Consolidation of Saturated Soils through a Different Prism

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Abstract. The founding principles of consolidation of soils was set by Terzaghi dating back to 1923. This theory focused on the interplay between pore pressure dissipation and effective stress increase on fully saturated soils, and works by means of the diffusion theory in one-dimensional flow conditions under constant static external loading. The theory remains to use still and works as the funding stone for all subsequent formulations in the international theory. It is noted, that all consolidation formulations reduce to the original Terzaghi's theory (as they comprise its extensions) for one-dimensional flow under constant static loading on fully saturated conditions. This paper revisits the consolidation theory and derives the governing equations via a rigorous integration of mass conservation, which accounts for instantaneous void ratio alterations within the continuity equation by including an additional term in the consolidation coefficient. This gives shorter consolidation times compared to Terzaghi's original expression. The derived equations reduce to Terzaghi's with appropriate manipulation. The paper compares the 1D consolidation coefficients to Terzaghi's and other formulations, and proposes a simplified solution of the consolidation equation based on regression, accompanied with a workaround to include the rigorous approach in commercial (FEM, FDM) codes within the consolidation governing equations.

Keywords: Consolidation · Continuity equation · Permeability · One-dimensional

1 Introduction

The founding principles for the consolidation of soils were embedded by Terzaghi (1923), scoping to understand the behavior of saturated soils. The theory describes the one-dimensional excess porewater pressure dissipation and subsequent settlement of fully saturated soils under static external loading, based on thermodynamic (heat diffusion) principles with temperature and heat energy per unit mass being replaced by porewater pressure and water content respectively. Terzaghi's one-dimensional consolidation theory extended to account for self-weight in geostatic stress distribution with depth by Fillunger (1936). The theory remains to use still, regardless of possible shortcomings stemming from the hydro-mechanical response of geomaterials.

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The consolidation equations were generalized to three-dimensional space by Rendulic (1936), giving the uncoupled consolidation theory, and Biot (1941), giving the coupled consolidation theory. Fredlung and Morgenstern (1977) proposed the superposition of two coincident stress fields to describe fully the current stress state of a saturated soil, within the extended framework of the consolidation theory for unsaturated soils. These governing equations describe soil settlement due to air and water flow within the soil skeleton. Numerous analytical and numerical studies address the integration of the partial differential governing equations of unsaturated soil consolidation for an ensemble of initial and boundary conditions.

All consolidation theories in the international literature however build, generalize and extend Terzhaghi's one-dimensional consolidation equation. For full saturation and static external applied loading, consolidation theories have been shown to reduce to Terzaghi's original one-dimensional expression (Sills 1975, Fredlund and Hasan 1979, Fredlund and Rahardjo 1993). As such, all consolidation theories may be bounded to approximations involved in the original integration of water mass conservation. The necessity to revisit the consolidation theory is in line with the De Boer's (2000) observations. This paper formulates the consolidation differential equations on full saturation by means of a rigorous integration of the (water) mass conservation equations similar to Thomas et al. (2009), giving a dependence on the void ratio even at an isotropic linear elasticity. The three-dimensional differential equations are reduced to one-dimensional conditions and compared to published one-dimensional consolidation expressions. The study works on a linear stress-strain relationship for comparative reasons with novel formulations founded on the same principles.

2 Revised 3-D Formulation of Soil Consolidation

The consolidation process in geomechanics describes alterations to the soil skeleton due to applied external tractions and body forces, by thus necessitating reformulation of the governing consolidation equations in the corresponding dimensionality (three-dimensional stress-space).

The mathematical formulation hereafter founds on the following assumptions: (a) saturation of the idealized soil medium (voids between the soil particles are filled with water); (b) the stress—train response is isotropic; (c) compression of the soil skeleton and associated hydraulic flow unfolds in all three dimensions; (d) water molecules and soil particles are assumed incompressible for the imposed stress range; (e) Darcy's law describes the water flow through the inter-particle voids; (f) the permeability coefficient is assumed constant in each flow direction, as the hydraulic coefficient is constant along the same flow axis. The permeability coefficient is different in all three directions to account for cross-anisotropy in the hydraulic conductivity (in each orthogonal axis); (g) the soil particles remain stationary as the water flows through the interparticle voids, which ensures that the velocity of the particles can be neglected; (h) self-weight distribution with depth is neglected; and (i) the effective stress is associated linearly with the void ratio (i.e., constant stiffness).

