Lecture Notes in Networks and Systems 172

Paola Magnaghi-Delfino
Giampiero Mele
Tullia Norando *Editors*

Faces of Geometry

II Edition

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This book is dedicated to the memory of our colleague and friend Paola Magnaghi Delfino.

Paola has been working for many years in our Department, employing the best of her efforts in planning and carrying out many initiatives. In particular, as a member of the FDS didactic laboratory, she was actively engaged in several projects dedicated to high school students, aiming at the improvement of their scientific knowledge and, in particular, at the study of mathematics through art. In particular, she took active part in the projects "Progettiamo con la Matematica" (Planning with Math), "In Action with Math" and "Matemartiamo" (Math & Art). All these initiatives were attended by hundreds of students, together with their teachers, throughout the years.

Paola was interested in all the fields in which Mathematics plays a role; another example is her collaboration with the company "Pacta dei.Teatri" for a series of theatre shows, inspired by mathematical concepts.

Yet another field in which Paola was engaged is the valorisation of the contributions made *by women to scientific research in Mathematics. In particular, Paola was deeply involved in the conception and organization of the international conference "Faces of Geometry, from Agnesi to Mirzakhani". The Proceedings of the second edition of this conference are published here.*

The whole staff at the Department of Mathematics of the Politecnico Milano, including of course myself, is deeply grateful to Paola for all the work she carried out with enthusiasm and competence. I hope that reading these proceedings can act as an incentive to the youngest researchers to undertake the difficult but important field of dissemination of Mathematical Knowledge in multidisciplinary contexts. And I am sure, that Paola would share this intent with us.

—Giulio Magli, Head of the Department of Mathematics, Politecnico Milano

Dear Paola,

We began to work together by chance, then we continued by choice. Sharing the same passion for Mathematics, we have decided to engage in experimenting with innovative forms of students training and dissemination. Paola, do you remember when I pushed you to go on stage to explain the mathematical arguments of theatre shows? You were frightened but you faced it with great courage. Not to mention the time when you arrived by car to set up the panels for the exhibition saying: "If the students can't come to the Politecnico, let's go to them!". And what about the time when we decided that art was the key to bringing scientific thinking into *the hearts of students, and you were telling me about how your father was talking to you about Architecture. That was the point when we connected, and you let me into that same world where you were putting all your heart in, and all your enthusiasm, determination, and will to overcome any obstacles.*

A few years ago, during a conference in Bratislava, we met Giampiero with whom we immediately bonded. Giampiero, an architect like your father, immediately became part of our group. We shared new ideas that linked Mathematics to Architecture first, and then to Art. A collaboration that quickly turned into friendship. True Giampiero?

Of course Tullia! You and Paola for me were true guardian angels, when I was in trouble you were there ready to support me. I shared many great moments with you. Mathematics and Architecture and then Art were like the glues for true spontaneous collaborations in which, each of us with our own specificity, entered to complete each other's work. In a few years we have worked a lot together and we have had and shared ideas and passions. The proceedings of this second Faces of Geometry conference are a proof of this. When I think about this time, I don't remember the fatigue but the pleasure of spending time together. Unfortunately, Paola left us suddenly and we didn't have time to do everything we had planned. Paola were a volcano of both sympathy and goodness. In her name and honour we will try to carry out all the projects we had created together even though we miss her. Paola, up there observing us, will enjoy seeing and hearing *our lucubration and will take us back as your usual.*

Dear Friend, we only ask you to intercede for us as only you know how to do when you want to get something.

You will always be with us.

—Tullia and Giampiero

Preface

Some introductory remarks about the reasons that motivated the choice of the topics of the conference Faces of Geometry. From Agnesi to Mirzakhani. We have two purposes, equally important. First, we have the intent of promoting interdisciplinary discussions and connections between theoretical researches and practical studies on geometric structures and its applications in architecture, arts, design, education, engineering and mathematics. Indeed, we believe that these related fields of study might enrich each other and renew common interests on these topics through networks of common inspirations. We invite researchers, teachers and students to share their ideas, to discuss their scientific opinions in teaching these disciplines, in order to enhance the quality of geometry and graphics education. Second, but not less important. We are sure that the scientific community and mathematics, in particular, need the contribution of women. Women have made significant contributions to science from the earliest times. Historians with an interest in gender and science have illuminated the scientific endeavours and accomplishments of women, the barriers they have faced, and the strategies they have implemented to have their work peer-reviewed and accepted in major scientific journals and other publications. The historical, critical and sociological study of these issues has become an academic discipline in its own right. In 2018, we celebrated, in Politecnico di Milano, the anniversary of Maria Gaetana Agnesi, Milanese mathematician, the first woman to write the first vernacular handbook of mathematics for learners.

In 2019, we celebrated the first Women in Mathematics Day, dedicated to Maryam Mirzakhani, the first woman who won the Fields Medal. The Turkish mathematician Betul Tanbay, in her tribute to Mirzakhani, recalled the classic geometric problem, called illumination problem, and compared Maryam Mirzakhani to candlelighting the path for others to follow. Quoting, she said "Maryam showed forever that excellence is not a matter of gender or geography. Maths is a universal truth that is available to us all". During the conference, we commemorate Giuseppina Biggiogero, the first woman who taught Descriptive Geometry in the Faculty of Architecture at Politecnico di Milano.

The Organizing Board of the Conference announced the birth of The International Association in Mathematics and Art—Italy (IAMAI), promoted by Italian scholars from various academic, disciplinary and cultural backgrounds. The mission of the association is the promotion of researches and the dissemination of results in the various application fields, in reference to national and international contexts, enhancing the plots and convergences between areas that link mathematics to art, opened to forms of collaboration and involvement of other subjects, institutions and organizations. Mathematics is the fruit of the thought both creative and logical, inspired and deeply linked to the beauty, recognizable in various expressions of art, from architecture to design and fashion, from painting to sculpture, from music to dance and theatre, including their digital and virtual expressions. For centuries, Italy has been a land of promotion and encounter between art and science and our country is full of signs of the Italian Cultural Heritage. The aim of the association is to give the maximum sharing to these witness through the appropriate communication and publishing channels.

The second edition of the conference Faces of Geometry was scheduled for 11 and 12 May 2020, the world day of Women in Science. Due to the situation generated by COVID-19 in large part of Europe, after careful consideration of the Organizing Committee, the congress has been postponed.

The speakers' response to our request of preparing their presentation for new scheduled dates has been in large part positive and for that we thank them.

Milan, Italy — Paola Magnaghi-Delfino
Novedrate, Italy — Giampiero Mele
Milan, Italy — Tullia Norando

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The Siege of Florence Between Chorography and Scenography

Maria Teresa Bartoli

Abstract The “Siege of Florence”, painted by Vasari to celebrate the event that restored Medicean power in Florence and made Cosimo’s lordship possible, shows a perspective image of Florence depicted in a vast landscape, from the eastern hills to the western Arno valley. The painting shows the marvelous natural basin in which Florence is situated, and its treasured buildings, castles and convents scattered over the hills occupied by the enemy. In his *Ragionamenti*, Vasari describes the steps he took to achieve his perspective: (1) a map of the surroundings of Florence, (2) the observation from the highest southern viewpoint. Perspective technique involves first drawing the plan and the profile and then intersecting the projecting rays. This would have enabled him to place “20 miles of territory” (about 35 km) in “a space of 6 arms” (3 m). Are these explanations enough? Is the image adequate? How accurate it is when compared with the actual geometric reality of the places? How realistic is the image of the River Arno? The answer to these questions requires a more detailed investigation of the actual geometric strategies implemented by the artist.

Keywords History of perspectives · History of science · Renaissance · History of land surveying

1 Introduction

At the end of the fourteenth century, the rediscovery of Ptolemy Almagest and Geography gave rise to a new area of research divided into three branches. Two of them, cosmography and geography, rested entirely on the sciences (mathematics and geometry, and therefore drawing). A third, chorography, namely the description of a region, depended on the first two and, to a significant degree, on the figurative, requiring the imitation of nature, familiarity with proportions and colours, and knowledge of the rules of architecture. In the 1400s and 1500s in Europe this manifested in the genre of views, often of a city, produced in the form of a printed

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Fig. 1 *The Siege of Florence* by G. Vasari, Palazzo Vecchio, Florence

drawing, which could be published in numerous editions. This figurative typology alluded to two scientific practices: geometric surveys and perspective [1]. Florence also took part in these activities: the first representations of the city were produced in the last quarter of the fifteenth century: they are the famous "Carta del Massaio" and the "Veduta della Catena," the former a sort of orthogonal projection, the latter a pseudo-perspective. There was an additional application similar to this type of action but more geared towards the scientific field: the survey of fortresses and territories for military purposes, used to obtain plans also by applying the *inverse procedure* to perspective drawings (obtaining plans from perspectives [2]). In Cosimo dei Medici's Florence, all these research areas existed and were pursued by men of science and action, in conjunction with their counterparts in the courts of Europe.

Vasari's *Siege of Florence*, in Palazzo Vecchio (painted between 1556 and 1562, in the Sala di Clemente VII in Palazzo Vecchio, Fig. 1), fits into this figurative trend, with some specific characteristics: (1) it is not a map, but rather a fresco that celebrates a historical event; (2) it represents a war story[1] which involved not only the city but also the nearby mountains and the Arno valley. A vast landscape following the river for around 35 km between Florence and Pistoia is depicted in a panoramic scene impossible to take in from a single viewpoint. The fresco is 4.72 m long and 2.07 m high. In it men are shown only as miniatures, participants in the events of a story full of martial and chivalric episodes, scattered about in an image dominated by the vastness of the scene.

[1]The siege depicted in the painting was brought upon Florence between 1529 and 1530 by the army of Charles V, who had made a pledge to the Medici Pope Clement VII to restore the family's power in the city, from which it had been banished a few years earlier. The siege ended after about a year with the famous defeat of Francesco Ferrucci in Gavinana.

2 The Urban Survey in Florence in the Sixteenth Century

The city within the walls, dominated by the dome, is drawn with accurate representations of the main monuments, major churches, squares and roads leading to the walls. In the "Fourth of his *Ragionamenti* of the second day" [3, IV], in which he describes the problems of the fresco, Vasari explains that he "portrayed it in a natural and measured way that does not stray far from the truth". He says nothing about this "way" in his *Ragionamenti*, almost as if he assumed the image was known.

The verisimilitude of the surrounding landscape is the technical innovation considered, as it represented the challenge the author had to overcome. A map was needed. By the time the siege occurred, the necessities of war had already driven Pope Clement VII to make a peculiar request to a highly skilled *maestro di levar piante* (plans surveyor) Benvenuto da Volpaia [3, IV, 61], a mathematician and astronomer: a survey of the city and the nearby hills, for one mile outside of the walls. The Pope's aim was to study the terrain in view of the impending siege. Benvenuto requested the help of a man of art versed in surveying, Tribolo; over the course of several months, working at night, they produced the survey in the form of a wooden model, as required by orography. The story of the survey is known to us from Vasari's account of it in the Life of Tribolo. The model was sent to the Pope who was very happy with it; it was returned to Florence after the Pope's death, but there is no other information about it.

Cosimo, who was extremely interested in knowledge of the territories to be conquered or defended, summoned to his court mathematicians, architects and engineers who refined the land survey and topography. In 1556 he commissioned Vasari to produce a perspective plan of Florence. Vasari responded by asking for clarification, but the outcome is unknown to us [2, note 15]. In the fourth *Ragionamento*, in describing the problems with the fresco, he mentions a map of the places around the city within one mile of the walls, which he produced using a compass. In 1559 Cosimo Bartoli, a humanist with wide-ranging interests at the court of the Duke, wrote "How to measure distances, surfaces, bodies, maps, provinces and all other earthly things" published in Venice in 1567. In the fourth book of the work, on provinces, he describes the construction of a compass (a circle with a diameter of 19.5 cm), necessary to survey territories. The circle of the compass is divided into 360°, each degree is divided into 5 min. A magnetic needle points North; a pointer identifies the position of the places to be surveyed with the help of a second vertical compass, integral with the pointer of the first (Fig. 2a). An illustration shows the drawing in relation to a practical example he had made: the survey of Florentine places from Giotto's bell tower (Palazzo Pitti, Torre degli Alessandri in Vincigliata, Fiesole, Villa la Petraia, Bellosguardo, Belvedere, Monte Uliveto, Bastione di San Giorgio, Mercato Nuovo, Fig. 2b): it seems to be the start of a survey of the city and its surroundings.

In around 1557 Cosimo hired the military architect Baldassarre Lanci to fortify the territory of Siena.

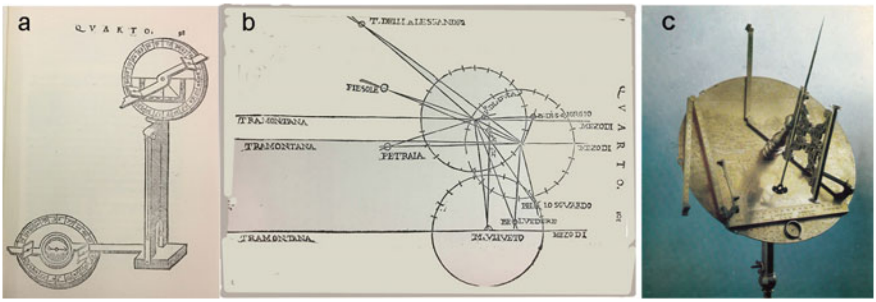

Fig. 2 **a** Cosimo Bartoli, the compass for surveying; **b** Cosimo Bartoli, example of topographical survey; **c** distances meter of Baldassarre Lanci. Museo Galileo - Istituto e Museo di Storia della Scienza, Florence

Lanci set about surveying buildings and territories of military interest and invented a tool to survey them (Fig. 2c). The instrument developed in 1557 is suitable for representing both landscapes in orthogonal projections and buildings in central projections, obtaining plans and elevations from them [2]. He also invented a perspective tool with a circular framework, which was praised by Daniele Barbaro in The Practice of Perspective (Venice 1568) and instead torn apart as a source of representation errors by the mathematician Danti (who also frequented Cosimo's court from 1567) in Vignola's *Two Rules of Practical Perspective* (Rome 1583), edited by him (Fig. 3). The instrument projects the view onto part of a cylinder, thereby transferring the images of real points onto it by means of a pointer coaxial to a finder placed on an axis in the centre of a portion of the cylindrical surface, which is then straightened on the plane. The device reduces the line of vision of an open view of 120° on the cylinder to an opening of 90° on the plane with the result that what would escape a motionless eye can be drawn as visible. Our vision, though, even if the head remains

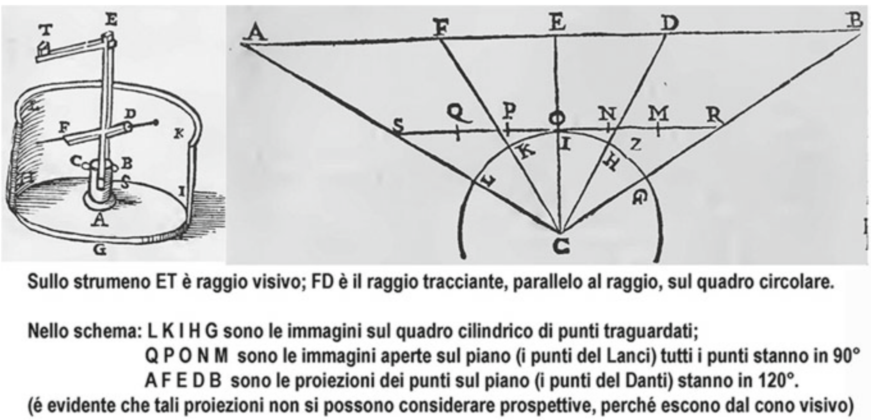

Fig. 3 The perspectograph of Baldasarre Lanci, drawn by Danti in Vignola Treatise

Fig. 4 The course of the Arno river between Florence and Empoli, according to Google Earth

in one position, allows us to turn our gaze to take in a wide scene. Lanci's image may be more similar to the view than Danti's, assembling the views of many main points. Lanci faced a new problem, the unitary and compact drawing of wide-ranging views. The great difficulty faced by Vasari exceeded the topics on the agenda. On Google Earth we see 20 miles of territory (Fig. 4). It was impossible to see them all from the same viewpoint. The image is not a panorama in a technical sense; however, it is unitary and persuasive. Vasari had only surveyed the territory within one mile of the walls. Do the representation and what is represented correspond?

3 Geometric Analysis of the Fresco

The analysis was conducted with reference to what Vasari did not have, namely a credible plan, constructing a perspective diagram of the places in his drawing. The viewpoint was located at the highest point of Pian dei Giullari, at the top of Via di Monteripaldi (height of 200 m), and at a higher height, which is not real because the perspective is a bird's eye view. The reference plan was the Florence Contour Map in Fantozzi's Guide to Florence, of 1848 (see Fig. 5, on which the radius circle one mile outside the walls and the graphic plan for the perspective construction were also marked). It was thought that the schematic perspective obtained (Fig. 6) provided better insights about what Vasari intended to communicate with his figures. The

Fig. 5 Detail from the Fantozzi's map of the surroundings of Florence. Bleu, the circle one mile outside the walls; red and green the plan drawing for the perspective scheme

horizon has been imagined above Fiesole, just above the haze of the Mugello region, because at that height the depth of perspective of Florence is compatible with that of the fresco.

The diagram only takes into consideration the salient points of the fresco, which have been associated with the elevations provided by Google, having deducted that of Florence. When superimposed on the fresco, it shows that the line of the walls, even if it differs from the real line, is acceptable in its extension. The inside of the city is described convincingly and is clearly the protagonist of the painting. The places on raised ground (all recognizable), compared with the results of the diagram, are a considerable distance from the geometric reality of the line of vision, with shortened or elongated distances and heights in order to emphasize, bring closer, move and group the many places portrayed within the space of the painting, always however managing to maintain a recognizable topological relationship between them.

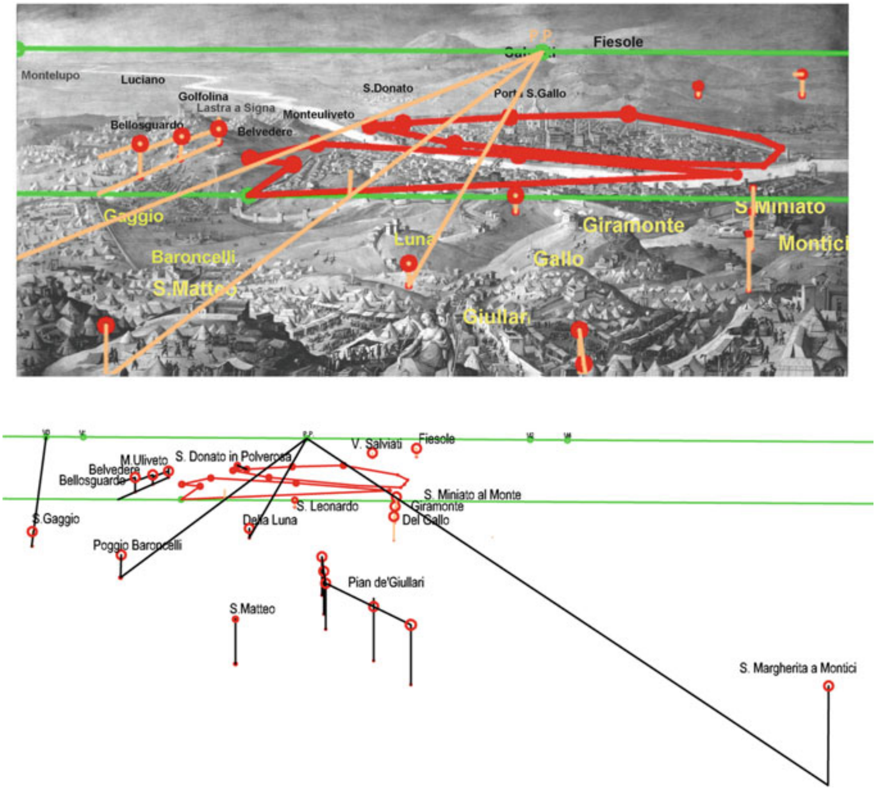

Fig. 6 Low, perspective of the scheme; high, the perspective put on the fresco

A comparison between the painting and the diagram therefore shows that in the image the horizon of the hills to the south is lower than that of the mountains to the north; the perspective cone towards Torre del Gallo, Giramonte and San Miniato (all closer to the city) would place the Tower higher, blocking our view of the others, while they are shown as visible in the painting, as if the vertex of the cone had been moved further east. The three hills in sequence in the southwest (Bellosguardo, Belvedere and Monte Uliveto) are brought closer to the city and pulled in towards the river, as if the eye had been moved to the west, while the convent of San Salvi in the northeast would like the eye on the hills to be moved east. The hills of Arcetri and Pian dei Giullari would be outside the fresco field and S. Margherita a Montici would be almost behind the eye's gaze. Fiesole and the hills to its right are moved to the west and brought closer to the city's most important monuments; the Mugnone stream rightly starts from Porta San Gallo, but the Porta is shown much further west than it actually is and placed strangely between the Bell Tower and the Dome of the Cathedral. Monte Morello, represented as an isolated cone, is seen from the east. In summary, the surroundings of the city are represented as closing in on it and ordered so as to make them visible in a small space, with a horizon that changes in height.

San Gaggio, on the left, deserves special observation. In the painting it is placed on Via Senese which exits Porta Romana, on a plane parallel to the painting. In order see it as positioned in the fresco, the viewpoint above Pian de'Giullari should rotate the gaze left, in an orthogonal direction to a plane passing through San Gaggio and the Gate: Landi's instrument would make this view compatible.

4 Towards a Chorography

The Arno leaves the walls with winding movements that would demand an even higher horizon. The river, however, would be seen as little more than an incision in the ground. How was it known? It was drawn in the early sixteenth century by Leonardo da Vinci [4]. A sheet of the Madrid codex traces its course between Florence and Pisa (Fig. 7). The names of the places written along its path allow a comparison with the real river and show that the drawing highlights all its meanders. Vasari's image is perhaps similar (smaller but emphasized) to the shape and proportions of its movements (Fig. 4). In the absence of images of urban settlements along the meanders, we can only attempt to associate them with toponyms: Lastra a Signa, Golfolina, Montelupo, Empoli; in the background, the mountains of Pistoia to the north descend towards the river and meet with the hills to the south behind which the river disappears. Cosimo brought renewed attention to the system of rivers, assigning to them two of the ten Captains of the Party, established in 1545 [5].

Later on, a team of master-builders skilled in surveying were appointed, whose maps that escaped the fires are conserved in the Piante di popoli e strade (A.S. Fi).

Fig. 7 The Arno river from Florence to Empoli by Leonardo da Vinci. Madrid codex, Biblioteca Nacional de España, Madrid

Fig. 8 The Arno in front of Signa

Many of them describe the course of the Arno; only the map that illustrates the meander below Signa remains (Fig. 8), of the stretch between the city and Empoli. It may however have been possible to create a model, whose perspective image was juxtaposed with that of the city. The urban settlements of Sesto, Prato and Pistoia are reported as between the river and the mountains to the north. The panorama was testimony of the ducal government's attention to the territory.

5 The Conclusion, a Scenography

Florence dominates the painting. The monuments, squares and streets within the walls are distributed in a plausible way and suggest that there was a map or that one was being worked on. But the perimeter of the city to the north of the Arno, outlined by the walls and the right bank of the river, has the strange shape of an almond within two circular arches and does not resemble the image of a correct perspective. In the sixteenth century, Baldassarre Peruzzi (died in 1536) produced a drawing of the walls of Florence (Gallerie degli Uffizi, Florence) very similar to those in the fresco (Fig. 9)

The city to the north of the river is contained between two circular arches with accentuated concavity (like the drawing of the Carta della Catena, Fig. 10); the gates of the north walls are distributed on the upper arch like those of Vasari. The city depicted in the fresco would seem to be drawn within those walls. In the Life of Peruzzi [6, III] Vasari praises his work as both an architect and a painter famous for creating scenic apparatus and perspectives, passionate about mathematics, and an expert in fortresses (between 1527 and 1529 he was engaged in fortifying Siena). In 1529, Pope Clement VII proposed that he help organize the camp for the siege of Florence; he refused for moral reasons, but the request reveals his worth as a land surveyor. The scene he designed for La Calandria, a famous comedy written by Cardinal Bibbiena (and an acclaimed literary work), was well-known in his time. Vasari's description enthusiastically summarizes its most noteworthy characteristics:

> Nor could one imagine how he, in such a narrow place, made room for so many streets, palaces, and various temples, balconies and cornices so well made that they seemed not imitations but very real, and the piazza not painted and little but real and very large. [6, III, 284]

Fitting many beautiful architectural works into a very narrow space reveals a special rhetorical meaning in the drawing and the scenographer's educated participation in the scholar's work (Fig. 11).

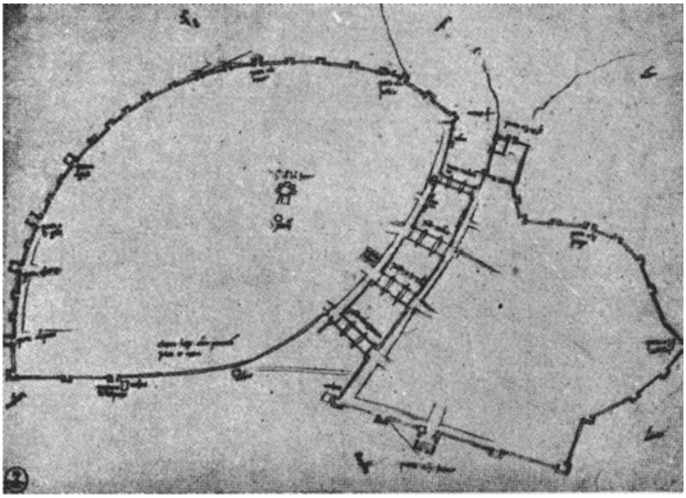

Fig. 9 Peruzzi, Pianta di Firenze, G.D.S. Uffizi (A.S. Fi, 595)

Fig. 10 View of Florence called "Carta della Catena", engraving attributed to Francesco Rosselli, 1470, Kupferstichkabinett Berlin

Fig. 11 Detail of *The Siege of Florence* by G. Vasari, Palazzo Vecchio, Florence

The drawing of a theatre scene (G.D.S. Uffizi U298A), which belonged to Vasari and was attributed by him to Peruzzi, shows the characteristics of the drawing (Fig. 12): it compresses into a small space, with meticulous scenic texture, a rich series of references that make the place recognizable (a square in Rome), highlighting them to the full [7]. Such was the spirit with which Vasari produced his painting on the Siege, free of the accuracy of the right angle but committed to showing, through consistent references, the well-known qualities of the Florentine landscape, the nobility of man-made constructions and the elegant lines of the narrow mountains around the city, followed by the undulating flow of the river in the distance. He did not seek an exact perspective reflecting reality, but rather a correct sequence of

Fig. 12 Drawing of Theatral scene, attributed from Vasari to Peruzzi. Gallerie degli Uffizi, Florence

images which in short best describe the work of nature and man, coming together to craft a scene in which the events of a tragic war are depicted, in small miniatures, in the places where they took place, "so that little strays far from the truth" [8, VI, 221] and so that every observer familiar with the place could recognize each site and each war event that took place there.

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From the Earth to the Moon: Two Stories of Women and Mathematics

Silvia Benvenuti and Linda Pagli

Abstract We will tell two stories with a common denominator: in both cases we talk about extraordinary women and challenges, successful thanks to a lot of mathematics and computer science. The first story is well rooted on the ground: it speaks of ENIAC, the first general purpose computer, built as part of a secret project of the US Defense during World War II. Six young women mathematicians designed its software, while programming languages, or even manuals and operating systems, were not available; without knowing the exact top secret architecture of the new computer, they became familiar with it and they performed complicated calculations of ballistic trajectories, founding this way the modern programming. The second story takes us to the Moon, where the protagonist made Apollo 11 land: it is the story of Margaret Hamilton's adventure the head of the project of the Apollo Guidance Computer software project, the on-board computer of the command and the lunar module. Two stories that, twenty years far from each other, present (fortunately) significant differences, but also surprisingly similar and curious experiences, starting from the methods with which the selection for hiring took place…

Keywords Mathematics · Computer science · War · Women

1 Introduction

A group of women in a male dominated environment; a bunch of equations to solve and several lines of programming; in the background, some kind of war: this is the common scenario of the two stories that we have decided to present together in this article, underlining their common aspects and sensitive differences.

The first adventure takes place at the end of World War II, when a group of engineers builds the hardware and a group of mathematicians designs the programs

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of ENIAC, the first electronic, digital, general purpose computer, part of a secret project of the US Defense. We remain in the United States but we make a jump forward of twenty years: we are then in the middle of another kind of war, the Cold War, but the goal is once again to reaffirm the US superiority, winning the race at the conquest of the Moon. Once again, «mathematics was a crucial weapon of the US arsenal, and women his secret practitioners» [1]. One woman, more precisely, a very young and bright one, at the head of a team of men but despite this not equally recognized by history.

Juxtaposing the two episodes naturally brings out a series of questions, which it is perhaps too difficult to answer, but which in our opinion can catalyze a fertile discussion on the role of women, on the (perverse?) usefulness of war as an instrument of social emergence, on the age-old question of the greater or less disposition of the so-called weaker sex for scientific subjects. Our purpose, in this work, besides of course explaining the mathematics and computer science underlying the two episodes, is to use the two stories to make these questions explicit, subtracting them from their historical dimension, to formulate them in the contemporary situation, in a perspective of gender disparity which is quite far from having found its definitive solution [2].

The two stories show that when women are given the opportunity (due to historical conditions) to do something great in science, they know how to respond with an intellectual and creative ability that of course is nothing less that of men. They also tell of other talents such as stubbornness, scrupulousness and the ability to work in a group that perhaps should inspire all research groups to never snub the other half of the sky!

2 Refrigerator Ladies

> It's 1.55 a.m., September 5th 1944, and I've been on the train just 45 minutes. The porter probably thinks that I've made at least twenty trips across the country before, because I'm acting as I'm a woman of the world.

Who's speaking is Doris Blumberg, 20 years old, traveling from her home town, in California to the Monroe Army Base, Philadelphia. She's one of the *computers* recruited from the US army to work as a mathematician on a top secret project to perfect aerial bombing.

> Just a few years earlier, this young bright women may have faced an open job as school teachers or secretaries. But in a world at war their mathematical skills proved invaluable to the United States military: like thousands of other women across the country, Doris and her twin sister Shirley had been recruited to work for the war effort: they worked as human computers, using their maths skills to compute shell trajectories, create ballistic tables and serve as programmers on ENIAC, the first electronic computer. When they signed on for the duration these young women hoped that their secret work would help win WWII. They did not know how it would change their own life, nor could they know that they were helping to usher in the computer age, that would change the world.

These paragraphs from the documentary *Top secret rosies* [1] bring us to the heart of the story we want to tell: in the next sections we will try to clarify its salient aspects.

2.1 Human Computers and Ballistics from World War II

During the Second World War, a conspicuous group of about two hundred people, mostly women, was employed by the American Ministry of Defense as *computers*, a word that in those days meant persons capable of performing complicated mathematical calculations without errors. Many men were at war, therefore the majority of the group was made up of women, young mathematicians selected in the universities of the Baltimore and Philadelphia area. The conviction of those times was that women were particularly suited to a complex but tedious and repetitive job, and that they did it more accurately and quickly than their male colleagues [3, 4] who were, however, not available due to the contingent situation. Human computers had to perform complex ballistic calculations for weapons of war, an example of which is clearly described by a newly hired 22-year-old girl, Kathleen McNulty [4]:

> An example of numerical integration is when you take, in this particular case, the path of a bullet from the instant of time when it comes out of the gun barrel until it touches the ground. It is a very complicated equation that requires about fifteen multiplications, a square root and I don't know what else. You have to describe where the bullet is from the moment it leaves the barrel, every tenth of a second, and you have to take into account all the things that can influence its path. The first thing is the speed with which the bullet was fired from the barrel, then also the angle from which it was fired out, and its size. All this incorporated into a given function, called the ballistic function. –As the bullet travels through the air, before reaching its highest point, it is constantly pushed down by gravity. It is then influenced by air pressure and also by temperature. [...] Then, finished all the calculations, you have to interpolate the values to understand what was the highest point and where it touched the ground....

The work of computers required a very high level of mathematical specialization, which also included the solution of nonlinear differential equations with many variables. Their task was to generate the so-called firing and bombing tables, to be used in ballistic calculations. Computers work was therefore critical to the war effort.

Let's clarify the terms of the problem better: the term *ballistic* (from the Greek "βάλλειν", or "to throw") indicates the study of the motion of a projectile, intended as a body equipped with an initial speed and without any type of self-propulsion. Computers, in particular, had to concentrate on the so-called *external ballistics*, the part that is concerned with the behavior of a mass launched with a certain initial speed, regardless of how this launch is made (with a catapult, with a crossbow, with a burst of a charge in the barrel of a firearm, etc.).

Of course the movement of the projectile is conditioned by the force of gravity. Moreover, in order to adequately estimate the behavior of the bullet, the friction of the physical medium in which it moves must be considered. We need also to take into account the wind, which can slow down or accelerate the progress of the projectile, and even deviate it from the launch plane. If we then want to consider the real situation and not just the ideal problem, we have to consider that the trajectory is influenced by many other factors: the angle of inclination of the shot, the pressure, the temperature, the humidity of the air (which determine density). Not to mention the so-called long range factors, which come into play when the flight time and the distance to travel are very long: the gyroscopic drift, due to the interaction of the mass and aerodynamics of the projectile with the atmosphere in which it moves; the Magnus effect which tends to move the projectile perpendicular to the direction of the wind, and those of Coriolis and Eötvös, due to the rotation of the Earth.

Hence, if the naive image that we all have in mind for the trajectory of a bullet is that of a parabola (which governs the behavior of a projectile in vacuum, subject to the acceleration of gravity g only), the real situation is modeled by much more complex equations. A simplified mathematical description of the flight of a projectile in the plane (and therefore without taking into account the wind, nor the effects of long range) can be obtained through the following differential equations [5, 6]:

$$\frac{d^2x}{dt^2} = -E\frac{dx}{dt};\ \frac{d^2y}{dt^2} = -E\frac{dy}{dt} - g;\ E = \frac{G(v)H(y)}{c}$$

In the equations x is the horizontal distance, y is the elevation, t is the time, g is the acceleration of gravity and much of the difficulty is hidden in E, which is an empirical function of considerable complexity, depending on the resistance $G(v)$ that the air opposes to the advancement of the projectile (quadratic with respect to the speed v) and the density of the air $H(y)$ (with respect to the density at sea level). The constant c, called the *ballistic coefficient*, incorporates factors depending on the physical and geometric nature of the projectile, estimated empirically by ballistics experts. For the function E there were at the time tables of pre-calculated values: these tables, then available for a wide range of mathematical functions, facilitated the calculations of the computers which, having consulted the tables relating to the function in question, were to replace their values in the formula, so as to obtain the specific result [5].

If we want to take into account the wind and the Coriolis effect, albeit in the simplifying hypothesis that the projectile is a particle, the equations to consider are three, as well as the directions in which the motion takes place [7].

Already setting the problem, as you can see, is not entirely trivial. Solving it, thus compiling the shooting tables that the armed forces needed, and with the urgency of World War II underway, was even more complex.

In order to calculate the trajectory resulting from the above equations, it is necessary to solve them numerically: firstly, therefore, it is necessary to discretize them, using for example the Heun method (a variation of Euler's numerical integration method). The only independent variable in the equations is time: a typical Δt, as Kathleen McNulty points out in the quotation above, is a tenth of a second. Hence, by interpolation we obtain the point of maximum height and the maximum distance. Each step of the method requires a certain number of sums and unfortunately also several multiplications/divisions, much more "expensive", in terms of time and complexity, for the table calculators (the most common of the Friden, Marchant or Monroe brands), unique aid of human calculators. Each step also requires consultation of the tabulated data of the empirical function. For each cannon (or rifle, or catapult,…) it is necessary to calculate as many tables as are the types of bullets that it can shoot. The processing of each table requires the calculation of 2000–4000 different trajectories, in addition to several other auxiliary and verification calculations. It is not surprising, with these premises, that the estimated calculation time for a 30 s trajectory was 20 h/man (or woman, in this case!). If you think about how boring this type of work had to be, in which, among other things, it was often necessary to manually record the results of auxiliary calculations and then re-enter them in the desktop calculator, it is not surprising that the calculations often contained errors of various gender.

To speed up the processes, the Ballistic Research Laboratory (BRL) for which computers worked also used a small differential analyzer, inspired by the first of its kind, designed in the 1930s by Vannevar Bush at the prestigious MIT: with this tool, the calculation of a 30 s trajectory took from 15 to 30 min, but the results were less precise than those obtained with manual calculators. Not to mention that the differential analyzer required a long and complex mechanical adjustment process to move from one trajectory to another, and therefore the presence of an expert throughout the calculation process.

Another problem, which made the use of more advanced calculation tools essential, is due to the behavior of the projectile at high speeds. This is related to the strong non-linearity of the friction coefficient in the transition from the sonic to the supersonic zone: for very high speeds, in fact, a small variation in speed determines a large variation in the friction coefficient. Moreover, when the projectile moves at supersonic speeds, the problem of integrating its trajectory becomes *stiff*, and therefore difficult and slow to solve with numerical methods. In particular, it becomes impossible to solve with an explicit integration method, because even if the Δt is reduced, the error does not converge. We therefore need a different method for integrating motion: a predictor/corrector system. The central idea of such a system is the following: integrated the value at time t, provide an estimate of the value at time $t + \Delta t$ with an explicit method (predictor); then correct the estimated value with an implicit method (corrector). Continuing this process it is possible, in principle, to arrive at the desired order of accuracy, but an enormous quantity of calculations is required.

For all these reasons, the responsibility of the human calculators was indeed enormous. The pressure under which the calculators were working was very strong, because the shooting and bombing tables were indispensable to be able to actually use the new weapons, which had already been made in quantity and distributed on the battlefields. To get an idea of the requests that came to the BRL, consider that during a week of August 1944, while the manual calculation of 15 firing tables had been completed, work was done on 74 others and 40 more remained pending. It was absolutely necessary to speed up the calculation procedures, inventing a tool capable of circumventing the bottleneck consisting of the use of mechanical or analogical calculation tools.

ENIAC was designed for this purpose. Thanks to this "mathematical Frankenstein", "electronic brain" or "mechanical Einstein" (as a sensationalistic press defined it at the time of its official presentation, in February 1946) times were drastically reduced: «ENIAC computes the trajectory faster than how long it takes the bullet to do it» says Marie Bierstein [3]. Without exaggeration, since a 30-s trajectory could in fact be calculated in less than this time, thanks to the new machine.

2.2 From Man to Machine: ENIAC Hardware and Software

The US government since 1943 financed a pioneering project for the construction of a calculating machine, which was able to perform the same manual calculations but faster and automatically. ENIAC was designed and built at the University of Pennsylvania's Moore School of Electrical Engineering for the BRL, a former US Army research center, where computer girls were employed, by a team of engineers led by J. Presper Eckert and John Mauchly. The designers managed to complete the design and construction of the machine in a relatively short time, while programs had to be written, that is, the software needed to solve problems had to be designed.

Within the group of human calculators, the six brightest were selected, all with a degree in mathematics and little more than twenty years of age. None of them after graduation had considered the possibility of a job as a teacher, they were looking for something different and had been attracted by the possibility of operating as a computer, thus being able to contribute to emergencies created by the war period. After a short training, they were used to write programs for the calculation procedures, previously computed by hand, so that they could be computed independently by the machine. Their task was very difficult: they had to adopt new mental patterns, invent ways of proceeding, cope with the physical limits imposed by the machine and other enormous impediments. In fact, they had no programming languages, neither manuals, nor an operating system. In addition, the ENIAC project had been commissioned and was to serve military purposes, thus remaining top secret.

The small group of girls succeeded in the venture, they were the first programmers ever to exist: they created the ENIAC software. To understand the type of problems they had to face and to fully appreciate the original work they were able to complete, it is necessary to describe with more detail the structure of ENIAC, initially completely unknown to the girls. By tests and experiments they were able to completely reconstruct the functioning of the machine. They got so successful in this reverse engineering process that they were eventually able to detect operating errors even at the level of the individual valve.

The ENIAC architecture scheme shown in Fig. 1 is composed of five different functional blocks. Since it is not a stored program computer, its architecture is not of the von Neumann type, i.e. it does not work according to the extracting-executing cycle od the sequence of the program instructions, but it can be considered as an architecture *data flow* type, if you want to use a specialist term. In a data flow architecture the execution flow depends on the data and allows you to program multiple instructions in parallel. The information is not yet coded in binary as in most modern computers but uses ten-base digits. The *memory uni*t is composed of

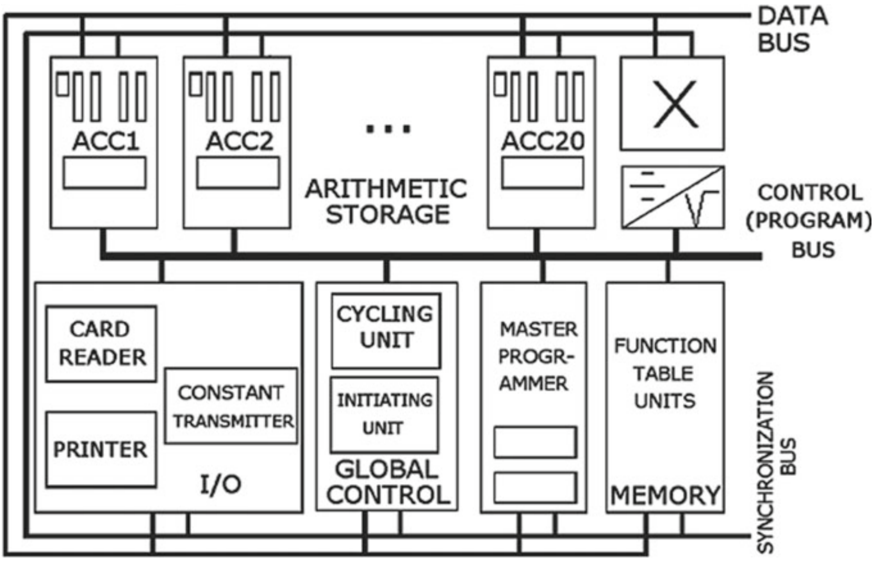

Fig. 1 ENIAC's architecture

only 20 registers for reading or writing a value of 10 decimal each. The registers also function as accumulators, i.e. are able to add a new value to the stored value and this operation can also be repeated several times. The subtraction can be performed with the sum of the complement to 10, while multiplication, division and square root can only be performed in one component of the memory (the one on the far right in the diagram), one at a time. Multiplications and divisions for small values can also be achieved by repeating sums and subtractions directly in the accumulators. The values on which to operate are entered through the *input/output unit*, equipped with both a punched card reader and a panel to directly input the constant values and finally a printer for the results.

So ENIAC could only work on the information accumulated on the 20 decimal registers, if you don't count the tables of numbers and constants and data entered by hand. Remembering the type of calculations described above, programming complex problems in this way seems an impossible mission. The control unit essentially contains the *timing unit* (cycling unit) which supplies the pulse trains to synchronize the operations, and the start unit. The control, unlike the subsequently designed computers, does not store the program to be executed.

Programming and changing programs was very difficult. The so-called *direct programming* was used, by means of which it was necessary to intervene directly in the connection of the units with the cables and change the settings of the units. It was like redesigning ENIAC to make it work every time like a computer dedicated to solving the specific problem. The *function table unit* is a read-only memory that contains the famous pre-calculated value tables of known functions, to be used during the calculation of a more complex function. The *master programmer* is used to program cycles or to start two or more different computations in parallel. The architecture is also equipped with two main communication lines, called buses, one for the transport of control commands (*control bus*) and one for the transport of data (*data bus*), which can carry a value of ten digits at a time by means of a ten-line connector. In addition to these buses, there is a synchronization one, with the task of sending impulses. The units are not connected to the buses in a fixed way, but connected each time in a functional way to the specific program. It is up to the programmer to find and arrange the appropriate interconnection.

Each accumulator can receive an input from 5 different channels each of which, regulated by a specific control signal, can be connected to one or more other accumulators or to any other useful unit (input, function tables, master programmer). Each of the accumulators also produces two possible values at the output: the stored value or its complement to be used for subtractions. The same operation can be repeated several times: an output pulse signals when the repetition of the operation ends.

2.3 Refrigerator Ladies

ENIAC was a parallel computer with all the problems and opportunities offered by a computer of this type, and if you also think about all the other difficulties, included not being able to keep the temporary results and that the connections and the settings had to be explicitly predisposed, the fact that our girls managed to make complex programs work perfectly is a kind of miracle.

For the girls, as they explain in [1], working at ENIAC was a fantastic experience: their task was very stimulating, they worked and lived all together, they had no hours, they were far from the family, therefore they also enjoyed an extraordinary freedom and a fairly good salary for the time.

In February 1946, the US Army organized a public demonstration to prove the strength of the new computer. The evening before the demonstration, the program worked perfectly except for the fact that it did not stop calculating when it should have been, that is, at the impact of the projectile with the ground. The girls worked until two in the morning trying and trying to solve the bug without success. During the night Elisabeth Snyder, one of the girls, had the enlightment, apparently during sleep. The next morning she changed a switch on the Master Programmer and the problem was solved. The program worked perfectly, managing to calculate the trajectory of a bomb in less time than it took the bomb to touch the ground, a thousand times faster than any previous calculation. The demonstration was perfect [3].

The girls were Frances Bilas, Betty Jean Jennings, Ruth Lichterman, Kathleen McNulty, Frances Elizabeth Snyder and Marlyn Wescoff. The documentary *Top Secret Rosies* [1] describes, through their own words and the photos that depict them at the time of the ENIAC adventure, their scientific and human story. They were well aware they were doing a difficult, urgent and very important job, and this awareness made the grueling shifts, to which they self-submitted, acceptable and almost amusing. Jean Jennings tells how, daughter of peasants, she did not want to marry a farmer and have a lot of children, but was looking for something different, that her mathematical skills allowed her to reach. She defines herself as «the luckiest person in the world» for having met, in addition to the girls, the designers of ENIAC, to whom she is grateful for «having given [her] life a meaning it could never have had in any other way». Their adventure did not end with the relatively few years they spent working at ENIAC: unlike many other women employed in wartime, and returning with peace to their previous life, all ENIAC girls continued to work in the world of programming, both inside and outside the US Army.

Their work has the dignity of a pioneering scientific research of a value comparable to the ENIAC hardware project itself. Their valuable contribution has been fundamental for the development of other important programs and puter models, their competence was the basis for the development of all subsequent software. At the time, they received a bit of recognition but always remained overshadowed by the glory of the engineer designers. They were not even invited to the gala dinner after the demonstration [1]. Their history had been lost, only that of the engineers who received many prizes and awards had remained. The girls remained freezed as the "refrigerator ladies" in the archive photos (Fig. 2): the image of a technological device next to young women was automatically interpreted with the girls taking care of the housework (albeit in a modern way). Until a young Harvard student, Kathy Kleiman, reconstructed their true story in 1985. She made interviews with the ENIAC programmers who were still alive, which allowed her to reconstruct their work at

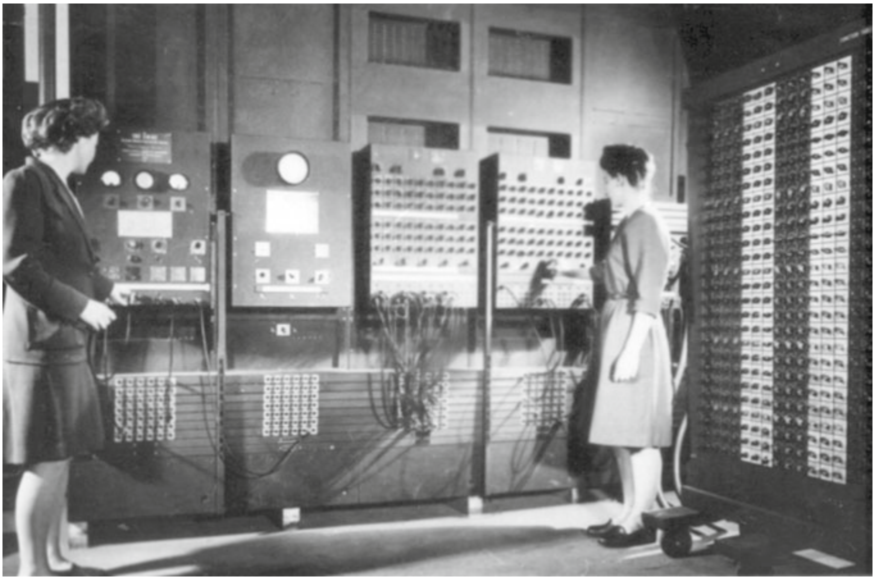

Fig. 2 Computers working at ENIAC: Betty Jennings (*left*) and Frances Bilas (*right*) *Credits* US Army [Public domain], via Wikimedia Commons

ENIAC and attribute the right value to it. From this experience the documentary *The computers* was born, presented at the film festival in Seattle in May 2014 [8].

3 From the Earth to the Moon

In the testimonies of those who wrote and made the first computer programs work, there are many similar and quite curious experiences, starting from the methods used in the selection for hiring. It was difficult to imagine what skills were needed to be a good computer programmer and so who was in charge of the selection invented creative solutions. To find the most suitable girls to work on ENIAC, fot examples, the applicant were asked even if they were afraid of electricity! Of course, that huge computer full of flashing lights, plugs and cables to be transported and connected could create fear, but the relationship between this question and the real inclination to program was not so evident [1, 9].

Subsequently, when the first programming languages were defined, the knowledge of one of these was sufficient to guarantee employment. Which is like saying that if you know Chinese you can write a novel in that language. This is because those who should have selected the inclination to program did not have the slightest idea of what the necessary requirements were, nor how to verify them [10, 11].

The hardware engineers, those who designed and physically built the computers, were not interested in the problems of programming which remained a mystery to them. They willingly delegated to others all the problems related to the software, that is the set of programs that make a computer work. The software was considered of less importance than the hardware and the field was left completely clear. For this reason, many women and many young people were hired. Sometimes, due to automatic tests, even African-American women were hired: the score obtained in the test exceeded all the prejudices of race or gender. So much the programming work was considered of secondary importance, only a support job, suitable for women, a sort of super specialized secretaries!

Another important factor was the complete freedom granted to young programmers: everyone worked starting from scratch, trying and experimenting with new techniques, learning from mistakes, relying on word of mouth, but without resorting to imposed or pre-established rules. What was required of them was only commitment and dedication, and above all that the programs worked. This situation, combined with the fact that the activity of programming is stimulating and fun and that wages were good, was the engine of groups of programmers very motivated and enthusiastic about their work, who masterfully succeeded in various noteworthy companies. At the time they did not have all the recognition due, only recently some stories have been rediscovered and evaluated in the right way.

The software programmers soon began to seek the attention of hardware engineers because even small changes in the hardware could have led to substantial improvements in the software. The voice of the programmers began to be heard more and more and even to enter fully even in the design of the hardware.

We have to wait for the end of the 1960s for programming to start getting the right attention. A wonderful book composed of various volumes (still in the completion phase) written by Donald Knuth, one of the world computer fathers, *The Art of Computer Programming* [12] contributed to this. The importance and diffusion of computers continues to grow and it is no longer sufficient for programs to work, they must be well written and documented and, above all, be efficient. In short, inspiration, intuition and originality are required, as well as technical skills. Programming becomes an art.

3.1 Margaret Hamilton

In 1959 Margaret arrives in Boston, with a degree in mathematics, to specialize with a doctorate. She has a husband and a young daughter that she always carries with her. After responding to an announcement from the prestigious Massachusetts Institute of Technology (MIT), she obtains temporary employment to develop weather forecast programs on the LGP-30 and PDP-1 computers and decides to postpone her doctorate. She becomes the reference programmer at the laboratory, she is involved in the Apollo project, for sending capsules to the Moon of which, in a short time, she becomes responsible [13].

In the photos of the time Margaret looks like a girl with her miniskirt, goggles and long, loose hair from the 70s; in reality she has very important tasks: for the Apollo 11 mission, she directs the Apollo Guidance Computer software project, the on-board computer of the command module and the lunar module, that is, a computer to be programmed for a critical and complex mission, with the risk of human lives. Moreover she has very limited tools and knowledge of that discipline available. She has already designed the software for previous missions such as Apollo 8, of which she always tells an anecdote: her six-year-old daughter was watching her while testing the astronaut software on the screen. The girl wanted to play the astronaut and suddenly the system crashed. The problem was caused by the fact that the daughter had pressed the P01 key, which started a program to be run before launch and not during the flight. The system did not allow the various phases to have parts in common. Margaret understood that the involuntary maneuver of her daughter had brought to light a serious problem, of which she immediately made her leaders aware. She was told that the astronauts were very well prepared people and would never, ever make mistakes like that, the different phases could remain distinct, because they would not interfere with each other… and, like the last famous words of the jokes, it was precisely that that happened: the P01 button was mistakenly pressed during the flight by an astronaut, causing an on-board computer shutdown. The spaceship had to be piloted from the base and it took hours to get everything right. In subsequent missions this was taken into account.

Margaret is scrupulous, obsessed with the possible mistakes that could happen, spends sleepless nights, clashes with the engineers who often block her solutions saying that they can not be achieved, just because nobody has ever done them before!

To explain the difficulties of the project, which is not a simple program, but a system of programs to coordinate, developed by different people, and perhaps to be better understood by hardware engineers, she defines the overall work as a software engineering project, starting a new discipline that later keeps this name [14]. Engineers, in their design, could be based on rules, software engineers could not, they had to solve problems never solved before, but they were not given the opportunity to be beginners. They were beginners and soon after experts and so they succeeded. In addition, computer resources, space and computing speed were very limited at the time. In addition, the software had to be ultra reliable, errors had to be detected and corrected in real time: the life of the astronauts was at stake.

Margaret and the skill of programmers in her group created the system of programs that was able to coordinate the flow of data coming from the gyroscopic navigation system, the telescope and two radars, providing astronauts with control over the engines and all on-board operations. It was thus possible that the first human being stepped on the lunar surface on July 20, 1969. But just when they were about to land another emergency occurred: the radar activation signal for the return of the spacecraft had erroneously appeared on the display. The designed software was able to ignore the error, which otherwise would have caused the on-board computer to overload, possibly compromising the landing operations. Similar situations to this one had been foreseen by Margaret who understood that it was necessary to organize the computer tasks by priority so that, in the event of a conflict of information during the moon landing, the higher priority operations were activated, excluding unnecessary ones.

3.2 Moon Mathematics

Leaving the Earth, getting in orbit and eventually landing on the surface of the Moon (but also getting back to the Earth!) are quite complicated tasks from a mathematical viewpoint.

The problem with landing is in essence a matter of solving real-time "stiff" non-linear differential systems: unfortunately at the time of the Apollo missions (even if twenty years had already passed since the stiff problems mentioned in paragraph 2.1, for which ENIAC was born) the calculation capacity available on board (but also on the ground) was still too limited, and therefore the consequent control problem posed enormous difficulties.

From the NASA reports of that period (now available to the public) it is evident a growing interest in the problem of the regularization of double collisions in the N bodies problem.

Just to give you a very naive idea, let us consider two bodies, the Moon and the spacecraft, modelized by particles of masses M and m. If we consider a coordinate system centered at the mass M and having its axis parallel to the axis of the inertial system, the differential equations of motions for the particle m with respect to the central mass M is given by the following set of equations:

$$\ddot{x} + \frac{K^2}{r^3}x = 0, \quad K^2 = k^2(M+m),$$

where k^2 is the universal gravitational constant and r is the distance between the two masses. Of course, the attraction is infinite if $r = 0$, i.e. if collision of the two bodies occurs. In other words, the equation is singular at the origin, since the Newtonian gravitational attraction of the central mass is infinite at that point. Unfortunately, this fact causes not only theoretical but also very umpleasant practical difficulties: if the particle approaches the central mass very closely, causing what is called a near collision, large gravitational forces and sharp bends of the orbit are produced. During a numerical integration the only way to overcome such a difficulty is to use small step length and many steps of integration during the phase of close approach, and because of truncation and round off errors the numerical precision after near collisions is necessarily very poor. That of course is not a problem in the issues of classical celestial mechanics, since collisions of the planetary bodies normally do not occur; on the contrary, any landing mission implies a close approach at the start and at its destination. Hence, we are interested in transforming singular differential equations into regular ones, and this is the process we refer to with the term *regularization*. The idea is to transform the singular vector field into a regular one so that when two bodies pass close enough (even if there is no collision) the numerical computations are more stable [see for instance 15]. We have then to deal with the "Levi-Civita regularization" (if we are working in the plane) or with the "Kustaanheimo-Stiefel regularization" (if we consider the situation in space).

From the point of view of the mathematics involved, the calculation of the orbits in a global sense, starting from their initial design, is also of extreme interest: at the time of the Apollo missions, the "patched-conics" technique was used to have a first approximation of the trajectory, and then the result was optimized to have a solution of the realistic problem. A now classic but no less interesting technique, described in all the books on space flight dynamics of Engineering courses (for example [16]).

Of course, Margareth was not in charge for those computations, which were rather the tasks of her fellow engineers. For this reason, we do not delve further into these issues here, referring the interested reader to [17].

3.3 *Put the Experience to Good Use*

In a recent invited speech given at an important software engineering conference, Margaret pulled the strings of her incredible programming experience that lasted for about 60 years, underlined the fundamental aspects, those that have changed over time and those that were maintained [14].

With her long experience she had the opportunity to confront, in her own words, with every kind of possible programming error; she soon realized that a software engineer's biggest problem is to produce reliable programs, which is an increasingly difficult task as systems are more complex. When there are more than one hundred people working on a project, the problem of coordination and of the responsibility, especially for updates that can erase useful data, is fundamental. With a strong sense of humour Margaret tells us how she had to learn to prevent errors: when your mistake blocks the whole system and the work of many people, who get angry at you screaming, you quickly learn to test your program locally, before adding it to the overall program! The documentation is also very important, it must be done day by day. “Always expect the unexpected” she recommends, the error comes in the most unexpected ways. They had devised various techniques to avoid mistakes in her group, one was to classify them by associating with each single type of error the photo of who had first committed it, which certainly had to remain impressed. Margaret had a great responsibility and a real obsession with mistakes, which kept her awake at night, but which led her to do something else: to provide a reliable system, capable of making some corrections, such as that of Apollo 11.

In the years following the Apollo missions, Margaret decided to put her incredible programming experience to good use, quit MTI and NASA and created her own company, The Hamilton Technologies. Her experience with lunar missions had taught her many things and, above all, that systems in nature are distributed, asynchronous and controlled by events. The languages they had used were completely inadequate. A programming language would have had to give the possibility to define these characteristics and make available mechanisms to implement them, thus avoiding having to explicitly organize the sequence of instructions for each possible event that occurred. She also understood that every system is inherently a system of systems.

Hamilton Technologies studied and developed a system definition language, The Universal System Language (USL), based on systems control theory. Unlike traditional languages, it is based on the philosophy of prevention: instead of identifying new ways of testing the error, which occurs too far during the life cycle of the program, it prevents errors immediately at the beginning, when defining the system. The language is not a real programming language, rather a formal language on the type of strictly theoretical ones.

The USL language is based on a series of axioms and formal rules. The same language is used to define different things like:

- All aspects of a system and its evolution.
- All functional architectures, resources and allocations, hardware, software and human products included.
- The documentation.
- All the definitions.

The language is independent of the syntax, implementation and architecture. Despite being formal and unlike other more mathematical formal languages it is user-friendly.

When a system is defined with a language of this type, it can immediately be automatically analysed and verified that it has been properly formalized; at this point, most of the project and the code itself can also be generated automatically. Thus the model obtained can be sent for execution and, if it does not meet the desired requirements, the definition can be revised until it is perfect. The philosophy of this language is to build reliable systems composed in turn of reliable sub systems. The recognized reliable systems can be reused and form the language library.

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Mathematics for Social Integration

Domenico Brunetto, Chiara Andrà, Nicola Parolini, and Marco Verani

Abstract This work presents preliminary research findings from the Italian project Teenagers Experience Empowerment by Numbers (TEEN), funded by the Politecnico di Milano through the Polisocial Award 2017. The project TEEN (www.teen.polimi.it) deals with the phenomenon of young immigrants, the so-called "teen-immigrants", and aims at promoting basic mathematical literacy as another fundamental right that may significantly increase the level of autonomy of teen-immigrants resorting to activities that have a mathematics root.

Keywords Mathematics education · App · Mathematical skill · Geometry · Out-of-school mathematics

1 Introduction

The project TEEN (Teenagers Experience Empowerment by Numbers http://www.teen.polimi.it), founded by Politecnico di Milano through the Poliscocial Award 2017, deals with the phenomenon of young immigration towards Europe. More precisely, such a phenomenon concerns the underage who leave their country without parents and relatives, we called them "teen-immigrants" [1, 2]. In Italy, the number of teen-immigrants has reached a peak of 18 thousands in 2017 [3]. Among them, the 93.3% ranges within 15yo and 17yo, while the majority of teen-immigrants are male (93.2% in 2017). Figure 1 shows the distribution of teen-immigrants towards Italy, the 69.9% of teen-immigrants come from Africa (e.g., Gambia 2202, Egypt 1807, Guinea 1752), a relatively low percentage of underage come from Albania (9.2%) and Bangladesh (4.7%).

Once teen-immigrants arrive in Italy, they are accommodated in communities for minors, that provide them for their basic needs (accommodation, food, health

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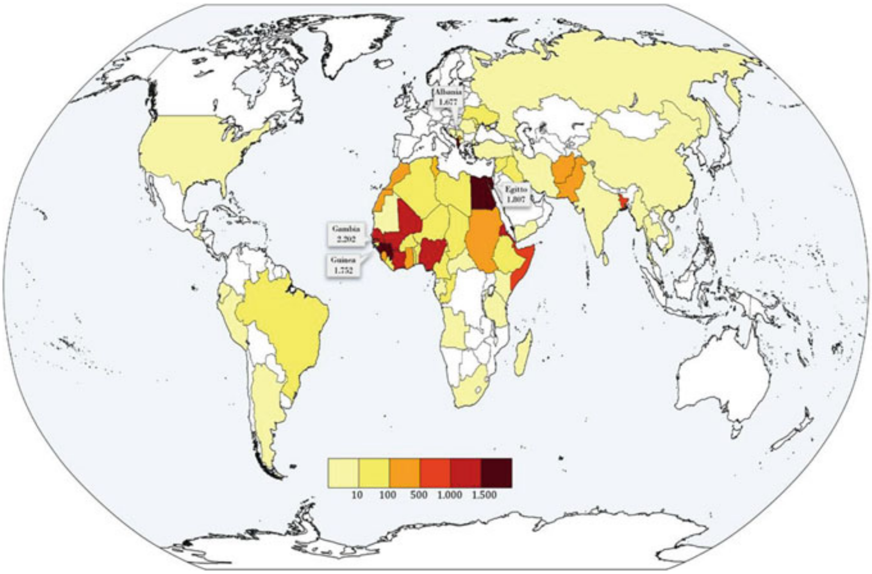


Fig. 1 Distribution of the origin countries of teen-immigrants in 2017. *Source* [3]

services and a language course). That turns out to be insufficient to deal with the requirements of the "real world" that they need to face early, considering that the protection guaranteed by the Italian legislation to unaccompanied minors ends on the day of their 18th birthday. The TEEN project aims at promoting basic mathematical literacy as another fundamental right that may significantly increase the level of autonomy of teen-immigrants resorting to activities that have a mathematics root.

The strength of the project is the idea that mathematics can be pivotal in the social integration of disadvantaged students [4–8]. The successful experiences in different contexts inform that a part of considering the social and ethical implications of teaching mathematics to a "minority" [6], students learn better when mathematics is related to their identity and their attitudes [5, 8] and students turn out to be more aware of the role of mathematics when teachers exploit the students' language and the way they interact to design the activities [7]. These two considerations reinforce the claim that it is necessary to promote a new idea of mathematics as valuable skills for social integration and to create alternative curricula with the purpose of showing mathematics as accessible (and useful) [9].

On the path of these previous experiences, the TEEN project aims at developing the mobile app *StreetMath* that can be used by teen-immigrants, outside the classroom and without the mathematics teacher. Moreover, in order to make the discipline more accessible and attractive, the app is designed to deal with everyday real problems, with non-academic language and, to be inclusive, with the least possible amount of written words through the extensive use of graphical support.

In this work, we present examples of activities focusing on geometry and preliminary research findings of the project from the experimental learning session with small groups of immigrants organized in the communities where they live.

2 StreetMath: The Learning Environment

StreetMath is the web app developed by the interdisciplinary team of the project TEEN. StreetMath is available for Android and can be also accessed by browser from any device (visit www.teen.polimi.it/sm). The activities proposed address situations that are familiar for teen-immigrants because connected to their identities and experiences. We called such situations as realistic scenarios, that are the units of the proposed educational project. On one hand, the scenarios allow learners to recognise mathematics as an accessible and valuable skills [9], on the other hand, learners resort to their own experiences to deal with it [5, 7, 8]. In particular, teen-immigrants can exploit previous knowledge (scholar and not) and attitudes towards a scenario, for instance, the motivation in dealing with a task.

The app contains two types of units collected in two sections:

1. the challenge units in home page (see Fig. 2a), and
2. the math-recap units in the "mathematics" page (see Fig. 2b).

The units are composed of tasks (open-end and multiple-choice) in Italian language and are supported by images that help the understanding of the text and provide information required to answer.

The tasks are built upon realistic scenarios, but in the real-challenge are more structured with the purpose of solving a problem identified by the units: for instance,

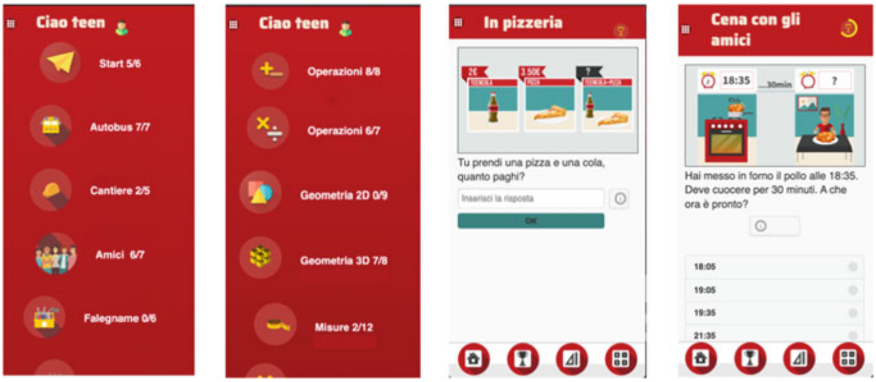


Fig. 2 Screenshots from the StreetMath: **a** list of the real-challenge, **b** list of math-recap, **c** example of an open-end task, **d** example of a multiple-choice task

the unit called "bus" is based on the experience of reaching the workplace from home to have a job interview, so the problem consists in planning the trip. Nevertheless, also the tasks of math-recap units resort to daily-life experiences to recap the main mathematical ideas.

The unit "start", which is the first one (see at the top of Fig. 2a), was designed with the purpose of introducing learners to the environment and addressing the instrumental genesis [10] of the app, with particular focus on the utilisation scheme. The unit is composed of 6 tasks, 4 are open-end tasks concerning elementary operations (addition, subtraction, multiplication and division), while 2 tasks are multiple-choice concerning the time estimation. In Fig. 2c, we report the first task as an example of the open-end task:

1. the title (at the top) identify the task and the scenario,
2. the image supports the understanding of the task,
3. a brief text reports the question and often it may be avoided,
4. the text form, in the middle of the page, allows writing the answer using a number-keypad,
5. the button "ok" submits the answer,
6. a system of feedback and help (the "i" button) is present to support the learners both in case of wrong and right answers.

The structure of the multiple-choice task (see Fig. 2d) is similar to the open-end, except for the input-form which is replaced by the list of answers.

The navigation bar at the bottom of the page is characterized by the three navigation buttons for (from left to right) "home", "progress", "mathematics" pages and one (the last on the right corner) that pop-ups a "calculator". We refer to "mathematics" and "calculator" buttons as tools to face with scenarios. This choice, in agreement with the idea of accessible math [9], unfolds the focus on the process and the mathematical thinking rather than the procedure and the computational skills, even though we recognise their importance in doing mathematics.

An important feature of the learning environment is the feedback system, that we intend as part of a range of practices that aid the learners' understanding [11]. The feedback system should address the goal, the progress and the quality of the performance of the learners, not only in terms of checking the correctness [1, 11]. To that end, we develop a feedback system composed of three parts (see Fig. 3):

A. the aforementioned "help" (button "i" in the middle of the page, see Fig. 2c–d) and provides details and suggestion about the task,
B. the "support" that popups in case of the wrong answer and provides further suggestions accordingly to common mistakes and misconceptions, and
C. the "remark" after the correct answer is given, it provides further information about and beyond the task as well as social messages.

We note that the "support" message is composed of two parts: the first one is the suggestion after the red word "Try again", while the second one is the actual answer to the task under the button "Show answer".

Fig. 3 The feedback system: example of a message of **a** “help”, **b** “support”, **c** “remark”

We design the three parts in order to address the three notions that underlie feedback systems [11] “feed up”, “feed back” and “feed forward” that work at four intertwined levels: the one the task, if they provide information about how well tasks are understood and/or performed; the level of the process; the level of self-regulation, if they concern self-monitoring, directing and regulating of actions; and the level of self, if they point to personal evaluations and positively affect learner’s identities [1, 11].

Moreover, the feedback system allows learners to support their experience of *empowerment*, that roughly speaking, is the process of learning and using problem-solving skills and the achievement of perceived or actual control [2, 12]. That suggests that experiences which provide opportunities to enhance perceived control help individuals to cope with stress and solve problems in their personal lives [2]. In the mathematical context, empowerment concerns the role of mathematics in daily activities and its impact both on the learning process at school and in social life [13]. Three different domains of empowerment have been identified: *mathematical*, *social*, and *epistemological*. The *mathematical* empowerment of power over the language, symbols, knowledge and skills of mathematics and the ability to confidently apply them in mathematical applications within the context of schooling, and possibly to a lesser extent, outside of this context. *Social* empowerment ranges from the straightforwardly utilitarian to the more radical ‘critical mathematical citizenship’. *Epistemological* empowerment concerns the individual’s growth of confidence not only in using mathematics, but also a personal sense of power over the creation and validation of knowledge [2].

StreetMath supports the empowerment thanks to its features: (i) reinforcing the motivation through the realistic scenario, (ii) aiding the text understanding using images, (iii) boosting the mathematical process exploiting the calculator, and (iv) driving towards the task resolution through the feedback system. We remark that all those features have the purpose to make learners feel as the active subject within the learning environment. In such a way teenagers can resort to the provided aids and their previous knowledge and skills in order to achieve control of the scenario as well as improve their level of mathematical, social and epistemological empowerment.

