h(t, u) = 0$, with W^{loc} having the following properties:

- *If $(s, x_s) \in W^{loc}$, there exists $\epsilon > 0$ such that $(t, x(t; s, x_s)) \in W^{loc}$ for $|t - s| < \epsilon$.*
- *W_t^{loc} is tangent to $\mathcal{R}(P(t))$ at $0 \in \mathbb{R}^n$.*
- *Any solution that remains in W^{loc} for the asymptotic time ranges specified in Definition II.4.1.2 satisfies the same asymptotic growth or decay rates.*

A local invariant manifold is $PC^{1,m}$-regular at zero *if*

- $z \mapsto h(t, z)$ *is* C^m *in a neighbourhood of* $0 \in \mathbb{R}^p$*;*
- *for* $j = 0, \dots, m$ *and all* $z_1, \dots, z_j \in \mathbb{R}^p$*,* $t \mapsto D_2^j h(t, 0)[z_1, \dots, z_j]$ *is continuous except at times* t_k *where it has limits on the left and, additionally, it is differentiable from the right everywhere.*

The $PC^{1,m}$-regular condition in addition to the tangency property implies that the function $h : \mathbb{R} \times B_\delta(0) \to \mathbb{R}^n$ has a Taylor expansion near $x = 0$ of the form

$$h(t, z) = \frac{1}{2!} h_2(t) z^2 + \frac{1}{3!} h_3(t) z^3 + \cdots + \frac{1}{m!} h_m(t) z^m + O(|z|^{m+1}), \quad \text{(II.4.5)}$$

for t fixed, and that the coefficients are differentiable from the right with discontinuities at impulse times t_k, where they have limits on the left.

Proving the existence of local invariant manifolds and their $PC^{1,m}$ regularity is formally equivalent to all of the work done in Chaps. I.5, I.6, and I.7 and is in fact implied by the relevant theorems therein. Indeed, taking the delay range $r = 0$ directly recovers the case of impulsive differential equations. As such, the following theorem need not be proven.

Theorem II.4.1.1. *Suppose the trivial solution* $x = 0$ *has exponential trichotomy. There exist local stable, centre, unstable, centre-stable and centre-unstable manifolds. These manifolds are* $PC^{1,m}$ *regular provided* (II.4.1)–(II.4.2) *if* PC^m*. The Taylor coefficients* $h_j(t)$ *in* (II.4.5) *are bounded, and the asymptotic form of that equation holds uniformly for* $t \in \mathbb{R}$ *provided* (II.4.1)–(II.4.2) *if* PC^{m+1}*.*

II.4.2 Invariance Equation for Nonautonomous Systems

The dynamics on any invariant manifold can be characterized by the abstract results in Sect. I.7.6. However, the situation here is a fair bit simpler because the projection matrices $P_j(t)$ onto the stable, centre and unstable fibre bundles are much more regular than the associated operators in the infinite-dimensional case.

This section will be devoted to the derivation of the *invariance equation* associated to a given local invariant manifold. Throughout, $P(t)$ will denote a projection onto one of the stable, centre, unstable, centre-stable or centre-unstable fibre bundles. The invariant manifold in question will be assumed to be $PC^{1,m}$-regular at zero and is represented in the form

$$W^{loc} = \{(t, x + h(t, x)) : t \in \mathbb{R},\ x \in B_\delta(0) \cap \mathcal{R}(P(t)) \subset \mathbb{R}^n\} \quad \text{(II.4.6)}$$

for $h : \mathbb{R} \times B_\delta(0) \to \mathbb{R}^n$ and satisfying $h(t, 0) = 0$ and $P(t)h(t, x) = 0$.

The following lemma characterizes the regularity of the projector $P(t)$. It will be needed in the subsequent sections.

Lemma II.4.2.1. *Let $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a matrix-valued function satisfying $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$. Then, P satisfies the matrix impulsive differential equation*

$$\dot{P}(t) = A(t)P(t) - P(t)A(t), \qquad t \neq t_k, \tag{II.4.7}$$

$$\Delta P(t_k) = B_k P(t_k-) - P(t_k)B_k, \qquad t = t_k. \tag{II.4.8}$$

More succinctly, at times $t = t_k$ we have $P(t_k)[I + B_k] = [I + B_k]P(t_k^-)$.

Proof. For ease of presentation, we will assume $A(t)$ is continuous on $[t_k, t_{k+1})$, so that $t \mapsto U(t,s)$ will be differentiable on each of $[t_k, t_{k+1})$, but the result remains true (in the sense of integrated solutions) under weaker PC^0 conditions. Let $t \in (t_k, t_{k+1})$. Then, we can write $U(t,t_k) = X(t,t_k)$ for X the Cauchy matrix of the continuous part, $\dot{x} = A(t)x$. This matrix is invertible, from which it follows that

$$P(t) = U(t,t_k)P(t_k)U^{-1}(t,t_k).$$

The right-hand side is differentiable, from which it follows that $P'(t)$ exists, with

$$\begin{aligned}\dot{P}(t) &= A(t)U(t,t_k)P(t_k)U(t,t_k)^{-1} + U(t,t_k)P(t_k)[-U(t,t_k)^{-1}A(t)]\\ &= A(t)P(t) - P(t)A(t),\end{aligned}$$

as claimed. As for the impulse times, since $U(t_k, t_k^-) = I + B_k$, the definition of P implies $P(t_k)[I + B_k] = [I + B_k]P(t_k^-)$. Rearranging gives (II.4.8). □

Suppose $x(t)$ is a solution on the invariant manifold. Then, at each time t we can write $u(t) = z + h(t,z)$ for some $z \in \mathcal{R}(P(t))$. Substituting this ansatz into the impulsive differential equation (II.4.1)–(II.4.2), we get

$$\begin{aligned}A(t)(z+h) + f(t,z+h) &= \partial_t h + [I + \partial_z h]\dot{z}, && t \neq t_k\\ B_k(z+h) + g_k(z+h) &= \Delta_t h(t, z+\Delta x) + \left[I + \int_0^1 \partial_z h(t_k^-, z + s\Delta z)ds\right]\Delta z, && t = t_k.\end{aligned}$$

Since $P(t)h(t,u) = 0$, we get $z = P(t)u(t)$. Applying Lemma II.4.2.1, one can check that

$$\dot{z} = A(t)z + P(t)f(t,z+h), \qquad t \neq t_k \tag{II.4.9}$$

$$\Delta z = B_k z + P(t_k)g_k(z+h), \qquad t = t_k. \tag{II.4.10}$$

Combining these two results, we arrive at the *invariance equation* for the invariant manifold:

$$A(t)h + (I - P(t))f(t, z + h) = \partial_z h[Az + Pf(t, z + h)] + \partial_t h, \qquad t \neq t_k, \tag{II.4.11}$$

$$B_k h + (I - P(t_k))g_k(z + h) = \left[\int_0^1 \partial_z h(t_k^-, z + sr_k)ds\right] r_k + \Delta_t h(t_k, z + r_k), \quad t = t_k, \tag{II.4.12}$$

where in the above $r_k = r_k(z, h) := B_k z + P(t_k)g_k(z+h)$, all unspecified time evaluations are at $t = t_k^-$ in (II.4.12), and we define $\Delta_t h(t_k, y) = h(t_k, y) - h(t_k^-, y)$.

The pair of Eqs. (II.4.11)–(II.4.12) defines an impulsive partial differential equation satisfied by the function $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ defining the invariant manifold.

II.4.3 Invariance Equation for Systems with Periodic Linear Part

When the linear part (II.4.3)–(II.4.4) is periodic, we can take advantage of Floquet theory to simplify the form of the invariance equation (II.4.11)–(II.4.12). More generally, we will assume a kinematic similarity as introduced in Sect. II.2.6.3. Let

$$x = Q_s(t)y_s + Q_c(t)y_c + Q_u(t)y_u, \tag{II.4.13}$$

be a real T-periodic change of variables of the form introduced in Corollary II.2.6.4. Each Q_j could be a chain matrix, a Floquet periodic matrix or some combination thereof, and the optimal choice will depend on the situation at hand. After completing the change of variables, (II.4.1)–(II.4.2) become

$$\dot{y}_s = \Lambda_s y_s + \tilde{f}_s(t, y_s, y_c, y_u), \qquad t \neq t_k \tag{II.4.14}$$

$$\dot{y}_c = \Lambda_c y_c + \tilde{f}_c(t, y_s, y_c, y_u), \qquad t \neq t_k \tag{II.4.15}$$

$$\dot{y}_u = \Lambda_u y_u + \tilde{f}_u(t, y_s, y_c, y_u), \qquad t \neq t_k \tag{II.4.16}$$

$$\Delta y_s = \Omega_s y_s + \tilde{g}_s(k, y_s, y_c, y_u), \qquad t = t_k \tag{II.4.17}$$

$$\Delta y_c = \Omega_c y_c + \tilde{g}_c(k, y_s, y_c, y_u), \qquad t = t_k \tag{II.4.18}$$

$$\Delta y_u = \Omega_u y_u + \tilde{g}_u(k, y_s, y_c, y_u), \qquad t = t_k, \tag{II.4.19}$$

with the nonlinearities

$$\tilde{f}_j(t, y, z, w) = Q_j^+(t)f(t, Q_s(t)y + Q_c(t)z + Q_u(t)w),$$
$$\tilde{g}_j(k, y, z, w) = Q_j^+(t_k)g_k(Q_s(t_k^-)y + Q_c(t_k^-)z + Q_u(t_k^-)w).$$

Recall that for a matrix M with linearly independent columns, the symbol M^+ denotes its left-inverse. The dynamics have been decoupled into stable (y_s), centre (y_c) and unstable (y_u) directions.

Denote σ_s, σ_c and σ_u the sets of Floquet multipliers of the decoupled linear parts, so that $\sigma(M_0) = \sigma_s \cup \sigma_c \cup \sigma_u$, with M_0 being the monodromy matrix of the original linear part (II.4.3)–(II.4.3). To derive the invariance equation for the invariant manifold $\mathcal{W}_f$, we will partition (II.4.14)–(II.4.19) as

$$\dot{y} = Uy + \tilde{f}_1(t, y, z), \qquad t \neq t_k, \tag{II.4.20}$$
$$\dot{z} = Vz + \tilde{f}_2(t, y, z), \qquad t \neq t_k, \tag{II.4.21}$$
$$\Delta y = R_k y + \tilde{g}_1(k, y, z), \qquad t = t_k, \tag{II.4.22}$$
$$\Delta z = S_k z + \tilde{g}_2(k, y, z), \qquad t = t_k, \tag{II.4.23}$$

where the linear part of the y equations has only the Floquet exponents σ_f, and the linear part of the z equations has only the Floquet exponents $\sigma(M_0) \setminus \sigma_f$. This partitioning is always attainable. The nonlinearities $\tilde{f}_i$ and $\tilde{g}_i$ will be some vectors involving those of (II.4.14)–(II.4.19).

In the (y, z) coordinates, the t-fibre $\mathcal{W}_f(t)$ of the invariant manifold is the solution set of the equation

$$z = \tilde{h}(t, y), \tag{II.4.24}$$

with $\tilde{h} : \mathbb{R} \times \mathbb{R}^{\dim X_f} \to \mathbb{R}^{n - \dim X_f}$ defined explicitly in terms of the function h in (II.4.6) by

$$\tilde{Q}(t)\tilde{h}(t, y) = h(t, Q_f(t)y),$$

where $\tilde{Q}(t)$ is the matrix $Q = \begin{bmatrix} Q_s & Q_c & Q_u \end{bmatrix}$ without the Q_f part. For example, if $Q_f = Q_c$, then $\tilde{Q} = \begin{bmatrix} Q_s & Q_u \end{bmatrix}$. These details are unimportant, since we can work directly with (II.4.24). Also, from this point on we will drop the tildes in (II.4.24) and simply write $z = h(t, y)$.

To derive the invariance equation, we substitute (II.4.24) into (II.4.20)–(II.4.23). Working first with the differential equation (II.4.21), we get

$$Vh(t, y) + \tilde{f}_2(t, y, h(t, y)) = \partial_t h(t, y) + \partial_y h(t, y)\dot{y}. \tag{II.4.25}$$

The next step would be to substitute (II.4.20) into (II.4.25) and replace all instances of z with $h(t, y)$. As for the jumps, substituting $z = h(t, y)$ into (II.4.23) gives the equation

$$S_k h(t_k^-, y) + \tilde{g}_2(k, y, h(t_k^-, y)) = h(t_k, y + \Delta y) - h(t_k^-, y).$$

We can write the right-hand side equivalently as

$$h(t_k, y + \Delta y) - h(t_k^-, y) = \Delta_t h(t_k, y + \Delta y) + \int_0^1 \partial_y h(t_k^-, y + s\Delta y)\Delta y ds,$$

where $\Delta_t h(t_k, v) = h(t_k, v) - h(t_k^-, v)$. Every instance of Δy can now be replaced with (II.4.22), and all appearances of z therein are replaced with $h(t_k^-, y)$. This entire discussion then leads to the complete invariance equation.

Theorem II.4.3.1. *The invariant manifold $\mathcal{W}_f$ in the (y, z) coordinates of system (II.4.20)–(II.4.23) can be expressed as the solution set of $z = h(t, y)$, where the function $h : \mathbb{R} \times \mathbb{R}^{\dim X_f} \to \mathbb{R}^{n-\dim X_f}$ is periodic in its first variable and satisfies the impulsive partial differential equation*

$$Vh + \tilde{f}_2(t, y, h) = \partial_t h + (\partial_y h)[Uy + \tilde{f}_1(t, y, h)], \qquad t \neq t_k \tag{II.4.26}$$

$$S_k h + \tilde{g}_2(k, y, h) = \Delta_t h(t_k, y + r_k) + \int_0^1 \partial_y h(t_k^-, y + sr_k) r_k ds, \quad t = t_k, \tag{II.4.27}$$

where $h = h(t, y)$ in the first equation, $h = h(t_k^-, y)$ in the second equation (unless otherwise specified), $\Delta_t h(t_k, v) = h(t_k, v) - h(t_k^-, v)$, and $r_k = r_k(y, h) := R_k y + \tilde{g}_1(k, y, h(t_k^-, y))$.

II.4.4 Dynamics on Invariant Manifolds

In the most general (nonautonomous) setting, the dynamics on a given invariant manifold can be derived from (II.4.9)–(II.4.10). Set $z(t) = \Phi(t)w(t)$ for $\Phi(t)$ a basis matrix for $\mathcal{R}(P(t))$ and some $w(t) \in \mathbb{R}^{\dim X_c}$. Then, the function w satisfies the impulsive differential equation

$$\begin{aligned} \dot{w} &= \Phi^+(t)P(t)f(t, \Phi(t)w + h(t, \Phi(t_k^-)w)), && t \neq t_k \\ \Delta w &= \Phi^+(t_k)P(t_k)g_k(\Phi(t_k^-)w + h(t_k^-, \Phi(t_k^-)w)), && t = t_k. \end{aligned}$$

The above system essentially describes the nonlinear part of the dynamics on the centre manifold. Indeed, the transformation $z = \Phi(t)w$ quotients out the linear part. However, this transformation is not generally uniformly bounded, so it is difficult to compare growth rates of solutions of the above equation with those on the invariant manifold.

The drawbacks described in the previous paragraph are remedied if the linear part (II.4.3)–(II.4.4) is periodic. In this case, the dynamics on the invariant manifold are topologically equivalent near the origin to

$$\dot{y} = Uy + \tilde{f}_1(t, y, h(t, y)), \qquad t \neq t_k \tag{II.4.28}$$

$$\Delta y = R_k y + \tilde{g}_1(k, y, h(t_k^-, y)), \qquad t = t_k. \tag{II.4.29}$$

Solutions of (II.4.28)–(II.4.29) near the origin are in one-to-one correspondence with those on the invariant manifold. For more information on notions of topological equivalence for impulsive systems, we refer the reader to [28] and the references cited therein.

II.4.5 Reduction Principle for the Centre Manifold

The centre manifold (at zero) contains several important classes of solutions, namely:

- all sufficiently small bounded solutions;
- all sufficiently small periodic solutions.

As a consequence, any small solution or attractor that is formed at a bifurcation point must necessarily be contained within the (parameter-dependent) centre manifold. The following theorem provides more detail.

Theorem II.4.5.1. *Suppose X_u is trivial. There exists a neighbourhood V of $0 \in \mathbb{R}^n$ such that any solution $x : [s, \infty) \to \mathbb{R}^n$ for which $x(t) \in V$ for $t \geq s$ converges exponentially towards $\mathcal{W}_c$. That is, there exists a solution $u(t) \in \mathcal{W}_c(t)$ such that $||x(t) - u(t)|| \leq K_1 e^{-\alpha_1(t-s)}$ for some positive constants K_1 and α_1.*

II.4.6 Approximation by Taylor Expansion

We have discussed a few ways to represent invariant manifolds in this chapter. In the periodic case, we can always express $\mathcal{W}_f$ as the (time-varying) graph of a function $h : \mathbb{R} \times \mathbb{R}^{\dim X_f} \to \mathbb{R}^{n-\dim X_f}$, where $t \mapsto h(t, v)$ is periodic and the Taylor expansion

$$h(t, v) = \frac{1}{2}h_2(t)v^2 + \cdots + \frac{1}{m!}h_m(t)v^m + O(||v||^{m+1})$$

holds uniformly in t near $v = 0$. Each of the coefficients h_j is periodic and differentiable from the right everywhere, with discontinuities only at the impulse times. The idea is to substitute the above Taylor expansion ansatz into the invariance equation (whichever is appropriate to the given situation) and compare powers of v, starting at degree two and proceeding higher until the desired expansion is computed. Since the Taylor coefficients of the invariant manifold are unique, this process yields a unique solution at each order of the expansion. Rather than develop this procedure abstractly, we will consider an example.

Example II.4.6.1. *Consider the following two-dimensional impulsive differential equation:*

$$\begin{aligned} \dot{u} &= -u + v^2, & t &\notin \mathbb{Z} & \qquad \Delta u &= 0.5u^3, & t &\in \mathbb{Z} \\ \dot{v} &= v - w^2, & t &\notin \mathbb{Z} & \qquad \Delta v &= -v, & t &\in \mathbb{Z} \\ \dot{w} &= \alpha uw, & t &\notin \mathbb{Z} & \qquad \Delta w &= 0, & t &\in \mathbb{Z}, \end{aligned}$$

where $\alpha \in \mathbb{R}$ is a parameter. We will determine the invariance equation for the centre manifold and obtain its Taylor approximation. The first thing to do is to transform the above system into the form (II.4.20)–(II.4.23). This is very nearly complete; the w component corresponds to the centre component for all values of α, while (u, v) corresponds to "leftover" components, z. However, the dynamics are not as simple as they could be, since the $z = (u, v)$ dynamics involve a singular stable direction (the v component) but the continuous-time portion in this direction is nonzero. To fix this, we can use a chain matrix for X_0. This is easily computed: $Q_0(t) = e^{[t]_1}$.

If we set $z = (u, Q_0(t)v)$ and $y = w$, then the above system becomes

$$\begin{aligned} \dot{y} &= \alpha y z_1, && t \notin \mathbb{Z} \\ \dot{z} &= \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} e^{2[t]_1} z_2^2 \\ -e^{-[t]_1} y^2 \end{bmatrix}, && t \notin \mathbb{Z} \\ \Delta y &= 0, && t \in \mathbb{Z} \\ \Delta z &= \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} z + \begin{bmatrix} 0.5 z_1^3 \\ 0 \end{bmatrix}, && t \in \mathbb{Z}. \end{aligned}$$

Compare to (II.4.14)–(II.4.19) for details. The centre manifold can be represented in the form

$$z_1 = h_1(t, y), \quad z_2 = h_2(t, y)$$

for a pair h_1, h_2 of scalar-valued functions that are periodic in their first variable. Writing $h = [\ h^1 \quad h^2\ ]^\intercal$, the invariance equation is

$$\begin{aligned} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} h + \begin{bmatrix} e^{2[t]_1} (\pi_2 h)^2 \\ -e^{-[t]_1} y^2 \end{bmatrix} &= \partial_t h + (\partial_y h) \alpha y \pi_1 h, && t \notin \mathbb{Z} \\ \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} h + \begin{bmatrix} 0.5 (\pi_1 h)^3 \\ 0 \end{bmatrix} &= \Delta_t h(t_k, y), && t \in \mathbb{Z}, \end{aligned} \tag{II.4.30}$$

where $\pi_1 h = h_1$ and $\pi_2 h = h_2$. Note: the function r_k from Theorem II.4.3.1 is identically zero, hence why the partial derivative $\partial_y h$ does not appear in the jump condition of the invariance equation.

Let us compute the fourth-order approximation of the centre manifold. We write

$$h(t, y) = \begin{bmatrix} h_{1,2}(t) y^2 + h_{1,3}(t) y^3 + h_{1,4}(t) y^4 \\ h_{2,2}(t) y^2 + h_{2,3}(t) y^3 + h_{2,4}(t) y^4 \end{bmatrix} + O(|y|^5)$$

for periodic functions $h_{i,j}$ of period one. Substituting the above into (II.4.30) and comparing y^2 coefficients, we get

$$\begin{bmatrix} -h_{1,2} \\ -e^{-[t]_1} \end{bmatrix} = \partial_t \begin{bmatrix} h_{1,2} \\ h_{2,2} \end{bmatrix}, \quad t \notin \mathbb{Z} \qquad \begin{bmatrix} 0 \\ -h_{2,2} \end{bmatrix} = \Delta_t \begin{bmatrix} h_{1,2} \\ h_{2,2} \end{bmatrix}, \quad t \in \mathbb{Z}.$$

The unique periodic solution is $h_{1,2}(t) = 0$ and $h_{2,2}(t) = e^{-[t]_1} - 1$. We can now update our expression for h

$$h(t, y) = \begin{bmatrix} h_{1,3}(t) y^3 + h_{1,4}(t) y^4 \\ (e^{-[t]_1} - 1) y^2 + h_{2,3}(t) y^3 + h_{2,4}(t) y^4 \end{bmatrix} + O(|y|^5).$$

Substituting this into (II.4.30) *and equating cubic terms* y^3*, the result is*

$$\begin{bmatrix} -h_{1,3} \\ 0 \end{bmatrix} = \partial_t \begin{bmatrix} h_{1,3} \\ h_{2,3} \end{bmatrix}, \quad t \notin \mathbb{Z} \qquad \begin{bmatrix} 0 \\ -h_{2,3} \end{bmatrix} = \Delta_t \begin{bmatrix} h_{1,3} \\ h_{2,3} \end{bmatrix}, \quad t \in \mathbb{Z}.$$

The unique periodic solution is the trivial solution $h_{1,3} = h_{2,3} = 0$*. Updating our expression for* h *yet again,*

$$h(t,y) = \begin{bmatrix} h_{1,4}(t)y^4 \\ (e^{-[t]_1} - 1)y^2 + h_{2,4}(t)y^4 \end{bmatrix} + O(|y|^5).$$

Finally, substituting into (II.4.30) *and equating coefficients on* y^4 *terms, the result is*

$$\begin{bmatrix} -h_{1,4} + e^{2[t]_1}(e^{-[t]_1} - 1)^2 \\ 0 \end{bmatrix} = \partial_t \begin{bmatrix} h_{1,4} \\ h_{2,4} \end{bmatrix}, \quad t \notin \mathbb{Z}$$
$$\begin{bmatrix} 0 \\ -h_{2,4} \end{bmatrix} = \Delta_t \begin{bmatrix} h_{1,4} \\ h_{2,4} \end{bmatrix}, \quad t \in \mathbb{Z}.$$

There is a nontrivial periodic solution: $h_{2,4} = 0$ *and*

$$h_{1,4}(t) = \frac{e^{-[t]_1 - 1}}{3(1 - e^{-1})}(e-1)^3 + e^{-[t]_1}\int_0^{[t]_1} e^s(e^{2s} - 2e^s + 1)ds. \qquad \text{(II.4.31)}$$

Note that $h_{1,4} > 0$*. The latter can be identified with the unique periodic solution of*

$$\dot{q} = -q + e^{2[t]_1}(e^{-[t]_1} - 1)^2.$$

To fourth order, the function h *representing the centre manifold is given by*

$$h(t,y) = \begin{bmatrix} h_{1,4}(t)y^4 \\ (e^{-[t]_1} - 1)y^2 \end{bmatrix} + O(|y|^5),$$

where $h_{1,4}$ *is the positive function from* (II.4.31)*. The dynamics on the centre manifold are topologically conjugate to those of* $\dot{y} = \alpha y h_1(t,y)$*. Substituting in the above expression for* h*, we get*

$$\dot{y} = \alpha h_{1,4}(t)y^5 + O(\alpha|y|^6).$$

Since $h_{1,4}$ *is positive, we conclude that the zero solution of the original impulsive system is* unstable *if* $\alpha > 0$*,* stable *if* $\alpha = 0$*, and* asymptotically stable *if* $\alpha < 0$*. These last two assertions follow by the reduction principle, Theorem II.4.5.1. See Fig. II.4.1 for a comparison.*

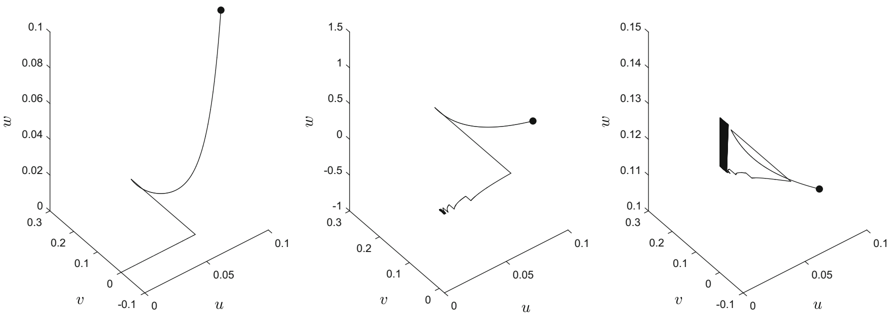

Figure II.4.1: Left to right: simulation from the initial condition $(u, v, w) = (0.1, 0.1, 0.1)$ at time $t = 0$ of the system from Example II.4.6.1 for parameters $\alpha = -100$, $\alpha = 0$ and $\alpha = 2$, respectively. The convergence rate for the case $\alpha < 0$ is incredibly small and numerically unstable, hence our decision to choose a large $\alpha = -100$. Time integration for the cases $\alpha < 0$ and $\alpha = 0$ was done for $t \in [0, 1000]$, and in the $\alpha > 0$ case for $t \in [0, 200]$. In all figures, the black dot denotes the initial condition

II.4.7 Parameter Dependence

In this section we will discuss how one can incorporate parameter-dependent systems into the invariant manifold framework. Suppose we have a system of the form

$$\begin{aligned} \dot{x} &= f(t, x, \epsilon), & t &\neq t_k(\epsilon), \\ \Delta x &= g_k(x, \epsilon), & t &= t_k(\epsilon), \end{aligned}$$

for a parameter $\epsilon \in \mathbb{R}^p$. We assume this system is periodic with period $T(\epsilon)$ with $q > 0$ impulses per period. Importantly, we assume the number of impulses per period *does not* change depending on the parameter. Suppose that $f(t, 0, 0) = g_k(0, 0) = 0$, so that 0 is an equilibrium point when $\epsilon = 0$. We will assume without loss of generality that $t_k(\epsilon) = 0$.

The first thing we will do is to perform a parameter-dependent rescaling of time so that the impulses occur on the integers. Specifically, set

$$t = t(\tau, \epsilon) = \left\{ \; t_k(\epsilon) + (\tau - k)(t_{k+1}(\epsilon) - t_k(\epsilon)), \quad \tau \in [k, k+1), \; k \in \mathbb{Z}. \right.$$

for rescaled time τ. Under this rescaling, $t = t_k(\epsilon)$ if and only if $\tau = k$. Moreover, $\tau \mapsto t$ is continuous, piecewise-linear and monotone increasing, so it has an inverse with the same properties. If we define $y(\tau) = x(t(\tau, \epsilon))$, then y satisfies the impulsive differential equation

$$\begin{aligned} \frac{dy}{d\tau} &= f(t(\tau, \epsilon), y, \epsilon)(t_{k+1}(\epsilon) - t_k(\epsilon)), & k < \tau < k+1 \\ \Delta y &= g_k(y, \epsilon), & \tau = k \in \mathbb{Z}. \end{aligned}$$

The above system is now periodic with period q, and q impulses per period. Moreover, it has the same level of regularity of the original system—if the original system is PC^m, so is the above. As such, we can always assume without loss of generality that a parameter-dependent system is in the form

$$\dot{x} = f(t, x, \epsilon), \qquad t \notin \mathbb{Z} \tag{II.4.32}$$

$$\Delta x = g_k(x, \epsilon), \qquad t \in \mathbb{Z}, \tag{II.4.33}$$

where the period is $q > 0$ and there are q impulses per period.

Next, we expand the state space of Eqs. (II.4.32)–(II.4.33) by taking ϵ as an additional state. The result is the system

$$\frac{d}{dt}\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} f(t,x,\epsilon) \\ 0 \end{bmatrix}, \quad t \notin \mathbb{Z} \qquad \Delta\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} g_k(x,\epsilon) \\ 0 \end{bmatrix}, \quad t \in \mathbb{Z}.$$

We can now apply the invariant manifold theory to the above system. Indeed, the above is equivalent to the semilinear form

$$\frac{d}{dt}\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} D_x f(t,0,0) & D_\epsilon f(t,0,0) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \epsilon \end{bmatrix} + \begin{bmatrix} F(t,x,\epsilon) \\ 0 \end{bmatrix}, \quad t \notin \mathbb{Z} \tag{II.4.34}$$

$$\Delta\begin{bmatrix} x \\ \epsilon \end{bmatrix} = \begin{bmatrix} D_x g_k(0,0) & D_\epsilon g_k(0,0) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \epsilon \end{bmatrix} + \begin{bmatrix} G_k(x,\epsilon) \\ 0 \end{bmatrix}, \qquad t \notin \mathbb{Z}, \tag{II.4.35}$$

with $F = f(t, x, \epsilon) - D_x f(t, 0, 0)x - D_\epsilon f(t, 0, 0)\epsilon$ and $G_k = g_k(x, \epsilon) - D_x g_k (0, 0)x - D_\epsilon g_k(0, 0)\epsilon$. It follows that $DF(t, 0, 0) = 0$ and $DG_k(0, 0) = 0$ as required.

II.4.7.1 Centre Manifolds Depending on a Parameter

System (II.4.34)–(II.4.35) always has a centre manifold of dimension at least p. If $x = 0$ in (II.4.32)–(II.4.33) at parameter $\epsilon = 0$ is nonhyperbolic with a c-dimensional centre fibre bundle X_c, then the centre manifold of $(x, \epsilon) = (0, 0)$ in (II.4.34)–(II.4.35) will be $(c+p)$-dimensional. Applying the transformation from Sect. II.4.3 and partitioning the equations appropriately, the result will be a q-periodic system in the form

$$\begin{aligned}
\dot{y} &= U_1 y + U_2\epsilon + \tilde{F}_1(t, y, z, \epsilon), && t \notin \mathbb{Z}\\
\dot{z} &= V_1 z + V_2\epsilon + \tilde{F}_2(t, y, z, \epsilon), && t \notin \mathbb{Z}\\
\dot{\epsilon} &= 0, && t \notin \mathbb{Z}\\
\Delta y &= R_1(k)z + R_2(k)\epsilon + \tilde{G}_1(k, y, z, \epsilon), && t \in \mathbb{Z}\\
\Delta z &= S_1(k)z + S_2(k)\epsilon + \tilde{G}_2(k, y, z, \epsilon), && t \in \mathbb{Z}\\
\Delta\epsilon &= 0, && t \in \mathbb{Z},
\end{aligned}$$

where the linear part of the y equation with $\epsilon = 0$ has only Floquet multipliers with absolute value equal to unity, and the Floquet multipliers associated to the linear part of the z component at $\epsilon = 0$ are disjoint from the unit circle. A local centre manifold of the above system at $(0, 0, 0)$ is locally representable by the solution set of the equation

$$z = h(t, y, \epsilon)$$

for $h : \mathbb{R} \times \mathbb{R}^c \times \mathbb{R}^p \to \mathbb{R}^{n-c}$ periodic in its first variable. In the ϵ direction, the dynamics on the centre manifold are trivial since there are no linear or nonlinear terms. However, in the y (centre) direction they are

$$\dot{y} = U_1 y + U_2 \epsilon + \tilde{F}_1(t, y, h(t, y, \epsilon), \epsilon), \qquad t \notin \mathbb{Z} \qquad \text{(II.4.36)}$$

$$\Delta y = R_1(k) y + R_2(k)\epsilon + \tilde{G}_1(k, y, h(t_k^-, y, \epsilon), \epsilon), \qquad t \in \mathbb{Z}, \qquad \text{(II.4.37)}$$

for ϵ fixed and sufficiently small. The *local parameter-dependent centre manifold* is the set with t-fibres

$$\mathcal{W}_{c,\epsilon}^{loc}(t) = \{(y, h(t, y, \epsilon)) : ||(y, \epsilon)|| < \delta\}.$$

The dynamics on this invariant manifold are topologically conjugate near $y = 0$ to those of (II.4.36)–(II.4.37), provided $|\epsilon|$ is small enough. The reduction principle (Theorem II.4.5.1) also applies to the parameter-dependent centre manifold, allowing one to derive bifurcation results.

II.4.8 Comments

Taylor approximation of invariant manifolds for nonautonomous ordinary differential in Banach spaces was developed by Pötzsche and Rasmussen [116]. The same authors also developed these techniques for nonautonomous discrete-time systems in [117]. The construction for impulsive differential equations with delays was completed by Church and Liu [33]. The computational (e.g. invariance equation and Taylor expansion) aspects of this chapter can be considered as a specification of the latter results to finite-dimensional systems.

Theorem II.4.1.1 is apparently new. The existence of invariant manifolds in the reversible hyperbolic case—that is, where X_c and X_0 are empty—has been known for some time. See for instance Theorem 6.8 of [9]. In the non-hyperbolic case, there was perhaps a good reason to believe such manifolds existed. Indeed, they can be identified with the forward time evolution of the associated invariant manifold of the time T map. Still, the concrete result of Theorem II.4.1.1 and the representation furnished by Eq. (II.4.6) remained absent.

Linear periodic systems are examples of *reducible* systems. Such systems can be transformed into block form by way of a bounded linear transformation with a bounded inverse, where the blocks induce a natural spectral decomposition. The transformation is called a *kinematic similarity*. A theorem of Siegmund [130] implies that such kinematic similarities always exist for linear ordinary differential equations $\dot{x} = A(t)x$ provided $A(t)$ is locally integrable. A suitable generalization of such a result to nonautonomous impulsive differential equations would permit the derivation of a concrete dynamics equation on the centre manifold analogous to (II.4.26)–(II.4.27) for general nonautonomous differential equations, not necessarily under periodic conditions.

Chapter II.5

Bifurcations

In this chapter we will specialize to periodic systems. We begin with a parameter-dependent system

$$\dot{x} = f(t, x, \epsilon), \qquad t \notin \mathbb{Z} \tag{II.5.1}$$
$$\Delta x = g_k(x, \epsilon), \qquad t \in \mathbb{Z}. \tag{II.5.2}$$

In what follows, M_0 will always denote the monodromy matrix for the linearization at parameter $\epsilon = 0$—that is, for the system

$$\dot{z} = D_x f(t, 0, 0)z, \qquad t \notin \mathbb{Z}$$
$$\Delta z = D_x g_k(0, 0)z, \qquad t \in \mathbb{Z}.$$

We denote by σ_c the set of eigenvalues of M_0 on the complex unit circle.

Following the results of Chap. II.4 and, in particular, Sect. II.4.7, we may assume that a suitable approximation of the centre manifold has been computed and that to some prescribed order the dynamics on the parameter-dependent centre manifold are given by a q-periodic system of the form

$$\dot{y} = U_1 y + U_2 \epsilon + f(t, y, \epsilon), \qquad t \notin \mathbb{Z} \tag{II.5.3}$$
$$\Delta y = R_1(k)y + R_2(k)\epsilon + g_k(y, \epsilon), \qquad t \in \mathbb{Z}, \tag{II.5.4}$$

where $D_y f(t, 0, 0) = D_y g_k(0, 0) = 0$, and the Floquet multipliers of the linear part at $\epsilon = 0$ are precisely σ_c. Some form of PC^m smoothness will always be assumed. The specific requirement on the degree m of smoothness will be stated when needed.

An important remark: The various transformations made to reach the dynamics on the centre manifold (II.5.3)–(II.5.4) will influence which of the

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matrices U_1, U_2, R_1 and R_2 is nonzero. The most important part is the form of the transformation

$$x = Q_s(t)y_s + Q_c(t)y_c + Q_u(t)y_u \tag{II.5.5}$$

that is originally used to split up the dynamics into stable, centre and unstable (linear) parts. In this regard, there are a few general statements we can make.

- If Q_c is a Floquet periodic matrix (i.e. from a Floquet decomposition $\Phi_c(t) = Q_c(t)e^{t\Lambda_c}$ for a basis matrix of X_c), then $R_1 = 0$ and $R_2 = 0$.
- If Q_c is a chain matrix for X_c, then $U_1 = 0$ and $U_2 = 0$.

For these reasons, in the subsequent sections we will state the optimal choice of the transformation that will make the bifurcation analysis as straightforward as possible. However, one can always make the appropriate transformation after the fact (i.e. on the system (II.5.3)–(II.5.4)) to eliminate either the U matrices or the R matrices, so there is really no loss of generality in assuming the optimal choice at the outset.

II.5.1 Reduction to an Iterated Map

Although it is perhaps an unsatisfying solution, the easiest way to analyze local bifurcations in (II.5.3)–(II.5.4) is to reduce to a discrete-time problem. Let $t \mapsto \phi(t; y, \epsilon)$ denote the solution of this impulsive differential equation (with parameter ϵ) satisfying $\phi(t; y, \epsilon) = y$.

Definition II.5.1.1. *The* time q map *is* $S : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ *defined*[1] *by* $S(y, p) = \phi(q; y, \epsilon)$.

Bifurcations from the fixed point $y = 0$ at parameter $\epsilon = 0$ in (II.5.3)–(II.5.4) can now studied by examining the equivalent parameter-dependent discrete-time system

$$y_{k+1} = S(y_k, \epsilon). \tag{II.5.6}$$

II.5.2 Codimension-one Bifurcations

In what follows, $\epsilon \in \mathbb{R}$ will be a real parameter. We will describe how the generic codimension-one bifurcations for discrete-time maps present themselves on the parameter-dependent centre manifold of an impulsive differential equation.

[1] We should be precise; S might not be defined on the entirety of $\mathbb{R}^n \times \mathbb{R}^p$. However, under PC^0 assumptions, it is at least defined on some open set containing $(0, 0)$. This is sufficient for applications.

II.5.2.1 Fold Bifurcation

Suppose $\sigma_0 = \{1\}$ with multiplicity one. Then, (II.5.3)–(II.5.4) are scalar. By Corollary II.2.6.1, we can choose Q_c in the transformation (II.5.5) to be a real, Floquet q-periodic matrix. Since $\Phi_c(t) = Q(t)e^{t\Lambda}$ has $\Lambda = \frac{1}{q}\log(\sigma_c) = 0$, this further implies $U_1 = 0$. As such, we may assume without loss of generality that $R_1 = 0$, $R_2 = 0$ and $U_1 = 0$, so that the dynamics on the parameter-dependent centre manifold are

$$\dot{y} = U_2\epsilon + f(t, y, \epsilon), \qquad t \notin \mathbb{Z} \tag{II.5.7}$$

$$\Delta y = g_k(y, \epsilon), \qquad t \in \mathbb{Z}. \tag{II.5.8}$$

The time q map takes the form

$$S(y, \epsilon) = \partial_y S(0,0)y + \partial_\epsilon S(0,0)\epsilon + \frac{1}{2}\partial_{yy}S(0,0)y^2 + \partial_{y\epsilon}S(0,0)y\epsilon + \frac{1}{2}\partial_{\epsilon\epsilon}S(0,0)\epsilon^2 + O(||(y,\epsilon)||^3).$$

This expansion requires PC^3 smoothness of the impulsive differential equation. To determine the coefficients of this Taylor expansion, we will iteratively apply Theorem II.1.2.1. If we denote $\phi(t; y, \epsilon)$ the solution of (II.5.7)–(II.5.8) satisfying $\phi(0; y, \epsilon) = y$, then $\phi(0; 0, 0) = 0$, $\partial_y\phi(0; 0, 0) = 1$, all other partial derivatives vanish at $t = 0$ and the partial derivatives

$$\phi_{\xi_1,\xi_2} := \partial_{\xi_1\xi_2}\phi(t; 0, 0), \qquad \xi_1, \xi_2 \in \{y, \epsilon\}$$

satisfy the impulsive differential equations

$$\begin{aligned}
\dot{\phi}_y &= 0, & t &\notin \mathbb{Z} & \Delta\phi_y &= 0, & t &= k \in \mathbb{Z}\\
\dot{\phi}_\epsilon &= U_2, & t &\notin \mathbb{Z}, & \Delta\phi_\epsilon &= 0, & t &= k \in \mathbb{Z}\\
\dot{\phi}_{yy} &= f_{yy}\phi_y^2, & t &\notin \mathbb{Z}, & \Delta\phi_{yy} &= g_{yy}\phi_y^2, & t &= k \in \mathbb{Z}\\
\dot{\phi}_{y\epsilon} &= f_{yy}\phi_\epsilon\phi_y + f_{y\epsilon}\phi_y, & t &\notin \mathbb{Z}, & \Delta\phi_{y\epsilon} &= g_{yy}\phi_y\phi_\epsilon + g_{y\epsilon}\phi_y, & t &= k \in \mathbb{Z}\\
\dot{\phi}_{\epsilon\epsilon} &= f_{yy}\phi_\epsilon^2 + 2f_{y\epsilon}\phi_\epsilon + f_{\epsilon\epsilon}, & t &\notin \mathbb{Z}, & \Delta\phi_{\epsilon\epsilon} &= g_{yy}\phi_\epsilon^2 + 2g_{y\epsilon}\phi_\epsilon + g_{\epsilon\epsilon}, & t &= k \in \mathbb{Z},
\end{aligned}$$

where $f_{\xi_1\xi_1}(t) = \partial_{\xi_1\xi_2}f(t, 0, 0)$ and $g_{\xi_1\xi_2}(k) = \partial_{\xi_1\xi_2}g_k(0, 0)$ in the above equations. Since $U_1 = 0$, we can solve these equations very easily.

$$\phi_y = 1, \qquad \phi_\epsilon = tU_2 \qquad \phi_{yy} = \int_0^t f_{yy}(s)ds + \sum_{0<k\leq t} g_{yy}(k)$$

$$\phi_{y\epsilon} = \int_0^t (f_{yy}(s)sU_2 + f_{y\epsilon}(s))ds + \sum_{0<k\leq t}(g_{yy}(k)kU_2 + g_{y\epsilon}(k))$$

$$\phi_{\epsilon\epsilon} = \int_0^t (f_{yy}(s)s^2U_2^2 + 2f_{y\epsilon}(s)sU_2 + f_{\epsilon\epsilon}(s))ds + \sum_{0<k\leq t}(g_{yy}(k)k^2U_2^2 + 2g_{y\epsilon}(k)kU_2 + g_{\epsilon\epsilon}(k)),$$

where in the above, empty summations are defined to be equal to zero. For example, we define $\sum_{0<k\leq 0.9} s_k = 0$ because the interval $(0, 0.9]$ contains no integers. Since $\partial_{\xi_1\xi_2} S(0,0) = \phi_{\xi_1\xi_2}(q)$, we can write the Taylor expansion of the time q map (nearly) explicitly by

$$\begin{aligned} S(y,\epsilon) = y + qU_2\epsilon + \frac{1}{2}\left[\int_0^q f_{yy}(s)ds + \sum_{k=1}^{c} g_{yy}(k)\right] y^2 \\ + \left[\int_0^q (f_{yy}(s)sU_2 + f_{y\epsilon}(s))ds + \sum_{k=1}^{q}(g_{yy}(k)kU_2 + g_{y\epsilon}(k))\right] y\epsilon \\ + O(\epsilon^2 + \|(y,\epsilon)\|^3). \end{aligned} \tag{II.5.9}$$

The $O(\epsilon^2)$ terms and the cubic terms above do not enter into the generic fold bifurcation conditions, and we arrive at the following theorem.

Theorem II.5.2.1. *Let (II.5.7)–(II.5.8) be PC^2 with $D_y f(t,0,0) = 0$ and $D_y g_k(0,0) = 0$. Suppose $a_0 := qU_2 \neq 0$ and $a_1 := \frac{1}{2}\left(\int_0^q f_{yy}(s)ds + \sum_{k=1}^q g_{yy}(k)\right) \neq 0$. Then, the discrete-time system (II.5.6) undergoes a fold bifurcation at parameter $\epsilon = 0$. More importantly, the impulsive system (II.5.7)–(II.5.8) undergoes a fold bifurcation of periodic solutions (of period q). There exists an interval $[-\delta, \delta]$ such that for $|\epsilon|$ sufficiently small, the following are true:*

- *If $a_0 a_1 \epsilon > 0$, (II.5.3)–(II.5.4) have no periodic solutions (or fixed points) in $[-\delta, \delta]$.*
- *If $\epsilon = 0$, $[-\delta, \delta]$ contains exactly one periodic solution, the trivial solution $y = 0$, and this solution is unstable.*
- *If $a_0 a_1 \epsilon < 0$, $[-\delta, \delta]$ contains a pair of nontrivial periodic solutions (of period q). These periodic solutions are*

$$y_\epsilon(t) = \pm\sqrt{-\frac{a_0\epsilon}{a_1}} + O(\epsilon).$$

The positive solution is stable (and asymptotically stable) if and only if $a_1 < 0$, and the negative solution is stable (and asymptotically stable) if and only if $a_1 > 0$.

Proof. The assertions concerning the existence of the periodic solutions follow by the fold bifurcation theorem for maps; see Wiggins [151]. The stability assertions are easy to check from (II.5.9). The only thing we need to verify is the claimed asymptotic for the periodic solution. To accomplish this, first suppose that $a_0 a_1 > 0$. Introduce the change of parameter $\rho = \sqrt{-\epsilon}$, defined for $\epsilon < 0$. In terms of the variable $\epsilon < 0$, $y_0(\epsilon) = \pm\sqrt{-\epsilon a_0/a_1} + O(\epsilon)$ is the lowest order approximation of the pair of periodic points of the iterated map derived from (II.5.9). Writing this in terms of ρ, we get

$$\tilde{y}_0(\rho) = \pm\, \rho\sqrt{\frac{a_0}{a_1}} + O(\rho^2).$$

The periodic solutions are $t \mapsto y_\epsilon = \phi(t; \tilde{y}_0(\rho), -\rho^2)$. A first-order Taylor expansion at $\rho = 0$ gives

$$y_\epsilon(t) = \phi_y(t; 0, 0)\tilde{y}_0'(0)\rho + O(\rho)^2 = \pm\sqrt{\frac{a_0}{a_1}}\rho + O(\rho^2) = \pm\sqrt{-\frac{a_0}{a_1}} + O(\epsilon),$$

as claimed. The case where $a_0a_1 < 0$ is similar. □

While not a codimension-one bifurcation in the strictest sense, we can state the analogous transcritical and pitchfork bifurcation theorems. The transcritical bifurcation theorem follows much the same lines as the previous theorem, while the pitchfork result will require a higher-order expansion of the time q map; see later Sect. II.5.2.2, where the differential equation for the cubic term is provided (albeit for a slightly different system). In what follows, we will be introducing a few quantities that will take the role of the Taylor coefficients of the time q map. To aid with the understanding of these theorems, the expansion of the time q map is assumed to take the form

$$S(y, \epsilon) = y + a_{01}\epsilon + a_{20}y^2 + a_{11}\epsilon y + a_{30}y^3 + \text{higher-order terms},$$

where the higher-order terms have no bearing on the transcritical or pitchfork bifurcation conditions (i.e. ϵ^2, mixed cubic terms not including y^3, and all higher-order terms).

Theorem II.5.2.2. *Let* (II.5.7)–(II.5.8) *be* PC^2 *with* $D_yf(t, 0, 0) = 0$ *and* $D_yg_k(0, 0) = 0$. *Introduce the constants*

$$a_{01} = qU_2, \quad a_{20} = \frac{1}{2}\left(\int_0^q f_{yy}(s)ds + \sum_{k=1}^{q} g_{yy}(k)\right),$$

$$a_{11} = \int_0^q (f_{yy}(s)sU_2 + f_{y\epsilon}(s))ds + \sum_{k=1}^{q}(g_{yy}(k)kU_2 + g_{y\epsilon}(k)).$$

If $a_{01} = 0$, $a_{20} \neq 0$ *and* $a_{11} \neq 0$, *and* $f(t, 0, \epsilon) = g_k(0, \epsilon) = 0$ *for* $|\epsilon|$ *sufficiently small (i.e.* $y = 0$ *is a stationary, trivial solution), the discrete-time system* (II.5.6) *undergoes a transcritical bifurcation at parameter* $\epsilon = 0$. *More importantly, the impulsive system* (II.5.7)–(II.5.8) *undergoes a transcritical bifurcation of periodic solutions (of period* q*). There exists an interval* $[-\delta, \delta]$ *such that for* $|\epsilon|$ *sufficiently small, the following are true:*

- *If* $\epsilon = 0$, $[-\delta, \delta]$ *contains exactly one periodic solution, the trivial solution* $y = 0$, *and this solution is unstable.*

- *If $\epsilon \neq 0$, $[-\delta, \delta]$ contains exactly two periodic solutions: the trivial solution $y = 0$ and a nontrivial periodic solution*

$$y_\epsilon(t) = -\frac{a_{11}}{a_{20}}\epsilon + O(\epsilon^2).$$

$y = 0$ is stable (and asymptotically stable) if and only if $a_{11}\epsilon < 0$. The solution y_ϵ is stable (and asymptotically stable) if and only if $a_{11}\epsilon > 0$.

