Geometrical Solution for the Trisection Problem

Paola Magnaghi-Delfino, Giampiero Mele, and Tullia Norando

Abstract The trisection problem date back to the Greeks and Arabs and it is related to the algebraic solution of third degree. It concerns construction of an angle equal to one third of a given arbitrary angle, using only two tools: unmarked ruler and compass. The problem is stated impossible to solve for arbitrary angles, as proved by Pierre Wantzel in 1837. In this article, we present some geometric or algebraic methods to solve the problem from the first one due to Greeks until Maria Gaetana Agnesi's algebraic-geometric effort. Then we propose a geometric approximation's method based only on straightedge and compass.

Keywords Mathematics · Geometry · History of Mathematics

1 Introduction

The ancient Greeks were particularly interested in the construction of angles of different sizes using only unmarked straightedge and compass. According to the problem of drawing regular polygons of given arbitrary number of sides. The trisection of the angle was one of the problems that employed mathematical scholars for a long time. Over the centuries, many scholars invented procedures for trisection of the angle but never it was possible to solve this problem using only straightedge and compass. The geometric trisection's problem became an algebraic problem, connected to the not solvable equations of third degree [1].

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2 Greek and Arab Methods

The Arab mathematician Abusaid Ahamed ibn Muhammad ibn Abd-al-Galil-as-Sigzi (about 951–1024) solves the trisection's problem as an intersection of a circle with an equilateral hyperbola. In XI century, Al-Kashi develops an iteration procedure that stands out for its simplicity and rapidity of convergence. His treatise *On the chord and the sine* was never found but is mentioned at the beginning of the book *Key to Arithmetic* with the hint to popularize the method used to calculate sen1°. In this treatise, the Persian mathematician exposed the equation of the angle trisection, based on two theorems due to Euclid and Ptolemy. Al-Kashi connects algebraic techniques to geometric methods due to ancient Greek scholars.

2.1 Euclid's Theorem

In a right-angled triangle, the square built on the height relative to the hypotenuse is equivalent to the rectangle whose sides show the projections of the two cathetuses on the hypotenuse [2].

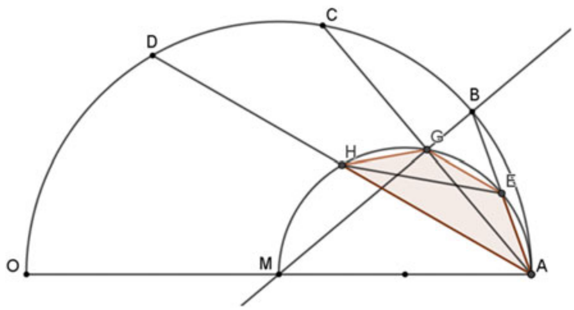
2.2 Ptolemy's Theorem

Given a quadrilateral inscribed in a circle, the sum of the products of the pairs of opposite sides is equal to the product of the diagonals.

2.3 Algebraic Equation

We consider now the semi circumference *ABO* with radius = *R*. The arcs *AB, BC, CD* are equal. We draw a semi circumference *AEM* with diameter *AM* (Fig. 1).

Fig. 1 .



We consider the chords *AB, AC, AD* and we note that the arcs *AE, EG, GH* are equal. Knowing the value of the chord *AH*, we can find the value of the chord corresponding to the arch *AE*.

We apply the Ptolemy's theorem to the quadrilateral *AEGH* and observing that $AE = EG = GH$ and $AG = EH$, we have

$$AE^2 + AE.AH = AG^2 \tag{1.1}$$

As $AG = GC$, using Euclid's theorem we have

$$AG^2 = BG(R - BG) \tag{1.2}$$

And, by secant theorem,

$$AB^2 = BG.2R$$

Substituting *BG* in (1.2), we find

$$AG^2 = 4AE^2 - 4AE^4 / R^2 \tag{1.3}$$

The equation for the trisection of an angle can be deduce substituting (1.3) in (1.1)