3 Examples of Activities

StreetMath provides activities concerning work scenarios. Among the most common jobs the teen-immigrants usually access, we identify those that require mathematical competences and knowledge. The main ones regard two intertwined domains, such as geometry and arithmetic, more precisely the teen-immigrants are asked to

1. measure physical quantities (length, mass),
2. read measures (e.g. from blueprint),
3. operate with measures (equivalences),
4. derive quantities (area, volume),
5. identify the proportion between quantities,
6. compute the percentage of quantities,
7. estimate measures and quantities.

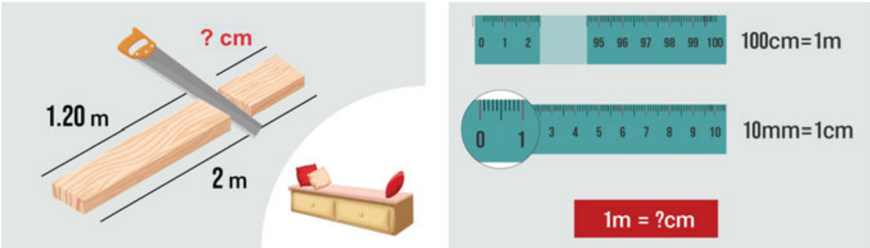
We report four tasks from four different units with the purpose of showing the approach we adopted to drive learners towards the achievement and development of such targets. More precisely, we report examples of tasks focused on geometrical aspects from real-challenge units that concern four work scenarios, coupled with four related tasks from math-recap units. The latter are meant as mathematical "tools" the learner may resort to deal with the four work scenarios.

Task 1. The first task we report, titled "Gear", is the last one (out of eight) from the unit named "Garage". This unit is mainly focused on the analysis of engines' efficiency that allows investigating the idea of percentage and proportion. Nevertheless, the task "Gears" is designed to improve the measures reading: the question is: "what is the length in mm of the gear?", the task is supported by the image in Fig. 4a which shows a gear (of 34 mm length) on the graph-paper sheet. Behind the evident request of identifying how many "big-square" (5 mm) and "small-square" (1 mm), learners need to resort their knowledge about the integer multiple and submultiple of a quantity. Once the learners have difficulties with this task, they can resort to the feedback system: (1) the "help" suggests to "look at the squares in background" and that big-square is 1 mm length. (2) the "support" provides further details on the dimension of the "small-square".

In case learners struggle with the task, they can resort to the unit named "Measuring". That unit belongs to the math-recap units, its aim is to introduce and recall the idea of measuring lengths and converting them. An example of a task is the one titled "Tablet" (see Fig. 4b) in which the learner is asked to measure the tablet length

(a) (b)

Fig. 4 Examples of images: **a** task "Gear" from "Garage" unit, **b** task "Tablet" from "Measuring" unit



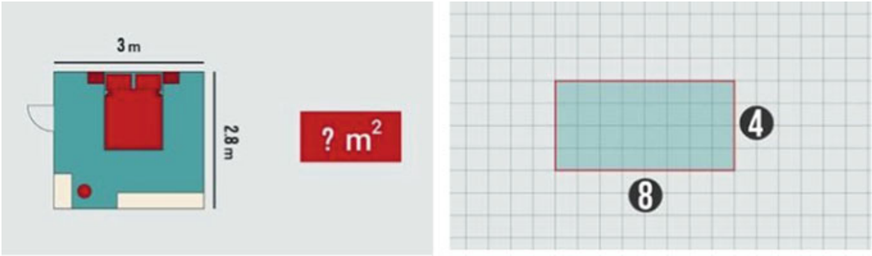
(a) (b)

Fig. 5 Examples of images: **a** task "Trim" from "Joiner" unit, **b** task "Equivalences" from "Measuring" unit

counting the numbers of red blocks. This approach allows to introduce the measurement as correspondence between two quantities: the tablet (to be measured) and the red block (the measure unit). The "remark" of this task concerns the introduction of the term "measure unit", that in the following tasks will turn in meter (adopting the international standard, SI).

Back to the task "Gear", we note that in the pop-up remark the measured quantity is called diameter and it is used to identify the gear dimensions in the automotive field. Those remarks are good examples of providing further information about the task, the topic and the scenario we are considering.

Task 2. The task named "Trim" is the fourth (out of six) of the unit "Joiner". In this unit the learner is asked to deal with tasks related to making a settle whose dimensions are 1.20 m $\times$ 0.40 m $\times$ 0.45 m. The main focus is on reading and converting measures, among those the task "Trim" asks to figure out what is the length of the discarded plank, more precisely the question is "You trim a piece of 1.20 m, how many cm of plank are discarded?" Figure 5a is the image that supports the task understanding, it shows a plank 2 m long trimmed at 1.20 m and the discarded piece labelled with the red text "? cm". The learners are asked to convert the measure to centimetres then make the subtraction: 200 cm $-$ 120 cm $=$ 80 cm.



(a) (b)

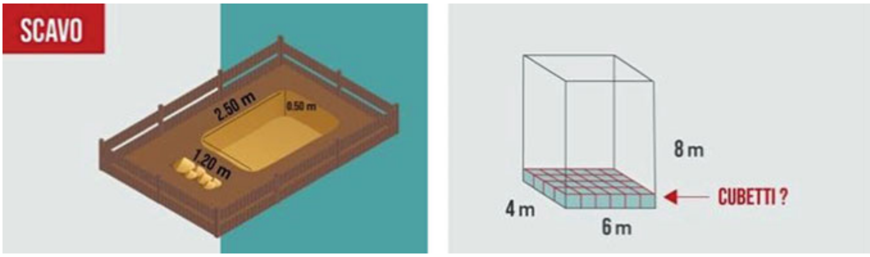
Fig. 6 Examples of images: **a** task "Floor" from "Painter" unit, **b** task "Rectangle area" from "Geometry 2D" unit

However, the feedback system suggests, through the help and support messages, to compute the subtraction 2 m − 1.20 m = 0.80 m, then converting it to centimetres. The reason why we drive in such direction is rooted both in the graphical representation of the task and in the way learners resort their previous knowledge and competences to convert measure units. Many teen-immigrants, as well as many Italian people, read the measure 1.20 m as "1 meter and 20", omitting centimetres for the second part. Therefore, from the image they are suggested to compute the subtraction, even using the calculator, then they read 0.80 m as "80" without unit, assuming it is centimetre. However, in case some of them have difficulties with the equivalences, the learner may exploit the aforementioned unit "measurement" in which we propose some task with such purpose (see Fig. 5b).

Task 3. The third unit we report is the "Painter", in which the learners are asked to compute how many litres of paint is needed to paint a room. The unit is composed of six tasks, mainly focused on computing the painting area (walls and ceiling). Here we report the second task titled "floor" whose supported image is shown in Fig. 6a. The question is "What is the dimension of your room?"

The feedback system drives the learner towards the solution: the "help" suggests to identify the shape of the room, then in the "support" we refer to the rectangle shape and its area. At that point, the learners may visit the math-recap unit labelled "Geometry 2D" where we design tasks focused on the area of rectangles (even square), triangles and circle. Figure 6b shows one of such tasks (the fourth out of nine) where it is required to compute the area of a rectangle exploiting the idea that the measure of the area is the number of small-square within the rectangle. The remark of the task unfolds that the measure unit is the square metre (m^2). A similar message is provided by the remark of the task "Floor" that also reports the use of the term "M squared" ("mq" in Italy).

Task 4. The last task is part of the unit called "Building site" composed of 5 tasks. The scenario concerns the preparation of the concrete for casting into a digging. We report the second task that concerns the employing of the digging volume. Its dimensions are 2.50 m × 1.20 m × 0.50 m, that are provided by the image in Fig. 7a, that is coupled with the question: "how many cubic metre of concrete is needed to



(a) (b)

Fig. 7 Examples of images: **a** task "Digging" from "Building site" unit, **b** task "Cubes" from "Geometry 3D" unit

fill the digging in?" We stress that the question is realistic, because it is similar to the one the supervisor may ask the worker. The feedback system suggests to compute the volume (the message in the help pop-up), then provides hints on how to do that. To support learners in this computation the unit "Geometry 3D" is provided within the math-recap section. Similar to the "Geometry 2D" we construct the idea of volume and its measurement starting from the idea of "cubes" (see Fig. 7b) inside the volume.

The StreetMath contains about 200 tasks so that the presented tasks are just a flavour of the activities available. However, from those, the reader can figure out the nature of the activities and our attitude in design and produce that. In particular, we want to highlight that each unit is the result of months of co-design with the teen-immigrants and the educator of the communities involved in the project. In the next section, we describe two episodes that contribute to achieving such goals.

4 Findings and Conclusions

The research project follows a design-based research methodology, which focuses on examining a particular intervention by continuous iteration of design, enactment, analysis, and redesign [1, 14]. To that end, the two-years long TEEN project planned a design phase composed of three main moments, that involved about 60 teen-immigrant hosted in six communities.

In the design phase of the TEEN project, we worked with small groups of teen-immigrants who volunteered to spend some time with us. The phase consists of different meetings that lasted 45 min on average (a minimum of 10 min to a maximum of 3 h), for a total of 30 h. The meetings provided us with important information about the learners, the scenarios and the learning environment. In the following, we report two episodes related to the tasks above, describing the process that emerged during the interaction with the app and the mathematics within the app, and focusing on the way(s) teen-immigrants empowerment increased.

In the first episode, Ali and Ken (fictional names), 17yo from Gambia and 18yo from Turkey respectively, were working on the “Joiner” scenario. After about 10 min, they approached Task 2 (titled “Trim”). Ken and Ali looked at the task for a while, then Ken said: “[it is] eighty!” The tutor (one of the authors) asked why then the dialogue continues as follows:

– Ken: “because two minus one meter and twenty is eighty”
– Ali: “zero point eighty” (*showing the pop-up calculator*)
– Ken: “yes! eighty” (*pointing the pop-up calculator*)
– Ali: “fine! put the number” (*pointing the app*).

Ken and Ali inserted the answer and got the remark, then said “nice!” At that point, the tutor explored the possibility of resorting to a different strategy, namely converting in centimetres (1.20 m = 120 cm, 2 m = 200 cm) and then computing the difference (200 cm − 120 cm = 80 cm). Ali and Ken did not object to the request of using a different strategy, however, at the first attempt they did not manage to employ by heart the conversion of the two measures contemporary. Somehow, the numbers “200” and “120” were too big and not natural for them: they forgot the first one (200) once they had computed the second one (120), so Ali wrote 120 in the calculator and then converted the first one again. But, at that point, they computed the difference 120 − 200 = −80. Ali and Ken were confused and started to struggle with the task, employing different computation (120 + 200, 120−2, etc.…). Eventually, they provided the correct answer complaining that the task was very hard for them.

From this episode, we are informed by two main findings. Firstly, the feedback system for this task mirrors the “natural” process teenagers enact to answer this type of task, which resorts to the natural language. The reader can argue that the second strategy would be better to unfold difficulties and misconceptions. However, the main purpose of StreetMath is to support teen-immigrants empowerment through mathematics. That leads us toward the second findings: the nature of the task motivates the learners making them “active”, the “natural” process drives the learners towards the correct solution, making them “happy”. Such dynamics can be read in terms of empowerment: Ali and Ken were able by their own to manage the task recognizing mathematics as useful and feeling the control on the scenario. On the other hand, the same guys, after they were asked to change strategy, felt powerless: they did not feel the sense of control on the “new” task and recognized the mathematics as “hard”.

The second episode concerns the “Building site” scenario (Task 4 titled “Digging”). We report one of the earliest experiences when the app contained the first draft of such activity and the feedback system was not well developed at all. The protagonists are two 17-yo guys, fictitiously named Ibra (from Ivory Coast) and Drissa (from Mali), Ibra and Drissa were stuck on Task “Digging” (see Fig. 7a): they did not figure out what was the request, so they asked for help. So the tutor (one of the authors of this paper) clarified that cubic meter is related to the volume of the digging, Drissa has started to struggle with the numbers writing on the paper sheet some computations while Ibra looked at him silently. After a couple of minutes, Drissa looked at Ibra to have feedback by him, but vainly. The tutor got that something missed, so asked them what is a volume, Ibra remained silent whilst Drissa replied: "the volume is measured in cubic meter, centimetre. It is a solid: length, width and height". But they did not know how to compute it, so the tutor provided further information constructing with them the idea of volume and its measure. After twenty minutes, Ibra and Drissa, using paper and pencil, addressed the task computing with the calculator the volume of the digging: $2.50\,\text{m} \times 1.20\,\text{m} \times 0.50\,\text{m} = 1.5\,\text{m}^3$. Finally, Ibra inserted the right number and Drissa exclaimed: "This was very difficult, but I think it is important because the technical jobs require this".

This episode is relevant for two main reasons. On one hand, the discussion with Ibra and Drissa informed us about the need for developing both the multilayer feedback system and the math-recap units, that were not planned at the beginning of the project. Moreover, the feedback system for the whole unit ("help", "support" and "remark") and the "Geometry 3D" unit were designed upon this episode in order to reflect the learning trajectory with Drissa and Ibra. On the other hand, the episode shed a light on the affective dimension: despite the starting stuck, Drissa and Ibra were motivated because they recognized the value of that task, they wanted to control that moment and with the right feedback they achieved the goal of empowering themselves.

We recall that StreetMath was the result of months of collaboration with the educators and the teen-immigrants involved in TEEN projects. These two episodes illustrate the co-design process [14] as well as the identification of the scenarios embedded and the way they were delivered. This coaction is the strength of the project.

In conclusion, we claim that this synergy makes StreetMath a learning environment in which learners can have experience of empowerment (mathematical, social and epistemological) thanks to everyday mathematics they are exposed through realistic scenarios.

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Stanley Kubrick's Perfection and the Divine Proportion

Franca Caliò and Samuele Picarelli Perrotta

Abstract In cinematography, Stanley Kubrick is universally known as one of the most significant proponents. He experimented several cinematographic styles: from noir to horror, from historic to science fiction, from psychological movies to war films. Obsessed about perfection, he was in absolute charge of every part of the making of his films (production, writing, filming and editing) [1–3]. In this paper, we underline how Kubrick loved to play with the symbolic meaning of his choices, with playful mechanisms and with perfect symmetries. His movies are also tightly bound to the other arts (painting, literature, music, theatre, architecture). Furthermore, one must not forget his strong relationship to mathematics, from which he takes logical rigour and expressive harmony. Through this research, we reveal an adherence to the sense of harmony and to the magic of the golden section in unforgettable and strongly symbolic shots which characterise masterpieces like *Barry Lyndon* or *A Clockwork Orange*.

Keywords Mathematics · Geometry · Golden number · Golden perfection · Golden ratio · Stanley Kubrick · Barry Lyndon · Cinema

1 Introduction

The connection between Kubrick's production and mathematics is well known. In this sense, we examined two aspects of his cinematography: his great passion for the symbolic meaning of numbers and his obsessive attention to shot composition.

From these observations, we came to consider that Kubrick was sensitive to the magic of the golden number with which he built some of his shots respecting the golden ratio.

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This paper is structured as follows: Sect. 1, we recall the meaning of the golden number and proportion, giving a brief historical reference with some examples of applications in different sectors. In Sect. 2, we consider some links between Kubrick's cinematography and mathematics to support the ideas developed in Sect. 3. Here we analyse some shots from the film *Barry Lyndon*, interpreting them as if paintings arranged according to the golden ratio.

2 Notes on the Concept of Golden Proportion

The magic of the golden number is part of today's language. It is known that the golden number is the ratio between two real numbers a and b with $a > b$, such that a is the average proportional between b and $(a + b)$.

The connection between the golden number and the harmony of proportions (which leads to the golden rectangle), in addition to the dynamic sequence of Fibonacci numbers, and finally the golden spiral (which interprets the repetitive duplication of the golden rectangle itself) is renowned.

We are used to seeing in reality, as in artistic expression of any kind, that the presence of harmony is given by the golden number. However, we would like to remind you about the historical evolution of the idea itself and giving examples made or interpreted in terms of the golden ratio. We do not intend to go into historical details, taking into account the large amount of literature already written about it—starting with the attribution of primogeniture—but paying attention to the historical statements that confirm the evolution of the concept of the golden number.

In fact, this was born from a relationship between different dimensions as in the case of the architectural elements in the Egyptian pyramid of Cheops and in the Greek facade of the Parthenon.

In the Greek era, in the period of abstracting capacity, the golden number becomes independent from the concept of a harmonious relationship between the proportions of an object, to enter, in Euclid's Elements, into the geometric study of forms such as the rectangle, the isosceles triangle and the pentagon.

Around one thousand years elapsed, from the decline that the golden section had had with the Hellenistic period, before it returned again to the minds of the mathematicians, who also found properties of an algebraic and geometric nature.

Fibonacci (thirteenth century) introduced for the first time the concept of recursive succession. A few centuries later its inextricable link to the golden section would be discovered.

A full and conscious re-evaluation of the golden ratio in geometric terms, with its application in all the noble arts, took place in the High Renaissance, in particular, with the eclectic figure of **Leonardo da Vinci** (fifthteenth century).

Leonardo believed that man was at the centre of his interests. The harmony of the human body was the starting point for the study of a perfect proportion, which he established to be the golden ratio. In all his artistic works the search for this proportion can be identified in the relationships between the parts: whether they are

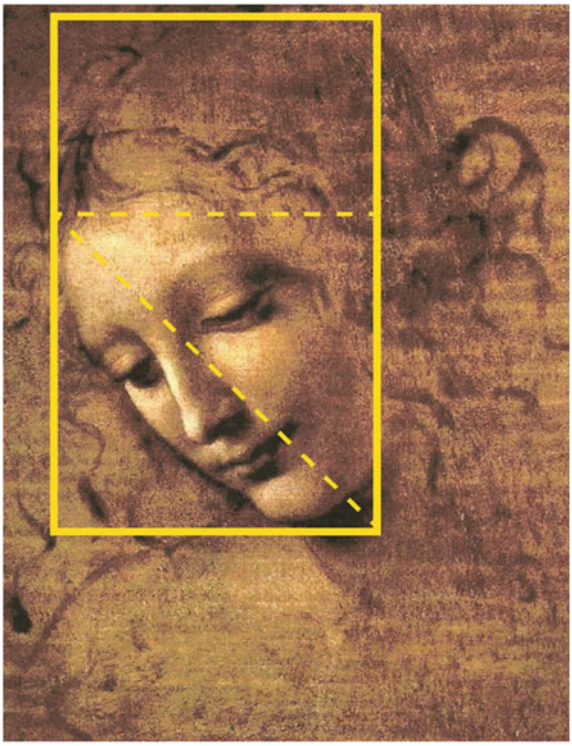

Fig. 1 La Scapiliata by **Leonardo Da Vinci**. *Credit* courtesy of Complesso Monumentale della Pilotta—Parma—Ministero per i Beni e le Attività Culturali e per il Turismo

components of a single subject (*Gioconda*, *La belle Ferronnière*, *Donna Scapiliata* in Fig. 1) or they are parts of a composition (*Annunciation*).

The awareness of the link between the golden number and the Fibonacci succession, which had previously been recognised by the Renaissance mathematician **Luca Pacioli** who called the golden number "the divine gold proportion" and linked it to the construction of the five Platonic solids, was officially given to **Johannes Kepler** (seventeenth century).

Kepler observed that the relationship between two consecutive numbers of the Fibonacci sequence gradually approximated, more and more precisely, to the golden number. Starting from these observations, Kepler, as an astronomer, proposed a heliocentric model in which the orbits of the planets were inscribed and circumscribed in Platonic solids and consequently connected to the divine proportion.

Therefore, it is interesting to see the juxtaposition of two characters such as Leonardo and Kepler: both, fixing the centrality of their own interests respectively in man and in the cosmos, "discovered" the same idea of harmony.

Highlighting the link between the golden ratio and the Fibonacci sequence into a geometric environment, we can find a geometric sequence of golden rectangles (or triangles) gradually smaller, which are obtained with a factor (golden number) of diminution applied to the outermost one.

Working on the sequences of golden rectangles obtained in this way, it is possible to find a sort of logarithmic spiral, called the golden spiral, linked to the homonymous golden section, but representing only a good approximation made by quarters of a circle.

However, the idea of a connection between recursion and the logarithmic spiral has been of great interest: in nature (phyllotaxis), in astronomy (structure of a galaxy—Fig. 2), in painting (*The Great Wave* by **Katsushika Hokusai**—Fig. 3), in poetic metrics, in fractal geometry … the logarithmic spiral spreads and flourishes forcefully.

Moving into the music environment, it is well known that music field is permeated by the concept of the golden ratio (from the structure of a piano keyboard to the structure of a violin, from the compositional structure to its interpretation).

For example, **Iannis Xenakis**, a Greek architect naturalised French, is indicated by some followers of the relationship between mathematics and music as an interpreter of the idea that the golden ratio, in different forms, intervenes powerfully.

Divine proportion also occurs in photography where, as an alternative to the traditional third-party rule grid, we find the so-called *Phi Grid* (Fig. 4). Both grids are intended to help the photographer obtain a harmonious composition of the subjects in view, but the peculiarity of the *Phi Grid* is that four of the nine rectangles that make it up are golden. The final effect is considered by many photographers to be more dynamic and interesting, forcing the viewer's eyes to a spiral path.

Wherever our eye is lead, taking into account now the contemporary, we can identify the presence of such harmony, which has been achieved either naturally or deliberately sought after: **Seurat**, **Gris**, **Severini**, **Pomodoro**, **Saarinen**, **Khankov**,

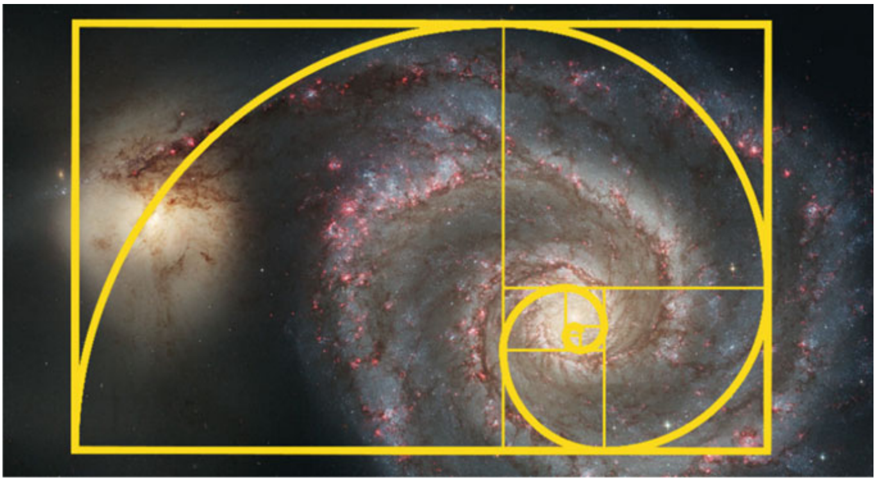

Fig. 2 A classic spiral galaxy located in the Canes Venatici constellation. *Credit* NASA, ESA, S. Beckwith (STScI), and The Hubble Heritage Team (STScI/AURA)

Fig. 3 The golden ratio appears in Under the Wave off Kanagawa, also known as The Great Wave by Katsushika Hokusai. *Credit* H.O. Havemeyer Collection, Bequest of Mrs. H. O. Havemeyer, 1929. CC0 1.0

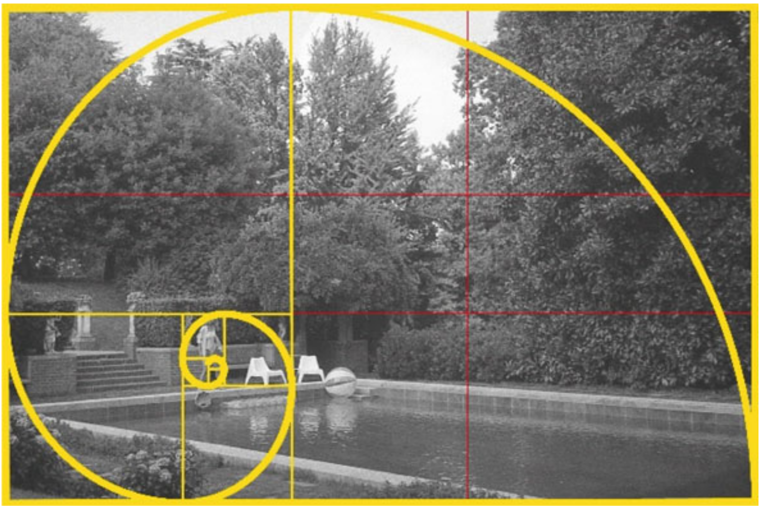

Fig. 4 Example of use of the Phi Grid for photographic composition. *Credit* Courtesy of photographer ©2019, **Marco Traverso**

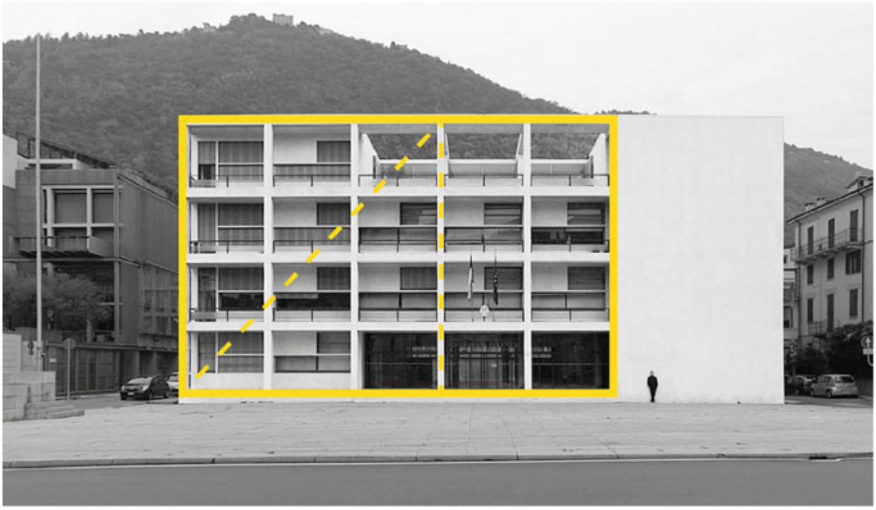

Fig. 5 Como, Casa del Fascio by **Giuseppe Terragni**. *Credit* **Danny Alexander Lettkemann**, CC BY-SA 4.0

Fig. 6 The National Geographic "golden logo".

Terragni (Fig. 5), the National Geographic logo (Fig. 6), electronic music, etc.… Therefore, in painting, architecture, sculpture, design, photography, music…

In this study, we decided to concentrate on a different artistic expression, represented by a refined character and a sharp observer and creator: cinematography and, in particular, that of Stanley Kubrick.

3 Kubrick and Mathematics

The connection between Kubrick and mathematics is clear. The widespread presence of symbolic numbers, the obsessive research of symmetries in his perfect shots and the frequent recall of the hexagon, the Platonic solids and other geometric elements is an open declaration of his passion towards the magic of mathematics in all its aspects.

The rigorous application of this subject becomes for him a means of communicating sensations and emotions [4].

The vision of hexagons and Platonic solids in more than one sequence in *2001: A Space Odyssey* is an evident message of the indescribable emotion of travelling from the earth to the extremes of the universe, getting lost into infinity [5].

In *The Shining*, the obsessive research of specular shapes and of strong symmetries in defining the environment, as well as the recurring numbers and the geometric shapes, pushes the main character towards exasperation (but also the audience), lost in an absurd labyrinth where there are no more references and logical relationships between the elements [6].

Furthermore, the film has two symbolic numbers that occur repeatedly: 42 and 237. The former has been related to the year of the Final Solution, for example, the use of a German typewriter by the character of Jack seems to be a reference to the genocide of the Jews during the Second World War [7]. We must not forget that Kubrick, born from a Jewish family, was interested in making a film about the Holocaust, and calling it *Aryan Papers*: a project that he never had the strength to face and complete [8]. According to another widespread theory, Stanley Kubrick was commissioned by NASA to direct the footage of the Apollo 11 moon landing, using the special effects developed for the creation of *2001: A Space Odyssey.* So, the number 237 could also refer to the moon's distance from the Earth (237,000 miles). Confirming this hypothesis, in a scene of the film Danny wears a sweater showing the Apollo 11.

Finally, it is weird to note that the two numbers taken into consideration also present a mathematical connection: the multiplication between the digits 2, 3, and 7 gives us 42.

These theories, not confirmed by the author or his collaborators, are shared by fans and several experts and collected in *Room 237*, a documentary by **Rodney Ascher** on the hidden meanings within the film.

Whether they are absurd coincidences, synchronicities as defined in the movie, or an actual veiled confession by the author, the presence of such elements scattered throughout his work makes us reflect, questioning the mind of an author considered by most people as simply "crazy".

Taking into consideration these observations, we oriented our research on a more experimental approach to Kubrick's films, determining tendencies and stylistic features which are affirmed and transformed with the progress of his career.

Shot after shot, from *2001: A Space Odyssey* to *Eyes Wide Shut*, the works of the filmmaker have been deeply analysed and dissected: we gathered and catalogued all the scenes distinguished by the use of regularity, perspectives and symmetries [9].

The results are stunning: more than 450 shots (Fig. 7) from his films from 1968 to 1999, with a more frequent presence in *The Shining* where there are more than 180 shots with a one-point perspective and central symmetries. A parabola that finds its conclusion in *Eyes Wide Shut* where the director returns to the balance of the

Fig. 7 Visual research in the graphic project Kubrick and the geometry of image (2006), made for the Design x Designers exhibition (Fuori Salone of Milan) by Samuele P. Perrotta.

starting point and the "special" shots are reduced to just twenty, used in a strategic and pleasing way.

In this multitude of shots, we were driven to think that the passion for numbers, their symbolic meaning, the golden number and all the geometries connected (from the golden rectangle to the platonic solids) may have sparked Kubrick's interest. We thought as well that his strong attraction to the one-point perspectives can be related to the use of the golden ratio and the dynamism of the golden spiral due to their classicism and perfection.

Ruler and calculator on hand, the intuitions which before just seemed an impression are confirmed to be real. The divine proportion reveals itself also in Kubrick, making of those that seem apparently well-composed and studied shots actually turn into geometric tables.

The frame below (Fig. 8) is so strong and indelible in our minds. It is taken from the opening of the film *A Clockwork Orange*. Here the research of the golden section is evident with the main aim, in our opinion, to give the image a strong symbolic meaning: the harmony to transfigure the torment, the composure to translate the restlessness, the peace to underline the interior conflict.

The shot captures the four characters inside the Korova Milk Bar, sitting still on a sofa while sipping a dose of "milk plus". Looking deeper, however, it is possible to see a hidden harmony. In fact, the composition of the four "droogs" follows a golden rectangle that circumscribes the characters from head to toe.

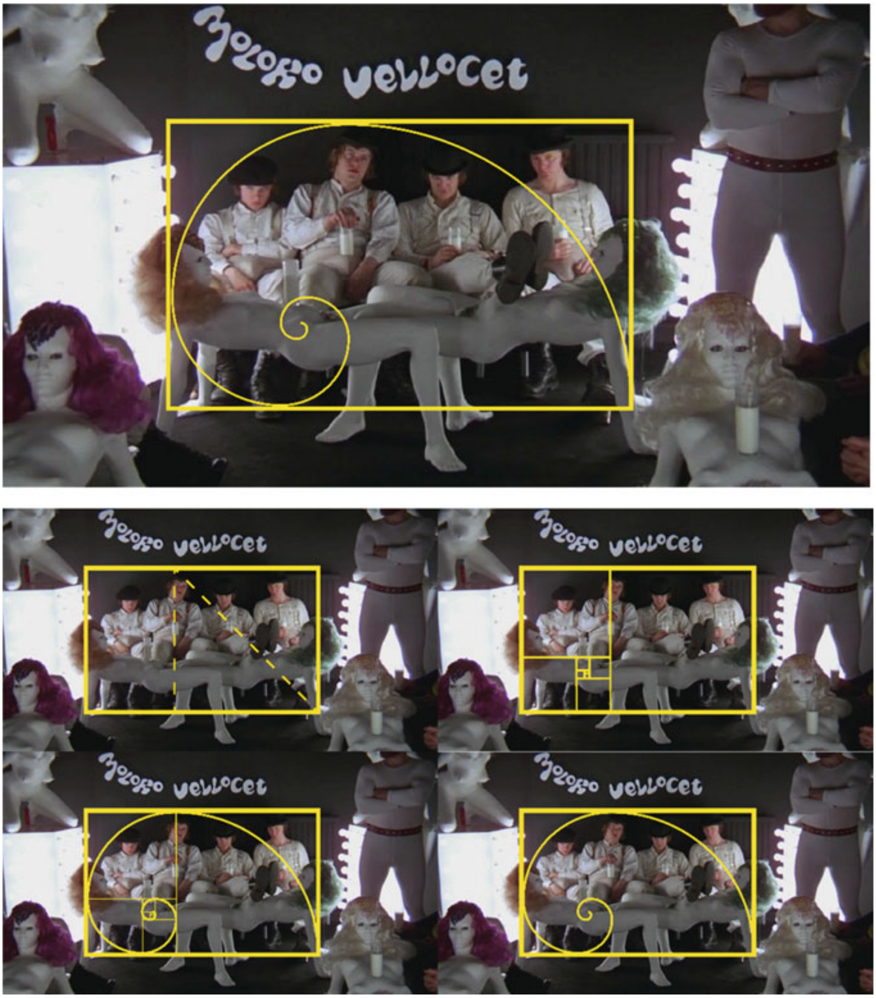

Fig. 8 Study of the golden proportion in A Clockwork Orange by **Stanley Kubrick** opening shot. *Credit* © 1971, Warner Bros. and Polaris productions Inc

The research in question does not limit itself to a single frame. Soon our attention has been drawn to another wonderful historical/psychological drama: *Barry Lyndon.*

Is this less famous and not so strong as his other films? It depends on the kind of observation. The succession of beautiful shots comparable to paintings makes the work of extreme interest. So, also our focus and our mathematical approach were attracted to it.

4 Kubrick and the Golden Number

Today *Barry Lyndon* is considered one of Stanley Kubrick's best films and one of the most important period movies ever produced.

The film, released in 1975, moves away from social, philosophical and political themes that characterise his previous ones (*Dr. Strangelove*, *2001: Space Odyssey*, *A Clockwork Orange*). This time, Kubrick decides to focus on realism, set in the Great Britain of the eighteenth century. In order to do this, he faces a detailed iconographical research of that period making use of historic paintings, prints and drawings. This made it possible for the film to win the Oscar for best photography (**John Alcott**), best costume design (**Ulla-Britt Soderlund** e **Milena Canonero**) and best production design (**Ken Adam**).

The film is strongly visual, rich in images and aesthetic references to eighteenth century landscape painters and becomes as such a sort of "cinematographic museum".

Among the European artists who inspired Kubrick to recreate the eighteenth century environments and costumes there were: **William Hogarth**, **Thomas Gainsborough**, **Joshua Reynolds**, **Jean-George Stubbs**, **Antoine Watteau**, **Johann Zoffany** and **John Constable** [10].

To reinforce this, Kubrick shoots the film without the aid of any artificial light, to give the film a realistic appearance, as if it were transformed into a canvas. The shots, therefore, use the natural set light and alternatively makes use of candles and oil lamps for night shots. This choice involved the use of very bright lenses, which were developed by Zeiss for NASA and specially adapted for the purpose of this film.

In *Barry Lyndon*, Stanley Kubrick uses a technique that he had already experimented with in his previous film (*A Clockwork Orange*), the so-called zoom out or reverse zoom. This technique was introduced to cinema by **Sergio Leone** in his "spaghetti westerns". But, unlike Kubrick, the dramatic effect was done in reverse, which means, with large landscape shots that were slowly reduced to an extreme close up of the protagonists of the duel. Stanley revives and adapts this innovation to underline some aspects of the story told: like fate, loneliness and the protagonist's illusion of having the total control on his life. It is more the visual strategies than the words which show us the limits and tragedy of each character. Many are the shots which start from a close-up of the main characters, in an almost stationary position, and they then broaden to reveal the landscape, often boundless, where they disappear into. In this way, the advantage of seeing the whole context of the action is given only to the viewer while the protagonists of the scene are metaphorically trapped in their emotions, and only being able to have a limited outlook of the situation [6].

During our research, from the observation of these types of shots, we noticed an added harmony emphasised by the composition of the scene. Everything is more elaborate than what one may think… In fact, perspectives, symmetries, and pictorial recalls are enriched from the introduction of the golden section. As well as in the opening scene of *A Clockwork Orange*, the case repeats itself and the golden number also pervades the compositions within the frames of *Barry Lyndon* (Figs. 9, 10, and 11).

Here are some visual examples with related comments.

The three studied frames are extrapolated from the *Barry Lyndon* Blu-ray edition. Through *Adobe Illustrator* graphic software, we analysed the proportions of the elements inside the images, with special attention to the composition of the people

Fig. 9 The presence of the golden rectangle and spiral in a first shot of Barry Lyndon by **Stanley Kubrick**. *Credit* © 1975, Warner Bros

Fig. 10 The presence of the golden rectangle and spiral in a second shot of Barry Lyndon by **Stanley Kubrick**. *Credit* © 1975, Warner Bros

Fig. 11 The presence of the golden rectangle and spiral in a third shot of Barry Lyndon by **Stanley Kubrick**. *Credit* © 1975, Warner Bros

sitting around the table. As in the case of *A Clockwork Orange*, we found that they are perfectly circumscribed by a rectangle of golden proportions. In particular, the resolution of the three frames is 1920 × 1072 px while the overlay rectangles have a size of respectively 734 × 455 px (Fig. 9) , 646 × 400 px (Fig. 10) and 651 × 404 px (Fig. 11).

So, in all three cases, the proportions of the sides of the rectangle are approximately 1.61 and therefore associated with the golden number (1,6180…). Consequently, the characters appear organised within golden rectangles.

Furthermore, there are several factors in common. Firstly, the scenes show indoor life situations of Lyndon's home where the protagonists are sitting around a table in an almost static state. Secondly, they begin with a fixed full shot of the room where the elements of the setting are already in their position and remain perfectly placed for the entire shot length, just as if it were a painting of that time. Finally, the "motionless situations" are followed by closer shots in which the subjects suddenly revive from their torpor and the action kicks off.

This specific structuring of the scenes suggests that the film director was able to arrange all the elements of the composition as he desired. In particular, the greatest focus seems to be placed on the human figures, harmoniously and precisely connected according to the evolution of the logarithmic spiral that ends right on the protagonist of the scene.

So, we can observe how, despite the fixity of the scene, a beautiful dynamism is created by the presence of the golden section within the composition of the image.

In the geometric constructions shown in Figs. 12, 13, and 14, the iteration process of the golden section is summarised for each shot, from the golden rectangle to the spiral.

Fig. 12 Visual studies of the golden ratio from the initial immobility framed by a golden rectangle and the finishing dynamism that takes place in a golden spiral for the first shot of Barry Lyndon by **Stanley Kubrick**. *Credit* © 1975, Warner Bros

Fig. 13 Visual studies of the golden ratio from the initial immobility framed by a golden rectangle and the finishing dynamism that takes place in a golden spiral for the second shot of Barry Lyndon by **Stanley Kubrick**. *Credit* © 1975, Warner Bros

Fig. 14 Visual studies of the golden ratio from the initial immobility framed by a golden rectangle and the finishing dynamism that takes place in a golden spiral for the third shot of Barry Lyndon by **Stanley Kubrick**. *Credit* © 1975, Warner Bros

These can only be suggestions, but the scrupulousness with which these bodies merge with each other and with the surrounding environment certainly cannot be just given by chance. Here is an excerpt from an interview taken from the documentary *Room 237*:

> One can always argue that Kubrick had only some or even none of these in mind. But we all know postmodern film criticism that author intent is only part of the story of any work of art. And those meanings are there regardless of whether the creator of the work was conscious of them.

5 Conclusions

There is not much to add to what we have just said and illustrated. The images must capture our attention and make us reflect. We think there is no doubt that we have picked out in some takes of *Barry Lyndon* the golden section in fictitious rectangles that limit, on the dark and deep background, static groups of people arranged to generate in its entirety "cinematic paintings". The fact that the arrangement of these characters follows a spiral evolution, repetitiveness of the golden rectangle, is also not questionable.

Now the matter is: was Kubrick completely aware of it or is it his artistic sense that unconsciously guided him? Whatever the answer is, the eclecticism and completeness of the film director make us conclude that mathematics is art.

> Nature is ruled by mathematics, and masterpieces of art are in consonance with nature; they express the laws of nature and themselves proceed from those laws. Consequently, they too

are governed by mathematics, and the scholar's impeccable reasoning and unerring formulae may be applied to art.

Le Corbusier—*Le Modulor*

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Filmography

Paths of Glory by **Stanley Kubrick**, 1957
Dr. Strangelove by **Stanley Kubrick**, 1963
2001: A Space Odyssey by **Stanley Kubrick**, 1968
A Clockwork Orange by **Stanley Kubrick**, 1971
Barry Lyndon di **Stanley Kubrick**, 1975
The Shining by **Stanley Kubrick**, 1980
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Eyes Wide Shut by **Stanley Kubrick**, 1999
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Room 237 by **Rodney Ascher**, 2012

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Roses in Architecture: One Symbol, Different Objects

Franca Caliò, Caterina Lazzari, and Elena Marchetti

Abstract Let's look carefully at some objects of Architecture and we'll find that in them we frequently can identify the same geometric shape. It is a particular flat geometric shape that can be presented in various ways, always maintaining regularity and dynamism: the rose window or rosette. We will stop our attention on two architectures: the Piazza del Campidoglio in Rome and the skyscraper nicknamed «The Gherkin» in London. History, times and architects are very different and yet, looking at them from particular observation points, we come to find in them a rose window of the same type and shape. Our work aims to intrigue the observations that are made, but also to suggest to a teacher to take the opportunity, following the succession of ideas, to talk to a student in geometric terms of rosettes, of geometric shapes that compose them, but also of analyze its symbolic and aesthetic meaning in the observed architectures.

Keywords Architecture · Parametric Geometry · Rose window · Symmetry

1 Introduction

Milan Expo 2015. The symbolic sculpture (Fig. 1) of the popular and highly successful event is the attraction of the occasion.

It is the *Tree of Life* that over the centuries has represented humanity in various ways.

Its shape is closely related to its meaning. The base, the roots of the tree and the terminal part, the branches of the tree, take on the appearance of a regular texture

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Fig. 1 Tree of Life (Milano, Italy). Photo taken by the authors

formed by elements similar to leafs and to water drops. The base and the top connect together in a vortex-like movement that seems to describe not only the shape of the tree, but also its slow and regular growth [1].

In Fig. 2 a virtual reconstruction highlights such sensations.

Let's look at it from above. The shape that appears to us is the one we see in the figure at the base of the tree. A schematic crown appears, formed by large leaves which, partly overlapping, are distributed in a regular corolla.

By virtually reconstructing the reflected image, we highlight an object called a rose window (Fig. 3).

Let's first try to give the idea of a rosette and then we will analyze our reconstruction to understand if it is a rosette and what type.

In current language, the rose window is a decorative element that, generally, is found on the facades of Romanesque or Gothic churches (Fig. 4), and is characterized by stylized floral elements that repeat themselves "chasing each other", but remaining within a circumference. In this sense, its shape is interpreted as a Christian symbol of the cyclical human life related to the infinity of God represented by the circle.

In mathematical language, however, the term "rose window" takes on a very precise meaning which will then characterize even very different objects.

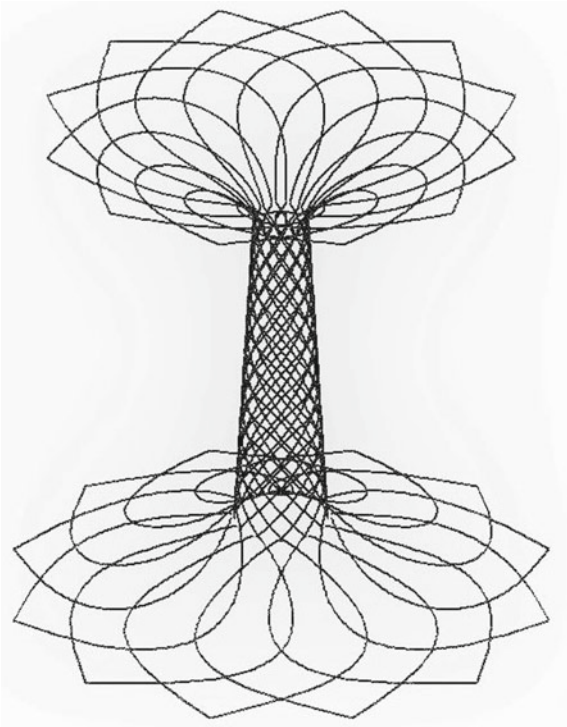


Fig. 2 Virtual reconstruction of the Tree of Life. By the authors

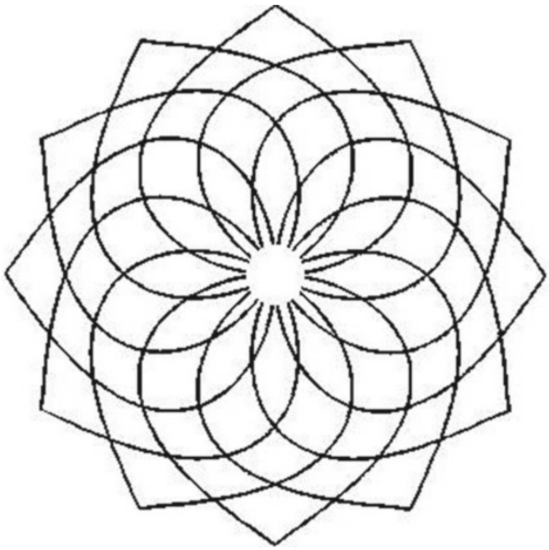


Fig. 3 Virtual projection of the Tree of Life. By the authors

Fig. 4 Rose window on the facade of the Church of San Francesco in Capranica (Viterbo, Italy).

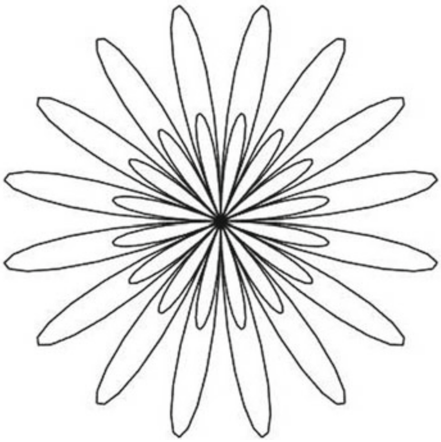


Fig. 5 The rose of the Grandi, from the name of the mathematician who provided the equations in the mid-eighteenth century. By the authors

The term "rose window" can be interpreted as "large rose" and therefore in a broader sense as a "large stylized flower"

Therefore, moving in geometry, we take as an example of "a flower" the geometric figure known in mathematics as a *Rose* or *Rhodonea Curve*.

A rose is represented in Fig. 5. By observing it, one can verify that it has characteristics of central symmetry and can be inscribed in a circumference. Based on these considerations, we take up a possible definition of a geometric rose.

First of all, remember that in geometry a figure has symmetry if there is a transformation that maps the figure onto itself. Furthermore, we remember that rotation and reflection are the transformations that, applied to a figure, maintain its dimensions [2, 3].

Hence it is deduced that a figure of finite dimensions, if it has some characteristic of symmetry, can be built starting from one part of it and applying a combination of rotations and reflections to it.

With these basic concepts in mind, let's try to define rosettes.

If a flat figure with central symmetry can only be obtained by means of n (with $n = 2, 3, \ldots$) rotations around the center of symmetry, with a rotation angle equal to $1/n$ of a turn angle, it is called a *cyclic rose.*

In Fig. 6 we have an example of a cyclic rose. It is one of the best known symbols of the Milan Cathedral: the "Raza" (or a radiant sun that crowns Biscione, heraldic symbols of the Visconti), one of the main elements of the very famous windows that is located inside the central window of the apse just behind the main altar.

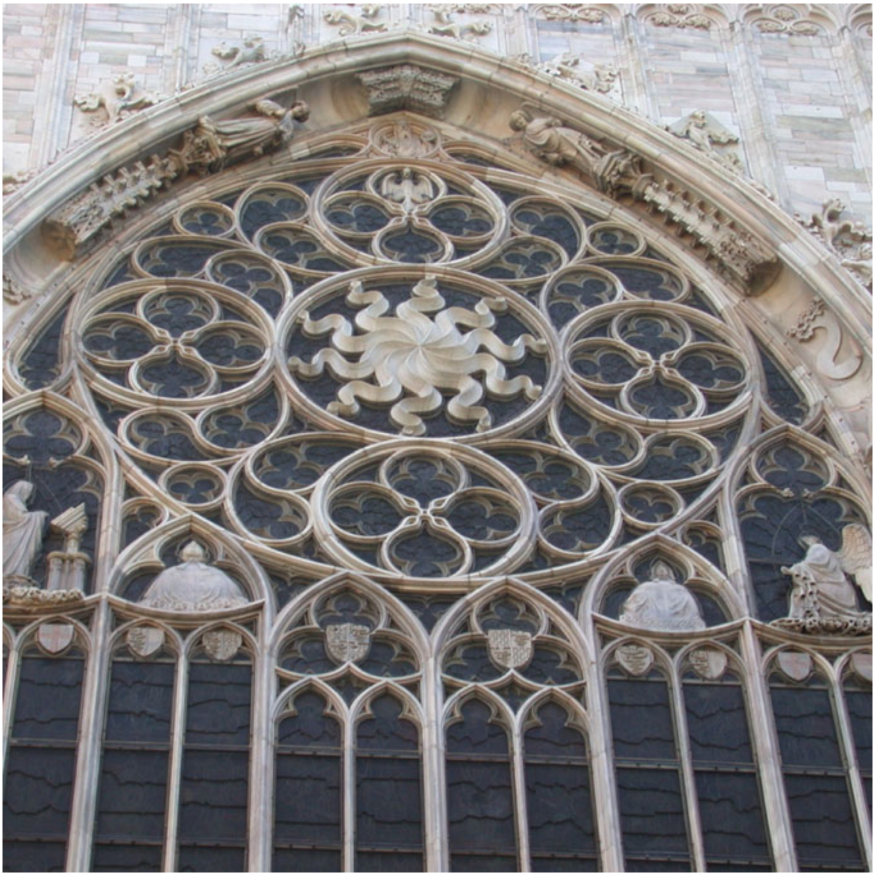


Fig. 6 Raza in Milan Cathedral. Photo taken by the authors

Figures 7 and 8 show a geometric reconstruction of it. In particular, successive rotations are applied to the base “petal” of Fig. 7, obtaining the rose of Fig. 8.

If instead a flat figure with central symmetry can be built by a combination of n (with $n = 2, 3, \ldots$) rotations around the center and n (with $n = 2, 3, \ldots$) reflections around axes of reflection passing through the center, it is called *dihedral rose*.

In Fig. 9 we have an example of a dihedral rose. It is another rose window of a stained glass window of the Milan Cathedral located in the apse of the southern transept.

Figures 10 and 11 show a geometric reconstruction of it. It can be seen that in the dihedral rose, in addition to the center of symmetry, there are also axes of symmetry, due to the reflection component of the global transform.

Let’s now look at the shape of Fig. 3: it is a flat figure with central symmetry. It is made up of twelve “leaves” which form a corolla contained in a circumference.

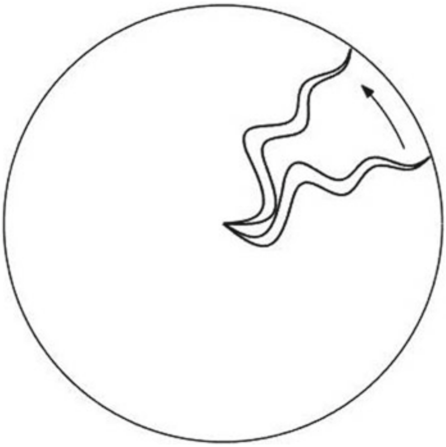


Fig. 7 Partial construction of a cyclic rose window. By the authors

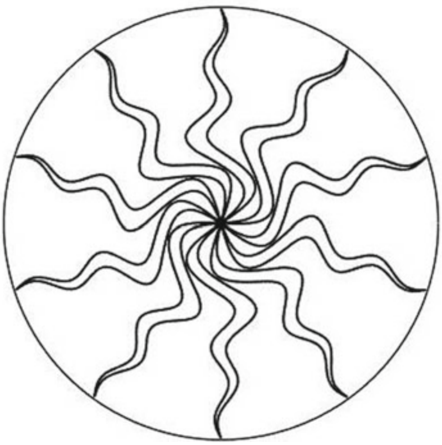


Fig. 8 Construction of a cyclic rose window. By the authors

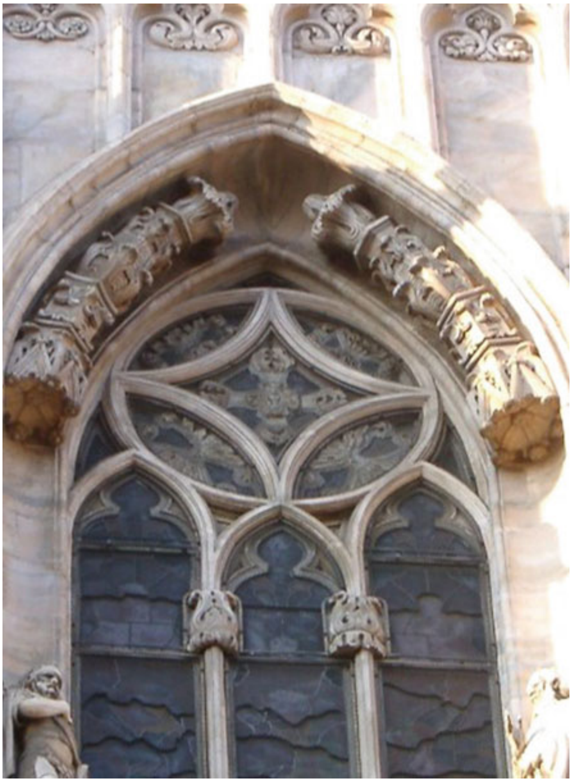


Fig. 9 Rose window in Milan Cathedral. Photo taken by the authors

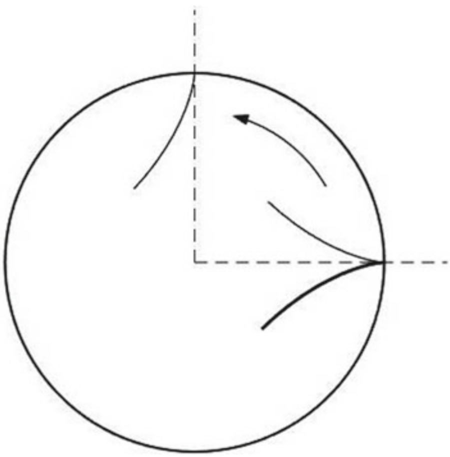


Fig. 10 Partial construction of a dihedral rose window. By the authors

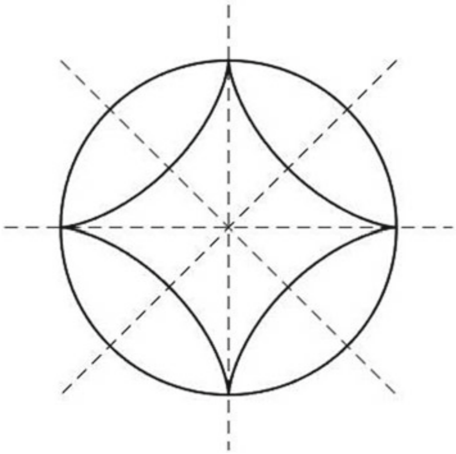


Fig. 11 Construction of a dihedral rose window. By the authors

Each leaf is symmetric with respect to an axis. The 12 axes of symmetry of the form converge in a point, center of symmetry (Fig. 12).

It is, therefore, a geometric form that can be generated by means of twelve "half-leaf" reflections with respect to the axes of symmetry and twelve rotations of amplitude one twelfth of a turn angle around the center of symmetry.

At this point we are able to conclude that the shape in Fig. 3 can be interpreted as a dihedral rose, in which each leaf is geometrically half of a Bernoulli lemniscate [2].

But what is a Bernoulli lemniscate? The lemniscate (Fig. 13) is a closed plane curve, with a double point in the center and a double axis of symmetry, which immediately recalls the mathematical symbol of infinity, thus adding symbolic charge to that intrinsic in the rose window.

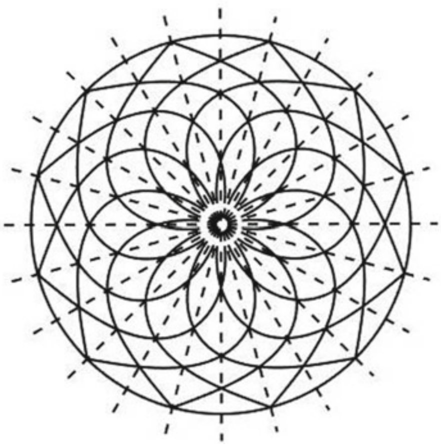
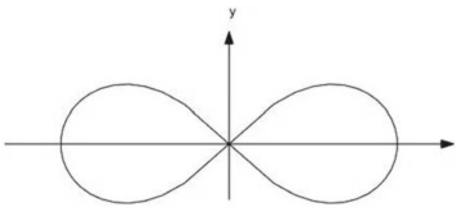


Fig. 12 Symmetries in the Virtual projection of the Tree of Life. By the authors

Fig. 13 Bernoulli lemniscate. By the authors



This particular dihedral rose will be an element of interest and meaning in Architecture.

In this work we offer two examples: two views from above of architectures designed by two architects: Michelangelo and Foster, temporally distant centuries and unmistakable, but united by the same symbol.

These are two zenith views. The first of the Piazza del Campidoglio in Rome, the second of Foster's famous *Gherkin* in London.

We have two roses in two completely different places and also different from the classic rose windows found on the front of the churches.

Let's try to observe them, find the geometric shape and interpret its symbolic meaning.

2 Observe and Combine

2.1 Piazza Del Campidoglio in Rome

Ruining the Capitoline Hill, the most famous of the seven hills of Rome, in a despicable state of abandonment, to the point of being invaded by grazing goats, Pope Paul III, around 1540, commissioned Michelangelo to restructure of the square above.

Michelangelo proposed and obtained a square open frontally and slightly trapezoidal in shape, maintaining the objective of turning the square, rather than towards the Roman Forums which were also in a state of neglect, mainly towards San Pietro, where the vitality of the city was developing again through Renaissance architecture.

Palaces designed or redesigned by Michelangelo overlook the square. Today the buildings house the Capitoline Museums and the Municipality of Rome (Fig. 14).

In the square, Michelangelo designed the famous stairways and, in place of the previous dirt road, the pavement which was actually completed, due to the slowness in carrying out the works, in modern times in compliance with the Michelangelo design (Fig. 15). In fact, only in 1940, four centuries later, Michelangelo's original idea was taken into consideration again and the square was paved according to Michelangelo's design, who had placed the famous gilded bronze statue of Marcus Aurelius in the center (currently placed in the Museums Capitoline and replaced by a copy).

The flooring is in dark stone with a white stone design. The motif represented is commonly defined as a twelve-pointed star inserted in an oval.

Fig. 14 Piazza del Campidoglio in Rome.

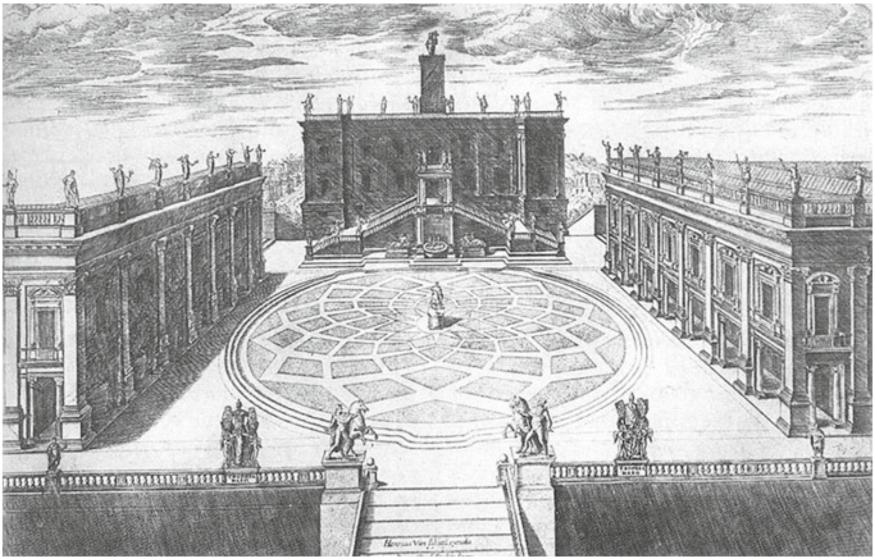


Fig. 15 Arrangement of the Campidoglio square by Michelangelo in an etching by Étienne Dupérac, 1569

Through this motif the pavement had to represent the rediscovered centrality of the place, considered sacred by the ancient Romans.

Michelangelo chose precisely for his design the symbol of life, which should have been reborn in that place.

We find in the motif (Fig. 16) the same idea that we presented at the beginning of the chapter. Here too, a sequence of portions of lemniscate chase each other by

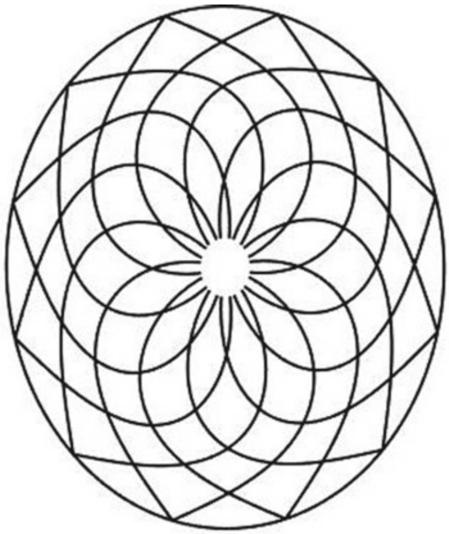


Fig. 16 The decorative motif of the floor of the Piazza del Campidoglio. By the authors

rotating, with a variant: the outline becomes an ellipse. In other words, the geometric rose window undergoes a deformation: an expansion in one direction.

The famous Michelangelo motif of the pavement of the square also appears on the two euro coins issued in 2007 by all Eurozone countries in commemoration of the fiftieth anniversary of the founding of the European Economic Community, whose treaty was signed in the Palazzo dei Conservatori on the Capitol.

2.2 *Skyscraper 30 St. Mary Axe London*

We change the environment and historical period. Let's go to London and observe an architecture designed by the studio of a contemporary architect: Norman Foster, whom we see at work in his hometown.

The building we are observing is London's 30 St Mary Ax Skyscraper, located in the City, informally better known as *The Gherkin* (Fig. 17).

The construction, 180 meters high, completed in 2004 and currently occupied by an insurance office, stands on the site of a building which was badly damaged in 1992 by a bomb detonated by the IRA. Due to the damage suffered by the building, it was decided not to restore it, but to build a new and completely different one, leaving designers free, in derogation of the strict London regulations.

The result is an original, but not bulky construction that offers a spacious, practical and bright interior.

The materials and organization of energy production make it an environmentally sustainable building.

Let's look at the shape. The nickname comes from its characteristic torpedo shape.

Fig. 17 Skyscraper 30 St Mary Axe.

Our geometric eye, however, prompted us to make two considerations.

First of all, we observe that it is a surface of rotation of a portion of circumference, around a vertical axis to which a cone-shaped surface is superimposed, closed in turn by a circle which is the shape of the closing lantern. In this sense, we propose (Fig. 18) a reconstruction [4].

Secondly, following the darker lines that cross the surface, we grasp an analogy and, at the same time, a profound difference with the Tree of life. The strip, following an archimedean spiral pattern projected on the surface of the building, wraps like the wooden components of the trunk of the Tree of life. Take a good look at the two figures!

From this consideration the difference emerges: the Tree opens in the roots and in the crown, the *Gherkin* closes at the base to adapt to the City, a place formed by

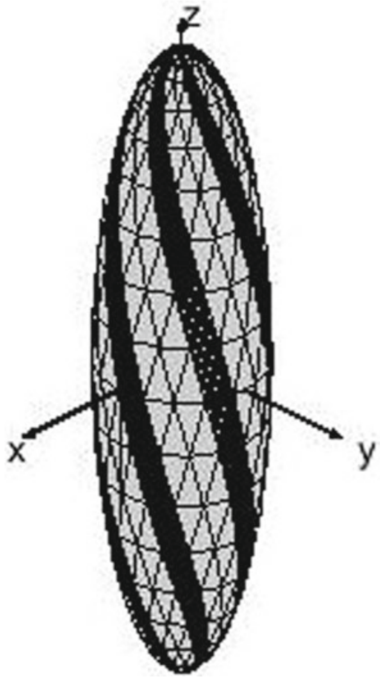


Fig. 18 Virtual reconstruction of Skyscraper 30 St Mary Axe. By the authors

narrow streets on which numerous buildings overlook and closes at the top "in a conical shape" so as not to offer the wind a too large surface.

The observation from above unites them: both offer a projection in an interesting "star" rose window, which in turn unites them to the view from the top of the Piazza del Campidoglio (Fig.15 and Fig. 19).

Our simulation (Fig. 20) identifies a rose obtained by rotating a half of Lemniscate by a rotation angle equal to one eighteenth of a round angle.

Has the architect deliberately researched this effect? Whatever the answer, this vision adds, in our opinion, an additional charm to a building considered among the most famous in the world.

We started by observing the Tree of life. It is the symbol of life. By extending this idea, the Piazza del Campidoglio takes this symbol to offer itself as the center of the new Renaissance life and the architectural object of Foster, which still offers this symbol, stands in a place where it represents the life that is rebuilt and renewed.

Fig. 19 The 30 St Mary Axe skyscraper seen from above. Courtesy of https://modulo.net/it/realizzazioni/30-st-mary-axe-sede-swiss-re

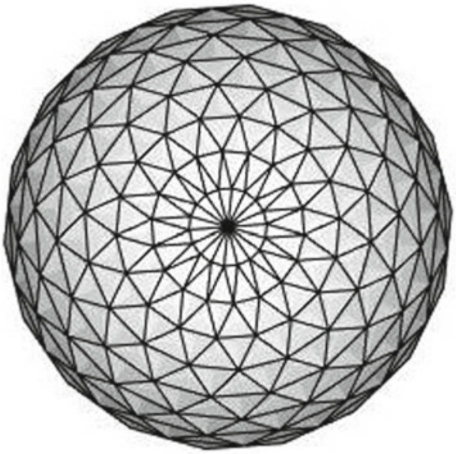


Fig. 20 Simulation of the top view of the 30 St Mary Axe skyscraper. By the authors

3 Conclusions

Beyond the curiosity of the observation that relates different objects by seeking their analogy, this work wants to leave a message: a vision, even superficial, but strongly interdisciplinary, of any topic helps to stimulate interest in every direction [4].

If the user is a student, he can take the opportunity to explore topics from different sectors [5]. This case can give rise to a more rigorous interest in certain aspects of geometry, while observing objects of architecture with particular attention, or it can stimulate looking for examples and applications in reality, theoretically studying parts of geometry or it can still complete each observation by paying attention on the symbolic meanings, which contribute to a participatory and emotional vision.

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The Fibonacci Numbers: A Link with Geometry, Art and Music

Giuseppe Conti and Raffaella Paoletti

Abstract In this paper we will show some properties of the Fibonacci numbers from the mathematical and the historical points of view. We will focus on their applications concerning art, music and geometry, especially in connection with the golden section. In fact, these important numbers, known in the West since the Middle Ages as a solution of the famous problem of the growth of rabbits, found numerous and important applications in many different fields.

Keywords Fibonacci numbers · Golden section · Fractals

1 Introduction

Fibonacci[1] numbers have numerous applications: in addition to mathematics itself, they are also found in fractals, in electronics (the Fibonacci heaps in Intel processors), in botany: many aspects of phyllotaxis, as Leonardo da Vinci and Kepler had already guessed, can be explained with the properties of Fibonacci numbers (see [1, 5, 7, 13]). They are also found in novels and poetry (see [11]).

We must take into account that these numbers were known in ancient India from the second century BC, as they provided the number of verses formed by the combination of long and short syllables; later, in the sixth century AD, the rule underlying the construction of these numbers was found (see [15]). Fibonacci probably knew these numbers through the mathematical works of the Arabs, who, in turn, had learned much of their knowledge from the Indians; knowing them, he used them to solve the famous problem of the number of pairs of rabbits (see [8]). Fibonacci numbers also find applications in economics; for example, R. N. Elliot (1871–1948) hypothesized

[1] For questions regarding Fibonacci and his numbers see [8, 10].

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that financial markets do not move randomly but following a very precise rule that can be explained using Fibonacci numbers (see [14]).

In this paper we want to show some applications of Fibonacci numbers to geometry, art and music.

2 Fibonacci: Geometry and Art

The Fibonacci sequence is strictly related to the golden section.[2] The first who sensed this relationship was Kepler (1572–1630), who probably did not know Fibonacci's work.[3]

In the Palazzo Vecchio in Florence the axis of symmetry of the tower does not coincide with the one of the façade of the building.

As Fanelli observed (see [9]), the axis of symmetry of the tower divides the façade of the building according to the golden proportions.

Looking at Fig. 1, we can also note that the number of façade windows, located at the top under the battlements of the building, are 13 and the number of corbels, including those at the ends, are 21 (they are both Fibonacci numbers). Furthermore, the point of the segment that divides the width of the façade according to the golden ratio is located near the fifth window from the right and the octave one from the left (ratio 5/8) and near the eighth corbel from the right and the thirteenth one from the left (ratio 8/13).

We note that the Fibonacci numbers are also on the sides of the building. In fact, the number of the side windows is 8 and the number of the corbels, including those at the ends, is 13. From this fact it can be deduced that the plan of Palazzo Vecchio is very close to a golden rectangle, as it can be seen from the relief shown in Fig. 2.

Finally, Fig. 1 shows that even the north side of the building has the shape of a golden rectangle and that the golden proportions are also found in the geometry of the tower.

[2]The Fibonacci numbers and the golden section are in close contact also with a very important curve: the logarithmic spiral. For properties and applications of this curve see [6].

[3]Kepler wrote in 1608 (see [11, p. 226]): "This proportion that today's people call divine is designed in such a way that the two minor terms of a nascent series taken together form the third and the last two, added together, form the next term, and so on indefinitely, since the same proportion remains unchanged the more you go from number 1, the more perfect the example becomes. Let 1 and 1 be the smaller terms; by adding them, the result is 2; add the previous 1 to this, and get 3; let's add 2 and get 5; let's add 3 and we have 8; 5 and 8 give 13; 8 and 13 give 21. As 5 stands at 8, so roughly 8 stands at 13, and as 8 stands at 13 so roughly 13 stands at 21". The mathematical proof of the convergence to the golden section of the ratios of a Fibonacci number with its successor is due to the Scottish mathematician Robert Simson (1687–1768).