Theorem II.5.2.3. *Let* (II.5.7)–(II.5.8) *be PC^3 with $D_y f(t, 0, 0) = 0$ and $D_y g_k(0, 0) = 0$. Introduce the constant*

$$a_{30} = \frac{1}{6}\int_0^q \left[f_{yyy}(s) + 3f_{yy}(s)\left(\int_0^s f_{yy}(\mu)d\mu + \sum_{0\leq k<s} g_{yy}(k)\right)\right] ds$$
$$+ \frac{1}{6}\sum_{k=1}^{q}\left[g_{yyy}(k) + 3g_{yy}(k)\left(\int_0^k f_{yy}(s)ds + \sum_{j=1}^{k-1} g_{yy}(j)\right)\right].$$

Let a_{01}, a_{20} and a_{11} be as defined in Theorem II.5.2.2. If $a_{01} = 0$, $a_{20} = 0$, $a_{11} \neq 0$, $a_{30} \neq 0$ and $f(t, 0, \epsilon) = g_k(0, \epsilon) = 0$ for $|\epsilon|$ sufficiently small (i.e. $y = 0$ is a stationary, trivial solution), then the discrete-time system (II.5.6) *undergoes a pitchfork bifurcation at parameter $\epsilon = 0$. More importantly, the impulsive system* (II.5.7)–(II.5.8) *undergoes a pitchfork bifurcation of periodic solutions (of period q). There exists an interval $[-\delta, \delta]$ such that for $|\epsilon|$ sufficiently small, the following are true:*

- *If $\epsilon = 0$, $[-\delta, \delta]$ contains exactly one periodic solution, the trivial solution $y = 0$ and this solution is stable (and asymptotically stable) if and only if $a_{30} < 0$.*

- *If $a_{11}a_{30}\epsilon > 0$, $[-\delta, \delta]$ contains exactly one periodic solution, the trivial solution $y = 0$ and this solution is stable (and asymptotically stable) if and only if $a_{11}\epsilon < 0$.*

- *If $a_{11}a_{30}\epsilon < 0$, $[-\delta, \delta]$ contains exactly three periodic solutions: the trivial solution $y = 0$ and a pair of periodic solutions*

$$y_\epsilon(t) = \pm\sqrt{-\frac{a_{11}\epsilon}{a_{30}}} + O(\epsilon).$$

The trivial solution $y = 0$ is stable (and asymptotically stable) if and only if $a_{11}\epsilon < 0$, and the other two periodic solutions are stable (and asymptotically stable) if and only if $a_{11}\epsilon > 0$.

Remark II.5.2.1. *If one of the above bifurcations occurs and* (II.5.7)–(II.5.8) *are the dynamics on the parameter-dependent centre manifold of a higher-dimensional system* (II.5.1)–(II.5.2), *then the bifurcating periodic solutions* *persist in the higher-dimensional system, as does their local uniqueness. The local stability properties of any such periodic solutions also transfer over to the higher-dimensional system provided M_0 has no eigenvalue with absolute value greater than one—that is, the unstable subspace X_u is trivial when $\epsilon = 0$. This is a consequence of the reduction principle; see Sect. I.5.5.*

II.5.2.2 Period-Doubling Bifurcation

Suppose $\sigma_0 = \{-1\}$ with multiplicity one. Then, the associated Floquet exponent is $i\pi$. As such, we cannot use a real Floquet q-periodic matrix for the Q_c part of the transformation (II.5.5) since one does not exist. We will therefore need to use a chain matrix instead if we wish to maintain the q-periodicity of the impulsive differential equation. We may without loss of generality assume that (II.5.3)–(II.5.4) take the form

$$\dot{y} = f(t, y, \epsilon), \qquad t \notin \mathbb{Z} \tag{II.5.10}$$
$$\Delta y = R_1(k)y + R_2(k)\epsilon + g_k(y, \epsilon), \qquad t \in \mathbb{Z}. \tag{II.5.11}$$

That is, $U_1 = 0$ and $U_2 = 0$. Also, the above equation is scalar since X_c is one-dimensional. Since the linear part at $\epsilon = 0$ must have -1 as its Floquet multiplier, the matrices R_1 must satisfy $\prod_{k=1}^{q}(1 + R_1(k)) = -1$.

We will need to compute some terms of the Taylor expansion for the time q map. Those needed for checking the conditions for the generic period-doubling bifurcation can be computed by solving the impulsive differential equations

$$\begin{aligned}
\dot{\phi}_y &= 0, & t &\notin \mathbb{Z} & \Delta\phi_y &= R_1(k)\phi_y, & t &= k \in \mathbb{Z}\\
\dot{\phi}_\epsilon &= 0, & t &\notin \mathbb{Z}, & \Delta\phi_\epsilon &= R_1(k)\phi_\epsilon + R_2(k), & t &= k \in \mathbb{Z}\\
\dot{\phi}_{yy} &= f_{yy}\phi_y^2, & t &\notin \mathbb{Z}, & \Delta\phi_{yy} &= R_1(k)\phi_{yy} + g_{yy}\phi_y^2, & t &= k \in \mathbb{Z}\\
\dot{\phi}_{y\epsilon} &= f_{yy}\phi_\epsilon\phi_y + f_{y\epsilon}\phi_y, & t &\notin \mathbb{Z}, & \Delta\phi_{y\epsilon} &= R_1(k)\phi_{y\epsilon} + g_{yy}\phi_y\phi_\epsilon + g_{y\epsilon}\phi_y, & t &= k \in \mathbb{Z}\\
\dot{\phi}_{yyy} &= 3f_{yy}\phi_y\phi_{yy} + f_{yyy}\phi_y^3, & t &\notin \mathbb{Z}, & \Delta\phi_{\epsilon\epsilon} &= R_1(k)\phi_{yyy} + 3g_{yy}\phi_y\phi_{yy} + g_{yyy}\phi_y^3, & t &= k \in \mathbb{Z},
\end{aligned}$$

satisfying $\phi_y(0) = 1$, with all other initial conditions zero. Define the convenience function

$$z(t) = \prod_{0<k\leq t}(1 + R_1(k)).$$

When the interval $(0, t]$ contains no integers k, the empty product is taken to be equal to unity so that, for example, $z(s) = 1$ for $s \in [0, 1)$. In terms of this convenience function, we can calculate

$$\phi_y(t) = z(t), \qquad \phi_\epsilon(t) = z(t) \sum_{0<k\leq t} \frac{R_2(k)}{z(k)}$$

$$\phi_{yy}(t) = z(t)\left[\int_0^t \frac{f_{yy}(s)}{z(s)}ds + \sum_{0<k\leq t} g_{yy}(k)\frac{z(k-1)}{1+R_1(k)}\right] \tag{II.5.12}$$

$$\phi_{y\epsilon}(t)=z(t)\left[\int_0^t \left(f_{yy}(s)\phi_\epsilon(s)+\frac{f_{y\epsilon}(s)}{z(s)}\right)ds+\sum_{0<k\leq t}\left(g_{yy}(k)\frac{\phi_\epsilon(k-1)}{1+R_1(k)}+\frac{g_{y\epsilon}(k)}{z(k)}\right)\right] \tag{II.5.13}$$

$$\begin{aligned}\phi_{yyy}(t) = z(t)\Bigg[&\int_0^t (3f_{yy}(s)\phi_{yy}(s) + f_{yyy}(s)z(s)^2)ds \\ &+ \sum_{0<k\leq t}\left(3g_{yy}(k)\frac{\phi_{yy}(k^-)}{1+R_1(k)} + g_{yyy}(k)\frac{z(k-1)^3}{z(k)}\right)]\end{aligned} \tag{II.5.14}$$

The coefficients needed to check the conditions of a generic period-doubling bifurcation are $\phi_y(q) = -1$, $\phi_{yy}(q)$, $\phi_{y\epsilon}(q)$ and $\phi_{yyy}(q)$. This is because the time q map has the expansion

$$S(y,\epsilon) = -y + \phi_{y\epsilon}(q)y\epsilon + \frac{1}{2}\phi_{yy}(q)y^2 + \frac{1}{6}\phi_{yyy}(q)y^3 + \text{higher-order terms},$$

where the higher-order terms contain some quadratic and cubic terms as well as fourth-order terms and above, all of which do not factor into the generic period-doubling bifurcation conditions. The following theorem can be proven using Theorem 4.3 of Kuznetsov [82] and similar arguments to the previous section.

Theorem II.5.2.4. *Let (II.5.10)–(II.5.11) be PC^3 with $D_y f(t,0,0) = 0$ and $D_y g_k(0,0) = 0$. Suppose the following nondegeneracy conditions are satisfied:*

- $s := \frac{1}{2}(\phi_{yy}(q))^2 + \frac{1}{3}\phi_{yyy}(q) \neq 0$;
- $\beta := \phi_{y\epsilon}(q) \neq 0$.

The discrete-time system (II.5.6) undergoes a period-doubling bifurcation at parameter $\epsilon = 0$. More importantly, the impulsive system (II.5.10)–(II.5.11) undergoes a period-doubling bifurcation of periodic solutions. There exists an interval $[-\delta,\delta]$ such that for $|\epsilon|$ sufficiently small, the following are true:

- *If $s\beta\epsilon \geq 0$, $[-\delta,\delta]$ contains exactly one periodic solution $t \mapsto y_0(t,\epsilon)$ of period q, satisfying $y_0(t,0) = 0$.*
- *If $s\beta\epsilon < 0$, $[-\delta,\delta]$ contains exactly two periodic solutions of period $2q$, exactly one periodic solution $y_0(t;\epsilon)$ of period q and no other periodic solutions.*

- *The q-periodic solution y_0 is stable (and asymptotically stable) if and only if $\beta\epsilon > 0$ or, if $\epsilon = 0$, it is asymptotically stable provided $s > 0$.*

- *The $2q$-periodic solutions are stable (and asymptotically stable) if and only if $\beta\epsilon < 0$, provided they exist.*

Remark II.5.2.2. *If a period-doubling bifurcation occurs and (II.5.10)–(II.5.11) are the dynamics on the parameter-dependent centre manifold of a higher-dimensional system (II.5.1)–(II.5.2), then the bifurcating periodic solutions of period q and $2q$ persist in the higher-dimensional system, as does their local uniqueness. The local stability properties of any such periodic solutions also transfer over to the higher-dimensional system provided M_0 has no eigenvalue with absolute value greater than one—that is, the unstable subspace X_u is trivial when $\epsilon = 0$.*

Special Case: $q = 1$

If there is one impulse per period, the constants s and β needed to check the nondegeneracy conditions of the period-doubling bifurcation can be computed more or less explicitly. They are

$$\begin{aligned}
\beta &= -\left(\int_0^1 f_{y\epsilon}(s)ds + g_{y\epsilon}(1)\right), \\
s &= \frac{1}{2}\left(F_{yy}(1) + g_{yy}(1)\right)^2 \\
&\quad - \frac{1}{3}\left(\int_0^1 \left(3f_{yy}(s)F_{yy}(s) + f_{yyy}(s)\right) ds - 3g_{yy}(1)F_{yy}(1) - g_{yyy}(1)\right),
\end{aligned}$$

where $F_{yy}(t) = \int_0^t f_{yy}(s)ds$.

II.5.2.3 Cylinder Bifurcation

Suppose $\sigma_0 = \{e^{i\omega}, e^{-i\omega}\}$, each with multiplicity one for $\omega \in (0, \pi)$. Then, (II.5.3)–(II.5.4) are two-dimensional. By Corollary II.2.6.1, we can choose Q_c in the transformation (II.5.5) to be a real, Floquet q-periodic matrix. We may therefore assume without loss of generality that $R_1 = 0$ and $R_2 = 0$.

To identify the matrix U_1, let ϕ be any eigenvector of M_0 such that $M_0\phi = e^{i\omega}\phi$. Then, $\Phi(t) = U(t, 0)[\ \Re(\phi) \quad \Im(\phi)\ ]$ is a basis matrix for X_c. It can be shown to satisfy

$$\Phi(q) = \Phi(0)\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix},$$

from which it follows that the Floquet decomposition $\Phi(t) = Q(t)e^{t\Lambda}$ has

$$\Lambda = \frac{1}{q}\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}. \tag{II.5.15}$$

Consequently, $U_1 = \Lambda$. The dynamics on the parameter-dependent centre manifold have the form

$$\begin{aligned} \dot{y} &= \Lambda y + U_2\epsilon + f(t,y,\epsilon), && t \notin \mathbb{Z} \\ \Delta y &= g_k(y,\epsilon), && t \in \mathbb{Z}. \end{aligned}$$

We will show that we may assume without loss of generality that $U_2 = 0$. Since Λ is invertible, define the parameter-dependent change of variables

$$y = \tilde{y} - \Lambda^{-1}U_2\epsilon.$$

Then, $\tilde{y}$ satisfies

$$\begin{aligned} \dot{\tilde{y}} &= \Lambda(\tilde{y} - \Lambda^{-1}U_2\epsilon) + U_2\epsilon + f(t,\tilde{y} - \Lambda^{-1}U_2\epsilon,\epsilon) = \Lambda\tilde{y} + \tilde{f}(t,\tilde{y},\epsilon), && t \notin \mathbb{Z} \\ \Delta\tilde{y} &= g_k(\tilde{y} - \Lambda^{-1}U_2\epsilon,\epsilon) = \tilde{g}_k(\tilde{y},\epsilon), && t \in \mathbb{Z}. \end{aligned}$$

Therefore, dropping the tildes, we may assume without loss of generality that the dynamics on the parameter-dependent centre manifold take the form

$$\dot{y} = \Lambda y + f(t,y,\epsilon), \qquad t \notin \mathbb{Z} \tag{II.5.16}$$

$$\Delta y = g_k(y,\epsilon), \qquad t \in \mathbb{Z}, \tag{II.5.17}$$

where f and g contain all terms of order 2 and above in y and ϵ.

Our goal is to determine conditions under which the time q map associated to (II.5.16)–(II.5.17) undergoes a Neimark–Sacker bifurcation. To accomplish this, we will first compute a third-order Taylor expansion of the time q map fixed at $\epsilon = 0$. Next, we will compute the partial derivative $\partial_{\epsilon y}S(0,0)$. These computations will grant a representation of the time q map as

$$\begin{aligned} S(y,\epsilon) = &\left[e^{q\Lambda}y + \frac{1}{2}B_0[y,y] + \frac{1}{6}C_0[y,y,y]\right] \\ &+ \partial_{\epsilon y}S(0,0)\epsilon y + O(\epsilon(\epsilon||y|| + ||y||^2) + ||(\epsilon,y)||^4), \end{aligned}$$

where B_0 and C_0 are symmetric multilinear maps. Note that there are no $O(\epsilon)$ terms since f and g_k satisfy $\partial_\epsilon f(t,0,0) = \partial_\epsilon g_k(0,0) = 0$. Since $\partial_y S(0,0) = e^{q\Lambda}$ does not have 1 as an eigenvalue, the implicit function theorem implies the existence of a unique $y^* = y^*(\epsilon)$ smooth such that $S(y^*(\epsilon),\epsilon) = y^*(\epsilon)$ for $|\epsilon|$ sufficiently small. By implicit differentiation, one can check that

$$\frac{dy^*}{d\epsilon}(0) = -e^{-q\Lambda}\partial_\epsilon S(0,0) = 0. \tag{II.5.18}$$

If we perform the change of variables $y = \tilde{y} + y^*(\epsilon)$, the time q map becomes

$$\tilde{S}(\tilde{y},\epsilon) = S(\tilde{y} + y^*(\epsilon),\epsilon) - S(y^*(\epsilon),\epsilon),$$

which implies (after a bit of algebra) the expansion

$$\tilde{S}(\tilde{y},\epsilon) = e^{q\Lambda}\tilde{y} + \partial_{\epsilon y}S(0,0)\epsilon\tilde{y} + \frac{1}{2}B_0[\tilde{y},\tilde{y}] + \frac{1}{6}C_0[\tilde{y},\tilde{y},\tilde{y}] + \text{higher-order terms},$$

where the higher-order terms include some mixed quadratic, cubic and terms of order higher than four that will not be important. Importantly, we have $\tilde{S}(0,\epsilon) = 0$ for $|\epsilon|$ sufficiently small. The above expansion will be sufficient in the verification of all nondegeneracy conditions for the Neimark–Sacker bifurcation.

As in the previous section, denote

$$\begin{aligned} f_{yy}(t) &= D_y^2 f(t,0,0), \quad & f_{yyy}(t) &= D_y^3 f(t,0,0),\\ g_{yy}(k) &= D_y^2 g_k(0,0), \quad & g_{yyy}(k) &= D_y^3 g_k(0,0). \end{aligned}$$

The second differentials are symmetric bilinear maps, and the third differentials are symmetric trilinear maps. Denoting $t \mapsto \phi(t;y,\epsilon)$ the solution of (II.5.16)–(II.5.17) satisfying $\phi(0;y,\epsilon) = 0$, the first two partial derivatives in the variable y at $(y,\epsilon) = (0,0)$ satisfy the impulsive differential equations

$$\begin{aligned} \dot{\phi}_y &= \Lambda\phi_y, & t &\notin \mathbb{Z} & \Delta\phi_y &= 0, & t &\in \mathbb{Z}\\ \dot{\phi}_{yy} &= \Lambda\phi_{yy} + f_{yy}[\phi_y]^2, & t &\notin \mathbb{Z}, & \Delta\phi_{yy} &= g_{yy}[\phi_y]^2, & t &\in \mathbb{Z}, \end{aligned}$$

with $\phi_y(0) = I$, and all other initial conditions equal to zero. We have used the shorthand $B[x]^2 = B[x,x]$ for bilinear maps B. It is clear that the right-hand side of the equation for ϕ_{yy} is a symmetric bilinear map, assuming ϕ_{yy} is itself bilinear. The third differential is slightly more difficult. If one computes $\partial_{yy}\phi(t;y,0)[u,w] := \psi[u,w]$ at an arbitrary point y, then the associated impulsive differential equation is

$$\begin{aligned} \dot{\psi}[u,w] &= \Lambda\psi[u,w] + f_{yy}[\phi_y u, \phi_y w] + f_{yy}[\phi, \psi[u,w]]\\ &\quad + f_{yyy}[\phi_y u, \phi_y w, \phi] + \frac{1}{2} f_{yyy}[\psi[u,w],\phi,\phi], & t &\notin \mathbb{Z}\\ \Delta\psi[u,w] &= g_{yy}[\phi_y u, \phi_y w] + g_{yy}[\phi, \psi[u,w]]\\ &\quad + g_{yyy}[\phi_y u, \phi_y w, \phi] + \frac{1}{2} g_{yyy}[\psi[u,w],\phi,\phi], & t &\in \mathbb{Z}, \end{aligned}$$

where $\phi = \phi(t;y,0)$. Then, if we take another y differential, evaluate at $y = 0$ and apply the linear map to an element h, then $\phi_{yyy}[u,w,h] = \partial_{yyy}\phi(t;0,0)[u,w,h]$ satisfies

$$\begin{aligned} \dot{\phi}_{yyy}[u,w,h] &= \Lambda\phi_{yyy}[u,w,h] + f_{yy}[\phi_{yy}[u,w],\phi_y h] + f_{yy}[\phi_{yy}[h,u],\phi_y w]\\ &\quad + f_{yy}[\phi_{yy}[w,h],\phi_y u] + f_{yyy}[\phi_y u, \phi_y w, \phi_y h], & t &\notin \mathbb{Z}\\ \Delta\phi_{yyy}[u,w,h] &= g_{yy}[\phi_{yy}[u,w],\phi_y h] + g_{yy}[\phi_{yy}[h,u],\phi_y w] + g_{yy}[\phi_{yy}[w,h],\phi_y u]\\ &\quad + g_{yyy}[\phi_y u, \phi_y w, \phi_y h], & t &\in \mathbb{Z}. \end{aligned}$$

This is indeed consistent; if ϕ_{yyy} is symmetric, then so is the right-hand side of the above impulsive differential equation. Suppressing the inputs (u,w,h) can lead to some ambiguity, so we refrain from doing this.

We can solve the impulsive differential equations directly. We get

$$\phi_y(t) = e^{t\Lambda}, \tag{II.5.19}$$

$$\phi_{yy}(t)[u,w] = e^{t\Lambda}\int_0^t e^{-s\Lambda} f_{yy}(s)[e^{s\Lambda}u, e^{s\Lambda}w]ds + e^{t\Lambda}\sum_{0<k\leq t} e^{-k\Lambda} g_{yy}(k)[e^{k\Lambda}u, e^{k\Lambda}w], \tag{II.5.20}$$

$$\begin{aligned}\phi_{yyy}(t)[u,w,h] = e^{t\Lambda}\int_0^t e^{-s\Lambda}\Big(&f_{yy}(s)[\phi_{yy}(s)[u,w], e^{s\Lambda}h] + f_{yy}(s)[\phi_{yy}(s)[h,u], e^{s\Lambda}w] \\ &+ f_{yy}(s)[\phi_{yy}(s)[w,h], e^{s\Lambda}u] + f_{yyy}(s)[e^{s\Lambda}u, e^{s\Lambda}w, e^{s\Lambda}h]\Big)ds \\ +e^{t\Lambda}\sum_{0<k\leq t} e^{-k\Lambda}\Big(&g_{yy}(k)[\phi_{yy}(k^-)[u,w], e^{k\Lambda}h] + g_{yy}(k)[\phi_{yy}(k^-)[h,u], e^{k\Lambda}w] \\ &+ g_{yy}(k)[\phi_{yy}(k^-)[w,h], e^{k\Lambda}u] + g_{yyy}(k)[e^{k\Lambda}u, e^{k\Lambda}w, e^{k\Lambda}h]\Big).\end{aligned} \tag{II.5.21}$$

The expression for ϕ_{yyy} in its complete explicit form is quite large, so we will be satisfied with the above expression in terms of ϕ_{yy}. Note that the above three are indeed symmetric linear, bilinear and trilinear maps, respectively. We therefore have

$$B_0 = \phi_y y(1), \qquad C_0 = \phi_{yyy}(1). \tag{II.5.22}$$

Next, we need to calculate $\partial_{y\epsilon}\tilde{S}(0,0)$. By the chain rule,

$$\partial_{y\epsilon}\tilde{S}(0,0) = \partial_{yy}S(0,0)\frac{dy^*}{d\epsilon}(0) + \partial_{y\epsilon}S(0,0) = \partial_{y\epsilon}S(0,0),$$

where we have used (II.5.18). The partial derivative $\phi_{y\epsilon}$ satisfies $\phi_{y\epsilon}(0) = 0$ and the impulsive differential equation

$$\begin{aligned}\dot{\phi}_{y\epsilon} &= \Lambda\phi_{y\epsilon} + f_{y\epsilon}\phi_y, && t \notin \mathbb{Z} \\ \Delta\phi_{y\epsilon} &= g_{y\epsilon}\phi_y, && t \in \mathbb{Z},\end{aligned}$$

which has the solution

$$\phi_{y\epsilon}(t) = e^{t\Lambda}\int_0^t e^{-s\Lambda} f_{y\epsilon}(s)e^{s\Lambda}ds + e^{t\Lambda}\sum_{k=1}^{q} e^{-k\Lambda} g_{y\epsilon}(k)e^{k\Lambda}. \tag{II.5.23}$$

To summarize, $\tilde{S}$ admits the Taylor expansion

$$\tilde{S}(\tilde{y},\epsilon) = e^{q\Lambda}\tilde{y} + \phi_{y\epsilon}(q)\epsilon\tilde{y} + \frac{1}{2}\phi_{yy}(q)[\tilde{y},\tilde{y}] + \frac{1}{3!}\phi_{yyy}(q)[\tilde{y},\tilde{y},\tilde{y}] + \text{higher-order terms}, \tag{II.5.24}$$

with the higher-order terms including mixed cubics and terms of order four and above. We remind the reader that $\tilde{S}(0,\epsilon) = 0$ for $|\epsilon|$ sufficiently small.

The eigenvalues associated to the fixed point $\tilde{y} = 0$ are the eigenvalues of the matrix

$$m(\epsilon) = e^{q\Lambda} + \phi_{y\epsilon}(q)\epsilon.$$

The condition that the eigenvalues of $m(\epsilon)$ should cross the unit circle $|z| = 1$ transversally at $\epsilon = 0$ is equivalent to the statement that their product $\gamma(\epsilon) = \det(m(\epsilon))$ satisfies $p'(0) \neq 0$. This can be computed by Jacobi's formula:

$$\frac{d}{d\epsilon}p(0) = \mathrm{tr}\left(\mathrm{adj}(m(0))\frac{d}{d\epsilon}m(0)\right) = \mathrm{tr}\left(e^{-q\Lambda}\phi_{y\epsilon}(q)\right),$$

Moreover, $p'(0)$ quantifies whether the eigenvalues exit the unit circle ($p'(0) > 0$) or enter the unit circle ($p'(0) < 0$) as ϵ increases through zero. The following theorem now follows by the generic Neimark–Sacker bifurcation [82] and arguments analogous to those for Theorem I.8.4.1.

Theorem II.5.2.5. *Suppose the following nondegeneracy conditions are met:*

- $e^{ik\omega} \neq 1$ *for* $k = 1, 2, 3, 4$*;*
- $\gamma(0) := \mathrm{tr}\left(\int_0^q e^{-s\Lambda} f_{y\epsilon}(s) e^{s\Lambda} ds + \sum_{k=1}^q e^{-k\Lambda} g_{y\epsilon}(k) e^{k\Lambda}\right) \neq 0$*;*
- *the first Lyapunov coefficient* $d(0)$ *(see* (I.8.40)*) associated to the map* (II.5.24) *is nonzero.*

Then, the equilibrium point at the origin of the nonlinear impulsive delay differential equation (II.5.16)–(II.5.17) *undergoes a bifurcation to an invariant cylinder (cylinder bifurcation) at the critical parameter* $\epsilon = 0$*. Specifically, for* $|\epsilon|$ *small, there is a unique periodic solution* $t \mapsto y_\epsilon(t)$ *that satisfies* $y_\epsilon \to 0$ *as* $\epsilon \to 0$*, in addition to a two-dimensional parameter-dependent invariant fibre bundle* $\Sigma_\epsilon \subset \mathbb{S}^1 \times \mathbb{R}^2$ *that exists for* $d(0)\gamma(0)\epsilon < 0$ *and is periodic. The* t*-fibre* $\Sigma_\epsilon(t)$ *can be locally realized as*

$$\Sigma_\epsilon(t) = \sigma_\epsilon(t) + O(\epsilon),$$

where $t \mapsto \sigma_\epsilon(t) \subset \mathbb{R}^2$ *is periodic with its image of a curve of diameter* $O(\sqrt{\epsilon})$*, and continuous in the Hausdorff metric except at integer times, where it is continuous from the right. Moreover,*

- y_ϵ *is asymptotically stable for* $\gamma(0)\epsilon < 0$*, stable for* $\epsilon = 0$ *and unstable for* $\gamma(0)\epsilon > 0$*, while* $\Sigma_\epsilon(t)$ *is attracting for* $\gamma(0)\epsilon > 0$ *provided* $d(0) < 0$*;*
- y_ϵ *is asymptotically stable for* $\gamma(0)\epsilon < 0$ *and unstable for* $\gamma(0)\epsilon \geq 0$*, while* $\Sigma_\epsilon(t)$ *is unstable for* $\gamma(0)\epsilon < 0$ *provided* $d(0) > 0$*.*

Moreover, the assertions concerning the stability and existence of the periodic orbit y_ϵ *for* $\epsilon \neq 0$ *are true regardless of the third nondegeneracy condition.*

Remark II.5.2.3. *If an invariant cylinder bifurcation occurs and* (II.5.16)–(II.5.17) *are the dynamics on the parameter-dependent centre manifold of a higher-dimensional system* (II.5.1)–(II.5.2)*, then the bifurcating periodic solution and the fibre bundle persist in the higher-dimensional system, as does their local uniqueness. In the higher-dimensional system, the image of the fibre bundle (as a periodic function* $\mathbb{R} \to \mathbb{R}^n$*) is a topological cylinder. The local stability properties transfer over to the higher-dimensional system provided* M_0 *has no eigenvalue with absolute value greater than one—that is, the unstable subspace* X_u *is trivial when* $\epsilon = 0$.

II.5.3 Comments

The reduction to a discrete-time map to analyze bifurcations in a finite-dimensional impulsive system seems to have been first completed by Lakmeche and Arino [84] for a pulse chemotherapy model. Once the discrete-time map has been approximated to a high enough degree, Lyapunov-Schmidt reduction is used to determine the bifurcation. Since then, this technique has been used numerous times to study bifurcations in impulsive differential equations; one needs to only glance at the hundreds of citations of the aforementioned paper to find them. To name a few in chronological order, Lu, Chi and Chen studied bifurcations in a susceptible–infected–removed (SIR) model with pulsed vaccination [102], as well as control strategies in pesticide models [103]. Jiang and Lu [76] study bifurcations in a state-feedback controlled predator-prey model. Impulsive harvesting in a predator-prey system and bifurcations were analyzed by Negi and Gakkhar [110]. Georgescu, Zhang and Chen [49] analyzed bifurcations to nontrivial periodic solutions in a pest management model. Bifurcations to nontrivial periodic solutions in an infectious diseases model with media coverage and pulse vaccination were considered by Li and Cui [86]. These articles all date prior to 2010 are some of the most-cited papers in which bifurcations to nontrivial periodic solutions in impulsive differential equations are explicitly proven using the reduction to discrete time. There are several others, and the method continues to have use today.

There are two ways the method described in the previous paragraph—which, for brevity, we will refer to as the Lakmeche–Arino method— fundamentally differs from the one we have advocated for in this chapter. First, we assume that the dynamics have already been reduced to the parameter-dependent centre manifold. Second, and most importantly, we approximate the period map *restricted to the centre manifold* and use the normal form theory for discrete-time maps to detect the bifurcations. With the Lakmeche–Arino method, one obtains an approximation (Taylor expansion) of the period map in the original phase space first and then applies Lyapunov-Schmidt reduction to reduce the dimension. The differences are subtle, but important. Since the Lakmeche–Arino method reduces the dimension of the map using

Lyapunov-Schmidt reduction, more complicated structures such as invariant cylinders cannot be detected. In particular, only bifurcations involving periodic solutions can be detected. An alternative to both the Lakmeche–Arino method and the one we have proposed in this chapter is to approximate the period map and subsequently apply the centre manifold reduction *for maps*. Church and Liu [30] provide a comparison between these two alternative methods for a few specific models.

We wish to point out that although cylinder bifurcations have only appeared in the literature in one publication to date [32] in an analytical context, they have come up in numerical simulations. Shuai and Qingdao [128] studied a three-species food-chain model with impulsive introduction of the middle predator. They used Lakmeche and Arino's method to prove the existence of a nontrivial solution and followed up with some numerical simulations. Their simulations very clearly show an invariant cylinder but the authors did not remark that this structure was the result of a particular bifurcation. Specifically, this cylinder results from a Neimark–Sacker bifurcation in the period map centred at a nontrivial periodic solution. A bit less verbose, it arises from a cylinder bifurcation.

Part III

Singular and Nonsmooth Phenomena

Chapter III.1

Continuous Approximation

In the theory of impulsive dynamical systems, impulses are often interpreted as idealized discrete jumps associated with a process that is continuous in time but occurs on a negligibly small time scale. The intuition is that one can ignore the transient small-time intermediate dynamics and consider only the change in state. It was demonstrated [38, Theorem 3.1] that under fairly mild conditions, this intuition appears to be correct in a pointwise sense, at least for linear systems. That is, under one formulation of an "approximating" continuous ordinary differential equation, taking the associated time-scale parameter to zero yields pointwise convergence to a given solution of the impulsive differential equation.

This intuition breaks down in any neighbourhood of an equilibrium point or periodic solution that has a Floquet multiplier on the unit circle. In this case, it is perhaps surprising that the stability of a given equilibrium point (or periodic orbit) in an impulsive system does not generally carry over to a suitable class of "approximating" ordinary differential equations, regardless of how small a time scale the impulse effect acts on. The interested reader may consult [38, Section 3.4.2] or [36, Section 3.2.1] for two concrete examples. Ultimately, the deficiency is due to a lack of hyperbolicity of the equilibrium point.

This observation has implications for bifurcations in these systems. Indeed, suppose a bifurcation is identified in a parameter-dependent system of impulsive differential equations. If this dynamical system is a model for a real-world process and an approximation to a continuous system with impulse effect acting on a short but nonzero time scale, it would be important to know that this bifurcation can be realized in reality, provided the time scale is small enough. Despite being of fundamental importance to the rigorousness of impulsive differential equations as mathematical models, this question of

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realizability of bifurcations has not yet been addressed. In this work, we will refer to this question as the *realization problem*.

An important **warning** to readers of this chapter: herein, we will be using the convention that solutions of (finite-dimensional) ordinary impulsive differential equations are continuous from the left. Our view is that at the level of continuous approximation, the convention of left-continuity is more physically appropriate—see Sect. III.1.1.5. This should not cause much discomfort, however, since as we have previously demonstrated in Sect. II.1.3 the two solution conventions—left-continuous and right-continuous—are essentially equivalent.

III.1.1 Introduction

We will be considering the situation of a bifurcation from the trivial equilibrium in the periodic impulsive differential equation

$$\dot{x} = f(t, x, \alpha), \qquad t \neq k \in \mathbb{Z} \tag{III.1.1}$$

$$\Delta x = g(t, x, \alpha), \qquad t = k \in \mathbb{Z}, \tag{III.1.2}$$

with $\alpha \in \mathbb{R}^m$ a parameter, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ being sufficiently smooth and periodic with integer period $q \geq 1$ in its first argument, $g : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ being sufficiently smooth and satisfying $g(k + q, \cdot, \cdot) = g(k, \cdot, \cdot)$ for all $k \in \mathbb{Z}$, and $f(t, 0, 0) = g(k, 0, 0) = 0$. Specifically, we assume that (III.1.1) and (III.1.2) are PC^ℓ for some $\ell \geq 1$ to be specified. However, in this context the continuity requirement is reversed.

Definition III.1.1.1. (III.1.1) *and* (III.1.2) *are* PC^ℓ (in the left-continuity sense) *if*

- $D^m_{(2,3)} f(t, x, p)$ *exist for* $m = 0, \dots, k$*, whenever* $(s_n, x_n, p_n) \to (s, x, p)$*, the limit*

$$\lim_{n\to\infty} D^m_{(2,3)} f(s_n, x_n, p_n)$$

  exists and, if s_n *is* ***increasing****, the limit is precisely* $Df^m_{(2,3)} f(s, x, p)$*;*

- g_j *and* t_j *are* C^k *for all* $j \in \mathbb{Z}$*.*

We will also modify somewhat our definition of solution.

Definition III.1.1.2. *A function* $x : [s, \omega) \to \mathbb{R}^n$ *is a* solution *of* (III.1.1)–(III.1.2) *with parameter* α *if it is continuous except at times* $t \in [s, \omega) \cap \{t_k : k \in \mathbb{Z}\}$ *and satisfies the integral equation*

$$x(t) = \begin{cases} x(s) + \int_s^t f(s, x(s), \alpha)ds + \sum_{s<t_k<t} g_k(x(t_k), \alpha), & s \notin \{t_j : j \in \mathbb{Z}\} \\ x(s) + g_j(x(s)) + \int_s^t f(s, x(s), \alpha)ds + \sum_{s<t_k<t} g_k(x(t_k), \alpha), & s = t_j,\ t > s. \end{cases}$$

Under the PC^1 assumption, to any initial condition $(s, x_0) \in \mathbb{R} \times \mathbb{R}^n$, there is a unique solution (in the above sense) $x : [s, \omega) \to \mathbb{R}^n$ satisfying $x(s) = x_0$, for each parameter α. See the related Theorem II.1.1.1 and subsequent remark.

Remark III.1.1.1. *In this chapter, we are taking the impulse times to be the integers. There is actually no loss of generality in doing this, since we can make an appropriate reparameterization of time. Suppose impulses occur at times t_k for $k \in \mathbb{Z}$. Define a reparameterization of time as follows:*

$$t = \{ \ t_k + (\tau - k)(t_{k+1} - t_k), \quad \tau \in (k, k+1], \ k \in \mathbb{Z}$$

for rescaled time τ. Under this rescaling, $t = t_k$ if and only if $\tau = k$. Moreover, $\tau \mapsto t$ is continuous, piecewise-linear and monotone increasing, so it has an inverse with the same properties.

III.1.1.1 Singular Unfolding of an Impulsive Differential Equation

If we wanted to maintain complete generality, we would address the realization problem by considering the functional differential equation

$$\dot{y} = f(t, y, \alpha) + G(t, y_t, \alpha, \epsilon), \tag{III.1.3}$$

where ϵ is a real positive parameter and $G : \mathbb{R} \times \mathcal{C}([-1, 0], \mathbb{R}^n) \times \mathbb{R}^m \times \mathbb{R}^+$ is characterized by the following properties that emphasize the idea that the above system is a continuous version of (III.1.1)–(III.1.2):

- $t \mapsto G(t, \cdot, \cdot, \cdot)$ is periodic with period q.
- The support of $t \mapsto G(t, \cdot, \cdot, \epsilon)$ is a subset of $\bigcup_{k \in \mathbb{Z}} [k - \epsilon, k + \epsilon]$.
- $(\phi, \alpha) \mapsto G(t, \phi, \alpha, \epsilon)$ is smooth.
- If $t \mapsto y(t; \epsilon)$ is any solution of (III.1.3) satisfying $y(t_0; \epsilon) = x_0 \in \mathbb{R}^n$ for all $\epsilon \in (0, 1)$ and some $t_0 \in I = \mathrm{Dom}(y(\cdot; \epsilon))$, and $t \mapsto x(t)$ is the solution of (III.1.1)–(III.1.2) satisfying $x(t_0) = y_0$, then for all integers $k \geq t_0$ such that $[k - \epsilon, k + \epsilon] \subset I$,

$$\lim_{\epsilon \to 0^+} \int_{k-\epsilon}^{k+\epsilon} G(s, y_s(\cdot; \epsilon), \alpha, \epsilon) ds = g(k, x(k), \alpha). \tag{III.1.4}$$

The purpose of G is to serve as a continuous-time replacement for the jump function g from (III.1.1)–(III.1.2). Taking ϵ as an unfolding parameter, this will permit us to study the realization problem. By definition, $\epsilon \mapsto G(t, x, \alpha, \epsilon)$ must have a singularity at $\epsilon = 0$, and for this and the previous reasons, we will call G a *singular unfolding* of g. By abuse of notation, we will also say that the functional differential equation (III.1.3) is a singular unfolding of (III.1.1)–(III.1.2).

To arrive at impulse extension equations [38], for example, one imposes the further constraint that G satisfies

$$G(t,\phi,\alpha,\epsilon)=\mathbb{1}_{\mathcal{S}(\epsilon)}(t)\varphi(t,\phi(\lfloor t\rfloor-t),\alpha,\epsilon)$$
$$\int_k^{k+a_j(\epsilon)}\varphi(s,x,\alpha,\epsilon)ds=g(k,x,\alpha),$$

with $\mathcal{S}(\epsilon)=\cup_{k\in\mathbb{Z}}[k,k+a_k(\epsilon))$ for $a_k(\epsilon)\in(0,1)$ a sequence of durations of impulse effect satisfying $a_k(\epsilon)\to 0$ as $\epsilon\to 0^+$. In that formulation, solutions are taken to be continuous from the left, so it is natural to pose that $t\mapsto G(t,\cdot,\cdot,\epsilon)$ has support in half-open intervals to the right of each impulse time.

To avoid complications associated with the infinite-dimensional phase space $\mathcal{C}([-r,0],\mathbb{R}^n)$ and how the system behaves in the singular limit $\epsilon\to 0^+$, we will mostly focus our attention on a very simple (candidate) singular unfolding:

$$\dot{y}=f(t,y,\alpha)+\frac{1}{\epsilon}\sum_{k=-\infty}^{\infty}g(k,y(k),\alpha)\mathbb{1}_{[k,k+\epsilon)}(t). \tag{III.1.5}$$

This is a specific impulse extension equation. The change in state due to the impulse effect is approximated by a piecewise-constant perturbation to the vector field, parameterized by the time-scale/unfolding parameter ϵ. It is chosen because, as we will see, it is easy to verify the limit property of the singular unfolding (even in the nonlinear setting), and the resulting linearization has nice spectral properties.

It should be remarked that one cannot generally replace $y(k)$ in Eq. (III.1.5) with $y(t)$. The resulting system would generally fail the limit condition of the singular unfolding, as can be demonstrated by way of scalar linear examples. In this sense, (III.1.5) may very well be the simplest general case singular unfolding that can be studied.

There are several equivalent ways of interpreting the dynamical system associated with (III.1.5). For each $\epsilon>0$, it defines a differential equation with discontinuous right-hand side and piecewise-constant arguments. Indeed, one can write $y(k)=y(t-t_k(t))$ for $t_k(t)=t-k$ for $t\in[k,k+1)$, so (III.1.5) is in fact a differential-difference equation with discontinuous right-hand side. Alternatively, one can multiply both sides by ϵ and view (III.1.5) as a singularly perturbed system

$$\epsilon\dot{y}=\epsilon f(t,y,\alpha)+\sum_{k=-\infty}^{\infty}g(k,y(k),\alpha)\mathbb{1}_{[k,k+\epsilon)}(t),$$

with intermittent periods of fast–slow dynamics in the intervals $[k,k+\epsilon)$ and $[k+\epsilon,k+1)$, respectively.

III.1.1.2 Preliminaries

Our first task will be to prove that solutions of singular unfoldings (III.1.3) converge pointwise to those of the impulsive system (III.1.1)–(III.1.2) as the unfolding parameter is taken to zero. This has been proven for unfoldings that coincide with impulse extensions [38], but only for linear systems. This is the subject of Sect. III.1.2. Such an analysis is necessary to support our claim that a singular unfolding of an impulsive differential equation really is a continuous approximation. Following this, we show that the candidate (III.1.5) truly is a singular unfolding of the impulsive differential equation, as claimed.

III.1.1.3 Time q Map

After having proven the pointwise convergence of solutions, we will begin to address the realization problem with respect to the simple (candidate) piecewise-constant unfolding from (III.1.5). We will define the time q map $P : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^n$ that maps an initial condition $x_0 \in \mathbb{R}^n$, system parameter $\alpha \in \mathbb{R}^m$ and unfolding parameter $\epsilon \in \mathbb{R}^+$ at time $t_0 = 0$ to its state at time q, where q is the period of the impulsive system (III.1.1)–(III.1.2). When $\epsilon > 0$, the evolution is governed by the singular unfolding (III.1.5), while if $\epsilon = 0$, it is the impulsive differential equation (III.1.1)–(III.1.2). We will study the smoothness of this map. In particular, we will prove that this map is at the very least C^3. This level of smoothness is sufficient for most typical applications. For instance, the normal form coefficients of all codimension-one bifurcations for iterated maps are determined by the cubic order terms and below. As for codimension-two, C^3 smoothness is sufficient for the analysis of cusp points, fold–flip points as well as strong resonances at Neimark–Sacker points.

III.1.1.4 The Realization Problem

With the conclusion that the time p map is C^3 smooth, we can apply the results of bifurcation theory to this map under the condition that the linearization of $x \mapsto P(x, 0, 0)$ at the fixed point $x = 0$. If the fixed point is hyperbolic, we have unique persistence of the fixed point for $\epsilon > 0$ and $|\alpha|$ sufficiently small from the implicit function theorem. If it is nonhyperbolic, there is generically a bifurcation curve along which the bifurcation of the impulsive system persists for $\epsilon > 0$ small. We study this by way of a few examples.

III.1.1.5 A Brief Discussion on the Continuity Convention

Consider the following "left-sided" functional singular unfolding for a scalar, linear impulsive differential equation:

$$\dot{x} = ax(t) + \sum_{k=-\infty}^{\infty} \mathbb{1}_{[k-\epsilon,k]}(t)b(t,\epsilon)x_t. \tag{III.1.6}$$

If $x(t;\epsilon) : [s-r,\omega) \to \mathbb{R}$ is a solution through a fixed initial condition $(s,x_0) \in \mathbb{R} \times \mathcal{RCR}$ with parameter ϵ, then the limit $x(t;0^+)$ upon taking $\epsilon \to 0^+$ will be an element of $\mathcal{RCR}([s-r,\omega),\mathbb{R})$ provided the functional $b(t,\epsilon)$ is regular enough to guarantee local existence and uniqueness of solutions for $\mathcal{RCR}$ initial data. As such, this form of a singular unfolding is suitable for the investigation of solutions in the right-continuous formalism for impulsive differential equations. However, the support of the forcing term on the right-hand side of (III.1.6) being to the left of the "impulse times" $k \in \mathbb{Z}$ is somewhat at odds with how such models are typically presented. In the formulation of such impulsive models, the impulse effect is usually introduced as follows:

> *There exists a sequence of times t_k at which* ***event*** X_k ***occurs****. These events occur on a time scale that is very small relative to the distance $\Delta t_k = t_{k+1} - t_k$ between events. As such, these events are modelled as occurring instantaneously.*

The question to ask is then, for the real-world process that this model should approximate, does the event X_k *start* at time t_k or *end* at time t_k? In the latter, we have a singular unfolding that looks like (III.1.6), and taking the unfolding parameter to zero imposes a right-continuous solution to the impulsive differential equation. For the former, we get a left-continuous solution. It is our stance that the former interpretation is more reasonable. This rationale comes from the heuristic observation that the phrase *event X_k occurs at time t_k* is nearly synonymous with the phrase *event X_k starts at time t_k*. There is of course some ambiguity if the description of event X_k involves some sort of temporal dependence, but for more coarse events that are equivalent to something of the form

$$X_k = \text{ change state } A(k) \text{ into state } B(k),$$

there is no such implied temporal dependence. For these reasons, we will be employing the left-continuity formalism for solutions of impulsive differential equations in this chapter.

III.1.2 Pointwise Convergence and the Candidate Unfolding

In this section we justify our earlier claim that a singular unfolding provides a continuous approximation of an impulsive differential equation. Following this, we prove that (III.1.5) is indeed a singular unfolding.

Proposition III.1.2.1. *Let $x_0 \in \mathbb{R}^n$. Let (III.1.1)–(III.1.2) be PC^1. Let $y(t;\epsilon)$ be any solution of the singular unfolding (III.1.3) satisfying the initial condition $y(t_0;\epsilon) = x_0$ for $\epsilon \in (0,\epsilon_0)$. Let $x(t)$ be the unique solution of the impulsive differential equation (III.1.1)–(III.1.2) satisfying $x(t_0) = x_0$. Let these solutions be defined on a maximal right-sided interval of existence $I = [t_0,\beta) \subset \mathbb{R}$. For all $t \in I$, we have $\lim_{\epsilon\to 0^+} y(t;\epsilon) = x(t)$.*

Proof. The proof is by strong induction. If $I \cap \mathbb{Z} = \emptyset$, then the result holds trivially. If $I \cap \mathbb{Z} = \{k\}$, then the limit is trivially true for $t \in [t_0,k]$. For any $\mu \geq 0$ such that $k+\mu \in I$ and $\epsilon > 0$ sufficiently small,

$$\begin{aligned} y(k+\mu+\epsilon;\epsilon) - x(k+\mu+\epsilon) &= \int_k^{k+\mu+\epsilon} \big[f(s,y(s;\epsilon)) - f(s,x(s))\big]ds \\ &\quad + \int_{k-\epsilon}^{k+\epsilon} \Big[G(s,y_s(\cdot;\epsilon),\epsilon) - \frac{1}{2\epsilon}g(k,x(k))\Big]ds, \end{aligned}$$

and we have suppressed the dependence on α. The condition on the maximal interval implies $\epsilon \mapsto y(t;\epsilon)$ is uniformly bounded on $[t_0,k]$. Taking this into account with the smoothness of f and the limit property (III.1.4) of the singular unfolding, it follows that we can take $\epsilon \to 0^+$ uniformly in μ. The result is the equation

$$y(k+\mu;0^+) - x(k+\mu) = \int_k^{k+\mu} \big[f(s,y(s;0^+)) - f(s,x(s))\big]ds.$$

The smoothness conditions on f and boundedness of $\epsilon \mapsto y(t;\epsilon)$ together with the above characterization imply that $\mu \mapsto y(k+\mu;0^+)$ is continuous. Defining the difference $h(\mu) = y(k+\mu;0^+) - x(k+\mu)$, it follows that for some local Lipschitz constant $L > 0$,

$$|h(\mu)| \leq L\int_0^{\mu} |h(s)|ds,$$

and applying Gronwall's inequality, it follows that $h = 0$. We conclude that $\lim_{\epsilon\to 0^+} y(t;\epsilon) = x(t)$ for all $t \in I$. Suppose now that $I \cap \mathbb{Z} = \{k_1,\dots,k_j\}$. Taking the intersection $I_1 = I \cap (-\infty,k_2)$, it follows from the base case that the limit relation holds on I_1. Taking the induction hypothesis that the limit relation holds for some $I_\ell = I \cap (-\infty,k_{\ell+1})$, one can prove in a similar manner that the limit relation holds on $I_{\ell+1}$, and it follows that the result holds true for $I \cap \mathbb{Z} = \{k_1,\dots,k_j\}$ for any j finite. The infinite case follows easily. □

Proposition III.1.2.2. *Let* (III.1.1)–(III.1.2) *be* PC^1. *Then,* (III.1.5) *is a singular unfolding.*

Proof. Similarly to Proposition III.1.2.1, we prove the result by induction. The first nontrivial case is when $I \cap \mathbb{Z} = \{k\}$ is nonempty. We have

$$\int_{k-\epsilon}^{k+\epsilon} G(k, y_s(\cdot;\epsilon), \epsilon))ds = \int_{k}^{k+\epsilon} \frac{1}{\epsilon} g(k, y(k;\epsilon))ds$$
$$= g(k, y(k;\epsilon)) = g(k, x(k)),$$

as required. By Proposition III.1.2.1, it follows that $y(t;\epsilon) \to x(t)$ for all $t \in I$. Take as an inductive hypothesis that for some $j \geq 1$, (III.1.4) holds for the integers $\{k_1, \ldots, k_j\} = I \cap (-\infty, k_j] \cap \mathbb{Z}$ and that $[k_{j+1} - \epsilon, k_{j+1} + \epsilon] \subset I$ for all ϵ sufficiently small. By Proposition III.1.2.1, the limit $y(t;\epsilon) \to x(t)$ holds for all $t \in I \cap (-\infty, k_{j+1}]$. Then,

$$\int_{k_{j+1}-\epsilon}^{k_{j+1}+\epsilon} G(k_{j+1}, y_s(\cdot;\epsilon), \epsilon)ds = \int_{k_{j+1}}^{k_{j+1}+\epsilon} \frac{1}{\epsilon} g(k_{j+1}, y(k_{j+1});\epsilon))ds$$
$$= g(k_{j+1}, y(k_{j+1};\epsilon)).$$

From the continuity of g, it follows that the above approaches $g(k_{j+1}, x(k_{j+1}))$ as $\epsilon \to 0$. The result follows by induction. □

III.1.3 Smoothness of the Time q Map

Throughout this section, we will assume that (III.1.1)–(III.1.2) be PC^ℓ for some $\ell \geq 1$. To define the time q map for unfolding parameter ϵ, it is useful to first denote $t \mapsto \Phi(t, s, x_0, \alpha) \equiv \Psi(t, s, \alpha)x_0$ the solution of the ordinary differential equation $\dot{x} = f(t, x, \alpha)$ satisfying the initial condition $x(s) = x_0$. Also, let $\Psi(t, k, x_0, \alpha, \epsilon) \equiv \Psi(t, k, \alpha, \epsilon)x_0$ denote the solution of the singular unfolding (III.1.5) satisfying the initial condition $y(k) = x_0$ for $k \in \mathbb{Z}$. Finally, define the *switching functions* $S_k : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^n$ by

$$S_k(x, \alpha, \epsilon) \equiv S_k(\alpha, \epsilon)x = \chi_{(0,1)}(\epsilon)\Psi(k + \epsilon, k, \alpha, \epsilon)x + \chi_{\{0\}}(\epsilon)(x + g(k, x, \alpha)).$$

We can succinctly write the time q map as follows:

$$P(x, \alpha, \epsilon) = \left(\prod_{k=0}^{q-1} \Phi(k + 1, k + \epsilon, \alpha) S_k(\alpha, \epsilon) \right) x, \qquad \text{(III.1.7)}$$

where the product denotes composition from right to left:

$$\prod_{k=0}^{q-1} A_k = A_{q-1} \circ \cdots \circ A_0.$$

Because of the PC^ℓ assumption, the C^ℓ smoothness of the time q map is equivalent to that of the switching functions $S_k(\cdot,\cdot)$ for $k = 0,\ldots,q-1$.