$$4AE^3 + R^2 \cdot AH = 3R^2 \cdot AE \tag{1.4}$$

Let α be the angle corresponding to the arc *AE* and 3α the angle corresponding to the arc *AH*, we have that $AH = R\ sin3\alpha$ and $AE = R\ sin\alpha$, so we find

$$sin3\alpha = 3sin\alpha - 4sin^3\alpha$$

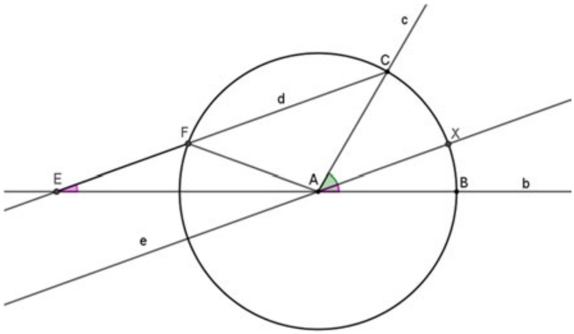
3 Archimedes's Method

In the solution proposed by Archimedes the ruler is used to report a length and, therefore, is thought of as a twice-notched straightedge. Suppose that we want to trisect the angle $C\hat{A}B$ so we draw the circumference Γ, with centre in *A* and radius $=r$. The circumference intersects the line *c* in *C* and the line *b* in *B*. Now we design a line *d*, passing through *C*. The line *d* intersects the line *b* in *E* and the circumference in *F* so that $EF = r$. We draw the line *e* parallel to *d* and passing through *A*. The line *e* intersects the circumference in *X*. The angle $X\hat{A}B$ is one third of the angle $C\hat{A}B$ (Fig. 2).

Hp: $EF = AF = AB = AC$

Th: $X\hat{A}B = \frac{1}{3}C\hat{A}B$

Fig. 2 .



The two triangles *EFA* and *CAF* are isosceles. The side *EF* is equal to the side *AF* and the side *AF* is equal to the side *AC*.

So, we have

$$F\hat{E}A \cong F\hat{A}E \text{ and } A\hat{C}F \cong A\hat{F}C$$

$C\hat{A}B$ is an external angle for the triangle EAC, then

$$C\hat{A}B \cong F\hat{E}A + A\hat{C}F \tag{2.1}$$

$A\hat{C}F \cong A\hat{F}C$, external angle for the triangle *EFA*, then

$$A\hat{F}C \cong F\hat{E}A + F\hat{A}E \cong 2F\hat{E}A \tag{2.2}$$

From (2.3) and (2.3), we have that

$$C\hat{A}B \cong F\hat{E}A + 2F\hat{E}A = 3F\hat{E}A$$

that is

$$F\hat{E}A \cong \frac{1}{3}C\hat{A}B \tag{2.3}$$

EF // AX and the angles $F\hat{E}A$ e $X\hat{A}B$ are corresponding angles then

$$F\hat{E}A \cong X\hat{A}B \tag{2.4}$$

From (2.3) and (2.4), we find that

$$X\hat{A}B \cong \frac{1}{3} = C\hat{A}B$$

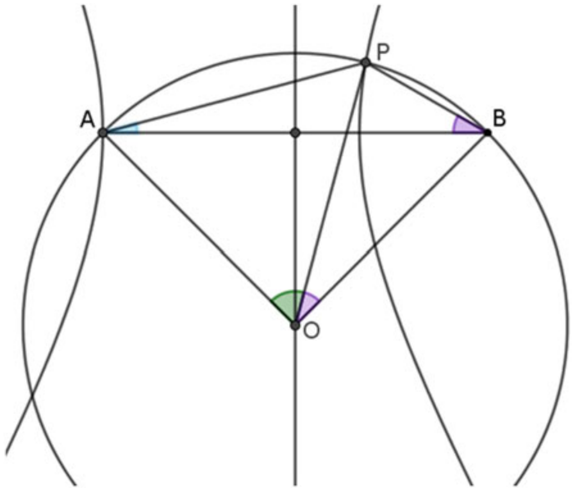


Fig. 3 Pappus' solution

4 Solutions Using Algebraic Curves

4.1 Pappus Solution

Pappus of Alexandria (290–350 AD) composed an opera in eight books entitled *Mathematical Collection*. In this work, Pappus solves the problem of trisection using the conics and referring to Apollonius [3, 4] (Fig. 3).