Extending Fulks et al. (1971) mathematical formulation of porous flow through an undeformable porous body, the consolidation theory of a deformable porous medium works on the water mass m_w definition within a an arbitrary, time dependent control volume $V(t)$ (that has a boundary $\partial V(t)$) as follows:

$$m_w = \int\limits_{V(t)} (\rho_w\, S_r\, \eta)\, dV \tag{1}$$

where "ρ_w" denotes the water density, "S_r" is the degree of saturation of the soil and "η" is the porosity.

The conservation of water mass flowing through the soil porous is described by means of a Lagrangian equation $\partial m_w/\partial t = 0$ (continuity equation), which considers a differential control volume that always accommodates/conserves the exact same volume of water (i.e. the control volume size changes). This means that while the volume of water remains intact, the control volume and the volume of solids changes within the time increment. Applying at a given time "t" the Leibniz-Reynolds transport theorem, the conservation of water mass flowing through an infinitesimal cubic element (i.e., the control volume) is expressed as follows:

$$\frac{\partial m_w}{\partial t} = \int\limits_{V(t)} \frac{\partial}{\partial t}(\rho_w S_r \eta) dV + \int\limits_{\partial V(t)} (\rho_w \eta)(\mathbf{v}_w \cdot \mathbf{n}) dS = 0 \tag{2}$$

where "$\mathbf{v}_w$" is the fluid velocity vector and "$\mathbf{n}$" is the unit vector vertical to the surface denoted "S". Bold face symbols indicate vectors, while the symbol "•" denotes the scalar dot product of the two vectors. Using Gauss's divergence theorem, Eq. (2) gives:

$$-\int\limits_{V(t)} \nabla \cdot (\rho_w \eta) \mathbf{v}_w dV = \int\limits_{V(t)} \frac{\partial}{\partial t}(\rho_w\, S_r\, \eta) dV \tag{3}$$

On fully saturated conditions ($S_r = 1$) and constant ρ_w (independent of time and space):

$$\nabla \cdot (\eta \mathbf{v}_w) = -\partial \eta / \partial t \tag{4}$$

Under the assumption of incompressible grains, any change of the total volume equals to the change in the volume of voids. Therefore, for the fully saturated soil (where $V_v = V_w$) it becomes $\partial V/\partial t = \partial V_v/\partial t = \partial V_w/\partial t$ and:

$$\begin{aligned} \nabla \cdot (\eta \mathbf{v}_w) &= -\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{V_V}{V}\right) = -\frac{1}{V^2}\left(\frac{\partial V_V}{\partial t} V - V_V \frac{\partial V}{\partial t}\right) \\ &= -\frac{(1-\eta)}{V}\frac{\partial V}{\partial t} \end{aligned} \tag{5}$$

Assuming that the solid particles cannot undergo volumetric deformation and the void ratio gives $e = V_v/V_s \Leftrightarrow 1/e = V_s/V_w \Leftrightarrow 1/(1+e) = V_s/V \Leftrightarrow V = (1+e)V_s$, the volume derivative with respect to time ($\partial V/\partial t$) can be computed by differentiating the volumetric strain $\varepsilon_{vol} = -\Delta V/V_0 = -(V - V_0)/V_0 = 1 - V/V_0$ as follows:

$$\frac{\partial \varepsilon_{vol}}{\partial t} = \frac{\partial}{\partial t}\left(1 - V/V_0\right) = -\frac{1}{V_0}\frac{\partial V}{\partial t} = -\frac{V}{V_0}\frac{1}{V}\frac{\partial V}{\partial t} = -\frac{\cancel{V_s}(1+e)}{\cancel{V_s}(1+e_0)}\frac{1}{V}\frac{\partial V}{\partial t} \tag{6}$$