At this stage, we will suppress all dependence on the parameter α. There is no loss of generality in doing so because one can always use the typical trick of expanding the state space to $\mathbb{R}^n \times \mathbb{R}^m \ni (x,\alpha)$. With this in mind, the switching functions admit the following integral representation, which will be helpful later:

$$S_k(x,\epsilon) = x + g(k,x) + \chi_{(0,1)}(\epsilon)\int_k^{k+\epsilon} f(s,\Psi(s,k,\epsilon)x)ds. \tag{III.1.8}$$

The first result we need is a proposition concerning uniform bounds on the solution of the singular unfolding and its Fréchet derivatives in a neighbourhood of $x = 0$.

Proposition III.1.3.1. *There exist positive constants C,δ and ρ such that for $|x| < \delta$ and $\epsilon < \rho$, one has, for $j = 0,\ldots,\ell$,*

$$||D^j\Psi(t,k,x,\epsilon)|| \leq C, \qquad t \in [k,k+\epsilon]. \tag{III.1.9}$$

Proof. We proceed by strong induction on j. For $j = 0$, we first show that $t \mapsto \Psi(t,k,x,\epsilon)$ is a fixed point of the nonlinear operator $F_0 : X^\delta_{\epsilon,0} \to X^\delta_{\epsilon,0}$ with datum

$$F_0\phi(t) = x + \frac{1}{\epsilon}\int_k^t g(k,x)ds + \int_k^t f(s,\phi(s))ds$$
$$X^\delta_{0,\epsilon} = \left\{\phi \in \mathcal{C}([k,k+\epsilon],\mathbb{R}^n\times\mathbb{R}^m) : \left|\phi(t) - x - \frac{t-k}{\epsilon}g(k,x)\right| < \delta\right\},$$

for appropriately chosen δ. To accomplish this, let $\eta(\delta) > 0$ be small enough so that for $|x| < \eta(\delta)$, we have $|x| + |g(k,x)| < \delta$. It follows that if $\phi \in X^\delta_{0,\epsilon}$ and $|x| < \eta(\delta)$, then $||\phi|| \leq 2\delta$. If we denote by L_μ a uniform (in t) Lipschitz constant for $f(t,\cdot)$ on the ball $B_\mu(0)$ in $\mathbb{R}^n \times \mathbb{R}^m$, one can check that F_0 is well-defined and, similarly, a contraction provided $\epsilon < 1/(2L_{2\delta})$. It follows that $\sup_{t\in[k,k+\epsilon]}|\Psi| \leq 2\delta$ provided $|x| < \eta(\delta)$ and $\epsilon < 1/(2L_{2\delta})$.

For $j = 1$, we define the nonlinear operator $F_1 : X^\epsilon_{\delta,1} \to X^\epsilon_{\delta,1}$ with

$$F_1\phi(t) = I + \frac{1}{\epsilon}\int_k^t Dg(x,k)ds + \int_s^t Df(s,\Psi(s))\phi(s)ds$$
$$X^\epsilon_{\delta,1} = \left\{\phi \in \mathcal{C}([k,k+\epsilon],\mathcal{L}(\mathbb{R}^n\times\mathbb{R}^m,\mathbb{R}^n)) : \left|\phi(t) - I - \frac{t-k}{\epsilon}Dg(k,x)\right| < \delta\right\}.$$

$D\Psi(t)$ is fixed point of F_1, which is well-defined and a contraction provided

$$\epsilon < \frac{1}{2L^1_{2\delta+1}(2\delta+1)},$$

where L^1_μ is the uniform Lipschitz constant for $Df(t,\cdot,\cdot)$ on $B_\mu(0)$. In this case, $\sup_{t\in[k,k+\epsilon]}|D\Psi(t)|\leq 2\delta+1$ provided $|x|<\eta_1(\delta)$, where the latter is small enough to guarantee that $||Dg(k,x)||\leq\delta$ for $|x|<\eta_1(\delta)$.

Suppose the conclusion holds for some $j\geq 1$. $\phi(t)=D_x^{j+1}\Psi(t,k,x,\epsilon)$ satisfies the fixed-point equation

$$\phi(t)=\int_k^t DD^j[f(s,\Psi(s)]ds+\frac{1}{\epsilon}\int_k^t D^{j+1}g(k,x)ds.$$

From the chain rule and the induction hypothesis, it follows that the above can be written in the form

$$\phi(t)=\int_k^t\left([Df(s,\Psi(s))]\phi(s)+R(s)\right)ds+\frac{t-k}{\epsilon}D_x^{j+1}g(k,x),$$

where $R(s)$ is a term that is uniformly bounded on $[k,k+\epsilon]$ for all $|x|<\eta$, for all ϵ sufficiently small, and contains all terms $D^r\Psi(s)$ for $r\leq j$. A similar fixed-point setup to before then yields the desired result. □

We begin by proving the continuity of the switching functions before moving onto smoothness. For brevity, let us denote

$$\Omega^+=\mathbb{R}^n\times(0,1),\qquad \Omega_0=\mathbb{R}^n\times\{0\},\qquad \Omega=\Omega^+\cup\Omega_0.$$

For a function $F:\Omega\to\mathbb{R}^n$ and $U\subset\Omega$, we will say that F is C^ℓ on U if it is C^ℓ on $U\cap\Omega^+$ and it is ℓ times Gateaux differentiable on U in the direction (x,ϵ) for all $x\in\mathbb{R}^n$ and $\epsilon>0$. This effectively means that along the boundary Ω_0, derivatives from the right with respect to ϵ are well-defined, continuous and commute with derivatives in x for all mixed order up to ℓ.

Lemma III.1.3.1. *The switching functions are continuous in a neighbourhood of* $0\in\Omega$*.*

Proof. The switching functions are clearly continuous on each of Ω^+ and Ω_0. So, let $(y,0)\in\Omega$. Without loss of generality, assume $\epsilon\neq 0$. Then, using Proposition III.1.3.1, we can get the estimate

$$|S_k(0)y-S_k(\epsilon)x|\leq|y+g(k,y)-(x+g(k,x))|+\epsilon CL_C$$

for some $C>0$, provided $\epsilon<\rho$ and $|x|<\delta$, where L_C is the uniform Lipschitz constant for $f(t,\cdot)$ on the ball $B_C(0)\subset\mathbb{R}^n$. Continuity follows upon taking $(x,\epsilon)\to(y,0)$. □

Lemma III.1.3.2. *The switching functions are continuously differentiable in a neighbourhood of* $0\in\Omega$*.*

Proof. We will prove that the partial derivatives D_xS_k and $D_\epsilon S_k$ exist and are continuous in a neighbourhood of $0\in\Omega$. For the latter, we will rather prove the sufficient condition that $D_\epsilon S_k$ is continuous on Ω^+ and can be uniquely extended to the boundary Ω_0 provided $|x|$ is small enough. Beginning with DS_k, one can verify from the $j = 1$ case of the proof of Proposition III.1.3.1 and Eq. (III.1.8) that we can write the derivative as

$$D_x S_k = I + Dg + \chi_{(0,1)}(\epsilon) \int_k^{k+\epsilon} Df(s,\Psi)D\Psi ds, \tag{III.1.10}$$

for any (x,ϵ), where we have suppressed several function inputs in Ψ and g. By Proposition III.1.3.1, the integrand is uniformly bounded for $|x|$ sufficiently small, and it follows that $(x,\epsilon) \mapsto DS_k(x,\epsilon)$ is continuous in a neighbourhood of $0 \in \Omega$.

Next, we note that for $(x,\epsilon) \in \Omega^+$, we have

$$D_\epsilon S_k(x,\epsilon) = f(k+\epsilon, \Psi(k+\epsilon,k,x,\epsilon)) + \int_k^{k+\epsilon} Df(s,\Psi)D_\epsilon\Psi(s,k,x,\epsilon)ds, \tag{III.1.11}$$

where $u(s) = u(s,k,x,\epsilon) = D_\epsilon\Psi(s,k,x,\epsilon)$ is the solution of the initial-value problem

$$\dot{u} = Df(t,\Psi(t,k,x,\epsilon))u(t) - \frac{1}{\epsilon^2}g(k,x), \quad t \in [k,k+\epsilon], \quad u(k) = 0.$$

From this, it follows that $D_\epsilon S_k$ is continuous on Ω^+. To deal with the limits on the boundary, we will first study the limiting behaviour of $f(k+\epsilon, \Psi(k+\epsilon,k,x,\epsilon)$ near $\epsilon = 0$. From the regularity assumptions on f and the functional form of $\Psi(k+\epsilon,k,x,\epsilon) = S_k(x,\epsilon)$, we can write

$$\begin{aligned} &f(k+\epsilon,\Psi(k+\epsilon,k,x,\epsilon)) \\ &\quad = f(k^+, x+g(k,x)) + f'(k^+,x+g(k,x))\epsilon \\ &\qquad + Df(k^+,x+g(k,x))\int_k^{k+\epsilon} f(s,\Psi(s,k,x,\epsilon))ds + O(||(x,\epsilon)||^2). \end{aligned} \tag{III.1.12}$$

As $f(s,\Psi)$ is uniformly bounded for $|x|$ small enough, (III.1.12) converges to $f(k^+,x+g(k,x))$ as $\epsilon \to 0$.

Working with the integral term in (III.1.11) is a bit more subtle. First, we need to compute $D_\epsilon\Psi$ explicitly. $y(s) = D_\epsilon\Psi(s,k,x,\epsilon)$ for $s \in [k,k+\epsilon]$ satisfies the integral equation

$$y(s) = \int_k^s Df(s,\Psi)y(s) - \frac{1}{\epsilon^2}g(k,x)ds,$$

which implies the solution

$$D_\epsilon\Psi(s,k,x,\epsilon) = -\frac{1}{\epsilon^2}\int_k^s X(s,\mu;x,\epsilon)g(k,x)d\mu, \tag{III.1.13}$$

$$X(s,\mu;x,\epsilon) = I + \int_\mu^s Df(r,\Psi(r,k,x,\epsilon))X(r,\mu;x,\epsilon)dr. \tag{III.1.14}$$

Using these expressions and Proposition III.1.3.1, for $|x|$ small enough, we can write

$$\int_k^{k+\epsilon} Df(s,\Psi)D_\epsilon\Psi ds$$
$$= -\frac{1}{\epsilon^2}\int_k^{k+\epsilon} Df(s,\Psi)\left[(s-k)I + \int_k^s\int_\mu^s Df(r,\Psi)drd\mu\right] gds + O(\epsilon^2), \tag{III.1.15}$$

where the $O(\epsilon^2)$ remainder terms are uniform with respect to $|x|$ small enough. Next we exploit the regularity of f to get the expansion

$$Df(s,\Psi) = Df\left(s, x+\frac{s-k}{\epsilon}g\right) + D^2f\left(s,x+\frac{s-k}{\epsilon}g\right)\left[\int_k^s f(r,\Psi)dr, I\right] + O(\epsilon^2).$$

Note that the $O(\epsilon^2)$ terms come from the squared norm of $\int_k^s f(r,\Psi)dr = O(\epsilon)$. Next, with ϵ small enough and $s\in[k,k+\epsilon]$, we can go one step further to get

$$Df(s,\Psi) = Df\left(k^+, x+\frac{s-k}{\epsilon}g\right) + (s-k)Df'\left(k^+, x+\frac{s-k}{\epsilon}g\right)$$
$$+ D^2f\left(k^+, x+\frac{s-k}{\epsilon}g\right)\left[\int_k^s f(r,\Psi)dr, I\right] + O(\epsilon^2).$$

Substituting into (III.1.15) and performing a few change of variables, we ultimately end up with

$$\int_k^{k+\epsilon} Df(s,\Psi)D_\epsilon\Psi ds$$
$$= -\epsilon\int_0^1 vD^2f(k^+,x+vg)\left[\int_0^v f(k^+,x+ug)du, g\right]dv - \int_0^1 Df(k^+,x+vg)vgdv$$
$$-\epsilon\int_0^1\left(\frac{1}{2}v^2Df(k^+,x+vg)g + v^2Df'(k^+,x+vg)\right)gdv + O(\epsilon^2). \tag{III.1.16}$$

From these observations, we conclude that for any $(x_n,\epsilon_n)\in\Omega^+$ with $(x_n,\epsilon_n)\to(x,0)\in\Omega_0$ and $|x|$ sufficiently small, the limit $\lim_{n\to\infty} D_\epsilon S_k(x_n,\epsilon_n) := D_\epsilon S_k(x,0)$ exists and depends only on $(x,0)$. □

Lemma III.1.3.3. *Let* (III.1.1)–(III.1.2) *be* PC^ℓ *for* $\ell\geq 2$. *The switching functions are* C^2 *in a neighbourhood of* $0\in\Omega$.

Proof. We proceed in a similar manner to the proof of C^1 smoothness. First, it is easily verified that D^2S_k exists and is equal to

$$D^2S_k(x,\epsilon) = D^2g(x) + \chi_{(0,1)}(\epsilon)\int_k^{k+\epsilon} D^2f(s,\Psi)[D\Psi,D\Psi] + Df(s,\Psi)D^2\Psi ds. \tag{III.1.17}$$

Similarly to C^1 smoothness, the above is continuous in a neighbourhood of $0 \in \Omega$. For the partial derivatives in ϵ, we can start from (III.1.11) to verify that $D_x D_\epsilon S_k(x, \epsilon)$ exists and is continuous for $(x, \epsilon) \in \Omega^+$. Then, we use the results from (III.1.12) and (III.1.16)

$$\begin{aligned} D_x D_\epsilon S_k &= Df(k^+, x+g)[I+Dg] + Df'(k^+, x+g)[I+Dg] \\ &\quad - \int_0^1 (D^2 f(k^+, x+vg)[I+vDg, vg] + Df(k^+, x+vg)vDg)dv + O(\epsilon), \end{aligned} \tag{III.1.18}$$

for $|x|$ small enough, which implies that the limit $D_x D_\epsilon S_k(x_n, \epsilon_n) \to D_x D_\epsilon S_k(x, 0)$ exists as $(x_n, \epsilon_n) \to (x, 0)$ and depends only on the argument x. Since one can easily check that $D_\epsilon D_x S_k(x, \epsilon) = D_x D_\epsilon S_k(x, \epsilon)$ for $(x, \epsilon) \in \Omega^+$, the other mixed partial derivative is similarly continuous on Ω and coincides with the first. As for $D_\epsilon^2 S_k(x, \epsilon)$, it is continuous for $(x, \epsilon) \in \Omega^+$ as can be inferred from (III.1.11). Near the boundary Ω_0, one may use (III.1.12) and (III.1.16) to show that $\lim_{n\to\infty} D_\epsilon^2 S_k(x_n, \epsilon_n) := D_\epsilon^2 S_k(x, 0)$ exists for $\Omega^+ \ni (x_n, \epsilon_n) \to (x, 0)$ and depends only on x sufficiently small. $\square$

Theorem III.1.3.1. *Let* (III.1.1)–(III.1.2) *be* PC^ℓ *for* $\ell \geq 3$. *The switching functions are* C^3 *in a neighbourhood of* $0 \in \Omega$. *The third-order Taylor polynomial of* S_k *at* $0 \in \Omega$ *is*

$$\begin{aligned} S_k(x, \epsilon) &= (I+Dg)x + \frac{1}{2}D^2 g[x,x] + \frac{1}{6}D^3 g[x,x,x] + \epsilon Df \cdot \left(I + \frac{1}{2}Dg\right)x \\ &\quad + \frac{1}{2}\epsilon\left(D^2 f[I, I+Dg] + \frac{1}{3}D^2 f[Dg, Dg] + \frac{1}{2}DfD^2 g\right)[x,x] \\ &\quad + \frac{1}{2}\epsilon^2\left(Df \cdot Df \cdot \left(I + \frac{1}{3}Dg\right) + Df' \cdot \left(I + \frac{2}{3}Dg\right)\right)x, \end{aligned}$$

where all function evaluations involving f *are at* $(k^+, 0)$, *and those involving* g *are at* $(k, 0)$.

Proof. The C^2 case was proven in Lemma III.1.3.3. The proof of C^3 smoothness is similar and omitted. We will, however, provide a fair bit of detail to demonstrate the computation of the cubic order terms. The first three terms come from the expansion of $x \mapsto x + g(k, x)$. For the order ϵ term, we can directly use Lemma III.1.3.2. Since $f'(\cdot, 0) = f(\cdot, 0) = g(\cdot, 0) = 0$, this term vanishes. The same is true of the second-order term ϵ^2. For the mixed term ϵx, evaluating (III.1.18) at $(x, \epsilon) = (0, 0)$ gives us the derivative

$$D_\epsilon D_x S_k(0,0) = Df \cdot (I + Dg) - \int_0^1 Df \cdot vDgdv = Df\left(I + \frac{1}{2}Dg\right),$$

which is equivalent to the term in the Taylor polynomial.

The cubic term ϵ^3 is obtained from $D_\epsilon^3 S_k(0,0)$. Starting from (III.1.11), taking two derivatives in ϵ and limits yields

$$\begin{aligned}
&D_\epsilon^3 S_k(0,0) \\
&\quad = f'' + 2Df' D_\epsilon S_k + D^2 f[D_\epsilon S_k, S_\epsilon S_k] + Df' D_\epsilon \Psi(k,k,0,0) \\
&\qquad + D^2 f[D_\epsilon S_k, D_\epsilon \Psi(k,k,0,0)] + Df \cdot \lim_{s\to 0^+} \frac{d}{ds} D_\epsilon \Psi(k+s,k,0,s) \\
&\qquad + D^2 f[D_\epsilon \Psi(k,k,0,0), D_\epsilon \Psi(k,k,0,0)] + Df \cdot D_\epsilon \Psi(k,k,0,0) \\
&\qquad + \lim_{t\to 0^+} \int_k^{k+t} D^3 f(s,0)[D_\epsilon \Psi, D_\epsilon \Psi, D_\epsilon \Psi] + 3D^2 f(s,0)[D_\epsilon \Psi, D_\epsilon^2 \Psi] + Df(s,0) D_\epsilon^3 \Psi ds,
\end{aligned}$$

where $D_\epsilon \Psi(s,k,x,\epsilon)$ is defined in Eq. (III.1.13). Taking note that $D_\epsilon(s,k,0,\epsilon) = 0$, many terms vanish. Also, one can similarly check that $D_\epsilon^j \Psi(s,k,0,\epsilon) = 0$ for any $j \geq 1$. Since $f''(\cdot,0) = 0$ and $D_\epsilon S_k(0,0) = 0$, we conclude $D_\epsilon^3 S_k(0,0) = 0$.

In a similar way, we can formally write down the mixed partial derivative $D_x^2 D_\epsilon S_k(0,0)$ as

$$\begin{aligned}
D_x^2 D_\epsilon S_k(0,0) = {} & D^2 f[I + Dg, I + Dg] + Df \cdot D^2 g \\
& + \lim_{\epsilon\to 0^+} \left(\int_k^{k+\epsilon} D^2 f(s,0)[D_x \Psi, D_x \Psi, D_\epsilon \Psi] ds \right. \\
& + \int_k^{k+\epsilon} D^2 f(s,0)[D_x^2 \Psi, D_\epsilon \Psi] + 2D^2 f(s,0)[D_x \Psi, D_x D_\epsilon \Psi] ds \\
& \left. + \int_k^{k+\epsilon} Df(s,0) D_x^2 D_\epsilon \Psi ds \right).
\end{aligned}$$

Again, many terms vanish, and we are left with

$$\begin{aligned}
D_x^2 D_\epsilon S_k(0,0) = {} & D^2 f[I + Dg, I + Dg] + Df \cdot D^2 g \\
& + \lim_{\epsilon\to 0^+} \int_k^{k+\epsilon} 2D^2 f(s,0)[D_x \Psi, D_x D_\epsilon \Psi] + Df(s,0) D_x^2 D_\epsilon \Psi ds.
\end{aligned} \tag{III.1.19}$$

We need to express each of $D_x \Psi$, $D_x D_\epsilon \Psi$ and $D_x^2 D_\epsilon \Psi$ in terms of the functions f and g. With $x = 0$, these functions satisfy for $t \in [k, k+\epsilon]$ the integral equations

$$\begin{aligned}
D_x \Psi(t) &= I + \int_k^t Df(s,0) D_x \Psi(s) + \frac{1}{\epsilon} Dg(k,0) ds, \\
D_x D_\epsilon \Psi(t) &= \int_k^t \left(Df(s,0) D_x D_\epsilon \Psi(s) - \frac{1}{\epsilon^2} Dg(k,0) \right) ds, \\
D_x^2 D_\epsilon \Psi(t) &= \int_k^t \left(2D^2 f(s,0)[D_x \Psi, D_x D_\epsilon \Psi] + Df(s,0) D_x^2 D_\epsilon \Psi(s) - \frac{1}{\epsilon^2} D^2 g(k,0) \right) ds.
\end{aligned}$$

In terms of the function $X(s,\mu) = X(s,\mu,0,\epsilon) = X(s,\mu,0,0)$ from (III.1.14), they can be written as

$$
\begin{aligned}
D_x\Psi(t) &= X(t,k)\left[I + \int_k^t X(t,s)Dg(k,0)\frac{1}{\epsilon}ds\right],\\
D_xD_\epsilon\Psi(t) &= -\frac{1}{\epsilon^2}\int_k^t X(t,s)Dg(k,0)ds,\\
D_x^2D_\epsilon\Psi(t) &= \int_k^t 2D^2f(s,0)\left[X(s,k) + \int_k^s X(s,\mu)Dg(k,0)\frac{1}{\epsilon}d\mu\,,\right.\\
&\qquad\left. -\frac{1}{\epsilon^2}\int_k^s X(t,\mu)Dg(k,0)d\mu\right]ds\\
&\qquad -\frac{1}{\epsilon^2}\int_k^t X(t,s)D^2g(k,0)ds.
\end{aligned}
$$

The integral $\int_k^{k+\epsilon} 2D^2f(s,0)[D_x\Psi, D_xD_\epsilon\Psi]ds$ can then be written as

$$
\begin{aligned}
&-\frac{2}{\epsilon^2}\int_k^{k+\epsilon} D^2f(s,0)\left[X(s,k) + \frac{1}{\epsilon}\int_k^s X(s,\mu)Dg(k,0)d\mu, \int_k^s X(s,\mu)Dg(k,0)d\mu\right]ds\\
=&-\frac{2}{\epsilon^2}\int_k^{k+\epsilon}\left(D^2f(s,0)\left[I + \frac{s-k}{\epsilon}Dg(k,0), (s-k)Dg(k,0)\right] + O(\epsilon^2)\right)ds\\
=&-\frac{2}{\epsilon^2}\int_k^{k+\epsilon}(s-k)D^2f(k^+,0)[I, Dg(k,0)]\\
&+\frac{(s-k)^2}{\epsilon}D^2f(k^+,0)[Dg(k,0), Dg(k,0)]ds + O(\epsilon)\\
=&-D^2f(k^+,0)[I, Dg(k,0)] - \frac{2}{3}D^2f(k^+,0)[Dg(k,0), Dg(k,0)] + O(\epsilon).
\end{aligned}
$$

Similarly, we can compute the integral $\int_k^{k+\epsilon} Df(s,0)D_x^2D_\epsilon\Psi ds$ by estimating it as

$$
\begin{aligned}
&\int_k^{k+\epsilon} Df(s,0)D_x^2D_\epsilon\Psi ds\\
&\quad= \int_k^{k+\epsilon} Df(s,0)\int_k^s X(s,t)\left[O\left(\frac{1}{\epsilon}\right) - \frac{1}{\epsilon^2}D^2g(k,0)\right]dtds\\
&\quad= -\int_k^{k+\epsilon} Df(s,0)\frac{s-k}{\epsilon^2}D^2g(k,0)ds + O(\epsilon)\\
&\quad= -\frac{1}{2}Df(k^+,0)D^2g(k,0) + O(\epsilon).
\end{aligned}
$$

Collecting the above results and substituting into (III.1.19), it follows that

$$
\begin{aligned}
D_x^2D_\epsilon S_k(0,0) &= D^2f[I+Dg, I+Dg] + Df\cdot D^2g - D^2f[I, Dg]\\
&\qquad - \frac{2}{3}D^2f[Dg,Dg] - \frac{1}{2}Df\cdot D^2g\\
&= D^2f[I, I+Dg] + \frac{1}{3}D^2f[Dg,Dg] + \frac{1}{2}DfD^2g,
\end{aligned}
$$

as claimed by the Taylor polynomial.

The last thing to check is the $\epsilon^2 x$ coefficient. After taking into account the vanishing terms in the same way as for the previous coefficients, we are left with

$$D_x D_\epsilon^2 S_k(0,0) = \lim_{\epsilon\to 0^+} R(\epsilon) + Df' \cdot (I + Dg) + Df \cdot D_x D_\epsilon S_k(0,0),$$
$$R(\epsilon){:=}Df(k+\epsilon,0)D_x D_\epsilon \Psi(k{+}\epsilon,k,0,\epsilon){+}\int_k^{k+\epsilon} Df(s,\Psi)D_x D_\epsilon^2 \Psi ds.$$

We know that $D_x D_\epsilon S_k(0,0) = Df(I + \frac{1}{2}Dg)$. As for the terms inside the limit, they can be written more explicitly as

$$\begin{aligned}
R(\epsilon) &= -\frac{1}{\epsilon^2} Df(k+\epsilon,0)\int_k^{k+\epsilon} X(k+\epsilon,s)Dg(k,0)ds \\
&\quad + \frac{2}{\epsilon^3}\int_k^{k+\epsilon} Df(s,0)\int_k^s X(s,t)Dg(k,0)dtds \\
&= -\frac{1}{\epsilon^2}(Df(k^+,0)+\epsilon Df'(k^+,0))\left(\epsilon I + \int_k^{k+\epsilon}\int_k^s Df(k^+,0)dtds\right)Dg(k,0) \\
&\quad + \frac{2}{\epsilon^3}\int_k^{k+\epsilon}(Df(k^+,0)+Df'(k^+,0)(s-k)) \\
&\quad \times \left((s-k)I + \int_k^s\int_t^s Df(k^+,0)dudt\right)ds + O(\epsilon) \\
&= -\frac{1}{\epsilon}DfDg - Df'Dg - \frac{1}{\epsilon^2}[Df]^2Dg\int_k^{k+\epsilon}(s-k)ds + \frac{2}{\epsilon^3}\int_k^{k+\epsilon}(s-k)DfDgds \\
&\quad + \frac{2}{\epsilon^3}\int_k^{k+\epsilon}(s-k)^2Df'Dg + [Df]^2Dg\int_k^s(s-t)dtds + O(\epsilon) \\
&= \left(-\frac{1}{6}Df\cdot Df - \frac{1}{3}Df'\right)Dg + O(\epsilon),
\end{aligned}$$

where starting from the third equality we have suppressed the (constant) inputs on f and g. In total, we get

$$\begin{aligned}
D_x D_\epsilon^2 S_k(0,0) &= -\frac{1}{6}Df\cdot Df\cdot Dg - \frac{1}{3}Df'\cdot Dg \\
&\quad + Df'\cdot(I+Dg) + Df\cdot Df\left(I + \frac{1}{2}Dg\right) \\
&= Df\cdot Df\cdot\left(I+\frac{1}{3}Dg\right) + Df'\cdot\left(I+\frac{2}{3}Dg\right),
\end{aligned}$$

as claimed by the associated coefficient of the Taylor polynomial. □

Corollary III.1.3.1. *Let* (III.1.1)–(III.1.2) *be* PC^ℓ *for* $\ell \geq 3$. *The time* q *map is* C^3 *in a neighbourhood of* $0 \in \Omega$.

We would conjecture that for any $\ell \geq 0$, the time q map is C^ℓ in a neighbourhood of $0 \in \Omega$ provided (III.1.1)–(III.1.2) is PC^ℓ. Proving it would seem a technical exercise in keeping track of the singular terms ϵ^{-k} that result from differentiating the switching functions. We do not attempt to prove such a result.

III.1.4 Sensitivity and Realization

From Theorem 3.1, we can write the switching function in the form

$$S(x,\epsilon) = x + g_3(k,x) + \epsilon r_2(k,x) + O(||(x,\epsilon)||^4),$$

where $g_3(k,\cdot)$ is the degree three Taylor polynomial of $g(k,\cdot)$ at zero, and $r_2(k,x) = O(x)$. The explicit form of the perturbation (i.e. no isolated ϵ terms) guarantees that the orbit structure associated with any generic bifurcation of the fixed point x^* in the iterated map

$$x \mapsto P(x,\alpha,\epsilon)$$

for x^* a nonhyperbolic fixed point for parameter $\alpha = \alpha^*$ and $\epsilon = 0$ is not affected by small perturbations in ϵ. In other words,

> *Under the reasonable smoothness assumptions, any generic bifurcation of a fixed point (or periodic solution) for the periodic impulsive system*
>
> $$\begin{aligned} \dot{x} &= f(t,x,\alpha), & t &\neq k \in \mathbb{Z} \\ \Delta x &= g(t,x,\alpha), & t &= k \in \mathbb{Z}, \end{aligned}$$
>
> *is locally realizable under the piecewise-constant unfolding*
>
> $$\dot{y} = f(t,y,\alpha) + \frac{1}{\epsilon}\sum_{k=-\infty}^{\infty} g(k,y(k),\alpha)\mathbb{1}_{[k,k+\epsilon)}(t).$$
>
> *That is, there exists a unique smooth curve $\epsilon \mapsto (x^*(\epsilon),\alpha^*(\epsilon))$ such that $x^*(\epsilon)$ is a nonhyperbolic fixed point of the time q map $x \mapsto P(x,\alpha^*(\epsilon),\epsilon)$ for all $\epsilon \geq 0$ sufficiently small, and the maps are topologically equivalent.*

With realizability dealt with, the next issue to consider is sensitivity. In many mathematical models involving impulsive differential equations, the impulse effect corresponds to a control designed to drive the system towards a particular equilibrium point of the continuously evolving part

$$\dot{x} = f(t,x,\alpha)$$

of the dynamical system. In these instances, the reference fixed point x^* is known and can without loss of generality be treated as the origin. We are interested in answering two questions:

1. (Sensitivity of stability) How is the stability of the fixed point x^* affected in the regime $0 < \epsilon \ll 1$?
2. (Sensitivity of bifurcation) If x^* is nonhyperbolic at some parameter $\alpha^*(0)$, how does variation of the unfolding parameter ϵ alter the critical bifurcation parameter $\alpha^*(\epsilon)$?

These questions are intrinsically linked. We begin with an assumption that will simplify the analysis somewhat.

Assumption 1. *The C^3 smoothness assumption of Theorem III.1.3.1 is satisfied, each of $f(t,\cdot,\cdot)$ and $g(k,\cdot,\cdot)$ is periodic with period 1 and there exists a periodic solution $t \mapsto x^*(t,\alpha)$ of (III.1.1)–(III.1.2) such that $x^*(0,\alpha) = 0$ and $g(k,0,\alpha) = 0$ for all $k \in \mathbb{Z}$ and $\alpha \in \mathbb{R}^p$.*

Under the above assumption, the time 1 map can be written in the form

$$P(x,\alpha,\epsilon) = P_0(\alpha,\epsilon)x + O(||x||^2)$$

for $P_0 \in \mathbb{R}^{n\times n}$ two times continuously differentiable in $U \times [0,\eta)$ for some $\eta > 0$ and $U \subset \mathbb{R}^p$ a neighbourhood of zero. By performing a linear invertible change of coordinates $x = \Gamma z$, we may assume without loss of generality that $P_0(0,0)$ is in the real Jordan canonical form. The stability of $x = 0$ is therefore determined by the eigenvalues of the matrix $P_0(\alpha,\epsilon) - I$. Define a function $W : (U \times [0,\eta)) \times (\mathbb{C}^n \times \mathbb{C}) \to \mathbb{C}^n \times \mathbb{C}$ by

$$W(\alpha,\epsilon;v,\lambda) = \begin{bmatrix} P_0(\alpha,\epsilon)v - \lambda v \\ v^*v - 1 \end{bmatrix},$$

where v^* denotes the conjugate transpose of v. Let λ_0 be a simple eigenvalue of $P_0(0,0)$. There exists some $v_0 \in \mathbb{C}^n$ such that $W(0,0;v_0,\lambda_0) = 0$. Namely, v_0 is a unit eigenvector of $P_0(0,0)$ associated with the eigenvalue λ_0. Then, we have

$$D_{(v,\lambda)}W(0,0;v_0,\lambda_0) = \begin{bmatrix} P_0(0,0) - \lambda_0 I & -v_0 \\ 2v_0^* & 0 \end{bmatrix}.$$

Since λ_0 is simple, $M := P_0(0,0) - \lambda_0 I$ has rank $n-1$ and $v_0 \in \ker(J)$. As such, $I - M^+M = v_0v_0^*$ is the orthogonal projection onto $\ker(J)$. Moreover, since M is in the Jordan canonical form and $Mv_0 = 0$ with λ_0 being simple, we know that $v_0^*M = 0$. It follows that $D_{(v,\lambda)}W(0,0;v_0,\lambda_0)$ has full rank and, specifically,

$$D_{(v,\lambda)}W(0,0;v_0,\lambda_0)^{-1} = \begin{bmatrix} M^+ & \frac{1}{2}v_0 \\ -v_0^* & 0 \end{bmatrix}.$$

By the implicit function, we have $(v,\lambda) = (v(\alpha,\epsilon),\lambda(\alpha,\epsilon))$ smoothly in some neighbourhood $U_0 \times [0,\eta_0)$ of $(0,0) \in U \times [0,\eta)$. Taking into account the appropriate change of variables to ensure the Jordan normal form, this proves the following lemma.

Lemma III.1.4.1. *Let $J(\alpha_0) \in \mathbb{R}^{n\times n}$ denote the real Jordan normal form of $P_0(\alpha_0,0)$, so that $P_0(\alpha_0,0) = \Gamma J(\alpha_0)\Gamma^{-1}$ for some $\Gamma \in \mathbb{R}^{n\times n}$. Let λ_0 be a simple eigenvalue of $P_0(\alpha_0,0)$, and let $v_0 \in \mathbb{R}^n$ satisfy $J(\alpha_0)v_0 = \lambda_0 v_0$ with $||v_0|| = 1$. $P_0(\alpha,\epsilon)$ admits a C^1 eigenvalue $\lambda(\alpha,\epsilon)$ for $(\alpha,\epsilon) \in U \times [0,\eta_0)$ with* *some neighbourhood U of $\alpha_0 \in \mathbb{R}^p$ and some $\eta_0 > 0$, satisfying $\lambda(\alpha_0, 0) = \lambda_0$ and*

$$\frac{d\lambda}{d\alpha}(\alpha_0, 0) = v_0^* \Gamma^{-1} \partial_\alpha P_0(\alpha_0, 0) \Gamma v_0, \qquad \frac{d\lambda}{d\epsilon}(\alpha_0, 0) = v_0^* \Gamma^{-1} \partial_\epsilon P_0(\alpha_0, 0) \Gamma v_0.$$

Keep in mind that the eigenvectors of the Jordan normal form $J(\alpha_0)$ correspond precisely to the standard basis vectors. The following definition is appropriate.

Definition III.1.4.1. *The* time-scale sensitivity *of the fixed point* 0 *at the parameter $\alpha \in \mathbb{R}^p$ is $s = s(\alpha) \in \mathbb{R}$ defined by*

$$s = \max_{j \in U(\Gamma)} |e_j^* \Gamma^{-1} \partial_\epsilon P_0(\alpha, 0) \Gamma e_j|, \tag{III.1.20}$$

where $P_0(\alpha, 0)\Gamma = \Gamma J(\alpha)$ for $J(\alpha)$ a real Jordan normal form of $P_0(\alpha, 0)$ with $\Gamma \in \mathbb{R}^{n \times n}$, and

$$U(\Gamma) = \{j = 1, \ldots, n : e_j \text{ is an eigenvector of } \Gamma^{-1} P_0(\alpha, 0)\Gamma \text{ with one-dimensional Jordan block.}\}$$

The sensitivity matrix with respect to Γ *is $S = S(\alpha; \Gamma) = \Gamma^{-1} \partial_\epsilon P_0(\alpha, 0)\Gamma$.*

Lemma III.1.4.2. *The time-scale sensitivity is well-defined. That is, it does not depend on the choice of matrix Γ satisfying $P_0(\alpha, 0)\Gamma = \Gamma J(\alpha)$, for $J(\alpha)$ any real Jordan normal form.*

Proof. We first prove that if $P_0(\alpha, 0)\Gamma_i = \Gamma_i J$ for some fixed Jordan matrix J and matrices Γ_i for $i = 1, 2$, then we must have $\Gamma_1 e_j = \rho_j \Gamma_2 e_j$ and $e_j^* \Gamma_1^{-1} = \rho_j^{-1} e_j^* \Gamma_2^{-1}$ for some constants ρ_j, whenever $j \in U(\Gamma)$. The columns of each of Γ_i form a basis for the generalized eigenspace associated with the matrix $P_0(\alpha, 0)$. Consequently, if $j \in U(\Gamma)$, then $\Gamma_1 e_j = \rho_j \Gamma_2 e_j$ because the associated Jordan block is one-dimensional. Similarly, the rows of Γ_i^{-1} form a basis for the generalized eigenspace of the transpose $P_0(\alpha, 0)^\intercal$, so that $e_j^* \Gamma_1^{-1} = \tilde{\rho}_j e_j^* \Gamma_2^{-1}$. Combining these two results together, one obtains $1 = \rho_j \tilde{\rho}_j$, which implies $\tilde{\rho}_j = \rho_j^{-1}$ as claimed. To conclude that the result is independent of the choice of Jordan normal form, one may recall that the Jordan normal form is unique up to the order of the Jordan blocks. □

Remark III.1.4.1. *The time-scale sensitivity of the fixed point* 0 *at the parameter $\alpha \in \mathbb{R}^p$ is the fastest linear-order speed at which the simple Floquet multipliers of the fixed point $x^* = 0$ can travel with respect to the time-scale parameter $\epsilon \ll 1$, provided the system parameter α is fixed. In this sense, the time-scale sensitivity partially answers the first question of this section. Since a generic matrix has all simple eigenvalues, we will be content with this answer.*

Let $\mu \mapsto \alpha(\mu) \in U_0$ be any C^1 curve with $\alpha(0) = 0$. Define $\Lambda : \mathbb{R} \times [0, \eta_0) \to \mathbb{R}$ by

$$\Lambda(\mu, \epsilon) = \lambda^*(\alpha(\mu), \epsilon)\lambda(\alpha(\mu), \epsilon) - 1, \tag{III.1.21}$$

with λ being the function from Lemma III.1.4.1. If (α^*, ϵ^*) satisfies $\Lambda(\alpha^*, \epsilon^*) = 0$, then the fixed point at zero for $x \mapsto P(x, \alpha^*, \epsilon^*)$ is nonhyperbolic. Suppose the fixed point is nonhyperbolic at $(\alpha, \epsilon) = (0, 0)$. Using the above results together with the implicit function theorem, it follows that the equation $\Lambda(\mu, \epsilon) = 0$ has a solution of the form $(\mu(\epsilon), \epsilon)$ for some smooth μ and $\epsilon \geq 0$ sufficiently small provided

$$\nu[\alpha'(0)] = \Re\left(\lambda_0^* v_0^* \Gamma^{-1} \partial_\alpha P_0(0, 0) \Gamma v_0 \alpha'(0)\right) \tag{III.1.22}$$

is nonzero, where $\lambda_0 = \lambda(0, 0)$. Note that $\partial_\mu \Lambda(0, 0) = 2\nu[\alpha'(0)]$. In this case, we can reparameterize the curve in terms of the time-scale parameter ϵ by $\alpha(\epsilon) := \alpha(\mu(\epsilon))$, and it admits the linear-order representation

$$\alpha(\epsilon) = -\epsilon\alpha'(0) \frac{\Re(\lambda_0^* v_0^* \Gamma^{-1} \partial_\epsilon P_0(0, 0) \Gamma v_0)}{\Re(\lambda_0^* v_0^* \Gamma^{-1} \partial_\alpha P_0(0, 0) \Gamma v_0 \alpha'(0))} + O(\epsilon^2).$$

The following lemma is therefore proven.

Lemma III.1.4.3. *Let λ_0 be a simple eigenvalue of $P_0(\alpha_0, 0)$ with unit modulus. Any C^1 curve $\alpha : \mu \mapsto \alpha(\mu)$ satisfying $\alpha(0) = \alpha_0$ and $\nu[\alpha'(0)] \neq 0$ can be reparameterized in terms of the time-scale parameter $\epsilon \geq 0$ in such a way that $|\lambda(\alpha(\epsilon), \epsilon)| = 1$ for $\epsilon \ll 1$. The reparameterization satisfies*

$$\alpha(\epsilon) = \alpha_0 - \epsilon\alpha'(0) \frac{\Re(\lambda_0^* v_0^* S(\alpha_0; \Gamma) v_0)}{\nu[\alpha'(0)]} + O(\epsilon^2), \tag{III.1.23}$$

where v_0 satisfies $\Gamma^{-1} P_0(\alpha_0, 0) \Gamma v_0 = \lambda_0 v_0$ and $||v_0|| = 1$.

Remark III.1.4.2. *If λ_0 is real, then $\lambda_0 = \pm 1$. The curve $(\alpha(\epsilon), \epsilon)$ then corresponds to either a fold (saddle-node) or a flip (period-doubling) curve. If λ_0 is complex, then $\lambda_0 = e^{i\omega}$ for some $\omega \in [0, 2\pi)$. The curve $(\alpha(\epsilon), \epsilon)$ then corresponds to a Neimark–Sacker (cylinder) curve. When α is a one-dimensional real parameter, Lemma III.1.21 provides a one-to-one correspondence between α in some one-sided neighbourhood of α^* and the time-scale parameter $\epsilon \geq 0$. These observations provide a partial answer to the second question of this section.*

Remark III.1.4.3. *The linear-order (ϵ) coefficient in (III.1.23) is scale-invariant with respect to $\alpha'(0)$, so one can always without loss of generality assume that $\alpha'(0)$ is a unit vector.*

If a higher-order reparameterization is desired, one can extend Lemma III.1.4.3 with further application of the implicit function theorem. However, the formulas for the implicit derivatives of the critical eigenvalues λ in Eq. (III.1.21) quickly become large, even for quadratic terms. As such, we will refrain from computing them here.

III.1.5 An Important Comment (Or Warning) Concerning Applications

Suppose we are interested in studying the sensitivity or bifurcation realization problem for a nonlinear system (III.1.1)–(III.1.2) in a situation where $t \mapsto x^*(t;\alpha)$ is a nontrivial branch of periodic solutions. It may seem reasonable to perform a parameter-dependent change of variables, shifting $x^*(t;\alpha)$ to the origin. This would give the system

$$\dot{y} = f(t, y + x^*(t;\alpha), \alpha) - f(t, x^*(t;\alpha), \alpha) := F(t, y, \alpha), \quad t \notin \mathbb{Z} \quad \text{(III.1.24)}$$
$$\Delta y = g(t, y + x^*(t;\alpha), \alpha) - g(t, x^*(t;\alpha), \alpha) := G(t, y, \alpha), \quad t \in \mathbb{Z}, \quad \text{(III.1.25)}$$

which satisfies $F(t,0,\alpha) = G(t,0,\alpha) = 0$ for all parameters α for which the branch exists. If we form the singular unfolding, we get the equation

$$\dot{z} = F(t, z, \alpha) + \frac{1}{\epsilon} \sum_{k=-\infty}^{\infty} G(k, z(k), \alpha) \mathbb{1}_{[k,k+\epsilon)}(t),$$

which has $z = 0$ as a fixed point for each α. We want to make conclusions about the singular unfolding associated with (III.1.1)–(III.1.2), so the logical step is to invert the change of coordinates we made at the beginning to place x^* at the origin; that is, we make the transformation $w = z + x^*(t;\alpha)$. This yields

$$\begin{aligned}\dot{w} &= F(t, w - x^*(t;\alpha), \alpha) + f(t, x^*(t;\alpha), \alpha) \\ &\quad + \frac{1}{\epsilon} \sum_{k=-\infty}^{\infty} \left[G(k, w(k) - x^*(k;\alpha), \alpha) + g(k, x^*(k;\alpha), \alpha)\right] \mathbb{1}_{[k,k+\epsilon)}(t) \\ &= f(t, w, \alpha) + \frac{1}{\epsilon} \sum_{k=-\infty}^{\infty} g(k, w(k), \alpha) \mathbb{1}_{[k,k+\epsilon)}(t),\end{aligned}$$

which is indeed the singular unfolding of (III.1.1)–(III.1.2). The problem is that the transformation $w = z + x^*(t;\alpha)$ now implies that $x^*(t;\alpha)$ is a periodic solution of the singular unfolding for every $\epsilon \in (0,1)$. This is clearly nonsense, as solutions of the singular unfolding are continuous and $t \mapsto x^*(t;\alpha)$ has (in general) discontinuities at the integers.

To identify where the flaw is in this argument, let us begin with the singular unfolding

$$\dot{w} = f(t, w, \alpha) + \frac{1}{\epsilon} \sum_{k=-\infty}^{\infty} g(k, w(k), \alpha) \mathbb{1}_{[k,k+\epsilon)}(t)$$

and assume that $w^*(t;\alpha,\epsilon)$ is a two-parameter branch of periodic solutions that exists for some range of α and for $0 < \epsilon \ll 1$. If we translate w^* to the origin by way of $w = u + w^*$, we get

$$\begin{aligned}\dot{u} &= f(t, u + w^*(t;\alpha,\epsilon),\alpha) - f(t, w^*(t;\alpha,\epsilon)) \\ &\quad + \frac{1}{\epsilon}\sum_{k=-\infty}^{\infty}\left[g(k, u(k) + w^*(k;\alpha,\epsilon),\alpha) - g(k, w^*(k;\alpha,\epsilon),\alpha)\right]\mathbb{1}_{[k,k+\epsilon)}(t).\end{aligned}$$

This is not the singular unfolding of any obvious impulsive differential equation, but for ϵ small, it is asymptotic to the $\dot{z}$ equation. The error therefore occurs in forming the singular unfolding of (III.1.24)–(III.1.25).

It is not possible to perform the analysis using the machinery of the previous section without modification if the reference periodic solution is nonconstant. The error was in shifting the periodic solution to the origin at the beginning, so the correct approach would be to compute a Taylor expansion of the Poincaré map of the singular unfolding near $(x^*(0;\alpha_0),\alpha_0)$ in the extended phase space, for some chosen parameter α_0. The main difference will be that whereas the $O(\epsilon^n)$ terms vanish in the expansion of the switching function from Theorem III.1.3.1, they will not vanish in this more general setting. We will not consider this more general setting here.

III.1.6 Example: Continuous-Time Logistic Growth with Pulsed Birth

Consider a single species undergoing intraspecific competition and birth pulses. Except at times t_k, the population dynamics evolve according to the scalar ODE

$$\dot{x} = -x(d + k_1 x).$$

The constants d and k_1 are positive. d represents baseline per capita death rate, while k_1 accounts for lower life expectancy due to competition over limited resources. At times t_k, a birth pulse occurs and the population gets reset according to

$$x \mapsto x + x(b - k_2 x),$$

where now b is baseline per capita birth rate and k_2 accounts for decreased fecundity due to competition.

If $t_k = kT$ for some period T, then after a suitable change of variables, we have a system of the form

$$\begin{aligned}\dot{x} &= -\delta x - k_1 x^2, \quad && t \notin \mathbb{Z} \\ \Delta x &= \beta x - k_2 x^2, && t \in \mathbb{Z}.\end{aligned}$$

One can readily check that when $(1+\beta)e^{-\delta} = 1$, the linearization at $x = 0$ has a single Floquet multiplier $\mu = 1$. Define the change of parameters

$\delta = \log(1+\beta) + \gamma$. We get

$$\dot{x} = -\log(1+\beta)x - \gamma x - k_1 x^2, \quad t \notin \mathbb{Z} \tag{III.1.26}$$
$$\Delta x = \beta x - k_2 x^2, \quad t \in \mathbb{Z}. \tag{III.1.27}$$

When $\gamma = 0$, the linearization at $x = 0$ has $\mu = 1$ as its only Floquet multiplier. We can compute the coefficients a_{20} and a_{11} of Theorem II.5.2.2 for the transcritical bifurcation. After performing the Floquet change of variables $x = (1+\beta)^{-[t]_1} y$ to get the system into the correct form, they are found to be

$$a_{20} = -\frac{k_1\beta}{\log(1+\beta)} - \frac{k_2}{(1+\beta)^2} < 0, \qquad a_{11} = -1 < 0.$$

It follows that for each $\beta > 0$ fixed, $x = 0$ in (III.1.26)–(III.1.27) undergoes a transcritical bifurcation of periodic solutions at $\gamma = 0$. The nontrivial periodic solution is asymptotically stable when $\gamma < 0$.