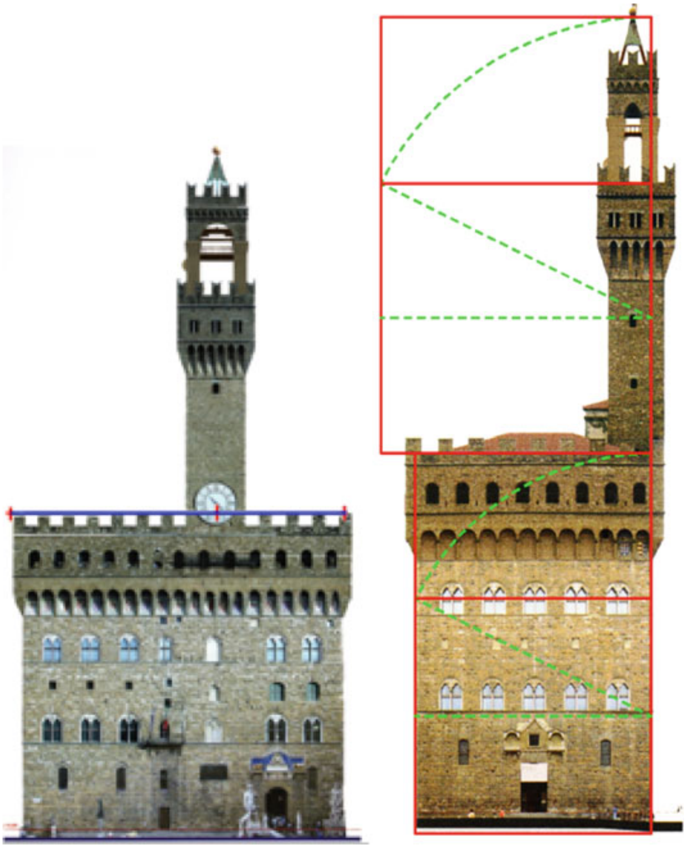

Fig. 1 Front elevation and west elevation of Palazzo Vecchio. The elevations of Fig. 1 are by Prof. G.Mele and are in [2]

Giuliano Maggiora [12, p. 150] states: "Is this due to a calculation or to an instinctive choice? It is difficult to accept the first hypothesis, because the golden number, which is not an integer, lends itself poorly to being used as a measure".[4]

We think that, in this case, this choice was desired. We must keep in mind that, first of all, the golden ratio can be approximated with fractions, each of them having a Fibonacci number as numerator and a consecutive one as denominator; furthermore, the geometric construction of the golden section can be done using the ruler and the compass, thus overcoming the problem of its irrationality.

The internal dimensions of the dome of Santa Maria del Fiore, according to the original plan of 1367, were 144 *arms*[5] (about 84 m) from the floor of the Cathedral to

[4]The presence of the golden section in art, botany, music and other disciplines is the subject of a profound debate among scholars. Undoubtedly this presence has been very often exaggerated, but it is undeniable that it frequently exists, especially for aesthetic reasons (see also [4, 11]).

[5]The Florentine *arm* ("braccio a panno") measured 0.5836 m.

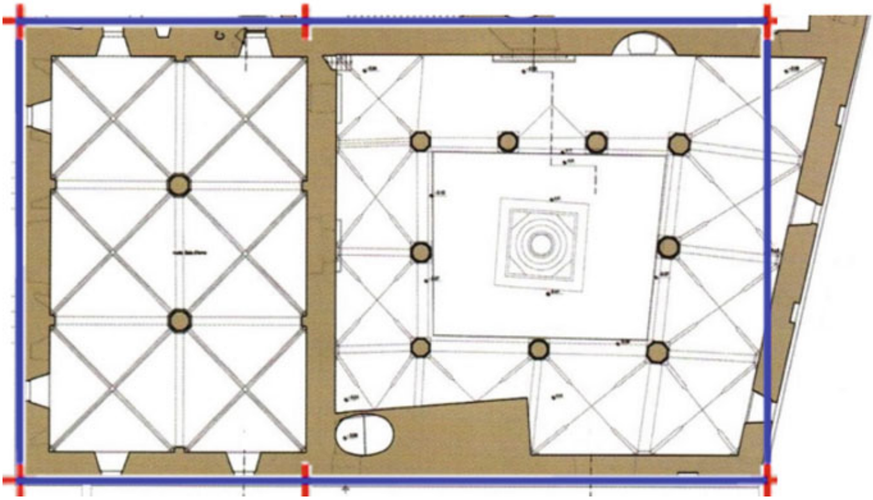

Fig. 2 Palazzo Vecchio's plan. The relief in Fig. 2 is taken from [2]

the top of the dome; the base of the dome was 89 *arms* (about 52 m) from the floor; the dome measured 55 *arms* (about 32 m) from its base to the top. We can observe that all of these are Fibonacci numbers.

Subsequently the drum under the dome, which measured 17 *arms* in height (about 10 m) was raised by about 5 *arms*, becoming about 13 m.[6]

In the left part of Fig. 3 we removed the part of the drum not foreseen in the original plan of 1367; since the ratio 55/89 is very close to the golden number, the dome is the golden section of the underlying part.

In the right part it's shown the dome after its construction; of course, the internal part no longer follows the golden ratios which, on the other hand, are respected externally.

Another very interesting example of the presence of Fibonacci numbers in art is found in the theater of Epidaurus.[7] Its *cavea* is made up of two sectors, divided by the service passage called *diazoma*: the lower one has 34 rows of seats and the higher one has 21 rows, for a total of 55 rows. Again, Fibonacci numbers are involved. The ancient Greeks almost certainly didn't know these numbers; however, we can reasonably assume that the purpose was to use the golden ratio, which, in fact, is approximated by 21/34 with an error less than 0.0004.

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[6]The height, width, shape of the dome and the insertion of the 10 m drum at its base had been decided in 1367 by a committee of "eight teachers and painters". In 1420 the master builder Giovanni D'Ambrogio decided to raise the drum three meters to allow the insertion of an external balcony (see [3, p. 115]).

[7]Epidaurus theater is located in the Argolis region, in the Peloponnese. Its construction, work of the architect Policleto the Younger, dates back to the fourth century BC. Epidaurus theater is also famous for its excellent acoustics.

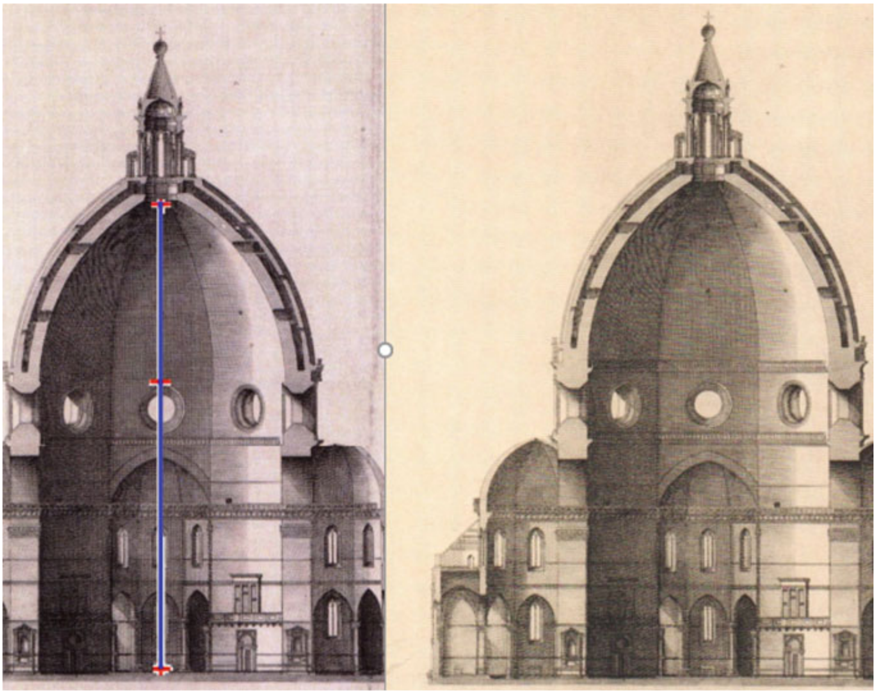

Fig. 3 The dome in the initial project (drum about 10 m high) and how it looks after its realization (drum about 13 m high). The drawings of Fig. 3 are taken from [16]

An Italian scholar, Professor Pietro Armienti of the University of Pisa (1957–2019), showed that some elements of the ornament on the façade of the church of San Nicola in Pisa follow the Fibonacci sequence. Armienti stated that, denoted by r the radius of the smallest circle present in the inlay, the other circles have radius $2r$, $3r$, $5r$, $13r$, $21r$, $34r$; finally, the circle surrounding the inlay has radius $55r$. According to the Professor, this inlay is a kind of abacus to represent irrational numbers, such as the golden ratio.

Starting from 1970, Fibonacci numbers are present in numerous works by Mario Merz[8] and sometimes they are connected to spiral curves.

One of his works, a blue neon light in the shape of a logarithmic spiral, is now in the Macro Museum in Rome (previously, in 2003, it had been installed in the Forum of Caesar). In the Vanvitelli station of the Naples metro line 1, since 2004 there is

[8]The painter and sculptor Mario Merz (1925–2003) was an exponent artist of Poor Art, an artistic movement born in Turin in the mid-sixties. Fibonacci's numbers and the spirals that derive from these have been sources of inspiration for his works. According to Merz, the spiral represents the organic growth of nature.

another work of the artist: it is a spiral of blue neon light, along which the Fibonacci numbers from 1 to 55 are inserted.[9]

Already in 1992, Merz had installed his work *The Philosophical Egg* in Zurich Central Station. It is a spiral of red light along which there are placed some Fibonacci numbers and silhouettes of deer and birds of prey.

In 2000 the artist, in the occasion of the *Luci d'artista* event, placed on the Mole Antonelliana in Turin an illuminated strip in which the first sixteen Fibonacci numbers were inserted. They light up every evening from dusk to midnight in winter and to one in summer. This work was called *Flight of Numbers.*

Merz used this idea in other works.

On the ancient medieval walls of San Casciano Val di Pesa (Florence), we can see some Fibonacci numbers made with blue neon light tubes which, together with an aluminum deer (initially it was stuffed), form a creation of Merz, made in occasion of the 1997 exhibition *Tuscia Electa*; unfortunately these numbers (55, 89, 144, 233, 377, 610) are rarely lit.

On a wall of the Palazzina of Forte Belvedere in Florence, the Fibonacci numbers from 1 to 55 appear with blue light but are accompanied by a caiman.

They are also illuminated in blue light the Fibonacci numbers from 1 to 987 that Merz has placed on the chimney of a former brewery (now Center for Art *Light International*) in Unna, Germany.

In 1994, the artist installed the Fibonacci numbers from 1 to 55 on the chimney of the Turku power plant (Finland), but in this case they glow red at night.

Of the same color are the Fibonacci numbers, from 1 to 89, located in the center of the Barceloneta, a seaside district of Barcelona. Merz inserted the above numbers under the pavement of a pedestrian area, protected by armored glass. We note that the distance between the windows where the numbers are inserted is not constant but increases proportionally to the value of the number inside. This work was inaugurated in 1992 in the occasion of the Olympic Games.

In 1972 Merz created the work *Without title (a real sum is a sum of people)*; it is a succession of eleven black and white photographs, each of them associated with a blue neon Fibonacci number (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55).[10] The characters in the photos are artists and art dealers from Milan. All the pictures were taken from the same position except the last one, that was taken from a different position. The number of characters in each photograph is the same as the number associated with the photo; the first photo is without characters. This work is placed in the permanent collection of the Museum of Modern Art in New York.

We conclude this roundup of Merz's works remembering his 1972 *Igloo di Marisa* (the name of his wife); it is a hemisphere-shaped structure, built with elements made

[9]The stations of metro line 1 in Naples welcome about 200 works of art created by world-renowned artists. They arose from a project by (at the time) mayor Bassolino who was advised by the art critic Achille Bonito Oliva. The first stations were inaugurated in 2001.

[10]Fibonacci sequence can be considered to start both from numbers 1 and 0; in the last case the sequence becomes 0, 1, 1, 2, 3, ….

of foam rubber, covered with white fabric (poor art) on which the Fibonacci numbers from 8 to 144 in blue neon are affixed.

The *McGinley Fountain* by Eric Ernstberger, in Purdue University (West Lafayette, Indiana) is based on Fibonacci's numbers: in this work there is a waterfall formed by 5 stone steps and above the waterfall there are 5 glasses, supported by a metal armor, on which the front garden is reflected, and the garden's path has the shape of a Fibonacci spiral. In addition, two stone benches in this path consist of 2 elements and another one of 3 elements. In the same Discovery Park it's located the *VOSS Model*, representing the solar system: a path, once again in the shape of a Fibonacci spiral, leads to a sculpture representing the sun and along this path there are models of the planets with explanations panels. The model is in scale: every foot along the spiral corresponds to about 5.4 million miles in real space!

The *Fibonacci Fountain* designed by the American sculptor Helaman Ferguson[11] is another fountain based on Fibonacci numbers. It is located in the Fibonacci Lake in the Maryland Science and Technology Center in Bowie and it consists of fourteen cannons launching water jets high into the air; the cannons are placed along a straight line at intervals proportional to the Fibonacci numbers, hence the name.

The same artist created a sculpture he called *Fibonacci box*. The choice of this name is because the "box" recalls a famous and apparent geometric paradox, based on a property of Fibonacci numbers, the identity of Cassini.

The covering of the splendid curved sail façade of the new *Center for nanoscience and quantum information* of the University of Bristol (built by Willmott Dixon Construction) is based on panels whose horizontal dimensions are proportional to the following sequence of Fibonacci numbers: 1, 1, 2, 3, 5, 3, 2, 1, 1.

The floor of the *Fibonacci Terrace* located in the *Science Center* of Singapore (2010) is also composed of elements that form figures whose sides are proportional to the Fibonacci numbers.

Portuguese artist Marisa Ferreira is the author of the work of art *Rear window* (from the same title of the 1954 movie directed by Alfred Hitchcock) on the façade of Oslo station (2016). It includes 59 vertical stripes based on coloured sequences, the lengths of which follow the Fibonacci sequence. This artist also created a series of paintings entitled *Homage to Fibonacci* (2012).

3 Fibonacci and Music

Fibonacci numbers can also be found in some music-related issues. Unfortunately, in this context sometimes there are statements made without any scientific validity. The most popular one, which we would immediately deny, concerns the musical intervals. Each musical interval (pair of two notes) is associated with a fraction which is given

[11] Many of Ferguson's works, in stone and in bronze, have themes related to mathematics. We cite, for example: *Fibonacci Tetrahedron, Aperiodic Penrose, Esker Trefoil Torus, Torus with Cross-Cap, Umbilic Torus, Umbilic Toroid in Desert Limestone.*

Table 1 Musical ratios obtained exclusively by Fibonacci numbers

Interval	Unison	Octave	Fifth	Major sixth	Minor sixth
Ratio	1/1	2/1	3/2	5/3	8/5

Table 2 Musical ratios involving also non-Fibonacci numbers

Interval	Major 2nd	Minor 2nd	Major 3rd	Minor 3rd	Fourth	Seventh
Ratio	9/8	16/15	5/4	6/5	4/3	15/8

by the ratio of the frequencies of the two notes (the largest frequency is put to the numerator and the smaller one to the denominator).[12]

Assuming that the first note of the interval is a C, in the natural scale we have that the ratio of the frequencies between the C of the next octave and the C taken as a reference (octave interval) is 2/1. The relationship between the G and C (fifth interval) is 3/2. Table 1 shows the ratios of some musical intervals.

On the other hand, there are other fundamental musical intervals whose ratios are not made exclusively of these numbers (Table 2).

We can see that some musical ratios are represented by Fibonacci numbers but not all of them! We believe that this is due to the fact that the musical ratios are formed by small integers and in the first ten natural numbers there are five Fibonacci numbers.

Again, the common observation that an octave of a piano keyboard consists of 8 white and 5 black keys (so 13 in total!) is only a consequence of the choice of the chromatic scale. As the Russian musician and composer Sofia Gubajdulina (born in 1931) pointed out, counting intervals taking a semitone as unit cannot give an interesting connection with Fibonacci numbers since the equal temperament is an artificial construction while the Fibonacci sequence is related with natural growth.

So Fibonacci numbers don't describe musical intervals but, however, they can enter musical issues in various ways.

[12]For questions regarding musical relationships and their links with mathematics and architecture, see [18].

Let's consider the *Vasilissa ergo gaude*[13] of the composer Guillaume Dufay.[14] In this study we take into account the number of notes of the musical composition.[15] The whole composition is made up of 306 notes; the first part is made up of 189 notes and the second part of 117 (189/306 = 21/34, 117/189 = 13/21). The first part is composed of a 72-note vocal prelude and a first *talea* (rhythmic pattern repeated throughout the composition) of 117 notes (72/117 = 8/13). The second 117-note part is made up of a second 45-note *talea* (45/117 = 5/13) and a 72-note ending (72/117 = 8/13). We can notice that the obtained ratios are formed by Fibonacci numbers; since the Fibonacci sequence tends to the golden section,[16] the structure of the motet is thus divided according to the golden proportions. Considering the messa *Ecce ancilla domini* by the same author, the mathematician and musician Sonia Cannas[17] observed that the ratio between the numbers of notes of the Credo (204) and of the Kyrie (126) it's equal to the golden number; moreover, the three sections of the Kyrie consist, respectively, of 48, 33, 45 notes so that the first part, the second with the third and the whole Kyrie are organized according to the golden proportion 48:78 = 78:126.

Recently, many scholars claimed to have found the golden section in the works of many old composers (Bach, Mozart, Beethoven …) but sometimes their analysis seems to be "forced". For example, let's consider the piano sonatas by Mozart: many scholars took the use of the golden section for granted but John Putz, a mathematician from Alma College, found some "reasonable" results only for Sonata no. 1 in C major k279.

The use of golden section, instead, is evident in the works of some modern composers, though most of them haven't explicitly mentioned it. One exception is given by Iannis Xenakis (1922–2001): the Greek architecture and music composer wrote down that the compositive structure of *Metastaseis* was based on Le Corbusier's *Modulor* which, in turn, was based on the golden section.[18] The percussionist and composer Greg Beyer analyzed Xenakis' rhythmic piece *Rebonds*[19] and, taking into

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[13]It's a four-voice isorhythmic motet, composed in 1420 at the court of Malatesta IV, called dei Sonetti, in Pesaro in the occasion of the wedding of his daughter Cleofe with Teodoro the Paleologo, son of the Byzantine emperor Manuele II Paleologo. This is considered to be the oldest work attributed to Dufay.

[14]Guillaume Dufay (1400–1474), of Franco-Flemish origin, was the greatest European composer of the fifteenth century; from the musical point of view, he ended the Middle Ages and began the Renaissance Age. He is famous for writing the motet *Nuper rosarum flores* in the occasion of the consecration of the dome of Santa Maria del Fiore (1436). In this composition the musician reports the proportions of the Cathedral and the Dome, which are also reflected throughout the use of the golden section.

[15]The considerations reported in this analysis are taken from *Enciclopedia della musica*, Giovanni Garzanti Editore, Milano 1996, p. 818.

[16]For further information on this topic, see [5].

[17]See http://www.mathisintheair.org/wp/2017/10/la-matematica-puo-aiutare-a-comporre-la-musica-parte-1/.

[18]See Temporelli G., http://www.sectioaurea.com/sectioaurea/S.A.&Musica.htm.

[19]See [17].

account the numbers of bars and of beats, he found a "Recursive System" analogous to the one used to generate the Modulor.

In an interview given to P. Odifreddi in 2004,[20] the composer Karlheinz Stockhausen (1928–2007) confirmed the use of Fibonacci sequence in the division of time in the piece *Klavierstück IX* and claimed its use in many other pieces, not only concerning the proportions in time but also, for instance, in the *"density of the mass of the sound"*, in the "*durations and frequencies*" of his music.

Going back in time, the Hungarian composer and pianist Béla Bartòk (1881–1945) has never declared the use of Fibonacci numbers in his compositions but, from the studies of some musicologists,[21] it seems likely that he deliberately used them. The first part of his famous piece *Music for String, Percussion and Celesta* (1936) is made up of 89 bars considering, as Lendvai writes, a whole-bar rest; the point of maximum sound intensity occurs after 55 bars (34 left at the end of the composition). At bar 5 the second voice enters (third and fourth violins); at bar 13 the fourth voice enters (second violins); at bar 21 the exposition of the theme ends (it's a fugue); at bar 34 the timpani enter and the strings remove the mutes; after 21 bars the point of maximum sound intensity is reached and the second part formed by 34 bars begins. Again, the structure here is divided according to the scheme 13, $13 + 8$. In the third movement, the dynamical division is very similar: $21 + 13$ bars for the two themes, $13 + 21$ (climax), $13 + 8$ (second and first theme). Lendvai's analysis focuses also on Bartok's preferred chords, namely the ones involving chromatic intervals of length 2, 3, 5, 8 corresponding to major 2nd, minor 3rd, perfect 4th and minor 6th. Bartok never mentioned the golden section but he wrote "*we follow nature in composition*" and it's well known his love for sunflowers and fair-cones.

Bartòk was deeply interested in Debussy's music (1862–1918) and some pieces of the latter author seem to involve Fibonacci numbers.[22] Analyzing the song *Spleen* (1888) and the piano piece *Mouvement* (from *Images*, 1905), Howat pointed out a symmetric structure based on Fibonacci numbers: in the first case, he recognized three main tonal modulations occurring at bars 13, 21 among a total of 34 with a tonal return at bar 17. In the second case, the opening section is 66 bars long and taking into account the main theme and a single bass the session can be divided in pieces at bars number 25, 33, 41, 66, which divided by two are almost the same numbers founded in *Spleen.* In three of the six pieces from *Images* there's a symmetric structure, the points of "climax" corresponds to a "Fibonacci" number of bars and these points are prepared and ended proportionally to Fibonacci numbers of bars. Even Debussy never mentioned the golden section in his works but it's known that he read about many subjects in literature, including numerology, cabbala and number construction in art, and he was surely influenced by the many related papers involving Fibonacci that were published at his time. Also Bartok was a "broad" reader and a careful observer of Nature so, following Howat, his use of golden section can't be "unconscious". We like to conclude this argument citing Gubajdulina: "*it's not so*

[20] https://www.piergiorgioodifreddi.it/wp-content/uploads/2010/10/stockhausen.pdf.

[21] See [20].

[22] See [19].

important if he [Bartok] or other composers were conscious to operate following that procedure because the breathing of nature in human acting is always present".

In contemporary music it's easier to establish the presence of Fibonacci numbers because often the authors themselves declare their intention, see for example the pieces *Fibonacci Mobile* by E. Krenek (1900–1991) or *Modulor* by G. Manzoni (1932).

In 2001, the American band *Tool* released the album *Lateralus*, which contains the single namesake, *Lateralus*. Here the *Tools* make extensive use of the first elements of the Fibonacci sequence: in fact, counting the syllables of the first stanza we get the sequence:

1, 1, 2, 3, 5, 8, 5, 3, 2, 1, 1, 2, 3, 5, 8, 13, 8, 5, 3.

Note that the song makes a continuous reference to the spiral figure ([…] *To swing on the spiral* […] *Spiral out*). The time signature of the song changes from 9/8 to 8/8 to 7/8. The drummer of the complex, Danny Carey, stated: *Originally it had been titled 9-8-7 for the time, then it turned out that 987 was the sixteenth number in the Fibonacci sequence. So it was cool.* In 2018 Joe Rogan, a friend of Tool's Maynard Keenan, described his writing process in a podcast: *He wrote a song for the Fibonacci sequence.*

Genesis assiduously used the Fibonacci sequence to build the harmonic and rhythmic structure of their compositions. The most emblematic case is the song *Firth of Fifth*, in which there are solos of 13 bars (played with the flute by Peter Gabriel), 34 bars (played on the piano by Tony Banks), 55 bars (played by guitarist Steve Hackett) and some of these are made up of 144 notes; moreover, counting the numbers of sixteenth notes of each of the main 7 sections,[23] we get the sequence

30, 30, 60, 90, 150, 240, 390

which, divided by 30, is the initial part of the Fibonacci sequence.

There have been other attempts to use the golden number φ in music by constructing special scales. Casey Mongoven[24] constructed musical scales by substituting the unit of the equal temperament, that is the chromatic semitone (which corresponds to the 12th root of 2 or to 100 cents) with units of type $1 + \varphi^n$; for instance, if $n = 5$, then the unit is valued approximatively 149 cents and there are 83 pitches in hearing range and 9 notes within one octave.

Another attempt[25] to construct an "octave" related to Fibonacci numbers was made choosing notes whose distances, in the chromatic scale, correspond to the numbers 1, 1, 2, 3, 5, 8, 13. So, for instance, starting from C_1 we get the "octave"

C_1, C#, D, E, G, C_2, Ab_3, A_4.

[23] See Temporelli G., http://www.sectioaurea.com/sectioaurea/S.A.&Musica.htm.

[24] See [21].

[25] See G. Goldstein in https://www.keyboardmag.com/artists/joseph-schillinger-a-brief-look-at-his-life-and-his-theoretical-system.

The composer David MacDonald, following a procedure similar to the one he already used in *Song for* π, numbered the notes of the E major scale from 1 to 8 adding 0 and 9 at each side as a natural extension of the scale. He encoded Fibonacci sequence in music playing a piano: with his right hand he created the melody playing the notes in the order given by the digits of the sequence and with his left hand he harmonized the melody.

Some musicians do not appreciate the choice made in 1955 to fix at 440 Hz the standard frequency for A_4 and many of them would prefer to come back to the frequency 432 Hz. There are many reasons for this request: vocal chords of lyric singers would be less stressed; the sound of music compositions would result to be more pleasant and relaxing; the frequencies of other notes (in particular the one of C_3) would be expressed as powers of 8 and that would give more connection with Nature. It's nice to observe that, in this case, we would get $E_3 = 162$ Hz, that is we would get a"phi-frequency"!

The golden section is used in the construction of some musical instruments. It's well known the case of Stradivari's violins, whose proportions (total length, case length, fingerbox length, form and positions of the F-holes of the case, …) are all related to the golden section. Even the form of the scroll seems to be a logarithmic spiral! Of course, this is not the only reason that makes these instruments so powerful but it probably contributes to their performances.

Around 2010 the brand *Pearl*, which constructs drums, launched the series of instruments named *Golden Ratio Air Vents,* characterized by the fact that the air vents of the drums were positioned so as to divide the total highness of the instruments according to the golden section. Reading the related drummer's blog, it seems that this solution slightly improved the quality of their sound, but this was not really relevant unless playing alone in a studio. At the end this more expensive series was given up!

Though not related to music, we like to end this paper remarking how "attractive" the Fibonacci name can be for sellers: there are different games involving this name, e.g. the *Fibonacci Nim* (a "ad hoc" variation of the classical nim), but some of them, e.g. the board game *Fibonacci,* despite the name have nothing to do with Fibonacci rules!

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The Importance of Geometrical Properties in Industrial Turbofan Design

Simone Cremonesi

Abstract Industrial fans are widely present in the world. The most common ones can be found in every industrial building for air exchange. This kind of fans, called centrifugal fans, move a huge flow rate through a small increment of pressure. They consist of a rotating part (the impeller) and a volute. This has to collect the air which flows from the impeller and turn its kinetic energy into potential energy. The shape of the volute is based on a spiral. Normally this kind of spiral is an Archimedean one, due to the simplicity of its construction. On the other hand, the logarithmic based spiral is highly efficient. Renè Descartes was the first to describe the Archimedean spiral, which was later called S*pira mirabilis*, the "marvelous spiral", by Jacob Bernoulli, equally fascinated by the mathematical beauty of the curve. The name "logarithmic spiral" is due to its equation in polar coordinates. Its efficiency is due to one of its geometrical properties: the angle between the tangent and the radius is constant in every point. These characteristics are extremely useful in the ventilator industry, since it is possible in every section to have the same optimal geometry and angle between the air and the volute. This is a very interesting testimonial of how the knowledge of the geometrical properties of a particular form can be extensively applied to create a more economical and ecological design.

Keywords Turbomachinery · Turbofan · Axial centrifugal fan · Volute housing logarithmic spiral · Optimization · Genetic algorithm

1 Beauty and Efficiency

Logarithmic spiral (aka "Spira mirabilis" in Latin) is recognizable for its wide presence in nature Fig. 1 [1].

Nature does not like to leave evolution to chance. Behind this apparent randomness are in fact hidden important criteria with one goal: energy optimization. Mastering

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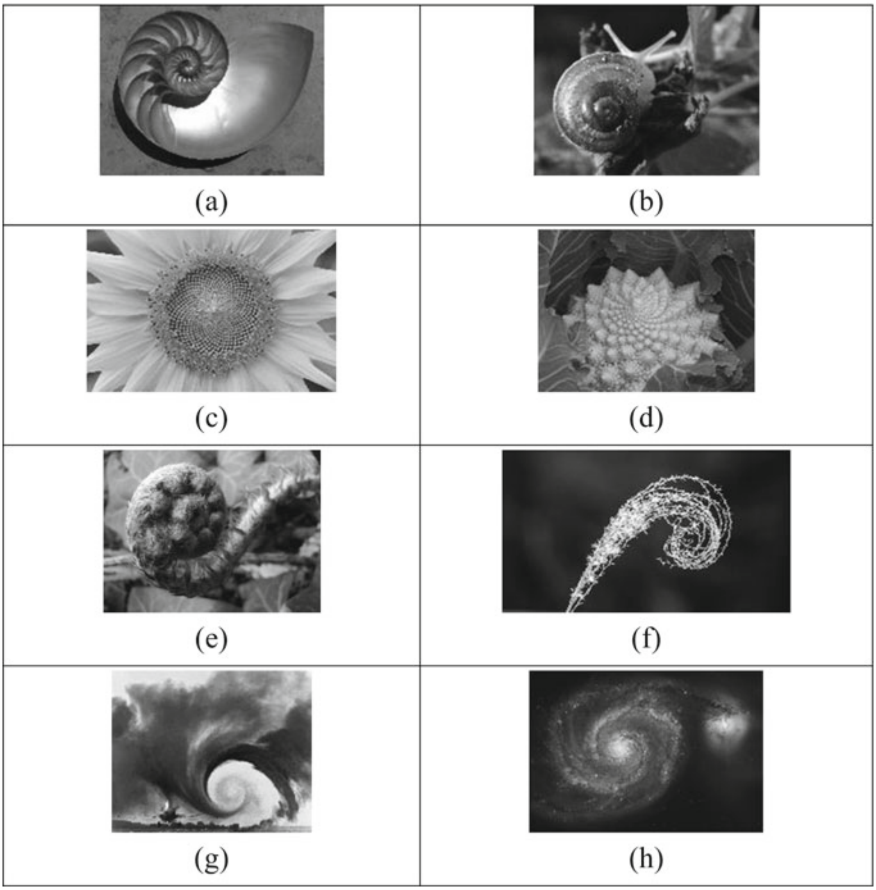


Fig. 1 Selection of different examples of logarithmic spiral in nature: (a) Nautilus pompilius, (b) snail, (c) sunflower, (d) broccoli plant, (e) fern, (f) grass, (g) wing tip vortex, (h) galaxy

the equations that underlie nature can thus help to support the engineering effort aimed at the best possible solutions in apparently unconnected fields.

2 Turbofan

A High Pressure High Volume fan, more briefly H.P.H.V. fan or turbofan, is a radial flow turbomachine used in industry to convey large quantities of gaseous fluid (typically clean air but also steam, i.e. *M.V.C. Mechanical Vapour Compression/M.V.R. Mechanical Vapour Recompression processes*), through the distribution system, in order to feed the processes that take place on site. As any pipeline affected by fluid flow is subject to mechanical energy dissipation phenomena (generally defined as

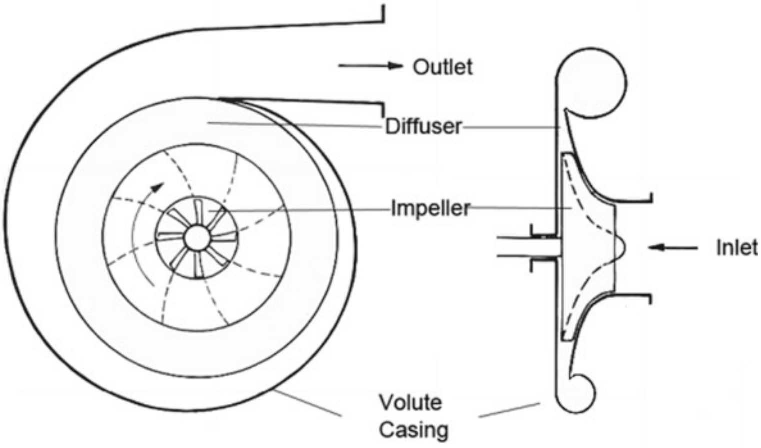


Fig. 2 Turbofan schematic diagram

concentrated/distributed flow resistances), the turbofan must overcome this loss by generating a pressure head (total pressure difference). In this concern the turbofan is the core unit in the served system.

The basic structure of a turbofan is practically identical to a centrifugal compressor, as shown in Fig. 1.

The energy transfer from the fan to the processed fluid occurs through a rotating bladed wheel (called rotor or impeller) Fig. 2, keyed on a transmission shaft (typically driven by an electric motor but also, in rare cases, by internal combustion engines or hydraulic motors), housed in spiral volute (or collector) Fig. 3 [2] and often coupled with a diffuser (vaneless, bladed—cascade or wedge—or piped) Figs. 4 and 5.

The role of the diffuser is to convert the kinetic energy of the fluid exiting the impeller into pressure energy, while the role of the volute is to collect the flow from the periphery of the diffuser, further expand it, and finally deliver it to the outlet pipe [20].

What makes H.P.H.V. fan extremely interesting for the process industry are two characteristics:

– its intrinsic ability to impart high energy transfers to the processed fluid (compression ratio from 5:1 to 8:1) in a wide range of flow rates.
– its compactness.

Since 1950 the aim was to raise up the punctual efficiency value (Fig. 5), but now, due to the fact that we are close to the physical limit of this technology, the tendency has been to extend the functioning limits at high efficiency to a wider area of the machinery operative curve without any change in their overall dimension and mass Fig. 6.

At present, most of the research conducted on the volute is focused on the improvement of the aerodynamic performance (although the volute does not take part in

Fig. 3 Shrouded turbo impeller [2]

Fig. 4 Single stage turbofan housing overview [3]

the energy transfer process onto the fluid, the pressure increase in the volute can be roughly equal to the pressure increase in the impeller [7] therefore all design considerations on it try to reduce the losses [8, 9]).

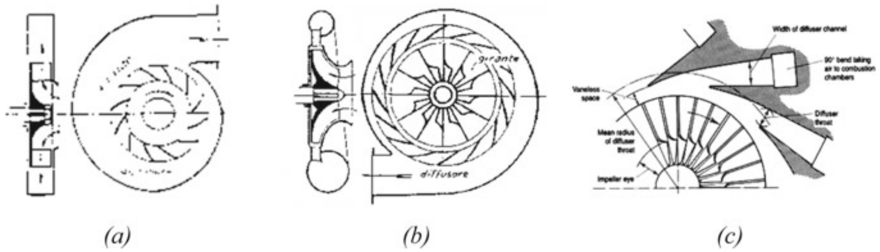


Fig. 5 Turbo fan with: (a) vaneless diffuser [4], (b) bladed diffuser [4] and (c) piped diffuser [5]

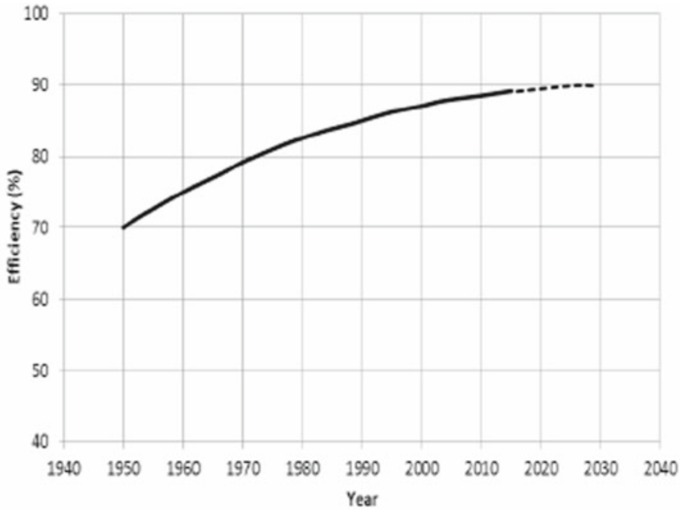


Fig. 6 Derived Centrifugal Compressor Efficiency Trend over the Years with Flow Coefficient Greater than 0.08 [6]

It is wrong to think that the volute operates independently of the diffuser and the impeller. The volute in fact determines the surroundings in which the impeller-diffuser group operates, affects the best efficiency point of the turbomachinery as much as the impeller [10] and, for that reason, can cause drastic performance drop of the whole machinery [11] (due to non-symmetrical pressure distribution in the upstream flow consequent to a poor design of the volute cross-sectional area) [12], besides a higher aeromechanical net radial force on the impeller [13].

3 Geometry

Turbofan volutes are always performed in a logarithmic spiral shape (constant circulation method [14]), but polycentric spiral models are usually adopted to approximate the ideal line Fig. 6. in production lines that create volutes from metal carpentry [15].

The volute cross sections can be divided in two main types: symmetric and overhung, both with different shapes Fig. 7. [7]

The spiral case theory in an ideal flow regime [16] allows deducing the volute cross-sectional shape that more than any others enables high levels of performance to be achieved, and it is the one shown in Fig. 8f. and even better in Fig. 8g. [7].

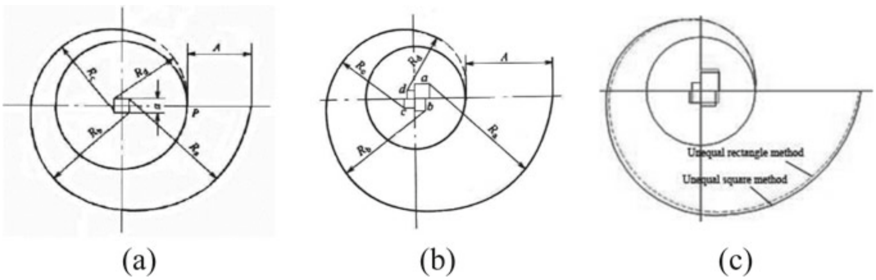


Fig. 7 Spiral plot: (a) equal square method, (b) unequal square method (c) unequal rectangle method

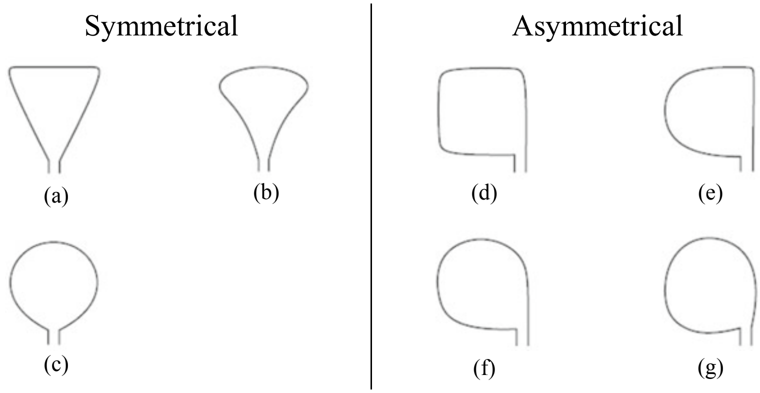


Fig. 8 Selection of different volute cross-sectional shapes

4 Considerations on the General Design of the Volute

Inlet, rotor and volute determine the most significant energy losses of a turbomachinery. Individual contributions, based on a centrifugal compressor, are shown in Fig. 9.

Analysing the graph it is possible to note that impeller and volute determine an equal contribution to the reduction of the isoentropic efficiency, although the former dominates the phenomenon at low flow rates and the latter at high flow rates.

In order to reduce the optimization process computational cost the development activity is normally focused on the impeller or the volute separately.

The volute design has a major impact on the stable operating range of a turbomachinery and can affect its performance in terms of operating range and efficiency (a proper volute design increases the polytropic efficiency [17]); it is studied preferentially with respect to the impeller.

The preliminary design of the volute through the application of simplified general theories does not allow defining geometric models capable of optimizing the functioning of the impeller-diffuser group in real conditions, especially if there are significant deviations from the basic hypotheses.

Since the problem is substantial, in literature it is possible to find numerous articles about the volutes optimizing process (both for hydrophores and pneumofores machinery) but the solution approach, in almost all cases, is based on two common elements: a massive use of CFD and the researchers' intuition [18].

As it is easily understood the problem is exceedingly complex (the flow entering the volute is strongly three-dimensional and in order to achieve high efficiency values at the design point the velocity must ensure a uniform static pressure distribution to the volute inlet [17]; this because the circumferential pressure distortion that occurs inside the volute could propagate upstream and cause flow fluctuations in the impeller resulting in efficiency drop in the whole compressor rotor-statoric stage [19]) and, without solution strategies based on robust criteria, the risk that the improvement

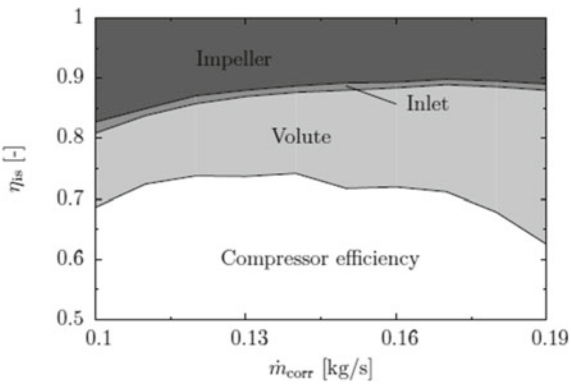


Fig. 9 Isentropic efficiency reduction determined by each component of the centrifugal compressor [7]

margin will not be fully exploited and the target achievement times will be long, is very high..

There are five geometrical parameters that are conventionally considered as decisive for the definition of the overall performances of a centrifugal turbomachinery: volute cross-sectional area, shape of the cross-section, tongue geometry, volute inlet location, and radial location of the cross-section [7].

The volute base spiral shape does not seem to be a relevant parameter. In turbomachinery, characterized by high degree of compactness or high specific energy, the hypothesis that the volute is enough displaced from the impeller-diffuser group (so that the deflections conditioned by the consideration of a finite number of blades can be ignored), cannot be fully respected. This leads us to necessarily consider the volute base spiral shape as an integral part of the research of the machinery optimal configuration.

5 Experimental Study

The volute, together with the impeller and the vaned diffuser, has been computationally investigated using the open-source tool for Computational Fluid Dynamics software known as OPENFOAM. The geometrical discretization of the compressor stage has been made using snappyHexMesh utility. The mesh shown in Fig. 10. consists of 36,6x10^6 cells.

The turbulence model used has been k-ω, while the MRF (Multiple Reference Frame) has been the method of computation for the flow stationary. The total pressure has been enforced as boundary conditions at the inlet. The flow in the inlet pipe is assumed normal to the inlet surface. The outlet boundary conditions have been applied by using a variable static pressure proportional to the kinetic energy at the

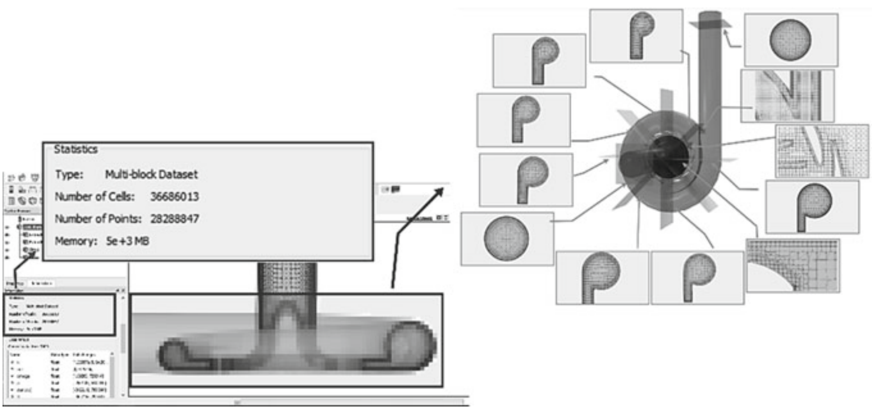
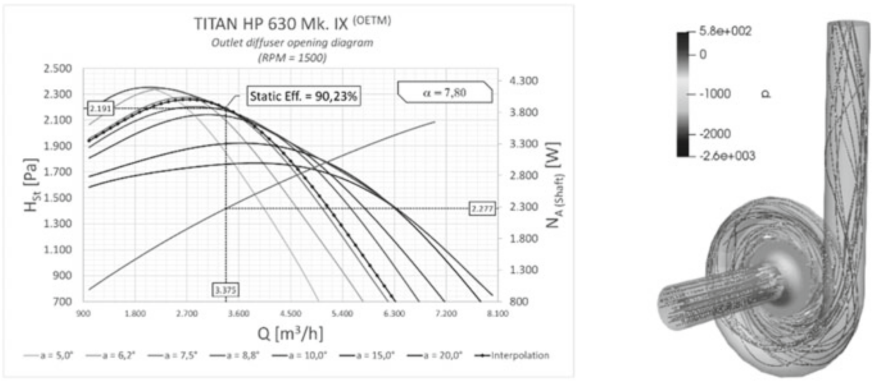


Fig. 10 The computational mesh, which includes: impeller, vaned diffuser and volute

outlet. Zero-slip boundary conditions have been imposed over the impeller blade, diffuser airfoils and all solid walls. The calculation is assumed to be convergent when the ratio between the sum of the residuals and the sum of the fluxes for a given variable in all cells is reduced to four orders of magnitude.

6 Results and Discussion

The purpose of the investigation has been to characterize the turbofan as a whole. Six fluid dynamic simulation campaigns have been set up. Each campaign had a different opening angle of the vaned diffuser (5°, 6.2°, 7.5°, 10°, 15° and 20°). For each opening angle, the machinery performance curve has been plotted (flowrate vs static pressure and flowrate vs shaft power). Each curve was an eight-point interpolated line. The obtained results have been interpolated both as a function of the flowrate and as a function of the vaned diffuser opening grade. The double date interpolation has been made possible to determine the overall machinery maximum efficiency Fig. 10. The results obtained are highly encouraging and further optimization activities can give rise to additional improvements (specific and global) (Fig. 11).



(a) (b)

Fig. 11 (a) Performance curves in function of the vaned diffuser opening degree; (b) Streamlines (case α = 8.8 [Deg.] - Q = 3.000 [m^3/h])

7 Perspectives of Future Research Activity Related to the Turbomachinery Volute Optimal Geometry

The work that has been presented constitutes the starting point of a future product development program focused on the systemic use of adaptive evolutionary procedures (Genetic Algorithms - GA) for the identification of the optimal solution.

Although this design methodology is the most powerful available today in R&D, it has not found a widespread use yet.

In turbomachinery field it has found its natural point of application in the impeller; where applied to the volute it has been used to identify its optimal shape in the sagittal plane, never in its coronal plane.

The idea is to use the parameters that define the geometry of the logarithmic spiral as chromosomes of the fitness function and assign the task of modeling the possible spirals to a numerical generator of geometries; to determine for each configuration the geometry's relative performance and the goodness degree of this; then to originate a new generation characterized by a higher level (on average) of the function that describes the optimum (fitness). All this applies for a number of generations high enough to be able to consider the problem as having come to convergence. Secondly, on the basis of the results obtained and those coming from a multi-criteria analysis to the desired whole, the process is supposed to be iterated.

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The Geometry of Beauty

Liliana Curcio

Abstract The inherent rules of Geometry, as well as the forms described by them, have always inspired artistic creation and suggested canons to achieve perfection in the creating of many works of art. Let us think about the beauty of some geometric shapes used in painting, architecture, sculpture. The natural forms have almost always inspired those built, where the aesthetic aspect is obtained through a deep dialogue between shape, function, materials, and technique. Everything tends to achieve a common goal; and thanks to this communion of knowledge the forms develop themselves differently, in relations with the historical period in which they are used within the various design choices. The dialogue between different fields is fundamental to find those optimal choices to achieve the beauty in each construction in progress. The aim of this work is to understand how it is possible to obtain even very complex three-dimensional surfaces starting from a simple geometric figure—such as a flat polygon—through synthetic processes and simple transformations. These surfaces, often described by structures of extraordinary beauty, are recognizable in the configuration of many natural growth's forms, in some admirable architectural constructions characterized by perfect and futuristic shapes and, at last, in the description of some contemporary and delicate beauty sculptures where the light is added to the shape to mark its beauty.

Keywords Geometry · Arts · Triangle · Golden rectangle · Polygon · Helicoid

1 The Triangle

The awareness that very simple geometric figures can generate works of considerable aesthetic value is incredible and, at the same time, wonderful. This is the purpose of our story. We will always start from a simple figure, a polygon, and show how, using geometric transformations, we can obtain more complex shapes that describe, through synthetic processes, the beauty of both natural and artificial

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growth: phenomena that can be admired in the expressions of art and architecture. We could say that our recurring motif is the following: at the beginning there is always a polygon!

Let us start with the polygon that has the minimum number of sides: the triangle.

This geometric shape is very present in our daily life; think of road signs and the fact that the triangular shape in this context indicates a danger and that attention must be paid. In this context, we do not want to dwell on the symbolic or religious aspect or on the various philosophical and psychological meanings; we will deal exclusively with the geometric figure, source of properties and principles but also the representation of a highly stable configuration. In three-dimensional space the solid consisting of triangles and with the minimum number of faces is the tetrahedron; a three-dimensional surface built through these forms is the tetrahelix, which is a self-supporting structure and which, theoretically, could extend itself into infinity.

The tetrahelix was studied for the first time in the 1950s by R. Buckminster Fuller (Milton, 1895–Los Angeles, 1983), one of the greatest exponents of American culture of the twentieth century. He was an architect, engineer, philosopher, and scholar of nature by which he was inspired by his projects, that had as their main objective the well-being and improvement of human existence. Let us go back to the triangle and remember that this, although being the simplest form, has a very important structure and in nature the triangular shape is the most stable that exists. Also remember that through the triangle many other forms can be described both on the plane and on the space. An evidence of what has been said comes from Plato himself who, in Timaeus, explains the natural elements starting from the triangle: "… the flat and straight surface is made up of triangles. And all triangles come from two triangles, each having a right angle and two acute angles. Of these triangles then, on each side, some have an equal part of right angles delimited by equal sides; others, however, have unequal parts divided by unequal sides".

We know that any triangle can be broken down from one of its heights into two right triangles and these, with the same process, into other triangles.

According to Plato these forms, assembling themselves, generate the surfaces on the space which will be subsequently identified by the name of Platonic solids (Fig 1).

These polyhedrons are associated by Plato to the natural elements: earth, air, water, fire and moreover, it is through these solids that the philosopher describes the architecture of the universe. The polyhedrons have overwhelmingly entered in the works of art; in addition to the famous representation of Platonic solids attributed to Leonardo da Vinci, to the portrait of Luca Pacioli attributed to Jacopo de' Barbari, let's consider, among other things, the Albert Dürer's Melancolie, the works of Salvator Dalì, and also the Alberto Giacometti's Cube.

The triangle is one of the most diffuse forms in varied artistic expression.

There are works, even recent ones, structured on compositions and transformations of triangles; an historical example, which is useful to remember, is the geodesic dome of R. Buckminster Fuller, a scholar we have just mentioned. Fuller obtained the American patent for its domes, although the first geodesic dome was designed after the end of the First World War by Walter Bauersfeld, chief engineer of the Carl Zeiss optical industries and built in 1922 to house a planetarium. In 1967 Fuller

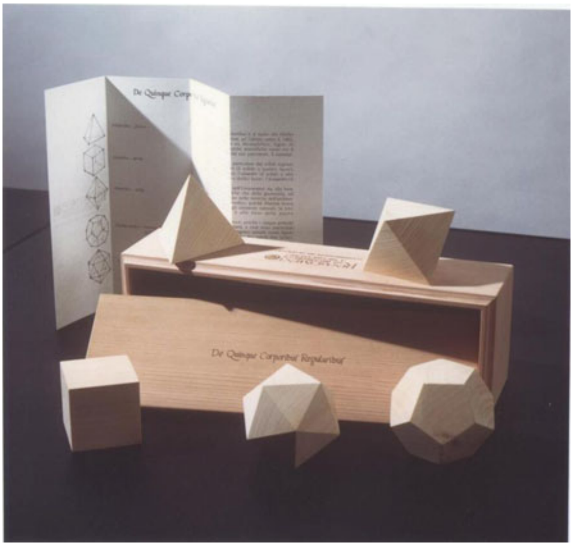

Fig. 1 "De quinque corporibus regolaribus" restitution of Platonic solids by Pierluigi Ghianda, 1994. Famiglia Ghianda Archive

presented his geodesic dome at the Montreal Expo in Canada; a strong, light, easy to assemble and disassemble structure (Fig. 2).

The construction principle is based on the projection of a regular icosahedron (polyhedron with twenty triangular faces which in the case of a platonic solid are equilateral triangles) on the surface of the sphere onto which it is placed. Each face of the polyhedron (therefore the triangle) is divided into smaller triangular faces, for example, divided into four or nine surfaces, etc. and then projected onto the sphere. The number of times the face is divided is called the "arc factor" or also "frequency" of the geodesic dome. The Montreal Dome has frequency 16! All the triangles in which the icosahedron has been divided approximate the spherical surface. Obviously, higher is the frequency then better is the approximation with the spherical surface. Furthermore, the intrinsic rigidity of the triangles guarantees an exceptional stability of the structure. The geodesic dome is one of the most important architectural solutions proposed around the middle of the last century; it contains the maximum volume with the minimum surface.

It should be remembered that in 1967 Fuller gave the city of Spoleto a geodesic dome which he called Spoletosfera (in the picture) (Fig. 3).

Inspired by the building principle of Fuller, in recent times there have been many architectonic applications of the same principle, especially in roofing; two examples are Fuksas' "Sailing" at the Milan Fair and Norman Foster's Great Court at the British Museum and Library in London (Fig. 4).

Fig. 2 Biosphere in Montreal (Canada), 1967

2 A Particular Polygon: The Golden Rectangle

Two congruent right triangles positioned such to have a common hypotenuse may describe a rectangle on the plane. We will deal with the golden rectangle.

That rectangle whose height is the golden section of the base is called golden. Or, if we prefer, the ratio between the major and minor base is the golden number, which is indicated by the Greek letter φ and its value is approximately 1.618… with infinite and not periodic digits.

The golden section is the reciprocal of the golden number (the ratio between the golden number and the golden section is equal to 1); its value is about 0.618… but, as we all know, both the golden section and the golden number can be expressed in perfect form only through an irrational number.

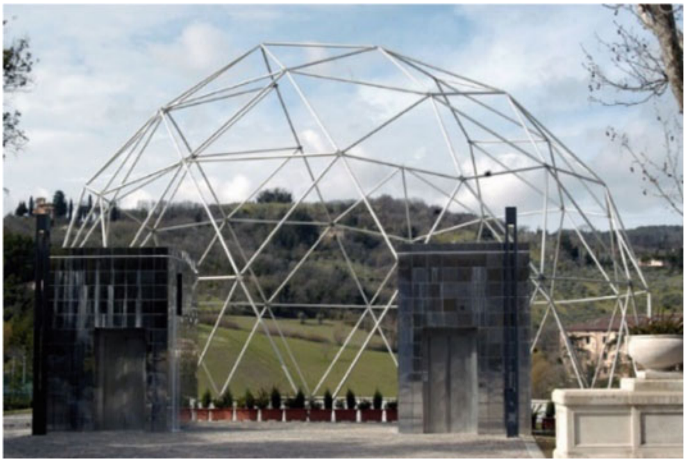

Fig. 3 Geodesic dome in Spoleto (Italy), 1967

We need to remember that we also find this notion described in Euclid's Elements (proposition XI, book II) in the following form: "Divide a given straight line ended according to the extreme and medium proportion".

We also recognize it in another form in the proposition XXX, problem X, book VI: "… rectangle area equal to the area of the square built on the largest part".

The golden rectangle is important because, as always, it is what, among other infinites, describes beauty. Many artists, in fact, have believed and still believe that among all the rectangles the most "pleasing to see" is the one in which the base and height respect the golden ratio. We will not dwell on the known link between the golden number and the Fibonacci's numbers. But, before continuing our story, we should remember that many works of art (in architecture, painting, and sculpture) were created respecting the rule of the golden section as a true canon of beauty [3].

Let us consider, among many examples, the studies of the Parthenon in Athens, on the Flagellation of Christ by Piero della Francesca, but also of the construction of Castel del Monte, etc. (Fig. 5).

Do not forget that Le Corbusier himself (whose real name was Charles-Édouard Jeanneret-Gris, 1887–1965) was inspired by the golden section and the Fibonacci's numbers to design his Modulor, in order to provide "a range of sizes harmonious to satisfy the human dimension, universally applicable to architecture and mechanical things". It was in his work Le Modulor, that Le Corbusier states: "Mathematics is the masterful building imagined by men to understand the universe. The absolute and the infinite, the graspable and the elusive, meet there. In front of them rise high walls in front of which you can pass and go over again without any result; sometimes a door is encountered; you open it, you enter it, you are in other places, where the gods are, where the keys of the great systems are. These doors are those of miracles. Go through one of these doors, it is no longer the man who works: it is the universe that the same man touches anywhere. In front of him, the prodigious rugs of limitless

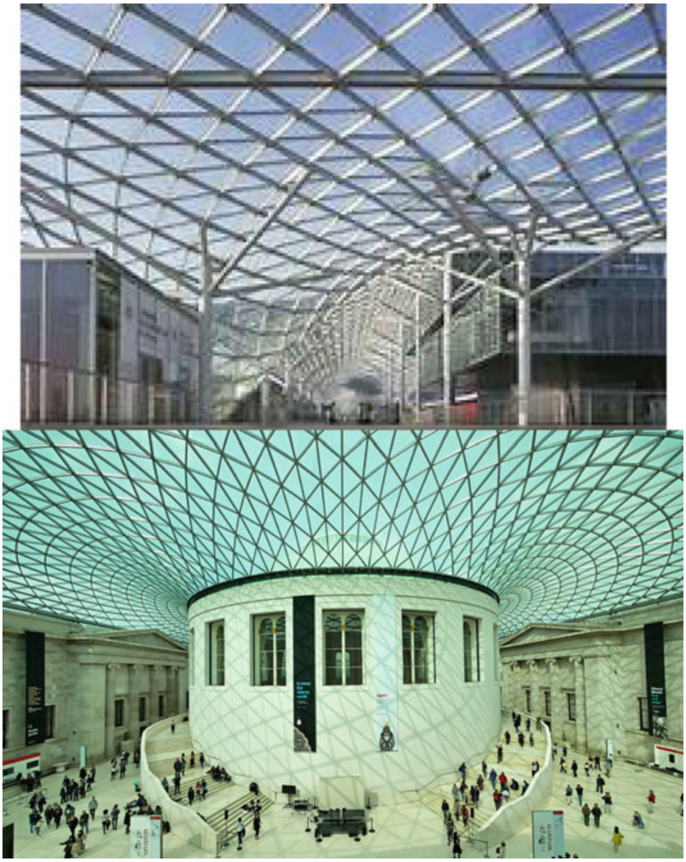

Fig. 4 Massimiliano Fuksas' "Sailing" at Milano Fair and the Great Court by Norman Foster at the British Museum and Library in London. L. Caruzzo Archive

combinations are unrolled and illuminated". The golden rectangle can be iterated, that is you can build, starting from the first, infinite rectangles that will always be golden. To iterate the golden rectangle, just project the smaller side onto the larger one, thus dividing the rectangle into two parts: a square and a rectangle which is, in turn, still a golden rectangle. By repeating the construction several times, then carrying out an iterative process, "infinite" golden rectangles are described (Fig. 6).

3 From the Golden Spiral to the Helicoid

Let's refer to a golden rectangle and its iterations as we consider the points where the squares and the new golden rectangles separate and join them, we can describe a curve, called the golden spiral, which is a particular logarithmic spiral. This type of

Fig. 5 Portal of Castel del Monte (Andria). Famiglia Gaggiotti Archive

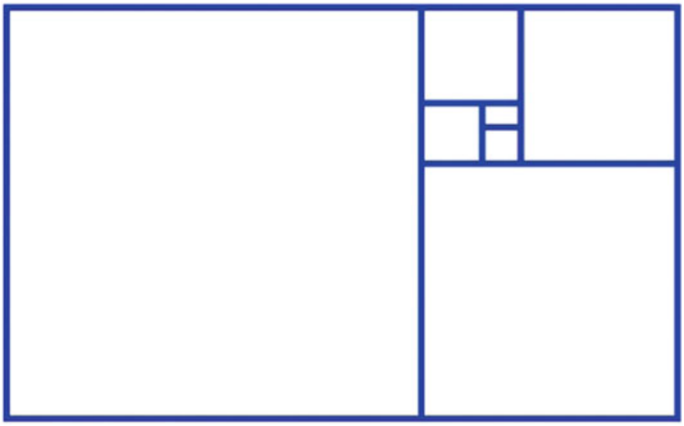

Fig. 6 Iteration of golden rectangles

spiral is the basis for the growth of many biological forms, both animal (for example the nautilus) and vegetable (as in the sunflower) (Fig. 7) [2].

If we now imagine rotating and translating the spiral around an axis, we obtain a conical helicoid (which is the shape of the Turritella and many other shells) (Fig. 8) [6].

The helicoid is one of the oldest and most fascinating forms employed in architectural structures, especially in stairs, in towers, but also in the twisted columns used as ornament and embellishment. Some examples of stairs, called "spiral" staircases,

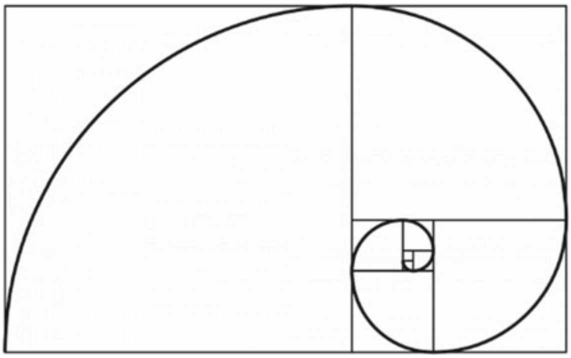

Fig. 7 Golden spiral in the golden rectangle

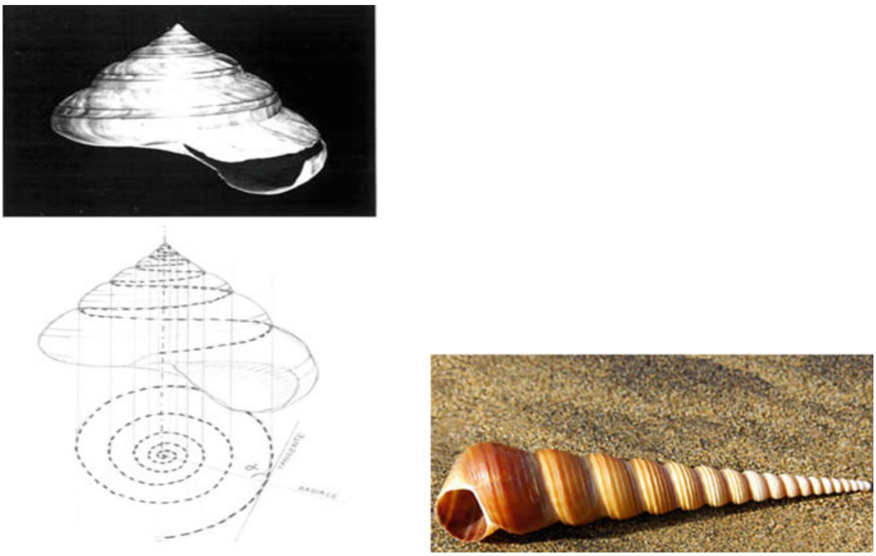

Fig. 8 Conical helicoid which is the shape of the Turritella and many other shells

are found in the towers of the Sagrada Familia in Barcelona, and the two staircases by Bernini and Borromini inside Palazzo Barberini in Rome. The spiral staircase that connects Piazza Mercatale to the Sanzio Theater in Urbino; the work of the Sienese architect Francesco Di Giorgio Martini and finally the spiral staircase in the Certosa di Padula (SA) in self-supporting white marble that allows access to the library (Fig. 9).

Apropos of towers, there are examples such as the Samarar Minaret (851 AD) in Iraq; Malwiyya, the helical minaret of the great Mosque of Caliph Al Mutawakkil, often mistaken for the legendary tower of Babel. In fact, many artistic representations of the tower of Babel are inspired by this form. The last example we are dealing with is the Lanternino of the church of Sant'Ivo alla Sapienza in Rome by Francesco

Fig. 9 Self-supporting white marble spiral staircase in the Certosa di Padula (SA). L. Curcio Archive

Borromini; a work of great courage for those times, 1640, with the intervention done on an existing structure. The criticism and doubts aroused by this work because of the great weight imposed on the structure that, according to all, would not have lasted long and the sensation was such that the Rector of La Sapienza "protests against himself regarding any damage that could occur in fifteen years, according to the legal provisions of the city"; Borromini is so sure of his work that he replies to the Rector that he did not intend to evade this obligation but rather intended to increase this obligation by extending it also to his heirs as long as the commissioned works were completed as requested by him. We can still admire the splendid Lanternino today.

Today the evolution of the tower has turned into the construction of skyscrapers with helical shapes; Zaha Hadid's Storto in Milan is in fact a helicoid but other buildings such as Marilyn Monroe towers in Toronto and DNA Tower in Abu Dabi are also helicoids, the latter is inspired by the helical shape of DNA (Fig. 10) [1].

Speaking of spirals and helicoids, I would like to conclude by recalling an interesting didactic intercourse which happened a few years ago involving classes and teachers from the Maxi-Experimental Art Institute of Monza (today Artistic high school "Nanni Valentini").

During this exchange an analysis of the form was carried out, arriving to the description of a spiral through the assembly of quadrilaterals of different sizes, positioned one on top of the other, and thus, of a helicoid through pseudo-pyramidal trunks also positioned one above the other. The path continues with the description of the shape of different types of shells and ends with the construction of three-dimensional models of the same performed in the laboratory. Here some passages. This time we start from an irregular quadrilateral, and on the minor base we draw another that has the smaller side of the first as the major side and, repeating the operation several times, we notice that the structure curves a little (Fig. 11).

Fig. 10 Francesco Borromini (1599–1667) Sant'Ivo alla Sapienza (Lanternino)—Rome. Zaha Hadid "Lo Storto", Milano. Marylin Monroe Towers, Mississauga—Toronto—2012—MAD Ltd and Burka Architects. L. Caruzzo Archive

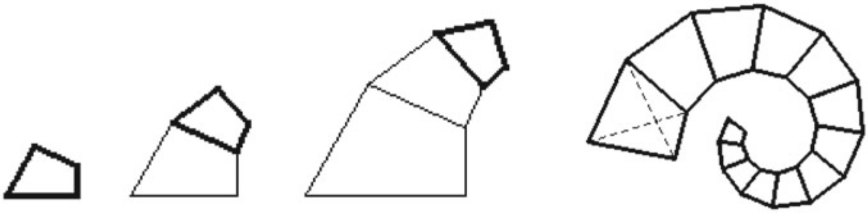

Fig. 11 Representation of a spiral starting from a quadrilater

Proceeding with the same construction and with other quadrilaterals we note that the figure tends to roll into itself assuming the course of an authentic logarithmic spiral even if, obviously, the discrete nature of this procedure only allows an approximation of the curve with a polygonal.

We can repeat the construction, just seen in plan, in three-dimensional space starting this time from a pseudo-pyramidal trunk, to then add onto the minor base another pseudo-pyramidal trunk having that as the major base (Fig. 12).

And by continuing to repeat the procedure, a three-dimensional structure is obtained which rolls into itself, one in which the generating principles of the shape of a shell can be recognized.

All constructions were carried out with the Cabri II software by Roberto Di Martino within the context of an educational project which is obviously much broader than these examples mentioned as can be read in the article proposed in point 5 of the Bibliography [5].

As a last step, from the descriptions made three-dimensional models of some shells were produced in the laboratory by Cinzia Tresoldi.

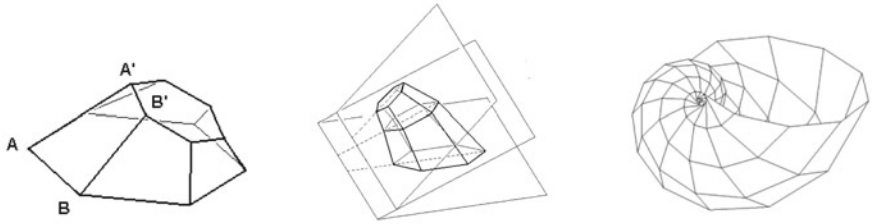

Fig. 12 Representation of a shell through the assembly of trunks of pyramids

In the photo we propose the models of two shells built with metal and plexiglas plates (Fig. 13).

Always starting from a polygon to arrive at Beauty.