Equation (6) can be expressed as a function of porosity by substituting the specific volume $v = (1 + e) = V/V_s = 1/(V_s/V) = 1/(1 - V_w/V) = 1/(1 - \eta)$:

$$\frac{\partial \varepsilon_{vol}}{\partial t} = -\frac{(1+e)}{(1+e_0)}\frac{1}{V}\frac{\partial V}{\partial t} = -\frac{(1-\eta_0)}{(1-\eta)}\frac{1}{V}\frac{\partial V}{\partial t} \tag{7}$$

which further gives $-\frac{(1-\eta)}{V}\frac{\partial V}{\partial t} = \frac{(1-\eta)^2}{(1-\eta_0)}\frac{\partial \varepsilon_{vol}}{\partial t}$. Substituting this expression in (5) gives:

$$\nabla \cdot (\eta \mathbf{v}_w) = \frac{(1-\eta)^2}{(1-\eta_0)}\frac{\partial \varepsilon_{vol}}{\partial t} \tag{8}$$

Assuming a linear relationship between the isotropic stress and the volumetric strain:

$$\sigma'_{oct} = K\,\varepsilon_{vol} \Rightarrow \partial \sigma'_{oct}/\partial t = K\,\partial \varepsilon_{vol}/\partial t \tag{9}$$

While it is well known that the stress–strain relationship is nonlinear even at small strain levels, this approach scopes in reproducing a relationship on the same principles as the novel equations produced by Terzaghi (1923, 1927), Davis and Raymond (1965), Tsytovich and Zaretsky (1969) and Gibson and Schiffman (1985) for comparative reasons. This approach will be extended to account for soil nonlinearity and is thus, not bounded by this approximation.

Using Eq. (9), expression (8) can be reformed as follows:

$$\nabla \cdot (\eta \mathbf{v}_w) = \left[(1-\eta)^2 \Big/ (1-\eta_0)K\right]\partial \sigma'_{oct}/\partial t \tag{10}$$

The water flow is assumed to follow Darcy's law, which holds (approximately) true if the flow is so slow that $\eta\,(\nabla \cdot \mathbf{v}_w)$ can be neglected and $\rho_w\,\eta\,\partial\,\mathbf{v}_w/\partial t$ (see Fulks et al. 1971) is small. Water velocity, $\mathbf{v}_w$, is related to Darcy's velocity, $\mathbf{v}_D$, by $\mathbf{v}_w = \mathbf{v}_D/\eta$, and therefore it is:

$$\mathbf{v}_w = \frac{\mathbf{v}_D}{\eta} = -\frac{\mathbf{k}\cdot\nabla h}{\eta} = -\frac{\mathbf{k}}{\eta}\cdot\nabla\left(z^* + \frac{u}{\rho_w g}\right) = -\frac{\mathbf{k}}{\eta\rho_w g}\cdot\nabla u \tag{11}$$

where "**k**" is the hydraulic conductivity tensor and "h" the hydraulic pressure given by $h = z^* + u/\rho_w g$, with "z^*" being the distance from the reference level and "u" the pore water pressure. Applying Eq. (11) in Eq. (10) gives:

$$\frac{(1-\eta_0)K}{(1\;-\;\eta)^2\rho_w g}\mathbf{k}\,\nabla^2\;\cdot\;(u)\;=\;-\frac{\partial\sigma'_{oct}}{\partial t} \tag{12}$$

which can be expressed in cartesian coordinates as follows:

$$\frac{(1-\eta_0)K}{(1-\eta)^2\,\rho_w g}\left(k_x\,\frac{\partial^2 u}{\partial x^2}\;+k_y\,\frac{\partial^2 u}{\partial y^2}\;+k_z\,\frac{\partial^2 u}{\partial z^2}\right)\;=\;-\frac{\partial\sigma'_{oct}}{\partial t} \tag{13}$$

Considering consistent hydraulic conductivity in all three dimensions, the hydraulic conductivity tensor is $\mathbf{k}\;=\;k\,\mathbf{I}$ and Eq. (11) can be rewritten as follows:

$$\frac{(1-\eta_0)K\,k}{(1-\eta)^2\,\rho_w g}\,\nabla^2 u\;=\;-\frac{\partial\sigma'_{oct}}{\partial t} \tag{14}$$