Let *AB* a line, we want to determine the locus of the points *P* such that $2P\hat{A}B = P\hat{B}A$.

It is shown that this locus is a hyperbola having eccentricity equal to 2, having a focus in *B* and the axis of the segment *AB* as a directrix. Considered as the centre the point *O*, we draw the circle passing through *A* and *B* and the hyperbola in such a way. The hyperbola intersects the circle in *P*. The segment *PO* trisects the angle $A\hat{O}B$. From the properties of the hyperbola, $2P\hat{A}B = P\hat{B}A$. The central angle of a circle is twice any inscribed angle subtended by the same arc then $2P\hat{A}B = P\hat{O}B$ that insist on arc *PB* and $2P\hat{B}A = P\hat{O}A$ who insist on the arc *PA*.

By combining the two relationships we get $2P\hat{O}B = P\hat{O}A$ that is, the angle $P\hat{O}B$ is the third part of the angle $A\hat{O}B$.

4.2 The Solution with the Conchoid of Nicomedes

Nicomedes, a contemporary of Archimedes, invented the curve called *conchoid* (Fig. 4).

The Cartesian equation of the curve is

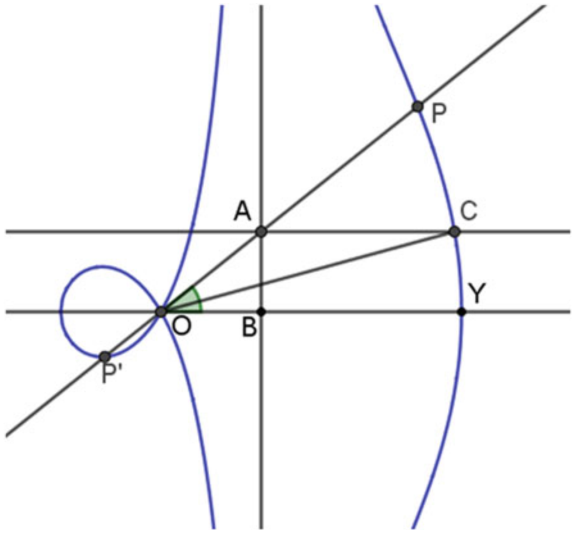


Fig. 4 The conchoid

$$\left(x^2 + y^2\right)(x - d)^2 = k^2 x^2$$

To obtain the conchoid, we fix a point *O* (pole) and a line distant *d* from *O*. Consider a second line passing through *O* that intersects the previous line in *A*. On this line, on opposite sides with respect to *A*, we consider two segments *AP* = *AP'* each of length *k*. The locus of the points *P* and *P'* obtained by rotating the line through *O* is called conchoid. Now let's see how to use the conchoid for the angle trisection problem. Let *AOB* be an angle, and consider the conchoid with $OB = d$ and $AP = k$. The parallel line to *OB*, through the point *A*, meets branch external of the conchoid in *C*. Joining *C* with *O* we prove that $A\hat{O}C = 1/3\ A\hat{O}B$.

4.3 Solution with the Use of Pascal's Snail

The conchoid of a circle for a fixed point on it is called *limaçon of Pascal*. The first part of the name picked by Étienne Pascal, father of Blaise Pascal, means snail in French.

The curve is simple to describe in polar coordinates as $r = b + d\cos(\theta)$, where *d* is the diameter of the circle and *b* is a real parameter. This can be converted to Cartesian coordinates and we obtain

$$\left(x^2 + y^2 - dx\right)^2 = b^2\left(x^2 + y^2\right)$$

The circle about *C* with radius *AC* is fixed. The line s rotates about *A* and the point *Q* is on *s* at the fixed distance from the circle $b = PQ$, and $d = 2AC$ (Fig. 5).