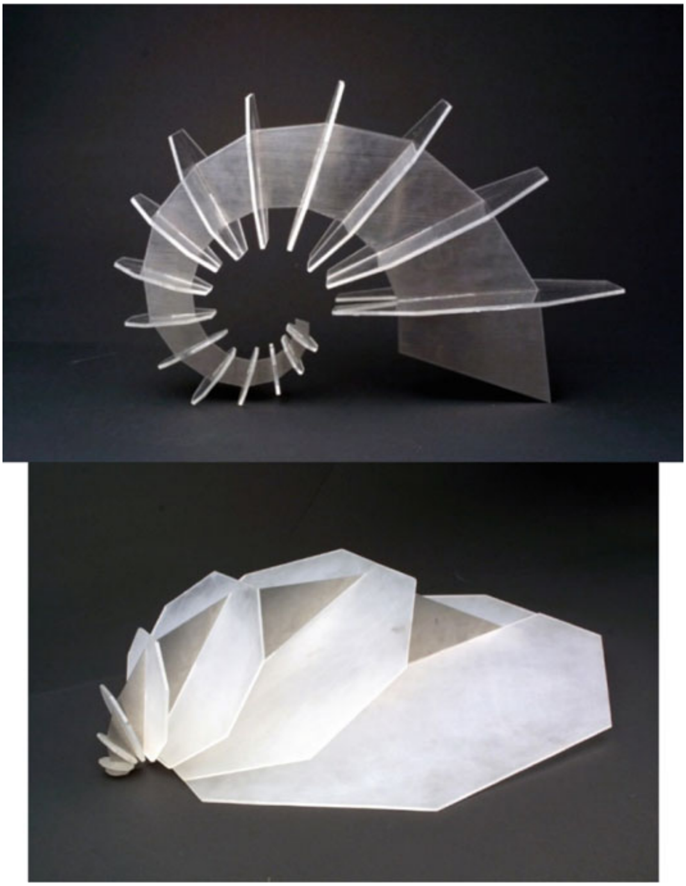

Fig. 13 Models of shell created in workshop

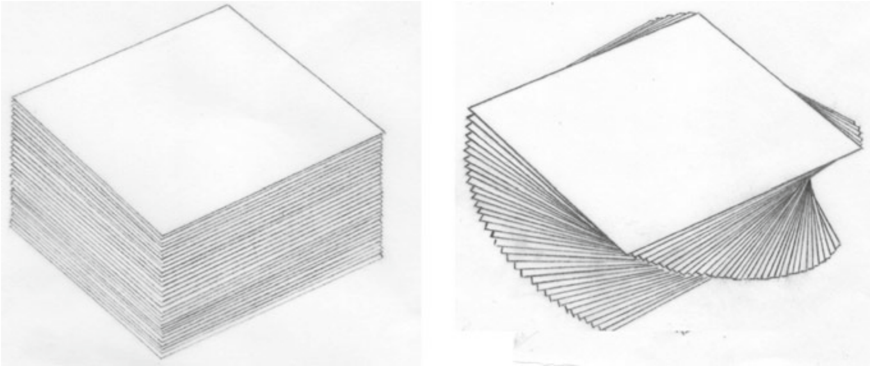

Fig. 14 Parallelepiped built by stacking square sheets. Helicoid obtained from the rotation of the parallelepiped

We would like to end our story by presenting a different type of artwork: wonderful sculptures that the artist creates from any polygon.

We are talking about the sculptures of a contemporary artist, Paolo Mazzuferi, who started his works thanks to a refined knowledge of geometry; in fact, he begins with regular polygons from which, through geometric transformations, he builds helicoids that unwind along the entire sculpture which, for obvious reasons, ends but which could continue to be generated infinitely while still retaining the shape.

It should be understood what has been said through simpler passages and comprehended, for example, how a helicoid can be generated from a simple square [4].

If we stack square pieces of paper, we obtain a parallelepiped; then if we rotate the sheets around a central axis, the lateral edges, which are straight line segments, turn into helices that wrap around the shape (Fig. 14).

Between the parallelepiped and this new form there are many more analogies than one might expect. The generating squares of the two forms maintain both equivalence and congruence, but above all, the lateral edges of the parallelepiped can be described as helices with a radius equal to zero. The movements that generate the two solids are therefore the translation and the roto-translation, but if in the parallelepiped we replace the lateral edges with helices having radius r, we will obtain forms in which the helices themselves will no longer be enveloping. The movements that generate these solids will then be translations and oscillations, with a very wide formal range, also due to the fact that they can be extended to all prismatic forms; we emphasize that translation and roto-translation are therefore configured as special cases. The sections perpendicular to the axis of this prism, which the artist defines as "prisma elicoforo" (a prism that brings helicoids), are transformed into quadrilaterals that lose congruence, but maintain equivalence (Fig. 15).

Once again, polygons play a central role and, as we have already said, retain equivalence. The new polygons generated by these operations are known in the literature as "midpoint polygons" studied, for the first time, by the American mathematician

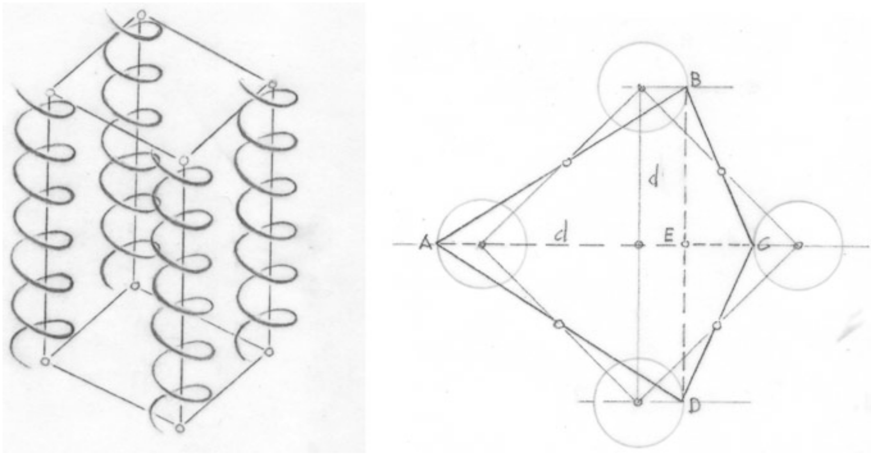

Fig. 15 Prism with helical edges

Edward Kasner (Martin Gardner's uncle) in the early 1900s ["The Group Generated by Central Symmetries, with Application to Polygons", American Mathematical Monthly, March 1903].

What we have described is the principle behind the sculptures that we present in the following photos. In the following examples, the sculptures generated have increasingly elaborate windings, and in one of them the polygonal generating forms are evident.

Working on this repertoire of forms, the author makes it the elements of an artistic language that seizes the opportunities offered by ever-new geometric procedures and translates them into narratives concerning the very origin of natural forms. The light that floods these sculptures becomes the absolute protagonist for the viewer, giving a surprising sense of unity and simplicity to these works despite their extraordinary constructive complexity (Fig. 16).

In conclusion, the pursued and hopefully achieved by us intent was precisely to show how even rational processes, typical of geometry, can lead to results of great aesthetic value that allow you to welcome without hesitation the works obtained within the field artistic in all its forms. What happened in the past happened in every epoch and is still happening today in the search of certain maximum expressions of creativity.

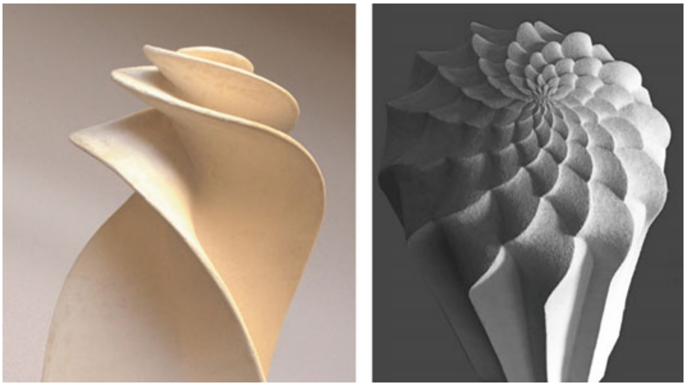

Fig. 16 Sculptures by Paolo Mazzuferi. T. Pelucchi Archive

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Resiliency in Geometric Aggregation and Social Connectivity: Anna Bofill Levi and the Taller de Arquitectura

Caterina Franchini

Abstract The Taller de Arquitectura—Barcelona reached the international stage with revolutionary works conceived between the mid-1960s and 1970s in Francoist Spain, endorsing the criticisms against the Modern Movement that had emerged after World War II. Notable works such as the Barrio Gaudí in Reus (1964–1968), Kafka's Castle in Sitges (1965–1968) and Walden-7 in Sant Just Desvern (1970–1975), among others, originated from a recursive process of isometric aggregations of "minimal cells", which allowed for a wide variety of adaptability characteristics. These realizations, as well as the unbuilt projects La Ciudad en el Espacio (1968–1972) and La Petite Cathédrale (1971–1972), can be identified as experimental tests of the "Theory of Form" (1974–1975) by architect and music composer Anna Bofill Levi (b. 1944). Her mathematical-geometric research offered a resilient design method for an inhabiting system with a high degree of porous spaces fostering social connectivity. She explored the isometric possibilities of parallelepipeds in the Euclidean three-dimensional affine space to intentionally generate "urban tissues" that can adapt to traditional or modern construction building techniques and ever-changing men and women's individual and social needs or behaviors. This paper investigates Anna Bofill Levi's early lesson by focusing on her multidisciplinary vision, the multi-folded concept of adaptability put into practice, and resiliency through voids spaces.

Keywords Anna Bofill Levi · History of architecture and construction · Gender and city's design · Geometry for sustainability · Resiliency

1 Thinking and Practicing Out of the 'Block': Anna Bofill Levi

Several internationally acclaimed works designed in the 1960–1970s by the Taller de Arquitectura Bofill (hereinafter T. de A., Barcelona) find their design logic in the

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geometric-mathematical method developed by architect and music composer Anna Bofill Levi (Barcelona, b. 1944) and presented in her doctoral thesis, *Contribución al estudio de la generación de formas arquitectónicas y urbanas* (Contribution to the study of the geometric generation of architectural and urban forms) at the Esquela Técnica Superior de Arquitectura (ETSAB) of the Universidad Politécnica de Barcelona.

Dated February 1975 (but defended in 1974), the thesis has opened up a new scenario of infinite possibilities for porous modular configurations of cubes and cuboids attempting to incorporate adaptability within the design process.

The problem faced was far beyond the systematic generation of complex three-dimensional shapes, as it might appear. From our point of view, the challenge of Anna Bofill's method consisted of made possible transformative aggregations of "minimal cells" (basic elements), "units", sometimes "nuclei", and "bodies" capable of forming and re-forming combinations of 'voids', that act not as disaggregation but as a concentration of nets of connectivity for the "urban tissues".

Since she expected future advances in computer science, her broader vision was the "almost automatic" generation of suitable morphologies for "urban tissues" in the Euclidean three-dimensional affine space (E^3) that she has defined in her thesis considering physical requirements of construction in the real space.

By endorsing post-war criticisms of the Modern Movement, Anna Bofill aimed to propose a "richest and more vital" organic alternative to "the rigid and cold urban planning of the 'blocks'" [1, transl. by the author] resulting from a formalist interpretation of the Athens Charter and dogmatic applications of its functional urban *zoning* principle.

Issued by the 1933 Congrès Internationaux d'Architecture Moderne (CIAM, International Congresses of Modern Architecture), the Charter proposition for unified block-like buildings had paralyzed the search for alternatives forms for years—according to leading architectural historians, including Kenneth Frampton. Moreover, Anna Bofill remarked that the division of soil uses by functional areas, called *zoning*, caused the "exacerbation of the separation of gender roles, increasing inequality of power, oppression and the destruction of social cohesion" in urban developments [2, transl. by the author].

Her organic concept of "urban tissues" challenges the traditional semantic and semiotic of the word Architecture. The Architecture is not an object autonomous within the city; instead, it is an "urban tissue" logically and mathematically generated, consistent with different scales of individual and collective inhabiting spaces.

The first stage of the T. de A. activity joined the reaction against the grand visions dreamt up by radical functionalists, reestablishing Architecture (in its broader meaning) as an instrument of political, social, and cultural critique.

The multidisciplinary team practice of the T. de A. met at several points the visions for novel ways of living and inhabiting, new individual and collective relationships and behaviors, proposed by "anti-architecture" or "radical design" groups such as, for example, the Archigram or the Metabolist movement. They all dared to question, rebel against, disrupt the rigid functionalist "block" and redefine architecture possibilities in the 1960–1970s.

Marked by widespread anti-war protests, the Berlin Wall, the second wave of feminism, women's liberation movement, rural communes and environmentalism, the first moon landing, Woodstock and the hippies, and so much more, the 1960s and 1970s occupy a central place in contemporary history. Meanwhile, Spain has gone through the last decade of Francisco Franco's Regime which "coincided with the first generation of women graduated from the Barcelona School of Architecture (ETSAB)" (1964–1975) to whom Anna Bofill belongs, being the 28th female graduated in Catalonia (1972) [3].

The year of her doctoral thesis is the same in which Franco died (1975, November 20th) and the Spanish transition to democracy opened, a key event for women's history in architecture [4].

The mainstream of architectural historiography has too often overlooked the crucial contribution and joint-authorship of Anna Bofill to some exceptional works by the T. de A., including the Barrio Gaudí in Reus (Tarragona, 1964–1968, Fig. 1a), Kafka's Castle in Villapineda at Sant Pere de Ribes (Sitges, 1965–1968, Fig. 1b) and Walden-7 in Sant Just Desvern (Barcelona, 1970–1975, Fig. 1c), as well as the not built projects La Ciudad en el Espacio (1968–1972) and La Petite Cathédrale (1971–1972).

As María Isabel Peña Aguado remarked in 2009, "the name Bofill has long been familiar to those in the field of international architecture. What is less well known, however, is that this name refers non only to a male architect [archistar Ricardo Bofill Levi, Anna's brother], but also to a woman architect and composer" [5, *Introduction*].

Anna Bofill's authorship has been highlighted recently by seminal researches such as the international project *Women's Creativity since the Modern Movement—MoMoWo 1918–2018* (financed by the European Union in 2014) [4, 6] and then by the national project *Women in Spanish (Post)Modern Architecture Culture, 1965–2000* (financed by the Spanish government) [3].

Fig. 1 Taller de Arquitectura. **a**. Barrio Gaudí (1970–1971). **b**. Kafka's Castle (1965–1968). **c**. Walden 7 (1970–1975); ©RBTA https://ricardobofill.com

1.1 Groundbreaking Multidisciplinarity for Freedom

Anna Bofill is a woman pioneer in architecture in Spain and an international groundbreaker in feminist urban planning, this latter intended as a relevant challenge to improve the human condition. She courageously moved her first steps in the study of architectural design during the misogynistic era of Franco's dictatorship "denoted for its sexism and gynopia" [3], when women were not allowed to hold a bank account and work without the legal authorization of the male householder.

She made her way in the men-dominated profession of architecture fostered by the family background, just as it occurred—according to MoMoWo investigation—to most of the women of her generation and previous ones [7]. Conversely, unlike most of them, she has embedded in her professional practices (architecture and music composition) a deep understanding of "everything related to the world of science" [8].

Anna Bofill's father, Emilio Bofill Benessat (1907–2000), was an architect, builder, and developer; he was a republican, liberal and progressive man and guided her into the world of construction. While Anna's mother, Maria Levi, a prominent patron of post-war Catalan literature, passed on her the passion for art and music, which prompted her to become a musician and composer.

In Anna Bofill's thought, architecture and music are very much inter-connected since, as she explains, "In architecture, you distribute volumes in space. And in music, you distribute sounds over time. These are very similar tasks and can follow the same patterns: you can place the volumes in space following the same structures that you used to locate the sounds in time" [9, transl. by the author]. The composition's architectural and musical structures, which she refers to, developed mainly through mathematical resources and methods, including algebra and theories of game, information, and probability. Regarding electroacoustic music, Josep Maria Mestres Quadreny—with whom she studied since the late 1960s—Luigi Nono and Iannis Xenakis are among the most influential figures.

Architect and music composer Xenakis had inspired Anna Bofill since her youth when she visited the famous Philips Pavilion he designed with Le Corbusier to create an immersive environment for the *Poème Electronique* by Edgar Varèse at the 1958 Brussels World Fair. The 'elective affinities' between the two are impressively recurrent. In his doctoral thesis (1976, May 18th), Xenakis proposed establishing a multidisciplinary new science of general morphology that encountered Anna Bofill's holistic vision of research. In the second phase of her life (opened in 1980), she translated Xenakis's book *Music and Architecture* into Catalan (1982), and she worked at Centre d'Études de Mathémathique et Automates Musicales—CEMAMu in Paris under his direction (1985).

Anna Bofill's power to handle different disciplines, moving disciplinary boundaries freely in a flexible space of knowledge, was already unfolded when she enrolled in architecture school (1964). At the same time, she began working at the Office that her father and brother, architect Ricardo Bofill Levi (Barcelona, 1939), had recently co-founded. They had envisaged an innovative concept of professional practice together with key people from the Catalan leftist progressive intelligentsia.

The T. de A. originated as a versatile and culturally lively laboratory to revitalize the conception of the built environment and question the ways of thinking and designing Architecture for a new democratic society. Grounded on both multidisciplinarity and experimentation, it targeted its investigations on the design process involving an open and forward-thinking group of architects, writers, artists, sociologists and mathematicians. Thus, it became a reference for a whole generation of Spanish architects which advocated freedom of expression, associative practices, democracy, and individual and social rights, reacting against the Francoist regime.

Among the initial members of the T. de A. (architects Ramón Collado, Joan Malagarriga, Dolors Rocamora, politician and writer Salvador Clotas, poet José Agustín Goytisolo, and actress Serena Vergano), Anna Bofill shared her interest in the geometric generation of forms with Manuel Núñez Yanowsky (Samarkand, 1942); who was a student of archaeology, history and dramatic arts, and later became an established architect.

The two approached the subject following a different method. According to Anna Bofill, Núñez Yanowsky "used intuition and trial and error, manipulation of volumes, cubes and parallelepipeds and others" [10] similarly—as one might advance—to those Froebel Building Gifts popularized in the design field by the Bauhaus and by masters of architecture, including Frank Lloyd Wright and Richard Buckminster Fuller. Differently, Anna Bofill "tried to find a logical/mathematical/geometric sense to it all", as she revealed in her 2017 interview.

The choices which structure her geometric-mathematical method—"Theory of Form"—rely on a groundbreaking multidisciplinary scientific approach. In addition to 1960–1970s modular architecture and megastructures principles (e.g. by Yona Friedman, and Kenzo Tange) and urban planning (e.g. perceptual form of urban environments by Kevin Andrew Lynch), Anna Bofill approach includes linguistics (e.g. by Noam Chomsky), generative grammar (e.g. by Gabriel Ferrater), sociology and psychology (e.g. by Burrhus Frederic Skinner), human geography (e.g. by Henry Lefebvre), and last but not least, mathematics (e.g. by Hermann Weyl).

She studied probabilistic mathematics with the mathematician Eduard Bonet Guiñó (b. Girona, 1936, her husband until the 1980s) and applied the Monte Carlo method to the composition of her first musical piece Esclat (1971) which also follows the generative geometric rules of Walden-7. Probabilistic mathematics is conceptually related to the "Theory of Form" presented in her thesis, although the text does not explicitly mention it. In architectural as in music composition, Anna Bofill ultimate will is creating the "highest possible degree of freedom" [5], while keeping the systematic control of mathematical procedures; by absorbing Bonet's lesson, she combined rigorous rules with the freedom of randomness.

2 Resiliency in Voids Full of Possibilities: Anna Bofill Levi's "Theory of Form"

The "Theory of Form"—explained by Anna Bofill in Part II of her thesis—is a mathematical-geometric method to generate complex sets of porous-cellular inhabiting spaces that have the potential to cope with changes of needs and expectations for the forming and re-forming of Architecture and society. For its intrinsic qualities of adaptability, we can now recognize the method as a 'mathematical tool' to design a resilient "architectural-urban space", called by Anna Bofill "urban tissue" using a biological metaphor.

The real, physical, architectural space is "the empty interior space (…) contained within the enclosure of walls (…), that surrounds it and that primarily serves as the environment, as the stage on which our life unfolds" [5], as Anna Bofill explains by quoting Bruno Zevi (*Saper vedere l'architettura*, 1st edn. 1948). Within the concept of "urban tissue", empty spaces are not only piazzas, squares, arenas, gardens, parks, streets, pathways, but also courtyards, loggias, patios, terraces, roof gardens, staircases, balconies, walkways, arcades.

According to the architect, in her systematic 'cellular aggregation', some of these out-door-spaces are the "counter-form" of indoor spaces, and both are "the total space" equally regulated by the same mathematical process of generation [1]. In other words, all voids in real physical space are not residuals, left-over spaces, since the configuration generated by the process is geometrically all-one in the Euclidean space (E^3).

To mathematically generate an "architectural entity", the architect assimilated the physical space to the E^3 associated with the vector subspace R^3. She established an affine space in search of volumetric settings suitable to fulfill a series of matters on tangible and intangible inhabiting needs and building techniques applications.

In her method, based on algorithms and rigid motions laws (isometries), every configuration of identical volumes results from the sum of motions applied to the initial geometric solid. Every inhabiting space is the position that the solid takes after the systematic application of predefined rigid movements.

Two fundamental choices are at the base of the method: using parallelepipeds as basic elements, called "minimal cells", and recursively using isometries to form "units", sometimes "nuclei", "bodies", and finally "urban tissues", in a growing progression.

The architect studied group and sub-groups of isometries to ensure a rigorous systematic generation of settings. In her thesis, she refers to the seminal book *Symmetry* (1952) by Hermann Weyl (one of the twentieth century's greatest mathematicians), which led her to a sort of 'biomimetic' composition strategy in some way. Concering the groups of symmetries, she also refers to Alan Bell and Trevor Fletcher (*Symmetry groups*, 1964), and she considers *Linear Algebra and Analytic Geometry* (1971) by Heinrich W. Brinkmann and Eugene A. Klotz as the most complete and precise classification of isometries for her purpose.

As far as concerning the mathematical analysis of the early works of the T. de A., we refer back to the study by María del Carmen Gómez-Collado and Macarena Trujillo Guillén from the University Institute of Pure and Applied Mathematics of the Polytechnic University of Valencia [10]. These mathematicians replicated the geometrical generative rules through Mathematica® software by assuming the doctoral thesis of Anna Bofill as the primary source. They tested the method using the cube as a primary generator, and by applying to it some main rigid transformations chosen by the architect—e.g., translational symmetry, rotation, helical motion—they have proved its materialization in several works by the T. de A.

Taking the result of this mathematical analysis as well-established, we discuss the geometric method as regards its meanings and implications within the multi-faceted concept of adaptability in terms of construction techniques, environmental and social sustainability. Once this adaptability set embodied in the T. de A. projects, it determines thespaces' resiliency to inhabit, from the micro to macro scale.

2.1 Adaptabilities to Inhabiting Needs, Buildings Techniques, Time and Cost Constraints

Anna Bofill chose isometries as generative geometric law to be applied to a "minimal cell" first and then to "units" generated from that cell; the cell serves as the "basic element" to originate the entire space, including voids. She chose parallelepipeds as geometric shapes for practical issues since she envisioned the cell as a living place. In her theory "*The basic element or residential module can be any parallelepiped whose dimensions or proportions are adequate for the respective residents and living conditions in general. Depending on the intentions of those involved in realizing a particular urban centre, one or another type of geometric body will be preferred*" [5].

For example, the cell of Walden-7—a built urban and mixed-use development for 1000 inhabitants—is a cube of 5.20 m side. In an ideal reference to Henry David Thoreau's experiment of individual self-sufficiency that he lived near Walden Pond, the cell's size is suitable to fulfill the essential dwelling needs for a single man or woman. Although every cell's geometrical volume is the same, the cells all differ from one another; each cell has a separate entrance and the location of the entrance door ensures visual privacy. The interior walls do not form conventional rooms that fix the functions into isolated compartments to respect differences in lifestyles of the inhabitant. Walden-7 has been conceived valuing individual freedom rather than individualism. Combining two, three, or four cells, horizontally or vertically, over two levels is possible when individuals wish to live together as a family or whatever else. The voids between those cells and their physical or visual interconnections are the places through which collective activities and individual human relationships can express themselves, and the sense of a lively community can come into being.

Walden-7 represents the built achievement of all previous research and experimentation, including the Barrio Gaudí and La Ciudad en el Espacio; this outstanding work marked the end of a T. de A. season.

2.2 Prefabrication, Industrialization or Traditional Local Building Techniques

Since Anna Bofill conceived her method as an operating "mathematical tool" for new architectural-urban configurations, the choice of parallelepipeds and isometries reverberates the practices and construction techniques of its time. It includes the possibilities offered by the spread of prefabricated elements and components that cannot undergo deformations in shape or size. As isometries on a built body, including its components, do not change perimeters, areas, and volumes, and only the position in the space changes, they were the most suitable for the construction field. The method allows the construction of complex shapes by maximizing the diversity of spaces while maintaining the highest degree of uniformity of the building elements and components.

Due to the possibility of replicating the same construction elements, components, construction techniques and technologies, construction based on cuboid modular cells can both keep construction costs low and offer various types of dwellings capable of adapting to changes in needs by the aggregation in "units". Besides, the geometric aggregation law applied to the "units" can provide that no unit type is the same as the near one, as it occurs in the Barrio Gaudí. For this "city in the city"—with a structure of reinforced concrete and brick cladding—the economic constraints and the lack of industrialized technology dissuaded an extensive use of prefabricated construction elements [11] (Fig. 2). The modular aggregation has proved to excellently adapt to more conventional techniques that were the less expensive, therefore the most suitable for affordable housing demanded by workers of the industrial town of Reus.

While perfectly meeting the requirements of prefabrication technology and the industrialization of buildings, the modular aggregation was also highly compatible with traditional local construction techniques, thereby allowing to cope with the scarcity of modern technologies in Spain at that time.

Apart from this specific situation, the use of techniques and construction elements of the regional vernacular architecture characterized a short testing phase of T. de

Fig. 2 Barrio Gaudí (1970–1971); ©HASXX_Tarragona http://arquitecturatarragona

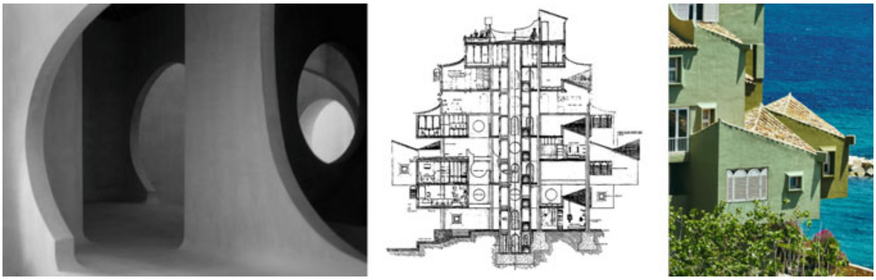

Fig. 3 Taller de Arquitectura, Xanadú (1966–1971); ©RBTA https://ricardobofill.com

A. research. According to art historian Christian Norberg-Schulz, the references to vernacular were a reaction against the International Style [12], as it occurred in the buildings of Plexus (1965–1966) and Xanadú (1966–1971); the latter, a prototypical experiment of a "vertical garden-city" (Fig. 3).

In both buildings of the tourist complex La Manzanera at Calpe (Alicante)—composed more by the phenomenological method of Manuel Núñez Yanowsky rather than by that of Anna Bofill [5]—cellular architecture accepts sloping-tiled roofs, in a hyperboloid variant, and the Mediterranean shutters on the windows. These traditional solutions have been deliberately used to integrate the new form within the costal's area traditional anthropic landscape.

Convesly to the realizations mentioned above, the use of prefabricated building elements characterized the conception of Kafka's Castle. The apartment complex with recreational facilities, sometimes labeled as "post-cubist", was meant as a 'cluster' of reinforced-concrete precast cubic modules every one designed for a specific function bathroom-bedroom or living room-dining room. The "dwelling-units" (apartments) are formed through the various possible aggregations of the functional cells, according to the users' needs, and interconnected at multiple levels creating porous spaces.

The use of precast building cellular-architecture has spread in the most experimental international architectural scene since at the 1967 World's Fair in Montreal Moshe Safdie built Habitat 67. Although the direct influence of this model community and housing complex on Kafka's Castle is not proven, the affinity between T. de A. and the Israeli-Canadian architect transpires from the American master Louis Kahn, of whom Moshe Safdie had been a disciple, and from whom the T. de A. drew lessons.

As regard to Anna Bofill's specific research interest in designing extended "urban tissues", construction elements allowing for industrialization fully met into the mathematical conception of macro-projects the Ciudad en el Espacio, a "city of the future" designed for Moratalaz near Madrid by recursive translations of the unit "T" (Fig. 4) and La Petite Cathédrale, a covered street about 300 longs and 70 m high, 20–60 m width for the new town of Cergy-Pontoise (northwest of Paris on the river Oise). This non-built project is an example of the geometric possibilities offered by the "U" shaped units to form a rectilinear succession of void covered spaces,

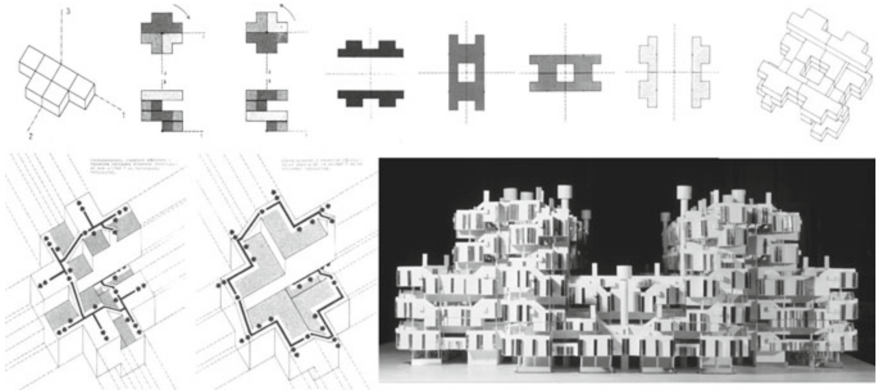

Fig. 4 Anna Bofill schemes of isometric movements of a "T" element referred to la Ciudad en el Espacio (project, 1968–1972) [1], and model from ©RBTA https://ricardobofill.com

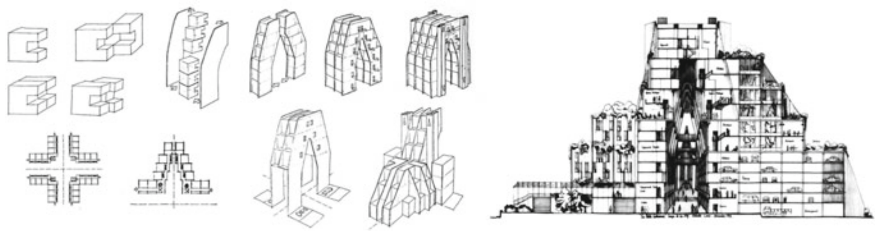

Fig. 5 Taller de Arquitectura, La Petite Cathédrale (project, 1970–1971). Schemes of aggregation of "U" elements, and cross-section; ©RBTA https://ricardobofill.com

like the streets, and uncovered, such as piazzas, and "cloisters", both suitable for the climate and morphology of the site. Besides, the aggregation of "U" well meets the symbolic image of great historic-cultural tradition in France, that of the Gothic cathedral (Fig. 5).

2.3 The Standard Time of Construction for Non-standarized Architectural Bodies

In extreme synthesis, we may claim that the process of recursive functions on which Anna Bofill isometries rely-on allowed maintaining standard construction costs and times despite the unconventional porous clustering of the built bodies in the real space by whichever building technique used.

Despite the complexity of the overall form, the mathematical rules have made it easy and fast construction. They allowed for the reduction of the number of the

Fig. 6 Anna Bofill's schemes of helical motion for Kafka's Castle [1]. Photos ©RBTA https://ricardobofill.com

ordinary constructive drawings, thereby facilitating and speeding up the builders' work.

As for the case of the Barrio Gaudí, the process has made it unnecessary technical drawings for variations at each level, and—according to James [11]—the construction of Xanadú not demanded plans or elevations, while only five sheets of paper held all the information needed to build that gigantic porous maze-like named Kafka's Castle. The complexity of its appearance comes from the helical plug-in scheme of the units' aggregation on the staircase-cores. The flight of the stairs originates an equation from which resulted in all the primary information technically represented in only one sheet of paper, and just four secondary technical drawings detailed the variations of the plug-in of each unit [11]. This residential estate's architectural body results in fact from applying to a "basic element" a helical motion in a recursive form, thus generating the vertical development, as Anna Bofill explains in her thesis (Fig. 6).

The helical movement creates different void networks within the cubic composition that offer users to enjoy shaded or sunny spaces and a variety of ever-changing openings on the cell walls creating a lively and healthy perceptual quality, from a psychological point of view, respecting privacy.

Due to the "concept of the plug-in of cells", Kafka's Castle is perhaps the work that most clearly shows the reference to the Archigram, as the supporting structure of the units is autonomous enough from the structure of the circulation bodies (staircases built out of structural bricks).

3 Porosity and Adaptability for Thermo-Hygrometric Control and Socio-Cultural Environmentalism in a Gender Perspective

According to Anna Bofill's theory, not all isometries applied to a "minimal cell" have the potential to form a "unit", just as not all units have the potential to become "nuclei" or "bodies" by the application of isometries. As a result of our empirical analysis, it appears that these potentialities are correlated to the voids, "interspaces", which have not to leave "residual areas" [5]. In other words, voids originate as places and not as "vague" spaces.

In continuous isometric motions, the specific condition for a cell to become a "unit" is to share a surface, edge, or vertex within its reproduced cell generated by the rigid movement. The same condition applies to “units” to become “nuclei” and then “bodies”. In general, we can say that the open configurations of cells (e.g. “L”, “T”, “X”, cross or double-stepped shaped “U”) are those with the potential to became units; specific conditions of the project, such as density or types of functions, will determine one arrangement over another.

The porosity (voids and interspaces) allows adaptation to the distinctive qualities of the anthropic or natural landscape, including the site’s morphology and climate. The voids and interspaces facilitate the design of a microclimate and passive thermo-hygrometric comfort regarding ventilation, shadow, and light control of each place (indoor and outdoor) to inhabit.

In the Barrio Gaudí, the “nucleus” forms around a patio, and several nuclei are joined to create more patios and diverse voids configurations. The voids facing north, northwest, and northeast orientations receive the interlockings zones of the nuclei and connections such as pedestrian walkways layered on all building levels rather than on the sole groundfloor. Moreover, the best sunlight exposure hosts communal spaces such as elevated gardens and squares, balconies, and terraces of each ‘nucleus’. This multi-layered outdoor spaces network has recreated the pleasant atmosphere of places for the encounter between people and families of the old towns’ Mediterranean architecture through a modern language.

The gender perspective underpinned the conception of Barrio Gaudí since in a housing neighbourhood for industrial workers, as this one is, women who did not work would have remained—alone or together with their children—the only inhabitants for most of the day time. Porosities in geometric aggregation generated those agreeable and safe outdoor spaces for women’s social life and children’s play. Talking about these spaces in an interview for the fifty years celebration of the Barrio Gaudí, Anna Bofill remembers: “I grew up in a house where clothes were hung on the terrace, and the children played, it was our meeting space in the building. We wanted to find this spirit of the community space typical of Mediterranean cities” [13].

The acknowledgment of gender differences proposed for the design of the Barrio Gaudí was groundbreaking in Francoist Spain. The recognition of women’s specific needs and desires for living in the city according to their different perceptions and experiences of the built environment has marked the new phases of Anna Bofill’s work as an urban planner since the 1980s.

In 2005 she stated: “A democratic society should allow each of its members [women and men] the possibility of developing their full potential and personal life project, including the possibility of acting on the society itself, contributing to its configuration and influencing its transformation. Our quality of life depends not only on ourselves but also on the relational fabrics that we can build with others and with the environment, and thus satisfy our needs and desires” [5].

The “Theory of Form” responded to the restrictions of freedom, social control, cultural and gender segregation, and the vulnerability of architectural and urban configurations exposed to the risk of lost democracy and rights of their inhabitants. The main Theory interest is to develop resiliency in social-connectivity that can create and recreate the space through a bi-univocal relationship between form and function.

As for the mathematical-geometric generation of "urban tissues", we claim to the conclusion that the novelty introduced by Anna Bofill is not in the aggregation of modules, but it is in the generation of voids. Complex nets of voids can change lives for the better, modify power relations between inhabitants, within porous spaces full of possibilities, multiple responses to individual and social human instances open-up. The systematic generation of these spaces is one of the most promising Anna Bofill Levi's legacies to design a built environment for a future of sustainable and equitable urban life.

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Christine de Pizan and Her Treatise on Fortifications (1410)

Damiano Iacobone

Abstract Christine de Pizan/ Cristina da Pizzano (1365–1430) is well known as poet and as author of *The Book of the City of Ladies* (1405), in which she describes female leaders in history. Instead it's quite unknown her book *Livre des fais d'armes et de chevalrie*, written around 1410, translated in English in 1489 and also in 1932 (*The Book of fayttes of armes and of chyualrye*). Apart from editorial events, this Treatise is important especially because it's the first to re-cognize the role of technology of medieval warfare and it's an important source for gunpower weapon technology, as well as for strategy and tactics. Based on Vegetius's theory on fortifications, she describes—in the second part—new forms and features of fortresses, also introducing parade grounds along the walls, which was an advanced system of defense for the XV century. She has been the first to talk about artillery and its importance to organize (also geometrically) fortresses in a Treatise, while other important authors, such as L. B. Alberti or Filarete, completely ignored this item in their books. The importance of the Book has been recognized by Italian military historians since the XIX century, for example Carlo Promis and Ignazio Calvi.

Keywords Christine de Pizan · Fortifications

1 Introduction

Christine de Pizan was born in Italy, Venice, in 1364 (as Cristina da Pizzano), but soon after she had to follow her father at the Court of Charles V of France in 1368, and she spent all her life in Paris. At the French Court she had the chance to receive a literary culture that would be of much importance further on in her life. In 1379 she married the notary and royal secretary Etienne du Castel. In few years her life changed completely: Charles V died in 1380, her father died in 1387 and her husband in 1390 of the plague. To support herself and her family, putting to good use her humanistic studies, Christine started to write ballads, poems and books so to become "the first

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professional woman of letters in Europe". Christine de Pizan wrote many successful books, but in this context it's important to quote: *Le Livre de fais et bonnes meurs du sage roy Charles V* (1405), in which she described Charles V as an ideal king to express her gratitude to him, and *Le Livre de la cité de dames* (The Book of the City of Ladies), published in 1405, surely her most famous work, appreciated everywhere [1–3].

2 The Book of the City of Ladies (1405)

The book aimed to refute unfavourable statements addressed to women expressed by authors as Oved, Cecco d'Ascoli, Cicerone and the last Jean de Meun with his popular Roman de la Rose, to reaffirm the importance of women throughout the history. So that, Christine decided "to build a city", a symbolic city, in which famous women of the past were housed. It's possible to recognize a double level of symbols: each woman with her role in history contributed as a "building block" to the construction and also with each experience to a better evaluation of women [4].

Moreover, there's an interesting shift of level between symbolic and realistic situations at the beginning of the book: even if it would be a symbolic city, there are references to real foundations and construction. The city had to be as a large walled city, a kind of citadel; the place had to be safe, fertile and rich, with rivers and plants, starting with the excavation of a ditch to define the external line of walls [5, 6].

These evaluations surely testify Christine's knowledge of principles and treatises on land surveying. At this point the second level takes over: Christine will be helped by three virtues: Lady Reason, Lady Rectitude and Lady Justice, organizing the book into three parts.

In the first one, Lady Reason tells Christine "to take the spade of intelligence and dig deep to make a trench all around the city" and She helps Christine to build the external walls of the city. Walls represented and symbolically realized by thirty-six women of the past, related for their importance. Just to mention a few of them: Semiramis, The Amazons, Medea, Circe, Minerva, till the last one: Lavinia.

In this part are described women in ancient history with political power: Semiramis who realized many cities and fortresses; Zenobia, queen of Palmyra, expert of warfare, or the noble queen Artemisia, but also those who gave a great contribution to culture as Sappho or Medea and Minerva who invented the art of building iron armours.

In Part II, Lady Rectitude showing a straight line as a sceptre to separate good behaviours from bad ones, gives it to Christine as a tool to measure and "to construct the houses and buildings inside the walls of the City of Ladies". The aim is to realize tall palaces, noble houses and towers for valiant ladies of great renown. After this first step of buildings thanks to few important women of the past, Rectitude invites Christine to choose inhabitants of the City, in particular with examples of women who loved their husband and acted virtuously.

So that, the second book tells the story—for example—of Agrippina the Elder, Julia, Xanthippe (Socrates' wife), Pompeia Paulina (Seneca's wife) and many others (in this second book Christine recounts of ninety-two women). In Part III Lady Justice invites Christine to finish the works with thirty-seven female saints (from Saint Catherine of Alexandria to Saint Lucy, from Saint Barbara to Saint Agnes of Rome) to celebrate the Queen of the City: The Virgin Mary [7–9].

3 The Book of Feats of Arms and of Chivalry (1410): A Short Historiography

There are at least three manuscripts (probably more) of the *Livre des fais d'arme et chevalerie*, written at about 1410: one is held at the Biblioteque Nationale de France, Manuscrits Francais 603; a second at the Biblioteque Royale de Bruxelles, ms. 10476 [10].

The book was printed for the first time in France in 1488 by the publisher Antoine Verard, with the title: *Cy après sensuit le liure des fais d'arme et de chevalrie.* And in the frontispiece: *L'art de chevalerie selon Vegece. Imprimé le XXV juin Mil CCC quatre vings et huit par Anthoine Verard libraire.* Verard presented the book as his own translation of the Roman author Vegetius, without any reference to Christine [11]. Just after one year the book was translated into English for Henry VII by William Caxton, who published it in 1490 with the title: *The Book of Fayttes of Armes and Chyvalrye* [10, 12]. It's not as unusual as it might seem Caxton introduced printing press in England in 1476. Working as a merchant and after as a diplomat, he spent many years in Bruges and after in Cologne, where he had the chance to appreciate the new printing industry. So that, he started a printing press in Bruges and after his return to England also at Westminster in 1476. The first book he reproduced was Chaucer's Canterbury Tales and translated into English many classical works and chivalric romances as *The Book of Feats of Arms.*

In the sixteenth and seventeenth century many works of De Pizan were published anonymously, without any reference to Christine as the author, and this contributed to her oblivion. However, it's in the second half of the nineteenth century that Christine has been 'rediscovered': in fact, in 1838 Raimond Thomassy published his book: *Essai sur les écrits politiques de Christine de Pisan,* giving new life to her knowledge [13].

In this period, we have to consider also the Italian contribution in this direction.

The architect and military historian Carlo Promis (1808–1873), author of relevant book *Dell'arte dell'ingegnere e dell'artigliere dalle origini sino al principio del XVI secolo*, published in 1841, in his essay referred to military engineers of the fifteenth and of sixteenth century born or at work in the area of Bologna (*Gli ingegneri e gli scrittori militari bolognesi del XV e XVI secolo*, 1863) dedicates the opening biography to Cristina (here named Chrestienne) da Pizzano, also because Pizzano is a place near Bologna. Calvi considered Christine's *Libro di Cavalleria* the best military essay written in the fifteenth century, also because in her military treatise she considered the introduction and the contribution of new weapons to warfare, going beyond ancient sources on this subject.

Promis first of all describes Christine's life and after gives his contribution to date *Le livre des fais d'armes et de chevalerie* to 1410 (She talked of the war between Genoa and France, started in 1409).

He asserts that Christine's book was printed for the first time in France in 1488 by Antoine Vérard (see above), as confirmed by following researches, to be translated into English by Caxton who recognized Christine as author of the book.

Promis also refers to a manuscript of Christine's book in the Royal Library of Turin (even if anonymous), that is probably the source of his knowledge of the book. In fact, Promis—with much detail- describes content and organization of the Treatise: the first part (with 29 chapters) referred to kings' management of wars, describing many military leaders of the past. In the second part, with 28 chapters, she talked about military engineers able to involve weapons into wars even if there were many problems related to their use: too many economic investments; difficulty to manage them etc. In the third part Christine wrote about war law and behaviours.

De Pizan referred to Frontino and Vegetius but—as Promis underlined—she talked also of systems of fortifications: for example, to realize parade-grounds along the walls to accommodate the artillery and particularly the guns. Promis considered this system surely what has mostly changed in the defence of a fortress.

These evaluations made by Christine were not considered in architectural treatises written later such as that of L. B. Alberti or that of Filarete and this is surely another great merit: she wrote about the reality of her time and not only of the past.

In fact, her Treatise became the source for other essays written in following periods: Promis refers, for example, to De Bueil's *Traicté du gouvernement monostique, économique, politique du Jouvencel*, printed in 1489 by the same Verard [11].

Another Italian military historian also wrote about De Pizan's book on warfare: Ignazio Calvi. Calvi published in 1943 the book: *L'architettura militare di Leonardo da Vinci*, the first essay on Da Vinci's contribution to the military field. He organized his book analysing which have been the previous contributions to this field (Vegetius, Valturio, Francesco di Giorgio) that influenced Leonardo's thought.

Calvi referred also to De Pizan's Treatise, named *Dei fatti d'armi e di cavalleria* (in Italian), as the first book to analyze weapons and fire defenses to protect fortress, updating the military strategy, while other authors didn't do (Alberti and Filarete). Calvi probably didn't know Promis text because in 1943 he considered Christine's Treatise unpublished (apart some extract- he says) also ignoring the English edition of 1932 [14].

Anyway, Calvi's consideration of De Pizan's work testifies her role in the field of fortifications and military history.

In 1932 Caxton's English edition has been re-published with the title: *The Book of Fayttes of Armes and of Chyvalrie*, edited by Alfred Thomas Plested Byles [15], who had already edited in 1926 *The Book of the Ordre of Chyvalry* by Ramon Lull and Adam Loutfut and translated by the same William Caxton.

More recently, in 1999, Sumner Willard, former professor at the U.S. Military Academy and Charity Cannon Willard, the preeminent Christine de Pizan scholar, have realized a new edition of the Treatise: *The Book of Deeds of Arms and of Chivalry*. It's not based on previous English translations, but on the manuscript held at the Bibliothèque Royale de Belgique, Bruxelles (Ms. 10476), with detailed references to her sources [10]. This latest edition has been adopted to arrange this essay, even if it would have been better to refer to a French modern transcription of De Pizan's work.

4 The Book of Feats of Arms and of Chivalry (1410): Topics and Issues Related to Fortifications

The first part of the Book is opened by the Prologue, in which Christine states to have written about "such exalted matter": the honorable office of arms and chivalry, not moved by presumption but to express this matter in the plainest possible language, avoiding refined words and expressions, also to let military and soldiers have knowledge of the Treatises of the past, she had recollected in a less detailed way and in a simple language (that is to say no more in Latin).

At the beginning she reaffirms that many books of the past have been collected to form her essay, in particular Vegetius and Frontinus. This approach is not unusual with Christine, rather it identifies her method of writing: to sum some "exempla" from different sources, explaining ideas, topics and issues in a simple language useful for much more people.

We can recognize that it's surely a great merit of her volume, even if -at the same time—she doesn't explain technical questions in a thorough way.

The Book of Fayttes of Armes and Chyvalrye is organized in four parts: in the first one she talked about the Just War, the reasons of wars and battles and the considerations of a king or prince should entertain in initiating wars, such as about the training of children to become valid men-at-arms.

In the following paragraphs, Christine deals with the issue of tactics: the way in which a commander will try to lodge his army, the way he will organize moving troops from one place to another, the way to cross a river, to spend the day before the battle, the ways of drawing up an army for combat. In many cases she refers directly to her main source, reaffirming in the title: according to Vegetius. "In the event that war was set in motion, after the prince's deliberation [..] the wise leader will, to begin with, order that frontiers be well garrisoned, with both good men-at-arms and artillerymen, as well as other necessary things and provisions. Cities and fortress should be prepared so that nothing further will be necessary (I, xii)" [10, p. 37].

In the case of a battle in the opened field, referring to the choice of a place and to its protection, Christine's words are slightly detailed in comparison to Vegetius's instructions. For the first matter, she writes: "He (the commander) will lodge his men as advantageously as possible, beginning by determining whether he can have a situation advantageous to him, and not to his enemy" [10, p. 39].

The question was much more complex and organized, as Vegetius' words testify: "Camps -especially when the enemy is near – should be built always in a safe place, where there are sufficient supplies of firewood, fodder and water, and if a long stay is in prospect choose a salubrious site, care must be taken let there be nearby mountain or high ground which could be dangerous if captured by the enemy. Thought must be given that the site is not liable to flooding from torrents and the army to suffer harm in this event. [….]" [10, p. 39 n.52; 16, I xxii, p. 108].

A second matter refers to the geometry of fortifications of a camp: also, in this case Christine derives from Vegetius the structure of such organization, but she doesn't understand fully some technical parts, for example the importance of the embankment to protect the camp. In fact, she writes: "The book of arms says further that if the army must remain for long in one place to await a great enemy host, the spot must be fortified with very good ditches, twelve feet wide and nine feet deep, or thirteen if the earth taken from the ditches is nearby […..]. But, if the army does not need to remain there for a long time, or if no great force is expected, there is no need for such fortifications, but rather it is sufficient, if ditches are desired, to have them eight or nine feet in width and seven in depth" [10, pp. 41–42].

Instead Vegetius wrote: "When there is no pressing danger, turves are cut from the earth and from them a kind of wall is built, three feet high above the ground, with the fosse from which the turves are lifted in front. Then there is a temporary fosse nine feet wide and seven feet deep. But when more serious forces of the enemy threaten it is advisable to fortify the perimeter of the cam with a proper fosse, 12 feet wide and nine feet deep below the 'line'. Above its revetments are built on either side and filled with earth that has been raised from the fosse, rising the height of four feet. The result is that the fosse is thirteen feet deep and twelve feet wide" [10, p. 42; 16, I xxiv, p. 110].

The main difference is represented by a dissimilar consideration of the part 'above the line', that is to say a kind of wall realized with the earth of the fosse to create protection from the outside, of three or four feet high.

Moreover, Christine resizes the role of the engineers of the legion, talking of men-at-army able to realise structures or useful things and not of engineers, carpenters, masons etc. as told by Vegetius [16, II xi, p. 150].

In Part II Catherine takes into consideration many 'examples' of the past, that is to say she analyses many military leaders and conquerors of the past to appreciate their teachings, referring in particular to Sextus Julius Frontinus' *Strategemata.*

So that are described deeds and stratagems of Scipio The African, Pericles, Alexander, Hannibal, Julius Caesar, Fabius Maximus and others.

From the 14th paragraph onwards, Christine talks about castles and fortresses and also of the techniques to defend them.

First of all, to build an enduring fortification it has to be followed five simple rules: to choose an elevated position, in a fertile and productive place with healthful air, avoiding hills and mountains close to the site.

Walls have not to follow a straight line, because in this case they would be easy to break with war machines or to be scaled. On the contrary, walls have to be curved, including angles defended by towers. Whenever possible, it's a good system to realize two walls: "the first would be elevated and so thick that passageways could be made on them with openings and slits for firing cannons and hurling stones and other sorts of missiles, and on each side there should be suitable well-built emplacements for projectile throwers, if these should be needed to defend the place […]" [10, p. 105].

In this sentence Christine goes beyond Vegetius' indications, referring to the use of 'cannons' (the early types of them) and the need of places for projectile throwers.

We have to say that fire defences worked better in the line of enemies (following a horizontal line of fire) and the vertical trajectory of this attack followed anyway medieval habits, not so useful in these cases.

Anyhow it's an attempt to give importance to new systems of defence, even if in a traditional organization of a fortress.

Before the siege, continues Christine, it's so important to have provisions of cannons, much gunpowder, stones, machines for firing different kind of objects. Christine describes with much details the quantities of the provisions for the defence of a fortress: "First of all at least twelve cannons throwing stones, two of which will be larger than the others to break up machines, mantelets, and other coverings, if necessary […]. Likewise, if it is thought that it will not be necessary to fire the cannons too frequently, a thousand pounds of powder should suffice, three thousand pounds of lead to make shot for the cannons […].

So, Christine wonders: what is necessary for attacking an important stronghold according to present-day custom? The answer is: a great deal of equipment (considering only the cannons), which is to say: "Item four large cannons: one called Garite, another Rose, another Seneca and the other Maye, the first firing four to five hundred pounds in weight, the second firing around three hundred pounds, and the other firing two hundred pounds or more. Item one more cannon called Montfort, firing three hundred pounds in weight, and according to the experts, this is the best one at all. Item one cannon called Artique, firing one hundred pounds in weight. Item twenty other ordinary cannons firing stones. Item other small cannons firing lead and ordinary stones of one hundred to a hundred and twenty pounds. Item two other large cannons firing three to four hundred pounds and four smaller ones, three other cannons, one large and two smalls. Item twenty-four other large cannons firing stones of two hundred to three or four hundred pounds and sixty other small ones. […] The total number of these cannons being two hundred forty-eight, which are named separately because they should be placed in various positions according to the situation of the fortress" [10, p. 118].

In the end Christine describes the equipment for a siege of her time (not only) with over two hundred cannons of different power, while -in the following paragraphs-she describes traditional assaults according to Vegetius.

The book will end with a third part referred to the rights of arms, according to written law; and a fourth part concerning of the law of arms with reference to safe conducts, treaties, letters of marque and of private combat [17].

Christine de Pizan in her *Book of Feats of Arms and Chivalry* refers many times to Vegetius, especially talking about military strategy and tactics. In few parts she writes of artillery in the siege of a fortress, also with many details (even if we don't know her source in this case), but with technical uncertainties.

It must be considered that artillery needed a very long period of experimentation to be used properly and this also happened with regard to the knowledge of its technology. For example, also in the manuscript written approximately in 1483 by Martial d'Auvergne: *Les Vigiles de la mort de Charles VII* (Biblioteque National de France, Ms. Fr. 5054), we find errors of representation or positioning of the artillery, although in France firearms were already used in the mid-fourteenth century [18].

In fact, bombards had been used by the English Army at the battle of Crecy in 1346, during the Hundred Years War (as testified by Giovanni Villani in his Chronicle of 1348).

Only in the mid-fifteenth century did cannons completely replace the old throw weapons, as evidenced in the French campaign to regain Normandy, which ended in one year (1449–1450) thanks to the gunners of Charles VII, up to the conquest campaign in Italy of Charles VIII (1494), represented on the march across the Peninsula with pieces of artillery pulled by horses that carry gunpowder.

Consequently, also military or architectural Treaties registered this ever more evident change in defence systems with the use of bombards and cannons, starting from the early fifteenth century with De Pizan's but also with Konrad Kyeser's Bellifortis (about 1405), which proposed wagons who carried small bombards or machines called machine guns.

It's with R. Valturio's treaty *De Re Militari* (1446–1455, published in 1472) [19] that artillery is fully understood in its technological components (Fig. 1) and in its application in defensive systems there changing, up to the full discussion made by Francesco di Giorgio Martini in his *Treaty of civil and military architecture* (1482),

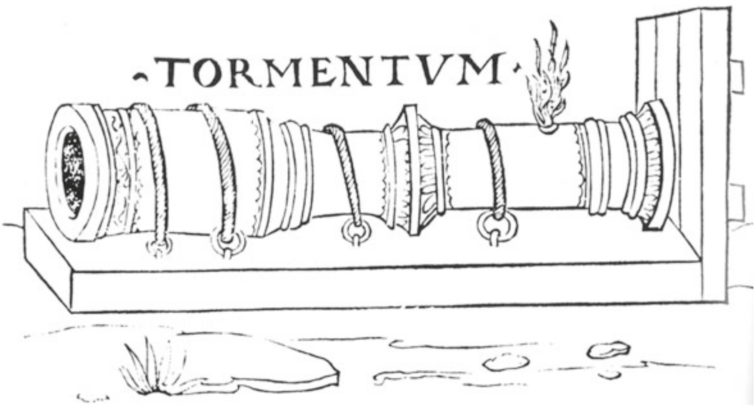

Fig. 1 R. Valturio, De Re militari (1st ed. 1472), 1534. Bombard of the fifteenth century [17]

who also attempts a classification of bombards and cannons, which is a prelude to mathematical studies of Niccolò Tartaglia applied to ballistics [20].

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Transposition of Biomimetical Principles into Generative Design: Example of the Species *Campanula patula L*

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Abstract This paper deals with the flower geometry analysis of the species *Campanula patula* L. (Campanulaceae) and attempts to use a freeform curve to transpose it into a surface model that can serve as a geometric pattern. Interest in geometric forms and processes that occur in nature and their implementation in modern design solutions are expanding thanks to the access to virtual technologies that have enabled the use of generic models in design. Relying on the approach of generating shape created as a product of transponding and decoding biological characteristics from a geometric point of view, using the widespread type of *C. patula*, an example was made in which the consistent geometry of a flower of this type served as the basis for obtaining a surface form. Based on the geometric analysis of the flower a form is presented and described as the reference of the freeform curve. The surface of the flower is represented as a frame that constitutes spatial curves, that is, a pattern that illustrates the carrier of the biological form. This form has been transformed into a model that is subject to manipulation and generation, for implementation in design solutions.

Keywords Biomimetics · *Campanula patula* L. · Freeform curve and surface · Generative design · Geometry pattern

1 Introduction

This research deals with biological design, more precisely it relies on the shapes that we encounter in nature and eco-design, a form that would later find applications in design items in accordance with user needs. By applying designer constructive tools, virtual computer programs, based on geometry of the objects, and on biological principles, it is possible to assume the characteristics that will affect the quality of the

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obtained form. From the point of view of geometry, generating forms is a process that results in the transformation of biological characteristics. In this process, the free forms of its biological references and through the formal relationship via morphological differentiation continues the process of generating through geometric features. This design process, digital morphogenesis, relies on a non-generative, algorithmic basis, and was introduced to generate generic forms that provide maximum optimisation of resources and exceed the restrictions imposed by the environment, materials and functionality [1].

Decoding of the natural form was carried out using the freeform curve on the *C. patula* flower, chosen for its simple and consistent flower form. It is additionally of interest because of its wide distribution. The results of the research were obtained by using design tools based on the generation of a large number of different inputs to obtain models of certain morphological characteristics that differ from the natural form. Using virtual designers, the aim was to achieve a morphological coherence between the natural model and the newly created structure, introducing into the design of quality that arises from the biological form, such as the evolutionary balance, the possibility of integration into the environment, the fluidity of the structure and its multi-functionality [2].

The flower, as the reproductive organ of the plant, can have three types of symmetry: actinomorphic (polysymmetric) in which, by placing the plane through the centre of the flower, an unlimited number of levels can be set; bisymmetric where two symmetry planes can be set; and zigomorphic (monosymmetric) with which, by placing the plane through the centre of the flower, it is possible to set only one plane that divides the flower into two identical halves. The flower of the *C. patula* is bisymmetric, and this feature can be used as the starting point for seeking the geometric pattern. One of the more common species in Europe, the spreading bellflower *C. patula* is used as the model in this paper [3]. Application of *C. patula* is a biennial herbaceous plant with an erect and roughened stem. The leaves are narrow with a short pedicel. Inflorescence is in a rich umbrella form with long lateral branches or a racemus form with only 1–3 flowers. Sepals are naked, sepal teeth are spear-like, shorter than the petals or the same length as the bell shaped petals. The petals are five lobed, widely funnelled along nearly half their length, and of pink or whitish colour [3]. This widespread and delicate species grows at forest edges and meadows from low to very high elevations all over Europe. Virtual frames of the *C. patula* flower are used in order to achieve a structure that can be integrated into the physical world. Instead of contrast, it is necessary to provide and establish a link between the artificial structures and the environment, which requires the development of knowledge regarding different levels of complexity of space, time, structure, behaviour, processes and geometry [1]. However, this paper deals with generative models that seek to establish a morphological connection with natural structures at the level of the geometric shape. The paper discusses issues related to the connection of natural forms and their applications in generating generic structures.

This paper aims to contribute to and answer the following:

- Research on the decoding of geometric patterns, natural forms using freeform curves and their application.
- How the newly created structures achieve the morphological quality of natural structures and how to implement these qualities through design.

1.1 Related Work

There are different approaches to the perception of this topic, on the basis of which numerous methods of reconstruction of plants and flowers have been carried out.

Based on one photo [4], offers a solution that does not require calibration of the camera, but photography in parallel projection is the only source of information, which makes the size of the flower negligible in relation to the distance between the flower and the camera. The design implies the detection of the overlapping of flower petals from the photograph and the integration of the petals using surfaces of revolution in the 3D model.

Another method for reconstructing flowers is by using a 3D scanner. Using a 3D scanner [5] captured the visible parts of the flower during the flowering period, represented by the 3D-point of the cloud-forming sequence. Flower reconstruction uses a pattern based on the dynamic tracking of the algorithm during a certain time period. Based on the physical and morphological characteristics of the flower, a synthetic animation is constructed which reconstructs the shapes and movements of the petals.

On the other hand, various methods of simulating the motion of the petals and leaves based on morphological features and changes based on biological theories are present. The simulation performed by [6] is a technique for the realistic deformation of a drying leaf. The leaf is presented as a structure composed of two layers of a Voronoi diagram, created by Delauney triangulation. Simulating a change in the concentration of the design solution controlled by the osmotic water flow through the leaf changes the parameters that affect the change of the shape of the sheet identically to the drying of the leaf.

2 Overview

Considering that the design is created under various influences, needs, creator viewpoints, environmental conditions and others, it is a complicated and synergistic product. However, if it is supported by the inspiration that emerged from nature, it leads to the evolution of a methodology using a biomimetic concept that offers an inexhaustible source of inspiration and solutions [7].

Starting from Watts's [8] view that natural forms are not made, but they are grown—from the inside to the outside, as parts of systems that are linked by simple to complex and which build their own whole, together with Thomson's approach to growth [9], we are bound to biomimetic principles and perceptions of form through the domain of generative art [8]. Producing an organic form from a natural context and making it available to generative forms in order to improve their morphological qualities can produce highly functional structures that are easily integrated into a space [1].

As *C. patula* has a large range of distribution, the petals oscillate in size (ecotypes). In nature, there is not quite the same organization of plant parts, but there are oscillations that can be simulated by computers by changing certain parameters. The basic form is based on the flowers in a natural environment. The shape of the flower model is represented by the surface formed by the contours of the flowers, presented as freeform curves, where the relationship between the congruence of the curve parameters and the facade of the model is direct. The logical sequence of the steps required for the solution leads us through the operations of finding the appropriate set of data, analyzing the boundary values, translating them and modifying the parameters, on the basis of which the visual composition is made. Finding a logical connection between elements can result in predicting and describing the behavior of the structure under certain conditions.

3 Method

In the modeling process, the geometric analysis of flowers and their proportions was applied, based on the freeform curves. Basically, a flower can be placed in a geometric pattern whose elements proportionally approximate each other (see Fig. 1).

The model is formed on the basis of the shape of a contour of the flower, presented in the form of spatial curves. One of the conditions is that the model is easily formed so it can quickly adapt to a configuration change. The structure of the model is oriented towards the reference point, which is the same as the starting parameter from which the central axis of the flower starts, which, in turn, determines how the projection is modeled in the space. The length of this axis is also the height of the flower, which also represents the height of the future model. The shape of the flower and the disposition of freeform curves are the basis of design and the surface carrier because they form the structure of the flower. The approach to the analysis of the geometry of the flower is limited to a photograph, with the position of the camera parallel to the centre of the flower. The shape of the petals varies slightly in relation to the curvature of the surface, and since the degree of curvature is precise, based on the photograph, it can’t be determined but only assumed. That is why we rely on a two-dimensional projection of the shape, obtained by cutting out the contour-column of the leaf from the photograph, simulating the contour of the bending petals. Based on the variation of the parameters curved through the control points and their imbalance, the effect of the curved surface of the petals of the flower model will be induced. The insertion of petals is also possible with the deformation of the surface of the model.

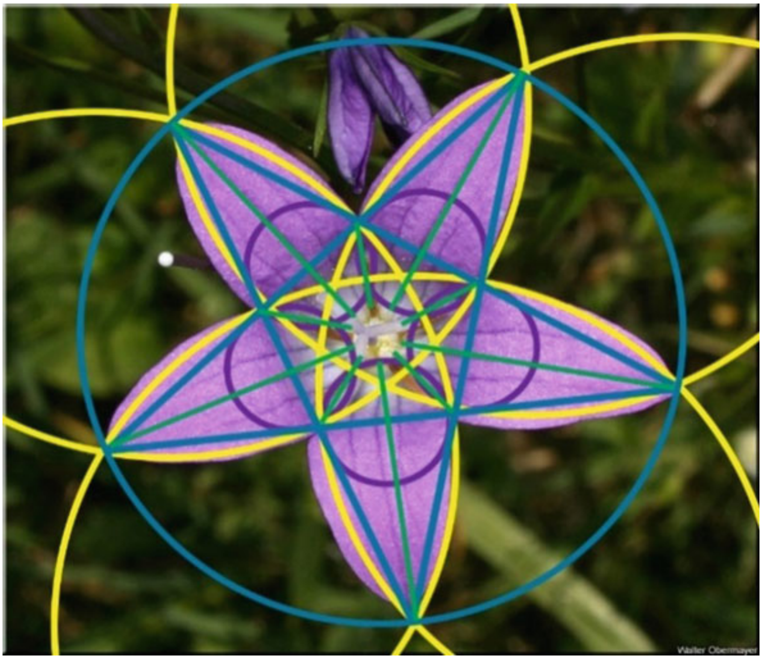

Fig. 1 Geometric pattern of flowers whose elements proportionally approximate

4 Freeform Curves

In order to translate the form of the *C. patula* flower into a geometric pattern that can be used in a later design process, the spatial freeform curve (curve of free forms) was used [10]. Freeform curves represent the widest used curves of free forms. Their geometric construction and some of their properties are based on the Casteljau-Bezier curve algorithm. In addition to Bezier's faults, B-splines and NURBS curves are often used. For all these curves, it is common to use a small number of control points that are related to the control polygon. The setting of the contours of the flower (in the photo) was done using a smooth curve which is performed automatically from the control points by a geometric algorithm. The control polygon serves to control the shape of the curve (i.e. to control the shape of the contour of the petals). With the change of the control polygon, the shape of the corresponding curve changes. The contours of the flower are plotted by entering several control points in the program, based on which algorithm the smooth curve is plotted. This method allows for a later change in the shape of the curve, whereby the shape change is facilitated by changing the positions of the control points (see Fig. 2).

Because the flower petals are limited by the surface, by introducing two more spatial curves representing the edges of the petals, the colour contour algorithm is defined. Modelling of the base of the flower is done in multi-step. The first step

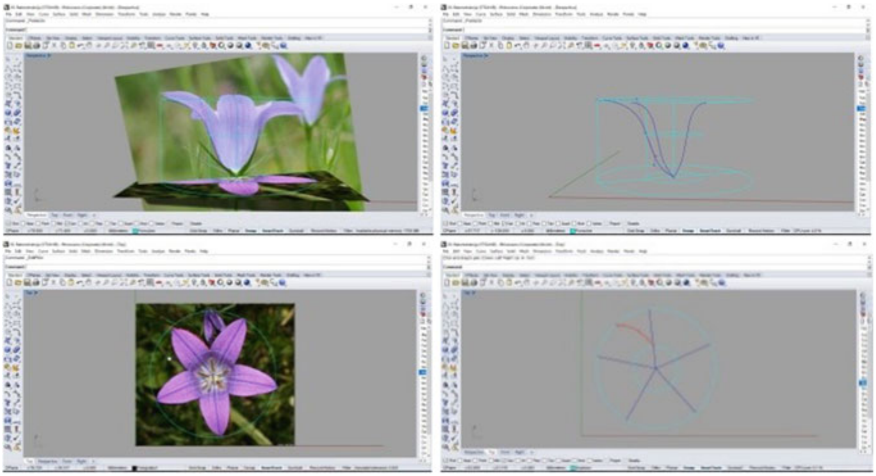

Fig. 2 Construction of freeform curves with checkpoints

involves the introduction of processed photos of flowers into the software. The photos show the appearance of the flower from the top and the appearance of the flowers on the side, and it is necessary to overlap these two photos so that the diameter of the flowers corresponds to both photos (Fig. 2). It should be noted that at this moment, the resolution of the photo is ignored, with the achievement of the particular size of the model done in the final step by scaling. Modeling the details of the inner parts of the flower to show the complete construction of the flower was done manually, based on photographs and drawings, although it will not be used in the later process.

4.1 Mathematical Background to Free Form Curves

Freeform cuves, surface, or freeform surfacing, [11] is used in CAD and other computer graphics software to describe the skin of a 3D geometric element. Freeform surfaces do not have rigid radial dimensions, unlike regular surfaces such as planes, cylinders and conic surfaces. They are used to describe forms such as turbine blades, car bodies and boat hulls. Initially developed for the automotive and aerospace industries, freeform surfacing is now widely used in all engineering design disciplines from consumer goods products to ships. Most systems today use nonuniform rational B-spline (NURBS) mathematics to describe the surface forms; however, there are other methods such as Gordon surfaces or Coons surfaces.

The forms of freeform surfaces (and curves) are not stored or defined in CAD software in terms of polynomial equations, but by their poles, degree, and number of patches (segments with spline curves). The degree of a surface determines its mathematical properties, and can be seen as representing the shape by a polynomial with variables to the power of the degree value. For example, a surface with a degree of 1 would be a flat cross section surface. A surface with degree 2 would be curved in one direction, while a degree 3 surface could (but does not necessarily) change once from concave to convex curvature. Some CAD systems use the term order instead of degree. The order of a polynomial is one greater than the degree, and gives the number of coefficients rather than the greatest [12]. The poles (sometimes known as control points) of a surface define its shape. The natural surface edges are defined by the positions of the first and last poles. (Note that a surface can have trimmed boundaries.) The intermediate poles act like magnets drawing the surface in their direction. The surface does not, however, go through these points. The second and third poles as well as defining shape, respectively determine the start and tangent angles and the curvature. In a single patch surface (Bézier surface), there is one more pole than the degree values of the surface. Surface patches can be merged into a single NURBS surface; at these points are knot lines. The number of knots will determine the influence of the poles on either side and how smooth the transition is. Free form curves are often constructed to satisfy one of three generic problems: interpolation, least squares approximations, and shape approximations.

Interpolation: Given an ordered sequence of (n + 1) points, possibly with associated tangent vector information, generate a curve that passes through the points in the given order, matching the provided tangent information, if any. In the simplest case, we want a single curve to interpolate a set of points. Creating a single curve through n + 1 points requires a curve of degree n. A curve of this type is not unique unless additional constrains are specified. This can be considered as accomplished by specifying parameter values at which the n + 1 point are interpolated. This can be formally defined as:

Let we have: n = 1…. n + 1 points: P0, …, Pn, n + 1 strictly increasing parameter values, t0, …, tn. They generate the unique single polynomial curve of degree n, P(t), such that P(ti) = Pi, $0 \leq i \leq n$.

An alternative approach generates no tone, but several pieces of different curves that, when they are placed end to end exactly interpolate all the point in order. Such piecewise curves are defined as splines. Each piece of the spline is called a span (or a segment). The spans meet "smoothly". Mathematically, we base measures on differential quantities (like tangent, vector, curvatures vectors, etc.). There are two common specific measures used for parametric curves, parametric continuity (C*) and the (usually) somewhat weaker geometric continuity (Gk). The measured can be defined recursively as follows: given curves (or curve spans) P(t) and Q(t): P and Q meet with C0 continuity if the end point of P is coincident with the start point of Q. They meet with Ck continuity if they meet with Ck−1 continuity and if the k-th derivate at the end of P exactly matches in length and direction the-kth derivative at the start of Q. Next also P meet continuity if the end point of P is coincident with the start point of Q (same as for C)0). They meet with Gk continuity if they meet with Gk-1 continuity, and if the unit vector in the direction of the k-th derivative at the end of P exactly matches that of the k-th derivative at the start of Q. Lets us note that C0 and G0 are synonymous. Both require only that the end point of P is coincident with the start point of Q. Note that C0 and G0 are synonymous. Both require only that the end point of P is coincident with the start point of Q. In font design, for example, some letters require sharp corners. We will see one way to get cusps when we consider B-Splines below [25].