Equation (14) works on an isotropic hydraulic conductivity in the three-dimensional space. Consequently, the governing equation of soil consolidation formulated in three dimensions of consistent permeability k, can be expressed as follows:

$$C^*\,\nabla^2 u\;=\;-\partial\sigma'_{oct}/\partial t\,,\;\;\text{where}\;C^*\;=\frac{(1-\eta_0)K\,k}{(1-\eta)^2\,\rho_w g}\;=\;\frac{(1+e)^2 K\,k}{(1+e_0)\rho_w g} \tag{15}$$

The mathematical formulation of three dimensional consolidation, shown in Eqs. (13) and (16), is derived based on a rigorous conservation of mass and accounts for instantaneous alterations of the void ratio at all times. The following section reduces the three-dimensional consolidation equation to a single dimension and compares the solution with published one-dimensional expressions.

3 Derivation of 1-D Revised Consolidation Equations

Derivations of one-dimensional consolidation in the international literature (e.g. Davis and Raymond 1965; Fillunger (1936); Gibson and Schiffman 1985; Gibson et al. 1967; Merchant 1939; Monte and Krizek 1976; Terzaghi (1923); Tsytovich and Zaretsky 1969) can be hindered by approximations involved in the continuity equation. As all formulations work on the conservation of fluid mass proposed by Terzaghi, they tend to neglect the void ratio term in the definition of the consolidation coefficient.

Alternatively, a coefficient of consolidation can be derived using Terzaghi's effective stress principle (16) into Eq. (15).

$$\Delta\varepsilon_{vol}\;=\;\Delta\varepsilon_z\;=\Delta\sigma'_{oct}/K\;=\;\Delta\sigma'_v/D\;\;\Rightarrow\;\;\partial\sigma'_{oct}/\partial t\;=(K/D)\partial\sigma'_v/\partial t \tag{16}$$

$$C_V\,\frac{\partial^2 u}{\partial z^2}\;=\;-\frac{\partial\sigma'_v}{\partial t}\;,\;\text{where}\;C_V\;=\frac{(1-\eta_0)D\,k}{(1-\eta)^2\,\rho_w g}\;=\;\frac{(1+e)^2 D\,k}{(1+e_0)\rho_w g} \tag{17}$$

The above differential equation uses the constrained modulus D and accounts for alterations in the void ratio, within the expression of the coefficient of one-dimensional consolidation C_V, based on a prompt derivation of water mass conservation flowing through the interparticle voids.

Next, we ensue few of the most commonly used novel expressions of the coefficient of consolidation in Eulerian form:

(a) $C_V = kD/\gamma_w$, from Terzaghi's original theory (1923).
(b) $C_V = kD/(1+e_0)\gamma_w$, from Terzaghi's revised theory (1927).
(c) $C_V = k(1+e_0)\,\sigma'_v/0.434\,C_C\,\gamma_w$, from Davis and Raymond's theory (1965).
(d) $C_V = kD(1+e_{av})/\gamma_w(1+e_0)$, from Tsytovich and Zaretsky's theory (1969).
(e) $C_V = kD(1+e)/\gamma_w(1+e_0)$, from the theory of Gibson and Schiffman (1985).

The proposed solution (shown in Eq. (17)) is based on an Eulerian description of the governing consolidation equations. The coefficient of consolidation is shown to differ considerably from the novel expressions in the international literature.

4 Solution of the Revised 1-D Consolidation Equations

This section studies the linear consolidation response of a h_0 (= $2H$) deep homogeneous soil deposit overlaying a permeable layer. Figure 1 compares the numerical solution of expression (17) (in black solid lines) to Terzaghi's linear elastic formulation (in grey dashed lines), for an initial void ratio e_0 = 0.6. Numerical analyses show that the effective stress builds faster on the revised formulation. Thus, the equivalent times are considerably smaller than those manifested by the classical formulation.