Taking $\beta > 0$ to be a fixed constant, we will investigate the sensitivity of the transcritical bifurcation with respect to the parameter γ in the context of the singular unfolding (III.1.5). By Theorem III.1.3.1, the switching function can be expressed in the form

$$S((x,\gamma),\epsilon) = \left[\begin{array}{c} (1+\beta)x - k_1x^2 - \epsilon\log(1+\beta)\left(1+\frac{\beta}{2}\right)x - \frac{1}{2}\epsilon\gamma(2+\beta)x + \frac{1}{2}\epsilon^2(\log(1+\beta))^2\left(1+\frac{\beta}{3}\right)x + O(||x||^2) \\ \gamma \end{array}\right].$$

For $t \in [\epsilon, 1]$, the dynamics of the singular unfolding are independent of γ. The time 1 map therefore admits the representation

$$\begin{aligned} x \mapsto \exp\big(-(1-\epsilon)(\log(1+\beta)+\gamma)\big)\Big[&1+\beta-\epsilon\log(1+\beta)\left(1+\frac{\beta}{2}\right) - \frac{1}{2}\epsilon\gamma(2+\beta) \\ &+ \frac{1}{2}\epsilon^2\log(1+\beta)^2\left(1+\frac{\beta}{3}\right)\Big]x + O(x^2) \\ := P_0(\gamma,\epsilon)x + O(x^2), \end{aligned}$$

for β fixed and $||(\gamma,\epsilon)||$ sufficiently small. The function $P_0(\gamma,\epsilon)$ satisfies

$$\partial_\gamma P_0(0,0) = -1, \qquad \partial_\epsilon P_0(0,0) = \frac{\beta\log(1+\beta)}{2(1+\beta)}.$$

Taking the reference parameter curve to simply be the constant $\mu \mapsto \gamma(\mu) = \mu$, Lemma III.1.4.3 implies that

$$\gamma = \epsilon\left(\frac{\beta\log(1+\beta)}{2(1+\beta)}\right) + O(\epsilon^2) := \epsilon\gamma_1 + O(\epsilon^2) \tag{III.1.28}$$

is a transcritical bifurcation curve for $|\epsilon|$ sufficiently small, for each β fixed. Namely, the following conclusions are valid for fixed β. There exists an open interval $J = J(\beta)$ containing $0 \in \mathbb{R}$ such that

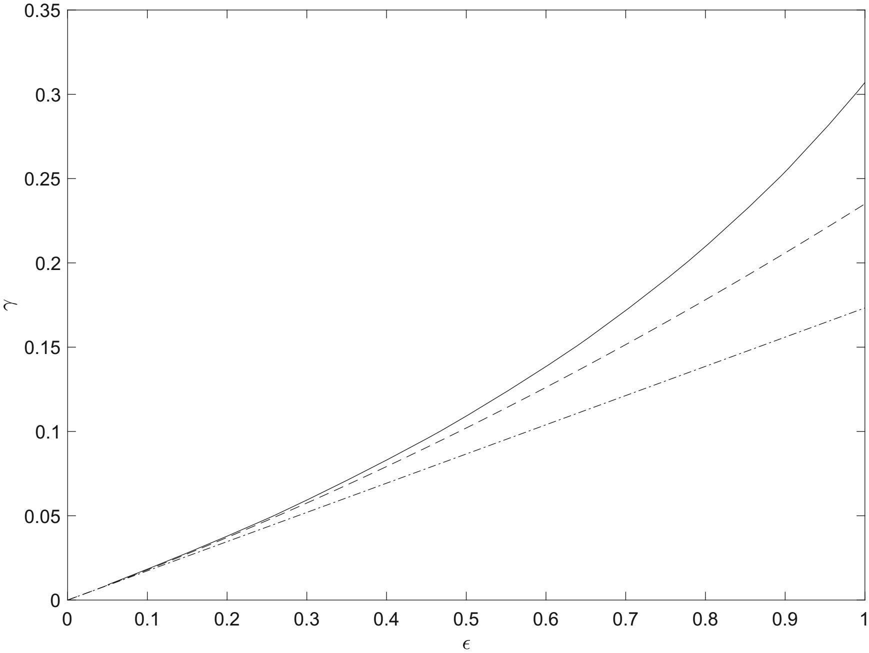

Figure III.1.1: Solid curve (top): plot of the bifurcation curve given by the solution γ of Eq. (III.1.30) at $\beta = 1$. Dashed curve (middle): plot of the quadratic approximation (III.1.29) to the bifurcation curve. Dashed–dotted line (bottom): the linear approximation (III.1.28)

- if $\gamma > \gamma_1 \epsilon + O(\epsilon^2)$, there are no nontrivial periodic solutions in J, and $x = 0$ is locally asymptotically stable;

- if $\gamma < \gamma_1 \epsilon + O(\epsilon^2)$, there is exactly one nontrivial periodic solution in J, and it is locally asymptotically stable, while $x = 0$ is unstable.

The linear approximation (III.1.28) provides a fairly good approximation to the true bifurcation curve when $0 < \epsilon \ll 1$. See Fig. III.1.1 for a plot of the true bifurcation curve along with the linear approximation.

As mentioned briefly at the end of the previous section, we can obtain a higher-order approximation to the bifurcation curve. Since $P_0(\gamma, \epsilon)$ is given explicitly to quadratic order, the quadratic term of (III.1.28) can be calculated. To expedite the process, we used symbolic differentiation and symbolic limit evaluations in MATLAB. The result is

$$\gamma = \epsilon \left(\frac{\beta \log(1+\beta)}{2(1+\beta)} \right) + \epsilon^2 \frac{\beta \log(1+\beta)}{2(1+\beta)} \left(\frac{1}{4(1+\beta)} + \frac{\log(1+\beta)}{3} \right) + O(\epsilon^3). \tag{III.1.29}$$

The quadratic truncation does provide a better fit to the bifurcation curve, as expected. See again Fig. III.1.1.

We claim that the bifurcation curve $\gamma = \gamma(\epsilon)$ is precisely the solution of the equation

$$\frac{e^{-\gamma}}{1+\beta}\left(1+\frac{\beta(e^{\epsilon(\log(1+\beta)+\gamma)}-1)}{\epsilon(\log(1+\beta)+\gamma)}\right)=1, \tag{III.1.30}$$

continuously extended so that the left-hand side is finite as $\epsilon \to 0^+$ and $\beta \to 0$. To see why this is the case, let $x \mapsto P(x, \gamma, \epsilon)$ denote the time 1 map of the canonical singular unfolding of (III.1.26)–(III.1.27). For $\epsilon \neq 0$, $\partial_x P(0, \epsilon, \gamma) = z(1)$, where $z : [0, 1] \to \mathbb{R}$ satisfies $z(0) = 1$ and

$$\frac{d}{dt}z(t) = -(\log(1+\beta)+\gamma)z(t) + \frac{\beta}{\epsilon}\mathbb{1}_{[0,\epsilon]}(t).$$

Solving this linear differential equation and evaluating at time $t = 1$ yield precisely the left-hand side of (III.1.30). The claimed result follows because the bifurcation curve satisfies $\partial_x P(0, \epsilon, \gamma) = 1$.

Chapter III.2

Non-smooth Bifurcations

III.2.1 Overview

In this chapter we will be interested in bifurcations that result from two "non-smooth" phenomena:

- perturbations in the sequence of impulses and
- crossings of discrete delays across impulse times.

These two situations have some elements in common, but the second of the two will typically involve explicit breaking of the overlap condition and will result in highly non-smooth bifurcations. A bifurcation involving this type of scenario will broadly be called an *overlap bifurcation*. The first one is slightly better behaved, and the lack of smoothness is more of a technical issue.

III.2.1.1 Bifurcations Involving Perturbations of Impulse Times

Consider the Hutchinson equation with impulsive harvesting:

$$\begin{aligned} \dot{x} &= rx(t)\left(1 - \frac{x(t-\tau)}{K}\right), & t &\neq kT \\ \Delta x &= -hx(t^-), & t &= kT, \end{aligned}$$

for $k \in \mathbb{Z}$. We will revisit this system in Chapter IV.2. The linearization at $x = 0$ has only the Floquet multipliers 0 and $\mu = (1-h)e^{rT}$. Since $x = 0$ is a fixed point for all parameter values, it is reasonable to suspect that a

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transcritical bifurcation could occur as T crosses through the threshold period

$$T^* = -\frac{1}{r}\log(1-h),$$

since $\mu = 1$ when $T = T^*$. Our previous framework for parameter-dependent centre manifolds is based on taking the parameter as an additional state and in the impulses must occur at fixed times. The required transformations are therefore first a rescaling of time to map the impulses onto the integers, followed by the introduction of the new state, T. After this is completed, we get the system

$$\begin{aligned}
\dot{x} &= rTx(t)\left(1 - \frac{x(t-\tau/T)}{K}\right), && t \notin \mathbb{Z}\\
\dot{T} &= 0, && t \notin \mathbb{Z}\\
\Delta x &= -hx(t^-), && t \in \mathbb{Z}\\
\Delta T &= 0, && t \in \mathbb{Z}.
\end{aligned}$$

The problem with the above system is that it contains a state-dependent delay (τ/T). The functional that defines the vector field is *not* smooth in any open subset of the extended state space, so our centre manifold theory does not apply. In Sect. III.2.2 we will give one method that allows this type of non-smooth formulation to be avoided while still providing a reasonable centre manifold theory.

Remark III.2.1.1. *For further justification as to why varying the period of impulse effect in an impulsive functional differential equation results in a non-smooth perturbation, consider the scalar equation* without delays

$$\begin{aligned}
\dot{x} &= 0, && t \neq kT\\
\Delta x &= x, && t = kT,
\end{aligned}$$

for $T \in (0,2)$, but with the phase space $\mathcal{RCR}([-2,0],\mathbb{R})$. The solution from the constant initial condition $x_0 = 1$ satisfies

$$x_t(\theta, T) = \begin{cases} 2^{\lfloor (t+\theta)/T \rfloor}, & t+\theta > 0\\ 1 & t+\theta \leq 0, \end{cases}$$

where we have included T as a function input to emphasize its dependence on the solution. Then for all $T \in (0,2)$, we can write

$$x_2(\theta, T) = 2^{\lfloor (2+\theta)/T \rfloor}.$$

This function (of θ) is piecewise-constant with discontinuities at $\theta \in T\mathbb{Z} - 2$. Since the location of these discontinuities depends on T, the function $T \mapsto x_2(\cdot, T)$ is not continuous as a function from $(-2,0)$ into $\mathcal{RCR}([-2,0],\mathbb{R})$. In fact, one can show that $||x_2(\cdot,T_1) - x_2(\cdot,T_2)|| \geq 1$ whenever $T_1 \neq T_2$. In other words, the solutions of an impulsive functional differential equation are generally not *continuous (in the phase space $\mathcal{RCR}$) with respect to the impulse times, even for fixed initial conditions.*

III.2.1.2 Bifurcations Involving Crossings of Impulse Times and Delays

The second phenomenon is most easily motivated with the following toy example:

$$\dot{x} = \log\left(\frac{3}{2}\right)x, \qquad t \notin \mathbb{Z} \tag{III.2.1}$$

$$\Delta x = -\frac{1}{2}x(t^-) + \frac{1}{4}x_{t^-}(-\omega), \qquad t \in \mathbb{Z}, \tag{III.2.2}$$

where $\omega \in [0,\Omega]$ is a real parameter. The state space is taken to be $\mathcal{RCR}([-\Omega, 0],\mathbb{R})$. This scalar equation essentially has a delayed jump of the form

$$\Delta x = \begin{cases} -\frac{1}{2}x(t^-) + \frac{1}{4}x(t-\omega), & \omega > 0 \\ -\frac{1}{2}x(t^-) + \frac{1}{4}x(t^-), & \omega = 0. \end{cases}$$

When $\omega \in [0,1]$, the situation is fairly simple. If we take $x(t) = \phi(t)e^{\lambda t}$ as a Floquet eigensolution ansatz, ϕ is a periodic solution (of period one) satisfying

$$\dot{\phi} = \left(\log\left(\frac{3}{2}\right) - \lambda\right)\phi, \qquad t \notin \mathbb{Z}$$

$$\Delta\phi = -\frac{1}{2}\phi(t^-) + \frac{1}{4}e^{-\lambda\omega}\phi_{t^-}(-\omega), \qquad t \in \mathbb{Z}.$$

Since $\omega \in [0,1]$, this equation can be explicitly solved with little effort. We find

$$\begin{aligned}\phi(1) &= \frac{1}{2}e^{\log(3/2)-\lambda}\phi(0) + \frac{1}{4}e^{-\lambda\omega}e^{(\log(3/2)-\lambda)(1-\omega)}\phi(0)\\ &= e^{-\lambda}\left(\frac{3}{4} + \frac{1}{4}\left(\frac{3}{2}\right)^{1-\omega}\right)\phi_0\end{aligned}$$

for each of the cases $\omega = 0$, $\omega \in (0,1)$ and $\omega = 1$. It follows that ϕ is periodic if and only if $\mu = e^{\lambda}$ satisfies

$$\mu(\omega) = \frac{1}{4}\left(3 + 1\left(\frac{3}{2}\right)^{1-\omega}\right).$$

Suppose $\omega = 1 + \widehat{\omega}$ for some $\widehat{\omega} \in (0,1)$. Taking a Floquet ansatz again, this time the function ϕ satisfies

$$\dot{\phi} = \left(\log\left(\frac{3}{2}\right) - \lambda\right)\phi, \qquad t \notin \mathbb{Z}$$

$$\Delta\phi = -\frac{1}{2}\phi(t^-) + \frac{1}{4}e^{-\lambda(1+\widehat{\omega})}\phi_{t^-}(-\widehat{\omega}), \qquad t \in \mathbb{Z}.$$

We can yet again solve this impulsive differential equation explicitly, and we find

$$\begin{aligned}\phi(1) &= \frac{1}{2}e^{\log(3/2)-\lambda}\phi(0) + \frac{1}{4}e^{-\lambda\omega}e^{(\log(3/2)-\lambda)(2-\omega)}\phi(0)\\ &= \left(e^{-\lambda}\frac{3}{4} + e^{-2\lambda}\frac{1}{4}\left(\frac{3}{2}\right)^{2-\omega}\right)\phi_0.\end{aligned}$$

In order for ϕ to be periodic, $\mu = e^{\lambda}$ must solve the equation

$$\mu^2 - \frac{3}{4}\mu - \frac{1}{4}\left(\frac{3}{2}\right)^{2-\omega} = 0.$$

This equation has two solutions, and they coincide with the nontrivial Floquet multipliers. Combining the result from the previous section, it follows that the nontrivial Floquet multipliers are

$$\mu(\omega) = \begin{cases} \frac{1}{4}\left(3 + 1\left(\frac{3}{2}\right)^{1-\omega}\right), & \omega \in [0,1] \\ \frac{3}{8}\left(1 \pm 3^{-\omega}\sqrt{3^{\omega}(2^{2+\omega}+3^{\omega})}\right), & \omega \in (1,2). \end{cases}$$

Notably, $\mu(1) = 1$, but $\mu(1^+) = \frac{3}{8} \pm \frac{1}{8}\sqrt{33} \approx \{1.093, -0.343\}$, so, in particular, the function $\omega \mapsto \max|\mu(\omega)|$ is discontinuous. If (III.2.1) and (III.2.2) correspond to the linearization of a particular nonlinear system at some equilibrium, a bifurcation could occur in the "smooth" regime $\omega \to 1^-$, while crossing over into $\omega > 1$ could completely destroy the local orbit structure. For example, consider the nonlinear system

$$\dot{x} = \log\left(\frac{3}{2}\right)x - \frac{1}{10}x^2, \qquad t \notin \mathbb{Z} \tag{III.2.3}$$

$$\Delta x = -\frac{1}{2}x(t^-) + \frac{1}{4}x_{t^-}(-\omega), \qquad t \in \mathbb{Z}. \tag{III.2.4}$$

For $\omega \in (0,1]$, the nontrivial Floquet multiplier is decreasing, there is a quadratic nonlinearity and $x = 0$ is a fixed point. It is therefore reasonable to suspect that a transcritical bifurcation might occur as $\omega \to 1^-$. However, the fixed point $x = 0$ is not a bifurcation point as $\omega \to 1^+$ since the Floquet multipliers are bounded away from 1 in absolute value. We will study the above nonlinear example in a bit more depth in Sect. III.2.3.

Roughly speaking, an *overlap bifurcation* is the resulting change in the orbit structure whenever a system parameter is varied causing the overlap condition to be violated. We will study some particular overlap bifurcation scenarios in Sect. III.2.3 by focusing on systems with delayed impulses, since these are slightly more amenable to analysis.

III.2.2 Centre Manifolds Parameterized by Impulse Times

In this section we consider a class of time-invariant delay differential equation systems undergoing impulses at a specific frequency $\frac{1}{p}$:

$$\begin{aligned} \dot{x} &= LSx_t + f(x_t), && t \neq kp \\ \Delta x &= BSx_{t^-} + g(x_{t^-}), && t = kp, \end{aligned} \tag{III.2.5}$$

for $k \in \mathbb{Z}$. The following hypotheses will be needed:

F.1 L and B are $n \times m$ matrices, and $S : \mathcal{RCR}([-r, 0], \mathbb{R}^n) \to \mathbb{R}^m$ is a bounded linear operator possessing a representation of the form $S\phi = \int_{-r}^{0}[d\eta(\theta)]\phi(\theta)$ with $\eta : [-r, 0] \to \mathbb{R}^{m\times n}$ of bounded variation and right-continuous.

F.2 There exists $p^* \in \mathbb{R}$ such that the periodic linear system

$$\begin{aligned} \dot{y} &= LSx_t, && t \neq kp^*, \\ \Delta y &= BSx_{t^-}, && t = kp^* \end{aligned}$$

has exactly $d > 0$ Floquet multipliers $\mu_1, \dots, \mu_d$ satisfying $|\mu_1| = \cdots = |\mu_d| = 1$, while all other Floquet multipliers μ_j satisfy $|\mu_j| \neq 1$. Also, $r < p^* < r^*$ for some r^*.

F.3 There exist $Z_1(p)$, $Z_2(p) \in \mathbb{R}^{n\times m}$ defined in a neighbourhood of p^* with the following properties:

- $Z_1(p^*) = L$, $Z_2(p^*) = B$.
- Z_1 and Z_2 are continuous.
- For each p, the periodic linear system

$$\begin{aligned} \dot{y} &= Z_1(p)Sx_t, && t \neq kp, \\ \Delta y &= Z_2(p)Sx_{t^-}, && t = kp \end{aligned} \tag{III.2.6}$$

  has exactly $d > 0$ Floquet multipliers $\gamma_1, \dots, \gamma_d$ satisfying $|\gamma_1| = \cdots = |\gamma_d| = 1$.

F.4 $f : \mathcal{RCR} \to \mathbb{R}^n$ and $g : \mathcal{RCR} \to \mathbb{R}^n$ are C^k for some k.

Remark III.2.2.1. *Hypothesis F.2 implies that the overlap condition holds at the critical period p^*.*

III.2.2.1 Dummy Matrix System and Robustness of Spectral Separation

Given hypotheses F.1–F.4, we introduce the quasilinear *dummy matrix system* (DMS),

$$\begin{aligned}
\dot{x} &= Z_1(p)Sx_t + M_1Sx_t + f(x_t), && t \neq kp \\
\dot{M}_1 &= 0, && t \neq kp \\
\dot{M}_2 &= 0, && t \neq kp \\
\Delta x &= Z_2(p)Sx_{t^-} + M_2Sx_{t^-} + g(x_{t^-}), && t = kp \\
\Delta M_1 &= 0, && t = kp \\
\Delta M_2 &= 0, && t = kp.
\end{aligned} \tag{III.2.7}$$

with $M_1, M_2 \in \mathbb{R}^{n\times m}$. Notice that if $M_1 = L - Z_1(p)$ and $M_2 = B - Z_2(p)$, then the DMS coincides with (III.2.5) extended trivially to the state space $\mathcal{RCR} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m}$.

The idea here is that since we cannot take p as a state variable, we will instead parameterize the linear part in such a way that for each *fixed* $p \approx p^*$, we have a centre manifold of the appropriate dimension. The introduction of a dummy (matrix) parameter eventually allows us to recover a particular invariant manifold of the original system. Our plan is as follows:

1. We show that under the assumptions F.1–F.4, the spectral separation of (III.2.6) near $p = p^*$ is robust (Lemma III.2.2.3).

2. Using the robustness of spectral separation and the analysis of the centre manifold construction, we prove that there exist $\epsilon > 0$ and a constant $\delta > 0$ such that the DMS has, for each $p \in (p^* - \epsilon, p^* + \epsilon)$, a centre manifold that contains all small solutions of size at most δ.

3. We prove that there exist $\nu > 0$ and $\delta > 0$ such that if $p \in (p^* - \nu, p^* + \nu)$, the family of centre manifolds parameterized by the parameter p and evaluated at $M_1 = L - Z_1(p)$ and $M_2 = B - Z_2(p)$ defines a parameter-dependent centre manifold for the original system (III.2.5) and contains all small solutions of size at most δ.

We begin with some notation. For given p, let $V(p) : \mathcal{RCR} \to \mathcal{RCR}$ denote the monodromy operator V_0 associated with the linear system (III.2.6). Also, let $U(t, s; p)$ denote the evolution family for (III.2.6) and $C(t, s; p) : \mathcal{RCR} \to \mathcal{RCR}$ denote the evolution family associated with the linear system without impulses $\dot{z} = Z_1(p)Sz_t$. Finally, let PC_0 denote the closed subspace of $\mathcal{RCR}$ consisting of functions that are continuous except at zero, where they have limits on the left.

Lemma III.2.2.1. *Let $V_0(p) : PC_0 \to \mathcal{RCR}$ denote the restriction of $V(p)$ to PC_0. $p \mapsto V_0(p)$ is strongly continuous at p^*—that is, for each $\xi \in \mathcal{RCR}$,*

$$\lim_{p \to p^*} ||V_0(p)\xi - V_0(p^*)\xi|| = 0.$$

Proof. From Lemma I.2.2.1,

$$\begin{aligned}
&||V_0(p) - V_0(p^*)|| \\
&\quad = ||\chi_0[Z_2(p) - B]SC(p, 0; p) - [I + \chi_0 BS](C(p^*, 0; p^*) - C(p, 0; p))|| \\
&\quad \leq ||Z_2(p) - B|| \cdot ||SC(p, 0; p)|| \\
&\qquad + (1 + ||BS||) \sup_{\xi \in PC_0, ||\xi||=1} |||[C(p^*, 0; p^*) - C(p, 0; p)]\xi||.
\end{aligned}$$

Continuity of Z_2 at p^* ensures that the first term above converges to zero as $p \to p^*$. Denote $z(t) = C(t, 0; p)\xi(0)$. Then,

$$\dot{z} = LSz_t + (Z_1(p) - L)Sz_t,$$

which implies by Theorem I.2.3.1 the decomposition

$$C(p, 0; p) = C(p, 0; p^*) + \int_0^p C(p, \mu; p^*)\chi_0 \left[(Z_1(p) - L)SC(\mu, 0; p)\right] d\mu.$$

We can then make the estimate

$$\begin{aligned}
||C(p^*, 0; p^*) - C(p, 0; p)|| &\leq ||C(p, 0; p^*) - C(p^*, 0; p^*)|| \\
&\quad + \int_0^p ||C(p, \mu; p^*)\chi_0 \left[(Z_1(p) - L)SC(\mu, 0; p)\right] ||d\mu.
\end{aligned}$$

The integral term converges to zero as $p \to p^*$ due to Lemma I.2.2.1 and the continuity of Z_1 at p^*. As for the other one, observe that due to hypothesis F.2, $C(p, 0; p^*)$ has range in $C([-r, 0], \mathbb{R}^n)$ for $|p - p^*|$ small enough. In the same way we proved Lemma I.3.1.1, one can show that $p \mapsto C(p, 0; p^*)$ is compact for $p \geq p^* - \epsilon$ for some small $\epsilon > 0$. It then follows (see for instance Lemma 4.22 of [43]) that $p \mapsto C(p, 0; p^*)$ is norm continuous at p^*. Combining the previous convergence results, the lemma is proven. □

Lemma III.2.2.2. *There exist $\epsilon > 0$ and constants $\alpha < 1 < \beta$ such that for all $p \in (p^* - \epsilon, p^* + \epsilon)$, any eigenvalue λ of $V(p)$ satisfies one of $|\lambda| = 1$, $|\lambda| < \alpha$ and $|\lambda| > \beta$.*

Proof. Since $V(p^*)$ is compact, assumption F.2 implies that its spectrum σ admits a decomposition $\sigma = \Sigma_u \cup \Sigma_c \cup \Sigma_s$, with

$$d(0, \Sigma_s) < \alpha_0 < 1, \qquad d(0, \Sigma_c) = 1, \qquad d(0, \Sigma_u) > \beta_0 > 1$$

for some constants α_0 and β_0. Note that $V(p)$ has range in PC_0, so any eigenvalue of $V(p)$ (and, by compactness, any nonzero element of the spectrum) must also be an eigenvalue of the restricted operator $V_0(p)$. But $V_0(p)$

is continuous at p^* from Lemma III.2.2.1, so by [Theorem 4–3.16, [79]] on semicontinuity of separated parts of the spectrum, there exists $\epsilon_1 > 0$ such that if $|p - p^*| < \epsilon_1$, the spectrum of $V(p)$ lies completely in the disjoint sets

$$\Sigma_1 = \{z \in \mathbb{C} : |z| < \alpha\}, \qquad \Sigma_2 = \{z \in \mathbb{C} : |z| > \alpha\}$$

for $\alpha = (1+\alpha_0)/2$, and the number of eigenvalues in Σ_2 is constant. Similarly, there exists $\epsilon_2 > 0$ such that if $|p - p^*| < \epsilon_2$, the spectrum of $V(p)$ lies completely in the disjoint sets

$$\Sigma^1 = \{z \in \mathbb{C} : |z| < \beta\}, \qquad \Sigma^2 = \{z \in \mathbb{C} : |z| > \beta\},$$

for $\beta = (1 + \beta_0)/2$, and the number of eigenvalues in Σ^2 is constant. Thus, for $\epsilon < \min\{\epsilon_1, \epsilon_2\}$, the number of eigenvalues in the annulus $\{z \in \mathbb{C} : \alpha < z < \beta\}$ is constant, and from the decomposition $\sigma = \Sigma_u \cup \Sigma_c \cup \Sigma_s$ of $V(p)$ and assumption F.3, this annulus contains only the d eigenvalues on the unit circle. □

Taking advantage of Lemma III.2.2.2, we obtain a parameter-uniform analogue of Theorem I.3.1.3. The proof is a trivial modification of the proof of the aforementioned theorem and is omitted.

Lemma III.2.2.3. *There exists $\epsilon > 0$ such that $U(t, s; p)$ is* uniformly spectrally separated for $p \in (p^* - \epsilon, p^* + \epsilon)$. *That is, $U(t, s; p)$ is spectrally separated for each $p \in (p^* - \epsilon, p^* + \epsilon)$ with projectors $(P_{s,p}, P_{c,p}, P_{u,p})$, the constants K, a and b appearing in Eqs.* (I.1.11)–(I.1.13) *can be chosen independent of p and there is a constant N independent of p such that*

$$\sup_{t \in \mathbb{R}} (||P_{s,p}(t)|| + ||P_{c,p}(t)|| + ||P_{u,p}(t)||) \leq N.$$

Denote $X = \mathbb{R}^n \times \mathbb{R}^{m \times m} \times \mathbb{R}^{m \times m}$. Let $\tilde{U}(t, s; p) : Y \to Y$ be the evolution family associated with the linear part of the DMS (III.2.7), for $Y = \mathcal{RCR}([-r, 0], X)$ the extended phase space. Note that this is simply

$$\tilde{U}(t, s; p)[\phi, x, y] = (U(t, s; p)\phi, x(0), y(0)),$$

so in the following we will abuse notation and identify $\tilde{U}(t, s; p)$ with $U(t, s; p)$. Lemma III.2.2.3 still holds for the evolution family on the extended phase space, and the projections $\tilde{P}_{i,p}$ (which we later identify with $P_{i,p}$) inherit block-diagonal structures $\tilde{P}_{i,p} = \mathrm{diag}(P_{i,p}, Z, Z)$, with $Z = I$ if $i = c$ and $Z = 0$ if $i \neq c$.

Due to Lemma III.2.2.3, there exist $\bar{\epsilon} > 0$ and a constant $\overline{K} > 0$ independent of p such that for all $\eta \in (0, \min\{-a, b\})$, the Lyapunov–Perron operator $\mathcal{K}_s^{\eta,p} : B^\eta(\mathbb{R}, X) \oplus B_{kp}^\eta(\mathbb{Z}, X) \to B^\eta(\mathbb{R}, Y)$

$$\begin{aligned}
&\mathcal{K}_s^{\eta,p}(F, G)(t) \\
&\quad = \int_s^t U(t, \mu; p) P_{c,p}(\mu)[\chi_0 F(\mu)] d\mu - \int_t^\infty U(t, \mu; p) P_{u,p}(\mu)[\chi_0 F(\mu)] d\mu \\
&\qquad + \int_{-\infty}^t U(t, \mu; p) P_{s,p}(\mu)[\chi_0 F(\mu)] d\mu + \sum_s^t U(t, t_i; p) P_{c,p}(t_i)[\chi_0 G_i] dt_i \\
&\qquad - \sum_t^\infty U(t, t_i; p) P_{u,p}(t_i)[\chi_0 G_i] dt_i + \sum_{-\infty}^t U(t, t_i; p) P_{s,p}(t_i)[\chi_0 G_i] dt_i
\end{aligned} \tag{III.2.8}$$

is well-defined, linear and bounded with norm $||\mathcal{K}_s^{\eta,p}||_\eta \le \overline{K}$, for all $p \in (p - \bar{\epsilon}, p + \bar{\epsilon})$.

III.2.2.2 Centre Manifold Construction

Define $\tilde{f} : Y \to X$ and $\tilde{g} : Y \to X$ to be the vector field and jump map associated with the DMS. Specifically, they are defined by

$$\tilde{f}(\Phi) = [\phi_1(0) S\psi + f(\psi), 0, 0]^T \qquad \tilde{g}(\Phi) = [\phi_2(0) S\psi + g(\psi), 0, 0]^T,$$

where $\Phi : [-r, 0] \to \mathbb{R}^n \times \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times m}$ is split into component functions via $\Phi = (\psi, \phi_1, \phi_2)$. Following Sect. I.5.1.3, given a bump function ξ, we can define the parameter-dependent smoothed nonlinearities $\tilde{F}_\delta^p$ and $\tilde{G}_\delta^p$ by replacing the projections in Eqs. (I.5.5)–(I.5.6) with the appropriate parameter-dependent ones described in Lemma III.2.2.3. Due to the robustness, there is a mutual Lipschitz constant L_δ for the nonlinearities that satisfies $L_\delta \to 0$ as $\delta \to 0$, for all $p \in (p^* - \epsilon^*, p^* + \epsilon^*)$ for some $\epsilon^* > 0$. The same is true for the parameter-dependent substitution operators $\tilde{R}_\delta^p$, with Lipschitz constant $\tilde{L}_\delta$.

Theorem III.2.2.1 (Centre Manifold: Dummy Matrix System)**.** *There exist γ and $\epsilon > 0$ such that for each $p \in (p^* - \epsilon, p^* + \epsilon)$, the dummy matrix system (III.2.7) possesses a centre manifold $\mathcal{W}_c^p$ with $(d + m)$-dimensional t-fibres*

$$\mathcal{W}_c^p(t) = \mathrm{Im}\,(C_p(t, \cdot)),$$

where $C_p : \mathbb{R} \times \mathcal{RCR}([-r, 0], X) \to \mathcal{RCR}([-r, 0], X)$. Moreover, the following are true:

1. *Define the map* $\tilde{C}_p : \mathbb{R} \times (\mathcal{RCR} \times \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times m})$ *by*

$$\tilde{C}_p(t,(\phi,y_1,y_2)) = \pi_{\mathcal{RCR}} C_p(t,(\phi, y_1 \cdot \mathbb{1}, y_2 \cdot \mathbb{1})),$$

   where $\pi_{\mathcal{RCR}} : \mathcal{RCR}([-r,0],X) \to \mathcal{RCR}$ *is the projection onto the first component of* $\mathcal{RCR}([-r,0],X)$ *for the product* $X = \mathbb{R}^n \times \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times m}$ *and* $\mathbb{1}(\theta) = 1$. *Then,* $\pi_{\mathcal{RCR}} C_p = \tilde{C}_p$.

2. $\mathcal{W}_c^p$ *is locally positively invariant under the process* $S^p(t,s) : \mathcal{M} \to Y$ *associated with the dummy matrix system* (III.2.7) *in the sense of Theorem I.5.3.1, for initial conditions smaller than* δ.

3. $\mathcal{W}_c^p$ *contains all mild solutions* $(\phi, y_1, y_2) = x : \mathbb{R} \to Y$ *of* (III.2.7) *satisfying the estimate* $||x||_\eta = ||\phi||_\eta + ||y_1|| + ||y_2|| < \gamma$, *for any* $\eta \in (0, \min\{-a, b\})$, *with* $a < 0 < b$ *the spectral separation exponents from Lemma III.2.2.3.*

4. $\mathcal{W}_c^p$ *is attracting in the sense of Theorem I.5.5.1, provided the unstable fibre bundle is empty,* f *is an ACR functional and the unique matrix* $t \mapsto Y(t;p)$ *satisfying* $P_{c,p}(t)\chi_0 = \Phi_{t,p} Y(t;p)$ *is continuous from the right with limits on the left, where* $\Phi_{t,p}$ *is an array whose columns form a basis for* $\mathcal{R}(P_{c,p}(t))$ *and such that* $\Phi_{t,p} = U(t,s;p)\Phi_{s,p}$ *for* $t \geq s$.

5. $C_p(t,\cdot) : \mathcal{RCR}([-r,0],X) \to \mathcal{RCR}([-r,0],X)$ *is* C^k *and uniformly (in* p*) Lipschitz continuous, as are its derivatives.*

Proof. Choose $\gamma > 0$ small enough so that $\tilde{L}_\gamma \overline{K} < \frac{1}{2}$. Note that $\tilde{L}_\gamma \overline{K}$ is independent of p, provided $|p - p^*|$ is small enough. The existence of the centre manifold then follows by the analysis preceding the statement of the theorem, together with Theorem I.5.2.1. Property 1 follows by the definition of the vector field $\tilde{f}$ and jump map $\tilde{g}$ of the DMS, while properties 2 and 3 follow by Theorem I.5.3.1 by taking the norm on X as $||(x,y,z)|| = ||x|| + ||y|| + ||z||$. To prove property 4, one emulates the proof of Theorem I.5.5.1. Smoothness of property 5 follows by Theorem I.5.6.1. □

With the above lemma at hand, we are ready to construct the parameter-dependent centre manifold for the periodic system (III.2.5).

Corollary III.2.2.1 (Parameter-Dependent Centre Manifold)**.** *Let* $S^p(t,s) : \mathcal{RCR} \to \mathcal{RCR}$ *denote the process associated with the periodic system* (III.2.5). *Consider the formal expression*

$$C(t,\phi;p) = \tilde{C}_p(t,(\phi, L - Z_1(p), B - Z_2(p))).$$

There exist $\nu > 0$ *and* $\delta >$ *such that* $C : \mathbb{R} \times \mathcal{RCR} \times (p^* - \eta, p^* + \eta) \to \mathcal{RCR}$ *is well-defined and enjoys the following properties:*

1. *For each* $p \in (p^* - \nu, p^* + \nu)$, *the nonautonomous set* $\mathcal{W}_c^p$ *with* t*- fibres* $\mathcal{W}_c^p(t) = \mathrm{Im}\,(C(t, \cdot; p))$ *is locally positively invariant under the process* $S^p(t, s) : \mathcal{M} \to \mathcal{RCR}[-r^*, 0]$, *for initial conditions* ϕ *satisfying* $||\phi|| < \delta$.

2. $\mathcal{W}_c^p$ *contains all mild solutions* $x : \mathbb{R} \to \mathcal{RCR}$ *of* (III.2.5) *satisfying* $||x||_\eta < \delta$, *for any* $\eta \in (0, \min\{-a, b\})$.

3. $\mathcal{W}_c^p$ *is attracting in the sense of Theorem I.5.5.1, provided the conditions outlined in part 4 of Theorem III.2.2.1 are satisfied.*

4. $\phi \mapsto C(t, \phi; p)$ *is* C^k *and uniformly (in* p*) Lipschitz continuous, as are its derivatives.*

Proof. Apply Theorem III.2.2.1 to obtain the centre manifold for the DMS as the nonautonomous set with fibres given by the images of $(\phi, y_1, y_2) \mapsto C_p(t, \phi, y_1, y_2)$ with $(\phi, y_1, y_2) \in Y$. Recall that the DMS at $M_1 = L - Z_1(p) := M_1(p)$ and $M_2 = B - Z_2(p) := M_2(p)$ coincides with the trivial extension of the process $S^p(t, s)$ to the phase space Y. Restricting the domain of the centre manifold function C_p to the hypersurface $\mathbb{R} \times \{(\phi, y_1, y_2) \in Y : y_1 = M_1(p) \cdot \mathbb{1}, y_2 = M_2(p) \cdot \mathbb{1}\}$, the invariance, inclusion of small mild solutions and attractivity properties of its image imply the same results for the projection onto the first component through $\pi_{\mathcal{RCR}} : Y \to \mathcal{RCR}[-r^*, 0]$, where we choose $0 < \eta \leq \epsilon$ small enough so that $||L - Z_1(p)|| + ||B - Z_2(p)|| < \frac{1}{2}\gamma$ for $p \in (p^* - \eta, p^* + \eta)$ and define $\delta = \frac{1}{2}\gamma$, where γ and ϵ are as described in Theorem III.2.2.1. By property 1 from Theorem III.2.2.1, the aforementioned projection onto the first component is precisely the image of $C(t, \cdot; p)$. Smoothness with respect to ϕ follows from Theorem I.5.6.1. □

Remark III.2.2.2. *One cannot conclude from the above construction that the parameter-dependent centre manifold is smooth (or even continuous) with respect to the parameter* p, *even under compatible (i.e.* C^k*) conditions on* Z_1 *and* Z_2. *The centre manifold associated with the DMS (Theorem III.2.2.1) in the extended phase space, however, is* C^k *for each fixed* p.

III.2.3 Overlap Bifurcations

In this section we will be interested in systems of the form

$$\dot{x} = Ax(t) + f(x(t)), \qquad t \notin \mathbb{Z} \qquad \text{(III.2.9)}$$

$$\Delta x = Bx(t^-) + Cx(t - \omega) + g(x(t^-), x(t - \omega)), \qquad t \in \mathbb{Z}, \qquad \text{(III.2.10)}$$

for f and g sufficiently smooth functions satisfying $f(0) = g(0, 0) = 0$, real $n \times n$ matrices A, B and C and $\omega \geq 0$.

A remark: when $\omega = 0$, we will interpret $x(t-\omega)$ in (III.2.10) as the left-limit: $x(t^-)$. Formally, the jump condition should be understood as

$$\Delta x = Bx(t^-) + C\mathrm{ev}_{-\omega}(x_{t^-}) + g(x(t^-), \mathrm{ev}_{-\omega}(x_{t^-})), \tag{III.2.11}$$

where $\mathrm{ev}_{-\omega}(\phi) = \phi(-\omega)$ is the evaluation functional at $-\omega$.

We have two related problems to investigate:

1. Characterize the Floquet spectrum for $\omega \in (1-\epsilon, 1)$, $\omega = 1$ and $\omega \in (1, 1+\epsilon)$ for small $\epsilon > 0$.
2. Describe the local orbit structure near $x = 0$ for $|\omega - 1| < \epsilon$ and small $\epsilon > 0$.

System (III.2.9)–(III.2.10) is essentially finite-dimensional. Indeed, with the help of a state transformation, one can eliminate the discrete delay for each ω fixed; it is introduced in the paper [30]. We will make use of this transformation in Sect. III.2.3.3 to set the stage for studying bifurcations.

Remark III.2.3.1. *The presentation of this section and the correctness of all results as stated depend crucially on the interpretation of the jump condition and on the limit convention used. For example, if the regulated left-limit* x_t^- *is used in* (III.2.11) *instead of the one-point limit* x_{t^-}*, many results will change.*

III.2.3.1 Floquet Spectrum

Here we will characterize the Floquet spectrum of (III.2.9)–(III.2.10) for $\omega \approx 1$. The analysis will be split into two stages. First, we will consider the case $\omega \le 1$. Next, we look at the $\omega > 1$ case. Neither proof is difficult.

Lemma III.2.3.1. *Let* $\omega \in [0, 1]$. λ *is a Floquet exponent for the linearization of* (III.2.9)–(III.2.10) *at* $x = 0$ *if and only if* $\mu = e^\lambda$ *satisfies the equation*

$$\det\Big((I+B)e^A + Ce^{A(1-\omega)} - \mu I\Big) = 0. \tag{III.2.12}$$

That is, μ *is an eigenvalue of* $(I+B)e^A + Ce^{A(1-\omega)}$.

Proof. Let $x(t) = \phi(t)e^{t\lambda}$ be a Floquet eigensolution. If $\omega \in (0, 1]$, the periodic function ϕ (complex-valued with period one) satisfies

$$\begin{aligned} \dot\phi &= (A - \lambda I)\phi, && t \notin \mathbb{Z} \\ \Delta\phi &= B\phi(t^-) + e^{-\lambda\omega}C\phi(t-\omega), && t \in \mathbb{Z}. \end{aligned}$$

At time $t = 1$, the solution with initial condition $\phi(0) = \phi_0$ satisfies

$$\phi(1) = \Big[(I+B)e^{A-\lambda I} + e^{-\lambda\omega}Ce^{(A-\lambda I)(1-\omega)}\Big]\phi_0 = \Big[(I+B)e^A + Ce^{A(1-\omega)}\Big]e^{-\lambda}\phi_0.$$

ϕ is periodic if and only if $\phi(1) = \phi_0$. This is equivalent to $\mu = e^{\lambda}$ satisfying

$$\det\left(\left[(I+B)e^{A} + Ce^{A(1-\omega)}\right]\mu^{-1} - I\right) = 0,$$

which is equivalent to (III.2.12). For the case $\omega = 0$, we get

$$\begin{aligned} \dot{\phi} &= (A-\lambda I)\phi, && t \notin \mathbb{Z} \\ \Delta\phi &= B\phi(t^-) + C\phi(t^-), && t \in \mathbb{Z}, \end{aligned}$$

so that $\phi(t)$ satisfies

$$\phi(1) = (I+B+C)e^{(A-\lambda I)}\phi_0.$$

The result follows by the same argument as the previous case. □

Lemma III.2.3.2. *Let $\omega \in (1,2)$. λ is a Floquet exponent for the linearization of* (III.2.9)–(III.2.10) *at $x = 0$ if and only if $\mu = e^{\lambda}$ satisfies the equation*

$$\det\left((I+B)e^{A}\mu + Ce^{A(2-\omega)} - \mu^2 I\right) = 0. \tag{III.2.13}$$

Proof. Let $x(t) = \phi(t)e^{t\lambda}$ be a Floquet eigensolution. The periodic function ϕ (complex-valued with period one) satisfies

$$\begin{aligned} \dot{\phi} &= (A-\lambda I)\phi, && t \notin \mathbb{Z} \\ \Delta\phi &= B\phi(t^-) + e^{-\lambda\omega}C\phi(t+1-\omega), && t \in \mathbb{Z}. \end{aligned}$$

Note that $1-\omega \in (0,1)$. At time $t = 1$, the solution with initial condition $\phi(0) = \phi_0$ satisfies

$$\begin{aligned} \phi(1) &= \left[(I+B)e^{A-\lambda I} + e^{-\lambda\omega}Ce^{(A-\lambda I)(2-\omega)}\right]\phi_0 \\ &= \left[(I+B)e^{A} + Ce^{A(2-\omega)}e^{-\lambda}\right]e^{-\lambda}\phi_0. \end{aligned}$$

ϕ is periodic if and only if $\phi(1) = \phi_0$. This is equivalent to $\mu = e^{\lambda}$ satisfying

$$\det\left(\left[(I+B)e^{A} + Ce^{A(1-\omega)}\mu^{-1}\right]\mu^{-1} - I\right) = 0,$$

which is equivalent to (III.2.13). □

From these two lemmas, we immediately conclude that for $\omega \in (0,1]$, there are at most n nonzero Floquet multipliers counting multiplicities, whereas for $\omega \in (1,2)$ there are at most n^2. There does not appear to be any general connection between the Floquet multipliers at $\omega = 1$ and the limit from the right, $\omega \to 1^+$.

III.2.3.2 Symmetries of Periodic Solutions

One observation that will be helpful later in analyzing bifurcations of periodic solutions is the following symmetry property.

Lemma III.2.3.3. *Let $x(t)$ be a periodic solution of (III.2.9)–(III.2.10) with period one, for delay parameter $\omega > 0$. Then, $x(t)$ is a periodic solution for the delay parameter $\omega + k$ for any $k \in \mathbb{Z}$ such that $\omega + k > 0$.*

Proof. Since x is periodic with period one, we have $x(t-\omega) = x(t-(\omega+k))$ whenever $\omega + k > 0$ and $k \in \mathbb{Z}$. The result follows. □

The reason for the strict inequalities $\omega > 0$ and $\omega + k > 0$ in the previous lemma is because when $\omega = 0$, the jump condition (III.2.11) reduces to

$$\Delta x = (B + C)x(t^-) + g(x(t^-), x(t^-)),$$

but when $\omega = k > 0$ for $k \in \mathbb{N}$, it is

$$\Delta x = Bx(t^-) + Cx(t-k) + g(x(t^-), x(t-k)).$$

Since $x(t^-) \neq x(t) = x(t-k)$, it is not possible to compare periodic solutions for $\omega = 0$ with $\omega \in \mathbb{N}$.

Take a note that this lemma applies to both linear and nonlinear systems. Consequently, it implies furthermore that whenever $\mu = 1$ is a Floquet multiplier for $\omega = 1$ or as $\omega \to 1^+$, the same is true for $\omega = k$ a positive integer or as $\mu \to k^+$.

III.2.3.3 A State Transformation that Eliminates the Delay

In order to analyze the local orbit structure near $\omega = 1$, we will introduce a delayed state transformation that eliminates the delay, producing a truly finite-dimensional system. The result will be an impulsive system whose impulse times depend on the parameter ω. For additional background on the transformation, see [30]. The explicit state transformation will be different depending on whether $\omega < 1$, $\omega = 1$ or $\omega > 1$. This should not be surprising considering the results of the previous section.

Lemma III.2.3.4 (Delayed State Transformation: $\omega < 1$)**.** *Suppose $\omega \in (0, 1)$, and consider the finite-dimensional impulsive differential equation*

$$\dot{x} = Ax(t) + f(x(t)), \qquad t \notin \mathbb{Z} \tag{III.2.14}$$

$$\dot{y} = 0, \qquad t \notin \mathbb{Z} - \omega \tag{III.2.15}$$

$$\Delta x = Bx(t^-) + Cy(t^-) + g(x(t^-), y(t^-)), \qquad t \in \mathbb{Z} \tag{III.2.16}$$

$$\Delta y = x - y, \qquad t \in \mathbb{Z} - \omega. \tag{III.2.17}$$

If $X:\mathbb{R}\to\mathbb{R}^n$ is a solution of (III.2.9)–(III.2.10)*, then*

$$(x(t),y(t))=(X(t),X(\lfloor t\rfloor-\omega)) \tag{III.2.18}$$

is a solution of (III.2.14)–(III.2.17) *defined for $t\in\mathbb{R}$ and vice versa. The period of the transformed system is one, with two impulses per period. Moreover, X is locally asymptotically stable (respectively stable, unstable) if and only if the same is true for the solution* (III.2.18)*.*

Remark III.2.3.2. *The transformation* (III.2.18) *applies only to solutions that are defined on the real line, as required by the lemma. If $X:[a,\infty)\to\mathbb{R}^n$ is a solution of* (III.2.9)–(III.2.10) *and $a\in\mathbb{R}$ is finite, then the transformation generates a solution $(x(t),y(t))$ of the transformed equation that is defined for $t\geq a+\omega$.*

Lemma III.2.3.5 (Delayed State Transformation: $\omega=1$)**.** *Suppose $\omega=1$, and consider the finite-dimensional impulsive differential equation*

$$\begin{aligned}
\dot{x}&=Ax(t)+f(x(t)), && t\notin\mathbb{Z} && \text{(III.2.19)}\\
\dot{y}&=0, && t\notin\mathbb{Z} && \text{(III.2.20)}\\
\Delta x&=Bx(t^-)+Cy(t^-)+g(x(t^-),y(t^-)), && t\in\mathbb{Z} && \text{(III.2.21)}\\
\Delta y&=x+B(x(t^-))+Cy(t^-)+g(x(t^-),y(t^-))-y, && t\in\mathbb{Z}. && \text{(III.2.22)}
\end{aligned}$$

This system enjoys the same property as the one from Lemma III.2.3.4, but there is only one impulse per period.