Different schemes employ different approaches to achieve desired smoothness goals.

a. Interpolating two or more (point, tangent) pairs One direct way to achieve C1 or (G1) continuity is to allow the designers to associate explicit tangent vectorship each point to be interpolated. Then each span begins at point i, matching the tangent there, and ends at point (i + 1), matching the tangent there. This way we get Hermite curves.
b. Interpolating collections of points without user-supplied tangents: Associating explicit vectors with the points is an ideal way to guarantee that all spans meet smoothly, and the vectors can be design handle. However requiring specification of vectors becomes an unreasonable burden on the designer when he or she simply wants a curve to be generated that passes through a given set of points. Hence most other common spline-based interpolation schemes attempt to automatically determine appropriate tangent vectors to associate with the points based strictly on the supplied point data. Two common such techniques are Catmull-Rom splines and Cardinal Splines. They differ in terms of how they use the adjacent points on a given span, and each employs a scalar parameter when doing so. The image below was created as a Catmull-Rom spline, but it could also have been generated as a Cardinal spline if the parameter were to be set appropriately.

Least Squares Curve Approximation: Given an ordered sequence of (n + 1) points and a desired polynomila degree d < n, generate a curve of degree d that minimize the sum of the squares of the distances between the (n + 1) points and the curve. This technique is oftentimes used in the context of digtizing a given shape because we frequently obtain a large number of points, but small degree cure suffices to capture the essence of the shape.

Shape approximation: Given an ordered sequence of (n + 1) points, generate a curve whose shape mimcs that implied by the ordered set of points [11, 12].

An example: A Bézier curve curve can be represented using a single polynomial of degree n, written as a function of the (n + 1) points (called control points). Two basic properties of Bézier curves are evident in these examples:

– The Bézier curve starts at the first control point and stops at the last control point. The vectore tangent to teh Bézier curve at the start (stop) is parallel to the line connecting the first two (last wto) control points. There are several other interesting properties of Beziér curves. Theer are also spline-based alternatives. Note that a B-spline curve may or may not interpolate any of its control points, including the first and the last.

Knots are the parameters values at which the spline spans meet. The collection of knots for s B-spline is called the knot vector and is typically written as t = (t0, …, tn + k). Thus each span will start at some knot, ti and end at knotti + 1. Knot vectors can be very simple. The knot vector for the B-Spline in the figure above, for example, is just a consecutive sequence of integers: i.e., ti = i. W can guess that each knot must be unique, but in fact that is not the case. We only require that ti + 1 ≥ ti. It turns out that repeating knots induces zero-length spans into the curve, but more significantly, it can force a B-Spline curve to interpolate any of its control points.

A Bézier curve [11, 13] is a parametric curve used in computer graphics and related fields. The curve, which is related to the Bernstein polynomial, is named after Pierre Bézier, who used it in the 1960s for designing curves for the bodywork. Other uses include the design of computer fonts and animations even in Renaults cars. Beziér curves can be combined to form a Bézier spline, or generalized to higher dimensions to form Bézier surfaces. The Bézier triangle is a special case o the latter. In vector graphis, Bézier curves are used to model smooths curves that can be saled indefinitely. „Paths“, as they are commonly reffered to an image manipulation programs are combinations of linked Bézier curves. Paths are not bound by the limits of rasterized images and are intuitivelyto modify. Bézier curve can be defined for any degree n and can be interpreted in more forms: recursive definition, explicite form even polynomial form.

Polynomial form: Sometimes it is desirable to express the Bézier curve as a polynomial instead of a sum of less straightforward Bernstein polynomials.

Voronoi diagram is a partition of a plane into regions close to each of a given set of objects. In the simplest case, these objects are just finitely many points in the plane (called seeds, sites, or generators). For each seed there is a corresponding region consisting of all points of the plane closer to that seed than to any other. These regions are called Voronoi cells, [13]. The Voronoi diagram of a set of points is dual to its Delaunay triangulation. The Voronoi diagram is named after Russian mathematician Georgy Voronov, and is also called a Voronoi tessellation, a Voronoi decomposition, Voronoi partition, or a Dirichlet tessellation. Voronoi cells are also known as Thiessen polygons. Voronoi diagrams have practical and theoretical applications in many fields like science, computer games, technology, visual art and design. Let {\textstyle X} be a metric space with distance function {\textstyle d}. Let {\textstyle K}be a set of indices and let {\textstyle (P_{k})_{k\in K}} be a tuple (ordered collection) of nonemptysubsets (the sites) in the space {\textstyle X}. The Voronoi cell, or Voronoi region, {\textstyle R_{k}} associated with the site {\textstyle P_{k}} is the set of all points in {\textstyle X} whose distance to {\textstyle P_{k}} is not greater than their distance to the other sites {\textstyle P_{j}}, where {\textstyle j} is any index different from {\textstyle k}. In other words, if {\textstyle d(x,\,A) =\inf\{d(x,\,a)\mid a\in A\}} denotes the distance between the point {\textstyle x} and the subset {\textstyle A}, then the Voronoi diagram is simple tuple of cells. Some of the sites can intersect and even coincide, but usually they are assumed to be disjoint. Infinitely many sites are allowed in the definition but again, in many cases only finite many sites are considered. In the special case where the space is finite-dimensional Euclidian space, each site is a point, there are finitely many points and all of then are different. The the Voronoi cells are convex polytopes and they can be represented in a combinatorial way using their vertices, sides, two dimensional faces, etc. Sometimes the induced combinatorial structure is refereed to as the Voronoi diagram. But let us note that the Voronoi cells may not be convex or even connected. It is possible t mention

Vornoi diagrams also with Voronoi tessellation and also Voronoi patterns. This leads to aggregations of cells, aggregations of regions as well as aggregations of Voronoi cells. This associative operation as aggregation is also used in design process as well as in pure mathematics [13].

5 Modeling Process

Generating an object implies changing its basic characteristics, or changing parameters such as length, width, height, shape of the base and geometry on the floor if it occurs. Complex software tools make it possible to generate very complex models. The most commonly generated elements appear in the form of full bodies, that is, three-dimensional models whose generation can be performed in any CAD software for three-dimensional modeling [14]. The possibilities of generating spatial structures are large. The accuracy of the executed forms, in this sense, always represent a challenge, limited by the configuration of the elements and the performance of the computer.

Flower petals are introduced into a spatial framework in the form of a surface structure that limits the contours of the petals presented in the form of freeform curves. The structure of this surface beside the border curves forms a central curve that mimics the contours of the flowers, which define the height and ratio of open-closed petals. This curve is the carrier of the flower form seen from the side. The surface of the flower is formed by filling the space between the central curve and the boundary curves that define the shape of the petals. This structure, with the surface definition, is presented as a unique model that reacts in its entirety to further changes in the parameters to simulate any further deformation of the petals.

The modeling precondition is the analysis of the proportion of flowers that derived a geometric pattern, which will serve as a basis for setting the freeform curve and will define the shape of the flower in the contours (see Fig. 3).

By introducing spatial curves into the model configuration, the process of modelling begins.

Firstly, a flower photograph which represents the appearance of the flower from the side in parallel projection, and the freeform curves, defines parameters such as length and the ownness of the flowers. The next step is to define one edge of a single petal and its mapping builds a universal boundary configuration of the contour of all the petals. By forming this curve with the control polygons we bring the curve to the desired position by defining it as one of the model parameters. Using a circular duplication of this structure, which represents the edges of the petals for the necessary number of petals, which in this case is five, we obtain the entire border structure of the flowers in the form of freeform curves. By changing the basic parameter referring to the universal petal, the change equally reflects on the shapes of the other petals.

The method we applied to model the flower is not a precise reconstruction but a quick way to transfer the approximate proportions of the flower to the model. From the standpoint that the whole and equal structure of the crown with the lobes that

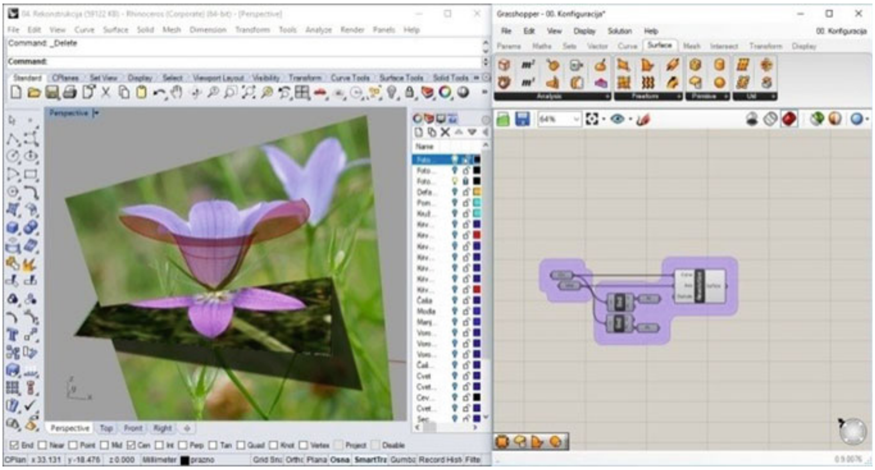

Fig. 3 Contours of flowers that are introduced in the form of spatial curves

define the petals, we have developed a method for modelling flowers with two parts: (1) the bell—which we form as a rotating surface (Revolve surface), and its cutting with the (2) raised flower contour (Extrude Curve). We obtained both objects with the help of freeform curves, as the basis for the initial definition of their contours. In order to construct the contours of the flowers, we used two photographs on which the flower is presented from two angles, viewed from the top and viewed from the side. The photo where the flower is viewed from above served to define the diameter of the flowers based on a central point—the centre of the flower, which we obtained using the intersection of the flower petals' symmetry. On the flower we distinguish two external diameters—one that uniquely defines the width and length of the flower, whose circle touches the peaks of the petals and the inner one—the smaller diameter that includes the points of use of the petals (see Fig. 4).

In the second step, using the photo that shows the flower from the side, from which we previously adjusted, with scaling, the dimension of the diameter of the flower from the previous step, we notice the length (instead of the height) of the flower. The circles we obtained in the previous step were distributed on the basis of their characteristic heights: (1) the length of the petals and (2) the position of the cut of the petals. In addition, on the basis of the centre point of the flower, we constructed the central axis of the future model, which, along with the other auxiliary lines (designed for an easier understanding of the proportions of flowers based on the aforementioned characteristics), served to draw freeform curves (see Fig. 5). The freeform curve construction is based on the perimeter of the flower with a minimum of 3 basic points and 2 control segments (with checkpoints). Basically, there are ten such lines (five pass through the centre of the circular leaves and touch the tops of the petals, and the other five pass through the places of use), so the curve is duplicated ten times from the flower centre, although only one curve is needed for the construction of the rotating surface.

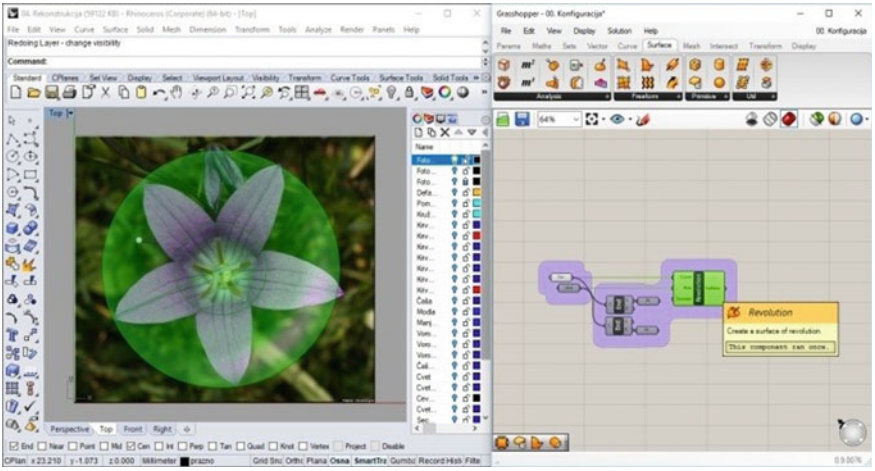

Fig. 4 Determination of the diameter of the flower and the construction of another freeform curve defining the outer contour of the flower

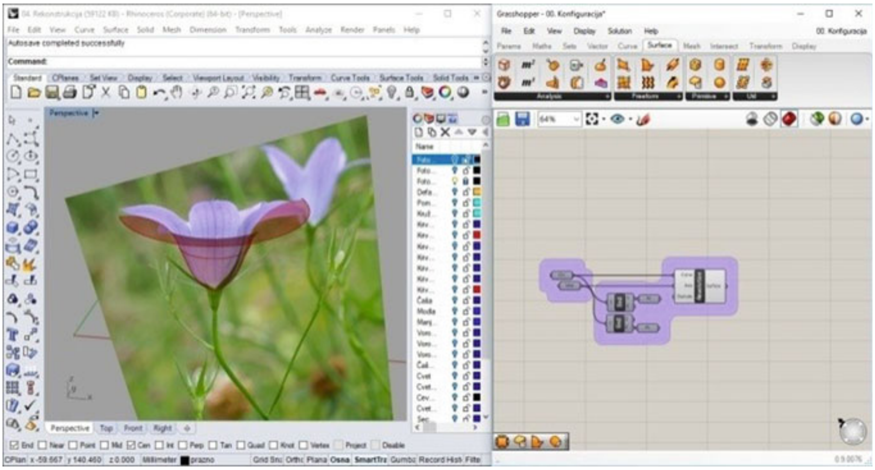

Fig. 5 Construction of the first freeform curve defining the length and degree of flower openness

For the construction of the model we need another curve, which we obtained in the third step. Looking again at the photo from above, we now notice cross-sections between the circles with which we defined the diameter of the flower and the freeform curve from the previous step to define the exterior limit of the flower. At the point between the curves, where the peaks of the petals touch the outer circle and the curves pass through the points of the intersect and cut off with the smaller circle, we have formed another spatial curve with four basic points and 3 control segments. This curve was formed as a two-dimensional curve that we first mapped and obtained two curves, and then multiplied them five times from the centre of the flower, in order to obtain the external contour of the petals, assuming there are fewer variations in nature.

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6 Results and Applications

On the basis of the *Campanula patula* flower form, characteristic starting curves were established as the basis for the future surface of the basic model obtained with the help of freeform curves. The layout of the curves forms a surface pattern whose structure corresponds to the cross-sectioned flower structure [15] arrangement of spatial curves influences the appearance of the shape of the flower model. The obtained model is the result of the application of the digital technologies of the 3D modeling program Rhinoceros and its plug-in Grasshopper, which, by means of mathematical and geometric relationships, forms the model, and whose outcome depends entirely on the parameters provided by the structure of the C*ampanula patula* flower. By manipulating some of the material elements, for example doubling the spatial curve within a frame that defines a flower, it is possible to obtain a completely new structure, the origin of which is the same as in the basic flower model (see Fig. 7).

The further process of the model processing is reduced to generating, based on the grammar of the form as the basic tool of the design process that allows the modification of the model through other elements and can affect the expansion, the morphological characteristics and proportions of the form, visually manifesting these changes, with variations in the relationship occurring in the basic model [16]. The arrangement of petals that form a flower can be placed in a five-cornered base. In this process, the flower, that is the surface structure (the polysurface) that represents the flower model, is presented as a building element on the basis of which we can form variations of the model.

There are several ways of modelling the transformation of this structure, one of which is parametric modelling, where the connection between the model and the configuration change is direct and visible in a realistic framework. By defining the surface of a model as a mesh, methods for modelling the surface of the model are opened up. One such structure is the Voronoi polygonal network that can be obtained

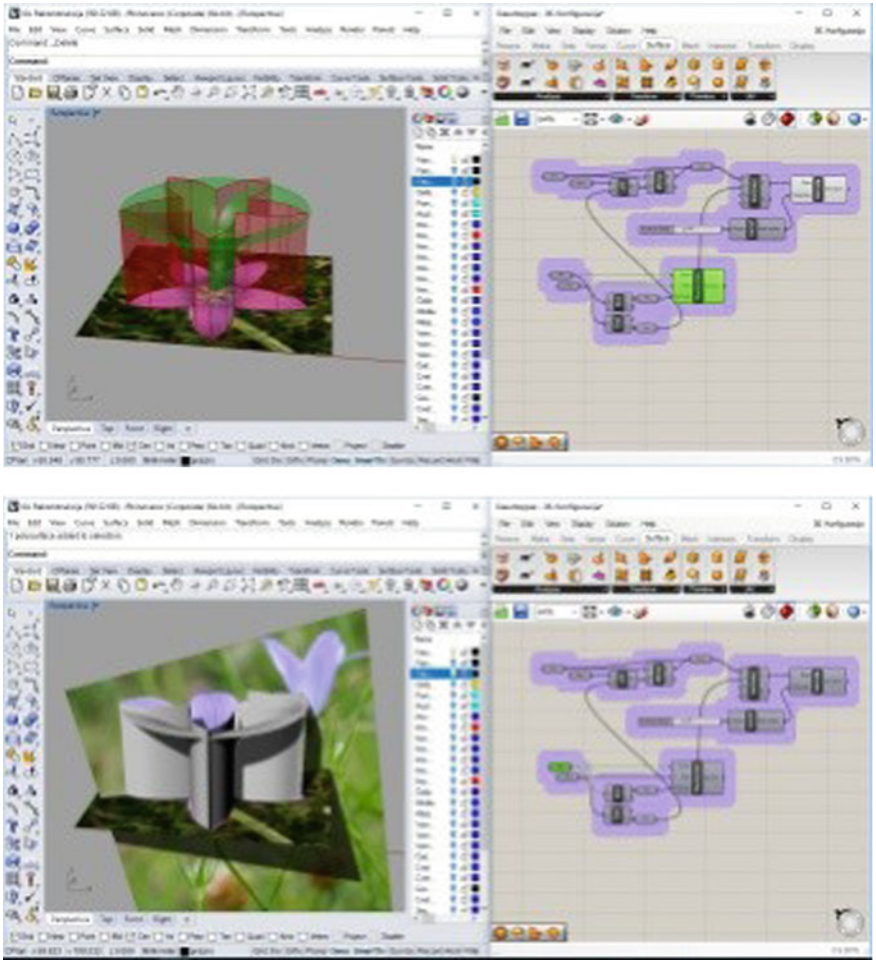

Fig. 6 Model of flower obtained with bell-shaped surface with raised outer flower contour

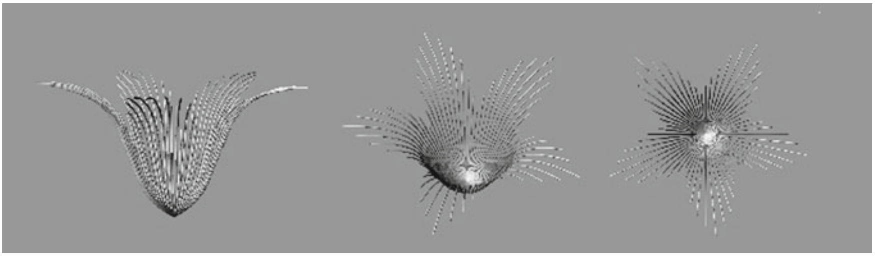

Fig. 7 Variation of the model obtained by multiplying the spatial curve

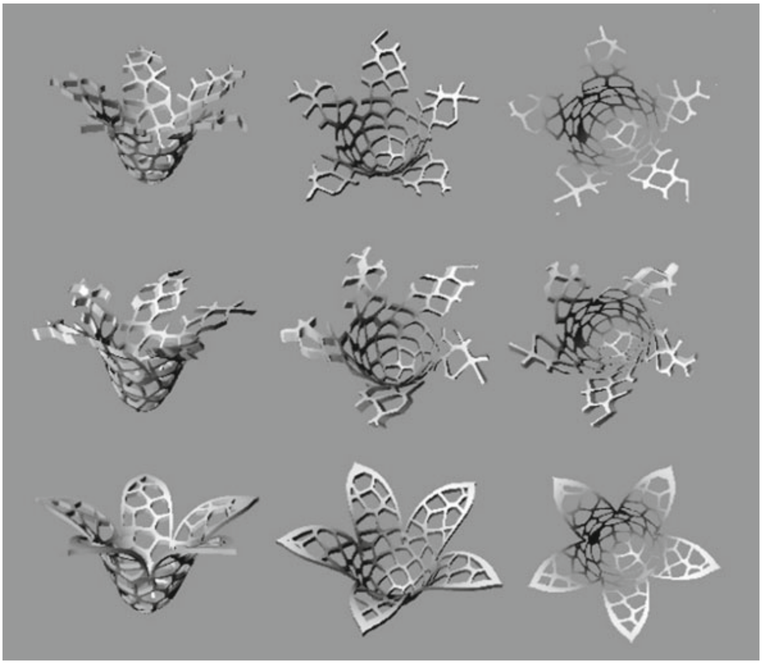

Fig. 8 Model based on Voronoi structure

by placing points on the surface of the model, which will represent the centres of the Voronoi cells. Such a structure is often referred to as "organic". From the layout, the number of points and the definition of their distance depend on the number and appearance of the perforations of the model (see Fig. 8). On the other hand, the manipulation of network parameters with real polygons will give the model a "geometric look" (see Fig. 9).

7 Discussion

Biomimetics sits between design and engineering, is used for one purpose or another, and brings equally subtle results. Accordingly, "bio-inspiration" can be used in the design of objects, urban mobiles, landscape-architectural elements, but also of larger structures and spaces. On the one hand, virtual technology allows us to examine the most suitable spatial configurations [2]. On the other hand, structures and objects of a smaller size, for example the elements of urban pergolas, benches, fountains, etc., can be fully implemented in a space (see Fig. 10).

Such objects are certainly possible on the basis of a flower pattern formed in the previous generation process. Manipulation with this template is possible in any 3D modelling software, whereby it is possible to stylise the form, determine the repetition

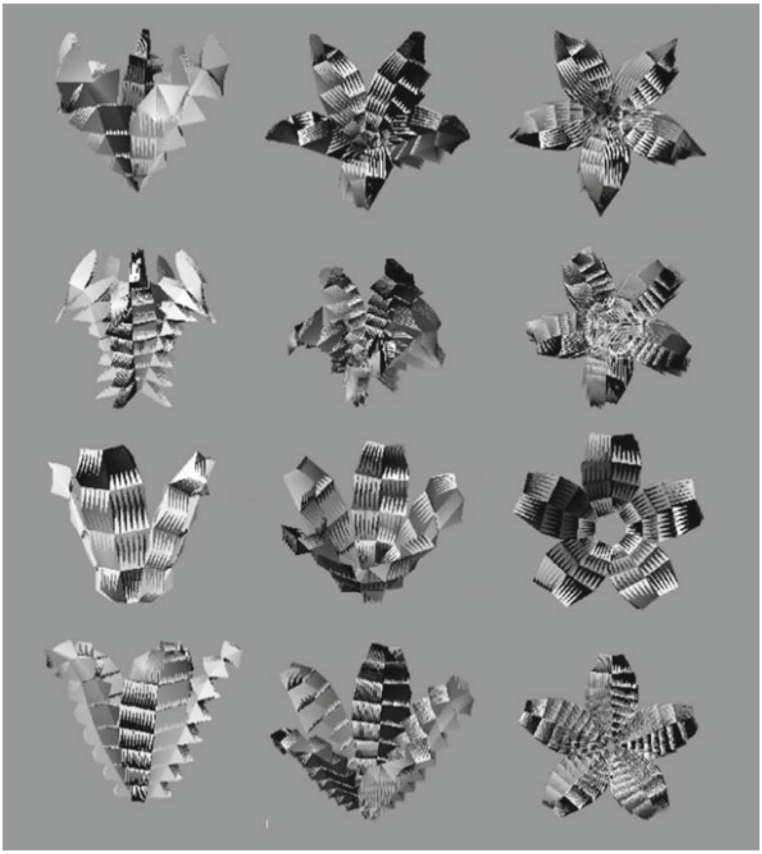

Fig. 9 Model variations based on rectangular network

Fig. 10 Application of the model in the case of fountains and pavilions

of elements or use tessellation to establish the method of matching elements to build a complex structure of a model that can be spatially implemented. This design, which is the carrier of biological reference, ensures a constant interaction of the object and the environment. A creative process involved in the design of generative systems shows how parameter manipulation and algorithm modification can result in a radical change in results. If we align the needs of a contemporary society with the direction of development of design and its tools, this will result in sustainable designs reduced to principles that rely on biological knowledge. Guided by the imitation of natural forms, structures, processes and strategies used by living beings, biomimics is considered an instrument of such design, in an effort to explore, utilize and learn from nature and adapt this knowledge to the needs and technologies of modern design [17]. In the process of learning from nature, consciousness is awakened of nature as a mentor, which enables the development of creative thinking to solve problems, which will result in creative results [18]. The translation of biological knowledge into design parameters leading to bio-inspiration can be a difficult process if the scientific understanding of these findings is limited, for which a complete understanding of biological, physiological and mechanical processes is required. The degree of understanding of these processes is a solution that fully or partially meets the sense of sustainability [17]. Based on the choice of species and its analysis, which can include the analysis of movement, behaviour, development, form, etc., we can conclude the characteristics of a space, its reactions to different stimuli and its environmental properties [18], and translate them into design parameters and use them under the same conditions. In this paper we use geometry, which is a universal analogue model that describes and simulates relationships in the structure of systems such as space and nature [19]. Thanks to a more open approach relying on biological research, the boundaries of technical capabilities and the development of innovative design ideas and engineering solutions [16] have been shifted. Analytical geometry has been used in generative design solutions based on algorithmic forms. Such solutions are distinguished by the diversity of geometric shapes [20], leading to the formation of complex models. In this process, the basic patterns evolve into more complex ones, changing their original appearance, but not the origin. Quite certainly this process is reminiscent of the traditional Japanese art of origami, which aims to create a higher form using basic paper bending techniques [21].

The fact is that elements in nature vary and it is difficult to fulfil and transfer these characteristics to a model. It is, therefore, clear that it is an approximation between the elements that we encounter in nature and those in the model. However, the mere idea that natural forms can serve as examples for design solutions is sufficient to fulfil a basic requirement, inspiration that came from nature. This attitude leads us through the modelling of the structures to which we are openly approaching, with the aim of expanding the possibilities of integration through different solutions and user needs. So far, we have tried to implement this approach in solutions that attempt to decode and use a natural design pattern. We have dealt with the geometry of natural forms in the example of the leaf of the species *Prunus domestica* L. and its correlation with the pattern of a particular cultural area using the Voronoi diagram, where we pointed out that the geometric interpolation of an existing pattern of a cultural landscape and precession elements can be combined with the geometric structure of the vein form of the *P. domestica* leaf by using a comparative method. We concluded that finding the correlation between the geometry of regions and smaller spatial units from geometric forms of bionic forms can be applied to the organisation of these spaces in order to improve the functioning of the area [15]. On the other hand, we copied the form and veins of the flower of the species *Ramonda nathalieae* Panč. te Petrović into the design pattern. On that occasion, a Voronoi pattern was used to model the flower petals, which is presented in the process as a building element that can be used in modelling by changing the parameters of the grammar of the form, while the achieved form retains the biological reference [15]. Similarly, in this paper we tried to translate the geometry of the *C. patula* flower and transfer it into a digital framework that we have presented as a design reference, which will serve to model the objects that are the bearers of its form.

A flower is reproductive and most certainly a plant organ from a systematic point of view. Unlike vegetative plant organs, a flower, as a reproductive organ, is less sensitive to the impacts of environmental factors. Design solutions whose shape is basically inspired by the flowering of flowers are the bearers of these same principles, which originate from reduced geometric shapes [22] and, in general, construct complex structures [23].

Such forms can be produced thanks to connections that smoothly overlap the spatial differences between structural elements, one of the basic characteristics of the generative design [24]. The use of biomimetic principles in the design with a geometric basis opens up the possibility for the development of generative models whose parameters have a natural origin and whose configuration is adapted to the requirements of a spatial structure [25].

Based on parametric models, generative design is used in algorithmic patterns that rely on geometric and mathematical relationships [7]. These models are the basis for the simulation of transformation and the analysis of outcomes and variations [21]. Finding the appropriate design configuration allows testing and analysing of the structure's response in a digital framework before completing the final design model. Also, the creation process almost coincides with the construction process, as the design configuration becomes constructive at the same time, allowing for a smooth transition to the real frame [26].

8 Conclusion

The purpose of this work is reflected in the research that deals with the shape of the *C. patula* flower in the field of biomimetics and generative design. In the process of the work we were guided by the idea that geometry represents a universal analog model that describes and simulates relationships in the structure of the system of space and nature. Virtual frames of the *C. patula* flower is used in order to achieve a structure that can be integrated into the physical world. Instead of contrast, it is necessary to provide and establish a link between the artificial structures and the environment, which requires the development of knowledge regarding different levels of complexity of space, time, structure, behavior, processes and geometry.

In this paper the starting points on which we rely are biological references that lead us to solutions that possess important characteristics. The structure of the flower forms the geometric basis that can be used to form different geometric patterns. This geometric basis is a design pattern, with a biological reference of the *C. patula*, which the future model will carry. Manipulation of this pattern takes place through a creative process that occurs in the domain of generative design. In this process, the basic pattern evolves, changing its former appearance, but not its origin. The shape of the flower model is represented by a set of spatial curves, characteristic of the species, which form the surface of the model where the relationship between the parameters and the facade of the model is direct. The obtained flower model represents an approximation of the elements that we encounter in nature because its performance is limited by the geometric transfer of elements from a photograph where, in addition to the variation of proportions in nature, there are discrepancies between the shape and the size of the elemental structure. The model, as a geometric pattern of the natural flower structure, depicts its shape and spatial characteristics. Through further manipulation of the model the process of the transformation of the basic parameters takes place, which results in the change of the facade of the model. Depending on the needs of the user, this transformation can have practical applications. This biomimetical approach used for biodesign represents novelty in education still very actual and should be more explored and could be used in many different areas trough interdisciplinary education.

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Primary Masterclass Maths

Thomas G. Lavery

Abstract It has been recognised for some considerable time that educational programmes for gifted and talented children are dominated by those from the higher socio-economic groups (47%) and that fewer than 9% come from families in the two lower socio-economic quintiles. There are, of course, other underrepresented groups (ethnic minorities, children from rural areas and inner cities, migrants). A small-scale research project has been undertaken in Australia which explores the experiences and engagement of a group of gifted lower secondary school students in a rural area and following a STEM programme designed around a local rural model which is using local knowledge as a means of developing scientific knowledge and understanding (Morris, Julia, Slater, Eileen, Fitzgerald, Michael T, Lummis, Geoffrey W, Van Etten, Eddie, Using Local Rural Knowledge to Enhance STEM Learning for Gifted and Talented Students in Australia, *Research in Science Education* [2019]) The Royal Institution of Great Britain has attempted to address some of these inequalities by providing "Masterclass" programmes in a range of STEM subjects. In conjunction with The Royal Academy of Engineering and Ulster University, the first primary Masterclass Maths event ever held on the island of Ireland took place between January and June 2019. Six primary schools each sent six pupils to a host secondary school. The pupils had been selected by standardised test and teacher observation. The pupils were accompanied by members of teaching staff for whom this was part of their CPD. The teaching was provided by staff from the Royal Institution for the first two classes and by staff from the maths department (of the host secondary school) for the final four sessions. Over the six month period of the programme (one half day per month), the pupils investigated Sierpinski Triangles, fractals, the special number Pi, probability, Graph Theory and the Bridges of Konigsberg. This presentation will provide the background to the programme held in Northern Ireland during the first half of 2019 and demonstrate some of the topics explored during the classes.

Keywords Euler · Serpinski · Fractals · Ciphers

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1 History and Organisation

The three institutions involved in the organisation and delivery of this specific Primary Masterclass programme were: The Royal Institution of Great Britain, The Royal Academy of Engineering and Ulster University. Below is a brief introduction to these three organisations. Further information may be obtained by clicking on to the websites listed at the end of the paper.

The Royal Institution (sometimes known as Ri) is an organisation based in the City of Westminster and devoted to science education and research. Founded by, among other notables, Henry Cavendish and George Finch (the 9th Earl of Winchilsea) in 1799, on the principles of diffusing scientific knowledge, helping to develop and improve mechanical inventions and support the application of science by way of teaching and experimentation. In "Our Mission and Vision" (http://www.rigb.org.uk/about/mission-and-vision), the Royal Institution declares that it is committed to "diffusing science for the common purposes of life".

The Royal Institution Masterclass programme enables students (and teachers) to explore maths, engineering, science and computer science through a series of activity workshops. The classes are led by teachers, staff of the Ri and specialists from industry and commerce. The topics are "challenging" and are designed to introduce students to the application of STEM in the "real" world and to stretch their abilities into investigations into areas beyond the set school curriculum. Through meeting activity leaders and assistants, it is hoped that students will obtain an understanding of potential careers and the education pathways that will enable them to gain appropriate qualifications needed in the twenty-first century workforce.

The Royal Academy of Engineering (RAEng) is the national academy of engineering in the United Kingdom. It was founded in 1976 (as the Fellowship of Engineering), granted a Royal Charter in 1983 and became the Royal Academy of Engineering in 1992. Its motto is "To bring engineering to the heart of society" and its purpose declared to be "To advance and promote excellence in engineering". To meet these objectives, the RAEng has, among many other initiatives, a very active education programme which includes the Connecting STEM Teachers' Programme (CSTP). This specific programme was established in 2011with the aim of creating a UK-wide network of support for teachers of STEM subjects, to provide them with relevant knowledge and to develop their confidence in the teaching of their subject specialisms and to engage a larger number of students and with a wide range of academic abilities.

The CSTP support activities are led by the team of over 40 Teacher Coordinators (TCs). TCs are experienced teachers and/or professionals in the education sector. They have five core responsibilities:

- To disseminate teaching and resources to the teachers in the TCs cohort of supported schools.
- To facilitate networking opportunities for teachers.
- To lead learning opportunities for teachers and pupils.
- To provide and promote collaborative opportunities for teachers.

- To assist in the development of learning resource activities.

Ulster University is a multi-campus public university located on four campuses in Northern Ireland. It is the largest university in Northern Ireland and, after the National University of Ireland (in the Republic of Ireland), is the second largest university on the island of Ireland.

Originally established at the New University of Ulster in 1968, this institution merged with Ulster College, the Northern Ireland Polytechnic, in 1984 to create the University of Ulster. However, one of the four campuses of the university (Magee in Derry/Londonderry) was founded as Magee College in 1865. The university rebranded as Ulster University in 2014.

The university has a progressive and dynamic schools' outreach programme located within the Access Digital and Distributed Learning (ADDL) department which has been highly supportive of this, and many other, activities designed to access groups underrepresented in higher education.

Glengormley High School agreed to host the six sessions and invited six of their feeder primary schools (which draw most of their students from families in the two lower socio-economic quintiles) to send students who had been identified by their teachers as talented and gifted in maths and could therefore benefit from the Masterclass Maths programme. Each of the primary schools sent the same six students (36 pupils in total) and a member of staff to the sessions. Two members of staff from the Royal Institution lead the first two classes and teachers from the maths department of the high school taught the final four sessions.

All necessary material and teaching were provided by the Royal Institution and Glengormley High School while coordination, supervision and transport were provided by the Royal Academy of Engineering and Ulster University. Ulster University sent two student teachers to observe and participate in the events.

2 The Activities

The Royal Institution declares that the resources are offered to (those) "who would like to inspire young people and enrich their mathematical knowledge. Taking in a variety of topics beyond the school curriculum, these activities will enhance young people's mathematical confidence, in and out of the classroom" (Mathematics Masterclass, Resources for the Primary Classroom, The Royal Institution of Great Britain, London).

There are approximately twenty activities from which the leaders of the programme can choose. These are:

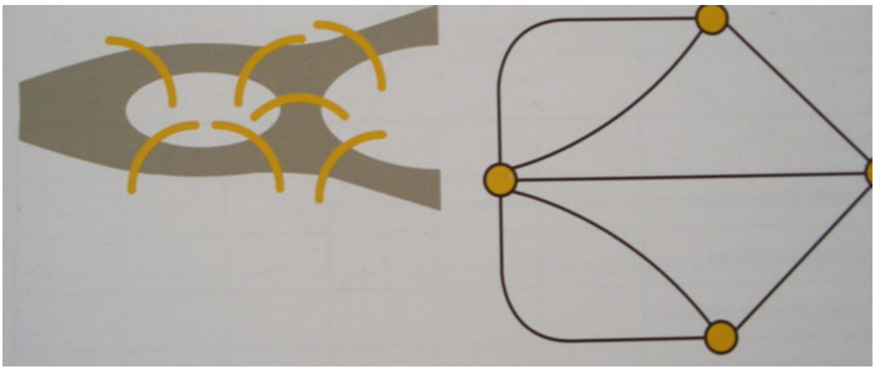
- Soap films and motorways
- Number shapes
- Sierpinski triangle investigation
- Mirrors and angles
- Big and small

- Star maths
- Curve stitching
- Calculating colours
- Codes and ciphers
- Folding fractals
- Mobius bands
- Number scripts
- Platonic solids
- Shape building in 2d and 3d
- The special number π
- Rabbits and sequences
- Numbers of nature
- Bridges of Konigsberg
- Kites and tiling
- Digital computers

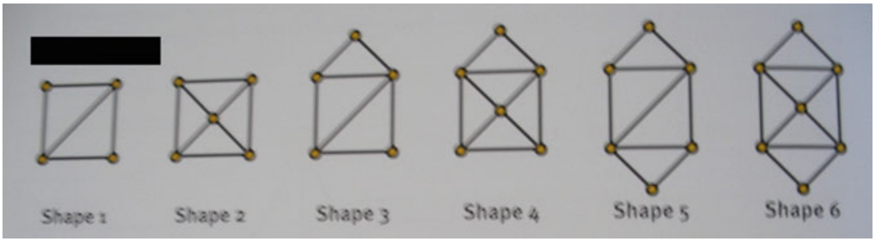
3 The Chosen Topics

a. The Bridges of Konigsberg.

The students are introduced to this activity by first undertaking some research into the town of Konigsberg, the dilemma posed by its residents and the solution proposed by the most celebrated mathematician of the day, Leonard Euler. The people of Konigsberg wanted to know if it would be possible to walk across their town and crossing each of its seven bridges only once. Euler simplified the map of the town as seen in Diag. 1. Land was replaced with dots and bridges with lines. Students can then attempt to draw the diagram without having to take their pencils off the page or crossing any line (bridge) more than once. It they can, the people of Konigsberg



Diag. 1 Leonhard Euler's simplified map of the bridges of Konigsberg



Diag. 2 Additional maps for student use in investigating the Bridges of Konigsberg

could walk across the seven bridges of their town in one continuous walk and without crossing any bridge more than once. If not, they cannot (Actually, it is impossible).

Students are then asked to try to draw more shapes in the same way and to say if it is possible to do so without lifting their pencils from the paper. Some examples are presented in Diag. 2.

Finally, pupils investigate applications of Euler's ideas. They will find that the London Tube map is a simplified version of the actual underground railway system and is based on the Euler principles. It is designed to be a simplified version of reality and relatively easy for passengers to use.

b. The Special Number π.

In this activity, students investigate the relationship between the diameter and the circumference of a circle, not by being given the equation, but by "discovering" the ratio by measuring the circumference and diameter of a range of circular objects (tins, tumblers, wheels etc.) and tabulating their measurements under the headings "c", "d" and "c/d". Students bring their measurements to a communal table and, hopefully, will show that the number obtained by dividing "c" by "d" is just over 3 no matter what size the circular object is. They are told that this is the number π.

This is also an excellent opportunity to revise vocabulary relating to the circle (radius, diameter, circumference etc.) and to introduce the idea that investigations involving practical measurement can often produce "errors" (not "mistakes"). It is highly unlikely that the students will obtain the same value of π but is again an opportunity to discuss the limitations of measuring using simple tools (ruler, string).

Students can undertake research on the history of the number π, that it was known as "the number that you get when you divide the circumference by the diameter", had been known to the Babylonians and Egyptians about 2000BC and the ancient Chinese.

Participants can also investigate why π is described as both "irrational" and "random".

One interesting and engaging activity following on from the students' investigation of irrational and random numbers is to determine where their birthdays are

Table 1 The value of π to 99 places

3	.	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	2	3	8
4	6	2	6	4	3	3	8	3	2	7	9	5	0	2	8	8	4	1	9
7	1	6	9	3	9	9	3	7	5	1	0	5	8	2	0	9	7	4	9
4	4	5	9	2	3	0	7	8	1	6	4	0	6	2	8	6	2	0	8
9	9	8	6	2	8	0	3	4	8	2	5	3	4	2	1	1	7	0	6

located (in the form: day/month/year) in the value of π (There are a number of websites that can be used in the activity) (Table 1).

c. Serpinski Triangle and Fractals

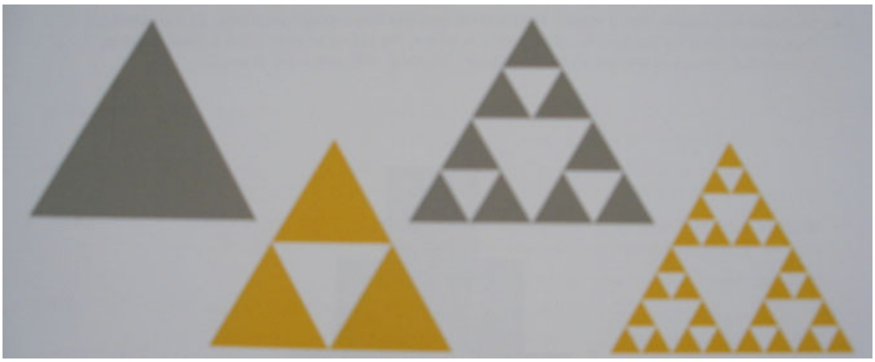
This activity introduces participants to the concept of patterns, in shapes and in numbers. It can begin with a very simple number sequence e.g. 1, 2, 3, 4 etc.; 1, 2, 4, 7, 11 etc. and students asked to explain what the rule is for that number pattern.

Once the students have understood number patterns and how to write a series of instructions to achieve that pattern, they can be introduced to the idea of patterns of repeating shapes and the Serpinski triangles which they will now create by drawing ever smaller triangles. They will than be asked to devise an instruction for the activity which results in this pattern.

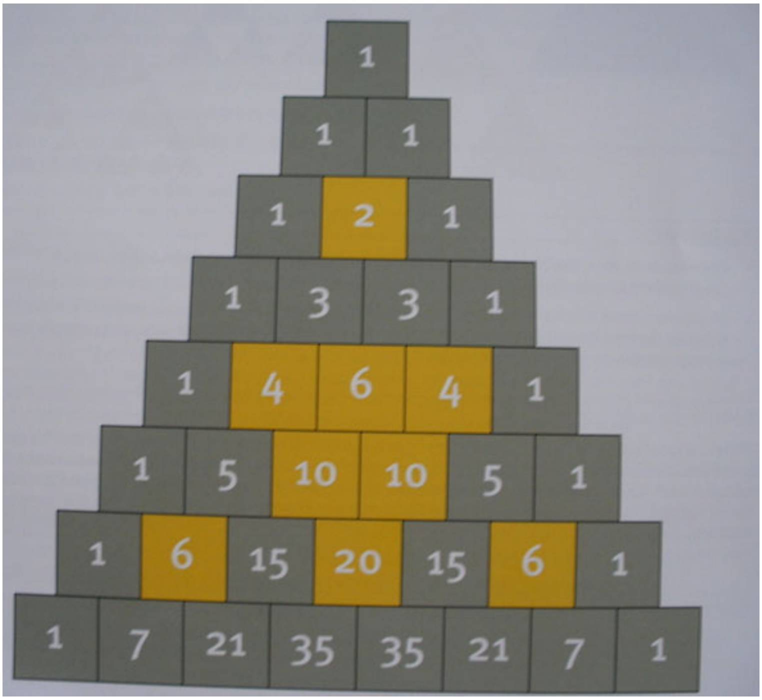
Diag. 3 Shows how this pattern of triangles could develop. Students are asked to think if their diagram will ever be finished. Is there a finite number of triangles that can be drawn or is the number infinite and only limited by the thickness of the pencil used to draw the shape?

This is a good point to discuss Pascal's triangle to the students. Similar to the exercise with the Serpinski triangles, participants try to produce an instruction for its construction. They should realise that the number on any block is the sum of the numbers on the two blocks above and touching it.

Diag. 4. Shows a sample Pascal triangle.



Diag. 3 Creating Serpinski triangles



Diag. 4 A Pascal triangle

Students can, at this point, be introduced to the concept of fractals (a shape repeating itself as it gets smaller and smaller).

Begin by handing out sheets of A4 paper and asking the students to fold the page progressively in half (the record is 12 times). Then show them a drawing of a Koch snowflake. Ask what shape was used as the starting point. Get them to draw their own snowflake by continuously adding triangles. Ask the same question you asked about the Serpinski triangles: is there a limit to the number of triangles that can be drawn on a Koch snowflake or number of times the page can be folded? What is the limit?

d. Codes and ciphers

Students are told that codes and ciphers are not the same. In a code, a word is changed to a different picture, word or phrase. In a cipher, each letter is represented by a different letter, number or symbol according to a set rule. Everyone sending or receiving a message must know the rule (the key) and reversing the rule will decode the ciphered message. Students are also told that anyone who can work out your rule can read your message! They can be set a task to investigate ciphers such as Morse Code, the phonetic alphabet and semaphore signals. They could even send messages using these methods.

x	y	z	a	b	c	d	e	f	g
B	C	D	E	F	G	H	I	J	K

Diag. 5 A four-shift encryption

A discussion about the need for hiding messages particularly in the age of Internet banking and Web-based buying and selling, not to mention social media, can make this activity of contemporary relevance.

Participants are introduced to "substitution ciphers" (where letters in your message are replaced by other symbols, numbers or letters. Shift ciphers are one example of this type of coding. (These are also known as "Caesar ciphers" as, it is popularly believed, Julius Caesar used this method of coding to communicate his orders to his generals in the field—in Latin of course!).

One way of encrypting your message would be to use a "shifted" alphabet e.g. in a two-shift alphabet, the letter in your message is substituted for the letter two ahead in the alphabet. To decrypt the message, reverse the rule. For example, the word "attack" would be encrypted as "CVVCEM" when encrypted using the 2-shift code. Students must also ensure that they keep to the convention that the actual message is in lower case while the encrypted message in in upper case.

Students are given tasks to encrypt and decrypt messages using different shifts. They can even use numbers, punctuation and signs and symbols rather than letters. This will reinforce the idea that encryption can make communication more secure and that both sender and receiver need the "key".

Diag. 5 shows a 4 - shift encryption.

e. Big or small

This activity introduces students to the idea that things cannot be described as big or small unless we know the nature of the context in which the comparison is made, that not all answers need accurate answers (often, an estimate is sufficient) and that we don't have to write big numbers by using lots of zeros (we can use the power of ten instead).

The students are first asked to compile two lists—"I am bigger than……." and "I am smaller than…..". Using these lists, students make assumptions about their height (they don't need to be accurate—"about" is fine in this context). Next, that have to make estimates quickly. For example, how would you imagine 100? Four classes of students or the number of toes on 10 human feet?

Wembley Stadium has a capacity of 90,000 so 1 million is about 10 and a bit Wembley Stadiums full (Picture 1).

Students are now introduced to the way of describing and writing big numbers. Begin simply by asking how could we write 10,000 as a power of 10? Obviously, 10^4 (or $10 \times 10 \times 10 \times 10$).

The pupils are also introduced to the correct math terminology and are told that the small number is known as an index.

Picture 1 How many geese are there in this photograph? (Approximately)

How do you multiply 10^4 x10? Ask students to write it down. How many zeros are there after the 1? Obviously, 6, so the answer is 10^5or one million (1,000,000).

In summary, the easiest way to keep track of large multiples of ten is to count the zeros and use an index to inform us of the number of zeros there are after the number 10.

4 Conclusion and Evaluation

This programme has been a wonderful example of how educational institutions can work jointly to provide learning opportunities for talented and gifted pupils from the lower socio-economic groups.

A second Primary Masterclass Maths event was organised for the first half of 2020. Despite 30 students and 6 teachers from six schools taking part, the programme had to be cancelled after only two classes due to the Covid 19 outbreak and the closure of schools in Northern Ireland.

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Geometric Structures as Design Approach

Cornelie Leopold

Abstract The creation processes of artists and architects are often seen as a throw of the genius. Looking for methods in designing remain a question for research in architecture and other creative disciplines. Geometry as the science to capture, develop and generate structures and forms can give a scientific methodical background for designing. Thinking in structures is part of the understanding of mathematics as a general structural science. Works of art and architecture developed on the basis of geometric structures had been analyzed. The chosen examples are related to tessellations and polyhedra. Those examples include studies on the works of the artists Olafur Eliasson and Gerard Caris as well as architectural works of Richard Buckminster Fuller, Konrad Wachsmann, and Anne Tyng. Exploring those examples reveal necessarily profound knowledge in the relationships and characteristics of tessellations and polyhedra. Students' research studies and design projects demonstrate an actual workability of such a geometric structural approach in architectural education.

Keywords Geometric structures · Art · Architecture · Design methods

1 Introduction

Referring to geometric structures could be a way to a systematic design approach in art, design and architecture. The Dutch artist Gerard Caris put the question "How to imagine something from nothing?" at the beginning of his art work.

Intuitively received forms and structures can be related to geometry which has to be seen as the science to capture, develop and generate structures and forms. In this way geometry is part of the understanding of mathematics as a general structural science according the project Bourbaki in the 1930s. The idea to use those geometric considerations for aesthetic creations has a long history, but especially structural approaches had been part of the concept of Ulm School of Design (1953–1968) with a mathematical-geometric visual methodology. The philosopher Elisabeth Walther,

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professor at University Stuttgart and lecturer at Ulm School of Design, described the aesthetics, developed by the philosopher Max Bense [1] as fundamental for techniques as well as architecture and art:

> Aesthetics, as Bense brings it into play, is the principle of order par excellence. Aesthetics is order, and order on the other hand is describable by mathematics. Therefore, aesthetics is important as structuring the world for techniques as well as architecture, literature, etc., for all what will be created. Whenever we take something out of the chaos of existing und assemble it new, we need an aesthetic foundation. [2], translated from German to English by the author

Digital design techniques provoke such a shift to thinking in geometric structures as a fundament in creative processes. Toni Kotnik [3] analyzed the relationship between actual digital design techniques and the necessarily changing way of thinking in structures. He speaks of a new form of structuralism based on mathematics as the science of structures. The described aesthetic approach of Bense had already anticipated the actual requirements and is of actual relevance. Such principles of order as an aesthetic foundation of art, design, and architecture can be found for example in the geometry of tessellations, patterns, transformations, and forms. Therefore, we will have a look at those geometric order principles before we will analyze works of art and architecture according those fundamentals of geometry.

2 Geometric Structures

Fundamental geometric structures can be seen for instance in plane and spatial tessellations. The regular and semiregular plane tessellations build the background for paintings and patterns. A next step could be looking for the principle of duality of these tessellations. The duals of the semiregular tessellations form tessellations out of congruent non-regular polygons. The number of lines in one vertex produces the number of vertexes of the resulted polygon. The duality principle of projective geometry is a mighty tool for creative steps from one structure to another. The counterpart in three-dimensional space are polyhedra and their duals. the Platonian solids are the fundament for developing the Archimedean solids and the dual Catalan solids. Approaches filling the space with solids give the basis for building structures. Tessellations, grids, and lattice structures can be found as the background for floor plans, connectivity of spaces as well as supporting structures in architecture.

Another point of view offers the symmetry considerations based on the notion of transformations. Patterns can be described and classified as symmetry groups. The symmetry concept gets expanded to more general transformations by this conception, which offers a process-oriented approach from single elements to complex structures, from one form to another. Transformations can be classified according March and Steadman [4] in identity, isometry, similarity, affinity, perspectivity, and topology transformations. These offer additional rich tools for designing, which are essential in digital design processes.

More sophisticated geometric structures like aperiodic tiling, fractals or Voronoi diagrams and its dual Delaunay triangulation supplement the repertoire of geometric structures which are effective for fundamental design processes. Further examples are described in Leopold, 2015 [5].

3 Examples of Geometric Structures in Works of Art

The work of the Dutch artist Gerard Caris shows a strong approach on the basis of tessellations with regular pentagons and in the sculptural spatial work on dodecahedra. Caris wrote in a statement of 2018 about his work:

> Geometric as well as non-algorithmic elements involved in the creation of these new works evoke a sense of unity in the viewing process in which aesthetics and mathematical logic merge together as one. The regular pentagon and the dodecahedron being the principal building blocks of this oeuvre …. [6]

After some explorations in tiling the plane with variations of irregular pentagons and hexagons, Caris had been fascinated by the universe of the regular pentagon with its systematical consequences. While trying to tile the plane with regular pentagons, he explored the different positions of the remaining rhombuses between them. Series of his works discover aesthetic results from variations, which suggest spatial interpretations by using bright-dark colors. The result of his explorations had been his Pentagrid which contains the manifold relationships and configurations resulting from the regular pentagon. It can be seen therefore as the repertoire for all possible derivable art works. Caris uses his Pentagrid in his creation processes. Figure 1 illustrates the resulted grid and figures from the tiling approaches by regular pentagons.

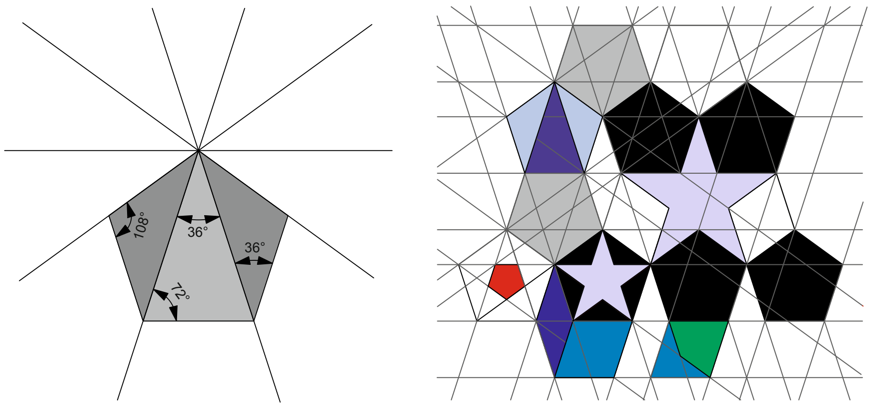

Fig. 1 Five-dimensional grid of the pentagon and figures in the Pentagrid

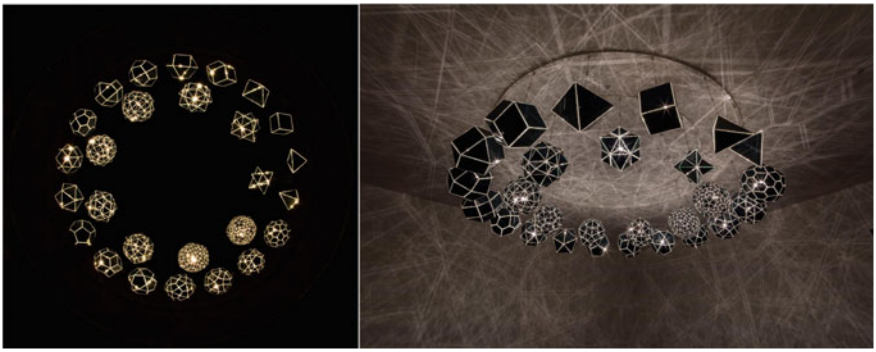

Fig. 2 "Your sound galaxy", 2012 by Olafur Eliasson, Red Brick Art Museum, Beijing 2018. Photo: Xing Yu (left), Anders Sune Berg (right). © Studio Olafur Eliasson GmbH [10]

A detailed analysis of his work also correspondingly of his sculptural and relief work on the basis of the dodecahedron is published by the author in the Handbook of the Mathematics of the Arts and Sciences [7].

A work of the Danish-Islandic artist Olafur Eliasson with his studio in Berlin [8] might serve as a second example for art based on geometric structures. Some examples of his works are strongly related to the relationships of polyhedra, some of them evolved in co-operation with the Icelandic architect Einar Thorsteinn [9]. The work "Your sound galaxy" of 2012 (Fig. 2), exhibited for example in the Red Brick Art Museum in Beijing, 2014, gives the chance to explore characteristics and relationships of polyhedra in an exceptional way. Eliasson described the work with its elements, relationships and materials as follows:

> Your sound galaxy suspends a group of twenty-seven polyhedrons from the ceiling in two horizontally concentric circles. Each polyhedron is made of a stainless steel frame clad in mirrored glass that has been turned inward so that the blue-grey backs of the mirrors act as the faces of the solid. A single halogen light mounted inside each polyhedron is multiplied by the reflective interior into a twinkling glow that escapes through the gaps in the frames. The polyhedrons are organizable into nine 'families' of three related forms. [10]

Our student Rahaf Nader analyzed and studied the relationships between the used polyhedra by remodeling this work in a seminar on polyhedral structures in 2020. Starting with two tetrahedra, because the tetrahedron is self-dual, the dual solids follow in the outer circle and get more complex and multi-faceted. In some parts she found additional geometric orders in the sequence of the polyhedra which led her to a deep research on the relationships between them. In her analysis in Fig. 3 the Platonic solids and their compound are shown in yellow, the Archimedean solids in red and their dual Catalan solids in blue. In the first four groups and in the sixth and seventh group the transition between the groups arises from the inner circle to the outer by forming the convex hull of the compound star polyhedron, marked with a green bent arrow. In this way we come from the star tetrahedron to the cube in the outer circle as its convex hull, from the compound of cube and octahedron to the Catalan solid rhombic dodecahedron and its dual Archimedean cuboctahedron,

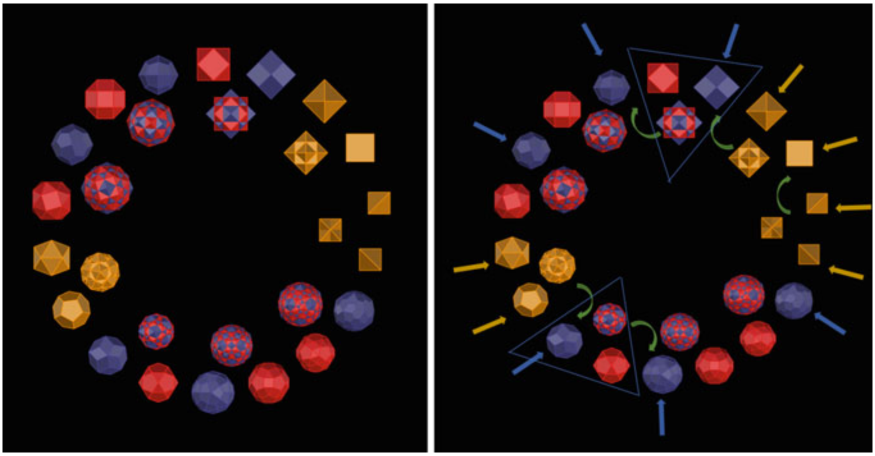

Fig. 3 Remodeling and study of Eliassons's galaxy by student Rahaf Nader

from their compound star solid by the convex hull to the Catalan solid deltoidal icositetrahedron and its dual Archimedean rhombicuboctahedron. These relationships explain why we step in the sequence first to the Catalan solid and then to the dual Archimedean solid, only with one exception in the last group. From the sixth group we have the transition from the convex hull of the icosahedron-dodecadron-star to the Catalan solid rhombic triacontahedron and it Archimedean dual icosidodecahedron, and one more transition from the compound of these two to the Catalan solid deltoidal hexecontahedron as their convex hull and from there again to the dual Archimedean rhombicosidodecahedron.

There remain artistic free decisions for selecting the solids and their sequence, but these studies on the relationships between the solids had been fascinating results in this research. It is interesting to see how geometric research can inspire artwork in Olafur Eliasson's studio. In the advanced geometry video of the livestream collection Anna Engberg-Pedersen, co-director of Studio Olafur Eliasson, says:

> What we are hoping to do, is demystify a little bit what it takes to do art works, where do the ideas come from …. [11]

It gets obvious that the art work had been a result of an intensive research on the characteristics and relationships between polyhedra.

4 Examples of Geometric Structures in Works of Architecture

Geometric structures can be used in many ways when designing and realizing architecture. We want to focus here on four examples related to architectural structures based on polyhedra.

The work of Richard Buckminster Fuller offers a lot to discover related to geometric structures. One of his most renowned inventions is the geodesic sphere, which resulted in built examples like the biosphere in Montreal, 1967. The idea for the geodesic sphere had been developed out of the icosahedron, the most spherical Platonic solid by subdividing and projecting onto the circumscribed sphere. Figure 4 shows the design and realization of a level 1 geodesic dome made from cardboard tubes, a waste product of A0 plots, and knots from laser cut MDF by our student Simon Kunzler in 2014.

By studying the various polyhedra it stands to reason taking another polyhedron with more faces for creating a dome. A suitable solid could be one of the Catalan solids. The faces are not regular, but they are congruent. The rhombic triacontahedron with 30 faces can be a suitable starting point. Our student Moritz Brucker had this idea and found a similar approach by Fuller in the Egg-Crate Basic Assembly Dome of 1950, realized in the University of Oregon Dome in 1953 with his students. Those explorations of Fuller and his students are well documented in the new published book by Daniel López-Pérez [12]. Moritz Brucker worked in 2020 with the rhombic triacontahedron, divided the rhombus in triangles, these in smaller rhombuses by connecting the midpoints of the triangles. By subdividing again, we get two different rhombuses, shown in green and red in Fig. 5. The faces with the rhombic pattern are then put on the faces of the rhombic triacontahedron, then projected onto the inscribed sphere. A Catalan solid has an inscribed sphere, not a circumscribed one. The rhombic sphere pattern is shown in Fig. 6 with the created 3D model.

A third example started with analyzing works of Konrad Wachsmann, especially the structure of the USAF Aircraft Hangar, 1951 [13]. Our student Arutiun Papikian created a triangular structure with irregular tetrahedra in 2020, stimulated by his research on Wachsmann's works, which should be usable for horizontal as well as vertical structures, therefore applicable for support structures and facade designs (Fig. 7).

The fourth example picks up the work of Anne Tyng [14]. She worked 1944 in the New York office of Konrad Wachsmann before joining Louis I. Kahn's office 1945 in Philadelphia. She had been fascinated by the work of Buckminster Fuller when she listened to his lecture in 1949. Her work, too, together with Louis I. Kahn, is permeated with the application of polyhedral structures, especially as space frame

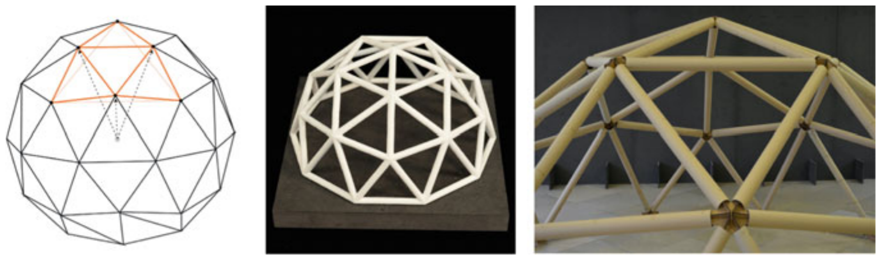

Fig. 4 Concept drawing, 3D-printed model and realized cardboard dome by Simon Kunzler

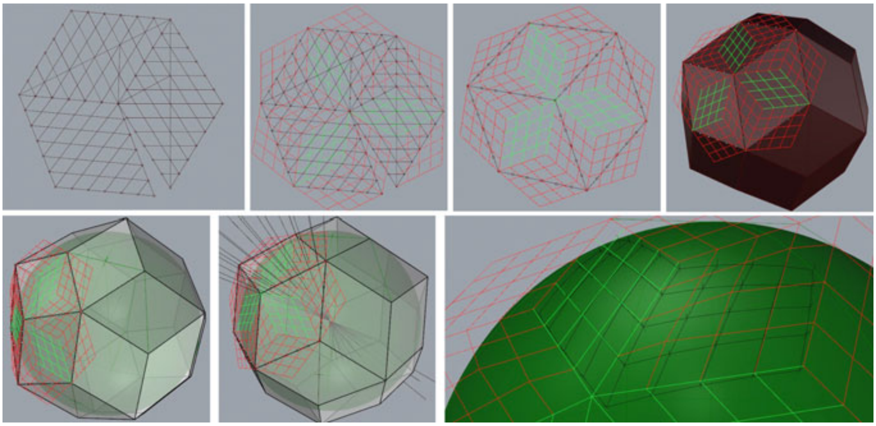

Fig. 5 Developing the sphere approximation from the rhombic triacontahedron by subdividing the rhombuses and projecting onto the inscribed sphere by Moritz Brucker

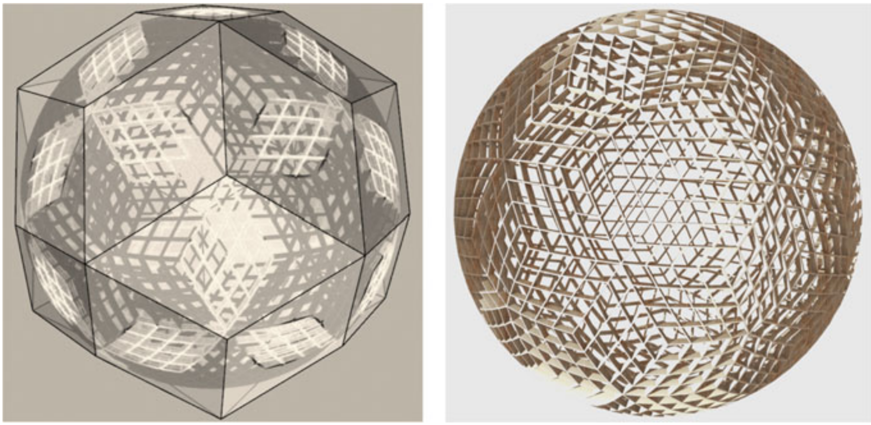

Fig. 6 Rhombic sphere pattern with the rhombic triacontahedron as starting solid and final 3D-model by Moritz Brucker

architecture. In her design of an Elementary School (1950–1951) for example she designed a structure out of octahedra and tetrahedra in several space layers that converge finally in a point and become a pillar. The article by Cristina Càndito includes a description of this work [15]. We can find similarities to Wachsmann's USAF Aircraft Hangar.

The space frame structure is variegated in the Philadelphia City Tower (1952–1957), designed together with Kahn. One layer is formed by three joined hexagons, packed with 18 tetrahedra, seven octahedra and four half octahedra. From level to level this structure is moved in six possible directions. Figure 8 shows the analysis of our student Benedikt Blumenröder in 2017 with physical models of the structure.

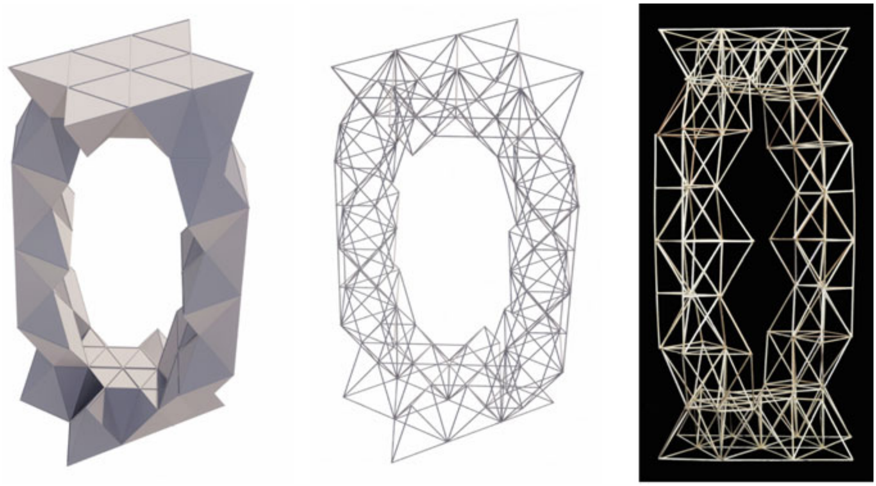

Fig. 7 Triangular structure based on irregular tetrahedra by Arutiun Papikian, digital and physical models

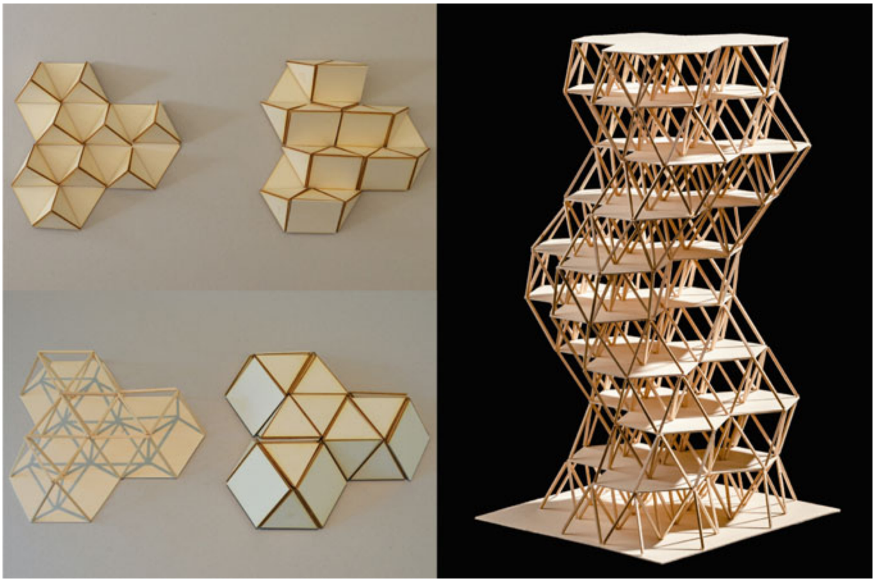

Fig. 8 Geometric structure of a layer and model of the Philadelphia City Tower by Benedikt Blumenröder, designed by Anne Tyng and L. I. Kahn, 1952–1957. Photo right ©Bernhard Friese

Anne Tyng assigned an important role to the Platonic solids in architectural design processes. She described them as a three-dimensional probability matrix.

> An icosahedron may be nested within the octahedron, and the dodecahedron may be built-out from a cube. It's a very elegant family of forms that expresses probability. It's a three-dimensional probability matrix …. [16, p. 130]

5 Conclusions

The shown examples can only give an idea of the power of geometric structures for creation processes in art and architecture. Digital design techniques provoke a new actuality of geometric structural attempts in designing and creating. The described background should be taken into account when forming curricula of such creative disciplines. Geometry has its place in design and should not be reduced to representation techniques. Those structural approaches facilitate design methodologies instead of intuitive throws of the genius and offer therefore ways for developing ideas.