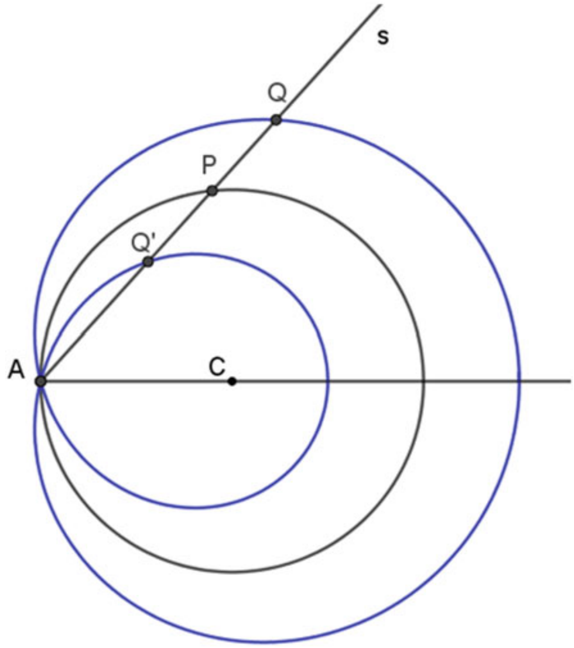


Fig. 5 Pascal's snail

Now let's see how to use the snail for the angle trisection problem. We draw the snail for which C belongs to the interior branch of the snail (Fig. 6).

We consider the angle $B\hat{C}Q$ draw and join Q with A. Let P the intersection point of AQ and the circle with centre C and radius AC. Since triangles APC and QPC are isosceles, $P\hat{A}C = A\hat{P}C = 2P\hat{Q}C$. Then for the triangle AQC we have:

$$B\hat{C}Q = P\hat{A}C + P\hat{Q}C = 3P\hat{Q}C$$

4.4 Solution Using the Maclaurin Trisectrix

The trisectrix is an algebraic curve of the third order, cubic with node that was studied by Colin Maclaurin in 1742. The Maclaurin trisectrix can be defined as locus of the point of intersection of two lines, each rotating at a uniform rate about separate points A and B, so that the ratio of the rates of rotation is 1:3 and the lines initially coincide with the line between the two points. This curve is notable because it can be used for trisecting the angles, indeed it follows from the property that when the line rotating about A has angle θ with the x axis, the line rotating about B has angle 3θ (Fig. 7).

If $A = (0{,}0)$ and $B = (a{,}0)$, the Cartesian equation is $2x\left(x^2 + y^2\right) = a\left(3x^2 - y^2\right)$