Proof. Let (x,y) be a solution of (III.2.19)–(III.2.22). By construction, $y(t^-)=x(t-1)$ whenever $t\in\mathbb{Z}$. Since the continuous-time dynamics (III.2.19) are the same as those of (III.2.9)–(III.2.10), and the impulse effect (III.2.21) is also the same upon replacing $y(t^-)$ with $x(t-1)$, we get that $x(t)$ is a solution of (III.2.9)–(III.2.10). The converse is similar. □

Lemma III.2.3.6 (Delayed State Transformation: $\omega>1$)**.** *Suppose $\omega\in(1,2)$, and consider the finite-dimensional impulsive differential equation*

$$\begin{aligned}
\dot{x}&=Ax(t)+f(x(t)), && t\notin\mathbb{Z} && \text{(III.2.23)}\\
\dot{y}_0&=0, && t\notin 2\mathbb{Z}-\omega && \text{(III.2.24)}\\
\dot{y}_1&=0, && t\notin 2\mathbb{Z}+1-\omega && \text{(III.2.25)}\\
\Delta x&=Bx(t^-)+Cy_{[t]_2}(t^-)+g(x(t^-),y_{[t]_2}(t^-)), && t\in\mathbb{Z} && \text{(III.2.26)}\\
\Delta y_0&=x-y_0, && t\in 2\mathbb{Z}-\omega && \text{(III.2.27)}\\
\Delta y_1&=x-y_1, && t\in 2\mathbb{Z}+1-\omega. && \text{(III.2.28)}
\end{aligned}$$

If $X:\mathbb{R}\to\mathbb{R}^n$ is a solution of (III.2.9)–(III.2.10)*, then $(x(t),y_0(t),y_1(t))$, with $x(t)=X(t)$ and*

$$\begin{aligned}
y_0(t)&=X(k-\omega), && t\in[k,k+2), && k\in 2\mathbb{Z}\\
y_1(t)&=X(k-\omega), && t\in[k,k+2), && k\in 2\mathbb{Z}+1,
\end{aligned}$$

is a solution of (III.2.23)–(III.2.28) *defined for* $t \in \mathbb{R}$ *and vice versa, and their stability is equivalent. The period of the transformed system is two, and there are four impulses per period.*

Proof. If (x, y_0, y_1) is a solution of (III.2.23)–(III.2.28), we have two observations. First, if $k \in 2\mathbb{Z}$, then $x(k - \omega) = y_0(k^-)$. Second, if $k \in 2\mathbb{Z} + 1$, then $x(k - \omega) = y_1(k^-)$. For a solution defined for all time, this implies that at any time $t \in \mathbb{Z}$, the delayed state $x(t - \omega)$ is one of $y_0(t^-)$ or $y_1(t^-)$. The remainder $[t]_2$ in (III.2.26) keeps track of the correct one. The argument then follows the same lines as the proof for Lemma III.2.3.5. The converse is similar.

To see that the period is two and there are four impulses per period, observe that although the sequence of impulses can be identified with the set $\mathbb{Z} \cup (\mathbb{Z} - \omega)$, which upon sequential ordering gives a sequence t_k satisfying $t_{k+2} = t_k + 1$—that is, period one with two impulses—the sequence of jump functions defining Eqs. (III.2.26)–(III.2.28) can be identified with a function G_k satisfying $G_{k+4} = G_k$. As such, there are four impulses per period and the period is two, since $t_{k+4} = t_k + 2$. □

If we are interested in locating bounded solutions—for example, periodic solutions in invariant cylinders—it is sufficient for us to study the period maps associated with the state transformed systems of the previous three lemmas. However, since the transformed system for $\omega \in (1, 2)$ is of period two, we will need an additional result if we wish to detect periodic solutions of period one.

Lemma III.2.3.7. *Suppose* $\omega \in (1, 2)$. (x, y_0, y_1) *is a periodic solution of* (III.2.23)–(III.2.28) *with period one if and only if* $y_0 = y_1$ *is constant.*

Proof. Suppose (x, y_0, y_1) is a periodic solution of period one. Then, $y_0(k^-) = x(k - \omega)$ for $k \in 2\mathbb{Z}$, while $y_1(k^-) = x(k - \omega)$ for $k \in 2\mathbb{Z} + 1$. Since x is periodic with period one, $x(k - \omega) = x^*$ is constant for $k \in \mathbb{Z}$. Since each of y_0 and y_1 is piecewise-constant and only has an impulse every 2 time units, we have $y_0(t) = y_1(t) = x^*$ for all $t \in \mathbb{R}$.

Conversely, suppose $y_0 = y_1 = x^*$ is constant. Then $x^* = y_0(k) = x(k^-)$ for $k \in 2\mathbb{Z}$ and $x^* = y_1(k^-) = x(k^-)$ for $k \in 2\mathbb{Z} + 1$. Consequently, $x(k) = (I + B)x^* + Cx^* + g(x^*, x^*)$ for all $k \in \mathbb{Z}$, whereas for $t \in [k, k + 1)$, x is determined by the autonomous dynamics (III.2.23). It follows that x is periodic with period one. □

III.2.3.4 Bifurcations of Periodic Solutions

Based on the description of the Floquet spectrum from Sect. III.2.3.1, we should expect some sort of local bifurcation of periodic solutions at $\omega = 1$ if one or both of the following occur:

T.1 at $\omega = 1$, there is a Floquet multiplier μ satisfying $|\mu| = 1$;

T.2 as $\omega \to 1^+$, some Floquet multiplier μ satisfies $|\mu| \to 1$.

The T.1 scenario corresponds to the situation in which some bifurcation occurs for $\omega \in (1 - \epsilon, 1]$. Such a bifurcation can be detected by computing a Taylor expansion in (x, y) with ω as a parameter of the time 1 map of the transformed system (III.2.14)–(III.2.17). To bridge the gap as $\omega \to 1^-$, we need the following lemma.

Lemma III.2.3.8. *Let* $P_1(\cdot, \omega) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ *denote the time* 1 *map of* (III.2.14)–(III.2.17) *for parameter* $\omega < 1$. *Then,* $\lim_{\omega \to 1^-} P_1(\cdot, \omega)$ *is well-defined, uniformly near* $0 \in \mathbb{R}^n \times \mathbb{R}^n$. *The continuous extension* $P_1 : \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n \times \mathbb{R}^n$ *is smooth (with the same regularity as the nonlinearities* f *and* g*) in a neighbourhood of* $(0, 0, 1) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, 1]$.

It can be proven by appealing to the time 2 map and the integral representation of solutions. Interestingly, the limit described in Lemma III.2.3.8 is completely unrelated to the time 1 map of the delayed state transformation at $\omega = 1$. We will observe this in Sect. III.2.3.5.

The second T.2 scenario suggests a possible bifurcation in the interval $[1, 1 + \epsilon)$. Formally, this can be justified by the following lemma, whose proof we omit.

Lemma III.2.3.9. *Let* $P_2(\cdot, \omega) : (\mathbb{R}^n)^3 \to (\mathbb{R}^n)^3$ *denote the time* 2 *map for* (III.2.23)–(III.2.28). *The limit* $\lim_{\omega \to 1^+} P_2(\omega, \cdot)$ *exists uniformly in a neighbourhood of* $0 \in (\mathbb{R}^n)^3$. *The continuous extension* $P_2 : (\mathbb{R}^n)^3 \times [1, 2) \to (\mathbb{R}^n)^3$ *is smooth (with the same regularity as the nonlinearities* f *and* g*) in a neighbourhood of* $(0, 1) \in (\mathbb{R}^n)^3 \times [1, 2)$.

For $\omega \in (1, 2)$ the time 2 map will be smooth with respect to all the variables. The limit property of the above lemma then justifies using bifurcation theorems in the interval $[1, 1 + \epsilon)$.

If scenario T.1 or T.2 explicitly does *not* occur, all Floquet multipliers should be away from the unit circle in the relevant ω regime: $(1 - \epsilon, 1]$ for the negation of T.1 and $(1, 1 + \epsilon)$ for the negation of T.2. Ideally, one would want to prove that this parameter regime does not contain any small (i.e. trivializing to zero as $\omega \to 1$) bounded solutions aside from the trivial solution $x = 0$. This in fact follows from very general static bifurcation theory applied to the period map—see [115], Proposition 2.6 for the relevant result. Taking into account Lemma III.2.3.3, we get the following theorem.

Theorem III.2.3.1 (Overlap Bifurcation)**.** *The following hold in a neighbourhood of the overlap point* $\omega = 1$ *for the general nonlinear system* (III.2.9)–(III.2.10)*:*

- *If scenario T.1 occurs, $x = 0$ may exhibit a bifurcation for $\omega \in (1-\epsilon, 1]$ and some small $\epsilon > 0$. That is, there may exist a nontrivial bounded solution x_ω satisfying $x_\omega \to 0$ as $\omega \to 1^-$.*
- *If scenario T.2 occurs, $x = 0$ may exhibit a bifurcation for $\omega \in [1, 1+\epsilon)$ and some small $\epsilon > 0$. That is, there may exist a nontrivial bounded solution x_ω satisfying $x_\omega \to 0$ as $\omega \to 1^+$.*
- *If scenario T.1 does not occur, there exist $\delta > 0$ and $\epsilon > 0$ such that the only bounded solution (defined for all time) contained in the ball $B_\delta(0)$ for $\omega \in (1-\epsilon, 1]$ is the trivial solution $x = 0$.*
- *If scenario T.2 does not occur, there exist $\delta > 0$ and $\epsilon > 0$ such that the only bounded solution (defined for all time) contained in the ball $B_\delta(0)$ for $\omega \in [1, 1+\epsilon)$ is the trivial solution $x = 0$.*

III.2.3.5 The Introductory Example Revisited

Let us return to the system

$$\dot{x} = \log\left(\frac{3}{2}\right)x - \frac{1}{10}x^2, \qquad t \notin \mathbb{Z} \tag{III.2.29}$$

$$\Delta x = -\frac{1}{2}x(t^-) + \frac{1}{4}x_{t^-}(-\omega), \qquad t \in \mathbb{Z}, \tag{III.2.30}$$

from the overview of this chapter. We saw that at $\omega = 1$, there is a Floquet multiplier $\mu = 1$, whereas for $\omega \in (1, 2)$ all Floquet multipliers have absolute value greater than one and, in particular, $\mu \not\to 1$ as $\omega \to 1^+$. This is a T.1 overlap bifurcation scenario, so we should compute a Taylor expansion for $\omega \in [1-\epsilon, 1]$ from the time 1 map of the delayed state transformation. For $\omega < 1$, the latter is

$$\dot{x} = \log\left(\frac{3}{2}\right)x - \frac{1}{10}x^2, \quad t \notin \mathbb{Z} \qquad \Delta x = -\frac{1}{2}x + \frac{1}{4}y, \quad t \in \mathbb{Z} \tag{III.2.31}$$

$$\dot{y} = 0, \quad t \notin \mathbb{Z} - \omega \qquad \Delta y = x - y, \quad t \in \mathbb{Z} - \omega. \tag{III.2.32}$$

The time 1 map admits an expansion of the form

$$z \mapsto c_1(\omega)z + \frac{1}{2}c_2(\omega)z^2 + O(z^3) = P_1(x, \omega)$$

for $z = [\ x \quad y\ ]^\intercal$, c_1 a linear map and c_2 a bilinear map, with $P_1(0, \omega) = 0$. From Lemma III.2.3.8, we know that the coefficients are smooth.

Computing $c_1(\omega)$ is relatively straightforward since it is simply the monodromy matrix of the linearization at $(x,y)=0$ of (III.2.32)–(III.2.31). One can check that

$$c_1(\omega) = \begin{bmatrix} \left(\frac{3}{2}\right)^{1-\omega}\left(\frac{1}{2}\left(\frac{3}{2}\right)^{\omega}+\frac{1}{4}\right) & 0 \\ \left(\frac{3}{2}\right)^{1-\omega} & 0 \end{bmatrix}. \tag{III.2.33}$$

For the coefficient c_2, we have $c_2(\omega) = Z(1)$, where Z is a symmetric bilinear map defined by $Z(0) = 0$ and

$$\begin{aligned} \dot{Z}[h_1,h_2] &= \begin{bmatrix} \log(3/2) & 0 \\ 0 & 0 \end{bmatrix} Z[h_1,h_2] - \frac{2}{10} e_1 h_1^\intercal E_{11} h_2 (3/2)^{2t}, && t \notin \{1, 1-\omega\} \\ \Delta Z[h_1,h_2] &= \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} Z[h_1,h_2], && t = 1-\omega \\ \Delta Z[h_1,h_2] &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ 0 & 0 \end{bmatrix} Z[h_1,h_2], && t = 1. \end{aligned}$$

Computing the solution and evaluating at $t = 1$, we get

$$\begin{aligned} &c_2(\omega)[h_1,h_2] \\ &\quad = -\frac{3h_1^\intercal E_{11} h_2}{20\log(3/2)} \underbrace{\begin{bmatrix} \left(\frac{3}{2}\right)^{\omega} - 1 + \frac{1}{2}\left(\frac{3}{2}\right)^{-\omega}\left(2\left(\frac{3}{2}\right)^{\omega}-3\right)\left(1+\frac{1}{2}\left(\frac{3}{2}\right)^{-\omega}\right) \\ \left(\frac{3}{2}\right)^{-\omega}\left(2\left(\frac{3}{2}\right)^{\omega}-3\right) \end{bmatrix}}_{\tilde{c}(\omega)}. \end{aligned}$$

The quadratic-order expansion of the time 1 map therefore takes the form

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \left(\frac{3}{2}\right)^{1-\omega}\left(\frac{1}{2}\left(\frac{3}{2}\right)^{\omega}+\frac{1}{4}\right) & 0 \\ \left(\frac{3}{2}\right)^{1-\omega} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \frac{3}{20\log(3/2)}\tilde{c}(\omega)x^2 + O(||(x,y)||^3),$$

uniformly for $0 < 1-\omega \ll 1$. Applying the Lyapunov–Schmidt reduction, we can uniquely solve the fixed-point problem associated with the second equation

$$y = 2^{1-\omega}x - \frac{3\left(2-3\left(\frac{3}{2}\right)^{-\omega}\right)}{20\log(3/2)}x^2 + O(||(x,y)||^3)$$

for $y = y(x,\omega)$ near $(x,\omega) = (0,1)$. The fixed point satisfies $y(0,\omega) = 0$. Restricted to the curve $y = y(x,\epsilon)$, x is a fixed point of the first equation if and only if

$$x = \left(\frac{3}{2}\right)^{1-\omega}\left(\frac{1}{2}\left(\frac{3}{2}\right)^{\omega}+\frac{1}{4}\right)x + e_1^\intercal \tilde{c}(\omega)x^2 + O(||x||^3) \equiv F(x,\omega)$$

uniformly for $|\omega - 1|$ small. We have

$$F(0,1) = 0, \qquad \partial_x F(0,1) = 1,$$
$$\partial_{x\omega} F(0,1) = -\frac{\log(3/2)}{4}, \qquad \partial_{xx} F(0,1) = \frac{3}{40\log(3/2)},$$

and $F(0,\omega) = 0$. A transcritical bifurcation therefore occurs at $\omega = 1$, and the x component of the branch of nontrivial fixed points satisfies

$$x^* = \frac{10}{3}\left(\log\left(\frac{3}{2}\right)\right)^2 (\omega - 1) + O((\omega - 1)^2).$$

Since the time 1 map we have computed only represents the dynamics of the system (III.2.29)–(III.2.29) for $\omega \in (1-\epsilon, 1)$ and some small $\epsilon > 0$, the nontrivial fixed point is only guaranteed to exist in this range. With the aid of Theorem III.2.3.1 and linearized stability, we can make the following conclusion.

Lemma III.2.3.10. *The trivial solution of* (III.2.29)*–*(III.2.30) *undergoes a transcritical bifurcation at $\omega = 1$ in the interval $(1-\epsilon, 1]$ for some $\epsilon > 0$ small. More precisely, there is a nontrivial, positive periodic solution $t \mapsto x_\omega(t)$ of period one that exists for $\omega \in (1-\epsilon, 1)$ for some $\epsilon > 0$ small and satisfies $x_\omega(0) > 0$ and $\lim_{\omega\to 1^-} x_\omega = 0$. There exists $\delta > 0$ such that no periodic solution apart from x_ω and the trivial solution exist in the ball $B_\delta(0)$. The nontrivial periodic solution is locally asymptotically stable for $\omega \in (1-\epsilon, 1)$, while the trivial solution is unstable.*

By the symmetry of Lemma III.2.3.3, we obtain the following corollary.

Corollary III.2.3.1. *For each positive integer k, the trivial solution of* (III.2.29)*–*(III.2.30) *undergoes a transcritical bifurcation at $\omega = k$ in the interval $(k-\epsilon, k]$ for some $\epsilon > 0$ small.*

Figure III.2.1 provides the plots of numerically computed solutions of (III.2.29)–(III.2.30) for two choices of delays near $\omega = 1$. We can see the nontrivial positive periodic solution for $\omega \in (1-\epsilon, 1)$, as well as a "large" nontrivial periodic solution for $\omega \in (1, 1+\epsilon)$. Performing a parameter continuation, it turns out that this "large" nontrivial periodic solution is created near $\omega = 2$ in the transcritical bifurcation predicted by Corollary III.2.3.1. The bifurcation diagram is provided in Fig. III.2.2.

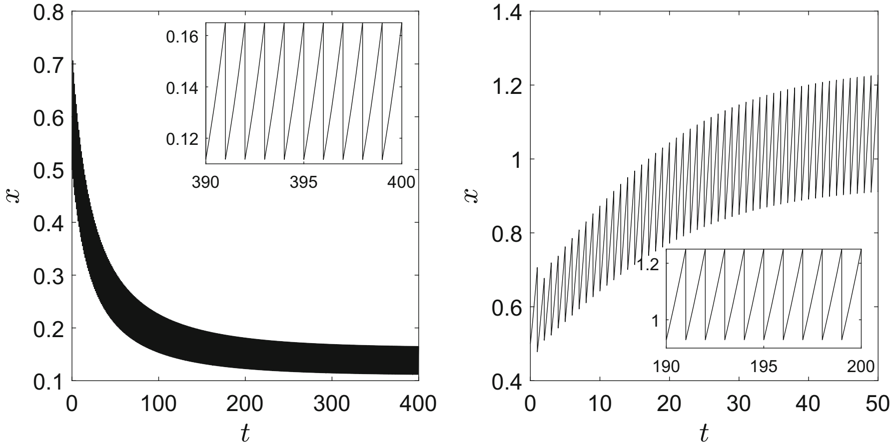

Figure III.2.1: Plots of numerical solutions from the constant initial condition $x_0 = 0.5$ for the scalar equation (III.2.29)–(III.2.30) with delay $\omega = 0.9$ (left) and $\omega = 1.1$ (right). Inset: windowing of the numerical solutions for various time arguments, once convergence to the periodic solution is (nearly) achieved

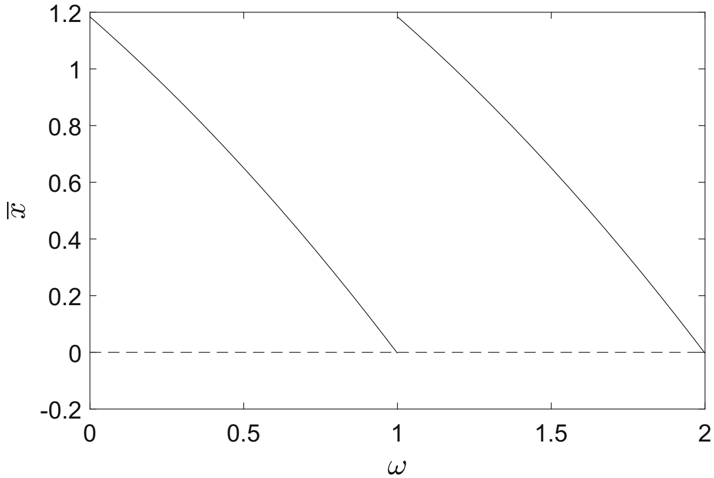

Figure III.2.2: Bifurcation diagram of (III.2.29)–(III.2.30) for $\omega \in [0, 2]$. Curves denote the mean value ($\overline{x}$) of a periodic solution. The nontrivial curves are asymptotically stable (solid curves), while the trivial zero fixed point is unstable (dashed line)

III.2.4 Comments

We have considered only two simple examples in this chapter of bifurcations driven by the perturbations to impulses and delay-impulse overlaps. Outside of this monograph, the former appears to have never been considered (when delays are involved). As for the later, Church and Liu [30] studied bifurcations in a logistic model with harvesting and census delay at a parameter configuration where the overlap condition was violated, but the bifurcation parameter was one of the smooth model parameters and not the delay parameter. On the whole, neither of these bifurcations—bifurcations driven by perturbations to impulse times or overlap bifurcations—is well-understood, and the methods we have developed here to analyze them are somewhat difficult to use.

In the example of Sect. IV.2, we needed to assume that $T^* > \tau$. If this assumption did not hold, we would be in the scope of an overlap bifurcation (if $T^* = \tau$) or at the very least need to modify the construction of the parameter-dependent centre manifold. Since the delay appears in the continuous part of the dynamics, however, the main conclusions of Sect. III.2.3 do not apply here in the case of an overlap bifurcation. How to analyze such a bifurcation scenario is unclear.

Part IV

Applications

Chapter IV.1

Bifurcations in an Impulsively Damped or Driven Pendulum

Consider the ordinary differential equation for the motion of a simple pendulum with drag

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta + F(\dot{\theta}) = 0.$$

θ represents the angle made by the rigid rod of the pendulum from the downward, vertical equilibrium position $\theta = 0$, g is acceleration due to gravity, ℓ is the length of the pendulum rod and $F(\dot{\theta})$ is drag. We will assume that

$$F(x) = \beta x^3 + O(x^4)$$

for some positive constant β. The reason we choose this function is threefold:

- we assume that the pendulum's bob is large, so that except at small velocities linear drag is negligible;
- the effect of air resistance is well-described by proportionality to $\left|\frac{d\theta}{dt}\right|\frac{d\theta}{dt}$ for a large bob [147];
- F provides a reasonable qualitative approximation to $x \mapsto \beta x|x|$ while being smooth.

We will consider two forms of damping and driving dynamics, both of which will be described by an impulse. First, suppose that for every unit

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of time $T > 0$ a damping or driving force is applied on a very short time scale such that the angular velocity is multiplied by some factor α. This corresponds to a discrete map $\dot{\theta} \mapsto \alpha\dot{\theta}$. Modelling this with impulsive differential equations, if we define $v = \dot{\theta}$ and drop the higher-order terms in the expansion for F, the resulting system is

$$\begin{aligned} \dot{\theta} &= v, & t &\neq kT & \Delta\theta &= 0, & t &= kT \\ \dot{v} &= -\frac{g}{\ell}\sin\theta - \beta v^3, & t &\neq kT & \Delta v &= (\alpha - 1)v, & t &= kT. \end{aligned} \tag{IV.1.1}$$

In this chapter we will refer to (IV.1.1) as *the model without delay* (for reasons that will become apparent soon).

The second form of damping/driving is similar to the first, except that this time the angular velocity is reset according to the average of the angular velocity over a small interval prior to damping/driving, multiplied by the parameter α. That is, the angular velocity is reset according to

$$\dot{\theta} \mapsto \frac{\alpha}{r}\int_{t-\tau}^{t}\dot{\theta}(s)ds = \frac{\alpha}{r}(\theta(t^-) - \theta(t-\tau))$$

at times $t = kT$ for $k \in \mathbb{Z}$, where we require $\tau \in (0, T)$. Formally, as $\tau \to 0$ the impulse effect of this model collapses to that of (IV.1.1). Writing the model in terms of impulsive *delay* differential equations, we get

$$\begin{aligned} \dot{\theta} &= v, & t &\neq kT & \Delta\theta &= 0, & t &= kT \\ \dot{v} &= -\frac{g}{\ell}\sin\theta - \beta v^3, & t &\neq kT & \Delta v &= -v + \frac{\alpha}{\tau}(\theta(t) - \theta(t-\tau)), & t &= kT, \end{aligned} \tag{IV.1.2}$$

where $v = \dot{\theta}$ like before. In this chapter we will refer to (IV.1.2) as *the model with delay.*

IV.1.1 Stability Analysis: The Model Without Delay

There are two equilibria in the model without delay: the downward rest position ($\theta = v = 0$) and the upward rest position ($\theta = \pi$, $v = 0$). We will analyze these two separately.

IV.1.1.1 Downward Rest Position

We want to determine conditions under which the equilibrium $\theta = v = 0$ is stable or unstable. Performing a linearization, we get the linear system

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{bmatrix} z, & t &\neq kT \\ \Delta z &= \begin{bmatrix} 0 & 0 \\ 0 & (\alpha - 1) \end{bmatrix} z, & t &= kT. \end{aligned}$$

The monodromy matrix M_0 is easy to calculate; it is

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} \cos(\rho T) & \frac{1}{\rho}\sin(\rho T) \\ -\rho\sin(\rho T) & \cos(\rho T) \end{bmatrix} = \begin{bmatrix} \cos(\rho T) & \frac{1}{\rho}\sin(\rho T) \\ -\alpha\rho\sin(\rho T) & \alpha\cos(\rho T) \end{bmatrix},$$

where $\rho^2 = \frac{g}{\ell}$. The characteristic polynomial (in the variable μ) of this matrix is

$$\mu^2 - \mu(\alpha + 1)\cos(\rho T) + \alpha. \tag{IV.1.3}$$

Computing the Floquet multipliers, we find that they are

$$\mu = \frac{(\alpha + 1)\cos(\rho T)}{2} \pm \sqrt{\frac{(\alpha + 1)^2}{4}\cos^2(\rho T) - \alpha}. \tag{IV.1.4}$$

There are two regimes we need to consider.

Case 1: $(\alpha + 1)^2 \cos^2(\rho T) \leq 4\alpha$

If $(\alpha+1)^2 \cos^2(\rho T) \leq 4\alpha$, then the term inside the square root is nonpositive. One can then check directly that $|\mu|^2 = \alpha$. Based on Theorem II.3.2.1, the stability assertions are summarized in Table IV.1.1.

Case 2: $(\alpha + 1)^2 \cos^2(\rho T) > 4\alpha$

In this case, the term inside the square root of (IV.1.4) is positive, so both Floquet multipliers are real, and the largest Floquet multiplier (in absolute value) satisfies

$$|\mu| = \frac{|\alpha + 1| \cdot |\cos(\rho T)|}{2} + \sqrt{\frac{(\alpha + 1)^2}{4}\cos^2(\rho T) - \alpha}.$$

It is useful to make the transformation $z = \alpha + 1$ and $\nu = |\cos(\rho T)|$. In these variables, the inequality $|\mu| < 1$ is equivalent to

$$r(z, \nu) := \sqrt{\frac{z^2\nu^2}{4} - z + 1} + \frac{|z|\nu}{2} - 1 < 0. \tag{IV.1.5}$$

	$(\alpha+1)^2\cos^2(\rho T)\le 4\alpha$	$(\alpha+1)^2\cos^2(\rho T)> 4\alpha$
Exponentially stable	$\alpha<1$	$\alpha>-1,\quad \rho T\notin\pi\mathbb{Z}$
Unstable	$\alpha>1$	$\alpha<-1$
Boundary, $\lvert\mu\rvert=1$	$\alpha=1$	$\alpha=-1$ **or** $\alpha>-1,\quad \rho T\in\pi\mathbb{Z}$
Floquet multiplier type	Complex conjugate	Real

Table IV.1.1: Parameter regimes for various stability classifications of the downward rest position $\theta=v=0$ in the model without delay (IV.1.1), with $\rho=g/\ell$. In the third row, the linearized stability theorem fails because there is a Floquet multiplier on the complex unit circle.

With respect to $\nu\in[0,\infty)$, $\nu\mapsto r(z,\nu)$ is increasing. As such, we will have $|\mu|<1$ if and only if $\nu<\nu^*$, where ν^* is the solution of the equation $r(z,\nu^*)=0$. Solving this equation is straightforward, and we end up with

$$\nu^*(z)=\frac{z}{|z|}=\operatorname{sgn}(z).$$

We therefore have $|\mu|<1$ if and only if $|\cos(\rho T)|<\operatorname{sgn}(\alpha+1)$. This will be the case if and only if $\rho T\notin\pi\mathbb{Z}$ and $\alpha>-1$.

The equality $r(z,\nu)=0$ corresponds to the cases where $|\mu|=1$. The relevant equation is

$$|\alpha+1|\cdot|\cos(\rho T)|=\alpha+1.$$

This can only occur if $\rho T\in\pi\mathbb{Z}$ and $\alpha>-1$ or if $\alpha=-1$. As for $|\mu|>1$, this corresponds to $r(z,\nu)>0$. If $z\neq 0$, the latter can only occur if $\alpha<-1$. If $z=0$—that is, $\alpha=-1$—we have $r(z,\nu)=1$, so $|\mu|=1$ for any ν. Based on Theorem II.3.2.1, the stability assertions are summarized in Table IV.1.1.

IV.1.1.2 Upward Rest Position

We can perform a similar analysis on the upward rest position $(\theta,v)=(\pi,0)$. The linearization is very similar to the previous case:

$$\begin{aligned}\dot z&=\begin{bmatrix}0&1\\ \frac{g}{\ell}&0\end{bmatrix}z, && t\neq kT\\ \Delta z&=\begin{bmatrix}0&0\\0&(\alpha-1)\end{bmatrix}z, && t=kT.\end{aligned}$$

One can then calculate the Floquet multipliers:

$$\mu(\alpha)=\frac{(\alpha+1)\cosh(\rho T)}{2}\pm\sqrt{\frac{(\alpha+1)^2}{4}\cosh^2(\rho T)-\alpha}. \tag{IV.1.6}$$

	$\alpha = -1$	$\alpha \neq -1$
Dominant $\|\mu\|$	1	> 1
Stability	Undecided to linear order	Unstable

Table IV.1.2: Modulus $|\mu|$ of the dominant Floquet multiplier for the upward rest position $(\theta, v) = (\pi, 0)$ in the model without delay (IV.1.1) along with stability classification, based on the parameter α. The classification is much simpler here than in the downward rest position. Also, the Floquet multipliers are real.

Notice the formal similarity to Eq. (IV.1.4), with the only difference being the presence of the hyperbolic cosine function.

At this stage, we will emulate the analysis of the previous section and break into cases.

Case 1: $(\alpha+1)^2 \cosh^2(\rho T) \leq 4\alpha$

This case is trivial. Indeed, if α satisfies the above equation, then $(\alpha+1)^2 \leq 4\alpha$, which is equivalent to $(\alpha-1)^2 \leq 0$. The only solution is $\alpha = 1$. However, if $\alpha = 1$, then the inequality $4\cosh^2(\rho T) \leq 4$ implies $\rho T = 0$, which in this model is not permissible since ρ and T are positive.

Case 2: $(\alpha+1)^2 \cosh^2(\rho T) > 4\alpha$

By exclusion, this is the only nontrivial case. The dominant Floquet multiplier μ satisfies

$$|\mu| = \frac{|\alpha+1|\cosh(\rho T)}{2} + \sqrt{\frac{(\alpha+1)^2}{4}\cosh^2(\rho T) - \alpha}.$$

We use a similar trick to the downward rest point case and denote $z = \alpha + 1$ and $\nu = \cosh(\rho T)$, so that the equation $|\mu| < 1$ is equivalent to $r(z,\nu) < 0$, with the function r the same one appearing in (IV.1.5). Again, $\nu \mapsto r(z,\nu)$ is nondecreasing for $\nu \in [1,\infty)$, so all we need to do is solve the equation $r(z,\nu^*) = 0$ for ν^*, since stability will be ensured for $\nu < \nu^*$, instability for $\nu > \nu^*$ and the boundary case when $\nu = \nu^*$. Just like last time, solving $r(z,\nu^*) = 0$ gives $\nu^* = \operatorname{sgn}(z)$ for $z \neq 0$, whereas when $z = 0$ we have $\alpha = -1$, and it is easy to check that $\mu(-1) = 1$. For the $z \neq 0$ case, exponential stability is impossible because $\nu = \cosh(\rho T) > 1$, so it is not possible to have $\nu < \nu^*$. We summarize the results in Table IV.1.2.

IV.1.2 Stability Analysis: The Model with Delay

We will use Theorem I.3.4.3 to compute the Floquet multipliers. Once again, we will analyze the downward and upward rest positions separately.

IV.1.2.1 Downward Rest Position

Linearizing at the downward rest position, we get the linear system

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{bmatrix} z(t), & t \neq kT \\ \Delta z &= \begin{bmatrix} 0 & 0 \\ \frac{\alpha}{\tau} & -1 \end{bmatrix} z(t^-) + \begin{bmatrix} 0 & 0 \\ -\frac{\alpha}{\tau} & 0 \end{bmatrix} z(t-\tau), & t = kT. \end{aligned}$$

Let $\rho = \sqrt{g/\ell} > 0$. Applying Theorem I.3.4.3, $\mu \neq 0$ is a Floquet multiplier if and only if it is an eigenvalue of the matrix

$$\begin{bmatrix} \cos(\rho T) & \frac{1}{\rho}\sin(\rho T) \\ \frac{\alpha}{\tau}(\cos(\rho T) - \cos(\rho(T-\tau))) & \frac{\alpha}{\rho\tau}(\sin(\rho T) - \sin(\rho(T-\tau)) \end{bmatrix}.$$

The characteristic polynomial (in the variable μ) of this matrix is

$$\mu^2 + \mu\frac{1}{\rho\tau}\Big(\alpha(\sin(\rho(T-\tau)) - \sin(\rho T)) - \rho r\cos(\rho T)\Big) + \frac{\alpha}{\rho\tau}\sin(\rho\tau). \tag{IV.1.7}$$

It is worth mentioning that as $\tau \to 0$, the coefficients of the polynomial (IV.1.7) converge to those of the polynomial (IV.1.3) of the monodromy matrix for the model without delays. Applying the Jury stability criterion, all Floquet multipliers μ satisfy $|\mu| < 1$ if and only if

$$\rho\tau + \alpha\sin(\rho\tau) \pm \Big(\alpha(\sin(\rho T) - \sin(\rho(T-\tau))) + \rho\tau\cos(\rho T)\Big) > 0 \tag{IV.1.8}$$

$$1 - \frac{|\alpha\sin(\rho\tau)|}{\rho\tau} > 0. \tag{IV.1.9}$$

More importantly for our analysis in the following section, we can determine sufficient conditions for the existence of a pair of complex-conjugate Floquet multipliers with absolute value of 1. The following lemma follows by a straightforward analysis of the roots of the quadratic polynomial (IV.1.7).

Lemma IV.1.2.1. *If $\alpha(\rho\tau)^{-1}\sin(\rho\tau) = 1$, the Floquet multipliers at the downward rest position in the model with delay are complex conjugate with nonzero imaginary part and have absolute value of* 1 *if the following inequality is satisfied:*

$$\left|\frac{\sin(\rho(T-\tau)) - \sin(\rho T)}{\sin(\rho\tau)} - \cos(\rho T)\right| < 2. \tag{IV.1.10}$$

If the Floquet multipliers are complex conjugate with nonzero imaginary part, then the absolute value of these multipliers is $|\mu| = |\alpha(\rho\tau)^{-1}\sin(\rho\tau)|$.

IV.1.2.2 Upward Rest Position

We can perform a similar analysis of the upward rest position. Linearizing, we get

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{bmatrix} z(t), \qquad t \neq kT$$

$$\Delta z = \begin{bmatrix} 0 & 0 \\ \frac{\alpha}{\tau} & -1 \end{bmatrix} z(t^-) + \begin{bmatrix} 0 & 0 \\ -\frac{\alpha}{\tau} & 0 \end{bmatrix} z(t-\tau), \qquad t = kT.$$

Applying Theorem I.3.4.3, $\mu \neq 0$ is a Floquet multiplier if and only if it is an eigenvalue of the matrix

$$\begin{bmatrix} \cosh(\rho T) & \frac{1}{\rho}\sinh(\rho T) \\ \frac{\alpha}{\tau}(\cosh(\rho T) - \cosh(\rho(T-\tau))) & \frac{\alpha}{\rho\tau}(\sinh(\rho T) - \sinh(\rho(T-\tau)) \end{bmatrix}.$$

The characteristic polynomial of this matrix is identical to (IV.1.7) except that all trigonometric functions are replaced with their hyperbolic variants. One can therefore derive stability criterion using the Jury test analogously to (IV.1.8)–(IV.1.9). However, we will not be analyzing bifurcations from the upward rest position in the model with delay, so we will omit this step.

IV.1.3 Cylinder Bifurcation at the Downward Rest Position in the Model Without Delays

To simplify the presentation, we will perform four transformations in order:

- a rescaling of time $Ts = t$,
- define $\rho^2 = \frac{g}{\ell}$ and $\sigma = \alpha - 1$,
- a change of variables $v = \rho\nu$ and
- define new parameters $\xi = \rho T$ and $r = \beta T \rho^2$.

After these transformations, if we replace s with t, then the impulsive differential equations for the model without delay become

$$\begin{aligned} \dot{\theta} &= \xi\nu & t &\notin \mathbb{Z} & \Delta\theta &= 0, & t &\in \mathbb{Z} \\ \dot{\nu} &= -\xi\sin\theta - r\nu^3, & t &\notin \mathbb{Z} & \Delta\nu &= \sigma\nu, & t &\in \mathbb{Z}. \end{aligned} \qquad \text{(IV.1.11)}$$

These transformations effectively reduce the number of parameters from five to three.

By far the easiest bifurcation to analyze is the one at $\sigma = 0$ ($\alpha = 1$), which we should suspect corresponds to a cylinder bifurcation since the critical Floquet multipliers are complex conjugate. The bifurcation at $\sigma = -2$ ($\alpha = -1$) has the critical Floquet multipliers $\mu = \pm 1$, which should generically lead to a codimension-two fold–flip bifurcation. For this reason, we will start with the analysis of $\sigma = 0$. This is a bifurcation of the downward rest position.

There is no need to perform a centre manifold reduction because (IV.1.11) is already in the form of (II.5.16)–(II.5.17). Indeed, to cubic order in the nonlinearities, we can write the system as

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \nu \end{bmatrix} = \begin{bmatrix} 0 & \xi \\ -\xi & 0 \end{bmatrix}\begin{bmatrix} \theta \\ \nu \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\xi}{6}\theta^3 - r\nu^3 \end{bmatrix} + O(|\theta|^5), \quad t \notin \mathbb{Z} \qquad \text{(IV.1.12)}$$

$$\Delta \begin{bmatrix} \theta \\ \nu \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma\nu \end{bmatrix}, \qquad t \in \mathbb{Z}. \qquad \text{(IV.1.13)}$$

The Floquet multipliers at $\sigma = 0$ are $\pm i\xi$. We can easily compute the crossing condition $\gamma(0)$ of Theorem II.5.2.5. We have

$$\gamma(0) = \text{tr}\left(\begin{bmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{bmatrix}\right) = 1.$$

The first two nondegeneracy conditions therefore pass provided $e^{ik\xi} \neq 1$ for $k = 1, 2, 3, 4$, and $\xi \notin \frac{\pi}{2} + \pi\mathbb{Z}$. The latter condition is included in the first ($e^{i4\xi} \neq 1$), so we may summarize the first two nondegeneracy conditions simply by the requirement that $e^{ik\xi} \neq 1$ for $k = 1, 2, 3, 4$.

Since there is no quadratic ν^2 term, the ν^2 coefficient in the stroboscopic map vanishes. As such, the third Lyapunov coefficient $d(0)$ can be computed according to

$$d(0) = \Re\left(e^{-i\xi}\frac{1}{2}\langle p, \phi_{yyy}(1)[q, q, \overline{q}]\rangle\right),$$

where the vectors $p, q, \in \mathbb{C}^2$ satisfy the normalization conditions

$$\begin{bmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{bmatrix} q = e^{i\xi}q, \quad \begin{bmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{bmatrix} p = e^{-i\xi}p, \quad \langle p, q\rangle = 1.$$

It is easy to check that the pair

$$p = q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

works. We can then simplify the expression for $d(0)$ by exploiting trilinearity of ϕ_{yyy} and the inner product. We have

$$d(0) = \frac{1}{8}\Re\left(e^{-i\xi}\left\langle \begin{bmatrix} 1 \\ i \end{bmatrix}, \phi_{yyy}(1)\left[\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right]\right\rangle\right) := \frac{1}{8}\Re\left(e^{-i\xi}\langle p, \tilde{\phi}_{yyy}\rangle\right).$$

This computation is very amenable to symbolic computation. We have

$$\tilde{\phi}_{yyy} = \begin{bmatrix} \cos\xi & \sin\xi \\ -\sin\xi & \cos\xi \end{bmatrix} \int_0^1 \begin{bmatrix} \cos(s\xi) & -\sin(s\xi) \\ \sin(s\xi) & \cos(s\xi) \end{bmatrix} \begin{bmatrix} 0 \\ \xi e^{i\xi s} - i6re^{i\xi s} \end{bmatrix} ds.$$

This equation follows from (II.5.21), the form of the third differential

$$f_{yyy}[u, w, h] = [\ 0 \quad \xi u_1 w_1 h_1 - 6ru_2 w_2 h_2\ ]^\intercal, \tag{IV.1.14}$$

and the following expressions for $e^{s\Lambda}q$ and $e^{s\Lambda}\overline{q}$:

$$\exp\begin{bmatrix} 0 & \xi s \\ -\xi s & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} e^{i\xi s} \\ ie^{i\xi s} \end{bmatrix}, \qquad \exp\begin{bmatrix} 0 & \xi s \\ -\xi s & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} e^{-i\xi s} \\ -ie^{-i\xi s} \end{bmatrix}.$$

Computing $d(0)$ using MATLAB's symbolic mathematics toolbox, we miraculously end up with the very clean expression

$$d(0) = -\frac{3r}{4}.$$

We can summarize the bifurcation result with the following theorem.

Theorem IV.1.3.1. *The impulsively forced/driven pendulum undergoes a cylinder bifurcation at the downward rest position $(\theta, \nu) = (0, 0)$ at forcing parameter $\sigma = 0$ as long as $e^{ik\xi} \neq 1$ for $k = 1, 2, 3, 4$. In this case, the following holds for $|\sigma|$ small enough:*

- *The rest position is locally asymptotically stable for $\sigma < 0$, stable for $\sigma = 0$ and unstable for $\sigma > 0$. When $\sigma > 0$, there is a family of small quasiperiodic oscillations parameterized by a cylinder in the space $\mathbb{S}^1 \times \mathbb{R}^2$ that is locally attracting.*

Figure IV.1.1 provides numerical solutions through a representative initial condition for the asymptotically stable ($\sigma < 0$) and stable ($\sigma = 0$) cases. After the bifurcation occurs ($\sigma > 0$), the phase portrait in the plane contains an attracting region that qualitatively looks like an annulus. However, the cylindrical structure is plainly visible in the correct coordinate system, as can be seen in Fig. IV.1.2.

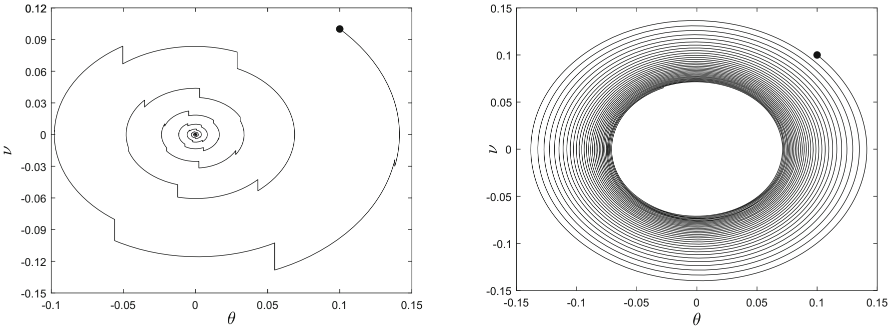

Figure IV.1.1: Left: solution through the initial condition $(\theta, \nu) = (0.1, 0.1)$ of the system (IV.1.11), plotted for $t \in [0, 200]$, with system parameters $\xi = 1$, $r = 1$ and $\sigma = -0.2$. Since $\sigma < 0$, the rest position at the origin is asymptotically (and exponentially) stable. Right: solution through the same condition and parameters, except that $\sigma = 0$. The rest point at the origin is stable, but the convergence is sub-exponential. In both figures, the initial conditions are indicated by black dots

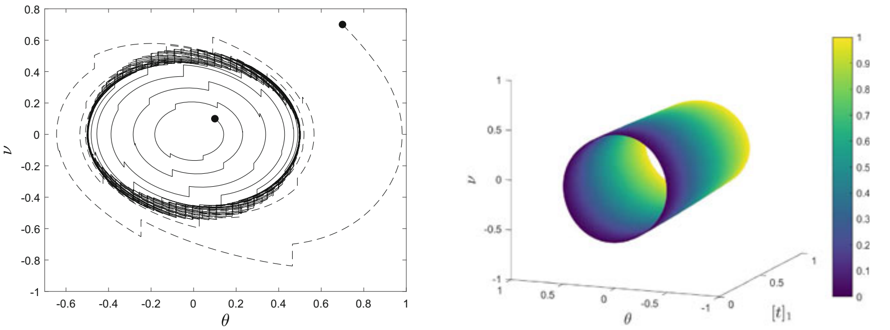

Figure IV.1.2: Left: solution (solid curve and dashed curve, respectively) through the initial conditions $(\theta, \nu) = (0.1, 0.1)$ and $(\theta, \nu) = (0.7, 0.7)$ of the system (IV.1.11), plotted for $t \in [0, 100]$, with system parameters $\xi = 1$, $r = 1$ and $\sigma = 0.2$. Since $\sigma > 0$, the rest position at the origin is unstable. In the plane, the attractor resembles an annulus. The initial conditions are represented by black dots. Right: the attractor in the space $\mathbb{S}^1 \times \mathbb{R}^2$. Specifically, the plot is the image of $t \mapsto ([t]_1, \theta(t), \nu(t))$ for the solution through $(0.1, 0.1)$, numerically integrated on the interval $[0, 2000]$ and then plotted for $t \in [90, 2000]$. The colours reflect the argument $[t]_1$ and help to provide a sense of depth to the figure

IV.1.4 Cylinder Bifurcation at the Downward Rest Position in the Model with Delay

Similar to the previous section, we will perform some transformations to reduce the number of parameters:

- a rescaling of time $Ts = t$,
- define $\rho^2 = \frac{g}{\ell}$ and $\kappa = \frac{\alpha}{\rho\tau}$,
- a change of variables $v = \rho\nu$ and
- define new parameters $\xi = \rho T$, $r = \beta T\rho^2$ and $\eta = \frac{\tau}{T}$.

After these transformations, if we replace s with t, then the impulsive delay differential equations for the model with delay become

$$\begin{aligned} \dot{\theta} &= \xi\nu & t &\notin \mathbb{Z} & \Delta\theta &= 0, & t &\in \mathbb{Z} \\ \dot{\nu} &= -\xi\sin\theta - r\nu^3, & t &\notin \mathbb{Z} & \Delta\nu &= -\nu + \kappa(\theta(t^-) - \theta(t-\eta)), & t &\in \mathbb{Z}. \end{aligned} \tag{IV.1.15}$$

These transformations effectively reduce the number of parameters from seven to five. Notice that in the new parameters, $\rho\tau = \eta\xi$.

IV.1.4.1 Floquet Multiplier Transversality Condition

Translating the statement of Lemma IV.1.2.1 into the transformed parameters, the parameter constraint that provides a candidate cylinder bifurcation at the downward rest position is $\kappa\sin(\eta\xi) = 1$ and

$$\left|\frac{\sin(\xi(1-\eta)) - \sin(\xi)}{\sin(\eta\xi)} - \cos(\xi)\right| < 2. \tag{IV.1.16}$$

Moreover, when the multipliers are complex conjugate with nonzero imaginary part, they have absolute value $|\mu| = |\kappa\sin(\eta\xi)|$. This proves the following lemma.

Lemma IV.1.4.1. *If (IV.1.16) is satisfied, the nontrivial pair of Floquet multipliers associated with (IV.1.15) at the downward rest position crosses the imaginary axis transversally as κ passes through $\tilde{\kappa} = \csc(\eta\xi)$. In particular, the absolute value $|\mu|$ of this pair of multipliers, taking κ as a variable and all other parameters fixed, satisfies*

$$\frac{d|\mu|}{d\kappa}(\tilde{\kappa}) = |\sin(\eta\xi)|\,\mathrm{sign}(\tilde{\kappa}). \tag{IV.1.17}$$

Lemma IV.1.4.1 provides a strong indication of the existence of a cylinder bifurcation at $\kappa = \tilde{\kappa}$ under the parameter configuration (IV.1.16). Indeed, the present scenario falls under the purview of Theorem I.8.4.1. In the terminology of that theorem, $\gamma(0)$ has the same sign as $\frac{d|\mu|}{d\kappa}(\tilde{\kappa})$, so the second nondegeneracy condition (G.2) concerning the transversal crossing of the Floquet multipliers is satisfied as long as $\eta\xi \notin \pi\mathbb{Z}$. To begin studying the other ones, recall from Sect. IV.1.2.1 that $\mu \neq 0$ is a Floquet multiplier if and only if it is an eigenvalue of the matrix

$$N(\kappa) := \begin{bmatrix} \cos(\xi) & \sin(\xi) \\ \kappa(\cos(\xi) - \cos(\xi(1-\eta)) & \kappa(\sin(\xi) - \sin(\xi(1-\eta)) \end{bmatrix}, \quad \text{(IV.1.18)}$$

where we have changed to the new parameters. If $\kappa = \tilde{\kappa}$ and inequality (IV.1.16) is satisfied, then $N(\tilde{\kappa})$ has a pair of complex-conjugate eigenvalues with nonzero imaginary part and absolute value of 1. Let $\tilde{\mu} = e^{i\omega}$ be the eigenvalue with positive real part. The first nondegeneracy condition (G.1) of the cylinder bifurcation is satisfied if $e^{im\omega} \neq 1$ for $m = 1, 2, 3, 4$.

IV.1.4.2 Computation of the First Lyapunov Coefficient

The last step is to verify that the first Lyapunov coefficient $d(0)$ is nonzero (condition G.3 of the cylinder bifurcation theorem). While certainly feasible to accomplish on paper, the computations are quite long. Even with the aid of symbolic algebra software, the setup is quite messy because of the computation of the matrix $Y(t)$ from Lemma I.8.4.3. To simplify the calculation, we will therefore exploit the fact that discrete delays in (periodic) impulse effects can always be removed by introducing additional state variables [30], since after one period of impulse effect, the dynamics are equivalent to an ordinary impulsive differential equation without delays. We saw a transformation of this type in Sect. I.6.4.2, where it was introduced to deal with failure in the overlap condition. It was also used in Sect. III.2.3.3 on overlap bifurcations. The following lemma is easily verified.