Acknowledgements Many thanks to the mentioned students in my courses for their engaged work. The artist Gerard Caris gave me the opportunity to study his works through discussions, studio visits and catalogues. The published material and visits of exhibitions of Olafur Eliasson enable deep insights into the creative processes in his studio.

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Geometrical and Fashion, the Case of Iris van Herpen

Marcella Giulia Lorenzi, Giampiero Mele, and Sara Zecchetto

Abstract Fashion design, in the construction of a piece of clothing, from sketch to modeling, is very close to geometry. Digital design software offers numerous applications. Some designers went beyond, using technology and innovation in a process, that sees in generative modeling software the key of new forms of design that use geometry as a creative tool, that is combined with 3D printing and innovative materials. One of the greatest exponents of this new way of doing fashion is Iris van Herpen.

Keywords Mathematics · Geometry · Fashion · 3D modelling

1 Geometry and Fashion—Clothing Modelling

Fashion is a more or less changing convention that represents an individual psychological aspect but becomes a social and cultural phenomenon of adherence to certain styles as well as value. The Italian term for fashion, "moda" is also used in statistics to define that value that, within a certain number of data, is presented more frequently.

Fashion has a fleeting and variable character in itself. Its enclosing habits and styles that can evolve and change even quickly makes the world of clothing almost ephemeral in the common imagination, certainly the opposite of a balanced world.

On the other hand, only if we look at the fashion system we can identify a character of rigor and precision that allows this area to become an important phenomenon at the social level, but also at the economic level.

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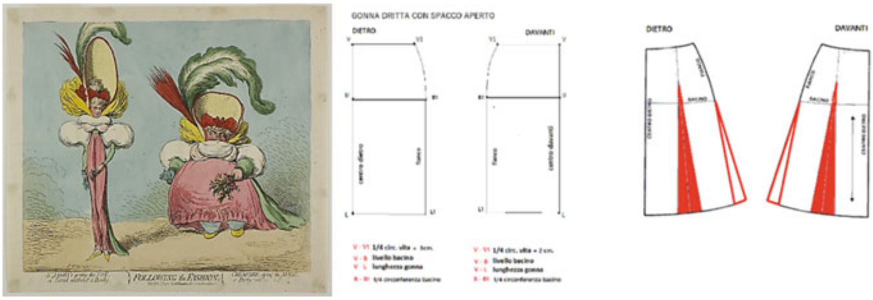

Fig. 1 (A) "Following the Fashion," a 1794 caricature by James Gillray. (B) Example to build a straight tube skirt with open split (www.buildcartamodelli.com)

Starting from the same construction of a piece of clothing, in its design we can identify a discipline known as modeling that by its nature and purpose is very close to geometry.

Modelling is a craft and an art together that determines the method of building a garment. It is the very creation of the model, not the creation understood as style but the building of all the components. Starting from the sketch you define the flat shape that will then be placed on the fabric to make the garment.

Shapes can be infinite, but they originate from geometric shapes that become the bases.

As we can see from the image above, we start from a simple geometric shape, which is built on the plane by drawing lines in the real measure (scale 1:1). These are created starting from arithmetic rules, with values of bodily measurements (in this case the waist circumference) operated with standard rules to build the line and therefore the desired shape. In Fig. 1, 1/4 of the waist measurement is used to create a line and then a rectangle because the straight skirt dress corresponds to the figure 1.

The fit is added as a number that allows to "dress" the body, that is to say, to the specific base shape are added numbers that can vary according to different factors (weight of the fabric and type of clothing, volume and adherence of the body to the body, corporate rules of comfort).

Starting from geometric shapes and always with arithmetic calculations it is possible to build increasingly complex shapes: as we can see in the picture above, for example, by adding triangles to the straight skirt we get a flared skirt.

The link between geometry and modelling is essential as a study of the figures (in this case shapes that reproduce the parts of the model) in space. The space is traditionally the fabric on which these figures will be placed, which once cut, assembled, packaged will give shape to the garment.

2 The Role of Geometry in the Fashion Project

For the Designer, currently, it is customary to deal with the sketch, which is then modelled, prototyped and put into production by the modeler. The role of the sketch, however, is not only exhausted in the idea of a way to represent the imaginary, but assumes a fundamental role in the part of the pattern, as a tool to describe in a punctual manner shape and size. In this last part, in that of the 2d pattern, geometry has a fundamental role, since it is the moment of drawing of the project filtered by the metaproject. This knowledge, closed and handed down in the traditional tailors' laboratories, has not had the same scientific development that has been found in other branches of knowledge; even today there are no unified standards for fashion design, everyone has a method to derive and represent their own paper model. There are different schools that teach and pass on various ways of doing things.

The first documents on cutting and sewing date back to the sixteenth century, a period in which specific treatise were developed. From an initial bibliographic survey carried out, numerous treatises were traced, of great interest because of their structure, that shows a great geometric knowledge for the development of the pattern, knowledge that was an integral part to the definition of the idea of project.

Among the first documents, in chronological order, there is the famous *Book of the Tailor* of 1540, preserved at the Fondazione Querini Stampalia in Venice, an exceptional collection of sartorial drawings belonging to Gian Giacomo del Conte, an important Milanese tailor who lived in the sixteenth century. The treatise dedicates space both to drawings, which we now call fashion sketches, and to pattern designs. Dates back 1550 the book *Crowns of the Noble and Virtuous Ladies* by Cesare Vecellio, where, in the First Book, it is demonstrated through various drawings that *all the kinds of demonstrations of cut points, points in the air, lattice and of every other species, as well as for friezes, as for laces and rosettes that with the needle are used today for all Europe. And many of which can still serve today for works in labs*. Here the geometry also defines the shapes of the applications and punctually describes the ideas of the various clothes. In 1590 Vecellio, at the time 70 years old, dedicated himself to the work *Old and Modern Garments from Different Parts of the World*, a volume that is a story of the costume resulting from an in-depth research that lasted many years [1, 2].

For this work he creates the drawings of the clothes, commenting on them with a description that does not stop at the clothing alone but becomes a reflection on society and economic, religious and political life. Again the role of the designed geometry is fundamental to the description of the clothes. Another important treatise is that of the Spaniard Juan de Alcega *Libro de Geometria, pratica y traça* (1589) (Book of Geometry, practice and pattern) [3].

Already from the title, the book reveals the importance of geometric knowledge for the design of the fashion project. The work collects, as for the volume of Conte, a significant collection of pattern designs combined with those of sketches.

All these historical texts document the stereotomic roots of the development of the model, showing special attention to knowledge and geometric creativity for the solution of the practical problem of transition from the design of the sketch to that of the pattern, that is, from the idea to the project. The 2D model design is based on theoretical and calculation reflections, on molded logical solutions, not on more or less random approximations. A pattern is a technical instrumentation that responds to certain schemes, implies a design pattern.

There are several ways to create a pattern, the common characteristic of all is the need to follow a strict principle of order and form, depending on the measures taken by the person. The geometric challenge is to create two-dimensionally the development of the designed shape. The 2D model designer must have the geometric expertise necessary to interpret the sketch design and transform it on the plan according to fundamental geometric shapes that derive from as many experimental geometric rules. This creative process, for too long entrusted to the practice of doing, needs to be investigated with up-to-date scientific tools and bases in order to identify new geometric models of reference that allow to broaden the basic range. Large quantities of models can come from the basic model. Cutting, rotating, translating, mirroring some of its parts, in fact, leads to only seemingly different formal solutions that find their reason in the genesis of the beginning (Fig. 2).

The geometric creativity of fashion design lies both in the identification of the fundamental geometric models of reference and in the ability of the model operator to identify formal variants to those of the beginning. It is therefore a kind of creativity that cannot and must not be unique to the modeler, but which must be included in the

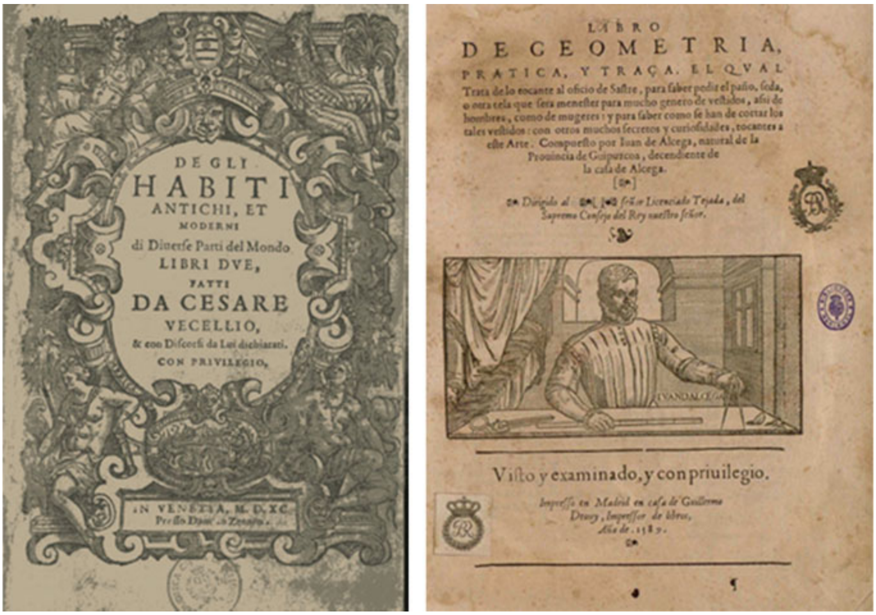

Fig. 2 (A)(B) Covers of the treatise digital creation of a period costume. (D) Rendering (Hughes-McGrail 2012)

basic knowledge of every fashion designer. It is no longer possible to create without having a notion of geometric creativity in fashion. Reforming the reference geometric models therefore means exploring new formalisms that produce innovative models [4].

3 Digital Creation

An important observation concerns digital fashion design software that, along with other applications, even if they improve productivity, do not help to automate the ideation process. The limitation of almost all these software, which also offer numerous and interested applications, is to start from the two-dimensional construction of the model according to the known methods and use parameters that allow to verify the fit of that model on a mannequin in the virtual environment. Therefore, the main and widespread aids for the fashion project are related to the modification of the model(pattern?), to the prototyping, not to the actual creation of the model itself. Hence the interest to go beyond in some designers, who have made geometry a real tool of creativity in fashion. The use of technology is not only intended as the use of innovative materials but it should be seen as a generative process that sees in various generative modeling software the key to the definition of new artistic forms of clothes that use geometry as a creative tool, that combined with 3D printing and the use of innovative materials, produces what is called tech couture. One of the greatest exponents of this new way of doing fashion is Iris van Herpen. The designer, with the aid of computer programs, uses generative geometry as a tool to support her specific design activity. Fashionable technology has a long established interest in the fusion of technology, fashion and apparel design, and a recognition of the relationships between virtual simulations and our everyday lives [5].

In their survey, Grant and Hughes-McGrail sustain that *many tools are specifically tailored and aimed at the fashion designers, with creation processes familiar to them, with pattern and seam identification at the heart of the 3D clothes modelling experience. There is a range of approaches to 3D image production: some* via *modelling, e.g. modelling with physics simulations, image based sculpting (displacement, vector displacement and bump), interactive sculpting, some with minimal modelling and an emphasis on creating images by pattern making with layered 'skins', procedural textures varied over times, producing renders in 3D software then painting and 'finishing' in Photoshop. Well-constructed and segmented geometry makes the trip between software if exported in the correct way, being mindful of object scale and orientation* [6] (Fig. 3).

The creation of 3D garments from 2D models may seem counterintuitive for the designer, who necessarily has to project the knowledge of the rules of garment construction through the abstract 2D image. A method, also used by character designers of games and animations, tends towards a 'sculpture' model: starting from a basic shape, all the folding and flow details in the fabric are sculpted on a 3D mannequin, thus of 3 fering a quick design, also through the use of graphics tablets,

Fig. 3 (A)(B)(C) digital creation of a period costume. (D) Rendering (Hughes-McGrail 2012)

and real-time exploration and modification of the lines and style of the garment, displaying the silhouette from all angles. Intuitive and iterative D modeling methods greatly accelerate the design phase at the same time as removing a layer of abstraction from the design process, thus making it easier and more playful for the designer to physically build a garment. Anyway, if digital sculpture tools may improve design conceptualization phase, the models created are not directly usable in the production of traditional clothing. More familiar methods of 3D digital clothing design are based on simulation in order to produce digital artefacts that can be used in traditional production practices, as, for example, starting with geometry pre-deformed by a cloth and gravity simulation (using a base mesh as a collision object).

According to Grant and Hughes-McGrail, two techniques, texture bombing and procedural surface sculpture, used by game and animation concept artists, are exemplary of the kind of 3D visualization techniques that may be of interest to the sketching fashion designer, since it is easy to generate limitless variations. They call the techniques ‘hybrid,’ as they work across software and involve a variety of techniques. And they use “the term ‘procedural’ for two reasons: (1) the steps to achieve the results abide by simple rule-sets where numerical input can be varied to achieve wildly varied results and (2) particular mathematically described textures, like ‘noise’, ‘cellular’, ‘dots’, ‘ripples’, ‘weaves’, ‘woods’, ‘checkers’, are used not only for pattern generation, but also surface perturbation and geometry displacement” [6].

According to Hughes-McGrail, the prior boundaries between the tasks (and roles) of designer, pattern cutters in fashion and costume design are more and more indistinct, enabling a more integrated process of design and of production by using digital sculpting tools and 3D pattern design tools. Iris van Herpen is a great example of this integration. “The powerful simulative qualities of 3D digital tools provide new methods that both challenge their art and enable them to push their designs further” [6].

The designer explains the potentialities of this approach: “Working with handcrafted techniques or a sewing machine gives you a lot of possibilities but also a lot of restrictions. 3D printing is an entirely different language. The complexity and detailing of it almost resembles old historic crafts. It lets me think in total three-dimensionality, instead of first imagining something in 3D, then drawing it on paper in 2D and then creating it for the body in 3D again. For me, it’s a dream” [7].

4 Iris van Herpen

An Iris van Herpen is described as “a modern alchemist rather than a fashion designer – a prolific decade of creations now charts her relentless material curiosity and experimentation” [8].

Iris van Herpen, was born in the Netherlands in 1984. She graduated from the ARTEZ fashion school (Arnhem) with a degree in Fashion Design and then she carried out internships with Claudy Jongstra and Alexander McQueen—whose influences pervade throughout in her style and techniques. In her unique style, she is able to create a very peculiar type of couture, combining the qualities of hand-worked materials with digital technology, continuously pushing the boundaries of fashion design [9].

Since 2007, the talented designer van Herpen combines the most traditional craftsman techniques and methods to the most innovative technologies into her unique aesthetic vision, with the clear intent of blending the past into the future. Due to this singular vision, combined with the complexity of her creations, since January 2011 van Herpen constantly shows on the Paris Haute Couture week. Her distinct aesthetic viewpoint made her dress many musicians, dancers and actresses. Van Herpen’s work has also been presented in various museum exhibitions, books and short films. The docu-video “Between the lines”, for example, casts a look into the creative process of the Between The Lines Couture collection, inside the atelier and behind-the-scenes at the show in Paris [10].

Though earlier works often relied on low tech solutions, because of van Herpen's increasingly interest in multidisciplinary approach to creation that goes beyond fashion design, she has often opened collaborations with various artists such as Jolan van der Wiel and Neri Oxman and architects such as Philip Beesley and Benthem and Crouwel Architects, contributing high-tech innovations to her collections. Her interest in science and technology has led to continuing collaborations with CERN (The European Organization for Nuclear Research) and MIT (Massachusetts Institute of Technology).

In Iris van Herpen's vision: *The body and female forms are my canvas to visualize the invisible, shaping a continuous dance between craftsmanship and innovation. Within my designs I search for symbiotic relationships, exploring the hidden beauty at the intersection of precision and chaos, art and science, the artificial and the organic, that are blending into infinite hybrids. This philosophy of duality is ground to all my designs. Within my search for transformation, one of the most influential things in my life has been my classical ballet practice. Those years of dancing I learned so much about the symbiotic relationship between the mind and body, the transformation and the 'evolution' of shape. […] My creative process is based on experiment and innovation, exploring new forms of craftsmanship that can transform traditional Couture to new grounds of innovation through a multi- disciplinary approach of collaborations with artists, architects and scientists. Through my eyes, fashion creates a dialogue between our insides and our outsides* [11].

As she underlined in many occasions, actually her dance experience was fundamental: through dance she became interested in the body and its relationship to space and movement [12].

Time-lapse motion, fractal like geometries, chronophotography, movement and fourth dimension, kinetic installations, generative design process, patterns of chaos and order, magnetism, Synesthesia, invisible rays (particularly electronics), the typical geometric and graphic patterns of Egyptian mummies, refinery smoke, magic and alchemy, are some sources of inspiration for the designer.

Here are some examples of her collections, which descriptions are given in her personal website [11] and we summarize hereafter:

CRYSTALLIZATION—July 2010, Amsterdam Fashion Week

Crystallization, launched at the Amsterdam Fashion Week in 2010, is the first fashion collection ever featuring 3D printed dresses. ARCAM (Architecture Centre Amsterdam) organized a collaboration between Iris van Herpen and Benthem Crouwel Architects. The latter's design for a new extension to Amsterdam's Stedelijk Museum was nicknamed the 'bath tub'. This inspired Van Herpen to design a dress that would fall around the wearer like a splash of water, like being immersed in a warm bath, and to express in the collection the different states, structures and patterns of water, see Fig. 4.

ESCAPISM—January 2011, Paris Haute Couture Week

Escaping from everyday reality through addictive digital entertainment incites in Iris van Herpen not only feelings of emptiness but also associations with the

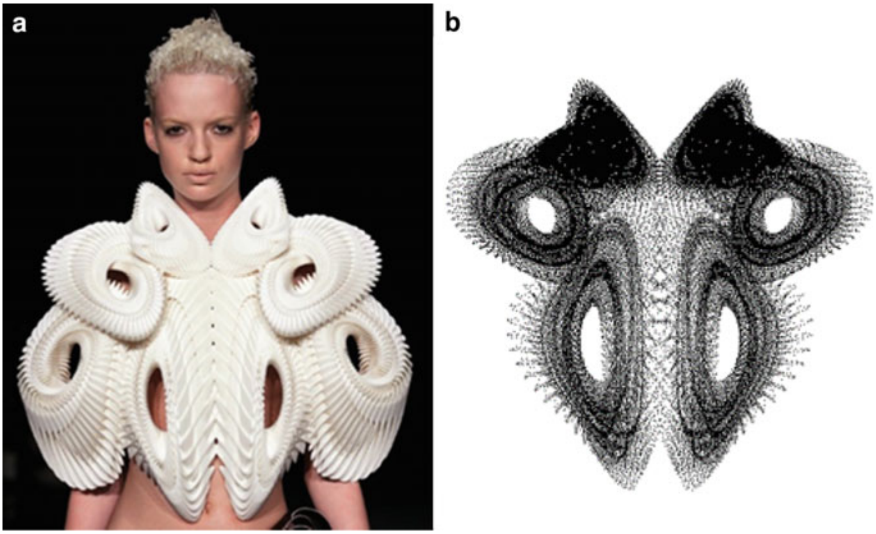

Fig. 4 (A) Crystallization. (B) Rendering gif from https://www.arch2o.com/crystallization-daniel-widrig-iris-van-herpen-mgx/

grotesque, the extreme and the fantastic. This was the source of inspiration, together with baroque and exuberant sculptures of Kris Kuksi, an American artist. Escapism is a continuation of the collaboration with the London based architect Daniel Widrig, un which they both attempted to further investigate possibilities and potentiality of advanced digital design techniques and computer aided manufacturing in the realm of haute couture fashion design, in which the geometric concept allowed to increase the wear ability of the pieces [13].

CAPRIOLE—July 2011, Paris Haute Couture Week

This new collection presented, in addition to a compilation of highlights from previous collections, five new models that reflect the extreme feelings experienced just before and during a free-fall parachute jump. "A 'leap in the air' (the meaning of the French word Capriole) that Van Herpen once in a while takes to reset her body and mind." Evokative Model n. 18 is constructed by uncountable triangles (Fig. 5).

EMBOSSED SOUNDS—October 2013, Paris Fashion Week

Starting from idea of the potential permeability between the senses, Iris Van Herpen has developed clothes that generate sounds by touch. "'Embossed Sounds' is the name of her orchestra of clothes which explore garments as electronic instruments that one can touch and play live" [11].

MAGNETIC MOTION—September 2014, Paris Fashion Week

After visiting at CERN the Large Hadron Collider, the designer was inspired by interplay of magnetic forces of attraction and repulsion. "I find beauty in the continual shaping of chaos which clearly embodies the primordial power of nature's performance," says Van Herpen describing the essence of the collection, for which she has been collaborating with the Canadian architect Philip Beesley, and the Dutch

Fig. 5 (A) Capriole luglio 2011. (B) Detail

artist Jolan van der Wiel. Beesley is a pioneer in responsive 'living' sculpture whose poetic works combine advanced computation, synthetic biology, and mechatronics engineering.

HACKING INFINITY—March 2015, Paris Fashion Week

For her Hacking Infinity collection, presented in Paris on March 10th, 2015 at the Palais de Tokyo, Iris van Herpen explores the possibility of new geographies and our place within them. "The desire to reconfigure space finds expression in light performative materials, which interact with the movement of the body, biomimetic structures and saturated spectral colours. The central geometry is the circle, in both silhouette and cut. The spherical shape of planetary bodies and the symbol of a boundless 'hackable' infinity unfolds before us in a constant flow of mandala-like forms. Hand plisséed geometries both follow and frame the body, whilst optical lighting film belts propose a polymorphic silhouette and challenge our perception of the figure in space" [11].

She goes on exploring 3D with the creation of a hand-woven textile together with designer Aleksandra Gaca. In addition, she continues her collaboration with the Canadian professor of architecture Philip Beesley creating digitally fabricated dresses showing fractal like geometries.

SEIJAKU—July 2016, Paris Haute Couture Week

For "Seijaku", the Couture collection presented in Paris on July 4th, 2016 Iris van Herpen explores the study of cymatics, which visualizes sound waves as evolving geometric patterns. "In cymatics, the higher the frequency of the sound wave, the more complex the visible patterns. […] The collection reflects circular shapes and geometric patterns that are common in Cymatics, which serve as the base for this collection's biomorphic volumes" [11].

Van Herpen didn't use actual sounds or compositions to create her plissé dresses, she clarified backstage; rather, known patterns, easily Googled, informed the manner in which she pleated her creations. The lines were straight, it was the pleating that created their snaking curves. Which brings us back to patterns and cymatics.

BETWEEN THE LINES, January 2017 Paris Fashion Week

"'Between the lines' explores the imperfection of systems and structure in both the physical and digital worlds. Van Herpen focuses on the gaps in between the structures of her materials, rather than the structures themselves, by shaping patterns that dissimulate the body's perspective or subtract it. By building up the patterns and then distorting them, the eye's perspective is tricked and challenged to see new patterns occurring in between. Linear shifts and sharp contrasts form the base of this innovative approach to material development and patternmaking and challenge us to "mind the system, but to find the gaps" [11].

During her show, the German artist Esther Stocker, known for her manipulation of dimensional geometries, used glitches, distortions, digital visuals, leading the mind to spatial illusions.

Among the innovations featuring in this collection, we find 3D hand-casted PU fabrics that are hand-painted through injection moulding and fine expandable laser-cut Mylar fabrics, reminding of digital glitches, in collaboration with architect Philip Beesley, see Fig. 6.

LUDI NATURAE—January 2018 Paris Fashion Week

"For this collection the designer examined the natural and manmade landscapes of our world from a bird's-eye view, tracing the laws of entropy […] The 21 silhouettes feature boundary-pushing construction and innovative material techniques like 'Foliage', a process initiated with the Delft University of Technology in which leaf-like patterns are 3D printed as thin as 0.8 mm and tulle is laid into the 3D printer to print directly onto the fabric, creating exceptional softness.

'Data Dust', in which parametric patterns are computationally distorted, foam-lifted, laser-cut and then heat-bonded onto an invisible silk tulle, create radiant glitches" [11].

HYPNOSIS—July 2019, Paris Fashion Show

For this collection, the designer finds inspiration in the hypnotic manifolds within our ecologies through the work of American artist Anthony Howe and the three-dimensional cyclical harmony of his kinetic sculptures. The idea of infinite expansion and contraction, representing the cycle of life, the relation with Nature, found in Howe's spherical sculpture 'Omniverse' is of great inspration for the collection. "As one of the key pieces of the collection, the finale 'Infinity' dress comes alive on the breath of a finely balanced mechanism. […] 'Hypnosis' reflects the beauty

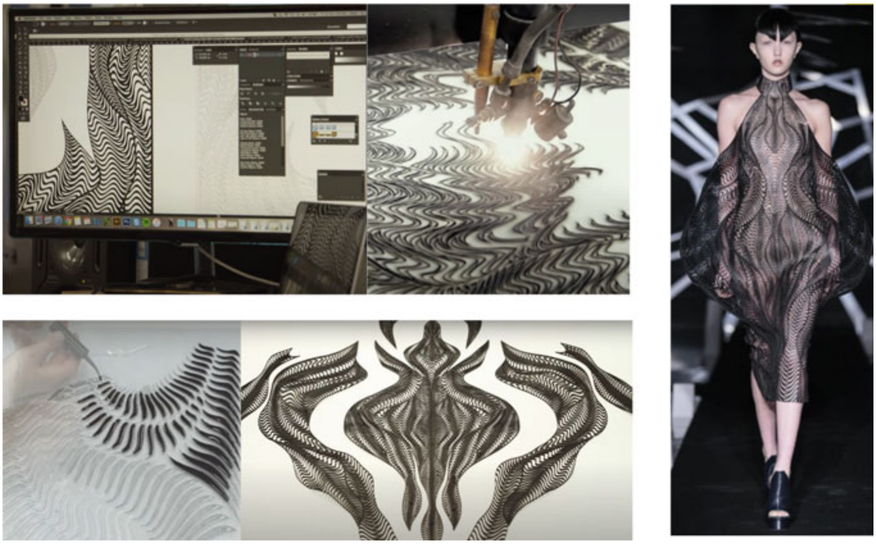

Fig. 6 Between the lines, Iris van Herpen

and complexity of our environment, exploring the patterns and structures within its fragile landscape" [4].

SENSORY SEAS—January 2020, Paris Fashion Week

The inspiration for this collection is an interplay between the sensory processes that happening in the intricate composition of the human body, and the fibrous marine ecology of our oceans: 21 silhouettes "illustrate a portrait of liquid labyrinths, […] colourful meshworks of cellular geometry are translucently layered to create deep-sea aquarelles" [11, 4].

The show is enriched of a light sculpture by Paul Friedlander, made of three waves transforming frequency into shape, from which the models emerge and are surrounded, as by a metaphorical sea of sensory waves.

5 Conclusions

Iris van Herpen in few years has become a world-renowned fashion designer. Her style is unique and innovative; anyway, talent itself is not sufficient, but a starting point to create a strong brand identity and an immaginific universe around it.

Yonson, analysing the creations of Iris Van Herpen, aims to identify the meaning and formation cause of a style, and the essential elements of style formation, through psychobiological research. As a result, it has been found that "the psychobiological causes to form a style stem from the action of 'long-term memory', which is consolidated by 'selective attention', 'perceptional subjectivity', the principle of the 'neuron's connection specificity and invariance', and the principle of a 'neuronal signal's unilateral flow'. With such action, Herpen could develop her own original composition techniques. The formative shapes created by such composition techniques are characterized by enumeration, superposition, and hanging. The study has also found that the essential elements for a designer to be able to form his/her own style include 'aesthetic originality' in which the designer views the property of a thing from his/her inherent perspective, and finds the uniqueness from the thing that only he/she can express, 'technical differences' that are creative and original, and 'formative specificity' that is summarized into one property through an impressive shape" [14].

Haute couture has always been an experimental laboratory of ideas and universes, though also bound to tradition; anyway, Iris van Herpen goes beyond in inspiration and innovation, unifying art and science in the inspiration, on the technical ground, in the production process and the construction of her shows.

As Lee points out, "the objects presented in Iris van Herpen's haute couture collections are classified into three categories: human body and nature, senses and emotions, and culture and technology. Iris van Herpen uses organic shapes, repetition, gradual change, and continuation to emphasize particular shapes. She finds new unfamiliar materials, and discovers her own unique structures and shapes through the process of finding solutions for them. Her innovative inclinations shown in her collections and techniques include the introduction of advanced technologies through collaboration, the combination of creative technology and materials, and the construction and realization of imagination based on technologies [11]. Personal expression on the worlds of art, architecture, science, nature, philosophy surrounding me. The garments generate a new perspective upon beauty, in which future, technology and craftmanship live together" [15].

Haute couture has always been an experimental laboratory of ideas and universes, though also bound to tradition; anyway, Iris van Herpen goes beyond in inspiration and innovation, unifying art and science in the inspiration, on the technical ground, in the production process and the construction of her shows.

In 2020, the coronavirus pandemic restrictions have brought disruptive innovations, forcing all fashion designers and fashion week organizers to rethink calendars and shows organization. New challenges lead anyway to new creative solutions to show the collections and not only: digital fashion weeks staged online, physical catwalks streamed online, podcasts, photo diaries and other contents, livestreaming of digital experiences and making of, shorts and movies, virtual and augmented reality, virtual fitting, 3D avatars as models, live stream shopping…

Digital innovations and "phygital" (physical-digital) approaches are now part of the shape of future fashion world, fostering creativity and freedom, extending the public by a global sharing: inspiring tools to support, rather than replace tradition.

A great example of a young woman, leader in her sector, one of the many "Faces of Geometry".

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Geometrical Solution for the Trisection Problem

Paola Magnaghi-Delfino, Giampiero Mele, and Tullia Norando

Abstract The trisection problem date back to the Greeks and Arabs and it is related to the algebraic solution of third degree. It concerns construction of an angle equal to one third of a given arbitrary angle, using only two tools: unmarked ruler and compass. The problem is stated impossible to solve for arbitrary angles, as proved by Pierre Wantzel in 1837. In this article, we present some geometric or algebraic methods to solve the problem from the first one due to Greeks until Maria Gaetana Agnesi's algebraic-geometric effort. Then we propose a geometric approximation's method based only on straightedge and compass.

Keywords Mathematics · Geometry · History of Mathematics

1 Introduction

The ancient Greeks were particularly interested in the construction of angles of different sizes using only unmarked straightedge and compass. According to the problem of drawing regular polygons of given arbitrary number of sides. The trisection of the angle was one of the problems that employed mathematical scholars for a long time. Over the centuries, many scholars invented procedures for trisection of the angle but never it was possible to solve this problem using only straightedge and compass. The geometric trisection's problem became an algebraic problem, connected to the not solvable equations of third degree [1].

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2 Greek and Arab Methods

The Arab mathematician Abusaid Ahamed ibn Muhammad ibn Abd-al-Galil-as-Sigzi (about 951–1024) solves the trisection's problem as an intersection of a circle with an equilateral hyperbola. In XI century, Al-Kashi develops an iteration procedure that stands out for its simplicity and rapidity of convergence. His treatise *On the chord and the sine* was never found but is mentioned at the beginning of the book *Key to Arithmetic* with the hint to popularize the method used to calculate sen1°. In this treatise, the Persian mathematician exposed the equation of the angle trisection, based on two theorems due to Euclid and Ptolemy. Al-Kashi connects algebraic techniques to geometric methods due to ancient Greek scholars.

2.1 Euclid's Theorem

In a right-angled triangle, the square built on the height relative to the hypotenuse is equivalent to the rectangle whose sides show the projections of the two cathetuses on the hypotenuse [2].

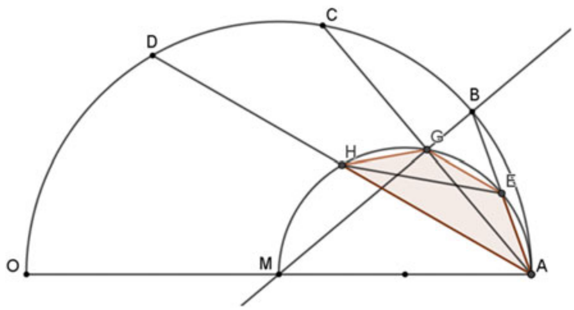
2.2 Ptolemy's Theorem

Given a quadrilateral inscribed in a circle, the sum of the products of the pairs of opposite sides is equal to the product of the diagonals.

2.3 Algebraic Equation

We consider now the semi circumference *ABO* with radius $= R$. The arcs *AB, BC, CD* are equal. We draw a semi circumference *AEM* with diameter *AM* (Fig. 1).

Fig. 1 .



We consider the chords *AB, AC, AD* and we note that the arcs *AE, EG, GH* are equal. Knowing the value of the chord *AH*, we can find the value of the chord corresponding to the arch *AE*.

We apply the Ptolemy's theorem to the quadrilateral *AEGH* and observing that $AE = EG = GH$ and $AG = EH$, we have

$$AE^2 + AE.AH = AG^2 \tag{1.1}$$

As $AG = GC$, using Euclid's theorem we have

$$AG^2 = BG(R - BG) \tag{1.2}$$

And, by secant theorem,

$$AB^2 = BG.2R$$

Substituting *BG* in (1.2), we find

$$AG^2 = 4AE^2 - 4AE^4/R^2 \tag{1.3}$$

The equation for the trisection of an angle can be deduce substituting (1.3) in (1.1)

$$4AE^3 + R^2 \cdot AH = 3R^2 \cdot AE \tag{1.4}$$

Let α be the angle corresponding to the arc *AE* and 3α the angle corresponding to the arc *AH*, we have that $AH = R\ sin3\alpha$ and $AE = R\ sin\alpha$, so we find

$$sin3\alpha = 3sin\alpha - 4sin^3\alpha$$

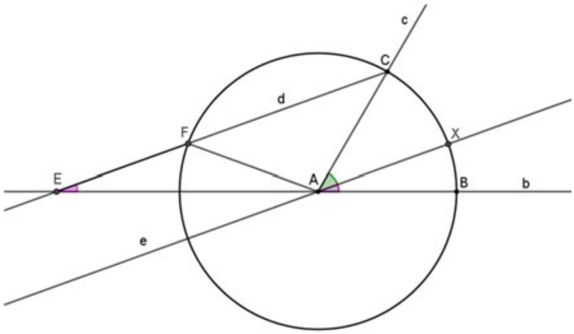
3 Archimedes's Method

In the solution proposed by Archimedes the ruler is used to report a length and, therefore, is thought of as a twice-notched straightedge. Suppose that we want to trisect the angle $C\hat{A}B$ so we draw the circumference Γ, with centre in *A* and radius $=r$. The circumference intersects the line *c* in *C* and the line *b* in *B*. Now we design a line *d*, passing through *C*. The line *d* intersects the line *b* in *E* and the circumference in *F* so that $EF = r$. We draw the line *e* parallel to *d* and passing through *A*. The line *e* intersects the circumference in *X*. The angle $X\hat{A}B$ is one third of the angle $C\hat{A}B$ (Fig. 2).

Hp: $EF = AF = AB = AC$

Th: $X\hat{A}B = \frac{1}{3}C\hat{A}B$

Fig. 2 .



The two triangles *EFA* and *CAF* are isosceles. The side *EF* is equal to the side *AF* and the side *AF* is equal to the side *AC*.

So, we have

$$F\hat{E}A \cong F\hat{A}E \text{ and } A\hat{C}F \cong A\hat{F}C$$

$C\hat{A}B$ is an external angle for the triangle EAC, then

$$C\hat{A}B \cong F\hat{E}A + A\hat{C}F \tag{2.1}$$

$A\hat{C}F \cong A\hat{F}C$, external angle for the triangle *EFA*, then

$$A\hat{F}C \cong F\hat{E}A + F\hat{A}E \cong 2F\hat{E}A \tag{2.2}$$

From (2.3) and (2.3), we have that

$$C\hat{A}B \cong F\hat{E}A + 2F\hat{E}A = 3F\hat{E}A$$

that is

$$F\hat{E}A \cong \frac{1}{3}C\hat{A}B \tag{2.3}$$

EF // AX and the angles $F\hat{E}A$ e $X\hat{A}B$ are corresponding angles then

$$F\hat{E}A \cong X\hat{A}B \tag{2.4}$$

From (2.3) and (2.4), we find that

$$X\hat{A}B \cong \frac{1}{3} = C\hat{A}B$$

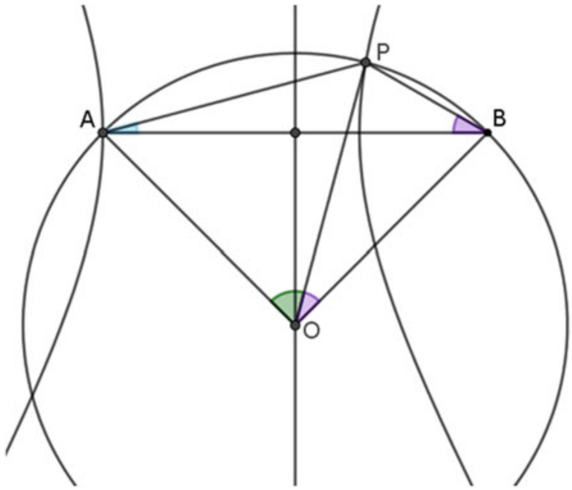


Fig. 3 Pappus' solution

4 Solutions Using Algebraic Curves

4.1 Pappus Solution

Pappus of Alexandria (290–350 AD) composed an opera in eight books entitled *Mathematical Collection*. In this work, Pappus solves the problem of trisection using the conics and referring to Apollonius [3, 4] (Fig. 3).

Let *AB* a line, we want to determine the locus of the points *P* such that $2P\hat{A}B = P\hat{B}A$.

It is shown that this locus is a hyperbola having eccentricity equal to 2, having a focus in *B* and the axis of the segment *AB* as a directrix. Considered as the centre the point *O*, we draw the circle passing through *A* and *B* and the hyperbola in such a way. The hyperbola intersects the circle in *P*. The segment *PO* trisects the angle $A\hat{O}B$. From the properties of the hyperbola, $2P\hat{A}B = P\hat{B}A$. The central angle of a circle is twice any inscribed angle subtended by the same arc then $2P\hat{A}B = P\hat{O}B$ that insist on arc *PB* and $2P\hat{B}A = P\hat{O}A$ who insist on the arc *PA*.

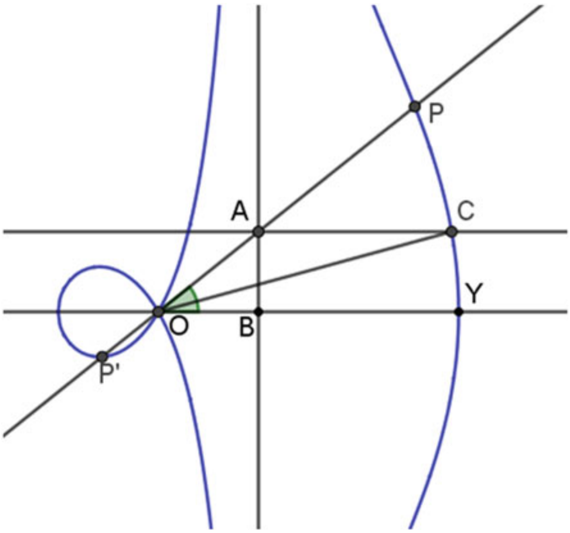
By combining the two relationships we get $2P\hat{O}B = P\hat{O}A$ that is, the angle $P\hat{O}B$ is the third part of the angle $A\hat{O}B$.

4.2 The Solution with the Conchoid of Nicomedes

Nicomedes, a contemporary of Archimedes, invented the curve called *conchoid* (Fig. 4).

The Cartesian equation of the curve is

Fig. 4 The conchoid



$$\left(x^2 + y^2\right)(x - d)^2 = k^2x^2$$

To obtain the conchoid, we fix a point *O* (pole) and a line distant *d* from *O*. Consider a second line passing through *O* that intersects the previous line in *A*. On this line, on opposite sides with respect to *A*, we consider two segments *AP* = *AP'* each of length *k*. The locus of the points *P* and *P'* obtained by rotating the line through *O* is called conchoid. Now let's see how to use the conchoid for the angle trisection problem. Let *AOB* be an angle, and consider the conchoid with $OB = d$ and $AP = k$. The parallel line to *OB*, through the point *A*, meets branch external of the conchoid in *C*. Joining *C* with *O* we prove that *AÔC = 1/3 AÔB*.

4.3 Solution with the Use of Pascal's Snail

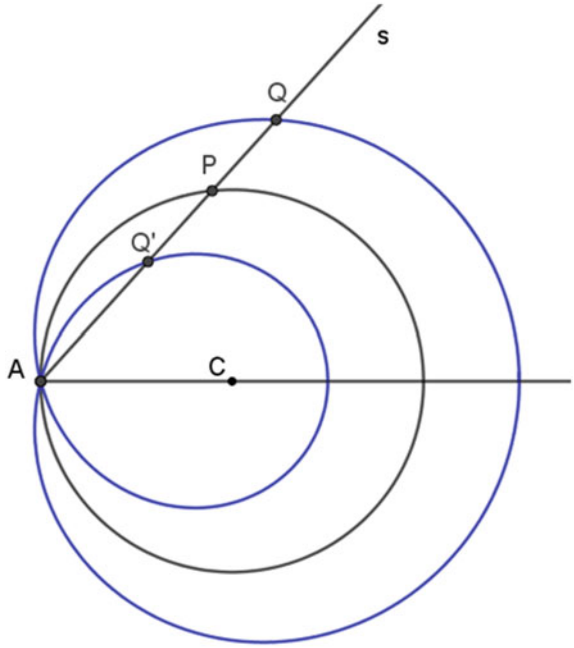
The conchoid of a circle for a fixed point on it is called *limaçon of Pascal*. The first part of the name picked by Étienne Pascal, father of Blaise Pascal, means snail in French.

The curve is simple to describe in polar coordinates as $r = b + dcos(\theta)$, where *d* is the diameter of the circle and *b* is a real parameter. This can be converted to Cartesian coordinates and we obtain

$$\left(x^2 + y^2 - dx\right)^2 = b^2\left(x^2 + y^2\right)$$

The circle about *C* with radius *AC* is fixed. The line s rotates about *A* and the point *Q* is on *s* at the fixed distance from the circle $b = PQ$, and $d = 2AC$ (Fig. 5).

Fig. 5 Pascal's snail



Now let's see how to use the snail for the angle trisection problem. We draw the snail for which C belongs to the interior branch of the snail (Fig. 6).

We consider the angle $B\hat{C}Q$ draw and join Q with A. Let P the intersection point of AQ and the circle with centre C and radius AC. Since triangles APC and QPC are isosceles, $P\hat{A}C = A\hat{P}C = 2P\hat{Q}C$. Then for the triangle AQC we have:

$$B\hat{C}Q = P\hat{A}C + P\hat{Q}C = 3P\hat{Q}C$$

4.4 Solution Using the Maclaurin Trisectrix

The trisectrix is an algebraic curve of the third order, cubic with node that was studied by Colin Maclaurin in 1742. The Maclaurin trisectrix can be defined as locus of the point of intersection of two lines, each rotating at a uniform rate about separate points A and B, so that the ratio of the rates of rotation is 1:3 and the lines initially coincide with the line between the two points. This curve is notable because it can be used for trisecting the angles, indeed it follows from the property that when the line rotating about A has angle θ with the x axis, the line rotating about B has angle 3θ (Fig. 7).

If $A = (0{,}0)$ and $B = (a{,}0)$, the Cartesian equation is $2x\left(x^2 + y^2\right) = a\left(3x^2 - y^2\right)$

P_1

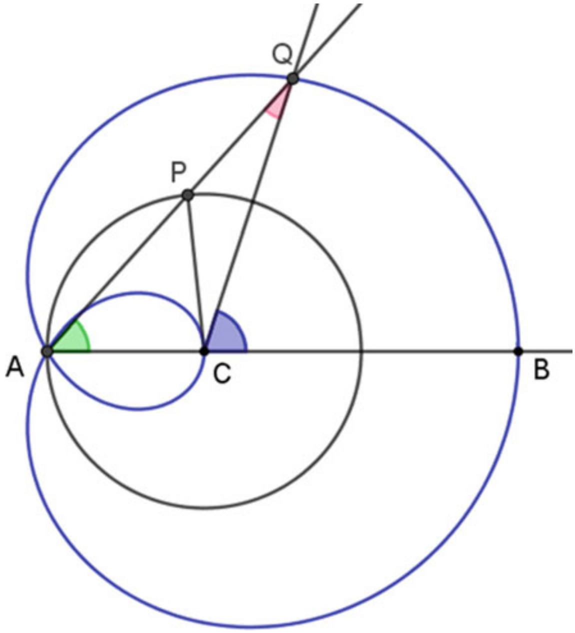


Fig. 6 Trisection with snail

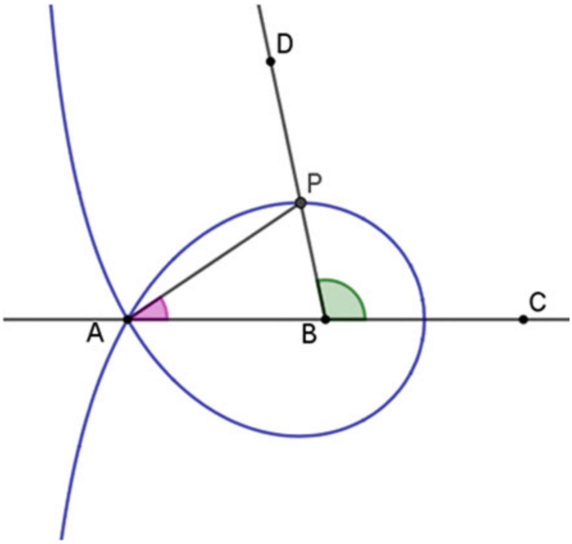


Fig. 7 Trisection's property

5 Letter from Jacopo Riccati to Maria Gaetana Agnesi (1751)

Jacopo Riccati maintained an intense correspondence with Maria Gaetana Agnesi from 1745 to 1752, that testifies the exchange of scientific ideas that arose around the writing and printing of *Instituzioni analitiche ad uso della gioventù italiana*. In 1751, among other suggestions he presents his own solution to the classical problem of angle trisection, hoping that Maria Gaetana will approve it [5].

After recalling that cubic equations often do not admit algebraic solutions, Riccati wonders if they cannot be handled with some unknown artifice.

He proposes an essay in the famous problem of angle trisection that cannot be satisfied analytically because the roots, even if real, appear as imaginary.

Consider the scalene triangle *ABC* and divide the angle *B* into three equal parts (Fig. 8).

We set

$$AB = a; BC = b;\ AC = c;\ AD = x;\ DE = y;\ EC = z \tag{4.1}$$

and suppose that $AB < BC$.

A well-known theorem applied to triangle *ABE* states that if the line *BD* divides the angle $A\hat{B}E$ into two equal parts, we have the following relation

$$AB.BE - DB^2 = AD.DE \tag{4.2}$$

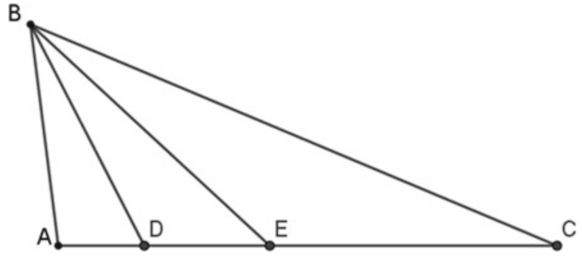
For the triangle *BDC*, in which the line *BE* divides the angle $D\hat{B}C$ into two equal parts, we have the following relation

$$BD.BC - EB^2 = EC.DE \tag{4.3}$$

From the angle bisector theorem, we also have that

$$AD : DE = AB : BE$$

Fig. 8 .



$$CE : ED = CB : BD$$

so

$$BE = \frac{ay}{x}, BD = \frac{by}{z} \tag{4.4}$$

From the two theorems, we get

$$\begin{cases} \frac{a^2 y}{x} - \frac{b^2 y^2}{z^2} = xy \\ \frac{b^2 y}{z} - \frac{a^2 y^2}{x^2} = zy \end{cases}$$

$$\begin{cases} a^2 z^2 - b^2 yx = x^2 z^2 \\ b^2 x^2 - a^2 yz = x^2 z^2 \end{cases} \tag{4.5}$$

From which we get the only equation

$$a^2 z^2 - b^2 yx = b^2 x^2 - a^2 yz \tag{4.6}$$

In this equation the side $AD = c$ is missing.

We need to introduce the AD side and at the same time eliminate the linear y. The (4.6) becomes

$$a^2 z^2 - b^2 xc + b^2 x^2 + b^2 xz = b^2 x^2 - a^2 zc + a^2 zx + a^2 z^2 \tag{4.7}$$

From which

$$-b^2 xc + b^2 xz = -a^2 zc + a^2 zx \tag{4.8}$$

where both x and z are linear.

We obtain

$$z = \frac{b^2 xc}{(b^2 - a^2)x + a^2 c}$$

we place

$$g^2 = b^2 - a^2$$

from which

$$z = \frac{b^2 xc}{g^2 x + a^2 c} \tag{4.9}$$

so

$$z^2 = \frac{b^4x^2c^2}{\left(g^2x + a^2c\right)^2}$$

From (4.5) we have

$$y = \frac{z^2\left(a^2 - x^2\right)}{b^2x} = \frac{x^2\left(b^2 - z^2\right)}{a^2z}$$

and so

$$\frac{z^2\left(a^2 - x^2\right)}{b^2x} = \frac{x^2b^2}{a^2z} - \frac{zx^2}{a^2} \tag{4.10}$$

Using the (4.9) we get

$$\frac{g^2x + a^2c}{a^2cx} - \frac{b^2cx}{a^2\left(g^2x + a^2c\right)} = \frac{\left(a^2 - x^2\right)b^2c^2}{x\left(g^2x + a^2c\right)^2}$$

Making the common denominator and collecting we obtain

$$g^4x^3 - b^2c^2x^3 + 3g^2a^2cx^2 + 3a^4c^2x = a^4c^3$$

We have come to a complete third-degree equation.

To simplify this expression we have to remove the term of maximum degree.

We set

$$g^2 = bc$$

that is $b^2 - a^2 = bc$.

We obtain

$$3g^2a^2cx^2 + 3a^4c^2x = a^4c^2$$

$$x^2 + \frac{a^2}{b}x = \frac{a^2c}{3b}$$

which is a second-degree equation with two real solutions of opposite sign.

$$x^2 + \frac{a^2}{b}x + \frac{a^4}{4b^2} = \frac{a^2c}{3b} + \frac{a^4}{4b^2}$$

$$\left(x + \frac{a^2}{2b}\right)^2 = \frac{a^2c}{3b} + \frac{a^4}{4b^2}$$

$$x = -\frac{a^2}{2b} \pm \sqrt{\frac{a^2c}{3b} + \frac{a^4}{4b^2}}$$

Recalling that $c = b^2 - a^2/b$
We obtain

$$x = -\frac{a^2}{2b} \pm \frac{a}{2b}\sqrt{\frac{4b^2 - a^2}{3}}$$

5.1 Geometric Meaning

Given the triangle *ABC*, with sides $AB = a$ (assumed fixed), $BC = b$ variable with the only condition $BC > AB$.

We get the value *c* of the side *AC* with the following proportion

$$b : (b - a) = (b + a) : c$$

derived from the condition $b^2 - a^2 = bc$

So given *a, b, c*, we get infinite triangles all well-defined since, even if we take two arbitrary values of *a* and *b*, we always have that

$$BC^2 - AB^2 = B.BA$$

So, we can trisect the angle *B* using a second-degree equation.

Angle $A\hat{B}C$ is always acute because $BC > CA$.

The construction does not consider obtuse angles, but this is not a limitation since if it trisects an acute angle in a similar way it can divide the supplementary.

The angle $B\hat{A}C$ is always obtuse because our operation leads to say that in the triangle *ABC* the following relation holds

$$AB^2 + AC^2 = a^2 + c^2 < b^2 = BC^2$$

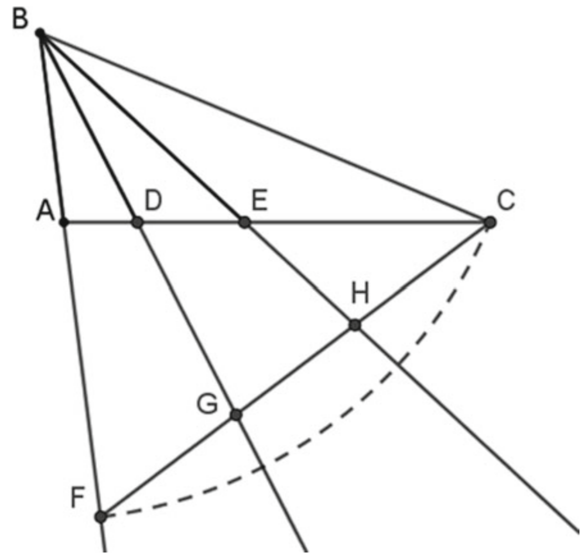
Given the triangle ABC with the above condition, we cut into equal parts the angle *ABC* with the two lines *BD, BE*. It follows that we can know *AD* using a quadratic equation (Fig. 9).

Consider another *CF* basis to vary the triangle. In this case the third-degree equation cannot be reduced unless $a = b$.

Then given the sides *AB, BC* between the infinite third sides that close the triangle only

$$AC = c = b - \frac{a^2}{b}$$

Fig. 9 .



it has a special property that allows us to obtain the result. In all other cases the algebraic way does not consent to solve the trisection problem.

6 Other Geometric Method: Using the Square

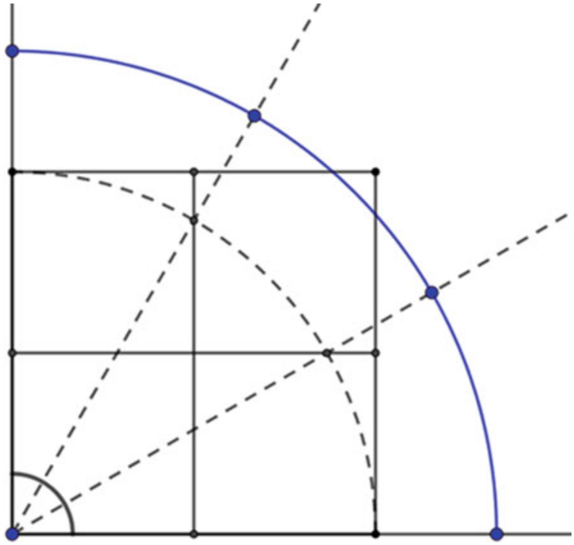
Many geometric methods were proposed for trisecting the right angle and arbitrary angles, but we focus on methods that require the use of straightedge and compass. In the case of the right angle we obtain the exact trisection, in the general case a good approximate trisection.

In the Phd thesis, G. Mele exposed his idea of trisecting right angle, based on his studies about medieval architecture.

Medieval architects studied the problem of the polygons' inscription in the square that is connected to trisection of the right angle. This problem exploits the relationship for the construction of a right-angled triangle having a base equal to 1/2, a height equal to $\sqrt{3/2}$ and hypotenuse equal to 1.

By drawing a square with its medians and considering a quarter of a circumference with a radius equal to the side of the square, we notice that the circumference intersects the medians in two points. By joining these points with the centre of the circumference, we obtain the division into three equal parts of the right angle. It is also possible to avoid tracing the circumference, and therefore the use of the compass, as it is sufficient to trace on the medians a measure equal to $L\sqrt{3/2}$, where L is the measure of the side of the starting square (Fig. 10).

Fig. 10 .



7 Trisection's Approximation of Arbitrary Angles Using Only Straightedge and Compass

The geometric approximation of the arbitrary angle's trisection, proposed by G. Mele in his Phd thesis [6], is obtained using only straightedge and compass. We show how to trisect an acute angle. This is enough, since we know it is possible to trisect a right angle using only a straightedge (without notches) and compass, and an obtuse angle is the sum of a right angle and an acute angle. Let α be the angle formed by the two half-lines s and t of origin A. We draw the circumference of vertex O and radius R. B and C are the intersections with s and t respectively. We divide the angle α into four equal parts using straightedge and compass (Euclid's Elements I, 9) [2].

Let *E* be the point on the circumference such that *BAE* = *1/4BAC*. Now we divide *AB* into three equal parts and consider *AF* = *1/3R*. We draw the circumference centered in *A* with radius *AF* and let H be the intersection of this circumference with *AE*. The *FH* arc is equal to *1/3* of the *BE* arc (Fig. 11).

Now we construct a rectilinear angle *EPG* equal to the rectilinear angle FAH on the straight line *PE* = *1/3R* (Euclid's *Elements* I, 23). Then we trace *EG* and report it on the circumference of radius *R* and center *A* obtaining *EK*. Truthfully, the *EG* chord is reported and we obtain EK which is good approximation of the arc length *EG*. On the circumference centered in *A* with radius R we obtain the arc *BK*, which corresponds to the angle β such that $\beta = 1/4\alpha + 1/3(1/4\alpha) = 1/3\alpha$ (Fig. 12).

The approximation error decreases when the angle measure decreases and its estimated value is done by the formula

$$\frac{x}{4} + \frac{2}{3} sin\left(\frac{x}{8}\right) \cong \frac{x}{3} - \frac{1}{72}\left(\frac{x}{4}\right)^3$$

Fig. 11 .

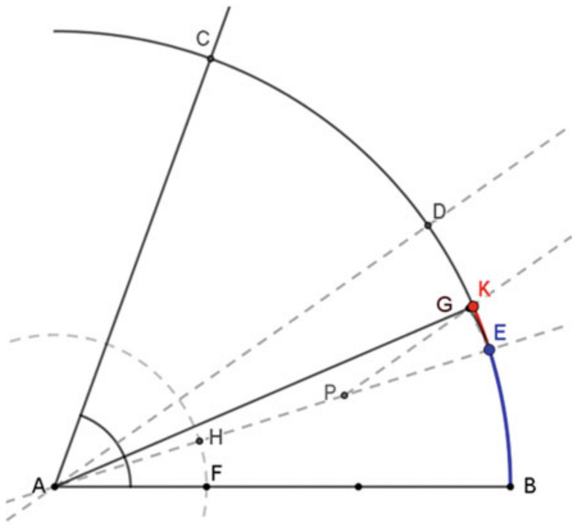
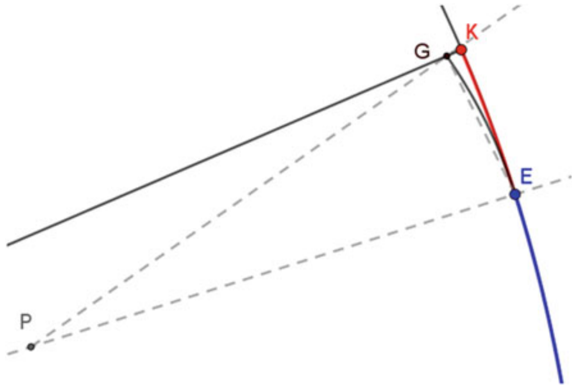


Fig. 12 .



where x $(0 < x < \pi/2)$ is the radiant measure of the arc, we want trisect. The approximation formula does not depend on the radius of the circumference and the error is always less than 0.1%.

8 Conclusions

Over the centuries, many scholars invented procedures for trisection of the angle but never it was possible to solve this problem using only straightedge and compass. The geometric trisection's problem became an algebraic problem, connected to the not solvable equations of third degree. This problem can be solved in a geometric way, with a very good approximation, as we noted in the previous paragraph and the method used can be extended now division into parts (in odd) 5 parts it is first divided into 2 parts, then each of them of 3 parts and then, with the method described above, division into 5 parts is obtained. This method is of great help for the construction of buildings, churches, and fortresses on the ground.

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Margherita Piazzolla Beloch in the Italian Mathematics Education Tradition

Paola Magrone

Abstract The paper focuses on Margherita Beloch's (1879–1976, married Piazzolla) approach to mathematical education, with particular attention to the education of prospective highschool teachers, which continued the approach by Federigo Enriques, as presented in his *Questioni riguardanti le matematiche elementari* (final edition, 1924–1927) written in collaboration with several university colleagues and highschool teachers. The educational role of classical geometrical matters regarding constructions was vindicated by her, from a personal approach including her attention to technology and applications, in her *Lezioni di Matematica Complementare* (1953).

Keywords Mathematics education · Geometric constructions · Paper folding · Women in science

1 From Algebraic Geometry to Applied Mathematics, from Rome to Ferrara

Margherita Beloch[1] (1879–1976), was the first of the two daughters of the German historian Julius Beloch [1] (1854–1929), professor of ancient history in the University of Rome since 1879, and the American Bella Bailey (her real name was Hannah [2] (1850–1918)). She grew up in a stimulating familiar cultural context, but her enrolment in the university was quite late: she only in 1909 graduated in Mathematics in the University of Rome (Table 1) with the thesis *On the birational transformation of the space* (Sulle trasformazioni birazionali nello spazio [3]), with advisor Guido Castelnuovo (1865–1952). She was a highschool teacher and voluntary assistant for the next ten years; she began her university career as assistant teacher in Pavia and

[1]The interest of studying Beloch's cultural profile was reported to Laura Tedeschini Lalli by Mario Fiorentini (born in 1918), professor at the University of Ferrara, who played an important role in Italian geometry in the 1960s and 1970s.

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Palermo and was appointed full professor of geometry at the University of Ferrara in 1927[2]: she thus became a pioneer, one of the first Italian woman mathematicians of the late modern age with a university position; her desire to come back to Rome was never fulfilled[3] [6, 7]. She become involved in applied mathematics, in quite new applications of geometry such as photogrammetry and medical radiology, for which she is mainly remembered, during the years of fascism[4]—for a Roentgen photogrammetric device, allowing precise measurements to be performed on organs of the human body which have their own movements, she won a prize assigned during the national exhibition devoted to Leonardo da Vinci and Italian Inventions in Milan in May–October 1938, and in 1942 she was again awarded a prize, by the CNR (National Research Council) for an aereophotogrammetric device—and after the end of World War II. In 1967 the Italian Society of Photogrammetry and Topography (SIFET, founded in 1951) published her selected works [8].

A description of the cultural profile of the 50-years-old "praised lady" ("lodata signora") was written by the Dean of the Faculty of Science, the chemist Filippo Calzolari (future Rector in years 1938–1945), summarizing her first three years at Ferrara.[5]

> Her publications speak for her the scientific activity in the last three years. With regard to her teaching activity, we noticed that she is a diligent teacher and full of love for the school, willing to make the most of her students who are also followed by her in the exercises related to the courses that she carries out, courses that are always very popular. In addition, she showed considerable interest in the greater development of the school by promoting the establishment of a Library for Mathematics and managing to obtain from the board of directors a special annual fund and a room suitable for the purpose. The direction of this library is entrusted to the praised lady who has always shown to perform her duty on every occasion with the satisfaction of all the colleagues and the students.

In her opening speech of year 1930–1931 at the University of Ferrara (Fig. 1), entitled *Mathematics in relation to its applications and its educational value* [9] (*La matematica in relazione alle sue applicazioni ed al suo valore educativo*), she shows a vision of mathematics rooted in its history, with a deep attention to its applications. She retraces "in a bird's eye like someone who flies over a vast region by airplane and stops only in the most important points" [9, p. 4] the history of mathematics, showing her knowledge of historical sources. This imaginary journey through the ages traces the development of mathematics and its countless applications. Her main point is to underline the value of mathematics related to its applications, and its value as a pure

[2]References to Margherita Beloch can be found in the website *Scienza a due voci* [4] and in Strikland [5, pp. 33–36]. A first interpretative profile, albeit short, is presented in [6, pp 3–5].

[3]During her career she wrote letters to the Ministry of Education to ask to be transferred in another university, such as Rome or Naples, in order to be closer to her family and to have more favourable work conditions (the letters are preserved in the Italian National State Archive, see [7]).

[4]She became member of the PNF (National Fascist Party) in 1926 (see [7]).

[5]Report on Margherita Piazzola Beloch's scientific and didactical activity during the years 1927–1930, in occasion of her confirmation in the role of full professor, December 6, 1930, written by F. Calzolari, Dean of the Faculty of Science, Ferrara University, my translation (document preserved in [7]).

Table 1 Chronology

1879	Born in Frascati (Rome)
1909	University of Rome: graduation with Guido Castelnuovo,
1909–1919	Rome: teaching in secondary schools; voluntary assistant of analytic and projective geometry in Rome University
1916	Marriage with Ruggero Piazzolla
1919	Pavia University: first appointment as teaching assistant (of descriptive geometry)
1920–1927	Palermo University: teaching assistant of descriptive geometry
1927	Ferrara University: full professor of analytic and projective geometry. Teaching: descriptive geometry, higher geometry, analytic geometry with elements of projective geometry
1935	For the first time she is in charge of the course Complementary mathematics
1938	Prize (silver ministry cup) at the Leonardo da Vinci and Italian Inventions exhibition
1942	Prize from the National Research Center (CNR) on the occasion of the "technical day"
1949	Retirement
1950	Foundation of the Higher Geometry Institute Preparation courses for the teaching qualification exams (in mathematics and physics) addressed to newly graduated
1953	*Complementary Mathematics (Elementary mathematics from a higher standpoint)*

science, which is in line with her being both an applied and a pure mathematician. On the one hand, she claims that "the habit of thinking with mathematical rigor will support the technician even where he has put aside the formulas and calculations and relies on experience" [9, p. 4]. Furthermore, to underline that pure, theoretical research, with no apparent immediate use, often has led to applications in many years, she claims "the mathematician in the search for truth animated by the flame that ignites his spirit lives in the abstract sphere of his speculations and the practical application interests him only as it confirms his theories and predictions", so "even if it approaches reality, science must fly high" [9, p. 17].

These words recall us the metaphor of the scientist as a traveler traced by Federigo Enriques (1871–1946) in his own opening address *The humanistic significance of science in national culture* (Il significato umanistico della scienza nella cultura nazionale) for the Congress of Mathesis[6] [11] in Livorno seven years before:

> […] the path of the sailor who opens new outlets to the activity of the homeland and brings to new people the torch of civilization, and the path of the thinker, who travels far from the rock of the surrounding reality, to conquer in his own way the wider world of truth […]. [11, p. 2]

[6]Mathesis, the Italian Society of Mathematical and Physical Sciences, was born in 1895 on the initiative of secondary school teachers, to improve and enhance the teaching of mathematics in schools of all levels, see [10].

Enriques speaks about the tight relationship between technical progress and theoretical progress and how scientific theories, where scientists' minds "venture away from any purpose", must not be neglected since it's the theoretical new ideas which nurture the technical ones. No doubt that the years of Beloch's mathematical education in Rome, her period as high school teacher and her first six years in the universities of Pavia, Palermo and Ferrara were marked by the deep cultural influence of Enriques in the Italian mathematics community, particularly in Rome (see [12]).

Fig. 1 Margherita Beloch's portrait, 1931. Courtesy of the Ministry of Cultural Heritage and Activities and Tourism, ACS 2020 aut.n. 1616/2020, it is prohibited to further reproduce or duplicate by any means [7]

2 The Educational Value of Mathematics

Since 1935 at the University of Ferrara—together with her course in geometry—Beloch engaged on the teaching of "complementary mathematics", a course founded in Italy to prepare prospective high school mathematics teacher [13]. After her retirement in 1949 an Institute of Higher Geometry was created, which allowed her to carry on her research and also teach courses addressed to the preparation of state exams for mathematics and physics graduates entering the public system of high schools.

In 1953 she published the litographied text of her lessons *Lezioni di Matematica Complementare. La matematica elementare vista dall'alto* (LMC, Lessons of Complementary Mathematics. Elementary mathematics from a higher standpoint [14]), a book of 440 pages divided in 11 chapters, each of them including a wide bibliography (see Table 2). The book is the result of collaboration with the principal of a Ferrara high school, Egidio Orzalesi, who helped her in the drafting and compilation of the didactic considerations.

> This book originates from the lessons of complementary mathematics I held for several years at the University of Ferrara. Conceived and written initially for my students only, it now presents itself to a wider audience, revised and expanded in its structure, in order to embrace all mathematics topics not covered by the first two years of university studies and contemplated by ministerial programs for competitions in the chair of secondary schools and teaching qualification. I hope to give candidates a valuable help in these competitions and a safe guide for the exams to be taken, along with valuable teaching advice for their future career as mathematics teachers. [14, preface]

The subtitle shows her attachment to Felix Klein's (1849–1925) views on "elementary mathematics from an advanced standpoint" [15] as a keystone in the training of teachers. Klein was a lasting influence among Italian mathematicians: he has been in touch with his senior Luigi Cremona (1830–1903), but specially his lessons on elementary mathematics were the root of Enriques' involvement in educational matters, with the editorial enterprise of the *Questioni riguardanti la geometria elementare* (*Issues regarding elementary geometry)* published in 1900 with the collaboration of Guido Castelnuovo with a paper on geometrical constructions, and of several high school teachers. In 1912, an enlarged edition was published under the title *Questioni riguardanti le matematiche elementari* [16] (QRME, *Issues regarding elementary mathematics*); a final edition was published in 1924–1927, while she was an assistant professor of descriptive geometry at the University of Palermo.

The QRME was a huge set of three books, divided in three parts, for a total of seven monographic chapters, each of them written by a different author, even if Enriques harmonized it with his own chapters. For example, in 1927 he included a new review chapter on geometrical constructions.

In the choice of the contents, the LMC follows the path traced by Enriques in his QRME, and articles taken from the QRME are the most frequently cited in the bibliographical notes. Enriques' work addresses the foundations of mathematics from an epistemological and philosophical point of view and the preparation of school teachers; the reading of this huge work may not be easy sometimes. The book by Beloch appears as aimed to collect the heritage of Enriques' approach, offering

Table 2 Index of the book *Lessons on Complementary Mathematics. Elementary mathematics from a higher standpoint* (1953)

Part one
The fundamentals of arithmetic and algebra
Chapter one
The concept of natural number and its first extensions
1. Integers, p. 5
2. Rational numbers, p. 17
3. Decimal numbers, p. 25

Chapter two
Elements of integer theory—congruences—continuous fractions—indeterminate analysis
1. Multiples and divisors—divisibility—prime numbers, p. 33
2. Congruences, p. 52
3. Continued fractions, p. 71
4. First degree indeterminate analysis—Pythagorean equation, p. 90

Chapter three
Third and fourth degree equations
1. General property of the roots of an algebraic equation, p. 112
2. Solving third and fourth degree equations with the Lagrange method, p. 121

Chapter four
Algebraic and transcendent numbers
1. Algebraic numbers, p. 131
2. Transcendent numbers, p. 147

Part two
Geometry
Chapter one
Geometric entities and fundamental postulates
1. The fundamental geometric entities, p. 160
2. Basic postulates, p. 171

Chapter two
Non-Euclidean geometries
1. Hyperbolic geometry, p. 182
2. Elliptic geometry, p. 210

Chapter three
Equality and equivalence
1. The concept of equality, p. 225
2. Equivalence, p. 235

Chapter four
Geometric transformations
1. Linear plane transformations, p. 242
2. Inversion or transformation by reciprocal vector rays, p. 262
3. Quadratic transformations, p. 282

Chapter five
Elementary geometric problems
1. Analytic and synthetic procedures, p. 292
2. The method of geometric loci, p. 305
3. The method of transforming figures, p. 323
4. Application of multiple combined methods and different artifices in solving geometric problems, p. 347

Chapter six
Third and fourth degree problems—the classic cyclotomy problems
1. Third and fourth degree problems, p. 353
2. Some particular curves, p. 358
3. Classical problems, p. 365
4. The cyclotomy, p. 378

Chapter seven
Geometric constructions
1. The ruler and compass in geometric constructions, p. 388
2. Constructions executable with other tools, p. 401
3. Simplicity and precision of geometric constructions, p. 419

a synthesis with a less elevated language perhaps, in order to be a support for new graduates, both for the preparation for the qualifying exams and to be kept alongside as a guide companion in the future teaching profession.