P_1

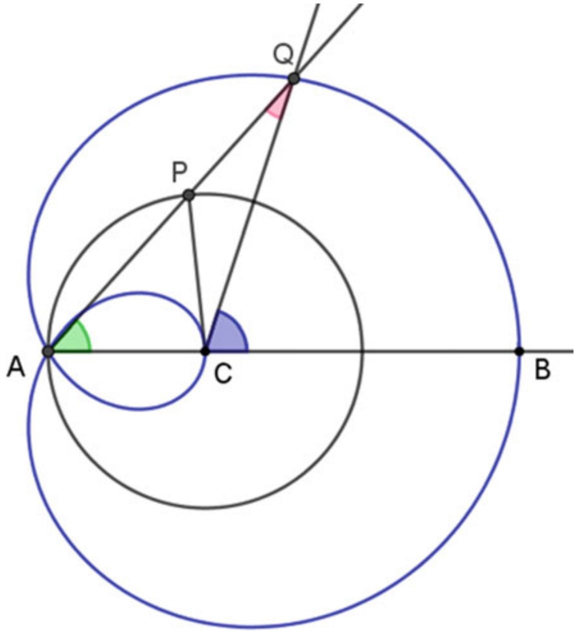


Fig. 6 Trisection with snail

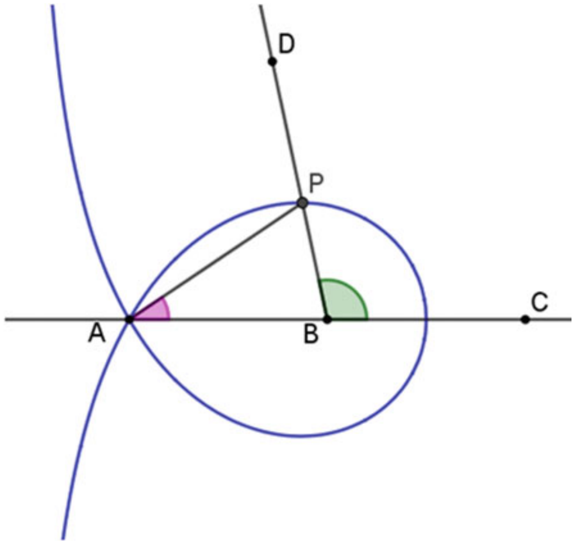


Fig. 7 Trisection's property

5 Letter from Jacopo Riccati to Maria Gaetana Agnesi (1751)

Jacopo Riccati maintained an intense correspondence with Maria Gaetana Agnesi from 1745 to 1752, that testifies the exchange of scientific ideas that arose around the writing and printing of *Instituzioni analitiche ad uso della gioventù italiana*. In 1751, among other suggestions he presents his own solution to the classical problem of angle trisection, hoping that Maria Gaetana will approve it [5].

After recalling that cubic equations often do not admit algebraic solutions, Riccati wonders if they cannot be handled with some unknown artifice.

He proposes an essay in the famous problem of angle trisection that cannot be satisfied analytically because the roots, even if real, appear as imaginary.

Consider the scalene triangle ABC and divide the angle B into three equal parts (Fig. 8).

We set

$$AB = a; BC = b;\ AC = c;\ AD = x;\ DE = y;\ EC = z \tag{4.1}$$

and suppose that $AB < BC$.

A well-known theorem applied to triangle ABE states that if the line BD divides the angle $A\hat{B}E$ into two equal parts, we have the following relation

$$AB.BE - DB^2 = AD.DE \tag{4.2}$$

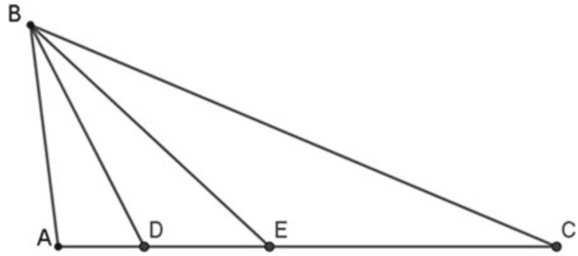
For the triangle BDC, in which the line BE divides the angle $D\hat{B}C$ into two equal parts, we have the following relation

$$BD.BC - EB^2 = EC.DE \tag{4.3}$$

From the angle bisector theorem, we also have that

$$AD : DE = AB : BE$$

Fig. 8 .



$$CE : ED = CB : BD$$

so

$$BE = \frac{ay}{x}, BD = \frac{by}{z} \tag{4.4}$$

From the two theorems, we get

$$\begin{cases} \frac{a^2 y}{x} - \frac{b^2 y^2}{z^2} = xy \\ \frac{b^2 y}{z} - \frac{a^2 y^2}{x^2} = zy \end{cases}$$

$$\begin{cases} a^2 z^2 - b^2 yx = x^2 z^2 \\ b^2 x^2 - a^2 yz = x^2 z^2 \end{cases} \tag{4.5}$$

From which we get the only equation

$$a^2 z^2 - b^2 yx = b^2 x^2 - a^2 yz \tag{4.6}$$

In this equation the side $AD = c$ is missing.