Lemma IV.1.4.2. *Any solution* $(\theta, \nu) : \mathbb{R} \to \mathbb{R}^2$ *of* (IV.1.15) *is uniquely identified with a solution* $(\theta, \nu, c) : \mathbb{R} \to \mathbb{R}^3$ *of*

$$\begin{aligned} &\dot{\theta} = \xi\nu, \\ &\dot{\nu} = -\xi \sin\theta - r\nu^3, \quad t \notin \mathbb{Z} \qquad \Delta\nu = -\nu + \kappa(\theta(t^-) - c(t^-)), \quad t \in \mathbb{Z} \\ &\dot{c} = 0, \qquad\qquad\qquad\quad t \notin \mathbb{Z} - \nu \qquad \Delta c = -c(t^-) + \theta(t^-), \qquad\quad t \in \mathbb{Z} - \eta, \end{aligned} \quad \text{(IV.1.19)}$$

and vice versa. Moreover, (θ, ν) *in* (IV.1.15) *is asymptotically stable (respectively, stable or unstable) if and only if the same is true for* (θ, ν, c) *in* (IV.1.19).

Linearizing (IV.1.19) at the fixed point $(\theta, \nu, c) = (0, 0, 0)$, we get the linear system

$$\dot{z} = \begin{bmatrix} 0 & \xi & 0 \\ -\xi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z(t), \qquad t \notin \mathbb{Z} \cup (\mathbb{Z} - \eta) \tag{IV.1.20}$$

$$\Delta z = \begin{bmatrix} 0 & 0 & 0 \\ \kappa & -1 & -\kappa \\ 0 & 0 & 0 \end{bmatrix} z(t^-), \qquad t \in \mathbb{Z} \tag{IV.1.21}$$

$$\Delta z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} z(t^-), \qquad t \in \mathbb{Z} - \eta. \tag{IV.1.22}$$

The monodromy matrix M_0 is precisely

$$M_0(\kappa) = \begin{bmatrix} \cos(\xi) & \sin(\xi) & 0 \\ \kappa(\cos(\xi) - \cos(\xi(1-\eta)) & \kappa(\sin(\xi) - \sin(\xi(1-\eta)) & 0 \\ \cos(\xi(1-\eta)) & \sin(\xi(1-\eta)) & 0 \end{bmatrix},$$

so if $\kappa = \tilde{\kappa}$, the Floquet multipliers are $e^{\pm i\omega}$ and zero. Note that the first 2×2 block corresponds precisely to $N(\kappa)$. The centre manifold is therefore two-dimensional. We will *not*, however, complete a centre manifold reduction. Doing so is unnecessary. Indeed, let $t \mapsto Z(t; \theta_0, \nu_0, c_0)$ denote the solution of (IV.1.19) satisfying the initial condition $Z(0; \theta_0, \nu_0, c_0) = (\theta_0, \nu_0, c_0)$. Let $S(\theta, \nu, c) := Z(1; \theta, \nu, c)$ denote the time 1 (stroboscopic) map. Since $\eta \in (0, 1)$, we have the identity $S(\theta, \nu, c) = S(\theta, \nu, 0)$ for all $c \in \mathbb{R}$. We can therefore consider the dynamics of the two-dimensional projected map

$$\tilde{S}(\theta, \nu) = (I_{\mathbb{R}^3} - \pi_3) S(\theta, \nu, 0)$$

with $\pi_3(x, y, z) = (0, 0, z)$ and compute the first Lyapunov coefficient of this map. If it is nonzero, this fact and the previously verified nondegeneracy conditions (G.1 and G.2) will give the cylinder bifurcation.

We cannot follow the cylinder bifurcation analysis of Sect. II.5.2.3 exactly, but the ideas carry over with minor modifications because the analysis is based on Taylor expansion of the period map. Since (IV.1.19) has no quadratic terms at $\kappa = \tilde{\kappa}$, the quadratic-order terms of the map $\tilde{S}$ are identically zero. It follows that at $\kappa = \tilde{\kappa}$, we have the expansion

$$\tilde{S}(\theta, \nu) = N(\tilde{\kappa}) \begin{bmatrix} \theta \\ \nu \end{bmatrix} + \frac{1}{6} C_0 \begin{bmatrix} \theta \\ \nu \end{bmatrix}^3 + O(||(\theta, \nu)||^4),$$

where we must compute the trilinear map C_0 from (II.5.22). From an analysis that is formally analogous to the derivation of (II.5.21), we can write

$$C_0[u, w, h] = X_{12}(1) \int_0^1 X_{12}^{-1}(s) D^3 f(0, 0)[X(s)u, X(s)w, X(s)h] ds, \qquad X_{12}(t) = jX(t)j^{-1}.$$

X is the fundamental matrix solution of (IV.1.20)–(IV.1.22) satisfying $X(0) = I_{\mathbb{R}^3}$, $j : \mathbb{R}^3 \to \mathbb{R}^2$ is the embedding $j(x,y,z) = (x,y)$ and $j^{-1} : \mathbb{R}^2 \to \mathbb{R}^3$ is the one-sided inverse $j^{-1}(x,y) = (x,y,0)$, and $f(\theta,\nu) = (0, -\xi\sin(\theta) + \xi\theta - r\nu^3)$. Note that f contains the nonlinear terms of the vector field relative to the fixed point $(0,0)$; it satisfies $Df(0,0) = 0$. The third differential of f at zero is

$$D^3 f(0,0)[u,w,h] = \begin{bmatrix} 0 \\ \xi u_1 w_1 h_1 - 6r u_2 w_2 h_2 \end{bmatrix}.$$

Explicitly computing the fundamental matrix solution, the projected block X_{12} satisfies

$$X_{12}(t) = \begin{cases} e^{\Lambda_\xi t}, & 0 \le t < 1 \\ N(\kappa), & t = 1, \end{cases} \qquad \Lambda_\xi = \begin{bmatrix} 0 & \xi \\ -\xi & 0 \end{bmatrix}.$$

To compute the first Lyapunov coefficient (I.8.40), we require an eigenvector q of $N(\tilde\kappa)$ with eigenvalue $e^{i\omega}$ and an eigenvector p of $N(\tilde\kappa)^\intercal$ with eigenvalue $e^{-i\omega}$ satisfying the normalization condition $\langle p, q, \rangle = 1$, where the inner product is the standard one on $\mathbb{C}^2$. A suitable choice is

$$q = \begin{bmatrix} \sin\xi \\ e^{i\omega} - \cos\xi \end{bmatrix}, \qquad p = \frac{1}{\zeta}\begin{bmatrix} \kappa(\cos\xi - \cos(\xi(1-\eta)) \\ e^{-i\omega} - \cos\xi \end{bmatrix},$$

with the normalization factor $\zeta = \kappa(\sin\xi)(\cos\xi - \cos(\xi(1-\eta))) + (e^{-i\omega} - \cos\xi)^2$. The Lyapunov coefficient is

$$\begin{aligned} d(0) &= \Re\left(e^{-i\omega}\frac{1}{2}\langle p, C_0[q,q,\overline{q}]\rangle\right) \\ &= \frac{1}{2}\Re\left(e^{-i\omega}\left\langle p, N(\tilde\kappa)\int_0^1 e^{-\Lambda_\xi s} D^3 f(0,0)\left[e^{\Lambda_\xi s}q, e^{\Lambda_\xi s}q, e^{\Lambda_\xi s}\overline{q}\right]\right\rangle\right). \end{aligned}$$

Computing this with symbolic algebra, the resulting output is *exceptionally* long, and there is no benefit in reproducing it here. Instead, we will choose a few values of the parameters η and ξ that satisfy the inequality (IV.1.16), substitute $\kappa = \tilde\kappa$ for these parameters and display the output of ω and $d(0)$ with $r = 1$. The result appears in Table IV.1.3, and some numerical simulations are provided in Fig. IV.1.3.

ξ	η	r	$\tilde{\kappa}$	ω	$d(0)$	Direction of cylinder	Stability of $(0,0)$
0.5	0.5	1	4.0420	0.4330	-0.2618	$\kappa > \tilde{\kappa}$	$\kappa < \tilde{\kappa}$
7	0.8	1	-1.5841	0.8801	-0.7961	$\kappa > \tilde{\kappa}$	$\kappa > \tilde{\kappa}$

Table IV.1.3: Parameter direction in which the invariant cylinder exists the fixed point $(0,0)$ is asymptotically stable near the bifurcation point $\kappa = \tilde{\kappa}$ for a few illustrative parameter values. The frequency ω of the orbits parallel to the cylinder at the bifurcation point and the first Lyapunov coefficient $d(0)$ are also provided. Equation (IV.1.17) is used to determine the sign of $\gamma(0)$, and the direction and stability computations are done using the value of $d(0)$ and Theorem II.5.2.5.

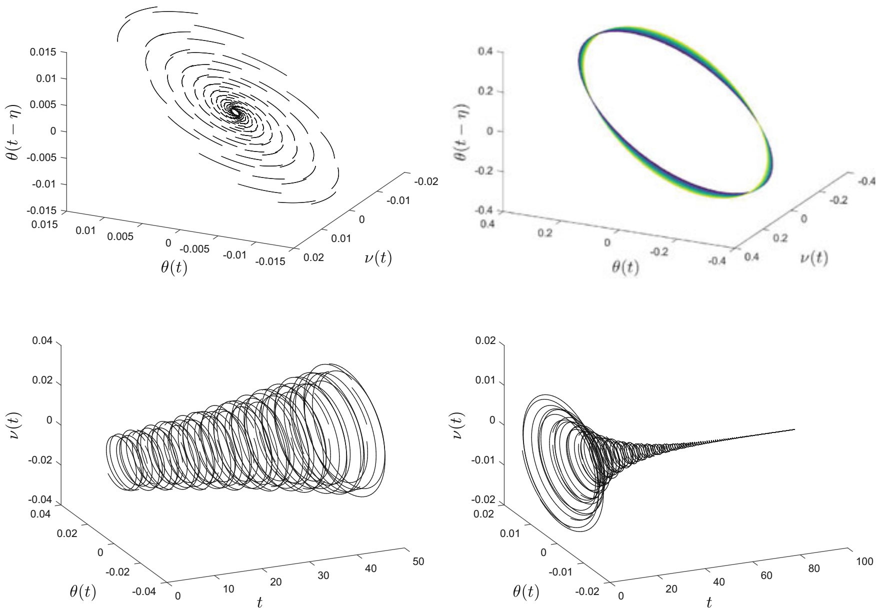

Figure IV.1.3: Top left: solution through the initial condition $(\theta, \nu) = (0.1, 0.1)$ of system (IV.1.15) with parameters from the first row of Table IV.1.3, except $\kappa = 3.9 < \tilde{\kappa}$. The solution is plotted for $t \in [0, 200]$, and we include the lagged argument $\theta(t-\eta)$. Asymptotic stability of the trivial solution is observed. Top right: same parameters as previous except $\kappa = 4.2 > \tilde{\kappa}$. The solution was computed for $t \in [0, 800]$, and the result was windowed to $[600, 800]$ to help visualize the attracting invariant cylinder. Points are coloured according to $[t]_1$, with dark purple corresponding to $[t]_1 = 0$ and light yellow to $[t]_1 \to 1^-$. Bottom left: time series plot with parameters from the second row of the table, with $\kappa = -1.65 < \tilde{\kappa}$ in the unstable region. Bottom right: same parameters as previous except $\kappa = 1.4 > \tilde{\kappa}$, in the stable region

Chapter IV.2

The Hutchinson Equation with Pulse Harvesting

The Hutchinson equation

$$\dot{x} = rx(t)\left(1 - \frac{x(t-\tau)}{K}\right)$$

is arguably one of the oldest and most well-studied delay differential equations. Under an appropriate change of variables and rescaling of time, the dynamics in the nonnegative section of the real line are equivalent to the dynamics of the exponential form of Wright's equation

$$\dot{y} = -\alpha(e^{y(t-1)} - 1),$$

for which characterizing the structure of the global attractor for $\alpha > 0$ remains an open problem [15]. Recent progress [75, 143] towards a complete understanding of Wright's equation includes proofs of the conjectures of Jones and Wright, which concern, respectively, the global attractor for $\alpha \in [0, \pi/2)$ and the existence and uniqueness of a branch of special periodic solutions for $\alpha > \pi/2$. See the references of the previous three papers for more background on Wright's and Hutchinson's equations.

Hutchinson's equation is motivated from mathematical ecology and is sometimes called the delay logistic equation. $K > 0$ is the carrying capacity of the species, $r > 0$ is the intrinsic growth rate and τ is a delay that takes into account such factors as maturation or gestation time. If some proportion $h \in (0, 1)$ of the species is harvested very quickly (i.e. approximated by an impulse) at the fixed multiples of time $T > 0$, we get the Hutchinson equation with harvesting

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$$\dot{x} = rx(t)\left(1 - \frac{x(t-\tau)}{K}\right), \qquad t \neq kT \tag{IV.2.1}$$

$$\Delta x = -hx(t^-), \qquad t = kT. \tag{IV.2.2}$$

This equation was studied with $\tau = 0$ (i.e. the logistic equation with linear impulsive harvesting) in [161]. For more background on the Hutchinson equation with impulse effect, see Section 4.2.1 of the reference [135] and the literature cited therein.

We will use the non-smooth centre manifold theory of Sect. III.2.2 to prove that the Hutchinson equation with impulsive harvesting undergoes a transcritical bifurcation of the zero solution (extinction equilibrium) at $T = T^*$, with

$$T^* = -\frac{1}{r}\log(1-h), \tag{IV.2.3}$$

provided $T^* > \tau$. To verify (IV.2.3), one can linearize (IV.2.1)–(IV.2.2) at $x = 0$ to get the linear system without delays

$$\begin{aligned} \dot{y} &= ry(t), && t \neq kT \\ \Delta y &= -hy(t^-), && t = kT. \end{aligned}$$

The only Floquet multipliers are zero (which comes from the compactness of the monodromy operator acting on $\mathcal{RCR}([-\tau, 0], \mathbb{R})$) and $\mu = e^{rT}(1-h)$, from which setting $\mu = 1$ gives the requirement that $T = T^*$ from (IV.2.3). The non-smooth theory is needed because, as explained in the beginning of Chap. III.2, the period of impulse effect does not behave as a "smooth parameter" at the level of the solution.

IV.2.1 Dummy Matrix System: Setup for the Non-smooth Centre Manifold

Following the method of Sect. III.2.2, our choice for the dummy matrix system is as follows:

$$\begin{aligned} \dot{x} &= -\frac{1}{T}\log(1-h)x + M_1 x - \frac{r}{K}x(t)x(t-\tau), && t \neq kT \\ \dot{M}_1 &= 0, && t \neq kT \\ \Delta x &= -hx(t^-), && t = kT \\ \Delta M_1 &= 0, && t = kT. \end{aligned}$$

Here, $Z_1(T) = -\frac{1}{T}\log(1-h)$ satisfies $Z(T^*) = r$. We have no need for a Z_2 in this example. If $M_1 = M_1^*(T)$, where

$$M_1^*(T) = r + \frac{1}{T}\log(1-h), \tag{IV.2.4}$$

the dummy matrix system reduces to (IV.2.1)–(IV.2.2) with an extra trivial component. The linearization of the DMS at $(x, M_1) = (0, 0)$ is

$$\begin{array}{llll} \dot{z}_1 = Z_1(T)z_1 & t \neq kT & \Delta z_1 = -hz_1(t^-), & t = kT \\ \dot{z}_2 = 0, & t \neq kT & \Delta z_2 = 0, & t = kT. \end{array} \tag{IV.2.5}$$

This system has only the Floquet multipliers 0 and 1. Our setup being complete, we can apply Theorem III.2.2.1 and its corollary to get the existence of a T-parameter-dependent centre manifold near $T = T^*$.

IV.2.2 Dynamics on the Centre Manifold

The dynamics on this centre manifold at each fixed T can be determined to quadratic order using Corollary I.6.1.1. To do this, we need the periodic array $Q_{t,T}$ from the Floquet decomposition $\Phi_{t,T} = Q_{t,T}e^{t\Lambda_T}$ of a basis array for the centre fibre bundle of the linearization (IV.2.5) at parameter T. We also need the matrix $Y_T(t)$ satisfying $P_{c,T}(t)\chi_0 = \Phi_{t,T}Y_T(t)$. Since the linearization is memoryless, we have $Q_{t,T}(\theta) = Q(t + \theta; T)$, where

$$Q(s;T) = \left[\begin{array}{cc} \exp\left(-\frac{[s]_T}{T}\log(1-h)\right) & 0 \\ 0 & 1 \end{array} \right].$$

Moreover, $Q_{t,T} = \Phi_{t,T}$. We will use the Riesz projection formula (I.3.4) to calculate the matrix $Y_T(t)$. The monodromy operator at parameter T is

$$V_t(T) \left[\begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right](\theta) = \left[\begin{array}{c} X(t+\theta+T,t)\phi_1(0)\chi_{[-T,0]}(\theta) + \phi_1(\theta+T)\chi_{[-r,-T)}(\theta) \\ \phi_2(0)\chi_{[-T,0]}(\theta) + \phi_2(T+\theta)\chi_{[-r,-T)}(\theta) \end{array} \right],$$

where X is the Cauchy matrix of the z_1 equation of (IV.2.5) interpreted as a finite-dimensional system. Using this information together with the observations that $X(t+T,t) = I$ and $X(t+T,s+T) = X(t,s)$, we can compute

$$(zI - V_t(T))^{-1}\chi_0(\theta) = \frac{1}{(z-1)z^m} \left[\begin{array}{cc} X(t+\theta,t) & 0 \\ 0 & 1 \end{array} \right],$$

whenever $-(m+1)T \leq \theta < -mT$ for some $m \in \mathbb{N}$. Since $\mathrm{diag}(X(t,s), I) = Q(t;T)Q(s;T)^{-1}$, we get

$$P_{c,T}(t)\chi_0(\theta) = \frac{1}{2\pi i}\int_\gamma (zI - V_t(T))^{-1}\chi_0(\theta) = \left[\begin{array}{cc} X(t+\theta,t) & 0 \\ 0 & 1 \end{array} \right] = Q_{t,T}(\theta)Q(t;T)^{-1},$$

which implies $Y_T(t) = Q(t;T)^{-1}$.

Combining these results, the dynamics on the (dummy) parameter-dependent centre manifold for the DMS are given for each T fixed by

$$\dot{u} = Q(t;T)^{-1} \left[\begin{array}{c} u_1u_2Q_{11}(t;T) - (r/K)u_1^2Q_{11}(t;T)Q_{11}(t-\tau;T) \\ 0 \end{array} \right] + O(||u||^3).$$

Now, observe that the function M_1^* defined in (IV.2.4) satisfies $M_1^*(T^*) = 0$. Setting $u_2 = M_1^*$ and ignoring the (trivial) second component of the dynamics on the centre manifold, we get

$$\dot{u}_1 = u_1 \left(r + \frac{1}{T} \log(1-h) \right) - \frac{r}{K} u_1^2 \exp\left(-\frac{[t-\tau]_T}{T} \log(1-h) \right) + R(u_1, T), \tag{IV.2.6}$$

accurate for $T \in (T^* - \epsilon, T^* + \epsilon)$ for some small $\epsilon > 0$, where the higher-order terms $R(u_1, T) = O(u_1^3)$ are C^∞ for each T fixed. Due to the uniform Lipschitz continuity of the parameter-dependent centre manifold, there exists a bound $|R(u_1, T)| \le C|u_1|^3$ valid for $|T - T^*| < \epsilon$. Moreover, $u_1 \mapsto R(u_1, T)$ and its derivatives are uniformly (in T) Lipschitz continuous near $u_1 = 0$.

IV.2.3 The Transcritical Bifurcation

Our goal is ultimately to prove that for $0 < |T - T^*|$ small, (IV.2.6) has a pair of periodic solutions: the trivial solution at $u_1 = 0$ and another small solution. The linear-order term will then dictate stability, and we will have proven the existence of a transcritical bifurcation in the original system (IV.2.1)–(IV.2.2). For each fixed T, (IV.2.6) is T-periodic, so the logical step is to compute a second-order Taylor expansion of the associated period map near $(u_1, T) = (0, T^*)$, treating T itself as a parameter in the expansion. However, doing this is risky because we are not guaranteed that the period map is, itself, even continuous. Recall that the higher-order terms $R(u_1, T)$ are not guaranteed to be continuous with respect to T, so we can make no assumptions about the period map. To avoid this technical issue, we will make use of the contraction mapping principle to find the bifurcating periodic solution, rather than relying on smoothness results. Since this last stage makes up the final argument in the proof of the transcritical bifurcation, we will state the result in the form of a theorem first.

Theorem IV.2.3.1. *Assume $T^* > \tau$. The fixed point $x = 0$ of the Hutchinson equation with periodic impulsive harvesting (IV.2.1)–(IV.2.2) exhibits a transcritical bifurcation as T passes through T^*. The following are valid for $|T - T^*|$ sufficiently small:*

- *If $T < T^*$, the fixed point $x = 0$ is locally asymptotically stable, and there is a nontrivial unstable periodic solution. There are no other small periodic solutions.*
- *If $T = T^*$, the fixed point $x = 0$ is conditionally stable, and there are no other small periodic solutions.*
- *If $T > T^*$, the fixed point $x = 0$ is unstable, and there is a nontrivial, locally asymptotically stable periodic solution. There are no other small periodic solutions.*

- *The nontrivial periodic solution x^* satisfies $x^* \to 0$ as $T \to T^*$.*

Proof. Let $x(t)$ be the solution of (IV.2.6) satisfying $x(0) = x_0$. For T fixed with $|T - T^*|$ small, we can write $x(t) = c_1(t,T)x_0 + c_2(t,T)x_0^2 + \tilde{R}(T, x_0)$ as a Taylor expansion with remainder near $x_0 = 0$. With a bit of effort, one can show that this expansion satisfies

$$x(T) = x_0 + (rT + \log(1-h))\, x_0 \\ - \left(\frac{r}{K}\int_0^T \exp\left(-\frac{[s-\tau]_T}{T}\log(1-h)\right) ds\right) x_0^2 + R(T, x_0).$$

If we write $R(T, x_0) = \tilde{R}(T, x_0)x_0^3$, then $\tilde{R}(T, x_0)$ is uniformly (in T) Lipschitz continuous in some neighbourhood of $x_0 = 0$.

If x is a periodic solution, x_0 satisfies the equation

$$x_0 = x_0 + \underbrace{(rT + \log(1-h))}_{m_1(T)}\, x_0 \\ - \underbrace{\left(\frac{r}{K}\int_0^T \exp\left(-\frac{[s-\tau]_T}{T}\log(1-h)\right) ds\right)}_{m_2(T)} x_0^2 + \tilde{R}(T, x_0)x_0^3.$$

This equation always has the trivial solution $x_0 = 0$, so x_0 is a nontrivial solution if and only if

$$0 = m_1(T) - m_2(T)x_0 + \tilde{R}(T, x_0)x_0^2.$$

This is equivalent to x_0 being a fixed point of the nonlinear map

$$y \mapsto G(y, T) = \frac{m_1(T)}{m_2(T)} + \frac{\tilde{R}(T, y)}{m_2(T)} y^2 := a_0(T) + a_2(T, y)y^2. \qquad \text{(IV.2.7)}$$

a_0 is continuous in a neighbourhood of T^* and, by its definition, satisfies $a_0(T^*) = 0$. Since $m_2(T) \geq Tr/K$, we can find another (uniform in T) Lipschitz constant for $y \mapsto a_2(T, y)$ valid in some closed δ-neighbourhood of $y = 0$ for $|T - T^*|$ small enough. Let L be such a Lipschitz constant valid for $|y| \leq \delta$.

Let $r_1 > 0$ be small enough so that $3Lr_1^2 < 1$. Define $r = \min\{\delta, r_1\}$. Then $r - Lr^3 > 0$ and we can choose $\epsilon > 0$ small enough so that $|a_0(T)| \leq r - Lr^3$ for $|T - T^*| < \epsilon$. Consider the nonlinear map $G(\cdot, T) : \overline{B_r(0)} \to \mathbb{R}$ defined by the right-hand side of (IV.2.7). For $|y| \leq r$,

$$|G(y, T)| \leq |a_0(T)| + |a_2(T, y)|y^2 \leq |a_0(T)| + Lr^3 \leq r,$$

so $G(\cdot, T)$ has range in $B_r(0)$. Also,

$$|G(x, T) - G(y, T)| \\ \leq |x^2 - y^2| \cdot |a_2(T, x)| + y^2|a_2(T, x) - a_2(T, y)| \leq 3Lr^2|x - y|.$$

Since $3Lr^2 < 1$, the map $G(\cdot, T)$ has a unique fixed point in $\overline{B_r(0)}$ by the Banach fixed-point theorem. This fixed point corresponds to the unique nontrivial periodic solution.

To see that this nontrivial periodic solution $t \mapsto x^*(t)$ trivializes to zero as $T \to T^*$, observe that the radius r of the ball in which the contraction mapping is defined can be made as small as desired by taking $|T - T^*|$ small enough. Indeed, the radius r of the ball is determined by the condition that $|a_0(T)| \leq r - Lr^3$ for $|T - T^*| < \epsilon$. For any $r > 0$ small enough so that $3Lr^2 < 1$, this can be achieved using the continuity of a_0 with $a_0(T^*) = 0$. □

Remark IV.2.3.1. *The assumption $T^* \neq \tau$ is needed to ensure that the overlap condition is satisfied for $|T - T^*|$ small enough. This condition is needed to get dynamics on the parameter-dependent centre manifold. We also need $T^* > \tau$ for hypothesis F.2 to be satisfied, and the latter is needed throughout this section.*

Figure IV.2.1 provides some numerical validation of Theorem IV.2.3.1. There we can see the transition from the stable fixed point $x = 0$ to a stable nontrivial periodic solution as T crosses through T^*.

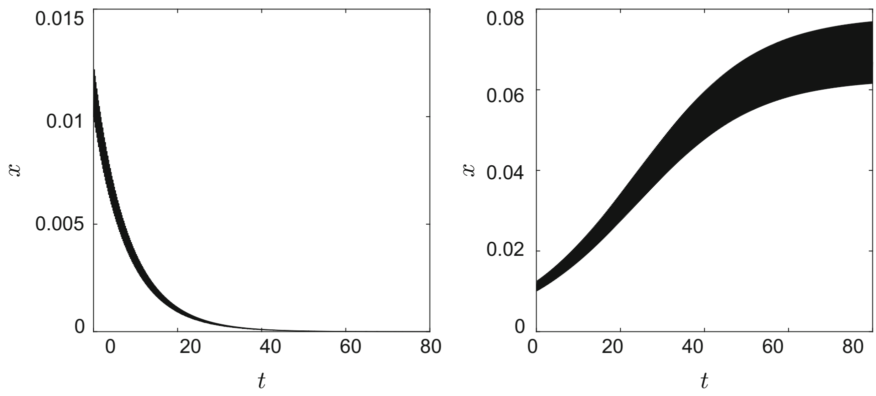

Figure IV.2.1: Plots of numerical solutions through the constant initial condition $x_0 = 0.01$ for the Hutchinson equation with impulsive harvesting (IV.2.1)–(IV.2.2). Parameters are $r = K = 1$, $\tau = 0.1$ and $h = 0.2$. The critical period is $T^* = -\log(0.8) \approx 0.2231$. Left: $T = 0.20$. Right: $T = 0.24$. The simulation was run until time $t = 80$, which results in 400 and 333 impulses, respectively, hence the lack of resolution of the solution curves in the plots

Chapter IV.3

Delayed SIR Model with Pulse Vaccination and Temporary Immunity

IV.3.1 Introduction

This chapter is an abridged version of the paper *Analysis of a SIR model with pulse vaccination and temporary immunity: Stability, bifurcation and a cylindrical attractor* [32] by Church and Liu, published in Nonlinear Analysis: Real World Applications in 2019. We encourage the reader to consult the journal version of this article for expanded scope and some omitted proofs.

Consider the system of impulsive delay differential equations

$$\dot{S} = \mu - \mu S - \eta f(I(t))S(t) + \gamma I(t-\tau)e^{-\mu\tau}, \qquad t \neq t_k \tag{IV.3.1}$$

$$\dot{I} = \eta f(I(t))S(t) - (\mu + \gamma)I(t), \qquad t \neq t_k \tag{IV.3.2}$$

$$\Delta S = -vS(t^-) + vS(t-\tau)e^{-\mu\tau}, \qquad t = t_k. \tag{IV.3.3}$$

This model represents a population consisting of three classes: susceptible (S), infected (I) and recovered/immune (the third class is decoupled and hidden). Here, $f(I)$ is a general nonlinear incidence rate (that is, a rate at which susceptible individuals become infected); infected individuals clear their infection at rate γ and acquire temporary immunity that lasts for τ time units. η is a recruitment rate and μ is a natural death rate, with birth rate scaled accordingly so that the total population (susceptible plus infected and the recovered/immune population, hidden) has a steady state of unity. As such, populations represent proportions. The incidence rate

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is assumed to satisfy the properties: $f(0) = 0$, $f'(0) > 0$, $f''(0) < 0$ and $\lim_{I\to\infty} f(I) = c < \infty$. The impulse effect corresponds to a pulse vaccination, and it is derived under the following assumptions:

1) At specific instants of time t_k for $k \in \mathbb{Z}$, any individuals who received their vaccine at time $t_k - \tau$ and are still alive lose their immunity and re-enter the susceptible cohort, at which point a fraction $v \in [0, 1)$ of the total susceptible cohort is vaccinated.

2) Vaccinated individuals are immune to infection for a period τ (the same immunity period as having recovered from infection) and are subject to the same natural death rate μ.

3) The sequence of vaccination times is periodic with shift of τ: there exists $q > 0$ such that $t_{k+q} = t_k + \tau$ for all $k \in \mathbb{Z}$.

It is known [77] that the model without pulse vaccination can exhibit a Hopf bifurcation. In impulsive systems, Hopf points generically lead to bifurcations to invariant cylinders; see Theorem I.8.4.1. As such, we should expect this model to exhibit a cylinder bifurcation under certain parameter ranges. To this end, we will perform a combination of analytical and numerical bifurcation analyses. For a derivation of the model justifying the discrete delay, see [32].

IV.3.2 Vaccinated Component Formalism

We have indicated that it is our goal to complete a bifurcation analysis on the system (IV.3.1)–(IV.3.3). However, there are some technical difficulties associated with this endeavour because the overlap condition (Definition I.5.4.1) is not satisfied, since each of $t_k - \tau$ is an impulse time and Eq. (IV.3.3) contains delayed terms. While the failure of the overlap condition does not complicate the stability analysis, it does complicate the bifurcation analysis. To remedy this, we will at times instead consider the following modification of the model (IV.3.1)–(IV.3.3):

$$\dot{x} = \mu - \mu x - \eta f(y(t))x(t) + \gamma y(t-\tau)e^{-\mu\tau}, \qquad t \neq t_k \tag{IV.3.4}$$

$$\dot{y} = \eta f(y(t))x(t) - (\mu+\gamma)y(t), \qquad t \neq t_k \tag{IV.3.5}$$

$$\dot{V}_j = 0, \qquad t \neq t_k, \tag{IV.3.6}$$

$$\Delta x = -vx(t^-) + (1-v)V_j(t^-)e^{-\mu\tau}, \qquad t = t_{j+qk} \tag{IV.3.7}$$

$$\Delta V_j = vx(t^-) - (1 - ve^{-\mu\tau})V_j(t^-), \qquad t = t_{j+qk}. \tag{IV.3.8}$$

In the above impulsive delay differential equation, j ranges from 0 to $q-1$, where q is the period of the sequence of impulse times as defined in assumption

(3). Taking note that $t_{j+qk} = t_{j+q(k-1)} + \tau$ and V is constant except at impulse times where it is continuous from the right, we see that for $t = t_{j+qk}$,

$$\begin{aligned}(1-v)V_j(t^-)e^{-\mu\tau} &= (1-v)V_j(t-\tau)e^{-\mu\tau} \\ &= (1-v)[vx((t-\tau)^-) + ve^{-\mu\tau}V_j((t-\tau)^-)] \\ &= v[(1-v)x((t-\tau)^-) + (1-v)V_j((t-\tau)^-)e^{-\mu\tau}] \\ &= vx(t-\tau).\end{aligned}$$

Substituting the above into the jump condition for x, the result is

$$\Delta x = -vx(t^-) + (1-v)V_j(t^-)e^{-\mu\tau} = -vx(t^-) + vx(t-\tau)e^{-\mu\tau}.$$

This is precisely the same functional form as the jump condition (IV.3.3) for the original model. Since the continuous-time dynamics are identical for both models, we can analyze bifurcations in (IV.3.1)–(IV.3.3) by equivalently studying bifurcations in the model (IV.3.4)–(IV.3.8) with explicit vaccinated components.

IV.3.3 Existence of the Disease-free Periodic Solution

When there is no disease—that is, on the invariant subspace $\{(S, I) : I = 0\}$—the nontrivial dynamics are determined solely by the linear, nonhomogeneous impulsive system

$$\dot{z} = -\mu z + \mu, \qquad t \neq t_k \tag{IV.3.9}$$

$$\Delta z = -vz(t^-) + vz(t-\tau)e^{-\mu\tau}, \qquad t = t_k. \tag{IV.3.10}$$

By the variation of constants (Theorem I.2.3.1), every solution $z(t)$ passing through an initial condition $\phi \in \mathcal{RCR}$ at time $t = 0$ can be written as

$$z_t = U(t,0)\phi + \int_0^t U(t,s)\chi_0\mu ds,$$

where $U(t, s)$ is the evolution family associated with the homogeneous equation

$$\dot{w} = -\mu w, \qquad t \neq t_k \tag{IV.3.11}$$

$$\Delta w = -vw(t^-) + vw(t-\tau)e^{-\mu\tau}, \qquad t = t_k. \tag{IV.3.12}$$

We can then prove the following lemma by means of the contraction mapping principle.

Lemma IV.3.3.1. *Suppose the trivial solution of the homogeneous equation* (IV.3.11)–(IV.3.12) *is exponentially stable. Then, the system* (IV.3.1)–(IV.3.3) *has a unique disease-free periodic solution* $(\tilde{S}, 0)$*, with period* τ*.*

Lemma IV.3.3.2. *The trivial solution of the homogeneous equation* (IV.3.11)–(IV.3.12) *is exponentially stable.*

Proof. Let $w(t) = \phi(t)e^{\lambda t}$ be a solution of (IV.3.11)–(IV.3.12) with ϕ periodic. Substituting this ansatz into the dynamical system, using the periodicity condition $\phi(t) = \phi(t-\tau)$ and cancelling exponentials, we arrive at the following impulsive differential equation for ϕ:

$$\dot{\phi} + \lambda\phi = -\mu\phi, \qquad t \neq t_k \tag{IV.3.13}$$
$$\Delta\phi = -v\phi(t^-) + v\phi(t)e^{-(\mu+\lambda)\tau}, \qquad t = t_k. \tag{IV.3.14}$$

The second equation is an implicit jump condition, but we can easily rearrange it to obtain the explicit condition

$$\phi(t_k) = \frac{1-v}{1-ve^{-(\mu+\lambda)\tau}}\phi(t_k^-).$$

Calculating the solution of the above impulsive differential equation at time τ given an initial condition at time $t = 0$, one obtains

$$\phi(\tau) = e^{-(\mu+\lambda)\tau}\left(\frac{1-v}{1-ve^{-(\mu+\lambda)\tau}}\right)^q \phi(0) := D(\lambda)\phi(0).$$

ϕ is periodic provided $\phi(\tau) = \phi(0)$, so we are left with describing the location of the solutions of the transcendental equation $D(\lambda) = 1$. Defining $z = e^{-\lambda\tau}$, it follows that λ is a solution of $D(\lambda) = 1$ if and only if z is a solution of

$$\begin{aligned} 0 &= f(z) + g(z), \\ f(z) &= 1, \\ g(z) &= -ze^{-\mu\tau}\left(\frac{1-v}{1-ve^{-\mu\tau}z}\right)^q. \end{aligned}$$

We will show that $|g(z)| < |f(z)|$ on the unit circle $|z| = 1$. We have

$$\begin{aligned} |g(z)| &= e^{-\mu\tau}\left(\frac{1-v}{|1-ve^{-\mu\tau}z|}\right)^q \leq e^{-\mu\tau}\left(\frac{1-v}{|1-|ve^{-\mu\tau}z||}\right)^q \\ &= e^{-\mu\tau}\left(\frac{1-v}{1-ve^{-\mu\tau}}\right)^q \leq e^{-\mu\tau} < 1 = |f(z)|, \end{aligned}$$

as claimed. By Rouché's theorem, the equation $f(z) + g(z) = 0$ has no solutions satisfying $|z| \leq 1$. Consequently, there are no Floquet exponents λ satisfying the inequality $|e^{-\lambda\tau}| \leq 1$. We conclude that all Floquet exponents have negative real part and the result follows by Corollary I.3.3.1. □

As a consequence of Lemmas IV.3.3.1 and IV.3.3.2, we are guaranteed a unique disease-free periodic solution that, in the absence of infection, is globally exponentially stable.

Corollary IV.3.3.1. *The model* (IV.3.1)–(IV.3.3) *has a unique disease-free periodic solution* $t \mapsto (\tilde{S}(t,v),0)$ *of period* τ. *Restricted to the disease-free subspace* $D_0 = \{(S,I) : I = 0\}$, *this periodic solution is globally exponentially stable.*

IV.3.4 Stability of the Disease-free Periodic Solution

Introduce the basic reproduction number

$$R_0 = \frac{\eta f'(0)}{\tau(\gamma+\mu)} \int_0^\tau \tilde{S}(t,v)dt. \tag{IV.3.15}$$

Note that if one denotes the average of $\tilde{S}$ over the interval $[0,\tau]$ by $[\tilde{S}]$, then one can equivalently write the basic reproduction number in the more suggestive form

$$R_0 = \frac{\eta f'(0)[\tilde{S}]}{\gamma+\mu}.$$

Then, the interpretation is that R_0 is the product of the average number of susceptibles, multiplied by the small-infection (i.e. near $I=0$) incidence rate, divided by the aggregate rate of leaving the infected class through death or clearance of the infection.

Lemma IV.3.4.1. $R_0 = 1$ *is an epidemiological threshold: if* $R_0 < 1$, *the disease-free periodic solution is locally asymptotically stable, while if* $R_0 > 1$, *it is unstable.*

Proof. The linearization at $(\tilde{S},0)$ produces the linear homogeneous impulsive system

$$\begin{aligned}
\dot{u}_1 &= -\mu u_1(t) - \eta f'(0)\tilde{S}(t,v)u_2(t) + \gamma e^{-\mu\tau}u_2(t-\tau), && t \neq t_k\\
\dot{u}_2 &= \eta f'(0)\tilde{S}(t,v)u_2(t) - (\gamma+\mu)u_2(t), && t \neq t_k\\
\Delta u_1 &= -vu_1(t^-) + vu_1(t-\tau)e^{-\mu\tau}, && t = t_k.
\end{aligned}$$

Notice that the second equation is decoupled from the first. Taking an ansatz Floquet eigensolution $u(t) = \phi(t)e^{\lambda t}$, we can examine the second component independently. Indeed, $\phi = [\ \phi_1 \quad \phi_2\ ]^T$ satisfies

$$\dot{\phi}_2 + \lambda\phi_2 = \eta f'(0)\tilde{S}(t,v)\phi_2 - (\gamma+\mu)\phi_2.$$

If $\phi_2 \neq 0$, then as ϕ is assumed to be periodic with period τ, the only possible Floquet exponent in this case is

$$\lambda_0 = -(\gamma+\mu) + \frac{\eta f'(0)}{\tau} \int_0^\tau \tilde{S}(t,v)dt. \tag{IV.3.16}$$

Conversely, if $\phi_2 = 0$, then Lemma IV.3.3.2 implies that the associated Floquet exponents all have negative real parts. Consequently, the Floquet spectrum includes the special Floquet exponent λ_0 and the remainder with strictly negative real part. The equilibrium is locally asymptotically stable provided all Floquet exponents have negative real part and is unstable if at least one has positive real part. Since λ_0 is real and the others are guaranteed to have negative real part, we obtain the conclusion of the lemma by noticing that $\lambda_0 < 0$ is equivalent to $R_0 < 1$ and that $\lambda_0 > 0$ is equivalent to $R_0 > 1$. □

IV.3.5 Existence of a Bifurcation Point

Before we can study bifurcations, we must establish the existence of a bifurcation point. The proof of the following lemma is a routine exercise taking advantage of the explicit form of $\lambda_0(v)$ and the intermediate value theorem.

Lemma IV.3.5.1. *Consider the critical Floquet exponent $\lambda_0 = \lambda_0(v)$ as defined in Eq. (IV.3.16). λ_0 is strictly decreasing. As a consequence, if $\lambda_0(0)\lambda_0(1) \leq 0$, there is a unique $v^* \in [0, 1]$ such that $\lambda_0(v^*) = 0$, that is, a critical vaccination coverage v^* at which $R_0 = 1$.*

IV.3.6 Transcritical Bifurcation in Terms of Vaccine Coverage at $R_0 = 1$ with One Vaccination Pulse Per Period

We will now take the vaccination coverage v as a bifurcation parameter and unfold the bifurcation at $v = v^*$. To simplify the analysis, we will assume that $q = 1$, so there is one vaccination pulse per period. That is, the sequence of impulse times is precisely $t_k = k\tau$ for $k \in \mathbb{Z}$. Then, from the previous section, we can explicitly calculate

$$\tilde{S}(t, v) = 1 - ve^{-\mu[t]_\tau}, \tag{IV.3.17}$$

which implies that $\tilde{S}(\tau^-, v) = 1 - ve^{-\mu\tau}$ and $\tilde{S}(0, v) = 1 - v$. We can also explicitly calculate the critical vaccination coverage where $R_0 = 1$. We find

$$v^* = \frac{\mu\tau}{1 - e^{-\mu\tau}}\left(1 - \frac{\gamma + \mu}{\eta f'(0)}\right). \tag{IV.3.18}$$

We will now pass to the equivalent system with vaccinated component (IV.3.4)–(IV.3.8). Define the changes of variables and parameters

$$X + \tilde{S}(\cdot, v) = x, \qquad V + \frac{v\tilde{S}(\tau^-, v)}{1 - ve^{-\mu\tau}} = V_0, \qquad Y = y, \qquad \epsilon + v^* = v.$$

The result is the following system of impulsive delay differential equations:

$$\begin{aligned}
\dot{X} &= -\mu X(t) + \eta f(Y)\big[\tilde{S}(t, v^* + \epsilon) + X\big] + \gamma Y(t-\tau)e^{-\mu\tau}, && t \neq k\tau \\
\dot{Y} &= \eta f(Y)\big[\tilde{S}(t, v^* + \epsilon) + X\big] - (\mu + \gamma)Y(t), && t \neq k\tau \\
\dot{V} &= 0, && t \neq k\tau \\
\dot{\epsilon} &= 0, && t \neq k\tau \\
\Delta X &= -(v^* + \epsilon)X(t^-) + (1 - (v^* + \epsilon))e^{-\mu\tau}V(t^-), && t = k\tau \\
\Delta Y &= 0, && t = k\tau \\
\Delta V &= (v^* + \epsilon)X(t^-) - (1 - (v^* + \epsilon)e^{-\mu\tau})V(t^-), && t = k\tau \\
\Delta \epsilon &= 0, && t = k\tau.
\end{aligned} \tag{IV.3.19}$$

Notice that $(X, Y, V, \epsilon) = (0, 0, 0, \epsilon)$ is an equilibrium whenever $v^* + \epsilon \in [0, 1]$. The change of variables has had the effect of translating the disease-free periodic solution to the origin. We will now follow the programme of Sect. I.8.6.

Linearization

We must linearize at a nonhyperbolic equilibrium—that is, an equilibrium at which the linearization has some Floquet exponents with zero real part. The origin is expected to be nonhyperbolic with a pair of Floquet exponents with zero real part, with the first zero exponent resulting from the nonhyperbolicity of $\tilde{S}$ at the critical vaccination coverage $v = v^*$, and the second zero exponent coming from the trivial dynamics equation for the parameter ϵ. The result is

$$\begin{aligned}
\dot{u}_1 &= -\mu u_1(t) - \eta f'(0)\tilde{S}(t, v^*)u_2(t) + \gamma u_2(t-\tau)e^{-\mu\tau}, && t \neq k\tau \\
\dot{u}_2 &= \eta f'(0)\tilde{S}(t, v^*)u_2(t) - (\gamma + \mu)u_2(t), && t \neq k\tau \\
\dot{u}_3 &= 0, && t \neq k\tau \\
\dot{u}_4 &= 0, && t \neq k\tau \\
\Delta u_1 &= -v^* u_1(t^-) + (1 - v^*)e^{-\mu\tau}u_3(t^-), && t = k\tau \\
\Delta u_2 &= 0, && t = k\tau \\
\Delta u_3 &= v^* u_1(t^-) - (1 - v^* e^{-\mu\tau})u_3(t^-), && t = k\tau \\
\Delta u_4 &= 0, && t = k\tau.
\end{aligned} \tag{IV.3.20}$$

Centre Fibre Bundle

Before we characterize the centre fibre bundle, we introduce a few convenience functions that will be useful both in this and subsequent sections. Define

$$\beta(t, s; \alpha) = \exp\left(\int_s^t (-\gamma - \mu + \eta f'(0)\tilde{S}(u + \alpha, v^*))du\right).$$

Then define the matrix $\mathbf{Z}_1(t,s;z,\alpha)\in\mathbb{C}^{2\times 2}$ for $t\geq s$ and $z\in\mathbb{C}\setminus\{0\}$ by

$$\mathbf{Z}_1(t,s;z,\alpha) = \begin{bmatrix} e^{-\mu(t-s)} & \int_s^t e^{-\mu(t-u)}(-\eta f'(0)\tilde{S}(u+\alpha,v^*)+\frac{1}{z}\gamma e^{-\mu\tau})\beta(u,s;\alpha)du \\ 0 & \beta(t,s;\alpha)\end{bmatrix}.$$

Then, set $\mathbf{Z}(t,s;z,\alpha)=\operatorname{diag}(\mathbf{Z}_1(t,s;z,\alpha),I_{2\times 2})$. Also define the matrix $B\in\mathbb{R}^{4\times 4}$:

$$B=\begin{bmatrix} 1-v^* & 0 & (1-v^*)e^{-\mu\tau} & 0 \\ 0 & 1 & 0 & 0 \\ v^* & 0 & v^*e^{-\mu\tau} & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Finally, the function β satisfies a few useful identities. They are clear from its definition:

$$\begin{aligned}\beta(t,s;\alpha)&=\beta(t,s,[\alpha]_\tau),\\ \beta(t,s;\alpha)&=\beta(t+\tau,s+\tau;\alpha),\\ \beta(t,s;\alpha)&=\beta(t+\alpha,s+\alpha;0).\end{aligned}$$

For convenience, we abuse notation and write $\beta(t,0;0)=\beta(t)$.

Since we have already determined that the dominant Floquet exponent of (IV.3.1)–(IV.3.3) at the disease-free periodic solution must be real—see Lemma IV.3.4.1—we take the ansatz that $u(t)$ is periodic with period τ. As a consequence, $u_2(t-\tau)=u_2(t)$, and (IV.3.20) reduces to an ordinary impulsive differential equation. If we denote $X(t,s)$ the Cauchy matrix of the resulting system, then $M=X(\tau,0)$ is a monodromy matrix. Specifically, $M=B\mathbf{Z}(\tau,0;1,0)$;

$$M=\begin{bmatrix} (1-v^*)e^{-\mu\tau} & (1-v^*)\kappa & (1-v^*)e^{-\mu\tau} & 0 \\ 0 & 1 & 0 & 0 \\ v^*e^{-\mu\tau} & v^*\kappa & v^*e^{-\mu\tau} & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}, \quad \kappa=e_1^T\mathbf{Z}_1(\tau,0;1,0)e_2.$$

The eigenvalues are 1, 0 and $e^{-\mu\tau}$. The periodic solutions are generated by the two-dimensional generalized eigenspace associated with the eigenvalue 1. The eigenvectors are

$$m_1=\begin{bmatrix} (1-v^*)\kappa & 1-e^{-\mu\tau} & v^*\kappa & 0\end{bmatrix}^T$$

and $m_2=e_4$. As a consequence, we can completely describe the centre fibre bundle.

Lemma IV.3.6.1. *The centre fibre bundle, $\mathcal{RCR}_c$, associated with the non-hyperbolic equilibrium $0\in\mathbb{R}^4$ of the system (IV.3.19) is two-dimensional.*

A basis matrix Φ_t, whose columns form a basis for the t-fibre $\mathcal{RCR}_c(t)$, is periodic with period τ and is given explicitly by

$$\Phi_t(\theta) = \mathbf{Z}([t+\theta]_\tau, 0; 1, 0)\begin{bmatrix} (1-v^*)\kappa & 0 \\ 1-e^{-\mu\tau} & 0 \\ v^*\kappa & 0 \\ 0 & 1 \end{bmatrix} := \begin{bmatrix} \Phi_{t,1}(\theta) & 0_{3\times 1} \\ 0 & 1 \end{bmatrix},$$

where $\Phi_{t,1}(\theta) \in \mathbb{R}^3$.