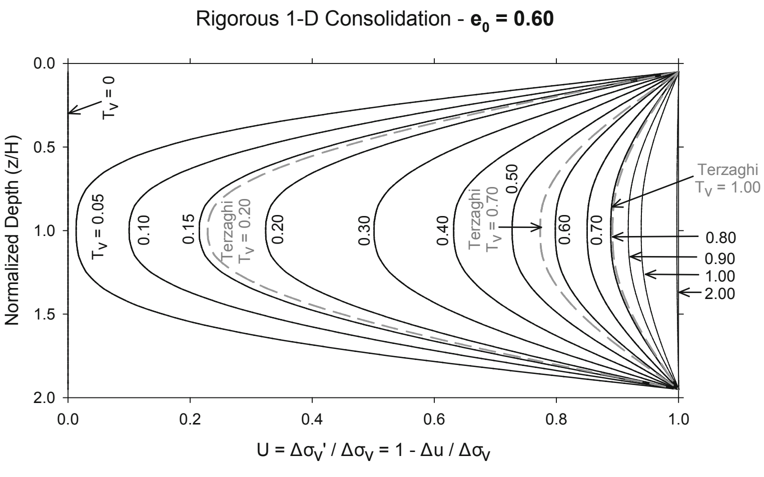

Fig. 1. Effective stress distribution with depth for a homogeneous layer overlaying a permeable deposit, for the rigorous analysis, implemented on the hypotheses of different values of initial void ratio

The degree of consolidation $U_{average}$ gives the ratio of the settlements $\delta(t)$ at time t to the settlements after full dissipation of the excess porewater pressures δ_∞ (Fig. 2) in terms of equivalent dimensionless time factor $T_V = C_V{}^*t/H^2$, where t is the time and H is the drainage length, for multiple values of initial void ratio e_0 ($e_0 = 0.6 \div 1.4$). For high initial void ratio values, the settlements predicted by the rigorous analyses occur in less than half the predicted times from Terzaghi's classical formulation, which has immense impact in the preliminary design of geotechnical projects. Considering that real consolidation times are greatly overestimated, this revisited consolidation theory provides the means of analysis for a more economic design of engineering projects.

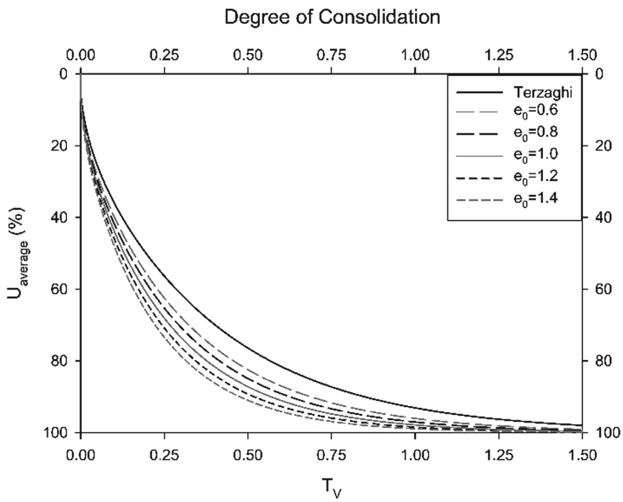

Fig. 2. Degree of one-dimensional consolidation $U_{average}$ with equivalent time T_V for a homogeneous layer overlaying a permeable deposit, compared to Terzaghi's formulation (black solid line)

The finite difference numerical analyses on Eq. (17) show a nominal influence on the $T_V - U$ curves of the hydraulic conductivity and constrained modulus as well as that of the depth of the soil deposit, for a given value of void ratio. Infinitesimal alterations to the degree of consolidation for a given initial void ratio are attributed to the difference in the integration time step employed throughout the finite difference scheme. Scoping to enhance the applicability of the integration method upon calculation of the surficial soil settlements, we propose an analytical solution upon implementation of constant void ratio e_0:

$$U_{average} \cong 17.667\left(1 - e^{-54.988(1+e_0)\,T_V}\right) + 82.221\left(1 - e^{-1.884(1+e_0)\,T_V}\right) \tag{18}$$

Application of Eq. (18) fits well to the numerical solution. The expression works on a modified time factor $T_V^* = T_V\,(1 + e_0)$. It is possible to include the proposed solution in F.E. and/or F.D. coupled consolidation analyses, working and extending Terzaghi's water mass conservation, by modifying the permeability as $k^* = k(1 + e)^2/(1 + e_0)$. This approach gives a prompt excess pore pressures evolution with time.