We need to introduce the AD side and at the same time eliminate the linear y. The (4.6) becomes

$$a^2 z^2 - b^2 xc + b^2 x^2 + b^2 xz = b^2 x^2 - a^2 zc + a^2 zx + a^2 z^2 \tag{4.7}$$

From which

$$-b^2 xc + b^2 xz = -a^2 zc + a^2 zx \tag{4.8}$$

where both x and z are linear.

We obtain

$$z = \frac{b^2 xc}{\left(b^2 - a^2\right)x + a^2 c}$$

we place

$$g^2 = b^2 - a^2$$

from which

$$z = \frac{b^2 xc}{g^2 x + a^2 c} \tag{4.9}$$

so

$$z^2 = \frac{b^4x^2c^2}{\left(g^2x + a^2c\right)^2}$$

From (4.5) we have

$$y = \frac{z^2\left(a^2 - x^2\right)}{b^2x} = \frac{x^2\left(b^2 - z^2\right)}{a^2z}$$

and so

$$\frac{z^2\left(a^2 - x^2\right)}{b^2x} = \frac{x^2b^2}{a^2z} - \frac{zx^2}{a^2} \tag{4.10}$$

Using the (4.9) we get

$$\frac{g^2x + a^2c}{a^2cx} - \frac{b^2cx}{a^2\left(g^2x + a^2c\right)} = \frac{\left(a^2 - x^2\right)b^2c^2}{x\left(g^2x + a^2c\right)^2}$$

Making the common denominator and collecting we obtain

$$g^4x^3 - b^2c^2x^3 + 3g^2a^2cx^2 + 3a^4c^2x = a^4c^3$$

We have come to a complete third-degree equation.
To simplify this expression we have to remove the term of maximum degree.
We set

$$g^2 = bc$$

that is $b^2 - a^2 = bc$.
We obtain

$$3g^2a^2cx^2 + 3a^4c^2x = a^4c^2$$

$$x^2 + \frac{a^2}{b}x = \frac{a^2c}{3b}$$

which is a second-degree equation with two real solutions of opposite sign.

$$x^2 + \frac{a^2}{b}x + \frac{a^4}{4b^2} = \frac{a^2c}{3b} + \frac{a^4}{4b^2}$$

$$\left(x + \frac{a^2}{2b}\right)^2 = \frac{a^2c}{3b} + \frac{a^4}{4b^2}$$

$$x = -\frac{a^2}{2b} \pm \sqrt{\frac{a^2c}{3b} + \frac{a^4}{4b^2}}$$

Recalling that $c = b^2 - a^2/b$
We obtain

$$x = -\frac{a^2}{2b} \pm \frac{a}{2b}\sqrt{\frac{4b^2 - a^2}{3}}$$

5.1 Geometric Meaning

Given the triangle ABC, with sides $AB = a$ (assumed fixed), $BC = b$ variable with the only condition $BC > AB$.

We get the value c of the side AC with the following proportion

$$b : (b - a) = (b + a) : c$$

derived from the condition $b^2 - a^2 = bc$

So given *a, b, c*, we get infinite triangles all well-defined since, even if we take two arbitrary values of a and b, we always have that

$$BC^2 - AB^2 = B.BA$$

So, we can trisect the angle B using a second-degree equation.

Angle $A\hat{B}C$ is always acute because $BC > CA$.

The construction does not consider obtuse angles, but this is not a limitation since if it trisects an acute angle in a similar way it can divide the supplementary.