Projection of χ_0 Onto the Centre Fibre Bundle

Another ingredient necessary in the centre manifold reduction concerns the projection of χ_0 onto the centre fibre bundle. Specifically, if $P_c(t) : \mathcal{RCR} \to \mathcal{RCR}_c(t)$ denotes the spectral projection, then there exists a unique $Y(t) \in \mathbb{R}^{2\times 4}$ such that $P_c(t)\chi_0 = \Phi_t Y(t)$. It is characterized as the solution of the equation

$$\Phi_t Y(t) = \frac{1}{2\pi i}\int_{\Gamma_1}(zI - V_t)^{-1}\chi_0 dz, \tag{IV.3.21}$$

where V_t denotes the monodromy operator associated with the linear delay impulsive system (IV.3.20), and Γ_1 is a simple closed counterclockwise contour in $\mathbb{C}$ such that 1 is the only eigenvalue of V_t contained in the closure of its interior. We must compute $Y(t)$. Therefore, to proceed we solve the equation

$$zy - V_t y = \chi_0 \xi \tag{IV.3.22}$$

for $y \in \mathcal{RCR}$, with $\xi \in \{e_1, e_2, e_3, e_4\}$. Our first task will be to obtain a representation of $V_t y$. Equation (IV.3.20) can be equivalently written as

$$\begin{aligned} \dot{u} &= A(t)u(t) + g(t), && t \neq k\tau \\ \Delta u &= (B-I)u(t^-), && t = k\tau, \\ A(t) &= -\mu(E_{11}+E_{22}) + \eta f'(0)\tilde{S}(t, v^*)(-E_{21}+E_{12}) - \gamma E_{22}, \end{aligned}$$

with standard basis matrices $E_{ij} = e_i e_j^T \in \mathbb{R}^{4\times 4}$ and $g(t) = \gamma e^{-\mu\tau}E_{12}u(t-\tau)$. Note that we have treated the delayed term as a nonhomogeneous forcing. If $U_0(t,s)$ denotes the Cauchy matrix associated with the (formally) homogeneous equation (without delays), we can use the variation of constants for ordinary impulsive differential equations (Theorem II.2.2.1) to write

$$u(t) = U_0(t,s)u(s) + \int_s^t U_0(t,r)\gamma e^{-\mu\tau}E_{12}u(r-\tau)dr.$$

Since $V_t y(\theta) = u(t+\tau+\theta; t, y)$, where $u(\cdot; t, y)$ is the solution with an initial condition $(t, y) \in \mathbb{R} \times \mathcal{RCR}$, we obtain the representation

$$V_t y(\theta) = U_0(t+\tau+\theta, t)y(0) + \int_0^{\tau+\theta} U_0(t+\tau+\theta, t+r)\gamma e^{-\mu\tau} E_{12} y(r-\tau)dr \tag{IV.3.23}$$

for $\theta \in [-\tau, 0]$, after a few changes of variables.

Returning to Eq. (IV.3.22), we notice that $zy(\theta) = V_t y(\theta)$ for $\theta < 0$. From the above representation, it follows that $\theta \mapsto V_t y(\theta)$ is differentiable except at times $\theta \in (-\tau, 0]$ where $t + \tau + \theta = k\tau$ for some $k \in \mathbb{Z}$, where it is continuous from the right. At $\theta = 0$, there is an external discontinuity because of the $\chi_0 \xi$ term in (IV.3.22). Taking this into account, we can take derivatives in θ on both sides of $zy(\theta) = V_t y(\theta)$ and compute jumps at those times where $\theta = -[t]_\tau$. We find that $y(\theta)$ is a solution of

$$y' = [A(t+\theta) + \frac{1}{z}\gamma e^{-\mu\tau} E_{12}]y, \qquad \theta \neq -[t]_\tau \tag{IV.3.24}$$

$$\Delta y = (B - I)y(\theta^-), \qquad \theta = -[t]_\tau \tag{IV.3.25}$$

for $\theta \in [-\tau, 0)$. Using the convenience function $\mathbf{Z}$ from earlier, we can explicitly write

$$y(\theta) = \begin{cases} \mathbf{Z}(\theta, -\tau; z, t)y(-\tau), & \theta < -[t]_\tau \\ \mathbf{Z}(\theta, -[t]_\tau; z, t)B\mathbf{Z}(-[t]_\tau, -\tau; z, t)y(-\tau), & \theta \geq -[t]_\tau. \end{cases} \tag{IV.3.26}$$

Since $y(-\tau)$ appears linearly on the right-hand side of the above, we will write it as a matrix product

$$y(\theta) = \mathbf{H}(\theta; z, t)y(-\tau). \tag{IV.3.27}$$

Next, from (IV.3.22), we have $zy(0) - V_t y(0) = \xi$. It is our goal to compute $y(0)$, and to facilitate this, we consider two separate cases. If $[t]_\tau = 0$, then we have $V_t y(0) = BV_t y(0^-)$, as can be verified via Eq. (IV.3.23). Since $V_t(\theta) = zy(\theta)$ for $\theta < 0$, it then follows that $V_t y(0) = Bzy(0^-)$. The equation $zy(0) - V_t y(0) = \xi$ is then equivalent to $zy(0) - Bzy(0^-) = \xi$. A similar argument in the case where $[t]_\tau \neq 0$ then implies that, in both cases, the end result is

$$y(0) = \frac{1}{z}\xi + \mathbf{H}(0^-; z, t)y(-\tau). \tag{IV.3.28}$$

Our final task is to solve for $y(-\tau)$. To do this, substitute (IV.3.28) into (IV.3.23) and set $\theta = -\tau$. Since $V_t y(-\tau) = zy(-\tau)$, the result is

$$zy(-\tau) = \frac{1}{z}\xi + \mathbf{H}(0^-; z, t)y(-\tau). \tag{IV.3.29}$$

Lemma IV.3.6.2. *$z \mapsto (zI - \mathbf{H}(0^-; z, t))^{-1}$ has a pole at $z = 1$. In particular, 1 is an eigenvalue of multiplicity two for $\mathbf{H}(0^-; z, t)$.*

We can now calculate $y = (zI - V_t)^{-1}\chi_0$. Solving Eq. (IV.3.29) and substituting the result into (IV.3.27), the following lemma is proven.

Lemma IV.3.6.3. *$(zI - V_t)^{-1}\chi_0$ has the explicit form*

$$(zI - V_t)^{-1}\chi_0(\theta) = \frac{1}{z}\mathbf{H}(\theta; z, t)(zI - \mathbf{H}(0^-; z, t))^{-1}. \tag{IV.3.30}$$

The next step is to explicitly calculate the contour integral in (IV.3.21). The following lemma provides just enough detail for later calculations.

Lemma IV.3.6.4. *There exist real constants a and b such that*

$$\frac{1}{2\pi i}\int_{\Gamma_1}(zI - V_t)^{-1}\chi_0 = \mathbf{H}(\theta; 1, t)\begin{bmatrix} 0 & ab & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{IV.3.31}$$

Finally, we can compute the matrix $Y(t)$.

Lemma IV.3.6.5. *The matrix $Y(t)$ appearing in the decomposition (IV.3.21) is*

$$Y(t) = \begin{bmatrix} 0 & (1 - e^{-\mu\tau})^{-1}\beta(-[t]_\tau, -\tau; t) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{IV.3.32}$$

Proof. Since the matrix $Y(t)$ appearing in (IV.3.21) is unique and therefore independent of the argument $\theta \in [-\tau, 0]$, we can evaluate both sides of the equation at $\theta = -[t]_\tau$ to simplify the computation. Using Lemmas IV.3.6.4 and IV.3.6.5, the result is the equation

$$\begin{bmatrix} (1 - v^*)\kappa & 0 \\ 1 - e^{-\mu\tau} & 0 \\ v^*\kappa & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \end{bmatrix} = \mathbf{H}(-[t]_\tau; 1, t)\begin{bmatrix} 0 & ab & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Explicitly calculating $\mathbf{H}(-[t]_\tau; 1, t)$, one immediately finds that the only nonzero entries of Y are Y_{12} and Y_{24}, the latter of which is $Y_{24} = 1$. The Y_{12} entry satisfies the equation

$$Y_{12}\begin{bmatrix} (1 - v^*)\kappa \\ 1 - e^{-\mu\tau} \\ v^*\kappa \\ 0 \end{bmatrix} = \mathbf{H}(-[t]_\tau; 1, t)\begin{bmatrix} ab \\ 1 \\ a \\ 0 \end{bmatrix}.$$

Comparing the entries in the second row, we find $Y_{12} \cdot (1 - e^{-\mu\tau}) = \beta(-[t]_\tau, -\tau; t)$, and the result follows. ∎

Dynamics on the Centre Manifold and Bifurcation

To quadratic order, the dynamics on the parameter-dependent centre manifold are driven by a scalar ordinary differential equation. The following follows by Corollary I.6.1.1.

Lemma IV.3.6.6. *The coordinate dynamics on the two-dimensional parameter-dependent centre manifold of the nonhyperbolic equilibrium* $0 \in \mathbb{R}^4$ *of the impulsive delay differential equation* (IV.3.19) *are, for* $||(w,\epsilon)||$ *sufficiently small,*

$$\begin{aligned}\dot{w} &= \eta(1-e^{-\mu\tau})\beta(-[t]_\tau,-\tau;t)\left(g(t)w^2 + f'(0)\partial_v\tilde{S}(t,v^*)\epsilon w\right) + R(t,w,\epsilon),\\ g(t) &= \tilde{S}(t,v^*)(1-e^{-\mu\tau})\beta(t)\left(\frac{1}{2}f''(0)(1-e^{-\mu\tau})\beta(t)\right.\\ &\qquad \left. + f'(0)\int_0^{[t]_\tau} e^{-\mu([t]_\tau - s)}(\gamma e^{-\mu\tau} - \eta f'(0)\tilde{S}(s,v^*))\beta(s)ds\right),\end{aligned} \tag{IV.3.33}$$

where $R(t,w,\epsilon)$ *satisfies* $R(t,0,\epsilon)=0$*, is periodic and right-differentiable in its first argument and is* C^∞ *in* (w,ϵ) *for fixed* t*. On the centre manifold, the evolution in the phase space* $\mathcal{RCR}$ *is determined by the time evolution rule*

$$t \mapsto \Phi_{t,1}w(t). \tag{IV.3.34}$$

With this lemma in place, we can finally state and prove our bifurcation theorem.

Theorem IV.3.6.1. *For a generic set of parameters, a transcritical bifurcation occurs in the model* (IV.3.1)–(IV.3.3) *along the disease-free periodic solution as* v *crosses through the critical vaccination coverage level* v^**. Specifically,*

$$\ell = \int_0^T \beta(-[t]_\tau,-\tau;t)\tilde{S}(t,v^*)\beta(t)g(t)dt$$

is nonzero on a generic subset of parameter space, and the following are satisfied for $|v-v^*|$ *small enough and in a sufficiently small neighbourhood of* $(S,I)=(\tilde{S}(t,v^*),0)$*.*

- *There are at most two periodic solutions: the disease-free solution and a second solution* $t\mapsto\xi(t,v)$ *that is exponentially stable when* $v<v^*$*, unstable when* $v>v^*$ *and satisfies* $\xi(t,v^*)=(\tilde{S}(t,v^*),0)$*.*
- *The unique periodic solution is conditionally stable when* $v=v^*$ *in some half-space.*
- $\xi(\cdot,v)$ *is positive (in both components) if and only if* $(v-v^*)\ell>0$*.*

Proof. The time τ (Poincaré) map associated with the ordinary differential equation (IV.3.33) is readily found to satisfy

$$\begin{aligned} w &\mapsto w + \eta(1-e^{-\mu\tau})[\ell w^2 + m\epsilon w] + h(w,\epsilon) \\ \epsilon &\mapsto \epsilon, \end{aligned}$$

where ℓ is as in the statement of the theorem, m is given by

$$m = \int_0^\tau \beta(-[t]_\tau, -\tau; t) f'(0)\partial_v \tilde{S}(t, v^*)dt$$

and $h(w,\epsilon) = \int_0^\tau R(t,w,\epsilon)dt$ is a C^∞ remainder satisfying $h(0,\epsilon) = 0$ and containing all terms of order 3 and above in (w,ϵ). Note that the mixed ϵw term, m, is strictly negative because $f'(0) > 0$, $\beta > 0$ and $\partial_v \tilde{S}(t,v^*) < 0$. As for the quadratic term, the equation $\ell = 0$ is unstable with respect to perturbations in $f''(0)$, as can be verified by the functional form of $g(t)$ appearing in (IV.3.33). Consequently, on a generic set of parameters, we have $\ell \neq 0$ and $m < 0$. From the transcritical bifurcation for maps, there exists a unique C^1 nontrivial fixed point $(w(\epsilon),\epsilon)$ for $|\epsilon|$ sufficiently small, satisfying $w(0) = 0$. From (IV.3.34), we obtain the claimed nontrivial periodic solution. The stability assertions follow by the reduction principle (Theorem I.5.5.1).

To see that $\xi(\cdot, v)$ is positive only when $(v - v^*)\ell = \epsilon\ell > 0$, we first remark that the fixed point satisfies the estimate $w(\epsilon) = -\epsilon\frac{m}{\ell} + O(\epsilon^2)$. This follows because of the properties of the remainder term h. Also, since $\xi(t,v) \to (\tilde{S}(t,v^*), 0)$ as $v \to v^*$, it suffices to consider only the sign of the second component. This is precisely

$$\begin{aligned} \operatorname{sign}(\xi_2(t, v^* + \epsilon)) &= \operatorname{sign}\left(-\epsilon\frac{m}{\ell}e_2^T\Phi_{t,1}(0)\right) \\ &= \operatorname{sign}\left(\epsilon\ell(1-e^{-\mu\tau})\beta(t)\right) \\ &= \operatorname{sign}(\epsilon\ell), \end{aligned}$$

which is what was claimed. □

IV.3.7 Numerical Bifurcation Analysis

In the previous section we proved that in the event there is only one vaccination pulse per period, the disease-free periodic orbit generically undergoes a transcritical bifurcation when the vaccination coverage crosses a critical threshold, v^*. We can approximate the endemic (i.e. having nonzero I component) periodic solution to linear order, but analyzing further bifurcations will require assistance from the numerical analysis.

In this section, we will use the illustrative parameter choices provided in Table IV.3.1, and to keep results consistent with the analysis appearing in [77, 83], we will use the incidence rate $f(x) = \frac{x}{1+x}$.

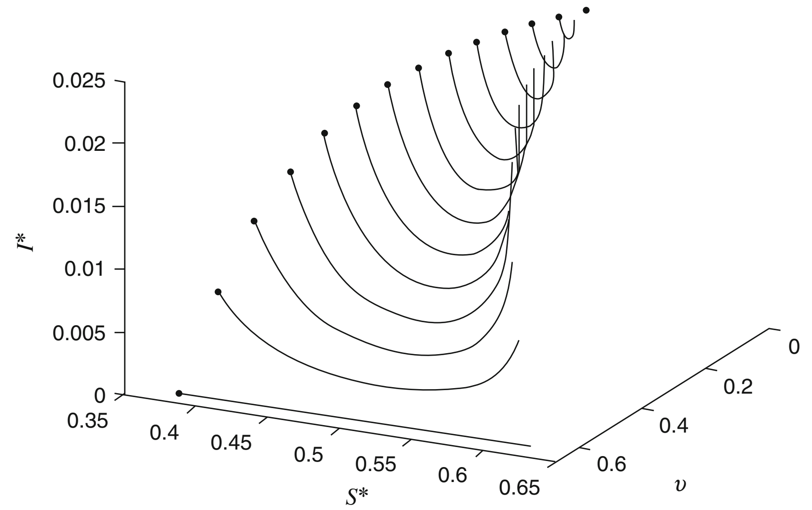

Figure IV.3.1: Plots of the periodic solution obtained by continuation for vaccine coverage $v \in [0, v^*]$. Dots indicate the "initial" points $(S^*(0,v), I^*(0,v))$ on each periodic solution, followed by evolution along the corresponding curve at each level v with time left implicit. The periodic solution is constant in the I variable at $v = v^*$ and collapses to a fixed point at $v = 0$. To improve visibility, only fourteen vaccination coverages in the interval $[0, v^*]$ are displayed

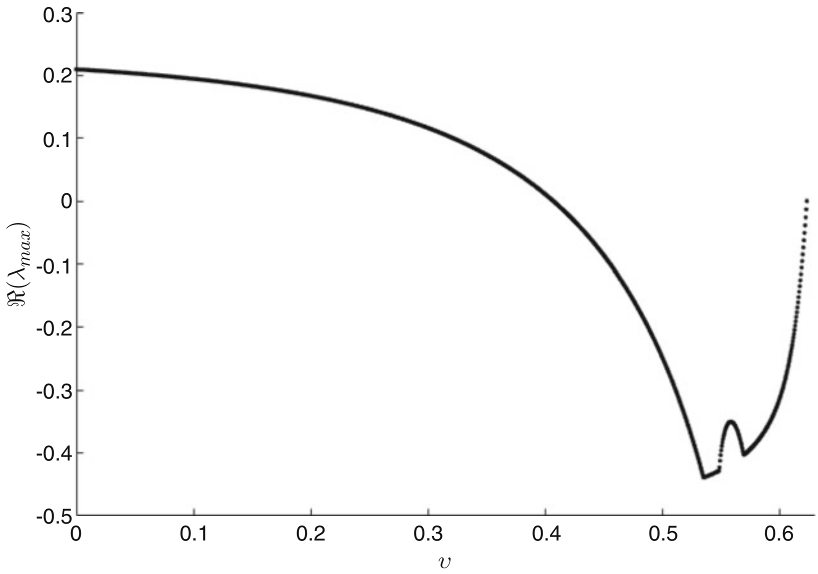

Figure IV.3.2: Plot of the real part of the dominant Floquet exponent. There is a crossing of the imaginary axis (0 on the vertical axis) at $v = v_c^* \approx 0.4063$, where there is a pair of complex-conjugate imaginary Floquet exponents

Parameter	Numerical value/range
μ	0.5
η	50
γ	25
τ	1
v^*	0.6227
v	$[0, v^*]$

Table IV.3.1: Parameters used for the numerical bifurcation analysis for the model (IV.3.1)–(IV.3.3)

By employing a boundary-value continuation scheme, the nontrivial bifurcating periodic solution was continued from $v = v^*$ to $v = 0$. This branch of periodic solutions is locally stable at least for v near v^*. Along this branch of periodic solutions, the dominant (i.e. with maximum real part) Floquet exponent was computed numerically. There is a single crossing point $v_c^* \approx 0.4063$ where $\Re(\lambda_{max}) = 0$ in the regime $[0, v^*]$, and the Floquet spectrum was computed there. The result was a Hopf point. This investigation is summarized in Figs. IV.3.1 and IV.3.2.

We should expect that a cylinder bifurcation occurs at $v = v_c^*$ due to Theorem I.8.4.1. From the Floquet spectrum at $v = v_c^*$, we can compute the rotation parameter θ, and we find $\theta = 1.9886$ and, in particular, $|e^{ik\theta} - 1| > 0.316$ for $k = 1, 2, 3, 4$, so the first nondegeneracy condition (G.1) passes. For the second nondegeneracy condition, we can infer from Fig. IV.3.2 that $\gamma(0) < 0$, since the dominant Floquet multiplier crosses the imaginary axis from right to left. In terms of the figure, the real part of λ_{max} is decreasing at $v = v_c^*$, which implies $\gamma(0) < 0$. As such, we can be confident that the second nondegeneracy condition (G.2) of the cylinder bifurcation succeeds. We do not bother computing the Lyapunov coefficient, and this task is quite onerous and does not improve the exposition.

Since $\gamma(0) < 0$, we should expect the invariant cylinder to be attracting when $\gamma(0)[v - v_c^*] > 0$, which is equivalent to $v < v_c^*$. Conversely, the endemic periodic solution should be asymptotically stable when $v > v_c^*$. This is consistent with the earlier analysis, since the bifurcating branch of periodic solutions appears to be locally stable in the regime (v_c^*, v^*). The complete bifurcation diagram is provided in Fig. IV.3.3. We indeed numerically detect an attracting invariant cylinder in the phase space; see Fig. IV.3.4. As v decreases from v_c^* to zero, the cylinder contracts until it collapses onto a periodic solution of the system without vaccination. This visual is provided in Fig. IV.3.5.

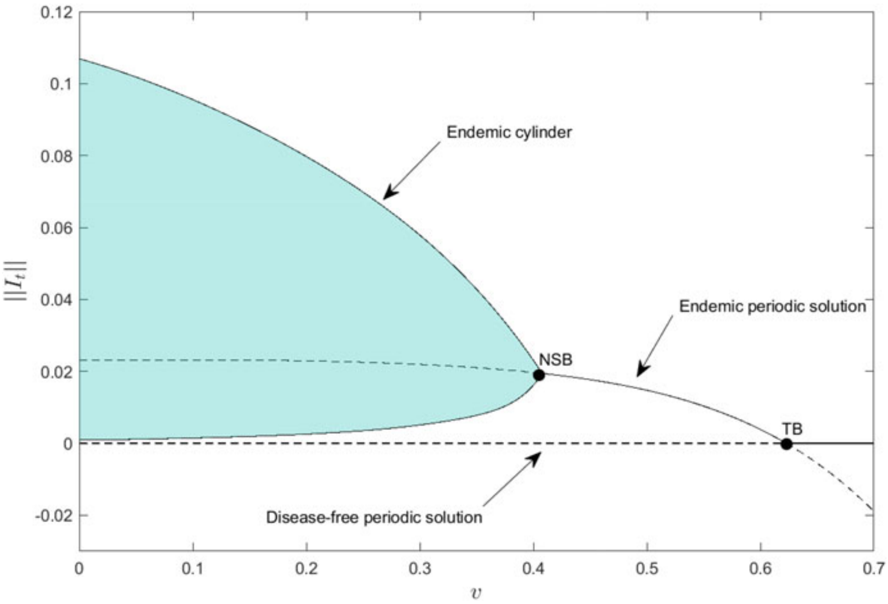

Figure IV.3.3: Bifurcation diagram for the model (IV.3.1)–(IV.3.3) with the parameters from Table IV.3.1. TB indicates a transcritical bifurcation, while NSB indicates a Neimark–Sacker (cylinder) bifurcation. The boundaries of the blue region denoted *endemic cylinder* correspond to the maximum and minimum values of the norm of the infected component for the bifurcating invariant cylinder over a long (100 time units) simulation range. Solid lines indicate asymptotically stable (or attracting) objects, while dashed lines indicate unstable objects

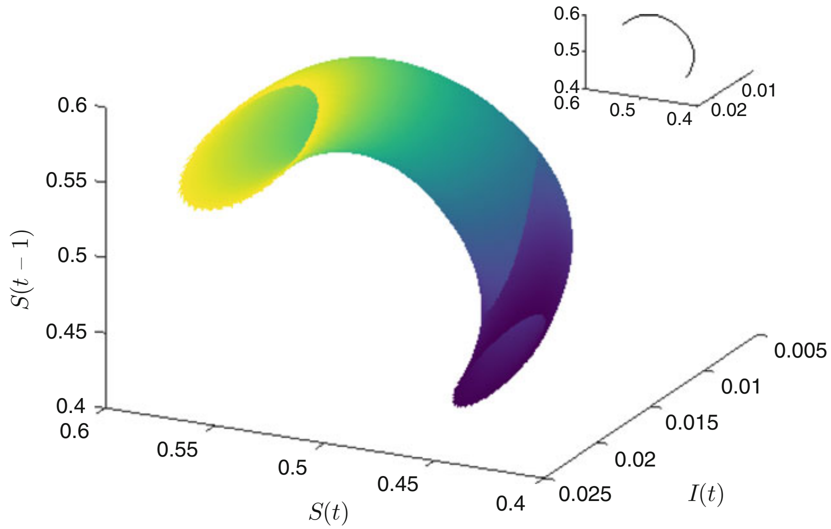

Figure IV.3.4: Plot of $t \mapsto (S(t), I(t), S(t-1))$ for $t \in [300, 1300]$ from a constant initial condition of $(S(0), I(0)) = (0.5, 0.5)$ for $v = 0.395$. Purple corresponding to arguments $t = k \in \mathbb{Z}$ and yellow to arguments $t \to k^-$. Inset: Plot of the image of the endemic periodic solution $t \mapsto (S^*(t, v), I^*(t, v), S^*(t-1, v))$

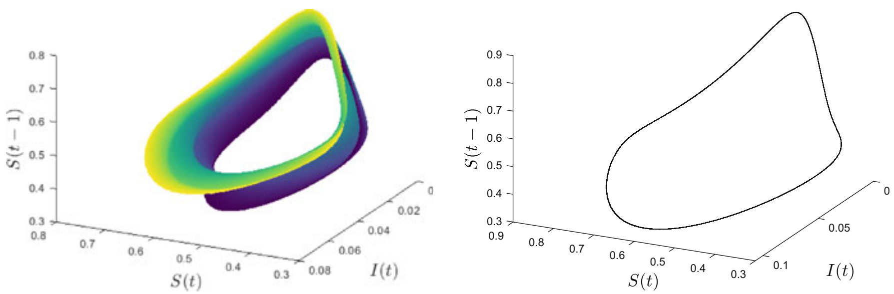

Figure IV.3.5: Plot of $t \mapsto (S(t), I(t), S(t-1))$ for $t \in [300, 1300]$ from a constant initial condition of $(S(0), I(0)) = (0.5, 0.5)$, for $v = 0.25$ (left) and $v = 0$ (right). Colours have the same interpretation as in Fig. IV.3.4

Chapter IV.4

Stage-Structured Predator–Prey System with Pulsed Birth

Predator–prey models have been studied for close to one hundred years, with perhaps the most well-known Lotka–Volterra model [101] being proposed in 1920. Jumping forward several decades, several authors have considered predator–prey models involving stage structure. Gourley and Kuang [54] consider the effect of maturation delay on a predator–prey system in which the predator class is divided into juvenile and adult classes. A model with Beddington–DeAngelis functional response and stage structure for the prey was studied by Chen and You [24]. We refer the reader also to the review article [123] for general discussion on predator–prey models with discrete delay. These references are very far from exhaustive, and predator–prey systems with numerous types of functional response, maturation delay and stage structures have been described in the literature with varying degrees of analysis.

The aforementioned models have considered continuous-time birth dynamics. Pulsed birth dynamics in ecological models have been considered by a few authors [97, 139]. More recently, Xiang, Long and Song [153] considered a delayed Lotka–Volterra model with prey birth pulse described by a discrete logistic update, with stage-structured predator population. In this chapter, we will consider a predator–prey model with prey birth pulse and stage structure, where the predator only has one life stage. The model takes

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the form of an impulsive functional differential equation with distributed and time-varying discrete delays:

$$x' = \rho\alpha x(t)y(t_k)\exp\left(-r_1(t-t_k) - \delta\int_{t_k-t}^{0} x(t+\theta)d\theta\right) - hx(t), \quad t \in [t_k, t_k+\tau), \tag{IV.4.1}$$

$$x' = -hx, \quad t \in [t_k+\tau, t_{k+1}) \tag{IV.4.2}$$

$$y' = -r_2 y - k_2 y^2, \quad t \neq t_k + \tau \tag{IV.4.3}$$

$$\Delta y = \alpha y(t-\tau)\exp\left(-r_1\tau - \delta\int_{-\tau}^{0} x(t+\theta)d\theta\right), \quad t = t_k+\tau. \tag{IV.4.4}$$

Note that the $y(t_k)$ term in (IV.4.1) can be interpreted as a discrete delay. Here, x represents the predator and y the adult prey. The juvenile prey population is factored into the birth pulse (Eq. IV.4.4)—this will be further explained in Sect. IV.4.1 where the model is derived from some biological assumptions. In Sect. IV.4.2 the stability of the extinction equilibrium is considered. The existence and stability of a predator-free periodic solution are studied in Sect. IV.4.3. The transcritical bifurcation involving these solutions is proven analytically in Sect. IV.4.4. Some additional discussion follows in Sect. IV.4.5.

IV.4.1 Model Derivation

We consider three population classes: predator (x), juvenile prey (y_1) and adult prey (y_2). The population dynamics are as follows:

- The hunting rate of juvenile prey by the predator is δ (units of (predator$\cdot$time)$^{-1}$) with a type I functional response. Juvenile prey undergo background death due to other causes at per capita rate r_1 and are not resource-limited.
- Predators are unable to hunt adult prey. The latter are subject to interspecific competition for resources with baseline death rate r_2 and interspecific competition coefficient k_2.
- The reproductive effort of prey (offspring per adult per birth cycle) is α. Juvenile prey are born at times t_k.
- Those juvenile prey that survive τ time units mature[1] into adults. The maturation delay satisfies $t_k + \tau < t_{k+1}$ for all $k \in \mathbb{Z}$, and the sequence $\{t_k : k \in \mathbb{Z}\}$ is periodic.

[1]Without any variance—that is, all juvenile prey mature at exactly the same time. This is a major simplifying assumption.

- The birth rate of predators conditional on hunting is ρ (units of (prey $\cdot$ time)$^{-1}$) with a type I functional response and death rate h.

Based on these assumptions, the number of juveniles at time t_k is $y_1(t_k) = \alpha y_2(t_k^-)$. The juvenile prey population then evolves according to the ordinary differential equation

$$\dot{y}_1(t) = -\delta x(t) y_1(t) - r_1 y_1(t)$$

for $t \in [t_k, t_k + \tau]$, whereas $y_1(t_k + \tau) = 0$ since all remaining juvenile prey mature simultaneously. We have

$$y_1(t_k + s) = y_2(t_k^-) \exp\left(-r_1 s - \delta \int_0^s x(t_k + \theta) d\theta\right) \tag{IV.4.5}$$

for $s \in [0, \tau)$. The right-hand side depends only on the adult prey population, and since all juveniles mature simultaneously, the result is the impulse effect (IV.4.4). Using Eq. (IV.4.5), one can then derive the predator growth rate from (IV.4.1). Since there are no juvenile prey available for hunting when $t \in [t_k + \tau, t_{k+1})$, we get (IV.4.2). The remaining differential equation (IV.4.3) is straightforward.

Remark IV.4.1.1. *The right-hand side of* (IV.4.1)–(IV.4.2) *can indeed be written as a functional on $\mathcal{RCR}$ in a sense that is compatible with, for example, conditions H.3/H.5 of Sect. I.4.1. With $x_t = \phi$ and $y_t = \psi$, it can be identified with the (time-dependent) functional $(\phi, \psi) \mapsto F(t, \phi, \psi) - h\phi(0)$, where*

$$F(t,\phi,\psi) = \begin{cases} \rho\alpha\phi(0)\psi(t_k - t)\exp\left(-r_1(t-t_k) - \delta\int_{-\tau}^0 \chi_{(t_k-t,0)}(\theta)\phi(\theta)d\theta\right), & t \in [t_k, t_k+\tau) \\ 0, & t \in [t_k+\tau, t_{k+1}). \end{cases}$$

IV.4.2 Stability of the Extinction Equilibrium

The extinction equilibrium is the trivial fixed point $(x, y) = (0, 0)$. The linearization of (IV.4.1)–(IV.4.4) at this fixed point takes the form

$$\dot{z} = \begin{bmatrix} -h & 0 \\ 0 & -r_2 \end{bmatrix} z(t), \qquad t \neq t_k + \tau \tag{IV.4.6}$$

$$\Delta z = \begin{bmatrix} 0 & 0 \\ 0 & \alpha e^{-r_1 \tau} \end{bmatrix} z(t - \tau), \qquad t = t_k + \tau. \tag{IV.4.7}$$

Lemma IV.4.2.1. *Let the period of the birth sequence be $T > 0$ with q births per period. That is, $t_{k+q} = t_k + T$ for all $k \in \mathbb{Z}$. Define the quantity*

$$R_0 = \left(1 + \alpha e^{(r_2 - r_1)\tau}\right)^q e^{-r_2 T}.$$

The extinction equilibrium is unstable if $R_0 > 1$ and locally asymptotically stable if $R_0 < 1$.

Proof. The first component of the linearization is an ordinary differential equation that is exponentially stable, contributing the Floquet multiplier e^{-hT}. As for the second component, any Floquet eigensolution $z_2(t) = \phi(t)e^{\lambda t}$ must satisfy

$$\begin{aligned} \dot{\phi} &= -(r_2 + \lambda)\phi(t), && t \neq t_k + \tau \\ \Delta\phi &= \alpha e^{-(r_1+\lambda)\tau}\phi(t-\tau), && t = t_k + \tau. \end{aligned}$$

Explicitly solving, we get

$$\begin{aligned} \phi(t_k + \tau) &= e^{-(r_2+\lambda)\tau}\phi(t_k) + \alpha e^{-(r_1+\lambda)\tau}\phi(t_k) \\ &= e^{-\lambda\tau}\left(e^{-r_2\tau} + \alpha e^{-r_1\tau}\right)\phi(t_k) \\ \phi(t_{k+1}) &= e^{-(r_2+\lambda)(t_{k+1}-(t_k+\tau))}e^{-\lambda\tau}\left(e^{-r_2\tau} + \alpha e^{-r_1\tau}\right)\phi(t_k) \\ &= e^{-(r_2+\lambda)(t_{k+1}-t_k)}\left(1 + \alpha e^{(r_2-r_1)\tau}\right)\phi(t_k), \end{aligned}$$

which then implies

$$\phi(t_q) = e^{-(r_2+\lambda)T}\left(1 + \alpha e^{(r_2-r_1)\tau}\right)^q \phi(t_0).$$

ϕ is a periodic solution (of period T) if and only if $\phi(t_q) = \phi(t_0)$. Explicitly solving for $e^{\lambda T}$, one finds $e^{\lambda T} = R_0$. In particular, R_0 is a Floquet multiplier. The result follows by linearized stability (Theorem I.7.7.1). □

Remark IV.4.2.1. *The quantity R_0 is the linear-order compounding ratio (per period T) of the prey population near extinction levels. By contrast, the linear-order continuous-time growth rate is given by the associated Floquet exponent: $\lambda_0 = \frac{1}{T}\log R_0$, which has the expression*

$$\lambda_0 = -r_2 + q\log\left(1 + \alpha e^{(r_2-r_1)\tau}\right).$$

Observe that if $r_2 > r_1$, the adult prey has a higher background mortality rate, and the time delay has a positive effect on fitness (that is, increasing τ will increase the growth rate). In the opposite case where $r_1 < r_2$, the juvenile prey has a shorter average life expectancy (equivalently, a higher death rate ignoring predation) than mature prey, with the time delay having a corresponding negative effect on fitness. This is consistent with intuition—overall fitness at low predator levels should be improved by lengthening the time spent in the juvenile stage only if the juvenile stage is "safer," in that the background death rate (ignoring predation) is lower than it is for those in the adult stage.

IV.4.3 Analysis of Predator-Free Periodic Solution

Any predator-free periodic solution $(0, y)$ must satisfy the impulsive delay differential equation

$$\dot{y} = -r_2 y - k_2 y^2, \qquad t \neq t_k + \tau \tag{IV.4.8}$$

$$\Delta y = \alpha y(t_k) e^{-r_2\tau}, \qquad t = t_k + \tau. \tag{IV.4.9}$$

Although this equation is explicitly solvable, it is difficult (or at least tedious) to analytically compute periodic solutions when there is more than one birth pulse per period. We will therefore opt for a more topological tactic for existence and uniqueness of the predator-free periodic solution, covering stability later.

IV.4.3.1 Existence and Uniqueness of the Predator-Free Solution

To find periodic solutions of (IV.4.8)–(IV.4.9), it suffices to look for solutions y that satisfy $y(t_0) = y(t_0 + T)$. To this end, the general solution $t \mapsto S(t, y_0)$ of the first equation (IV.4.8) satisfying $y(0) = y_0$ is given by

$$S(t, y_0) = \frac{-r_2 \xi(y_0)}{k_2 \xi(y_0) - e^{r_2 t}}, \quad \xi(y_0) = \frac{y_0}{r_2 + k_2 y_0}. \tag{IV.4.10}$$

Since this differential equation is autonomous, the solution $t \mapsto z(t)$ satisfying $z(s) = z_s$ is given by $z(t) = S(t - s, z_s)$. Any solution of (IV.4.8)–(IV.4.9) satisfies

$$\begin{aligned} y(t_k + \tau) &= S(\tau, y(t_k)) + \alpha e^{-r_2\tau} y(t_k) \\ y(t_{k+1}) &= S(t_{k+1} - t_k - \tau, S(\tau, y(t_k)) + \alpha e^{-r_2\tau} y(t_k)). \end{aligned}$$

Define for convenience

$$g_{k+1}(z) = S(t_{k+1} - t_k - \tau, S(\tau, z) + \alpha e^{-r_2\tau} z). \tag{IV.4.11}$$

It follows that the solution $t \mapsto y(t)$ of (IV.4.8)–(IV.4.9) with $y(t_0) = Y$ is periodic if and only if Y is a solution of the fixed-point equation

$$Y = g_q \circ \cdots \circ g_1(Y) := G(Y). \tag{IV.4.12}$$

Lemma IV.4.3.1. *The function G is*

- *twice continuously differentiable on $(-\epsilon, \infty)$ for some $\epsilon > 0$,*
- *increasing, with $G'(0) = R_0$ and*

- *strictly concave on* $[0, \infty)$.

Proof. Twice continuous differentiability of G follows from its definition. Observe that the function $y \mapsto S(t, y)$ in (IV.4.10) is strictly concave on the nonnegative semiaxis, $Y \geq 0$. This can be verified by computing its second derivative and checking $\partial_y^2 S(t, y) < 0$. It is also increasing. Since $\alpha e^{-r_2\tau} \geq 0$, it follows that $y \mapsto S(\tau, y) + \alpha e^{-r_2\tau} y$ is strictly concave and increasing and, subsequently, that each g_k for $k = 1, \ldots, q$ is strictly concave and increasing. The same is therefore true for the composition $G = g_q \circ \cdots \circ g_1$. As for the claim concerning differentiability at zero, define $z(t) = \frac{\partial}{\partial y_0} y(t; 0)$ for $t \mapsto y(t; y_0)$ the solution of (IV.4.8)–(IV.4.9) satisfying $y(t_0) = y_0$. Theorem I.4.2.1 implies that this function is well-defined and, in particular, satisfies

$$\begin{aligned} \dot{z} &= -r_2 z, & t &\neq t_k + \tau \\ \Delta z &= \alpha z(t_k) e^{-r_2\tau}, & t &= t_k + \tau, \end{aligned}$$

with initial condition $y(t_0) = 1$. Following (the proof of) Lemma IV.4.2.1, we get $z(t_0 + T) = R_0$. Since $G(Y) = y(t_0 + T; Y)$, the result follows. □

Lemma IV.4.3.2. *A nontrivial, nonnegative predator-free periodic solution exists if and only if* $R_0 > 1$. *In this case, the predator-free periodic solution is unique.*

Proof. Suppose $R_0 \leq 1$. Then $G'(0) = R_0 \leq 1$ by Lemma IV.4.3.1, but since G is strictly concave, it must satisfy $G(y) < y$ for all $y > 0$. Consequently, there are no positive solutions of the equation $G(Y) = Y$ and no nonnegative, nontrivial predator-free periodic solutions.

If $R_0 > 1$, then $G'(0) = R_0 > 1$ implies the existence of some $a > 0$ such that $G(y) > y$ for $0 < y \leq a$. We also know that $G(0) = 0$. Since G is strictly concave on $[0, \infty)$ and increasing, $\lim_{y\to\infty} G(y)$ exists. It follows that there exists $b > a$ such that $G(b) < b$. Applying Theorem 3.3 from Kennan [80], G has a unique positive fixed point. We conclude that there is a unique nontrivial, nonnegative predator-free periodic solution. □

IV.4.3.2 Stability

The next step in our analysis is the investigation of stability of the predator-free periodic solution. Here our result will be a bit implicit, since an analytical expression for the predator-free periodic solution is unavailable.

Theorem IV.4.3.1. *Suppose* $R_0 > 1$. *Define the quantity*

$$R_0^* = \exp\left(-hT + \frac{(1 - e^{-r_1\tau})\rho\alpha}{r_1} \sum_{k=0}^{q-1} y^*(t_k)\right),$$

where y^* is the prey component of the unique predator-free periodic solution. The predator-free periodic solution is unstable if $R_0^* > 1$ and locally asymptotically stable if $R_0^* < 1$.

Proof. The linearization of (IV.4.1)–(IV.4.4) at the predator-free periodic solution is

$$\begin{aligned}
\dot{z}_1 &= (\rho\alpha y^*(t_k)e^{-r(t-t_k)} - h)z_1(t), && t \in [t_k, t_k+\tau) \\
\dot{z}_1 &= -hz_1(t), && t \in [t_k+\tau, t_{k+1}), \\
\dot{z}_2 &= -(r_2 + 2k_2 y^*(t))z_2(t), && t \neq t_k+\tau \\
\Delta z_2 &= \alpha e^{-r_1\tau}\left(z_2(t_k) - y^*(t_k)\delta\int_{-\tau}^{0} z_1(t_k+\theta)d\theta\right), && t = t_k+\tau.
\end{aligned}$$

Observe that the continuous-time portion (the first three equations) generates a diagonal system of ordinary differential equations—that is, the z_1 equation is independent of z_2 and vice versa. We can formally write the linearization in block form

$$\dot{z} = \begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{bmatrix} z(t), \qquad t \neq t_k+\tau \tag{IV.4.13}$$

$$\Delta z = \begin{bmatrix} 0 & 0 \\ B_1(k) & B_2(k) \end{bmatrix} z_{t^-}, \qquad t = t_k+\tau, \tag{IV.4.14}$$

for some matrices $A_1(t)$ and $A_2(t)$ and some functionals $B_1(k)$ and $B_2(k)$. Lemma I.2.2.1 implies that the evolution family satisfies

$$U(t_q+\tau, t_0+\tau) = \prod_{k=q}^{1}\left(\begin{bmatrix} I & 0 \\ \chi_0 B_1(k) & I + \chi_0 B_2(k) \end{bmatrix} C(t_k+\tau, t_{k-1}+\tau)\right),$$

where C is the evolution family for the diagonal system generated by (IV.4.13), and the product is from left to right. The above is a monodromy operator for the linearization, and it admits a decomposition of the form

$$V\phi := V_{t_0+\tau}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} V_1 & 0 \\ V_2 & V_3 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

for compact operators V_1, V_2 and V_3 on $\mathcal{RCR}([-\tau, 0], \mathbb{R})$.

Suppose ϕ is an eigenvector of V with eigenvalue (the Floquet multiplier) μ. If $\phi_1 = 0$, then by definition $V_3\phi_2 = \mu\phi_2$. It follows that ϕ_2 is a nonzero eigenvector of the monodromy operator associated with the system

$$\begin{aligned}
\dot{z}_2 &= -(r_2 + 2k_2 y^*(t))z_2(t), && t \neq t_k+\tau \\
\Delta z_2 &= \alpha e^{-r_1\tau} z_2(t_k), && t = t_k+\tau.
\end{aligned}$$

This is precisely the linearization at y^* in the predator-free system (IV.4.8)–(IV.4.9), from which it follows that $\mu = G'(y^*(t_0))$. Since G is strongly concave, increasing and $G'(0) = R_0 > 1$—see Lemma IV.4.3.1—we conclude that $0 < G'(y^*(t_0)) < 1$, so $\mu \in (0, 1)$.

The remaining possibility is that $\phi_1 \neq 0$. If this is the case, ϕ_1 must be a nonzero eigenvector of the monodromy operator associated with

$$\begin{aligned} \dot{z}_1 &= (\rho\alpha y^*(t_k)e^{-r(t-t_k)} - h)z_1(t), && t \in [t_k, t_k + \tau) \\ \dot{z}_1 &= -hz_1(t), && t \in [t_k + \tau, t_{k+1}). \end{aligned}$$

There is only one nontrivial (i.e. nonzero) Floquet multiplier, given precisely by the Floquet multiplier of the above scalar equation interpreted as an ordinary differential equation. With a bit of effort, one can verify that this nontrivial Floquet multiplier is R_0^*. The result then follows by Theorem I.7.7.1. □

IV.4.4 Bifurcation at Extinction

The conclusions of Lemma IV.4.2.1 and Lemma IV.4.3.2 strongly suggest that at $R_0 = 1$, there is a transcritical bifurcation of the predator-free periodic solution with the extinction equilibrium. In order to complete the proof, we must demonstrate that there are no other small periodic solutions for $|R_0 - 1|$ small, apart from the extinction state and the predator-free solution. To do this, we will perform a centre manifold reduction[2] at $R_0 = 1$ and verify that the quadratic coefficient of the associated discrete-time map does not vanish. This will guarantee that any parameter variation resulting in a transversal crossing through the surface $R_0 = 1$ (in parameter space) will induce a transcritical bifurcation.

From (the proof of) Lemma IV.4.2.1), we know that when $R_0 = 1$, the centre fibre bundle $\mathcal{RCR}_c(t)$ is one-dimensional and is spanned by the singleton $q_t := (0, z_t)$, where z is a nontrivial periodic solution of

$$\dot{z} = -r_2 z, \qquad t \neq t_k + \tau \tag{IV.4.15}$$

$$\Delta z = \alpha e^{-r_1\tau} z(t - \tau), \qquad t = t_k + \tau. \tag{IV.4.16}$$

Also, the unstable fibre bundle is trivial. From this point forward, we will take z to be a fixed nontrivial periodic solution. We may without loss of generality assume $z > 0$.

To apply Corollary I.6.1.1 concerning the quadratic-order dynamics on the centre manifold, we must compute the 1×2 matrix $Y_c(t)$ satisfying the equation $P_c(t)\chi_0 = q_t Y_c(t)$ (note: $\Lambda = \frac{1}{T}\log(R_0) = 0$ since we have assumed $R_0 = 1$), for $P_c(t)$ the Riesz projection operator onto $\mathcal{RCR}_c(t)$ along the monodromy operator. It is therefore necessary to solve the equation

$$\xi\phi^{(i)} - V_t\phi^{(i)} = \chi_0 e_i \tag{IV.4.17}$$

[2]This will essentially give an alternate proof of Lemma IV.4.3.2 with additional details for the case $R_0 < 1$, as well as ruling out the existence of other small periodic solutions.

for $\phi^{(i)} = \phi^{(i)}(\xi, \cdot) \in \mathcal{RCR}$ and $t \in \mathbb{R}$, with e_i the standard basis vectors for $\mathbb{R}^2$ and $i = 1, 2$. After completing this calculation, we will have

$$P_c(t)\chi_0 = \frac{1}{2\pi i}\int_\gamma \left[\ \phi^{(1)}(\xi,\cdot) \quad \phi^{(2)}(\xi,\cdot)\ \right] d\xi, \tag{IV.4.18}$$

for γ a sufficiently small counterclockwise curve enclosing $1 \in \mathbb{C}$. We can then use the above to construct $Y_c(t)$.

IV.4.4.1 Calculation of the Matrix $Y_c(t)$

If ϕ solves Eq. (IV.4.17), then examining the first component gives the equation

$$\xi\phi_1 - V_t^{(1)}\phi_1 = e_1^\intercal \chi_0 e_i,$$

where $V_t^{(1)}$ is the monodromy operator for first component of the linearization: $\dot{z}_1 = -hz_1$. If $i = 1$, evaluation at $\theta = 0$ gives the equation

$$\xi\phi_1(0) - e^{-hT}\phi_1(0) = 1,$$

so that $\phi_1(0) = (\xi - e^{-hT})^{-1}$ and, subsequently, $\phi_1(\theta) = \xi^{-1}e^{-h(T+\theta)}(\xi - e^{-hT})^{-1}$ for $\theta < 0$. Conversely, if $i = 2$, then we get $\phi_1(\theta) = 0$. Since $\xi = 1$ is not a pole of either of these functions, we conclude that the first row of the 2×2 matrix (IV.4.18) is identically zero.

Next we proceed with the second row of $P_c(t)\chi_0$. Again, if ϕ solves (IV.4.17), then the second component gives the equation

$$\xi\phi_2 - V_t^{(2)}\phi_2 = e_2^\intercal \chi_0 e_i, \tag{IV.4.19}$$

where $V_t^{(2)}$ is the monodromy operator for (IV.4.15)–(IV.4.16), the (decoupled) second component of the linearization (IV.4.6)–(IV.4.7). Suppose first that $i = 1$. Then $e_2^\intercal \chi_0 e_1 = 0$, and the above equation reduces to

$$\xi\phi_2 - V_t^{(2)}\phi_2 = 0.$$

From (the proof of) Lemma IV.4.2.1, the only eigenvalues of $V_t^{(2)}$ are 0 (the trivial eigenvalue shared by all compact operators) and $R_0 = 1$. As such, the above equation has a nontrivial solution if and only if $\xi = 1$. Thus, in a punctured neighbourhood of $\xi = 1$, we have $\phi_2 = 0$, and we conclude that the $(2, 1)$-entry of $P_c(t)\chi_0$ is zero.

To complete our calculation of $P_c(t)\chi_0$, we need to study the second row of (IV.4.17) with $i = 2$. This is a bit more involved. Let $s \mapsto w(s)$ denote the solution (IV.4.15)–(IV.4.16) satisfying $x_t = \phi$, for given $t \in \mathbb{R}$ and some $\phi \in \mathcal{RCR}([-\tau, 0], \mathbb{R})$. That is, $w_{t+T} = V_t^{(2)}\phi$ with $V_t^{(2)}$ the same operator as appearing in (IV.4.19). We will handle two cases separately.

Case 1: $t \in [t_k + \tau, t_{k+1}]$ for some $k \in \mathbb{Z}$

We claim that $w(s) = z(s)z(t)^{-1}\phi(0)$ for $s \geq t$. Indeed, this function is a scalar multiple of the fixed, nontrivial periodic solution of (IV.4.15)–(IV.4.16) and satisfies $w(t) = \phi(0)$ as required, while the restriction of ϕ to $[-\tau, 0)$ has no effect on the future dynamics because of the initial time condition t. Since $T - \tau > 0$, we are guaranteed $t + T + \theta > t$ for $\theta \in [-\tau, 0]$. We can then express $V_t^{(2)}\phi$ as

$$V_t^{(2)}\phi(\theta) = w(t + T + \theta) = z(t + T + \theta)z(t)^{-1}\phi(0) = z_t(\theta)z(t)^{-1}\phi(0).$$

Evaluating (IV.4.19) at $\theta = 0$ results in the algebraic equation $\xi\phi(0) - \phi(0) = 1$, which we readily solve to get $\phi(0) = (\xi - 1)^{-1}$. Evaluating (IV.4.19), we can directly compute $\phi(\theta)$. The end result is

$$\phi(\xi, \theta) = \frac{1}{\xi - 1}\left\{ \begin{array}{ll} 1, & \theta = 0 \\ \xi^{-1}z_t(\theta)z(t)^{-1}, & \theta < 0, \end{array} \right. , \quad \frac{1}{2\pi i}\int_\gamma \phi(\xi, \theta)d\xi = z_t(\theta)z(t)^{-1}. \tag{IV.4.20}$$

Case 2: $t \in (t_k, t_k + \tau)$ for some $k \in \mathbb{Z}$

For $\theta \neq 0$, Eq. IV.4.19 implies that we can write $\phi(\theta) = \frac{1}{\xi}w(t + T + \theta)$. Consequently,

$$\begin{aligned} \frac{d}{d\theta}\phi(\theta) &= \frac{1}{\xi}\frac{d}{d\theta}w(t + T + \theta) \\ &= \frac{1}{\xi}\left(-r_2 w(t + T + \theta)\right) \\ &= -r_2\phi(\theta) \end{aligned}$$

when $\theta \neq 0$ and $t + T + \theta \notin \{t_k + \tau : k \in \mathbb{Z}\}$—equivalently, when $t + \theta \notin \{t_k + \tau : k \in \mathbb{Z}\}$ due to periodicity of the sequence $\{t_k : k \in \mathbb{Z}\}$. We will take a further ansatz on ϕ and assume that it is continuous from the left at $\theta = 0$, so that, in particular, we have $\phi(t_k - t) = e^{-r_2(t_k - t)}\phi(0)$. Under this ansatz, we have for $s \geq t$ that the function w can be written in the form

$$w(s) = z(t)z(t_k)^{-1}e^{-r_2(t_k - t)}\phi(0).$$

However, since $z(t_k)^{-1}e^{-r_2(t_k - t)} = [z(t_k)e^{-r_2(t - t - k)}]^{-1} = z(t)^{-1}$, the function w reduces to precisely the same one as in case 1. Solving for ϕ then proceeds in the exact same way, so that (IV.4.20) remains valid for $t \in (t_k, t_k + \tau)$. Since (IV.4.19) is guaranteed to have a unique solution for $\xi \neq 1$, we are satisfied.