The number and length of articles and chapters devoted to geometry in QRME clearly shows the importance that Enriques confer to the realm of geometry (his main area of research), when dealing with the training of future teachers. Foundations of geometry and classical problems of synthetic geometry receive a wide attention. Almost two thirds of Beloch's LMC are devoted to elementary geometry from a modern viewpoint (see Table 2).

3 A Variety of Devices to Solve Elementary Geometric Problems

In the 1924–1927 final edition of Enriques' QRME includes 440 pages devoted to geometrical constructions, with 7 chapters written by seven different authors, including Castelnuovo who considers the subject from the analytical viewpoint. This section is closed by Enriques himself, with the chapter entitled "Some general observations on geometrical problems" (Alcune osservazioni generali sui problem geometrici [16, pp. 575–596]); a wide new chapter consisting in 156 pages on elementary methods for solving geometric problems was entrusted to the highschool teacher Alfredo Sabbatini ("Sui metodi elementari per la risoluzione dei problem geometrici" [16, pp. 1–156]) (Table 3).

Beloch's treatment of classical geometric constructions is included in the seventh and last chapter of the second part (31 pages) of LMC (see Table 2). Her exposition combines a general theoretical approach with an attention to the instruments as "drawing machines".

Her approach is centered on the epistemological categories of *simplicity* and *precision*. She refers to the classification proposed by the French mathematician Émile Lemoine (1840–1902) at the turn of the nineteenth century, when classic geometry was receiving less and less attention [17, 18]. His classification is based on counting the number of 5 elementary operations:

- R1 placing a straight edge on a given point
- R2 drawing a line with the straight edge
- C1 placing a compass end on a given point
- C2 placing a compass end on a given line
- C3 drawing a circle.

Table 3 Chapters concerning geometric constructions, from the second part of QRME [16]

• On elementary methods for solving geometric problems (Alfredo Sabbatini)
• On solving geometric problems with a compass (Ermenegildo Daniele)
• On solving geometric problems with rulers and linear tools: contribution of projective geometry (Amedeo Giacomini)
• On the solvability of geometric problems with elementary tools: contribution of analytic geometry (Guido Castelnuoyo)
• Third degree problems: duplication of the cube—trisection of an angle (Alberto Conti)
• On transcendent problems and in particular on the quadrature of the circle (Benedetto Calò)
• Some general observations on geometrical problems (Fedengo Enriques)

Constructions can be classified as more or less "simple" according to the resulting number of elementary operations performed (the lesser the number, the simpler the construction). Enriques mentions Lemoine's research in the above mentioned chapter [16, pp. 575–596].

In order to obtain a more accurate classification, Beloch takes into account the diversity between different operations, such as, for example, "placing the compass" and "tracing a line", introducing a weight coefficient; and considers also the average time needed to perform an operation.

After classifying the constructions made with the classical ruler and compass, one could contemplate other drawing tools, and compare the length and difficulty of the procedures. Following Enriques' discussion, she considers a qualitative classification including the mechanical simplicity of the instruments, the geometric simplicity of the drawn lines, and finally the analytical simplicity of the equations connected to the problem. All these reasonings lead to a more complete and deep comparative assessment of instruments. Moreover, she adds new "machines" to the tools presented in Enriques' work, showing her interest in technical and scientific applications of mathematics with other ones. She introduces *linkages* (Fig. 2) such as the Kempe's translator and reversor, Scheiner's pantograph, and the skew pantograph of Sylvester.[7]

In recent years her treatment of geometrical constructions by paper folding has received great attention.[8] While drawing tools connect geometrical constructions to the technical drawing of the architect and the engineer, paper folding connects it to recreational mathematics. A classical essay on this topic was published in 1893 in Madras [21], in English, by the Indian mathematician Sundara Row (born 1853). He was inspired by hands-on activities for the kindergarden inspired by Friedrich Fröbel (1782–1852) and enjoyed considerable attention in the subsequent years.[9] Klein mentioned this book, together with Hermann Wiener's (1857–1939) contributions, in his *Vorträge über ausgewählte Fragen der Elementargeometrie* (1895) (see the English translation [23, p. 42]).

> A method that can be a useful tool for solving geometric problems is that of paper folding. It should be noted that this procedure, in addition to being applicable to a larger set of problems than that related to classical rulers and compasses, because it allows solving third and fourth degree problems as well, solves some problems with much simpler operations of

[7]Alfred Bray Kempe (1849–1922), James Joseph Sylvester (1814–1897), Christoph Scheiner (1573–1650).

[8]With Valerio Talamanca, I have discussed her contribution in [19]. See also Michael Friedman essay [20, pp. 318–340]. Beloch contributions went unnoticed for many years, until the interest on this subject was rekindled by the First International Meeting of Origami Science and Technology, held in Ferrara in December 1989, and organized by the Italian Center for Diffusion of Origami (Bologna), with Humyaki Huzita as main promoter; Luigi Pepe opened the conference with a remembrance of Beloch, and the Proceedings begin with the reprint of some pages from LMC on paper folding and her publications on solvability of any geometrical problem of 3rd degree by paper folding.

[9]Fröbel's educational geometrical activities are part of a group of nineteenth century contributions trying to transform the rote learning of written arithmetic by the introduction of intuitive geometry (solid objects, linear drawing, and so on), see [22].

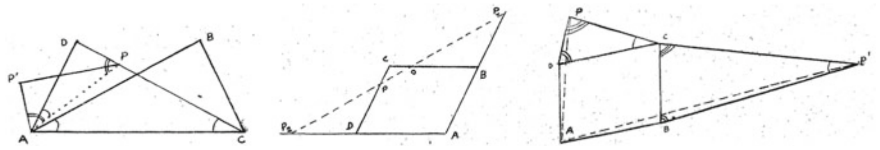


Fig. 2 Linkages in LMC, from left: Kempe's reversor, Scheiner's pantograph, the skew pantograph of Sylvester [14, pp. 408–410]

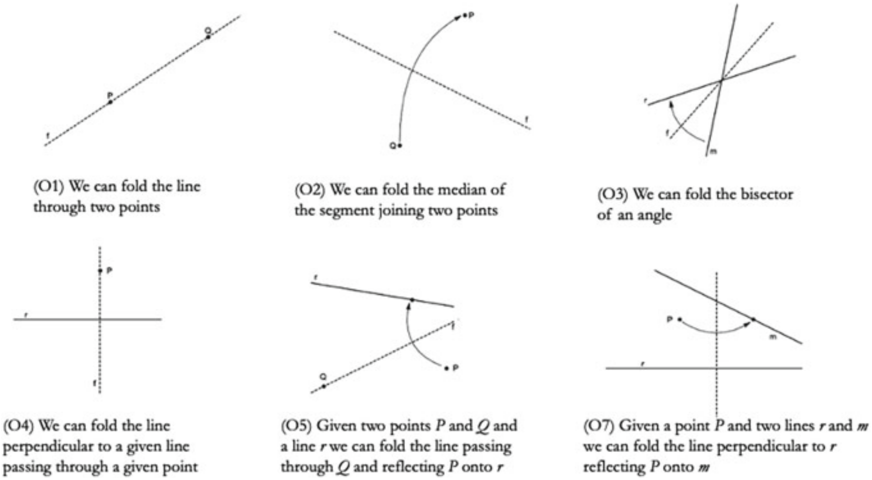


Fig. 3 The basic folds of origami geometry are seven. In this picture six of them are listed. Pictures from [19], courtesy of authors

> those necessary when turning to other tools, while the accuracy of the results is, in many cases, not less than that of the solutions obtained with the use of the ruler and compass. [14, p. 413]

As in the case of Lemoine's classification, Beloch—initially driven by an educational interest—gives a new look to a classic Euclidian geometry subject, considering it from a modern viewpoint. In fact, from an axiomatic viewpoint, a short number of basic folds can be identified, which correspond to Euclid postulates or the modern axiomatic versions of Euclidean geometry. The first five basic folds were used by Row, but were not available as a complete and organized list. In a paper dating 1934 [24]) Beloch presented a 6th fold (also recalled in LMC) known nowadays as Beloch's fold. A 7th fold has subsequently been introduced[10]: the seven basic folds are known today as "origami axioms" (see Fig. 3).

The fold O6, known as "Beloch's fold" is the following (Fig. 4).

[10]Fold seven was announced by Koshiro Hatori in 2002, but it had been already mentioned by Jacques Justine in the Proceedings of the First International Meeting of Origami Science and Technology in Ferrara in 1989 [25, see note 5].

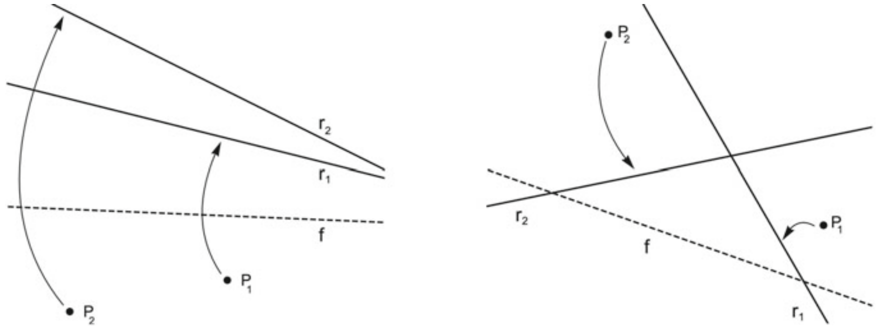


Fig. 4 Two possible configurations of the sixth origami basic fold (O6), also called Beloch's fold. Pictures from [19], courtesy of authors

Given two points P_1 and P_2 and two lines r_1 and r_2 then, we can fold (when possible) the line reflecting P_1 onto r_1 and P_2 onto r_2.

In order to visualize and understand the action performed by fold O6, it is better to have a look at O5 first. Take a rectangular sheet of paper, place a point P in any position, for example as in Fig. 5.

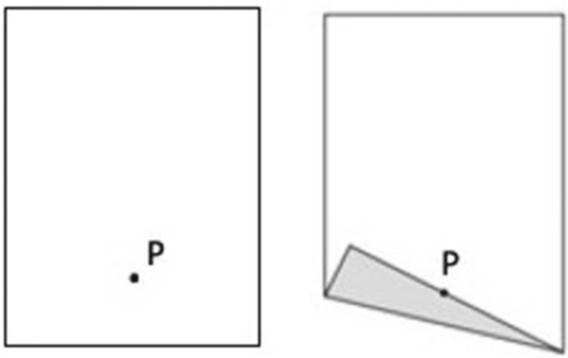
Iterating the fold, one gets the following picture, which recalls the shape of a parabola. Indeed all the folded lines are actually tangent lines to the same parabola (this is quite easy to prove) having P as focus and the edge of the paper as directrix.

Figures 5 and 6 are nothing but fold O5, where the line r is the edge of the paper. So the geometrical interpretation of O5 is "to fold a tangent line to a parabola", given its focus and directrix.

Regarding O6 (Figs. 4 and 7), since there are two given points and two given lines, there are two identified parabolas: the fold produces a line which is tangent to both curves, when the fold is possible.

Beloch's scholarship in classical elementary geometry, developed along the lines of nineteenth century and early twentieth century Italian schools of geometry, offered a fresh encompassing approach, which consider it as a bedrock of mathematical

Fig. 5 Fold the edge of the sheet on the marked point. Pictures from [19], courtesy of authors



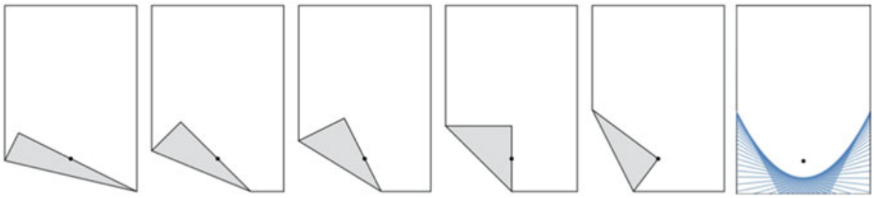
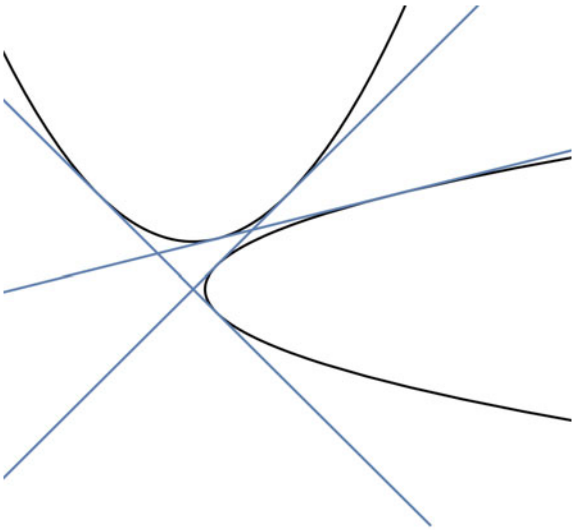


Fig. 6 Iterate the folds, and a parabola appears. Pictures from [19], courtesy of authors

Fig. 7 Two parabolas with three common tangent lines. Picture from [19], courtesy of authors



culture, with its many links—from technology to play—that make it a crucial aspect of the education of future teachers.

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Grete Hermann and Effective Methods in Geometry

Mirella Manaresi

Abstract Sometimes in the history of science we can find scientists who were not acknowledged at their time and only later the scientific community has recognized the importance of their work. Among them there is Grete Hermann, who worked in mathematics, philosophy, and physics during the early to mid twentieth century. To mathematicians she is known as the first Ph.D. student of Emmy Noether in Göttingen and a pioneer of modern computer algebra, but she also discovered a flaw in John von Neumann's proof of the impossibility of hidden variables theory in quantum mechanics and she did further interesting work on the foundations of quantum mechanics. Her collaboration with Emmy Noether and the philosopher Leonard Nelson in Göttingen, her involvement with Heisenberg's group in Leipzig, some aspects of her work in philosophy and education, her activities in socialist groups and in the anti-Nazi underground movement make of Grete Hermann a very interesting intellectual of last century. In this note we will illustrate Hermann's work and discuss one of her mathematical results, which has had many improvements and now is much cited.

Keywords Computer algebra · Effective methods · Algebraic geometry · History of mathematics

1 Who Is Grete Hermann?

Grete Hermann (2 March 1901–15 April 1984) was born in Bremen (Germany), in a middle-class protestant family. In 1921 she acquired teaching qualifications for secondary schools and from 1921 to 1925, with one year interruption when she moved to the University of Freiburg, she studied mathematics and physics at the University of Göttingen, where she was the first doctoral student of Emmy Noether. She registered for her doctoral examination in pure mathematics as major and physics and philosophy as minors and obtained a Ph.D. in Mathematics in 1926.

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During 1926 and 1927 Grete Hermann (see Fig. 1) worked as a private assistant to the neo-Kantian philosopher Leonard Nelson, whose seminars and classes she had attended since her first semester in Göttingen with an interruption of two years.

Speaking about the University of Göttingen at that time Giulia Paparo writes in her thesis [22]: "*Hilbert can be seen as the central figure in the faculty of mathematics, trying, and partially succeeding, to direct the philosophy department towards his own interest, which he did first with the selection of Husserl, and then with the professorship of Leonard Nelson. He was aware of the important role philosophy would play in the elaboration of his axiomatic method; philosophy could offer at the same time a useful tool for the clarification of the meaning of mathematical symbols, and justification for the axioms. Nelson's philosophy, or critical mathematics, represented an example of the systematic and mathematically oriented philosophy that Hilbert wished for. The Hilbert programme in a broader sense was not limited to mathematics, but settled on an interdisciplinary ground, including philosophy and physics, and assigned to the relation between mathematics and critical philosophy a particularly crucial role. It had been Hilbert (together with Klein) who in 1915 invited*

Fig. 1 Grete Hermann

Emmy Noether to Göttingen and it was Hilbert again who interceded numerous times to ensure a professorship for Leonard Nelson."

In a letter to L. van der Waerden (24 January 1982) Grete Hermann remembers with these words (see [27]) that Emmy Noether (see Fig. 2) was irritated for her decision to work with the philosopher: "*Schon Emmy Noether sagte grollend, als ich nach meinen Examina die Assistentin meines anderen Göttinger Lehrers, des Philosophen Leonard Nelson, wurde: Da studiert sie vier Jahre lang Mathematik, und auf einmal entdeckt sie ihr philosophisches Herz!.*" Emmy Noether had planned for Grete a position of assistant at the University of Freiburg.

In 1926 Nelson (see Fig. 3) founded a radical socialist youth group that Grete joined, the Internationaler Sozialistischer Kampfbund (ISK). Member of this group was also the pedagogue Minna Specht (1879–1961). The ISK was supported by important European intellectuals, including Albert Einstein. Grete became an active socialist and together with Leonard Nelson and Minna Specht, she tried to introduce a new non-authoritarian form of education in Germany, opening the school Landerziehungsheim Walkemühle near Kassel for both adult and childhood education. This school was closed by the Nazis in 1933.

In 1932 she published a work on a system of philosophy of ethics and education. In the same year she started to work as an editor for the newspaper "Der Funke". The socialist group ISK contributed to the formation of a united front against the Nazis

Fig. 2 Emmy Noether

Fig. 3 Philosopher Leonard Nelson

and was very active in the resistance during World War II. Grete Hermann was also involved in the Internationaler Jugendbund (International Youth Federation) (IJB) founded by Leonard Nelson.

From 1934 Grete Hermann worked in Leipzig with prominent physicists like Werner Heisenberg and Carl Friedrich von Weizsäcker. At the time Leipzig was one of the most important centers for the study of quantum mechanics and also the mathematician B. L. van der Waerden (who had studied with Emmy Noether in Göttingen) was present there in that period. During these years there was a radical change in thinking in science. In much of the interpretation of the physical world classical Newtonian physics was being supplanted by quantum mechanics and Hermann's work in physics was mainly related to the interpretation of quantum mechanics. She contributed to the historic debates on causality in quantum mechanics, on the completeness of quantum mechanics and its description of reality.

In 1935 Grete Hermann discovered a logical mistake in John von Neumann's proof of the impossibility of hidden variables in quantum mechanics, that he had published three years before. Her refutation of von Neumann's proof was largely ignored for over 30 years.

In 1936 she was awarded the Richard Avenarius Prize by the Academy of Sciences of Saxony in Leipzig for her work on the significance of quantum theory and field theory of modern physics for the theory of knowledge, starting from a question posed by Werner Heisenberg.

For Grete Hermann (see Fig. 4) an academic career under the Nazi regime in Germany was impossible, however, she held classes for members of the resistance. During 1935/1936 many members of the ISK group were forced to leave Germany for persecution. By 1936, Hermann left Germany for Denmark, where her friend Minna Specht had opened a school for children similar to the Walkemühle School. The threat of war and the risk of a German occupation of Denmark pushed Grete Hermann to leave Denmark. First she went to Paris, then in 1938 she went to London and did a marriage of convenience with Edward Henry, that permitted her to acquire British citizenship. During World War II she lived in exile in England, was involved with the underground resistance against the Nazi regime and continued her activities in contact with Minna Specht and other colleagues. She dedicated herself completely to political work in the ISK and in the Union Deutscher Sozialistischer Organisationen in Großbritannien (UNION), that brought together several socialist groups in a socialist united front.

In 1946, after World War II, she returned to Germany, and divorced from Edward Henry. She contributed to cultural and educational politic of post-war Germany. She worked on the rebuilding and development of the Pädagogische Hochschule in Bremen, and in 1947 she became its head. From 1950 to 1966 she had a professorship and taught in the faculty of philosophy and physics. She was involved in educational

Fig. 4 Grete Hermann

Fig. 5 Grete-Henry-Strasse, Göttingen Photo by Axel Manegold, Göttingen

and cultural activities with the Social Democratic Party of Germany (SPD) and in particular she was a founding member of the Gewerkschaft für Erziehung und Wissenschaft (GEW), a trade union in education and science. During her later years she was interested in politics, philosophy and education, rather than mathematics and science. From 1954 to 1966, she was a member the German Committee for Upbringing and Education, continued her work in philosophy on the critique and further development of Nelson's ethics. In later years, Grete Hermann remembered Emmy Noether with great reverence and cheerful memories. Grete Hermann died on April 15, 1984 in Bremen, her home city. A part of her Nachlass is in the Archive of Social Democracy of the Friedrich Ebert Foundation in Bonn. A street in Göttingen, the Grete-Henry-Strasse (see Fig. 5), was named after her.

2 Hermann's Mathematical Studies in Göttingen

Grete Hermann studied mathematics in Göttingen, one of the main centers of mathematics in the world at that time, for the presence of great mathematicians including Felix Klein, David Hilbert, and Emmy Noether.

Fig. 6 Emmy Noether and some colleagues

Emmy Noether (1882–1935) is one of the most important twentieth-century mathematicians, who studied, taught and researched in Göttingen from 1915 until 1933. Her researches contributed to the development of modern abstract algebra. In 1907 she obtained the Ph.D. cum laude in Erlangen with a dissertation on invariant theory prepared under the supervision of Paul Gordan. Hilbert's Basis Theorem (1888) had given an existence result for finiteness of the ring of invariants in n variables. Gordan was looking for constructive methods which could give the same result and Noether's dissertation was very different from the papers which made her famous later. After the Ph.D. Emmy Noether started a more abstract approach following Hilbert's results. In 1915 she moved to Göttingen by invitation of Hilbert and Klein, as a specialist in invariant theory. In 1919, after the end of World War I, she could obtain the habilitation with a dissertation with title "Invariante Variationsprobleme", in which she proved a famous result, which now has her name, on the connection between symmetries and conservation laws in physics. After habilitation Emmy Noether's interests moved from invariant theory to ideal theory. Emanuel Lasker (see Fig. 7) had proved that every ideal in a polynomial ring has a primary decomposition, that is a decomposition as intersection of primary ideals. This can be considered as a generalization to polynomial rings of the fundamental theorem of arithmetics for integers. Lasker's proof used a difficult computational argument, but Emmy Noether discovered that the key condition behind the result is the fact that every ascendent chain of ideals becomes stationary, or equivalently, every ideal of the ring is finitely generated. Emmy used this fact to give a shorter proof of a much more general theorem, which is valid in all rings that satisfy the ascending chain condition on ideals. Now these rings are called Noetherian rings in her honor. In those years many important mathematicians

Fig. 7 Emanuel Lasker

came to Göttingen to study with Emmy Noether (see Fig. 6), among them Emil Artin (1921/1922), Pavel Alexandrov (1923/1924), Bartel L. van der Waerden (1924). In 1930 van der Waerden publishes his "Moderne Algebra", perhaps the most important book in algebra of last century, and in 1932, Emmy Noether and Emil Artin receive the Alfred Ackermann-Teubner Memorial Prize for the Advancement of Mathematical Knowledge and in the same year Emmy Noether gives a plenary lecture to the International Mathematical Congress in Zürich, the first woman in the history of the Congress. In 1932/1933 Wolfang Gröbner studied with E. Noether in Göttingen, in 1933 Emmy Noether was expelled from the University of Göttingen and in 1934 she left Germany for the States.

In 1922 Emmy Noether had published under Kurt Hentzelt's name a paper on elimination theory. Kurt Hentzelt was a Ph.D. student of Ernst Fischer in Erlangen, who never finished his Ph.D., since he died in the World War I, and the paper contains the results of his research written by Emmy Noether. The style of that paper is similar to Lasker's papers and in a footnote Emmy Noether writes that the part of Hentzelt's dissertation on computation in a finite number of steps will be treated in another paper. This is the problem that she gave to Grete Hermann for her Ph.D. thesis.

Since her first semester in Göttingen Grete Hermann attended not only courses in mathematics, but also Nelson's seminars and classes in philosophy. She completed her doctorate under Emmy Noether and she was her first Ph.D. in Göttingen. She discussed her dissertation in February 1925 with E. Noether and E. Landau at the age of 24. In that period Emmy Noether was already looking toward abstraction, while Hermann's thesis had a computational approach, inspired by Kurt Hentzelt's paper [11]. In her thesis, Hermann showed that Emanuel Lasker's approach and Noether's proof of the Lasker-Noether theorem could be turned into an effective procedure (today we say an efficient algorithm) for computing primary decomposition in polynomial ideals. Her paper presents the first examples of effective procedures for a variety of computations in multivariate polynomial ideals, hence for many of the basic problems of abstract algebra. Mathematicians of Hermann's time were unfamiliar not only with the concept of computers but even with our contemporary concept of what an algorithm is, so she anticipated by some 39 years the birth of computer algebra, that, for the mathematical community, starts in 1965 with Buchberger's algorithm for the computation of Gröbner bases.

Macaulay in [18] indicated a way based on Lasker's work [15] to compute the associated prime ideals associated to an ideal and their exponents, but he had no upper bounds on the number of steps necessary for the computation.

After her thesis Grete Hermann decided to dedicate herself to philosophy with Leonard Nelson, disregarding Noether's wishes and she didn't publish any other paper in mathematics.

3 The Doctoral Thesis of Grete Hermann and Some Developments

Grete Hermann published her doctoral thesis as "Die Frage der endlich vielen Schritte in der Theorie der Polynomideale. (Unter Benutzung nachgelassener Sätze von K. Hentzelt.)" [The question of finitely many steps in the theory of polynomial ideals. (With the use of posthumous propositions by Kurt Hentzelt.)] in Mathematische Annalen in 1926, see [6].

It seems to be her only mathematical publication that initially received little attention, although it had been cited in the books by van der Waerden (Moderne Algebra, 1930/1931 [31]) and Krull (Idealtheorie, 1935 [14]). The paper found the recognition and appreciation it deserves only later, starting in mid-twentieth century, when mathematicians became interested in carrying out computations with polynomial ideals by hand and by using computers. At the same time some gaps and inaccuracies in the paper had been noticed in addition to one discovered already by van der Waerden [30]. The papers by A. Seidenberg [28], M. Reufel [26], C. Veltzke ([23] and [24]), D. Lazard [16], W. Krull [13], A. Fröhlich and J. C. Shepherdson [4] contain remarks on some small gaps in Hermann's proofs.

Nowadays G. Hermann's thesis is considered the foundational paper for computer algebra and is much cited. The significance and the contents of Grete Hermanns doctoral thesis are shortly described in the introduction to the English translation by M. Abramson "*Grete Hermann's 1926 paper […] is an intriguing example of ideas before their time. While computational aspects of mathematics were more fashionable before the abstractions of the twentieth century took hold, mathematicians of that time certainly knew nothing of computers nor of today's idea of what an algorithm is. The significance of the paper can be found on the first page, where we find* (in translation): *The claim that a computation can be found in* finitely many steps *will* *mean here that* an upper bound for the number of necessary operations for the computation *can be specified. Thus it is not enough, for example, to suggest a procedure, for which it can be proved theoretically that it can be executed in finitely many operations, if no upper bound for the number of operations is known. […] The computational procedures which are presented in this paper include multivariate polynomial factorization, polynomial system solving, least common multiples, greatest common divisors, ideal quotients, divisibility of one ideal by another, fundamental ideals, norms, elementary divisor forms, associated prime ideals, primary decomposition, and isolated components.*"

The computations in Hermann's work are based on ideal theory and elimination theory developed by Emmy Noether and Kurt Hentzelt in [11] and G. Hermann shows that all standard objects in the theory of polynomial ideals over fields, including the prime ideals associated to a given ideal, can be determined by means of computations involving finitely many steps. She also gave explicitly upper bounds for the number of steps necessary for any computation. From a footnote in [11] it seems that originally Emmy Noether planned to write herself a paper on the question of finitely many steps.

Grete Hermann works in the polynomial ring $k[x_1,\dots,x_n]$ over a field k, where ideals are generated by a finite number of polynomials. An ideal is *given* if a basis of the ideal is known, an ideal is *computable* if a basis can be computed. Grete Hermann used elimination of variables, based on resultant techniques, to reduce the computation of the primary decomposition and the radical of a given ideal to the case of polynomials in one variable, where ideals are principal. For polynomials in one variable the computation of radicals reduces to the computation of the square-free part of polynomials and the computation of primary decomposition reduces to factorizing polynomials into prime factors and this is done with a method suggested by Kronecker.

Here is not the place to explain all Hermann's computations for polynomial ideals. We limit ourselves to the discussion of an important result of G. Hermann on the generators for the solutions of systems of linear equations over polynomial rings (Satz 2 and Satz 3 in Hermann's paper [6]) and how this leads to an effective Nullstellensatz and to explicit bounds for the ideal membership problem.

Hilbert's famous Nullstellensatz relates sets of solutions of a system of polynomial equations to ideals in polynomial rings over algebraically closed fields.

If C is the (algebraically closed) field of complex numbers and $f_1, \dots, f_s$ are polynomials in n variables $x_1, \dots, x_n$ with coefficients in C, that is,

$$f_1, \dots, f_s \in C[x_1, \dots, x_n],$$

an *algebraic variety* in C^n is the common zero locus of $f_1,\dots,f_s$, in symbols
$V(f_1,\dots,f_s) := \{(z_1,\dots,z_n) \in C^n \mid f_i(z_1,\dots,z_n) = 0, i = 1,\dots,s\}$
or, in other words, the set of solutions of the polynomial system

$$\begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ \quad \ldots \\ f_s(x_1, \ldots, x_n) = 0. \end{cases} \tag{1}$$

The so-called weak version of Hilbert's Nullstellensatz states that the polynomial system (1) has no solutions in C^n if and only if there are polynomials $g_1, \ldots, g_s \in C[x_1, \ldots, x_n]$, such that

$$g_1 f_1 + \cdots + g_s f_s = 1.$$

However it does not tell us how to construct the polynomials $g_1,\ldots,g_s$ and up to which degree we should go in the computation. Grete Hermann provided a method of construction together with a degree bound.

Let us begin our discussion of Hermann's degree bound with the simple case in which the polynomials $f_1,\ldots,f_s$ are linear

$$f_i(x_1, \ldots, x_n) = a_{i,1}x_1 + \cdots + a_{i,n}x_n + a_{i,n+1}, \ a_{i,1} \in\in C, i = 1, \ldots, s, j = 1, \ldots, n+1.$$

and set

$$A_{s,n} = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ \ldots & \ldots & \ldots \\ a_{s,1} & \ldots & a_{s,n} \end{pmatrix}, \ A_{s,n+1} = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n+1} \\ \ldots & \ldots & \ldots \\ a_{s,1} & \ldots & a_{s,n+1} \end{pmatrix}$$

It is well-known from linear algebra that the linear system (1) has no solutions if and only if the rank of $A_{s,n}$ is strictly smaller than the rank of $A_{s,n+1}$ or equivalently, computing the rank by Gaussian elimination, if and only if there are complex numbers $g_1,\ldots, g_s$ such that $g_1 f_1 + \cdots + g_s f_s = 1$. Thus zero is a degree bound in the linear case.

Now let us consider another simple case in which $f_1 = f_1(x_1), \ldots, f_s = f_s(x_1)$ are nonconstant univariate polynomials, that is, $n = 1$.

Set $d = \max\{\deg f_1,\ldots, \deg f_s\}$ and $f = \gcd(f_1,\ldots,f_s)$ the greatest common divisor of $f_1,\ldots,f_s$.

Then $V(f_1,\ldots,f_s) = V(f)$ and the polynomial system (1) has no solutions if and only if f is a constant nonzero polynomial, that is, $V(f)= \emptyset$.

The extended Euclidean algorithm for polynomials produces both the greatest common divisor of f_1 and f_2 and the coefficients of Bézout's identity, which are polynomials h_1 and h_2 such that

$$h_1 f_1 + h_2 f_2 = \gcd(f_1, f_2)$$

and

$$\deg h_1 < \deg f_2 \leq d, \deg h_2 < \deg f_1 \leq d (\text{we set} \deg 0 = -1).$$

If $s > 2$, since $f = \gcd(f_1,\dots,f_s) = \gcd(\gcd(f_1,\dots,f_{s-1}), f_s)$, the extended Euclidean algorithm can be used inductively to compute $f = \gcd(f_1,\dots,f_s)$ and the Bézout coefficients $g_1,\dots, g_s$.

Then the Bézout coefficients and their degrees depend on how $f_1,\dots,f_s$ are ordered, but it is easy to see that $\deg g_i < (s - 1)d$ for all $i = 1,\dots,s$. Summarizing, for univariate polynomials $f_1,\dots,f_s$ the system (1) has no solutions if and only if there exist polynomials $g_1, \dots, g_s \in C[x_1]$ such that

$$\deg g_i < (s-1)\, d (i = 1, \dots, s) \text{ and } g_1f_1 + \cdots + g_sf_s = 1.$$

For the general case of multivariate polynomials $f_1,\dots,f_s$ in the system (1), it was Grete Hermann who first gave a bound on the degrees of $g_1,\dots, g_s$. She solved the more general problem to bound the degree of solutions of linear systems with coefficients in the ring $R = C[x_1,\dots,x_n]$, that is, a system

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,s} \\ \dots & \dots & \dots \\ a_{t,1} & \dots & a_{t,s} \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ \dots \\ X_s \end{pmatrix} = \begin{pmatrix} b_1 \\ \dots \\ b_t \end{pmatrix} \tag{2}$$

with $b_i, a_{i,j} \in C[x_1,\dots,x_n] = R$ $(i = 1,\dots, t, j = 1,\dots, s)$ and solutions in R^s.

Then a crucial result in Hermann's thesis (Satz 2 and Satz 3), with Hermann's bound corrected by Masser and Wüstholz [20], can be formulated as follows, see Bayer and Stillman [2]:

Theorem [G.Hermann] *Let* $d = \max_{i,j}\{\deg(a_{i,j})\}$ *and* $b = \max_i\{\deg b_i)\}$, *taking* $\deg 0 = 0$. *If the system* (2) *has a solution, then there exists a solution* $(g_1,\dots, g_s) \in R^s$ *such that*

$$\deg(g_i) \le b + 2(sd)^{2^{n-1}} (i = 1, \dots, s).$$

Furthermore, when the system (2) is *homogeneous, that is* $b_1 = \cdots = b_t = 0$, *then the* R-module of *solutions of* (2) *is generated by elements* $g_1,\dots, g_s$ *with*

$$\deg(g_i) \le 2(sd)^{2^{n-1}} (i = 1, \dots, s).$$

If $t = 1$, then we set $a_{1,1} = f_1,\dots, a_{1,s} = f_s$, $b_1 = f$ and obtain a degree bound for the ideal membership of a polynomial f to the ideal generated by $f_1,\dots,f_s$.

Corollary (Ideal membership) *Given* $f, f_1, \dots, f_s \in C[x_1, \dots, x_n] = R$, *there are polynomials* $h_1, \dots, h_s \in R$ *such that* $f = h_1f_1 + \cdots + h_s f_s$ *if and only if there are polynomials* $g_1, \dots, g_s \in R$ *such that*

$\deg(g_i) \le 2(sd)^{2^{n-1}}$ $(i = 1, \dots, s)$ and $f = g_1f_1 + \cdots + g_s f_s$.

If in addition $f = 1$, then we get the following effective Nullstellensatz:

Corollary (Effective Nullstellensatz) *The polynomial system* (1) $f_1(x_1, \dots, x_n) = \cdots = f_s(x_1, \dots, x_n) = 0$ *has no solutions if and only if there are polynomials*

$g_1, \ldots, g_s \in C[x_1, \ldots, x_n]$ such that $\deg(g_i) \leq 2(sd)^{2^{n-1}}$ $(i = 1, \ldots, s)$ and $g_1 f_1 + \cdots + g_s f_s = 1$.

Hermann's bound is doubly exponential in the number of variables n. Hence it leads (almost always) to unpracticable linear systems for the extremely large number of unknown coefficients of generic polynomials g_i of degree $2(sd)^{2^{n-1}}$.

Examples of E. W. Mayr and A. R. Meyer show (see Bayer and Stillman [2]) that the doubly exponential dependence of the degree bound on n is unavoidable. However, for the Bézout identity in the Nullstellensatz the bound has been improved considerably by W. D. Brownawell [1] and others, who arrived at single exponential bounds that do not depend on s. If $d > 2$ then d^n is a bound (see J. Kollár [12]) and if $d = 2$ then one can take 2^{n+1} (see M. Sombra [29]).

The linear systems arising from Hermann's bound are enormous so that today *Gröbner bases* techniques are used in order to perform computations with polynomial ideals. The concept of a Gröbner basis together with an algorithm for its computation was introduced by B. Buchberger in his Ph.D. thesis in 1965. An analogous notion for power series was introduced by H. Hironaka under the name of *standard basis* in his famous paper on the resolution of singularities in 1964. We refer to [23] and [24] for historical remarks and a complete bibliography on the subject up to 1980 and to [19] for more recent developments.

As early as 1913 the Russian mathematician Nikolai M. Günther (also transliterated as Nicholas M. Gunther or N. M. Gjunter) had introduced the Gröbner basis concept, called by him *canonical form*, of an ideal generated by forms of the same degree. His papers on the subject, published in various Russian mathematical journals, had been forgotten until their rediscovery in 1987 by Bodo Renschuch et al.

A Gröbner basis *G* of a polynomial ideal $I = (f_1, \ldots, f_s)$ is a *good* generating set of polynomials $h_1, \ldots, h_r$ for I. That means in particular that the polynomial systems (1) $f_1 = \cdots = f_s = 0$ and $h_1 = \cdots = h_r = 0$ are equivalent.

A Gröbner basis *G* refers to and depends on an ordering of the monomials in $C[x_1,\ldots,x_n]$ and is, by definition, such that the ideal generated by the leading terms of polynomials in I equals the ideal generated by the leading terms of *G*. For example, if $f_1,\ldots,f_s \in C[x_1]$ are polynomials in one variable, then $G = \{\gcd(f_1,\ldots,f_s)\}$ is a Gröbner basis of $I = (f_1,\ldots,f_s)$, which we compute by Euclid's algorithm. Gröbner basis computation can be seen as a multivariate, non-linear generalization of both the Euclidean algorithm and the Gaussian elimination for linear systems.

If a Gröbner basis *G* of the ideal $(f_1,\ldots,f_s)$ in $C[x_1,\ldots,x_n]$ is known (with respect to any monomial ordering), then the condition in the weak version of Hilbert's Nullstellensatz can be checked easily: $f_1,\ldots,f_s$ have no common zeros in C^n if and only if some non zero complex number belongs to *G*.

Fig. 8 John von Neumann

4 Grete Hermann's Work and the Physicists' Community

Although in 1935 Grete Hermann found a decisive mistake in John von Neumann's "proof" (see Fig. 8) of the impossibility of hidden variables in quantum mechanics and she showed that his proof was not correct, that "proof" continued to be cited and Hermann's work was ignored.

For nearly 30 years the scientific community continued to accept and believe in von Neumann's incorrect 'proof'. In 1952 David Bohm independently realized that von Neumann's theorem could only be relevant for a limited class of hidden variable theories. Bohm created an interpretation of quantum mechanics that included hidden variables. However, "hidden variables" were such an untouchable subject that Bohm's work was largely ignored by the physicists' community. Bohm's work inspired John Bell to check the no-hidden-variables "proof" and rediscover the same mistake in von Neumann's proof already found by Grete Hermann. Even after John Bell for some years some physicists continued producing impossibility "proofs" with closely related errors. Two generations of physicists seem to ignore Hermann's work and today John Bell's result is much better known than Grete Hermann's original work. This is very strange, since at least Heisenberg and von Weizsäcker knew Grete Hermann's work. Kay Herrmann writes in [9] that many eminent physicists, including


Fig. 9 Werner Heisenberg

Heisenberg (see Fig. 9), Schrödinger, Courant, de Broglie, Einstein, von Laue, knew Grete Hermann's discovery since 1935.

In [10] Caroline L. Herzenberg supposes which can be the reasons. In particular she says that

1. von Neumann was a well known scientist, while Grete was a female young and unknown mathematician,
2. many physicists accepted von Neumann's work and quoted it without studying it deeply, on the basis of his authority, since the paper was difficult to read,
3. Grete Hermann was not really part of the physicists' community, since she had a background in mathematics and philosophy. Her work in physics was at the boundaries between physics and philosophy and mathematics. Moreover she wrote her works in physics in German and she published them in journals and books with limited distribution.

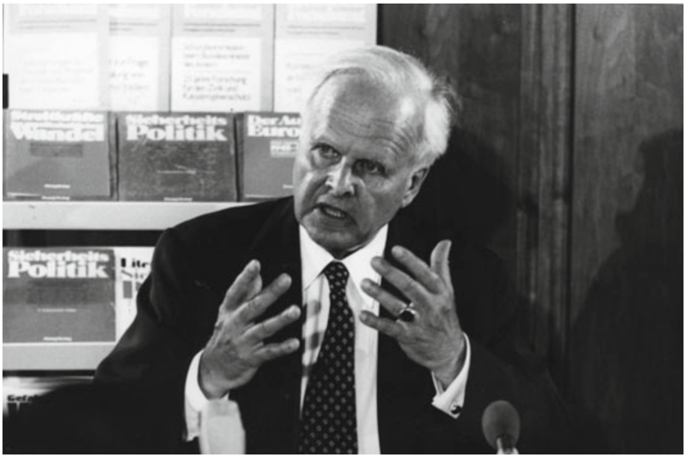

Fig. 10 Carl Friedrich von Weizsäcker

4. At Grete's time almost all academics were male, and females were so discriminated that even Emmy Noether never obtained a professorship position in Göttingen.
5. Grete Hermann was a socialist and a dissenter in an academic world which was conservative. For this reason an academic career in Germany was impossible during the Nazi period. When she came back to Germany after the exile she finally obtained a position of professor of physics and philosophy and taught physics courses, but she did not do research in physics anymore, hence she was not known to young physicists engaged in research.

Probably all these reasons are true, but they are not all the reasons. In [17] Konrad Lindner interviews Carl Friedrich von Weizsäcker on the period in Leipzig and von Weizsäcker says (see Fig. 10) "*Ich erinnere mich an eine Episode. Da kam einmal eine Diskussionstagung zustande, von einer Dame namens Grete Hermann, die selbst eine Philosophin war und eine naturwissenschaftlich ausgebildete. Sie hielt einen Vortrag über die philosophische Interpretation der modernen Physik. Danach gab es eine Diskussion, in der die Philosophen alle zu Wort kamen. In dieser Diskussion war die Grete Hermann all den Philosophen überlegen. Also die Philosophen in Leipzig waren nicht übel, aber sie waren auch nicht so gut.*" (I remember an episode. Once there was a discussion meeting with a lady called Grete Hermann, philosopher and scientist. She gave a talk on the philosophical interpretation of modern physics. After the talk there was a discussion, where all the philosophers said their opinions. In that discussion Grete Hermann was superior to all other philosophers, hence philosophers in Leipzig were not so bad, but also not so good). On pages 12–13 Konrad Lindner asks: "*Demnach war der Arbeitsstil von Grete Hermann, die extra nach Leipzig in die„Linnestr. 5" zu Heisenberg ging, die auch mit lhnen diskutiert hat, doch ein gewisses Novum?*" (hence the style of work of Grete Hermann, who went to Leipzig into the "Linnerstr. 5" to Heisenberg, who also discussed with you, is something new?) and Carl Friedrich von Weizsäcker answers: "*Genau. Das hat Heisenberg sehr interessiert. Grete Hermann schickte, glaube ich, einen Text an ihn. Heisenberg, weil er wußte, daß ich an Philosophie sehr interessiert war, gab ihn mir, ich möchte dazu was sagen. Ich glaube, ich war dann derjenige, der Grete Hermann geschrieben hat, das fänden wir nun eigentlich sehr interessant. Wir seien zwar nicht überzeugt davon, daß es genau so sei, wie sie sage, aber sie hätte offenbar im Unterschied zu anderen Philosophen die Physik verstanden*" (Right. Heisenberg was very interested. I think that Grete Hermann sent him a paper. Since Heisenberg knew that I was very interested in philosophy, he gave me the paper, in order that I can say something on it. I think I was the person, who Grete Hermann wrote, we would actually find the paper very interesting. We are not convinced that it is exactly as she says, but, differently from other philosophers, she really had understood physics).

Grete Hermann sent to Heisenberg the first complete version of her paper, published as a booklet [7] with the description of the gap in von Neumann's paper, then, in a good journal, she published only a shorter version [8], where the gap was not mentioned anymore.

E. Crull and G. Bacciagaluppi [3] speaking of [7] say: "*... This essay, written after a protracted visit to Heisenberg's research group in Leipzig, is possibly the most remarkable and complete early philosophical analysis of quantum mechanics. After criticising known arguments for the completeness of quantum mechanics, including von Neumann's proof, Hermann argues that the notions of causality and predictability have to be separated, and that only the latter but not the former is lost in quantum mechanics. Indeed, she argues using the Heisenberg microscope as her prime example, that causal chains can be reconstructed after the fact for any quantum mechanical effect, so that quantum mechanics is in fact already causally complete. She goes on to analyse Bohr's views on complementarity, and to sketch the natural-philosophical picture provided by quantum mechanics along the lines of a neo-Kantian (neo-Friesian) transcendental idealism.*"

With Chapter 10 "Quantum Mechanics and Kant's Philosophy (1930–1934)" in his book "Physics and Beyond: Encounters and Conversations", Werner Heisenberg [5] created a permanent memorial to Grete Hermann, talking about the discussions in Leipzig with Grete and Carl von Weizsäcker on the philosophical analysis of quantum mechanics.

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Perspective with Nature: Daniele Barbaro, Jean François Niceron and a Device of Giovanni Battista Vimercati

Cosimo Monteleone

Abstract In 1565 Giovanni Battista Vimercati published in Venice a treatise on gnomonics in which there is a device for making copies of sundials. This tool must be exposed under the Sun and contains: two styli; a drawing; and a blank sheet on which to draw the new delineation. The way it works is simple. While one stylus retraces the drawing with its shadow, the other projects an equal shadow that makes the user able to draw a copy. Vimercati did not explain the geometric rules underlying his device and did not indicate relations with other scientific or practical fields. A few years later, Barbaro will explain the link between perspective and gnomonics, just mentioning the tool of Vimercati. In his treatise Barbaro demonstrates also why we should consider the rays of light like physical elements that embody the geometric process of vision. In the following century Jean Fraçois Niceron understood the true potential of Vimercati's tool by projecting the shadow of a stylus on all kind of surfaces. The Minim friar used this gnomonic device to obtain anamorphoses. So, Vimercati's tool epitomizes the physical connection between light and visual rays, acting as a bridge between sundials and vision.

Keywords Mathematics · Geometry · Gnomonics · Perspective

1 Filology and Gnomonics in the Renaissance

After the ancients, gnomonics did not develop until the Renaissance. The new studies were made by astronomers like: Regiomontanus, Sebastian Münster, Petrus Apianus, Francesco Maurolico, Giovanni Battista Vimercati. As happened with other disciplines, the philological passion drove scholars towards a renewal of gnomonics. They studied the treatises of Greek and Roman scientists, who had successfully realized sundials able to describe the motion of the sky. The astronomer Geminus of Rhodes indicated the simplest models of the universe with the term *sphairopoiïa*, referring to the supracelestial sphere [1], while Vitruvius has handed down some devices

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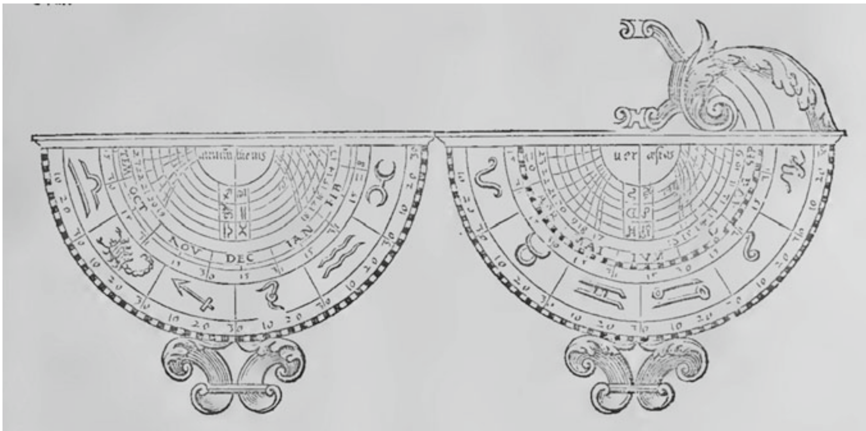

Fig. 1 D. Barbaro, Vitruvius' *Ten Books of Architecture* (1556), semicircular hanging sundial

designed to mark the hours of the day as well as the most important dates of the year. So, we should not be surprised by the philological work carried out by Daniele Barbaro in his comments to Vitruvius's Book IX, because he wished to bring again to life (at least in a printed book) some devices, in his mind lost and forgotten over the centuries (Fig. 1).

One would expect to find the same scientific attempt also in the second edition of the comments to Vitruvius but, in this case, Barbaro eliminated all the reconstructions of the ancient gnomonic devices [2]. This drastic simplification is only apparently in contrast with the author's philological purposes, since: "The sundials discovered by ancients and listed here by Vitruvius can be imagined by those who well understand the circles of the sphere and know the rationale of analemmas because then each can be accommodated to whatever form is desired" [3]. These words indicate that the decision to remove in the second edition images and descriptions of ancient sundials is, however, profoundly humanistic. Barbaro preferred to deal with something more important, namely the explanation of the scientific principles of gnomonics. To do this he mentioned Federico Commandino, who had interpreted and published two fundamental Greek works: *Planisphaerium* [4] and *Liber de Analemmate* [5] by Claudius Ptolemy. While Daniele Barbaro is explaining the gnomonic theory of the ancients, he clarifies that there are common geometric principles underlying different disciplines, such as astronomy, geography and linear perspective. Federico Commandino had made a fleeting reference to this connection in the introductory letter to Ptolemy's *Planisphaerium.* Where he specifies that he drew inspiration from the practices in use at the painters' workshops, to solve problems relating to the representation on a plane of geographical parallels and meridians. But Barbaro goes further, explaining this underlying mathematical link and specifying that it is necessary to know the theory of conic sections of Apollonius, before studying these disciplines.

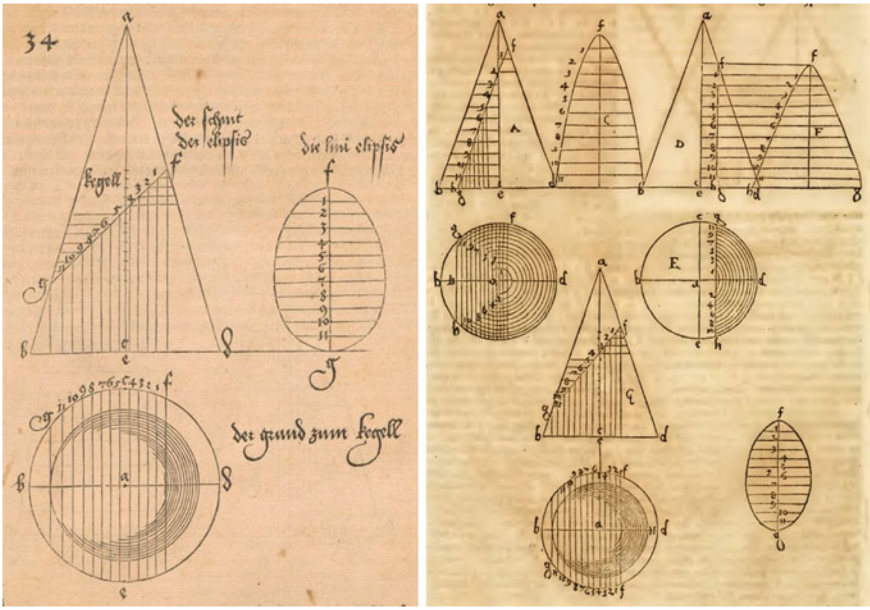

Fig. 2 A. Dürer (left), *Underweysung der Messung* (1525); D. Barbaro (right), *Vitruvius' Ten Books of Architecture* (1567). Conic sections

Always careful to avoid long theoretical disquisitions, he makes use of "the method proposed by Albrecht Dürer, although there are also other ways" [6]. As usual, recognizing perfection in the work of other authors, Barbaro copied the figures published in the *Underweysung der Messung*, where an ellipse, a parabola and a hyperbole, have been represented using three projections of the cone and the construction with 'horizontal sections' (Fig. 2).

Finally, perhaps because aware that the connection between the sections of the cone and the gnomonics may not be easily understood by all readers, Barbaro warns that: "Now, in order that you know the reason why we have explained these figures, I say that the sun, revolving from day to day, sends its rays to the gnomon, the tip of which we imagine as being the apex of a cone; the circle that the sun makes is the base of the cone; and the rays that go from the body of the sun are that line which, turning around, describes the cone. If we think carefully about this effect that the sun makes with its rays in the gnomon, we will see that it makes a conic surface because it is a surface made of two surfaces on opposite sides of the apex of a cone: one is the circle that the sun makes above the tip of the gnomon, the other is one going down from the tip of the gnomon in the opposite direction, which would go on infinitely if it were not opposed by a plane. Since this plane is opposed in different ways [i.e., lies at different angles], and cuts this rays of the lower conic surface, it is necessary to consider the properties of those cuts because they make different lines" [7].

Gnomonics and linear perspective have not only common geometric basis; similarities can also be found in their practical approach. Despite a dimensional difference due to the fact that astronomers work in space, while the painters refer to a plane, the way to sight the celestial bodies by means of moving circles, as suggested by Federico Commandino in the *Liber de Analemmate*, is very similar to that which the artists perform when they transport on the canvas the points of intersection between the visual ray and the pictorial plane. We also find in Barbaro's treatise that the astrolabe, one of the most popular astronomical instruments, requires that its analemma must be built using the rules of perspective. In this case we should proceed in the manner of geographers, as Commandino teaches in his comments to Ptolemy's *Planisphaerium*, that is, by projecting the circles of the celestial sphere on a plane from the South Pole.

2 Conic Section, Gnomonics and Linear Perspective

After these considerations I would like to describe the problem from a geometric point of view, referring to a horizontal sundial. Every day of the year the Sun rises and sets above the horizon, outlining a circular trajectory in the sky (red line). This arc constitutes the base of a double cone, with its apex at the tip of the gnomon. One of these cones is made of light, because the lines that generate its surface are the rays of the Sun, while the other cone is made by shadow rays. This cone, made of shadow, is cut by an oblique plane, if we refer to the axis of this cone, and the section that this cut generates is usually a hyperbole (blue line). However, two days a year, during the equinoxes, it happens that the section of the shadow cone is a straight line, because the circular trajectory of the Sun and the tip of the gnomon are contained in the same plane, named equinoctial (Fig. 3).

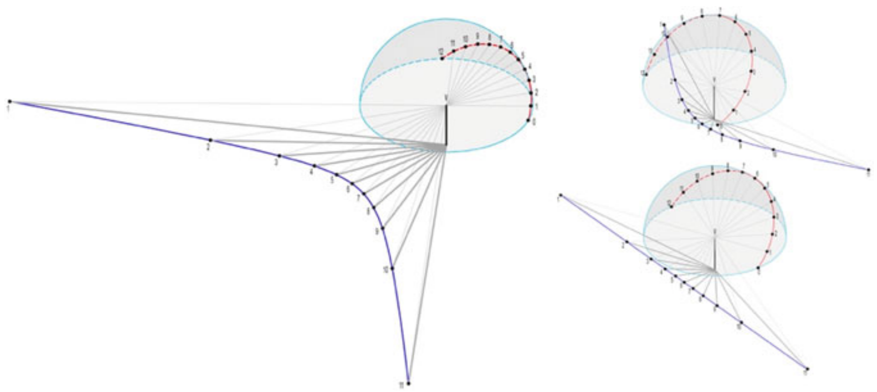

Fig. 3 Conical sections of the celestial geometries: right up, Tropic of Cancer (summer solstice); right down, Aries and Libra (equinoxes); left, Tropic of Capricorn (winter solstice)

The geometric link between conic sections, gnomonic and perspective, is made even more explicit in the description of Daniele Barbaro's *horario universale*, a device that works as a sundial and can be used to mark the time and to make sundials [8]. The illustration, provided by the Venetian nobleman, shows a sequence of disembodied eyes, emanating straight lines, that are set along the Tropic of Cancer. So, the visual rays replace both the luminous rays of the Sun and those of the shadows. Barbaro's instrument is an object similar to an armillary sphere because it is a simplified representation of the celestial sphere plus: the poles; the polar circles; the two coluri; the horizon; the tropic of Cancer; and the tropic of Capricorn. Thus composed, according to the definition given by Geminus, it is still a simple *sphairopoiïa*, that is an interpretation of the cosmos, but Barbaro transforms this tool into a sundial by adding 24 portions of meridians. These arches divide the two tropics into equal parts and fix the astronomical hours.

The perfect coincidence between light-shadow rays and visual ones reaches in these illustrations an adamant clarity. The only problem is that it raises an implicit issue linked to the position that the terms of a Renaissance perspective normally occupy. The painters had the habit to consider the pictorial plane interposed between the eye of the observer and the object to be drawn. But, relying on the rules of gnomonics, Daniele Barbaro shows that it is possible to revolutionize the spatial arrangement of these elements. In the art of tracing sundials, the eye is replaced by the tip of the gnomon while the pictorial plane is beyond the object to be represented. Indeed, the spatial sequence eye-pictorial plane-object is shuffled into another order: eye-object-pictorial plane. The effects that Barbaro's universal sundials will have in the following years, after the publication of *La pratica della perspettiva*, should not be sought in the field of gnomonics since, although innovative, the instrument does not break the traditional patterns of this discipline. Instead, it is appropriate to turn our attention to the changes that occur in the development of perspective theory.

About 75 years after Barbaro's universal sundial appeared, the Minim friar Jean François Niceron published in Paris a Latin work, titled *Thaumaturgus opticus* [9], a scholarly reinterpretation of a previous book in French, *La perpsective curieuse* [10]. In the Latin text some of the principal innovations introduced by Daniele Barbaro found full maturity. Niceron's Figure XXXVIII illustrates a mixing of the elements of a perspective, which are arranged in the order described by Barbaro [11] (Fig. 4).

The object of the representation is set between the pictorial plane and the observer. It is a diaphanous square that allows visual rays to cross its contours and project them

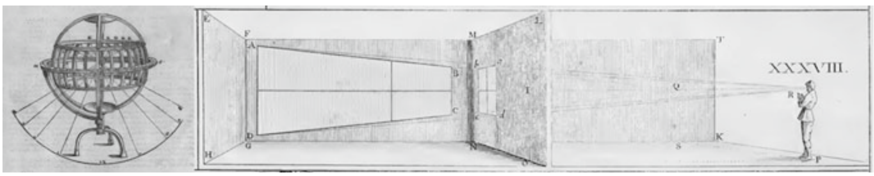

Fig. 4 (left) D. Barbaro, *La pratica della perspettiva* (1568). Horario universale. (right) J. F. Niceron, *Thaumaturgus opticus* (1646). Optical link between square and trapezoid

onto a plane that is very inclined referring to the axis of the visual pyramid. The perspective image that this projection produces is an anamorphosis, an elongated isosceles trapezoid, geometrically different from a square, but optically identical to it, if perceived by the *punctum optimum*.

Inspired by the practice of astronomers, who were able to draw sundials on any kind of surface, the object-picture inversion will free the seventeenth-century artists from outlining perspectives exclusively on a plane, since: "There is no doubt that, if you have well understood the way of forming the sundials in the plane of the horizon, you will be able to draw the sundials in the other straight, hollow, bent or whatever planes" [12]. This is the reason why the famous anamorphoses of Niceron on cones and pyramids, can be traced back to the projected shadows of a gnomon. These strange drawings amaze the observer who, after having positioned his eye in the privileged point of view, is able to grasp their true sense.

3 Drawing with Shade and Shadows

If we consider the method that Daniele Barbaro suggests for outlining an anamorphosis we have to conclude that he had not yet fully understood the difference between cylindrical and conical projections. Indeed, he invites the reader to project a drawing using the light of the Sun instead of a candle. But, Giovanni Battista Vimercati's skiagraphic machine, described in Part Nine of Barbaro's *La pratica della perspectiva*, proves that at least Renaissance astronomers knew well the difference between the two projections. This device, which will be illustrated for the first time by Niceron in his *Thaumaturgus opticus* [13], is an invention of the Milanese monk, described in the *Dialogo de gli horologi solari*, a book published in Ferrara in 1565 [14] (Fig. 5).

Barbaro discovered Vimercati's device probably during his search for perspective tools and trying to complete his knowledge of sundials, a topic that interested him. Describing this skiagraphic machine, he explains that: "With the help of the Sun one drawing of a specific size can be transported to another with that proportion, which a man desires, and he can copy a sundial, a fortress, a human figure, and any

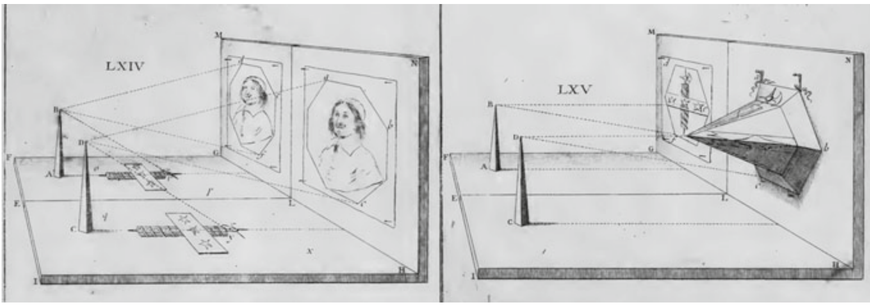

Fig. 5 J. F. Niceron, *Thaumaturgus opticus* (1646). Vimercati's Skiagrafic tool

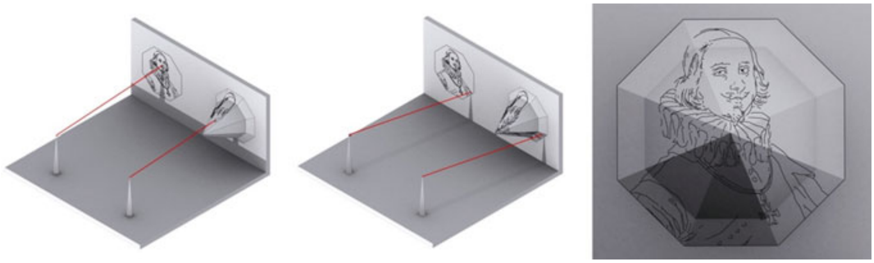

Fig. 6 Skiagrafic tool working and view of the anamorphosis from the tip of the gnomon

other type of thing, as the Milanese Reverend D. Gianbattista Vimercati shows in his book of sundials" [15]. This tool consists of two orthogonal planes; the horizontal one contains two gnomons, that rise on it, while the vertical plane has a drawing and a white paper on which to copy. In practice, starting from a given prototype, a copy of a drawing can be generated, using the projection of the shadow made by a stylus. This skiagraphic machine must be exposed under the parallel rays of the sun rotating it continuously. Following these guidelines, while the shadow generated by the first stylus follows the contours of the prototype-image, the shadow produced by the other gnomon copies this drawing. The draftsman can mark and join the points of the projection on the white paper. From a geometric point of view, Vimercati's instrument transforms parallel rays (cylindrical projection) into rays converging at the vertices of the two gnomons (conical or central projection) (Fig. 6).

In order for the device to work, the straight lines, that carry the shadow, have always to keep the same direction; this occurs only when the light source produces parallel rays, as in the case of the Sun. So, to produce a central projection, emulating the eye vision, it is necessary to rotate the entire system making sure that the shadow follows the drawing you want to copy.

Maybe the time was not yet ripe for Daniele Barbaro to realize that Vimercati's skiagraphic machine was useful not only in the production of equal copies, but also in the production of anamorphoses. However, a doubt arises in favor of his awareness, reading his words: "one can do the same things without the sun, and without a candle, and without the hole-punched paper, and first with the rules set out in the second part about the description of planes, and plans. then with the tools, which I will discuss in the last part" [16]. Talking about anamorphoses, does Barbaro refer to a possible reinterpretation of the Dürer's perspective machine, to the Vimercati's skiagraphic instrument or to the *camera obscura*? Unfortunately, this passage is too vague to formulate a precise hypothesis and in Part Ninth of his treatise Barbaro does not indicate how to use perspective machines to generate anamorphoses.

Many years later, Niceron will pick up all the opportunities offered by the first two prospective devices mentioned. As we said, the Minim friar will describe the function of Vimercati's skiagraphic machine in *Thaumaturgus oprticus*, extending its function for delineating anamorphoses. This important outcome was achieved by applying the projection of the second gnomon on any type of surface "whether they are continuous, discontinuous, sunken, convex, equally extended, protruding, rough, cut, curved, hollowed out and in others of any kind" [17]. All this is possible because the deformed drawing appears identical to the starting one, if the observer places his eye in the vertex of the second stylus, regardless of the geometry of the surface on which the drawing has been projected. Vimercati's illustration of the skiagraphic device clearly shows that Niceron knew that the parallel solar rays after having intercepted the tip of the gnomon change their path creating a conical projection.

While thinking about the way in which nature emulates the mechanism of vision, the Minim friar was not alone, he could count on the help of eminent scientists, gathered in Paris around the philosopher Marin Mersenne. Among them we remember the brother Emmanuel Maignan, physicist and gnomonist, author in Rome of two sundials that use the reflections of the Sun's rays to measure time. Niceron and Maignan experimented together the rules of conical projection in artistic practice.

Along the corridors on the first floor of the convent of Trinità dei Monti in Rome we can admire two extraordinary anamorphic paintings, about 15 meters long each: *Saint Francis of Paola* in *grisaille* [18], work of Maignan; and *Saint John the Evangelist writing the Apocalypse*, painted with tempera colors by Niceron and discovered only in very recent times [19]. Only if the observer, crossing the corridors, conquers the privileged position the anamorphoses the two Saints become intelligible. In any other place, the visitor perceives only a trail of lines, similar to the roughness of a landscape, within which Maignan and Niceron have disseminated respectively: single episodes of the life of Francis of Paola; and the apocalyptic visions received by the Evangelist (Fig. 7).

The way to obtain these anamorphical delineations is well illustrated in Niceron's *Thaumaturgus opticus*. But this method could be very long and boring, because you have to project on the wall each single point through wires (Fig. 8).

If Niceron had wanted to avoid this problem he could have used a device for the anamorphic delineation of his *Saint John The Evangelist*, such as Vimercati's tool. First of all, we should focus on the light source, indeed, this wall painting is an art work created inside a building. Maybe this source was a candle, whose light radiations had been artfully directed to create a set of parallel rays. The control of a light source, transforming it from a central to parallel projection, could have been easily achieved by means of a parabolic mirror with the candle placed in its focus. The

Fig. 7 J. F. Niceron, *Saint John the Evangelist* (left) and two details: the pen/falling water (center) and the book/plowed field (right)

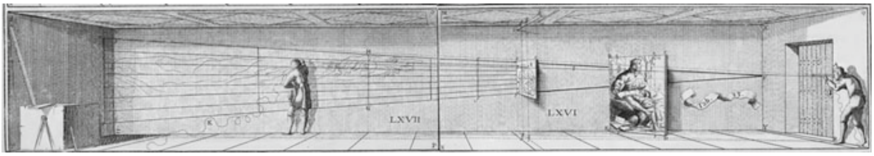

Fig. 8 Niceron, *Thaumaturgus opticus* (1646). Way to trace an anamorphosis

'magical' applications of the mirrors had already been described in the *cogitationes privatae* of Descartes and illustrated by Athanasius Kircher [20]. So, it is plausible to assume that Niceron was perfectly able to transform a point source into a parallel light thanks to a parabolic mirror. To recreate the effects of Vimercati's device, he could have set two obelisks, one in front of the perspective drawing of *Saint John*, and the other in the privileged point of view. In this way, moving and rotating the mirror, the shadow generated by the tip of the first stylus retraces the contours of the drawing, while the shadow produced by the second stylus reaches the wall, where a collaborator can trace the anamorphosis. Such a skiagraphic device would have quicken the tedious identification of each single point, solving also the problem of precision linked to the use of long ropes (Fig. 9).

4 Conclusion

A new interest in Ancients' science led Renaissance scholars to wonder about the common principles that lie under different disciplines. Daniele Barbaro is among the first scholars to point out that conical sections match geographical representation, sundials and linear perspective. From a practical point of view The Venetian scholar makes this connection explicit by listing the instrument of the gnomonist Giovanni Battista Vimercati among the machines useful for copying drawings.

Almost a century later, Jean François Niceron collects the legacy of Barbaro, he illustrates the instrument of Vimercati in his *Thaumaturgus opticus* and perhaps uses it for the realization of the anamorphosis, titled *Saint John the Evangelist writing the Apocalypse*. This means not only that in the seventeenth century scholars know the difference between conical and parallel projection but that they are also able to bend natural phenomena forcing the Sun, or a candle, to draw in their place.