The angle $B\hat{A}C$ is always obtuse because our operation leads to say that in the triangle ABC the following relation holds

$$AB^2 + AC^2 = a^2 + c^2 < b^2 = BC^2$$

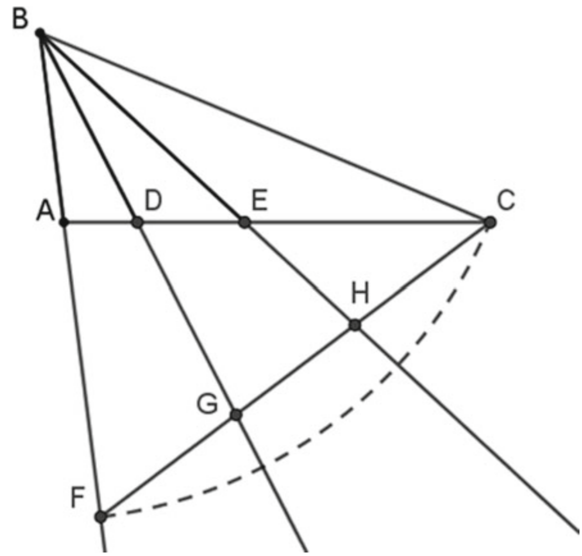
Given the triangle ABC with the above condition, we cut into equal parts the angle ABC with the two lines *BD, BE*. It follows that we can know AD using a quadratic equation (Fig. 9).

Consider another CF basis to vary the triangle. In this case the third-degree equation cannot be reduced unless $a = b$.

Then given the sides *AB, BC* between the infinite third sides that close the triangle only

$$AC = c = b - \frac{a^2}{b}$$

Fig. 9 .



it has a special property that allows us to obtain the result. In all other cases the algebraic way does not consent to solve the trisection problem.

6 Other Geometric Method: Using the Square

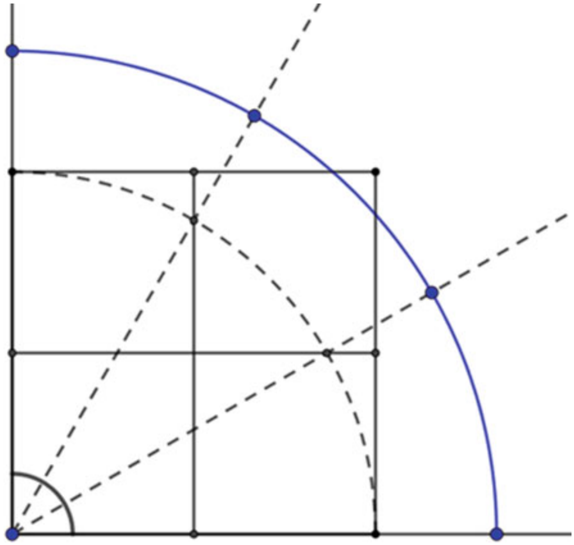
Many geometric methods were proposed for trisecting the right angle and arbitrary angles, but we focus on methods that require the use of straightedge and compass. In the case of the right angle we obtain the exact trisection, in the general case a good approximate trisection.

In the Phd thesis, G. Mele exposed his idea of trisecting right angle, based on his studies about medieval architecture.

Medieval architects studied the problem of the polygons' inscription in the square that is connected to trisection of the right angle. This problem exploits the relationship for the construction of a right-angled triangle having a base equal to 1/2, a height equal to $\sqrt{3/2}$ and hypotenuse equal to 1.

By drawing a square with its medians and considering a quarter of a circumference with a radius equal to the side of the square, we notice that the circumference intersects the medians in two points. By joining these points with the centre of the circumference, we obtain the division into three equal parts of the right angle. It is also possible to avoid tracing the circumference, and therefore the use of the compass, as it is sufficient to trace on the medians a measure equal to $L\sqrt{3/2}$, where L is the measure of the side of the starting square (Fig. 10).

Fig. 10 .



7 Trisection's Approximation of Arbitrary Angles Using Only Straightedge and Compass

The geometric approximation of the arbitrary angle's trisection, proposed by G. Mele in his Phd thesis [6], is obtained using only straightedge and compass. We show how to trisect an acute angle. This is enough, since we know it is possible to trisect a right angle using only a straightedge (without notches) and compass, and an obtuse angle is the sum of a right angle and an acute angle. Let α be the angle formed by the two half-lines s and t of origin A. We draw the circumference of vertex O and radius R. B and C are the intersections with s and t respectively. We divide the angle α into four equal parts using straightedge and compass (Euclid's Elements I, 9) [2].