5 Concluding Remarks

The paper develops a revised three-dimensional consolidation theory to account for instantaneous alterations of the void ratio implemented upon a rigorous integration of incompressible fluid mass. The consolidation equation is reduced to the 1-D conditions for horizontally stratified media.The consolidation times predicted are significantly lower than those working on Terzaghi's formulation. Thus, corresponding settlements were realized faster, in less than half of Terzaghi's equivalent times.

The paper proposes also an analytical solution to describe the solution of the one-dimensional consolidation equation based on regression analyses. The proposed analytical formulation works on a modified time factor T_V^*. It is possible to account for the exact solution in a Finite Element and/or Finite Difference Scheme by introducing an equivalent hydraulic conductivity, $k^* = k(1 + e)^2/(1 + e_0)$ dependent on the current specific volume.

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References

Biot, M.A.: General theory of three-dimensional consolidation. J. Appl. Phys. **12**, 155–164 (1941)

Davis, E.H., Raymond, G.P.: A non-linear theory of consolidation. Geotechnique **15**(2), 161–173 (1965)

Fillunger, P.: Erdbaumechanik?. Selbstverlag des Verfassers, Wien (1936)

De Boer, R.: Theory of porous media: Highlights in historical development and current state. Springer Science & Business Media (2000)

Fredlund, D.G., Hasan, J.U.: One-dimensional consolidation theory: Unsaturated soils. Canad. Geotech. J. **16**(3), 521–531 (1979)

Fredlund, D.G., Morgenstern, N.R.: Stress state variables for unsaturated soils. J. Geotech. Eng. Div. **103**(5), 447–466 (1977)

Fredlund, D.G., Rahardjo, H.: Soil mechanics for unsaturated soils. John Wiley & Sons (1993)

Fulks, W.B., Guenther, R.B., Roetman, E.L.: Equations of motion and continuity for fluid flow in a porous medium. Acta Mech. **12**(1), 121–129 (1971)

Gibson, R.E., Schiffman, R.L.: A theory of consolidation of soft sediments. GIBSON (ROBERT E) FERRING (ENGLAND) (1985)

Gibson, R.E., England, G.L., Hussey, M.J.L.: The theory of one-dimensional consolidation of saturated clays. Geotechnique **17**(3), 261–273 (1967)

Merchant, W.: Some theoretical considerations on the one-dimensional consolidation of clay. S. M. Thesis, Department of Civil and Sanitary Engineering, Massachusetts Institute of Technology, Boston (1939)

Monte, J.L., Krizek, R.J.: One-dimensional mathematical model for large-strain consolidation. Geotechnique **26**(3), 495–510 (1976)

Rendulic, L.: Porenziffer und porenwasserdruck in Tonen. Der Bauingenieur **17**, 559–564 (1936)
Sills, G.C.: Some conditions under which Biot's equations of consolidation reduce to Terzaghi's equation. Geotechnique **25**(1), 129–132 (1975)
Terzaghi, K.: Die Berechnug der Durchlassigkeitsziffer des Tones aus dem Verlauf der hydrodynamischen Spannungserscheinungen (The computation of the permeability of soils from the hydrodynamic pressure gradients), Academie der Wissenchaften in Wien, Sitzungsberichte, Mathematusch-naturwissenschaftliche (Math.-naturwiss) Klasse IIa, 132 (3/4): 125–138 (1923)
Terzaghi, K.: Principles of final soil classification. Public Roads **8**(3), 41–53 (1927)
Thomas, H.R., Cleall, P.J., Li, Y., Harris, C., Kern-Luetschg, M.: Modelling of cryogenic processes in permafrost and seasonally frozen soils. Geotechnique **59**(3), 173–184 (2009)
Tsytovich, N.A., Zaretsky, Y.K.: The development of the history of soil consolidation in the U.S.S.R., 1917–1967. Geotechnique **37**(3), 357–375 (1969)