Factoring $P_c(t)\chi_0$ and Identifying $Y_c(t)$

Combining the previous results, it follows that $P_c(t)\chi_0$ can be written in the form

$$P_c(t)\chi_0 = \begin{bmatrix} 0 & 0 \\ 0 & z_t(\cdot)z(t)^{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ z_t(\cdot) \end{bmatrix} \begin{bmatrix} 0 & z(t)^{-1} \end{bmatrix}.$$

The row vector on the far right-hand side is precisely $Y_c(t) = [\begin{matrix} 0 & z(t)^{-1} \end{matrix}]$.

IV.4.4.2 Centre Manifold Quadratic Dynamics and Bifurcation

The data to be input into the quadratic-order dynamics equation on the centre manifold from Corollary I.6.1.1 are as follows: $\Lambda = 0$, $Y_c(t) = [\begin{matrix} 0 & z(t)^{-1} \end{matrix}]$, $Q_t = [\begin{matrix} 0 & z_t \end{matrix}]^\intercal$ and

$$f(t,\phi,\psi) = \begin{bmatrix} F(t,\phi,\psi) \\ -k_2\psi(0)^2 \end{bmatrix}, \quad g(\phi,\psi) = \begin{bmatrix} 0 \\ \alpha\psi(-\tau)\exp\left(-r_1\tau - \delta\int_{-\tau}^0 \phi(\theta)d\theta\right) \end{bmatrix},$$

where F is as defined in Remark IV.4.1.1. The end result is the scalar ordinary differential equation

$$\dot{u} = -k_2 z^2(t)u^2 + O(u^3). \tag{IV.4.21}$$

We initially assumed without loss of generality that $z > 0$, and we now see that even if we had taken the opposite choice $z < 0$, this would not alter the conclusion. Since $k_2z^2(t) > 0$, the quadratic coefficient $a_{20} = -k_2\int_0^T z^2(t)dt$ of the associated time T map does not vanish and is strictly negative. It follows that the extinction equilibrium undergoes a transcritical bifurcation with the predator-free solution whenever parameter variation results in a transversal crossing of R_0 through unity. There are no other small (i.e. near $(x,y) = 0$) periodic solutions near $R_0 = 1$. The predator-free periodic solution is locally asymptotically stable (for $|R_0 - 1|$ small) if and only if $R_0 > 1$, while the extinction equilibrium is locally asymptotically stable if and only if $R_0 < 1$. In the opposite cases, they are unstable.

IV.4.5 Discussion

The analysis of stability for the extinction equilibrium and the existence and stability of the predator-free periodic solution in this chapter were quite simple. This is primarily because the impulsive delay differential equations decouple in convenient ways upon linearization, and the study for existence of predator-free solutions allows us to consider a scalar equation. It should be emphasized, however, that as simple as the proofs concerning existence and stability of the predator-free periodic solution were, it was necessary to employ the invariant manifold theory to rule out the existence of other solutions near the bifurcation point, $R_0 = 1$.

$R_0^* = 1$ represents another possible point of analysis, and we would expect a transcritical bifurcation to occur there as well. The analysis is a bit more challenging, however, since the predator-free periodic solution is needed to express R_0^* analytically. Moreover, the linearization at $(0, y^*)$ is a fair bit more complicated, as can be seen from the proof of Theorem IV.4.3.1. The first equation is still decoupled from the second, so computing the matrix $Y_c(t)$ needed to complete the bifurcation analysis may not be overly difficult. We leave the problem of analyzing the bifurcation at $R_0^* = 1$ open.

Chapter IV.5

Dynamics of an In-host Viral Infection Model with Drug Treatment

The final chapter of this part of the monograph will quickly demonstrate the difficulty in analyzing systems of impulsive functional differential equations without analytically available "reference states" such as periodic solutions. The majority of the analysis will be completed with the help of numerical methods, and even then we will not delve too deeply into it.

Broadly, our starting point here is the dynamics of human immunodeficiency virus (HIV). Here, there is a small time delay between the time when a $CD4^+$ T cell is infected by a HIV-1 virus and when it begins producing virus particles. In the infectious disease modelling literature, this is sometimes modelled by a discrete delay; see for instance [87, 111, 134]. The impact of protease inhibitors and reverse transcriptase inhibitors—two drugs that serve to decrease viral load—has been considered [133] in a model without delay, with the drug doses being modelled by impulses. Reverse transcriptase inhibitors—as their name implies—inhibit reverse transcriptase. The latter is an enzyme responsible for generating complementary DNA from an RNA template, a process that retroviruses including HIV rely on to replicate. In some sense, reverse transcriptase inhibitors "immunize" cells against viral infection. Protease inhibitors disrupt the production of proteins necessary to correctly build virus particles in an infected cell.

There are two interesting mechanisms we wish to study here. First, there is the "vaccination" action of drug treatment on viral target cells. There

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is also the finite time delay between infection of a cell and production of virions. We will propose here a general in-host viral infection model that takes into account these two features. Although the inspiration is in-host HIV dynamics, the model is fairly general and could be applied in other situations. The model is as follows:

$$\dot{T} = \hat{s} - \mu T(t) - \beta T(t)V(t) - rT(t)R(t) + mT_r(t), \qquad t \neq kq \tag{IV.5.1}$$

$$\dot{T}_r = rT(t)R(t) - mT_r(t) - \mu T_r(t), \qquad t \neq kq \tag{IV.5.2}$$

$$\begin{aligned}\dot{I} &= \beta T(t-\tau)V(t-\tau)e^{-\mu\tau}\exp\left(-r\int_{-\tau}^{0} R(t+\theta)d\theta\right) + mI_r(t)\\ &\quad - rI(t)R(t) - \alpha I(t), \qquad t \neq kq\end{aligned} \tag{IV.5.3}$$

$$\begin{aligned}\dot{I}_r &= \beta T(t-\tau)V(t-\tau)e^{-\mu\tau}\left(1 - \exp\left(-r\int_{-\tau}^{0} R(t+\theta)d\theta\right)\right) - mI_r(t)\\ &\quad + rI(t)R(t) - \alpha I_r(t), \qquad t \neq kq\end{aligned} \tag{IV.5.4}$$

$$\dot{V} = \alpha(\gamma I(t) + \nu I_r(t)) - \delta V(t) - \beta T(t)V(t), \qquad t \neq kq \tag{IV.5.5}$$

$$\begin{aligned}\dot{R} &= -r\eta\left(\int_{-\tau}^{0} \beta T(t+\theta)V(t+\theta)\exp\left(\theta\mu - r\int_{\theta}^{0} R(t+\xi)d\xi\right)d\theta\right)R(t) - \rho R(t)\\ &\quad - r\eta(T(t) + I(t))R(t), \qquad t = kq,\end{aligned} \tag{IV.5.6}$$

$$\Delta R = Q, \qquad t = kq, \tag{IV.5.7}$$

for $k \in \mathbb{Z}$. T denotes the population of susceptible cells that have not absorbed the drug, T_r denotes the population of susceptible cells that have absorbed the drug, I denotes the infected cells that have not absorbed the drug, while I_r denotes the cells that have. V is the population of virions (virus particles), and R is the amount of drug. These quantities are typically measured in terms of concentrations. The fate of each individual cell/particle in addition to its population dynamics is as follows.

- Susceptible cells (T) are produced at a constant rate $\hat{s}$ and die at rate μ. They can also absorb drug at rate $rR(t)$ or become infected at rate $\beta V(t)$. Those cells that had absorbed drug clear it at rate m.
- Cells that have absorbed drug (T_r) either clear it from their intracellular compartment or die.
- Infected cells (I) become productive τ units of time after coming into contact with virus, provided they do not die (at rate μ) or absorb drug (at rate $rR(t)$) during this intermediate phase, hence the exponential terms. These cells die at rate α.

- Infected cells that have absorbed drug come from two sources: from those infected cells that absorb drug while productively infected or by absorbing drug after infection but before becoming productively infected τ time units later. The latter is accounted for in the distributed delay term, while the former is the semilinear term $rI(t)R(t)$.

- Virus particles (V) are produced by infected cells at an effective rate of $\alpha\gamma I(t)$ and drug-affected cells at rate $\alpha\nu I_r(t)$. Virions degrade at rate δ and infect susceptible cells at rate $\beta T(t)$.

- The dose size of the drug is Q, and doses occur at times kq for $k \in \mathbb{Z}$ (or, for biological significance, on some positive half-line of the integers). The drug is metabolized (i.e. removed) at rate ρ and is absorbed into susceptible cells at rate $\eta r T(t)$ and productively infected cells at rate $\eta r I(t)$. The absorption rate of drug by those infected cells that have not yet become productively infected (during the τ time unit intermediate phase) is captured by the distributed delay term. It is assumed that doses occur infrequently relative to the delay, so $\tau < q$. η is a conversion ratio representing the average quantity of drug absorbed per cell before it becomes effective.

Remark IV.5.0.1. *In the case of HIV, some experimental evidence [111] suggests $\tau \approx 1$, whereas HIV drugs are often taken once per day or multiple times per day. This is therefore one of the seemingly rare cases where the delay appearing in the model feasibly could be* larger *than the time between impulses.*

The distributed delay terms arise naturally from the presence of the discrete time delay between infection and production of virions. A derivation of these terms is provided in Sect. IV.5.1. The complete model (IV.5.1)–(IV.5.7) is very difficult to analyze by hand. Indeed, we have no analytical expression for even a disease-free periodic solution and to establish the existence of such a solution, we will need to use indirect methods—see Sect. IV.5.2. As such, after proving well-posedness of the model and boundedness of solutions in Sect. IV.5.3, we will immediately proceed to the numerical bifurcation analysis.

In this section, we will denote $\mathcal{RCR}^+ = \mathcal{RCR}([-\tau, 0], \mathbb{R}^6_+)$—that is, the phase space consists of the right-continuous regulated functions that are non-negative.

IV.5.1 Derivation of the Delayed Terms

During the short time interval $[t-\tau, t-\tau+dt]$, an amount $I_0 = \beta T(t-\tau)V(t-\tau)dt$ of cells is infected. At time t, some of these cells will have died, while some of them will have absorbed drug. Let $x(s)$ denote the amount of cells that have not died or absorbed drug for $s \in [0, \tau]$. Since these cells exhibit the same death rate and absorption rate of the drug, we have

$$\frac{dx}{ds} = -rx(s)R(t - \tau + s) - \mu x(s),$$

with $x(0) = I_0$. Solving the linear ODE, we get

$$\begin{aligned} x(\tau) &= I_0 \exp\left(-\mu\tau - r\int_0^\tau R(t - \tau + s)ds\right) \\ &= \beta T(t - \tau)V(t - \tau)e^{-\mu\tau} \exp\left(-\int_{-\tau}^0 rR(t + \theta)ds\right) dt. \end{aligned}$$

Dividing by dt, this is precisely the rate at which cells become productively infected, and it accounts for the distributed delay in (IV.5.3).

We also need to determine the amount of cells that absorb drug after infection but before becoming productively infected and survive this intermediate period. This can be determined by first observing the total amount of cells that have not died by time $s \in [0, \tau]$, in the sense of the previous derivation, which is given by

$$y(s) = \beta T(t - \tau)V(t - \tau)^{-\mu s} dt.$$

If we denote by $z(s)$ the amount of cells that absorbed drug by time $s \in [0, \tau]$, then from conservation of mass, $y = x + z$. Consequently,

$$z(\tau) = \beta T(t - \tau)V(t - \tau)e^{-\mu\tau}\left(1 - \exp\left(-\int_{-\tau}^0 rR(t + \theta)ds\right)\right) dt.$$

This accounts for the distributed delay term in (IV.5.4).

Finally, we need to determine the rate of change or the drug. From the model assumptions, this is consumed by

- cells that have not been infected ($T(t)$, rate constant ηr),
- infected cells ($I(t)$, rate constant ηr),
- those cells that have been infected but are not yet producing virus (rate constant ηr) and
- linear decay (rate ρ).

The third item on the list will require a bit of work to compute. We know that

$$I_u = \beta T(t - \tau + u)V(t - \tau + u)dt$$

is the amount of cells infected in the small interval $[t - \tau + u, t - \tau + u + dt]$ for some arbitrary $u \in [0, \tau]$. Similar to the first paragraph,

$$x_u(s) = \beta T(t - \tau + u)V(t - \tau + u) \exp\left(-\mu s - r\int_0^s R(t - \tau + u + z)dz\right) dt$$

is the amount of cells that have not absorbed drug or died at time $t-\tau+u+s$, given an initial small quantity of infected cells $x_u(0) = I_u$ at time $t-\tau+u$. We want to count all of the cells that were infected at any point in the interval $[t-\tau, t]$ but have not absorbed drug prior to time t or died. This is given precisely by the integral

$$\begin{aligned}\int_0^\tau x_u(\tau-u)du &= \int_0^\tau I_u \exp\left(-\mu(\tau-u) - r\int_0^{\tau-u} R(t-\tau+u+z)dz\right)du \\ &= dt \cdot \int_{-\tau}^0 \beta T(t+\theta)V(t+\theta)\exp\left(\theta\mu - r\int_\theta^0 R(t+v)dv\right)d\theta.\end{aligned}$$

Therefore, the differential of $R(t)$ is

$$\begin{aligned}dR(t) = -r\Bigg(T(t) + I(t) + \int_{-\tau}^0 \beta T(t+\theta)V(t+\theta) \\ \times \exp\left(\theta\mu - r\int_\theta^0 R(t+v)dv\right)d\theta + \frac{\rho}{r}\Bigg)R(t)dt,\end{aligned}$$

which is equivalent to (IV.5.6).

IV.5.2 Existence of a Disease-free Periodic Solution and a Disease-free Attractor

The disease-free subspace, that is, the set

$$\Omega = \{\phi = (T, T_r, I, I_r, V, R) \in \mathcal{RCR} : I = I_r = V = 0\},$$

is invariant under (IV.5.1)–(IV.5.6). The dynamics restricted to Ω are given by the system of ordinary impulsive differential equations

$$\dot{T} = \hat{s} - \mu T - rTR + mT_r, \qquad t \neq kq \tag{IV.5.8}$$

$$\dot{T}_r = rTR - (m+\mu)T_r, \qquad t \neq kq \tag{IV.5.9}$$

$$\dot{R} = -\rho R - \eta rTR, \qquad t \neq kq \tag{IV.5.10}$$

$$\Delta R = Q, \qquad t = kq. \tag{IV.5.11}$$

Observe that the positive orthant $\Omega^+ = \{(T, T_r, R) \in \mathbb{R}^3 : T, T_r, R \geq 0\}$ is positively invariant. If we define the sum $Z = T + T_r + Q$, it satisfies the differential equality

$$\begin{aligned}\dot{Z} &\leq \hat{s} - \min\{\mu, \rho\}Z, && t \neq kq \\ \Delta Z &= Q, && t = kq\end{aligned}$$

whenever $(T, T_r, R) \in \Omega^+$. One can then prove with little difficulty that $Z(t)$ is uniformly ultimately bounded. That is, for any initial condition $Z(0) \geq 0$ and $\epsilon > 0$, there exists $t^* > 0$ such that

$$0 \leq Z(t^*) \leq \epsilon + \frac{Q}{1 - e^{-q\min\{\mu,\rho\}}} + \frac{\hat{s}}{\min\{\mu, \rho\}}.$$

Moreover, this bound can be achieved uniformly for $T(0) \in B$ for any bounded set $B \subset \Omega^+$. Define the simplex

$$\Omega^*(\epsilon) = \left\{(T, T_r, R) \in \mathbb{R}^3 : 0 \leq T + T_r + R \leq \epsilon + \frac{Q}{1 - e^{-q\min\{\mu,\rho\}}} + \frac{s}{\min\{\mu, \rho\}}\right\},$$

and let $S : \Omega^+ \to \Omega^+$ be the time q map associated with (IV.5.8)–(IV.5.11). It follows that $\Omega^*(\epsilon)$ is a compact, absorbing set for the discrete-time dynamical system $S : \Omega^+ \to \Omega^+$ for each $\epsilon > 0$. By the theory of dissipative systems[19], we then get the following result.

Lemma IV.5.2.1. *The disease-free subspace Ω contains at least one periodic solution. The dynamics restricted to this subspace admits a unique global attractor. We refer to it as the* disease-free attractor*.*

We can obtain slightly more information concerning the structure of the disease-free attractor. When $Q = 0$, the system degenerates to (IV.5.8)–(IV.5.10), which has the global attractor consisting of a single, hyperbolic fixed point

$$X^*(0) = \left(\frac{\hat{s}}{\mu}, 0, 0\right).$$

There are no other fixed points. Since this fixed point is hyperbolic, it persists uniquely for $Q > 0$ small enough. Denote this fixed point by $X^*(Q)$, and let $t \mapsto x^*(t; Q)$ denote the associated periodic solution.

Corollary IV.5.2.1. *Exactly one of the following holds:*

- *for each $Q > 0$, the periodic solution $x^*(\cdot; Q)$ is locally asymptotically stable;*
- *there exists $Q > 0$ such that $x^*(\cdot; Q)$ is nonhyperbolic—equivalently, $X^*(Q)$ is a nonhyperbolic fixed point of the discrete-time system $S : \Omega^+ \to \Omega^+$.*

This corollary will later justify our numerical approach of searching for the disease-free periodic solution by parameter continuation in the Q parameter.

IV.5.3 Well-Posedness and Boundedness

In this section we consider some fundamentals concerning the complete model. Namely, we will prove that the nonnegative phase space $\mathcal{RCR}^+$ is indeed positively invariant and that every solution remains bounded for all (future) time.

Lemma IV.5.3.1. *For any initial condition* $(\phi, s) \in \mathcal{RCR}^+ \times \mathbb{R}$*, the unique (local) solution* $t \mapsto S(t,s)\phi$ *of* (IV.5.1)–(IV.5.7) *satisfies* $S(t,s)\phi \in \mathcal{RCR}^+$ *as long as this solution exists.*

Proof. Suppose, by way of contradiction, that there exists $t^* > s$ such that $S(t^*, s)\phi \notin \mathcal{RCR}^+$. Let $t \mapsto x(t)$ denote the solution in the Euclidean space $\mathbb{R}^6$. Then, there exists $t^{**} \geq s$ with the following properties:

- $x(t^{**})$ is on the boundary of $\mathbb{R}^6_+$—that is, at least one component of $x(t^{**})$ is zero;
- at least one component of $x(t)$ that was zero at $t = t^{**}$ is negative for all $t \in (t^{**}, t^{**} + \epsilon)$ for some small $\epsilon > 0$;
- there is no $t \in [s, t^{**})$ with these properties.

Let $x(t^{**}) = (T, T_r, I, I_r, V, R)$. If $T = 0$, then from (IV.5.1), we can see that $\dot{T} > 0$, so this component cannot decrease through zero. If $R = 0$, then from (IV.5.6) we have $\dot{R} = 0$, so the R component cannot decrease through zero. If $T_r = 0$, then $\dot{T}_r = rTR$. Since T and R are bounded away from zero, we conclude that T_r cannot decrease through zero.

Now, suppose that $I = 0$ and that the I component is negative for $t \in (t^{**}, t^{**}+\epsilon)$. It follows that one of I_r and $V(t^{**}-\tau)$ is nonpositive. Since they both cannot be zero on $(t^{**}, t^{**} + \epsilon)$, one of them must be strictly negative. However, it cannot be $V(t^{**}-\tau)$ because this would contradict the definition of t^{**}. Thus, $I_r < 0$. But this once again contradicts the definition of t^{**}. It follows that the I component cannot go negative. The same type of argument can be used to show that neither I_r nor V goes negative, thereby completing the proof. □

Lemma IV.5.3.2. *The solutions of* (IV.5.1)–(IV.5.7) *remain bounded for all time. That is, to any* $(\phi, s) \in \mathcal{RCR}^+ \times \mathbb{R}$*, there exists* $K > 0$ *such that* $||S(t,s)\phi|| \leq K$ *for all* $t \geq s$*.*

Proof. Without loss of generality, let $s = 0$. Consider the sum $Z_1 = T + T_r$. Z_1 satisfies the differential inequality

$$\dot{Z}_1 \leq \hat{s} - \mu Z_1,$$

from which it follows that each of $T(t)$ and $T_r(t)$ is asymptotically bounded above by $\mu^{-1}\hat{s}+\epsilon$ for any $\epsilon > 0$. In a similar manner, R satisfies the impulsive differential inequality

$$\begin{aligned} \dot{R} &\leq -\rho R, && t \neq qk \\ \Delta R &= Q, && t = qk. \end{aligned}$$

The upper solution of this differential inequality converges exponentially to a periodic solution. In particular,

$$R(t) \leq \frac{Q}{1 - e^{-\rho q}} + \epsilon$$

asymptotically for any $\epsilon > 0$.

Next, we aim to show that any given solution with an initial condition in $\mathcal{RCR}^+$ remains bounded for all (positive) time. Assuming to the contrary, at least one of the three components, I, I_r or V, must be unbounded since we already know that the remaining components are uniformly bounded. Suppose V is unbounded. Since the sum $Z_1 = T + T_r$ remains bounded and satisfies the differential equation

$$\dot{Z}_1 = \hat{s} - \mu Z_1 - \beta T(t)V(t).$$

V can only be unbounded if $T(t)V(t)$ is bounded. Now, define $Z_2 = I + I_r$. Then,

$$\begin{aligned} \dot{Z}_2 &\leq \beta T(t-\tau)V(t-\tau)e^{-\mu\tau} - \alpha Z_2, \\ \dot{V} &\leq \alpha \max\{\gamma, \nu\} Z_2 - \delta V. \end{aligned}$$

By Gronwall's inequality, Z_2 remains bounded (since both $T(t-\tau)V(t-\tau)$ and R are bounded). We can then apply Gronwall's inequality to get boundedness of V, which is a contradiction. Therefore, V remains bounded. Since V is bounded, we then have $\dot{Z}_2 \leq C - \alpha Z_2$ asymptotically for some constant C, which implies that Z_2 is bounded. Since each of Z_1, R, V and Z_2 is bounded and all quantities are positive, the solution itself remains bounded for all time. □

The well-posedness of the model is proven, and we will now move onto the numerical results. We would ideally like to prove that solutions are uniformly ultimately bounded so that with a bit of effort we could again apply the theory of dissipative systems to get the existence of a global attractor. From the proof of the previous lemma, it is not difficult to obtain uniform bounds on the T, T_r and R components, but the infected and virus components, I, I_r and V, are difficult to bound uniformly. As such, we will not continue along these lines here.

IV.5.4 Numerical Bifurcation Analysis: Preamble

There are two tasks we must accomplish in order to complete even a non-rigorous numerical bifurcation analysis. They are as follows:

- transform the system (IV.5.1)–(IV.5.7) into a form that is amenable to numerical integration—for crude parameter continuation of periodic solutions;

- derive an efficient method of discretizing the monodromy operator associated with a linearized periodic solution—for bifurcation detection.

Following this, we will locate a disease-free periodic solution, perform parameter continuation and attempt to identify bifurcation points.

IV.5.4.1 Model Transformation

It is necessary to perform some transformations to the model as most numerical integration packages do not support distributed delays out of the box and these appear in (IV.5.3), (IV.5.4) and (IV.5.6). For instance, MATLAB's `dde23` solver supports discrete (and state-dependent discrete) delays, but not distributed delays. To accomplish this, define the functions

$$\kappa_1(t) = \exp\left(-r\int_{-\tau}^{0} R(t+\theta)d\theta\right), \tag{IV.5.12}$$

$$\kappa_2(t) = \int_{-\tau}^{0} \beta T(t+\theta)V(t+\theta)\exp\left(\theta\mu - r\int_{\theta}^{0} R(t+\xi)d\xi\right)d\theta. \tag{IV.5.13}$$

These functions appear in (IV.5.3), (IV.5.4) and (IV.5.6). Each one is continuous and differentiable from the right everywhere. By rewriting the integral terms, we can determine a system of delay differential equations in the variables, R, κ_1 and κ_2. Namely, writing equivalently

$$\kappa_1(t) = \exp\left(-r\int_{t-\tau}^{t} R(\theta)d\theta\right),$$
$$\kappa_2(t) = \int_{t-\tau}^{t} \beta T(\theta)V(\theta)\exp\left(\mu(\theta-t) - r\int_{\theta}^{t} R(\xi)d\xi\right)d\theta,$$

one can verify that these functions satisfy

$$\dot{\kappa}_1(t) = r\left(R(t-\tau) - R(t)\right)\kappa_1(t), \tag{IV.5.14}$$

$$\dot{\kappa}_2(t) = \beta T(t)V(t) - \beta T(t-\tau)V(t-\tau)e^{-\mu\tau}\kappa_1(t) - (\mu + rR(t))\kappa_2(t). \tag{IV.5.15}$$

Appending these to (IV.5.1)–(IV.5.7) and substituting κ_1 and κ_2 in the appropriate places in the original functional differential equations, the resulting system can now be simulated using MATLAB's `dde23`, with the impulses handled using `events`. For ease of reference later, we write down the complete system now:

$$\dot{T} = s - \mu T(t) - \beta T(t)V(t) - rT(t)R(t) + mT_r(t), \quad t \neq kq \tag{IV.5.16}$$

$$\dot{T}_r = rT(t)R(t) - mT_r(t) - \mu T_r(t), \quad t \neq kq \tag{IV.5.17}$$

$$\dot{I} = \beta T(t-\tau)V(t-\tau)e^{-\mu\tau}\kappa_1(t) + mI_r(t) - rI(t)R(t) - \alpha I(t), \quad t \neq kq \tag{IV.5.18}$$

$$\dot{I}_r = \beta T(t-\tau)V(t-\tau)e^{-\mu\tau}\left(1-\kappa_1(t)\right) - mI_r(t) + rI(t)R(t) - \alpha I_r(t), \quad t \neq kq \tag{IV.5.19}$$

$$\dot{V} = \alpha(\gamma I(t) + \nu I_r(t)) - \delta V(t) - \beta T(t)V(t), \quad t \neq kq \tag{IV.5.20}$$

$$\dot{R} = -r\eta\kappa_2(t)R(t) - \rho R(t) - r\eta(T(t)+I(t))R(t), \quad t = kq, \tag{IV.5.21}$$

$$\dot{\kappa}_1 = r(R(t-\tau) - R(t))\kappa_1(t), \quad t \neq kq \tag{IV.5.22}$$

$$\dot{\kappa}_2 = \beta T(t)V(t) - \beta T(t-\tau)V(t-\tau)e^{-\mu\tau}\kappa_1(t) - (\mu + rR(t))\kappa_2(t), \quad t \neq kq \tag{IV.5.23}$$

$$\Delta R = Q, \quad t = kq. \tag{IV.5.24}$$

One must be careful with the modified system (IV.5.16)–(IV.5.24). The initial conditions $\phi_1, \phi_2 \in \mathcal{RCR}$ for the functions κ_1 and κ_2 must be chosen in such a way that

$$\phi_1(0) = \exp\left(-r\int_{-\tau}^{0} R_0(\theta)d\theta\right), \tag{IV.5.25}$$

$$\phi_2(0) = \int_{-\tau}^{0} \beta T_0(\theta)V_0(\theta)\exp\left(\mu\theta - r\int_{\theta}^{0} R_0(\xi)d\xi\right)d\theta, \tag{IV.5.26}$$

where $T_0, V_0, R_0 \in \mathcal{RCR}$ are the respective initial conditions for T cells, virus and RTI, respectively. This is needed to ensure compatibility with the definition of κ_1 and κ_2.

Remark IV.5.4.1. *The system* (IV.5.1)–(IV.5.7) *is not equivalent to* (IV.5.16)–(IV.5.24)*. For example, even though every periodic solution of the first system uniquely determines one in the second system, the opposite is not true. Indeed, This is mainly due to the fact that κ_1 and κ_2 are essentially defined through differentiation and subsequent integration of the distributed delay terms and so are not determined uniquely. A periodic solution must have its κ_1 and κ_2 components satisfy the conditions* (IV.5.12)–(IV.5.13)*. Additionally, although one can study the stability of a periodic solution of the original system using the new one* (IV.5.16)–(IV.5.24)*—and we will indeed do this—the latter will always have $\mu = 1$ as a Floquet multiplier. See [106] for additional details.*

IV.5.4.2 Monodromy Operator Discretization

Once a (numerical) periodic solution (of period q) has been identified, we can symbolically linearize the system at this solution. The linearization at this periodic solution does not actually have any impulse effects, since the Fréchet derivative of the (constant) right-hand side of (IV.5.7) vanishes. As such, abstractly the linearization will take the form of a linear delay differential equation

$$\dot{z} = A(t)z(t) + B(t)z(t-\tau), \tag{IV.5.27}$$

where the $n \times n$ ($n = 8$ for (IV.5.16)–(IV.5.24)) matrices $A(t)$ and $B(t)$ are q-periodic, continuous and differentiable except at times $t \in q\mathbb{Z}$ where they are continuous from the right. This delay differential equation can be solved using the method of steps, and for continuous A and B, Gilsinn and Potra [50] used the Chebyshev spectral collocation to discretize the monodromy operator for systems of the type (IV.5.27) and obtained some convergence guarantees. Church and Liu [32] proposed a quadrature-based approach to discretize a monodromy operator for a linear impulsive delay differential equation but did not derive any convergence guarantees. Since the latter method is quick to implement, the exposition is fairly simple and we will only be using it as a crude test for possible bifurcation points, we will use it as our discretization scheme and outline it here.

Let $V_0 = U(q, 0)$ be the monodromy operator. Let M be the largest integer such that $M\tau \leq q$. Roughly speaking, the method is divided into $M + 1$ stages:

- Stage 0: Given an initial condition (function) z_0, compute (approximate) z_τ.
- Stage $j = 1, \ldots, M-1$: Compute (approximate) $z_{(j+1)\tau}$ given $z_{j\tau}$.
- Stage M: Compute (approximate) $z_q = V_0 z_0$ given $z_{M\tau}$.

Each of $z_\tau, \ldots, z_{M\tau}$ and z_q can be computed sequentially by the method of steps. We will begin with steps $1, \ldots, M$.

Let $0 \leq j \leq M-1$. For notational simplicity, denote $z_{j\tau} = \phi$. The solution through the initial condition $(z_0, 0)$ satisfies, for $t \in [j\tau, (j+1)\tau]$, the ordinary differential equation

$$\dot{u} = A(t)u(t) + B(t)\phi(t - (j+1)\tau).$$

Let $X(t,s)$ denote the Cauchy matrix for the ODE $\dot{x} = A(t)x$. Then, define $Z(t,s) \equiv X(t,s)$ for $t \geq s$, while $Z(t,s) = 0$ for $t < s$. Let $\nu \in [-\tau, 0]$. For brevity, denote $\bar{t} = j\tau$. By the variation-of-constants formula,

$$\begin{aligned} u(\bar{t}+\tau+\nu) &= X(\bar{t}+\tau+\nu,\bar{t})\phi(0) + \int_{\bar{t}}^{\bar{t}+\tau+\nu} X(\bar{t}+\tau+\nu,s)B(s)\phi(s-\bar{t}-\tau)ds \\ &= X(\bar{t}+\tau+\nu,\bar{t})\phi(0) + \int_{-\tau}^{\nu} X(\bar{t}+\tau+\nu,\bar{t}+\tau+s)B(\bar{t}+\tau+s)\phi(s)ds \\ &= Z((j+1)\tau+\nu,j\tau)\phi(0) + \int_{-\tau}^{0} Z((j+1)\tau+\nu,(j+1)\tau+s)B((j+1)\tau+s)\phi(s)ds. \end{aligned}$$

Writing the above in the phase space $\mathcal{RCR}$, we conclude that

$$\begin{aligned} z_{(j+1)\tau}(\theta) = Z((j+1)\tau+\theta,\tau j)z_{j\tau}(0) \\ + \int_{-\tau}^{0} Z((j+1)\tau+\theta,(j+1)\tau+s)B((j+1)\tau+s)z_{j\tau}(s)ds. \end{aligned} \tag{IV.5.28}$$

We discretize/approximate (IV.5.28) as follows. Let some natural number N be given, assume $\theta \in \{s_1,\dots,s_N\}$ for $-\tau < s_1 < \cdots < s_N < 0$ the standard Gauss–Legendre points translated to the interval $(-\tau,0)$ and let $w_1,\dots,w_N$ denote the corresponding weights. For a function $f:[-\tau,0]\to\mathbb{R}^n$, define the $(N+1)\times 1$ vector array

$$\tilde{f} = \begin{bmatrix} f(s_1) \\ \vdots \\ f(s_N) \\ f(0) \end{bmatrix}.$$

Denote $Z_{j+1}(t,s) = Z((j+1)\tau+t,(j+1)\tau+s)$ and $B_{j+1}(s) = B((j+1)\tau+s)$. Since the integrand in (IV.5.28) is continuous except at finitely many points, the Gaussian quadrature justifies the approximation

$$\begin{aligned} \tilde{z}_{(j+1)\tau} &\approx \begin{bmatrix} Z_{j+1}(s_1,s_1)B_{j+1}(s_1)w_1 & \cdots & Z_{j+1}(s_1,s_N)B_{j+1}(s_N)w_N & Z_{j+1}(s_1,-\tau) \\ \vdots & \ddots & \vdots & \vdots \\ Z_{j+1}(s_N,s_1)B_{j+1}(s_1)w_1 & \cdots & Z_{j+1}(s_N,s_N)B_{j+1}(s_N)w_N & Z_{j+1}(s_N,-\tau) \\ Z_{j+1}(0,s_1)B_{j+1}(s_1)w_1 & \cdots & Z_{j+1}(0,s_N)B_{j+1}(s_N)w_N & Z_{j+1}(0,-\tau) \end{bmatrix} \tilde{z}_{j\tau} \\ &\equiv \tilde{K}_{j+1,N}\tilde{z}_{j\tau}, \end{aligned}$$

with convergence (in some appropriate sense) as $N\to\infty$, and the multiplication is interpreted as the product of a block matrix with an array (with the correct dimensions). Successively iterating, it follows that

$$\tilde{z}_{m\tau} = \left(\prod_{j=1}^{M}\tilde{K}_{j,N}\right)\tilde{z}_0$$

with the indexed product multiplying from right to left: $\prod_{j=1}^{M}\tilde{K}_j = \tilde{K}_M\cdots\tilde{K}_1$.

The final stage is to compute/approximate z_q. This is very similar to the previous derivation. In particular, if one defines $Z_q(t,s) = Z(q+t,(M+1)\tau+s)$ and $B_q(s) = B((M+1)\tau+s)$, we can write

$$z_q(\theta) = \begin{cases} Z_q(\theta,-\tau)z_{M\tau}(0) + \int_{-\tau}^{0} Z_q(\theta,s)B_q(s)z_{M\tau}(s)ds, & q+\theta \geq 0 \\ z_{M\tau}(q+\theta), & q+\theta < 0. \end{cases}$$

Let $\xi = \max\{i = 1,\ldots,N : q+s_i < 0\}$. Then, we get the approximation

$$\tilde{z}_q \approx \left[\begin{array}{c|c|c} 0_{\xi n\times(N-\xi)n} & I_{\xi n\times \xi n} & 0_{\xi n\times n} \\ \hline \multicolumn{3}{c}{D} \end{array}\right] \tilde{z}_{M\tau} \equiv \tilde{F}_N \tilde{z}_{M\tau},$$

where D is the $(N+1-\xi)n \times (N+1)n$ matrix

$$D = \begin{bmatrix} Z_q(s_{\xi+1},s_1)B_q(s_1)w_1 & \cdots & Z_q(s_{\xi+1},s_N)B_q(s_N)w_N & Z_q(s_{\xi+1},-\tau) \\ \vdots & \ddots & \vdots & \vdots \\ Z_q(s_N,s_1)B_q(s_1)w_1 & \cdots & Z_q(s_N,s_N)B_q(s_N)w_N & Z_q(s_N,-\tau) \\ Z_q(0,s_1)B_q(s_1)w_1 & \cdots & Z_q(0,s_N)B_q(s_N)w_N & Z_q(0,-\tau) \end{bmatrix}.$$

We can now define the *approximate monodromy matrix*. It is the $n(N+1)\times n(N+1)$ matrix

$$\tilde{V}_{0,N} = \tilde{F}_N \cdot \prod_{j=1}^{M} \tilde{K}_{j,N}. \tag{IV.5.29}$$

It is reasonable to suspect that as $N\to\infty$, every eigenvalue of $\tilde{V}_{0,N}$ approaches some eigenvalue of the operator V_0. This heuristic is far from rigorous, and we make no effort to prove it here. See also the comments from Sect. I.3.5.

Remark IV.5.4.2. *For computational efficiency, it is worth noting that in general, the nonzero entries $Z_q(s_i,s_j)$ can be equivalently written as $Z_q(s_i,s_j) = Z_q(s_i,-\tau)[Z_q(s_j,-\tau)]^{-1}$, so it is only necessary to compute $Z_q(s_i,-\tau)$ for $i=1,\ldots,N$. This of course assumes that the matrix inversion is numerically stable, which, as it turns out, is not the case for the analysis of the model from this section.*

The reader interested in how this method can be adapted to accommodate for impulse effects (with or without delay) in the linearization is encouraged to consult Appendix B of [32].

IV.5.4.3 Parameters

The model parameters are chosen for illustrative purposes. For the bifurcation analysis, we will look at one parameter that is expected to have an influence on the stability of the disease-free periodic solution. This will be Q, the drug dose size. The complete list of parameters and ranges is provided in Table IV.5.1

Parameter	Numerical value/range
s	100
μ	0.1
r	5.61
η	0.001
m	4.16
ρ	5
δ	3
β	0.0032
τ	1
α	0.5
γ	125
ν	125
Q	[350,450]
q	0.5

Table IV.5.1: Parameters used for the numerical bifurcation analysis for the system (IV.5.1)–(IV.5.7)

IV.5.5 Transcritical Bifurcation from the Disease-free Periodic Solution

The bifurcation parameter being Q, the disease-free periodic solution with $Q = 350$ was computed by solving the boundary-value problem

$$\dot{T} = 100 - 0.1T - 5.61TR + 4.16T_r, \qquad T(0) = T(0.5) \tag{IV.5.30}$$
$$\dot{T}_r = 5.61TR - (4.16 + 0.1)T_r, \qquad T_r(0) = T(0.5) \tag{IV.5.31}$$
$$\dot{R} = -5R - 0.001 \cdot 5.61TR, \qquad R(0) = R(0.5) + Q \tag{IV.5.32}$$

with `bvp4c` in MATLAB. We then increased Q by increments of 1 and used the prior solution as the new guess for the solver, terminating at $Q = 450$. This branch of solutions was stored for later calculations.

The result was then used to define the branch of disease-free periodic solutions that is passed to the monodromy operator discretization scheme. This requires redefining the R component of the periodic solution so that

$$R(0.5) = R(0.5^-) + Q = R(0),$$

thereby correcting the impulse condition, in addition to computing $\kappa_1(t)$ from its definition. This is accomplished by periodically extending the function R. No changes to T and T_r are needed apart from the periodic extension.

Next, we use the monodromy operator discretization scheme to approximate the Floquet spectrum along the branch of periodic solutions. We use $N = 40$ quadrature points and `ode45` with standard integration tolerances for the Cauchy matrix calculations. Since $0.5 = q < \tau = 1$, only the final stage of the scheme is necessary—that is, we have $M = 0$ in the notation of Sect. IV.5.4.2. The spectral radius of $\tilde{V}_{0,N}$ was calculated at each point Q along the branch of periodic solutions, and the value of Q at which the spectral radius crossed unity was identified. This was taken as the approximate bifurcation point, which we denote by $\tilde{Q}$.

IV.5.5.1 Results

Figure IV.5.1 provides the main result of the "algorithm" outlined earlier. The (numerical) critical dosage $\tilde{Q} = 366$ was identified. Since the relevant Floquet multiplier crosses the unit circle transversally (or at least this is observed numerically, since $Q = 366$ appears to be a regular point on the curve from the figure) and is real at the crossing point, we should expect a transcritical bifurcation of periodic solutions to occur there.

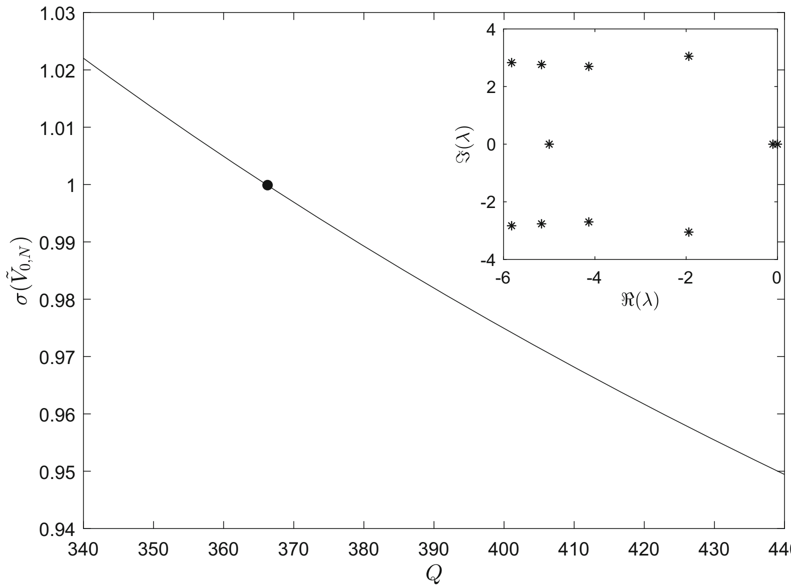

Figure IV.5.1: Plot of the spectral radius of the discretized monodromy operator versus the drug dose size Q along the disease-free periodic solution. The crossing point at unity spectral radius is indicated by a dot. Note the small numerical range of the spectral radius: for this range of Q arguments, it is $[0.9494, 1.0220]$. Inset: the Floquet spectrum (set of Floquet exponents) at the numerically estimated critical dosage size $\tilde{Q} = 366$, windowed to the strip $-6 \leq \Im(z) \leq 1$

To provide some additional verification of this bifurcation, we have simulated the system (IV.5.1)–(IV.5.7) using the discrete delay representation from Sect. IV.5.4.1 at $Q = 361$ and $Q = 376$ (the convergence rate is far too slow even at $Q = 376$ to provide a reasonable figure demonstrating stability of the disease-free periodic solution). This can be seen in Figs. IV.5.2 and IV.5.3. As expected, for $Q = 361$, the viral component settles into an endemic (nonzero) state, while at $Q = 376$ the viral load is decreased to zero.

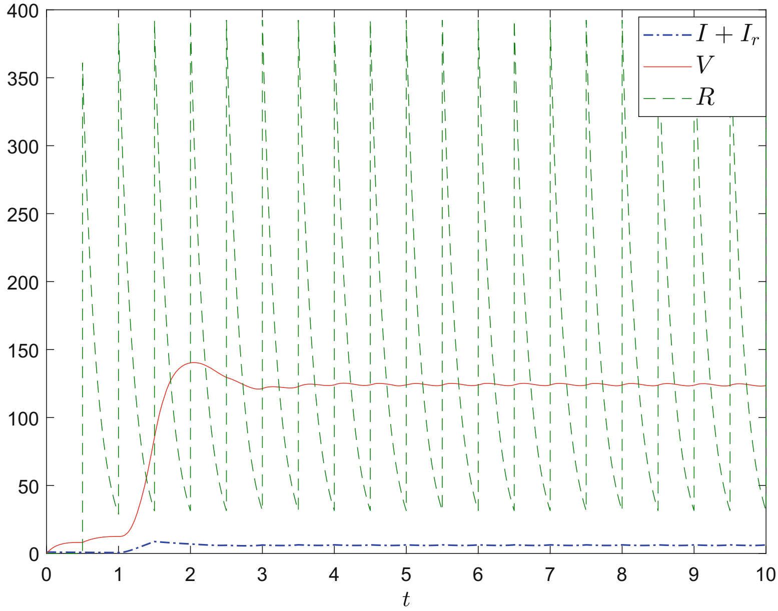

Figure IV.5.2: Plots of the virus (V), drug (R) and total infected cell ($I + I_r$) components of the system (IV.5.1)–(IV.5.7) versus time (t) from initial conditions $T(0) = 1000$ and $V(0) = 1$ (all others zero), with $Q = 361$. The susceptible cell component ($T + T_r$) is not plotted because it is on the order of 1000 and is not as informative insofar as verifying the stability of the disease-free solution. The viral component provides a good proxy for verifying the stability or instability of the disease-free periodic solution, and we see that this component does not go to zero

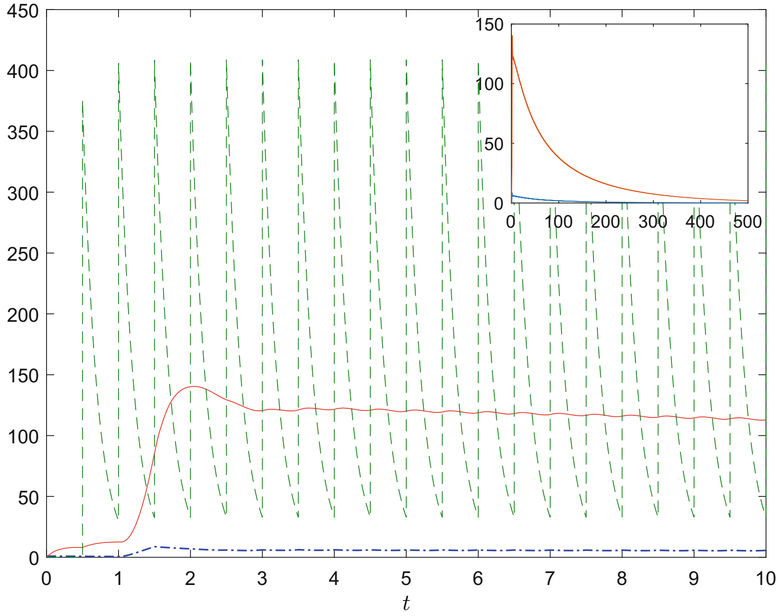

Figure IV.5.3: A plot analogous to the one in Fig. IV.5.2 except with $Q = 376$. The colours and line styles are the same as in the previous legend. Compared to the previous case $Q = 361$, the viral component is decreasing near $t = 10$, albeit very slowly. Inset: simulation until time $t = 500$ with drug component not plotted. Here the exponential decay of virus and infected cell is more clearly seen

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Index

t-fibre, 9

A
Assumption
  C, 111
  H, 21, 55, 56, 89, 90, 103
Attracting, 62, 217
  uniformly, 62, 218

B
η-bounded, 68
Basis array, 101
Basis matrix, 208
Bifurcation
  codimension-one, 153
  cylinder, 173, 247, 311, 339
  fold, 159, 161, 237
  Hopf, 176, 326
  Neimark-Sacker, 166, 244
  nonsmooth, 279
  overlap, 282
  period-doubling, 241
  pitchfork, 240
  saddle-node, 159
  transcritical, 159, 239, 350

C
Cauchy matrix
  impulsive differential equation, 201
  ordinary differential equation, 199
Chain matrix, 212
Characteristic
  equation, 48
  matrix, 48

D
Delay
  discrete, 22, 53, 127, 153, 161, 176, 279, 326, 344, 357
  distributed, 22, 344, 357

E
Evolution family, 23
Exponential trichotomy, 203, 218

F
Fibre bundle, 11
  centre, 11
  invariant, 25
  reversible, 208
  stable, 11
  unstable, 11
Floquet
  decomposition, 43, 206
  eigensolution, 47
  exponent, 45
  multiplier, 38, 205
  spectrum, 45

H
History, 12
  one-point left-limit, 12
  regulated left-limit, 12
Hutchinson equation, 279, 320
Hyperbolic, 11, 65, 218

K. E. M. Church, X. Liu, *Bifurcation Theory of Impulsive Dynamical Systems*, IFSR International Series in Systems Science and Systems Engineering 34, https://doi.org/10.1007/978-3-030-64533-5

I
I_X, 23
Impulse extension equation, 256
Invariance equation, 119, 225

L
Linearization, 61, 218

M
Manifold
  centre, 76, 222
  centre, Euclidean representation, 114
  centre, hyperbolic part of, 105
  centre, parameter-dependent, 153, 233, 288
  centre, reduction, 84
  centre-stable, 146, 222
  centre-unstable, 144, 222
  extended hierarchy, 149
  stable, 143, 222
  unstable, 142, 222
Monodromy matrix, 205
Monodromy operator, 36
  discretization, 53

N
Nonautonomous set, 9
  invariant, 11

O
Overlap condition, 80, 111, 133

P
$PC^{1,m}$-regular at zero, 101, 223
Pendulum, 303
Periodic, 35, 98, 111, 205
Pettis integral, 17
Process, 9
  generated by impulsive RFDE, 58
  linear, 10
Projection
  Riesz, 38

R
$\mathcal{RCR}$, 12
Regulated, 12

S
Singular unfolding, 255
Solution
  classical, 57
  classical (linear), 22
  integrated (linear), 22
  mild, 56
Spectrally separated, 10, 25
Stable, 32, 62, 203, 217
  asymptotically, 62, 218
  exponentially, 32, 62, 203, 218
  uniformly, 62, 217
  uniformly asymptotically, 62, 218
Stage structure, 343
Stroboscopic map, 160
Substitution operator, 75
Switching functions, 260

T
Time-scale sensitivity, 271
Two-parameter semigroup, 9

U
Unstable, 32, 203

V
Variation-of-constants, 27
Variation-of-constats, 201
Variational equation, 62, 218

Z
χ_0, 23