Let *E* be the point on the circumference such that *BAE* = *1/4BAC*. Now we divide *AB* into three equal parts and consider *AF* = *1/3R*. We draw the circumference centered in *A* with radius *AF* and let H be the intersection of this circumference with *AE*. The *FH* arc is equal to *1/3* of the *BE* arc (Fig. 11).

Now we construct a rectilinear angle *EPG* equal to the rectilinear angle FAH on the straight line *PE* = *1/3R* (Euclid's *Elements* I, 23). Then we trace *EG* and report it on the circumference of radius *R* and center *A* obtaining *EK*. Truthfully, the *EG* chord is reported and we obtain EK which is good approximation of the arc length *EG*. On the circumference centered in *A* with radius R we obtain the arc *BK*, which corresponds to the angle β such that $\beta = 1/4\alpha + 1/3(1/4\alpha) = 1/3\alpha$ (Fig. 12).

The approximation error decreases when the angle measure decreases and its estimated value is done by the formula

$$\frac{x}{4} + \frac{2}{3} sin\left(\frac{x}{8}\right) \cong \frac{x}{3} - \frac{1}{72}\left(\frac{x}{4}\right)^3$$

Fig. 11 .

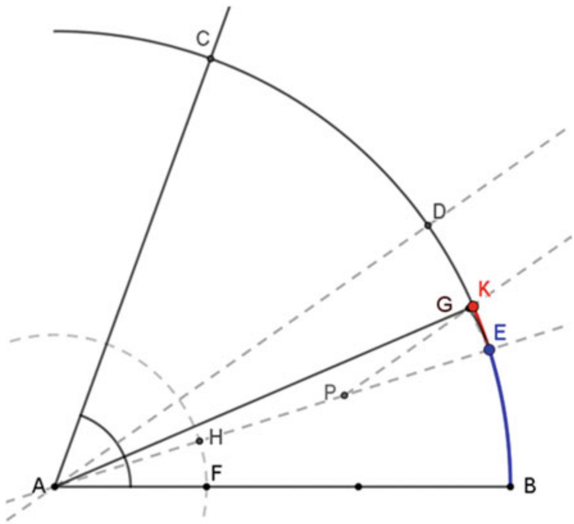
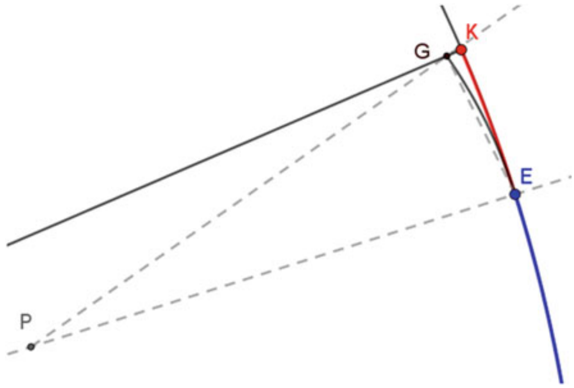


Fig. 12 .



where x ($0 < x < \pi/2$) is the radiant measure of the arc, we want trisect. The approximation formula does not depend on the radius of the circumference and the error is always less than 0.1%.

8 Conclusions

Over the centuries, many scholars invented procedures for trisection of the angle but never it was possible to solve this problem using only straightedge and compass. The geometric trisection's problem became an algebraic problem, connected to the not solvable equations of third degree. This problem can be solved in a geometric way, with a very good approximation, as we noted in the previous paragraph and the method used can be extended now division into parts (in odd) 5 parts it is first divided into 2 parts, then each of them of 3 parts and then, with the method described above, division into 5 parts is obtained. This method is of great help for the construction of buildings, churches, and fortresses on the ground.

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