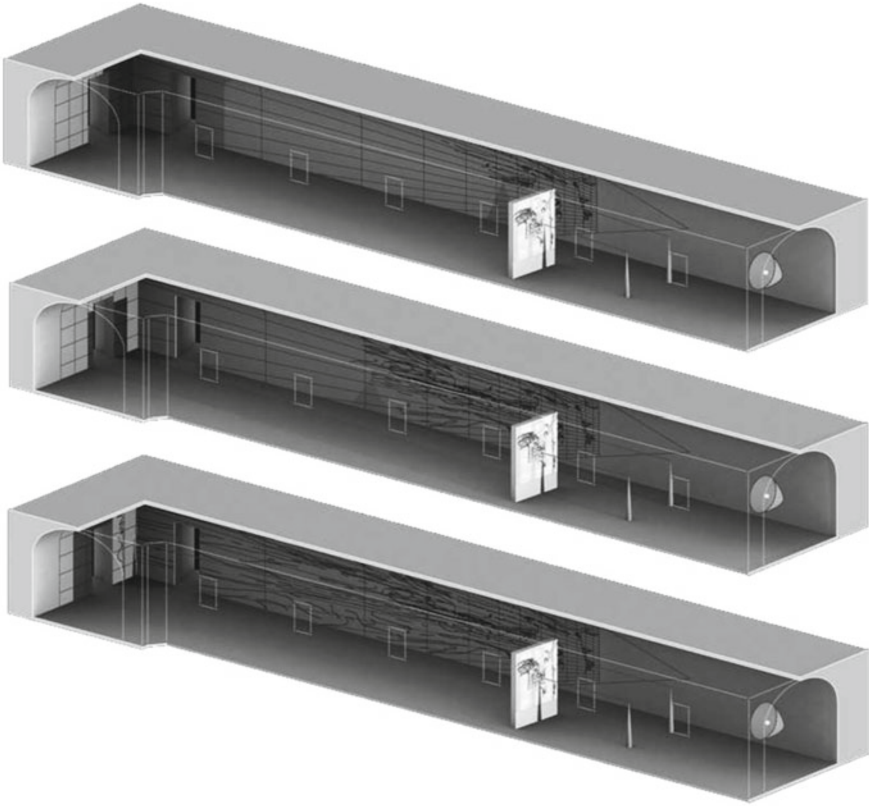

Fig. 9 Hypothesis of application of Vimercati's skiagraphic tool for creating Niceron's anamorphosis of *Saint John the Evangelist* in the corridor of Trinità dei Monti in Rome

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Geometrical Discretization for Differential Equations

Carlo Augusto Pasquinucci

Abstract Navier-Stockes equations can be solve only via numerical calculus. One of the most used technique is the finite volume method. The equations are solved discretizing the geometrical domain into volumetric cells (called mesh). A perfect discretization is not simple to achieve, in particular in industrial applications. The main is that is not simple to understand the influence of the discretization on the results

Keywords Geometrical discretization · Computation fluid dynamics · Numerical methods

1 Finite Volume Methods

Finite volume method is a numerical method for calculating partial differential equations (such as Navier-Stokes equations) in the form of algebraic equations [1]. Volume integrals are converted to surface integrals, using the Gauss divergence theorem. These terms are then evaluated as fluxes at the surfaces of each finite volume. In reality, the equations are calculated in the centroid of the finite volume and only in a second time, the surface fluxes are reconstructed.

$$\int_V \nabla a * dV = \oint_{dS} a * dS \tag{1}$$

In this method, there are three main approximations, a numerical and two geometrical ones:

1. The numerical method used to interpolate the surface fluxes from the centroid values. The results are not so sensitive to this approximation if it is use a correct numerical method.
2. The "definiteness" of the cells, i.e. the cells have a finite volume instead of an infinitesimal one. This approximation, obviously, introduces an error. On the

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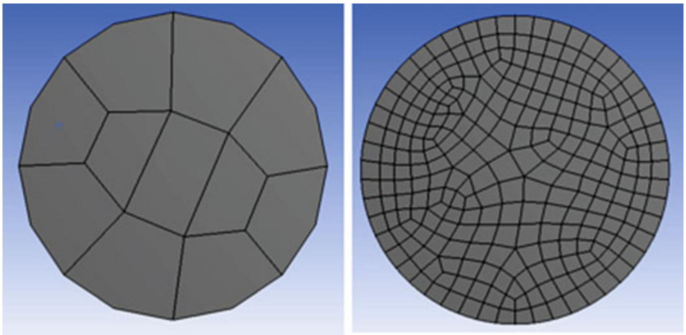


Fig. 1 Discretization of a circle—It is possible to understand the influence of the mesh size on the correct definition of the sectional area

other hand, not always a reduction in the cell dimensions gives a better result. This approximation can be avoid doing different simulations with different cell dimensions, but this method is very time consuming and not possible to be used in industrial simulations.

3. The solution is influenced also by the type and the quality (the regularity, uniformity) of the cells. A hexahedral cell gives normally better results instead of a tetrahedral one, due to a better alignment of the faces.
4. The solution is in the end also affected by another relevant geometrical approximation. The domain is creating approximating the original geometry using finite volumetric cells. For this reason, the volume and the surfaces of the domain is not the same of the original geometry. This can be a relevant approximation in particular when surfaces are involved, such as, when it would like to calculate the flow injection a determined mass flow on a circular pipe (it is not possible to reproduce a circle using quadrats) or when it would like to calculate forces on surfaces (see Fig. 1).

2 Mesh Forms and Quality

The numerical method based on finite volume cells is very robust and quite every kind of cells can be used for decompose the domain.

For the calculus can be used for example tetrahedrons, hexes, prisms, pyramids and polyhedrons. The only requirement is that the elements need to be convex and the faces that made up the control volume need to be planar. Obviously, the volume should be a positive number. Furthermore, the method need to know which control volumes are internal and which control volumes lie on the boundaries.

Also if the method can accept quite any kind of cell, the result given can be very different.

Two opposite requirements need to be satisfied.

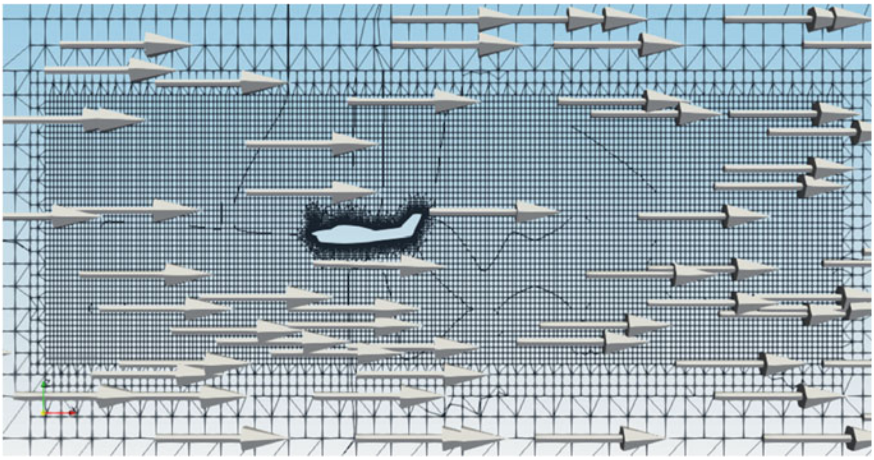


Fig. 2 Sectional view of the discretization of the fluid volume around a plane. It is possible to see that the cells are aligned with the flux (indicated by vectors)

1. Accuracy of the results, given by the homogeneity of the cell type and dimensions
2. Accuracy of the original geometry reproduction, given by different types and cell dimensions.
3. Reduction of the request time, depending of the cell number and so of the cell dimension.

For the first requirement, it is quite easy to understand the same cell type and the same cell dimension can help a lot the interpolation between cells.

In particular, it would be very nice is a cell has a minimum number of neighbour cells.

Furthermore, if the cells are hexahedral and also aligned with the flux, it is possible to understand the interpolation would be simpler (see Fig. 2).

For the second requirements, it is easy to understand that different cell type and cell dimension can help a lot in a good approximation of the original geometry.

In particular, the most common problem is the decomposition of cylinders. This form is very common in the industrial filed. It is possible to find cylinders in pipelines, in rotative machineries, but also in electronic devices (i.e., diodes, capacitors or lights).

It is very hard to decompose cylinder using hexahedral cells (see again Fig. 1).

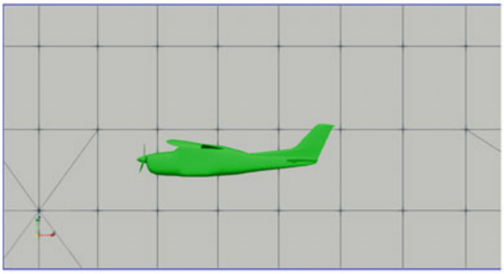
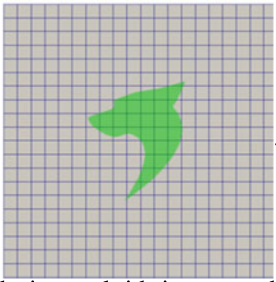
3 Example of Mesher—SnappyHexMesh

Lots of different meshing software are available, both commercial and non-commercial. One of the most studied is the software open-source snappyHexMesh.

Since it is released with an open-source license, it is possible for everyone to access the code and understand what the software does [2, 3].

The idea behind this software is quiet simple. To explain it, two examples are here shown: the mesh of an air plane and of simplified shark.

1. It starts with a background mesh of hexahedral cells that fills the entire region that has to include the surface that one want to mesh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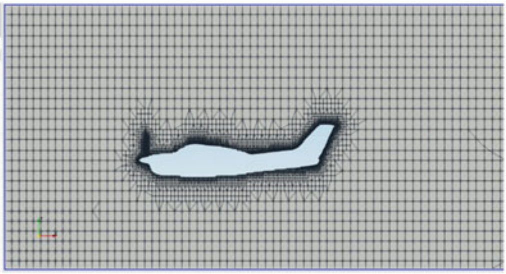
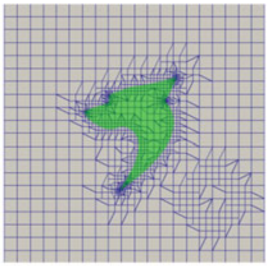
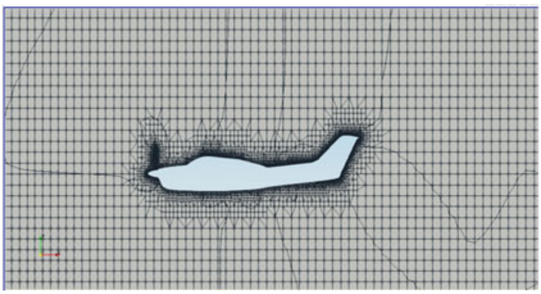
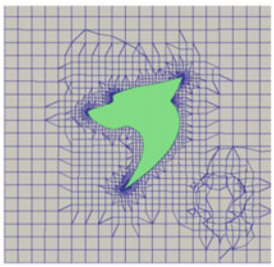
2. The cells that are intersecting the surface are split and the internal side is removed. It is possible to control the number of refinement level, i.e. how man time is the smallest cells dimensions compared to the original one.
   Since the cell dimension is splitted by 2 every time, it is easy to understand that the cell dimension decrease exponentially.

$$\delta_{cell} = \frac{\delta_{original}}{2^n} \tag{2}$$

   Once the cell splitting is complete, a process of cell removal begins. The region in which cells are retained are simply identified by a location point within the region, Cells are retained if 50% or more of their volume lies within the region.

3. After deleting the cells in the region specified and refining the volume mesh, the points are snapped on the surface to create a conforming mesh. This is a valid snapped or body fitted mesh that can be used for a simulation.

4 Mesh Optimization

An interesting method for create a very nice mesh in the mesh optimization.

Since it is possible to have some numerical value for the mesh quality, it is possible to optimize it.

On the other hand, quite always the mesh quality is also evaluated looking at its beauty. This criterion cannot be numerical evaluated.

The goal is to create a mesh with the minimum number of cells, but the best quality (in this case, the lowest value of non-orthogonality, i.e. how much the faces of the cells are not orthogonal).

In Fig. 3, the results of a mesh optimization study are shown.

It is possible to see that a huge difference in the number of cells and in the quality of the mesh can be reached, this means that can be necessarily to do lots of different mesh, also if it would take normally around 20–40 min each.

The best mesh is the one on the bottom-left side, i.e. with the minimum number of cells and the lowest non-orthogonal value. This mesh is the one that should be used in the simulation, but nobody can say the one that would give the most correct result.

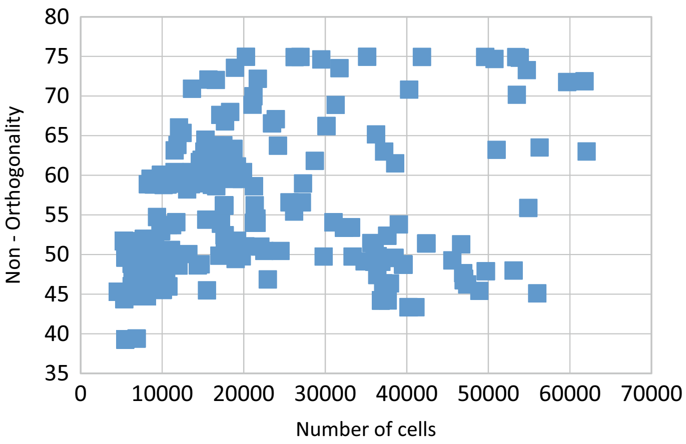


Fig. 3 Mesh optimization study. Number of cells vs non-orthogonal quality. It is possible to see that the best mesh in the bottom-left side, i.e. the mesh with the fewest cells and the low-est non-orthogonality

5 Mesh Independence Study

The only way to know which is the discretization that can give the most correct results is to perform several simulation and compare it with experimental results.

This seems to be a no-sense, but normally, it is used to create some best practice and use similar settings also when experimental data are not available (Figs. 4 and 5).

For this case, the mesh dimension is validated comparing the results of a CFD simulation of the "Ahmed body" geometry (that represents a simplified car) with experimental data. The particular goal of this benchmark is to compare the Drag Coefficient (Cd), with the real experimental measurement in the wind tunnel [6].

$$C_D = \frac{2 * Resitance}{\rho * V^2 * A} \tag{3}$$

where ρ is the air density (1.2 kg/m^3), V the car velocity and A the car frontal area.

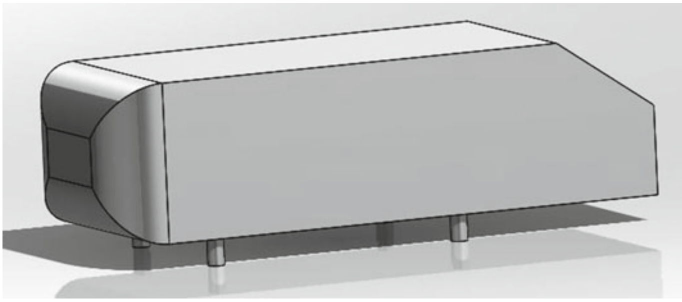


Fig. 4 Ahmed body geometry

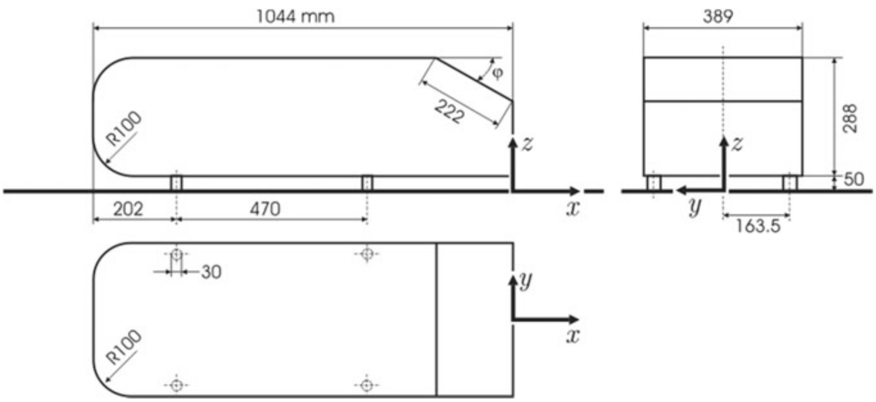


Fig. 5 Ahmed body main dimension [4, 5]

The Drag Coefficient is a very important total value, which is a measure of the object's resistance to the airflow. According to measurement in the wind tunnel [6], the drag coefficient of the Ahmed body, with the slant angle 25º, is equal to 0.285.

A hexahedral block mesh (5 m high, 11 m long, and 6 m wide) was used for mimic the virtual tunnel. The Ahmed body model itself is 1.044 m long, 0.338 m high, and 0.389 m wide (see Figs. 3 and 4). For the mesh, different main mesh size, from 0.1 m to 0.001 m, were used for the background geometry. An internal box, with a mesh size of a half of the main one, was built around the car, in order to have a better definition of the Ahmed geometry.

In Figs. 6 and 7, the mesh on the symmetry plane is shown, respectively for a mesh dimension of 0.05 m and 0.01 m.

The differences in the calculated Cd are shown in the graphic.

It is possible to see how different can be the results, with a range of error from 3 to 0.1% between the calculated Cd and the experimental data.

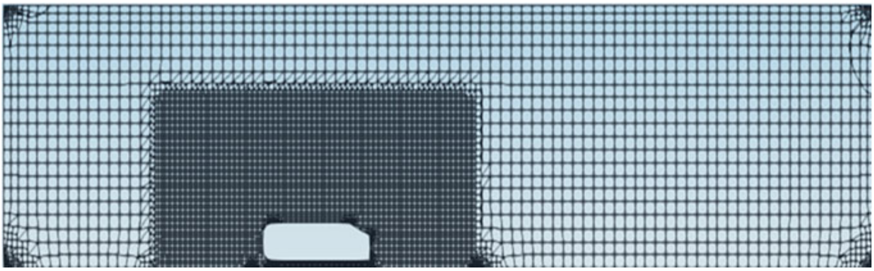


Fig. 6 Mesh on the symmetry plane for a mesh size of 0.05 m

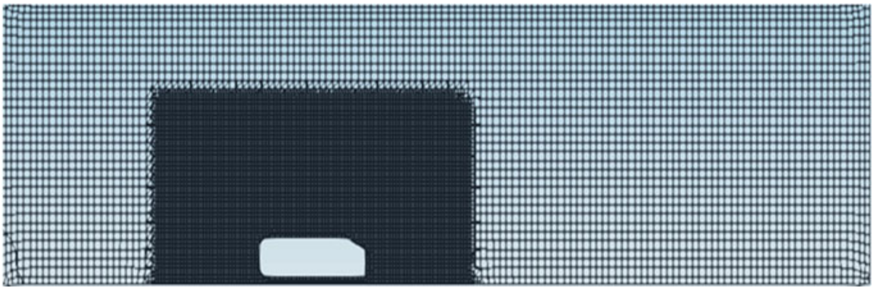
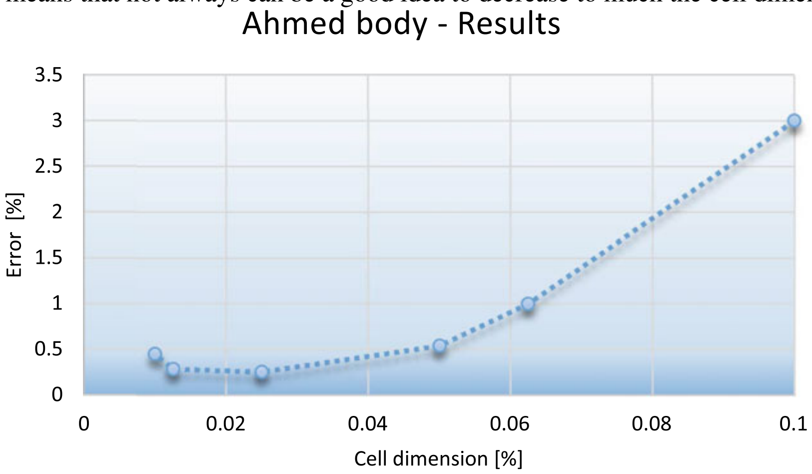


Fig. 7 Mesh on the symmetry plane for a mesh size of 0.05 m

It is also possible to see that the best result is not given by the finest mesh. This means that not always can be a good idea to decrease to much the cell dimensions.



6 Final Considerations

In this paper, it is shown how complex can be the discretization of a geometry also if quite simple as the Ahmed body. Since CFD simulations tend to reach a high level or complexity, it would be necessary in the short future to have more and more robust automatic algorithms that create good discretization.

This is field of the geometry that should be more investigated.

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For Representation, a New Reality: Hybrid Immersive Models

Adriana Rossi, Lucas Fabian Olivero, and António Bandeira Araújo

Abstract In alignment with the relevant disciplinary tools and methods, we analyse the innovative potential of cubical perspective combining the uniqueness of hand-made sketches with their digital navigation. The investigated model (called hybrid for the above mentioned reason) offers the scholar a twofold opportunity: on the one hand, the possibility to exploit the intuitive immediacy of the creative idea which is made objective by the graphical signs, and, on the other hand the possibility of manipulating the communication and configuration features of immersive techniques. Over the last few years, the authors have delved into the scientific fundamentals of cubical perspective and verified the peculiarities of the solutions through experimental applications pertaining to the context of architectural drawing. The case studies presented were pedagogically tested and were aimed at promoting the knowledge of the presuppositions that inform the work of good formal structure and conformation designers.

Keywords Innovative representation strategies · Cubical perspective · Hybrid immersive models · Survey culture · Design culture

1 Introduction

It is currently possible at any time to explore virtual environments through web browsers. The backgrounds of such environments are made up of mosaic like images 'stitched' together so as to completely encompass the observer and offer a total field of vision (i.e. 360° around the vertical axis and 180° from zenith to nadir). The users/observers, with the aid of ad hoc devices (such as stereoscopic glasses VR)

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can perceive the digital scenes inside these ‘bubbles’ or ‘ideal cubes’ as if they were actually and physically there. They can also focus on details, enlarge them or obtain close ups via simple head movements. This type of fruition has made it possible, to visit existing places that are not normally accessible or too far away, as well as hypothetical places that do not exist as in the case of designs plans. In any case full immersive experiences have changed the way we conceive space and have triggered significant cultural and socio-economic impacts.

The idea of this kind of encompassing visual experience and the quest to realize it can be traced back to Robert Barker’s Panorama” (from the Greek words “pan”—all + “horama” view; “horan”—see) patent. A few years later (1801), in compliance with Barker’s building specifications Robert Mitchell, designed a circular theatre that was to be built in Leicester Square in London on a circular surface. As repeatedly mentioned in several contexts, the London panorama enabled painting to overcome “the physical boundaries of the frame” and as a result it deeply influenced the customs and lifestyle of the Victorian age. Even bearing in mind the differences, the innovation underlying the building/painting panoramas of two centuries ago re-propose the most interesting aspect of the contemporary interactive panoramas.

More recently, however, its potential has been exploited to a limited extent for educational purposes to provide information or collaboration spaces for “culture” consumers and to a greater extent for entertainment purposes as in the case of backgrounds for videogames. The latter uses have become so popular that they have obscured what we consider the most important potentiality of this technology and that is the impact on the way architecture is conceived and used.

The fruition of immersive models is useful in survey analysis and design synthesis. Not only does it allow to control-communicate what has been conceived and elaborated, but it also enables us to use representation tools and methods to guide knowledge and orient conformation hypotheses [1]. These are the reasons that prompt us to delve into the innovation potential of cubical perspective sketches and their immersive digital views.

Hubert Damisch explained that Brunelleschi’s experimentations (with the cathedral and in Signoria square in Florence, around 1401 and 1409 respectively) represented the first step towards a ‘biased’ manipulation of the point of view. The scholar highlighted that the proven match between the perspective image and the objective image paved the way for an interpretation of the elements of an architectural composition order which was made objective in the graphical schemes [2, p. 108].

More explicitly Giulio Carlo Argan argued that “man perceives as his mind conceives” [3, p. 5]. In other words, it is the mind’s eye that conveys the knowledge of the intermixing of the empirical space with the ideal space. “The painters and the architects of the renaissance did not create works but rather a relationship with the new reality (…) Using the new perspective, they tried to mathematically reconstruct reality through geometry.” Today a similar effort is required but with reference to a reality that Virilio calls stereoscopic: in the present space the concrete presence is mixed with the virtual presence [4]. A balance must be struck between the real and artificial realities, which are both integral parts of our daily existence. Just like Leonardo and Michelangelo before us, we will become aware that we do not live in an artificial (ideal or platonic) world, nor do we live in a real physical world that can be understood through immediate perception only. We live in a technical scientific system that mixes both worlds [5].

In this paper we explore the properties of the so called 'hybrid' model to highlight the criteria and principles that can prove to be the keys to further speculation on the idea of space. If representation techniques and their multimedia potential are employed critically, they promote the acquisition of the knowledge and presuppositions that inform the ideation processes, which are indispensable for both the scientific dissemination of research in the field of architectural survey and representation and also for the professional development of designers.

The exercises and results presented in the paper, have been produced as activities carried out during the *Advanced Representation Techniques-Product* (Prof. A. Rossi) course of the Master's degree (unicampania). Through the detailed study of classical themes of descriptive geometry [6]—such as the plane representation of volumes and photogrammetry [7] the didactic activities of the course derive and develop the rules of cubical perspective with further aim of experimenting with a certain number of real and hypothetical cases of functional solutions to an original methodology of digital cooperation (VR, AR). Starting from original hand drawn sketches, prototypes are constructed. When the fruition of latter occurs in immersive mode it verifies holistic analyses.

2 The State of the Art

Among the various immersive perspectives, we choose the cubical perspective because it is best suited to our purposes. Cubical perspective methods are associated with linear drawings methods, which recall a familiarity among architects and designers. On the other hand, perspective deformation in every face of the cube is an intuitive, convenient distortion, which does not always apply due to the considerably curved warping of other spherical perspectives. For a more in-depth comparison between cubical perspective and other immersive perspectives check [8, 9].

The current repertoire of cubical applications consists mainly of technical applications in the CGI (Computer Generated Imagery) field. There are also some intuitive procedures from the artistic field and just a few examples of applications in the architectural field [10, 11].

The first group, better known as 'cubical mapping', uses the geometry of a cube for scenes creation with ludic-artistic purposes, e.g. in videogames [12]. Starting with the Environmental Mapping of Ned Greene in 1986, these scenes (better known as 'worlds') revolutionized CGI increasing rendering and lighting performance thanks to the use of the cube. Since then, several improvements and variations were proposed in a large number of methods: standard cube map, QSC (Quadrilateralized Spherical Cube), continuous, tangent adjustment, UniCube and others.

The second group of artistic applications re-uses cubical mapping for illustration-artistic purposes. Here we can find either several web tutorials and semi-automatic solutions, such as Oniride 360 Art Plugin.

In architecture, we find cubical mapping applications for minor editing (e.g. deleting the tripod or for point cloud colouration) of 360° photography during architectural surveys. In this case, a software tool converts and switches back between the equirectangular and the cubical map format. Other cases get a cubical mapping creating first a 3D model and then using an automated plugin. In both cases, the designer depends on software availability and the programmers' decisions about its implementation. As we can see, this peripheral approach is only moderately useful. We see that current applications call on cubical mapping just as a secondary resource (e.g. a last aesthetical intervention in post-production) undervaluing its full potential. Therefore, we aim to consider cubical mapping as a complete instrument for both architecture survey and design.

Hence, the trial-and-error knowledge of the current panorama does not give more than a 'quick and friendly how-to' without going into the essence of the representation: the study and properties of the projections on the cube's surface, both with existing and non-existing scenes. To get at this essence it is necessary to master the mathematical fundaments of the cubical model.

One main problem of cubical perspective is how to efficiently solve the fragmentation of a line representation [9, pp. 33–34]. Going beyond the existing approaches' limitations, the authors developed two methods. In the first method, we face the fragmentation problem considering the disjoint union of six linear perspectives [10]. In the second one, we treat cubical perspective as a particular case of spherical perspective [8, 9], which allows us to apply the general strategy for solving such perspectives proposed by A. B. Araújo in [13]. We have chosen to follow the second method since:

- it solves all the possibilities of representation (the first method was truncated just to lines parallel to the cube's faces);
- it proposes a compact solution, which implies having all the vanishing points and their auxiliary constructions within the confines of the paper;
- it integrates our work within the general theory developed for other spherical perspectives, both fisheye [13] and equirectangular [14].

3 Method

We introduced the theory of cubical perspective through practice activities consisting in exercises of increasing difficulty. The first seven (E-I–E-VII) were aimed at the solution of basic problems applied to elementary architectural spaces (architectural cell) carried out and tested as the didactic activities of the above-mentioned curricular course (Fig. 1). The designers (in our case the students) were able to test their work using plugins and open source software. In all the stages of the representation process, the anamorphosis constructed in the plane development of a cube can be visualized in an interactive modality (Figs. 1, 2 and 3). We intend to focus on two different

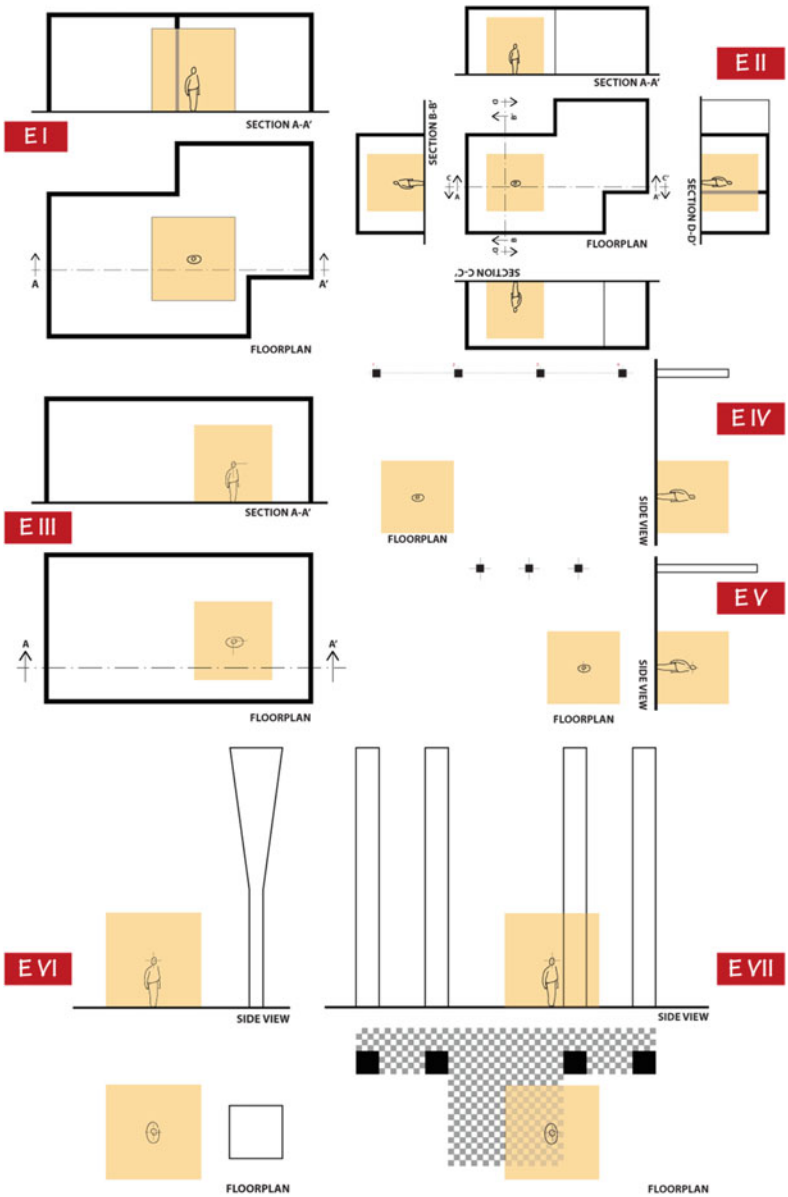

Fig. 1 Geometrical definitions for exercises E-I–E-VII. Didactic activities of the TAR course, Master's Degree Prof. A. Rossi, academic year 2019–2020. E-I Single room using panoramic faces. E-II Representing hidden walls. E-III Using superior and inferior faces. E-IV Repetition of elements symmetrically organised. E-V Repetition of elements asymmetrically organised. E-VI Representation of an element with variable-section using a grid. Lecture and repetition of modules and tiles. Author and tutor Lucas Fabian Olivero

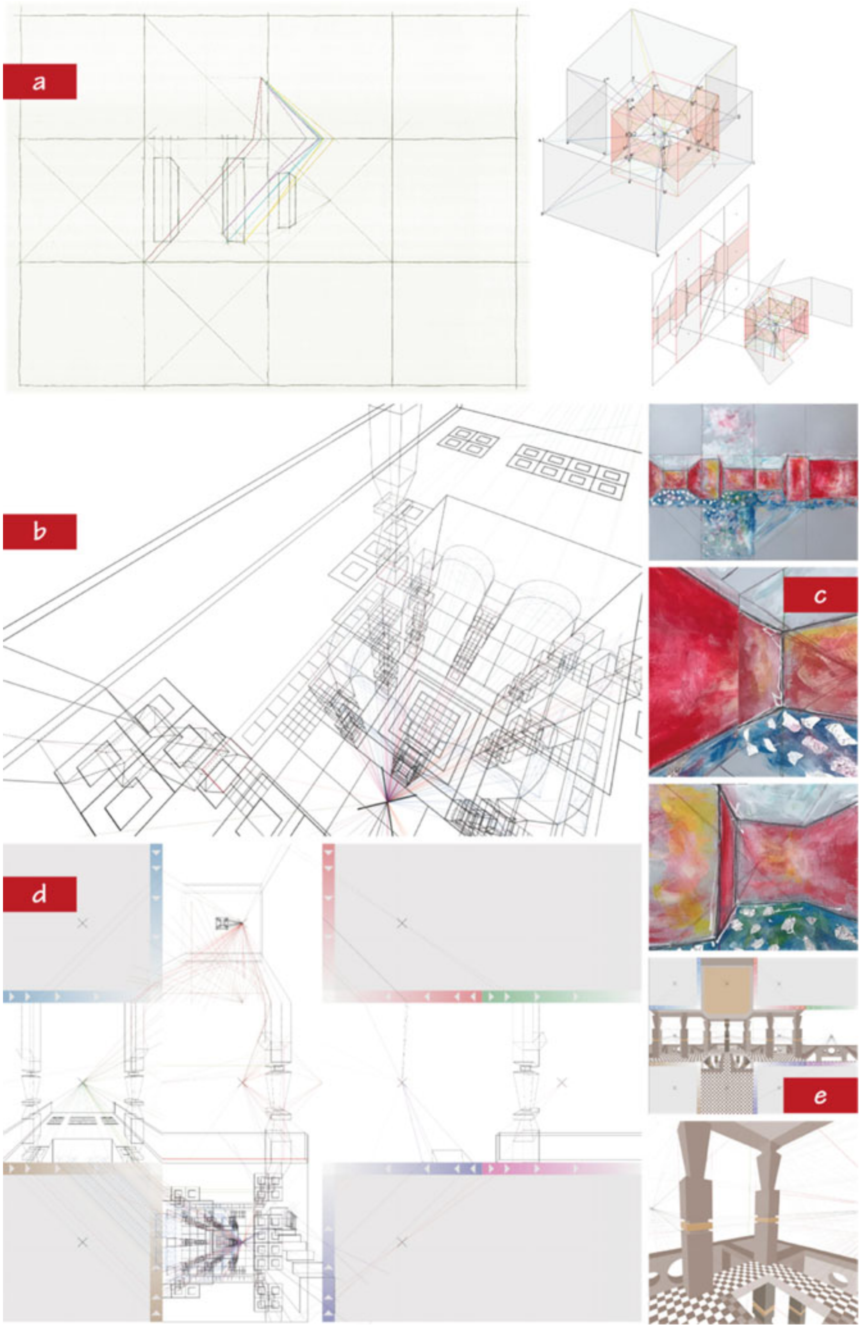

Fig. 2 **a** Graphical applications of E-I–E-VIII. **b** Immersive navigation of E-VIII. **c** Handmade sketch resolution of E-I and its immersive navigation. **d** Cubical perspective E-VIII. **e** Cubical construction and immersive navigation of E-VIII. Authors **a**, **c** Teresa Di Palma, **b**, **d** Ibtissam Jayed and **e** Assia D'Alessio

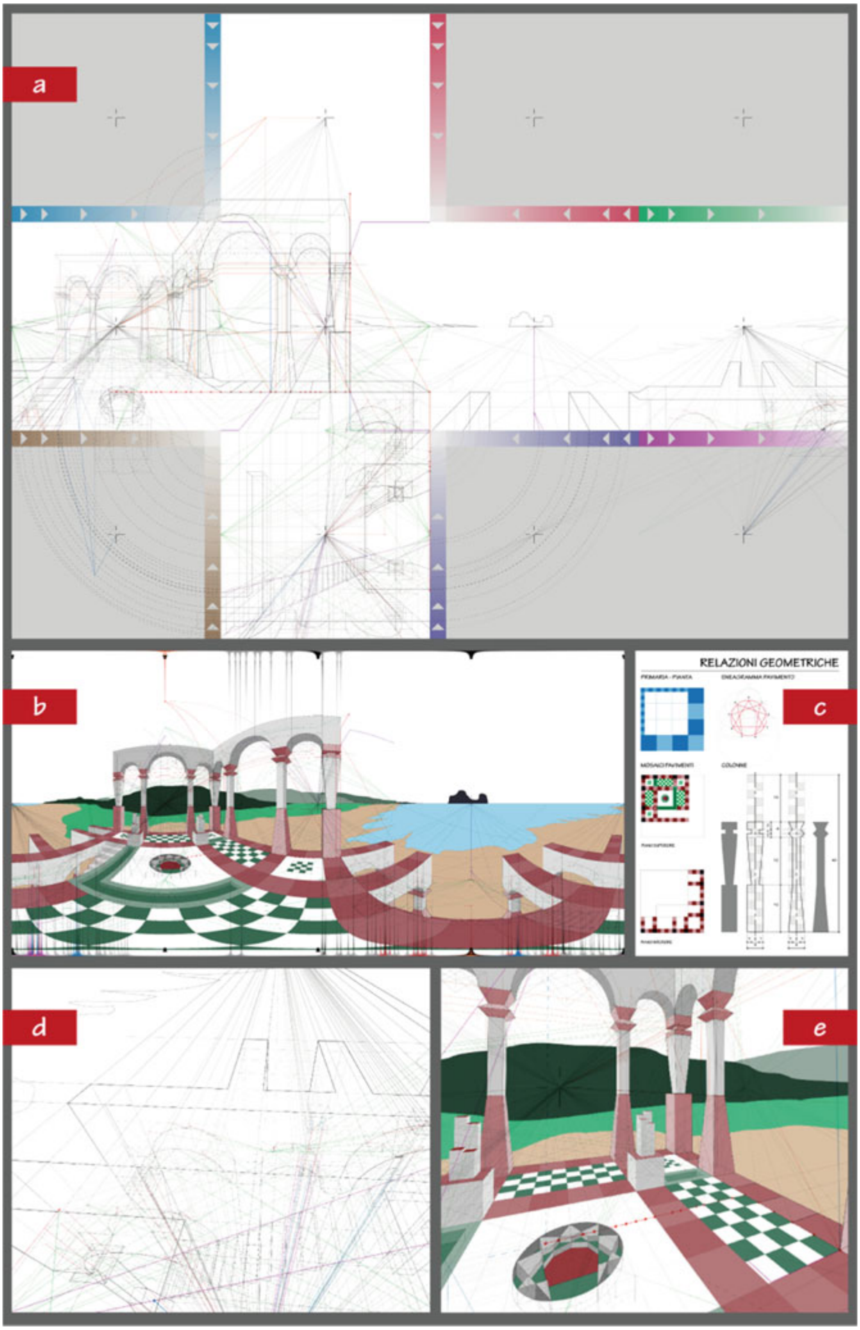

Fig. 3 First synthesis exercise (design culture) E-VIII. **a** Cubical perspective; **b** Equirectangular panorama; **c** Geometrical proportions of the project (**d**); immersive navigation (**e**, **f**). Author Lucas Fabian Olivero

sets of synthesis exercises which require the application of the acquired rules and allow us to guide cognitive processes so as to operatively experiment with solutions. The two approaches are both interactive, circular and convergent towards the same objectives.

The first synthesis exercise E-VIII starts with the analysis of a geometrical structure hypothesized in plan and elevation (design culture) (Fig. 3). Then the operator employs the knowledge gained from the exercises E-I–E-VII to work out, from the structure of the cubical perspective, the criteria and reasons of the ratios and proportions that define the shapes of and relationships between the hypothesized and realized architectural elements. In this way the operators, drafters of the (2D image) product, can perceive and experience the hypothesized space (3D simulation). Thus, they can also verify the correctness of their design before embarking on its detailed development (i.e. the construction of the 3D model). Furthermore, another advantage is that it is also possible to offer a hypothetical client an immersive experience of the first drafts of the design so as to assess its appeal.

The second set of exercises, which we consider much more interesting and significant due to its pertinence to survey culture, starts with E-IX analysing the state of the building (Fig. 4). The workflow begins with the acquisition of giga panoramas that cover a field of view of 380° × 180° from a selected position. From every point, we shot single photos with a manual or motorized panoramic head (such as Nodal Ninja o Gigapan Epic Pro).

Then, a stitching software assembles these multiple photos in a single equirectangular panorama. In our case, we worked optimally yielding a photogram juxtaposition in the order of 40% with Kolor Autopano Giga.

Next, we used this panorama to digitally texturize an ideal sphere. The observers are then virtually projected into the centre of such a sphere, 'placing' their eye in the same position as the photo-camera's nodal point, which allow them to inspect the coherence of the construction. We can reproduce a similar experience and spread analogous graphical information but in the internal faces of a cube: if we substitute the sphere with an equally centred cube, we can immersively navigate the environment with the advantages of using cubical mapping.

In E-IX, students can identify and select elements in the cube map that are suitable for their survey analysis and learning objectives. The exercise presents different levels of complexity according to its aim and is also used to introduce the more complex case of geodesic constructions i.e. when two points of a line are on adjacent faces [9].

In E-X we analyze the cubical perspective by tracing on it a skeleton of horizontal and vertical reference planes (Fig. 5). These can be translated back to floorplan and section to obtain a modular grid of the scene. We can then execute project variation on the elements thus obtained, and in turn visualize these alterations immersively.

Therefore, both exercises focus on the uniquely human ability to free the form of its materiality so as to probe the conformation principles of ideas with the aid of geometrical drawing. Far from being arbitrary, such ideas are guided by a geometrical grid and oriented towards the analysis of what exists. Hybrid models confirm to be the key to interpret existing space and alternatively to conform digital space. The

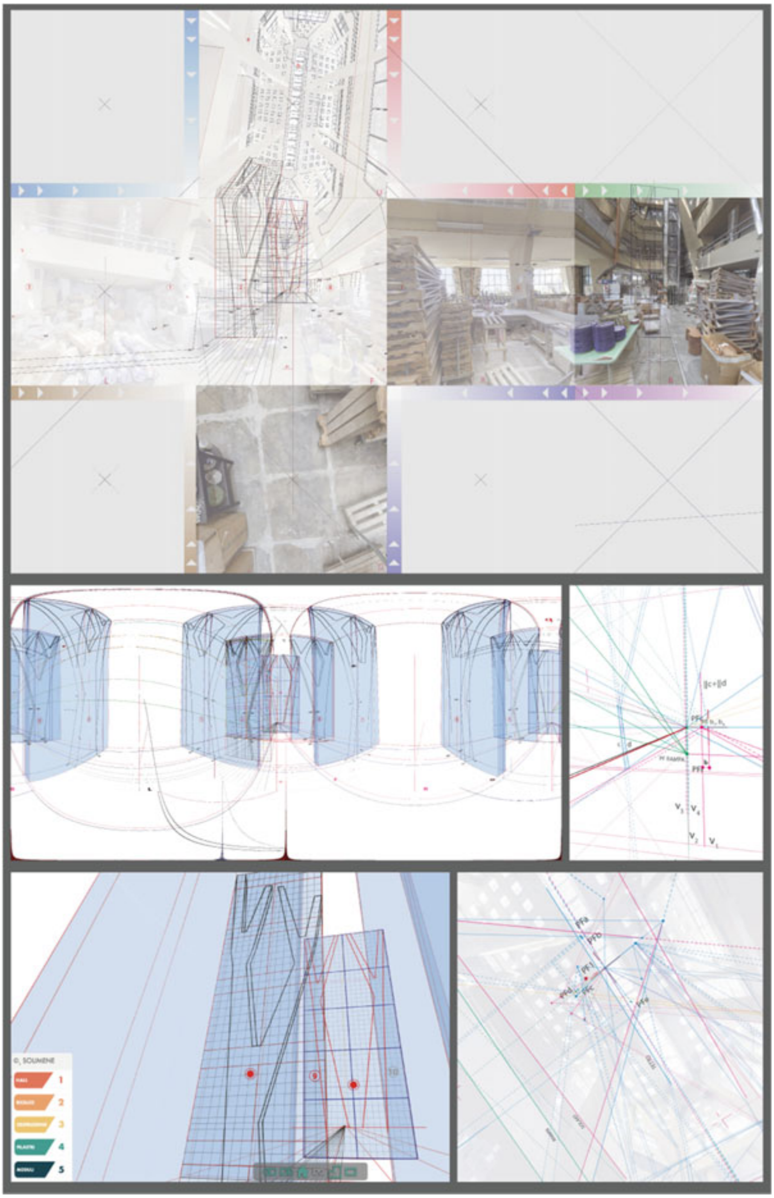

Fig. 4 Second synthesis exercise (survey culture) E-IX and E-X. Internal view of the Solimene factory [16]. Selected and rendered elements in cubical perspective with analysis of vanishing points and proportions. Equirectangular panorama and immersive views of the average geometry and selected criteria. Authors Assia D'Alessio, Teresa Di Palma, Caterina Crispino, Marta Campanile, Maria Petrillo, Lorenzo Villani and Lucas Fabian Olivero

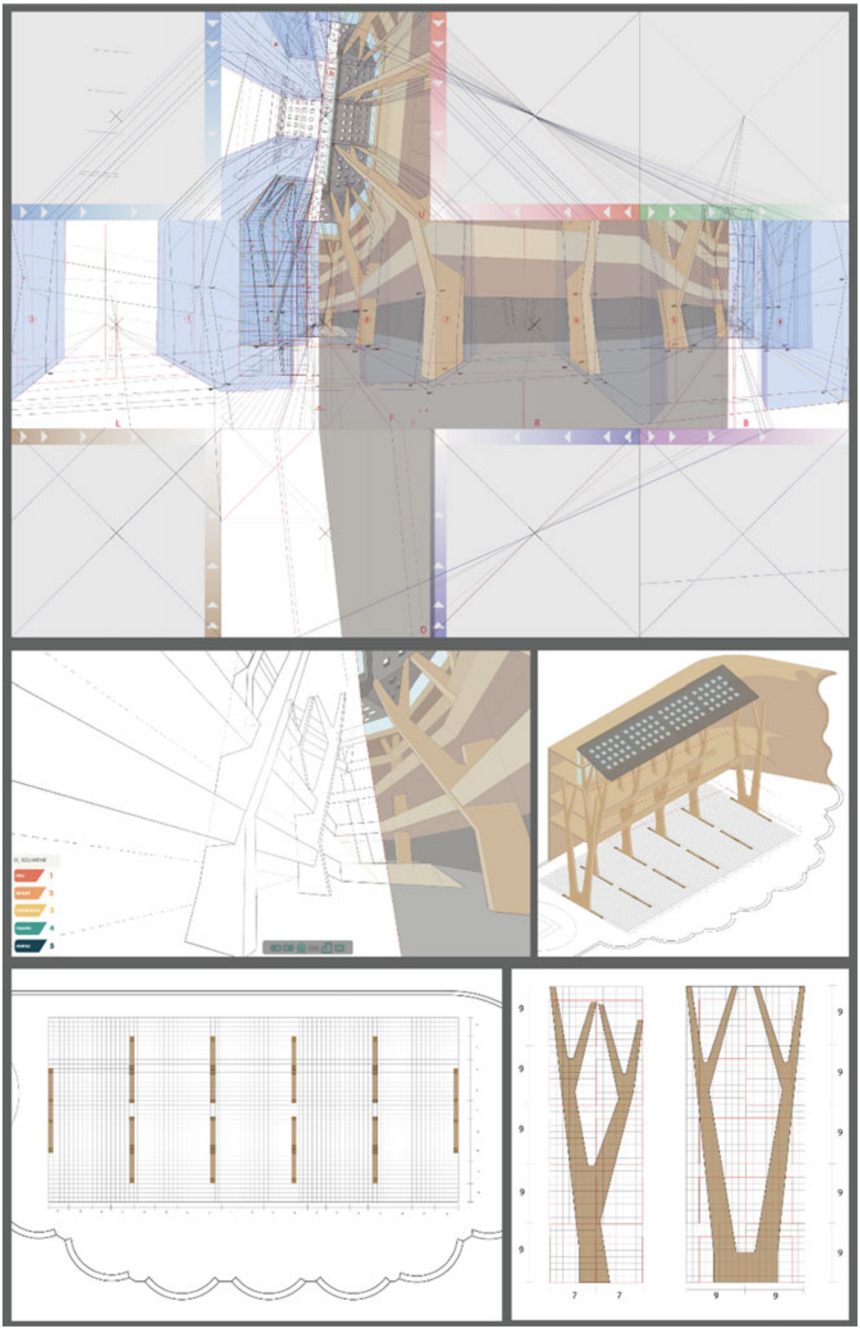

Fig. 5 Second synthesis exercise (survey culture) E-IX and E-X. Internal view of the Solimene factory. Modulation proposal in floorplan, section, in cubical perspective and immersive view of the final rendering drawing. Authors Assia D'Alessio, Teresa Di Palma, Caterina Crispino, Marta Campanile, Maria Petrillo, Lorenzo Villani and Lucas Fabian Olivero

analysis, which in the former case refers to cognition and in the latter case to ideation, “allows to approach the limit between what can and what cannot be done” [15, N. 3].

Finally, a communicative and interactive design project realized using Panotour Pro, presents the results of both sets of activities connecting panoramas, information and models (geometrical, graphical, etc.) thorough hotspots.

4 Deductions

We have shifted the observer from an external and centrally static position to a fully immersive one. The fixed axis and limited field of view of classical perspective has been replaced by a moving visual cone, like that of a camera rotating freely around a point. The traditional scheme bounded by a frame has become totally explorable thanks to the use of immersive devices (VR glasses).

The ambiguity generated between empirical space and theoretical space [4] prompts the association of ideas and knowledge whether one starts from a modular grid to derive a composition of elements that can be immersively (E-VIII) explored thanks to cubical perspective, or one starts from the immersive representation of reality to derive a composition grid (E-IX–E-X). The circularity that through drawing has always connected survey culture and design culture, is here re-proposed in a mixed form, that, unlike the previous model, mixes ‘the empirical’ with the ‘theoretical’ to emphasize that analogical and digital, far from being complementary or alternatives, are integrated into representation to offer a lever to ideation and cognitive thought [7].

Up to date with the times, the plurality of points of view identified and discussed with reference to Albrecht Dürer’s famous painting *Underweysung der Messung*’ (1525) take on a new light.

5 Expected and Achieved Results

The applications discussed in the paper have given us the possibility to verify at the operative critique level the theoretical bases of cubical perspective that we studied previously. The analyzed case studies have allowed us to introduce one of the most interesting aspects of our current research in the field of architectural representation.

The applications discussed in the paper have given us the possibility to verify at the operative critique level the theoretical bases of cubic perspective that we studied previously. The analyzed cas studies have allowed us to introduce on of the most interesting aspects of our current research in the field of architectural representation. Immersive fruition of environments constructed in cubic perspective or of environments rendered through its rules have been used as a system to explore, verify and communicate architectural composition exercises (design culture) and interpretation of the built environment exercises (survey culture). In alignment with the disciplinary tools and methods we analyzed the innovative potential of the hybrid model which combines the originality of the composition schemes or vice versa the survey drafts with their digital navigation. Thanks to its scientific nature, the comparable and verifiable procedure traces a further step ahead in the integration of methods and languages of graphic and of visual and multimedial communication. The procedure, in fact, offers the scholar a twofold opportunity: on the one hand, the possibility to exploit the originality and immediacy of by the graphical signs, and, on the other hand, the possibility to take advantage, from the speculative point of view, of the fruition efficacy of immersive techniques that allow the operator to experience digital scenarios. The model built using the studied procedure has advantages.

The first consists in allowing for immersive fruition of ideal spaced that do not require the addition of details and therefore 3D formalization. The second refers to that immersive fruition enhanced the study of architectural space and thus guides an intellectual re-foundation of what has been studied and classified.

How the method and the AR/VR applications can be further developed still needs to be verified. Amongst the examined possibilities there is the definition of algorithms for the construction or re-construction in cubical perspective of extracted and re-contextualized elements, so as to devise an automatization of landscapes starting from the sketches of the immersive environment or vice versa from the extraction of the relevant plans and elevations. The advocated software should ensure the user friendliness of a procedure which is to date restricted to a small circle of experts. It should be able to integrate the features of rapidity and interpretation of hand drawn sketches (necessarily analogical) with the archiving and management potential of data typical of informatics applications.

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The Fatal Birth of Architecture: The Obligation of Order

Renato Saleri

Abstract Creative processes in architecture and design are often accompanied today by a complete set of assistance tools inspired by biomimetic heuristics. Based on processes that took billions of years to set up, some formalisms now allow us to quickly explore vast areas of solutions. For example, they are able to optimize the optimum of a function in a reasonable time when there is no exact method or when the constraints given at an input are antagonistic. When not in a logic of multi-criteria optimization, one can also look for, among the innumerable creative supports offered by geometry or mathematics, those that will be best able to accompany the designer during the initial phases of conceptual exploration. We could of course rely on purely combinatorial logics while being sure to find—among the infinite possibilities offered by a universe number for example (Π, $\sqrt{2}\ldots$)—a solution to our problem but in temporalities that would go far beyond the life of the cosmos itself. Among the experiments conducted by the MAP-Aria laboratory for nearly 20 years, many have relied on bio-inspired heuristics, others have tested combinatorial logic—without much hope—and finally some combined the most recent advances in Computer Vision and shape recognition to try to understand the artist's gesture and the notion of reproducibility or ownership of an artwork. These crucial questions consider the essence of the creative process itself and search the remaining place of the human being in the reduced free space left by the "thinking machines" of today and tomorrow.

Keywords Digital generative tools · Formal grammars · 3D modeling

1 Context

Over the past twenty years, the considerable growth of digital tools has enabled the emergence of technologies capable of imitating and reproducing human behavior in an increasingly autonomous way. Initially conceived as artifacts capable of repeating tedious tasks over and over in order to give humans complete freedom to focus on

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more interesting activities, our contemporary societies are seeing the emergence of a large number of tools capable of assisting us in our daily actions. They are now equipped with behavioral autonomy and are increasingly able to make decisions for us. We've come a long way since the first automatons, designed to reproduce a sequence of predetermined actions, and the devices capable today of potentially replacing humans cognitive faculties. But although the idea of "artificial intelligence" already emerged in the early 1950s in Alan Turing's now famous paper "Computing Machinery and Intelligence" (Turing, 1950), it is certain that—even today—one wonders whether a machine is really capable of thinking.

> Poetry is not only in the verb. The poetry of facts is even stronger. Objects that mean something and are arranged with tact and skill create a poetic fact. [1]

2 The Fatal Birth of Architecture. the Obligation of Order [1]

Architects did not wait for computers to rely on formalisms able to accompany the creative process: "*The regulator layout is an insurance against arbitrariness. It is mind soothing*" [1]. Beyond this assertion, which in our opinion frames the act of thinking a little too closely, it is true that not everyone can or wants to systematically resort to the "*deregulation of the senses to reach the unknown*" (from A. Rimbaud, letter to G. Izambard—1871: in French: "*Il s'agit d'arriver à l'inconnu par le dérèglement de tous les sens*"). "*The regulator layout is a path; it is not a recipe. Its choice and its modalities of expression are an integral part of architectural creation […] by establishing the preeminence of the initiatory trace, bearing the seed of the form and capable of guiding it in its slow maturation towards its finalization*" [1]. The deep harmony of a work cannot, however, be summed up in the beams that seem to fit happily around the guidelines of its composition, and besides "*it would be quite possible, by multiplying the points of reference and without specifying the acceptable margin of error, to always find what one presupposes*" [2].

We must understand that the regulator layout is a mediator between the abstract space of geometry and the concrete space of the project. Set aside the countless contemporary ramblings that force the function within the form, for the sake of the purity of the mathematical or geometrical expression source of its birth. Many of them flourish abundantly on the covers of fashionable magazines, but very few of them delight the users who borrow them. Moreover, one cannot be extremely sure either that the final project—however finite it may be—has reached its full maturity: one could only be happy with a state of "satisfactory incompletion" that best meets the programmatic and functional constraints defined at a given time, but that is another debate…

Thus, if we consider N. Salingaros short essay "*The visceral experience of architecture: object affordance and our need to grasp our surroundings*", we can multiply the geometrical abstractions that will seduce the observer's gaze, but we will have to look elsewhere for the rules of composition that will allow us to satisfy our need for inner order:

> Human beings respond to spaces, surfaces, detail, and ornament viscerally, which determines how a built structure will actually be used independently of whatever the architect intended. [3]

But how to detach the reasoned perception of the architectural object from what could trigger a most profound accomplishment?

> By analogy, we can identify primary elements of architecture that are responsible for exerting the strongest influence of form and space on users. What we normally perceive as "architecture" is the surface of nested layers of a complex cognitive/response system. Less obvious but still primary (or primal) aspects of a building or a space influence our physiological and psychological responses: the success of a building or an urban space depends much more on these primal elements than on an intellectual analysis of its visible structure. [3]

And, according to Rudofsky:

> Without mentioning the first fifty centuries of its existence, historians present us with a well-ordered panorama of accepted and recognized architectures, which is as arbitrary a way of conceiving the art of building as if, for example, we were to trace the history of music back to the birth of the symphony orchestra. The forms of certain houses, sometimes handed down through a hundred generations, seem to be eternally valid, as are the forms of the basic tools. [4]

Let us here quote Voltaire who, with his famous sentence: "*Never use a new word unless it has these three qualities: necessary, intelligible, and sonorous*", denounces the gratuitous substitution of a word of use by another word which would have only the merit of novelty: "*it is not enriching the language*," he says, "*it is spoiling it*."

In architectural design, creativity is both a myth and a taboo. For a long time, many researchers have been interested in the inadequacy of design assistance tools in terms of creativity and autonomy. To quote J. P. Chupin who himself invokes the work of D. Shön:

> …architects are far from paying equal attention to process and product. If the introduction of information technology does not certainly increase the architect's creativity in his mission, most CAD software behaves like over-equipped drawing assistants: they presuppose both the maturity of the designer and that of the object of his design. To make full use of the potential of digital tools it is not enough to increase their ability to simulate materiality, but at the same time it is important to take over the relationship with the body they anaesthetize. [5]

The essence of the black box at the origin of the creative process cannot thus be altered other than by the mobilization of "*situations to think*", the only ones capable of stimulating the creative process by "successive jumps of intuition". Again, according to D. Shön:

This does not mean that computers are of no use, no assistance in design. Instead, we suggest that research should focus on computer environments that increase the designer's ability to capture, store, manipulate, organize and reflect on what they see. [5]

Beyond cognitive faculties, a question that arises today concerns precisely the ability of an artificial system to assist us in "creative" disciplines. Without wishing to supplant inspiration, we are now seeing the emergence of many tools capable of accompanying conceptual exploration, a fragile phase if ever there was one, because it comes from a set of cognitive processes that would be able to understand and produce an indefinite number of new processes. Serendipity, which is frequently used in the creative context to designate a form of intellectual availability, fortuitously brings rich teachings from unexpected discoveries or errors. Moravec's paradox establishes that often what is difficult in robotics is easier for man (and inversely, what is difficult for humans seems quite easy to computers…): we enter here into the dark space of a black box in which even the most optimistic predictions do not foresee an artificial intelligence supremacy before many decades. Let us consider instead the phenomena that are still poorly understood concerning the interpretation of intelligible data by the human brain, and in particular those with which, in our field, it is interesting to play. It is not a question of replacing creativity with an artificial cognitive process, but of understanding the levers by which a digital medium is capable of opening the doors of perception.

Malevitch's arkhitectons. From 1923 to the early 1930s, Kasimir Malevich produced several three-dimensional models, assemblages of abstract forms which appear similar to models of skyscrapers, called "arkhitektons". The drawings accompanying the construction of the models are called "planits". The arkhitektons are mostly white plaster models made up by several rectangular blocks added one to another. Usually a central bigger block is the main compositional element and smaller parallelepipeds are progressively added to it. No function is shown or translated into form, the final shape being the pure result of assembling abstract masses vertically or horizontally. With their spatialization of abstraction and their formal non-objectivity, the arkhitektons embody Malevich effort to translate the suprematist principles of composition to three-dimensional forms and architecture.

> In a series of prismatic, quasi architectural sculptures (which he called 'Arkhitektons') [he] sought to demonstrate the timeless laws of architecture underlying the ever changing demands of function. (…) Malevich's Arkhitektons resemble early De Stijl compositions in which ornament is non-figural and 'form' and 'ornament' are differentiated only by scale. These studies are purely experimental and the buildings have no function and no internal organization. [6]

Responding to the issues stated by Malevitch in his supremacist manifesto, these formal games not only appeal to interpretative shifts due to their plastic ambiguity but also herald—perhaps unintentionally—the rise of those recursive formalisms that today are called "fractals".

> At the same time, we are discovering that traditional wisdom embedded in the built environment contains many of the design answers we now seek. Our ancestors who built towns and cities had an intuitive idea of which environments were more accommodating emotionally, and more healing (Alexander, 1979; Alexander et al., 1977). The tools they used to evaluate them were their own direct senses. [3]

The tools we manipulate for conceptual support should therefore be more attuned to the deep resonances of our sensory biology. We can sense that juxtaposed relative quantities or judiciously superimposed tracings are at the origin of unexpected perceptive phenomena—such as optical illusions for example—or visually satisfying phenomena; this perception seems to be universally shared by all observers since time immemorial. Certain harmonic relationships, whether musical or geometric, arouse common feelings in most human beings and it has been shown that certain sensory stimuli seem to favor the development of plant species.

The observation of the growth phenomena of living organisms can thus largely inspire generative processes designed to be attuned to their ecosystem, whether natural or artificial. In this, we can draw on the cumulative experience of thousands of generations that have succeeded in adapting their evolution to the constraints of the environment in which they evolved. Is it not a universal mechanism to be able to adapt to the environment around us? Bjarke Ingels, from the BIG agency, sees the possibility of a kind of "**architectural Darwinism**" emerging within the decision-making groups that bring the idea to its concrete conclusion. Let us broaden the field of application of this fundamental notion and imagine that all the solutions generated by humanity to respond to the constraints linked to its evolution are evaluated, cross-referenced, hybridized or discarded before being adopted and put to the test of time.

> Long before the advent of architects, architecture emerged from the place, the time and what you have on hand to build it. It is more a form of community production, not the product of a few intellectuals or specialists, but of the spontaneous and continuous activity of a whole people, custodian of a common heritage and obedient to the lessons of a common experience. [7]

More interestingly, the survival of an idea will depend on its capacity to undergo during its existence a set of transformations likely to harmonize its functioning with the transformation of uses. If we want to generalize this concept, we could say that this is valid with regard to the "phenotype"—an object, a form or a building capable of surviving and "reproducing" (being imitated, or even copied)—as much as the "genotype" that is at the origin of it: an idea, a function or a path of thought or debate that remains and remains applicable whatever the circumstances. The very persistence of an aesthetic expression—a baroque and emergent expression of form—may even be at the origin of its survival when its functional support has disappeared. All this generates an infinity of persistent figures and signs that in turn shape our predisposition to find in them a subjectively personal and familiar meaning.

3 Bio-Mimetic Processes and Optimization of Generative Strategies

Combinatorial strategies, although capable of creating an infinity of formal solutions, are not able to produce interesting solutions in a temporality compatible with human existence. According to this point, current research focuses on those mechanisms that have regulated the generative processes of our biosphere for billions of years. Some of them were theoretically described several decades ago and still constitute today a solid exploratory basis for the ongoing investigations in this very field. For example, L-systems—described during the early 60' by the Hungarian biologist, Aristide Lindenmayer—make it possible to model in space and time some growth phenomena that mimic the growth dynamics involved, e.g., in plants evolution.

> Designers long have dreamed of buildings that behave like living things. Frank Lloyd Wright defined "organic architecture" as "building the way nature builds." In 1963, Archigram envisioned a "Living City"—community as organism. And now the Cascadia Green Building Council has issued a Living Building Challenge as the next stage of evolution toward "true sustainability." The challenge: "Imagine a building designed and constructed to operate as elegantly and efficiently as a flower." [8]

The recursive and auto-similar properties of their structure will allow them to be displayed with incremental levels of detail able to produce evolutionary shapes that model, for example, the transformation of an architectural ensemble built over time. L-systems are based on axioms and rule mechanisms whose formal expression is easily applicable to constructed elements with redundant and self-mimetic characteristics that actually imitates evolutionary process with a recursive regeneration of auto-similar patterns (Fig. 1).

Experience shows that the structured use of shape variables with a proper set of geometric transformations make it a very efficient 2D and 3D generator. Applied to topological germs specific to architecture, an infinite number of formal varieties can be obtained. Ongoing experiments aim to demonstrate the formal versatility of this model in generating the most disparate morpho stylistic varieties.

> But how does a flower grow? It might be time to shift the conversation from product to process. What if buildings could be created in the same way a cell develops into a plant—from the bottom up instead of the top down? Technology may point the way. Automated processes are changing every aspect of design and construction, and it's only a matter of time before self-assembly completely takes over. [8]

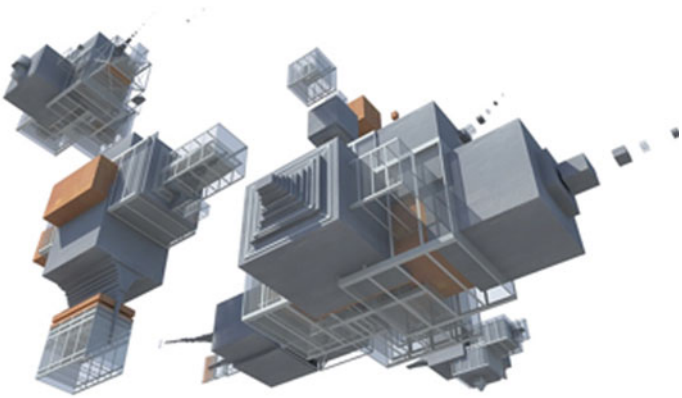

Fig. 1 Author's early L-System generator (2008)

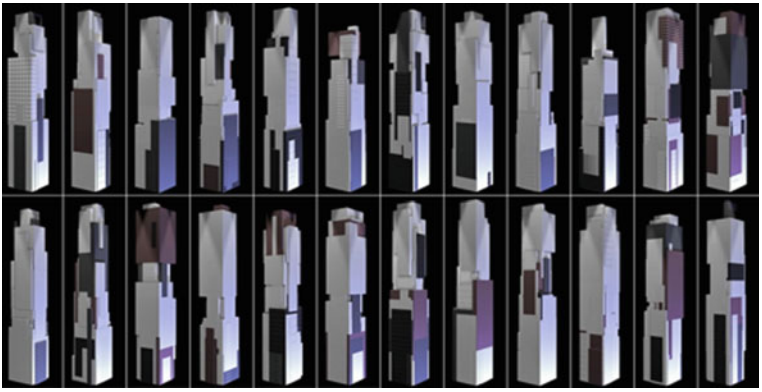

Fig. 2 The Arkhology generator – Biennale d'architecture de Lyon (2017)

This formal distopia resonates with the issues related to the use of combinatorial formalisms that have interested our research since the beginning. Following the chimera described by Borges in his famous account "The Library of Babel", it is tempting to imagine a device capable—in the field of architecture—of sweeping away, according to a combinatorial logic, all past, present and future architectural production of mankind. Of course, we are not at all in the same generative paradigm and it is highly likely that the vocabulary elements involved in such a generator would exceed the 24 characters used to populate the Borges library (Fig. 2).

A very modest attempt was made by the MAP on the occasion of the first Lyon Architecture Biennale in 2017, during which we left a machine running for only 5 days during the event and which was responsible for randomly producing a mapped projection on the two pillars surrounding the reception area. During the exhibition, nearly 300,000 digital arkhitektons were produced in pairs, most of which went unnoticed, either because of the inattention of the spectators or because the projection took place at night during the closing of the venue.

> The 'un-authored' object idea; objects formed out of chance and accident, that are then noticed and valued in equivalent ways to the art object is theoretically interesting because it questions the technical relationship between seeing and creating, and more broadly the dynamic of the viewer and the artist. [9]

We like the idea of these machines working tirelessly to produce images that no one will be able to see. A little in this logic, we have reproduced an artistic online installation that already dates back about ten years: a matrix of pixels turned on or off according to a binary logic; when a pixel has finished its cycle (turned on and off) the adjacent pixel changes state. For practical reasons all pixels are disposed in a square matrix, the logic remaining the same for the "line" pixel as for the adjacent pixel. According to this very simple rule, the occurrence of the state change for each pixel of the matrix will depend on its position in the row and will follow an exponential temporality. Nevertheless, left alone, this system will sweep away all existing binary

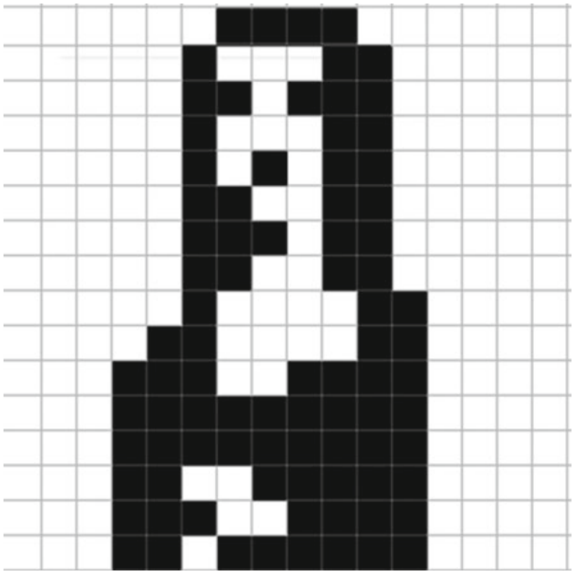

Fig. 3 One amongst the 2^{256} display possibilities of a 16×16 b&w px matrix

possibilities and will eventually produce some happier solutions beyond a universe of meaningless configurations (Fig. 3).

Following the same logic, we have developed and experimented with a tool displaying a numerical sequence in the form of grey pixels distributed on a matrix whose l/h ratio can be modified. This tool makes it possible to easily highlight the periodicity of a real number: the sensitivity of the eye to moiré phenomena makes it possible in this precise case to appreciate the repetition of the decimals by simply observing the image produced. By modifying the l/h ratio it is even possible to vary the aspect of the display and thus to look for more interesting configurations. Is there however a hidden order in the series of decimals that make up the substance of irrational numbers? We could certainly find, following the example of the quest of the Aleph according to Borges, the sequence of numbers forming any past, present and future Instagram© picture but still it would be necessary to locate it in the infinite sequence of PI decimals: we would not have more probabilities to find its location than to find the novel "Les Misérables" by Victor Hugo in the famous library mentioned previously.

Without looking so far, it is possible to find in the already mentioned L-Systems harmonic phenomena that occur after a large number of iterations. Are they due to the particular recursive properties of these formalisms on which all types of rules can be applied? (Fig. 4)

To do this, it is necessary to act with great finesse on the factors applied recursively in order to observe unexpected alignments when the major axes of composition approach particular angular ratios: it is fascinating to note to what extent the recursive functions applied to geometry are capable of modelling the formal expression of a large number of more primitive plant and animal species. Is the initial search for

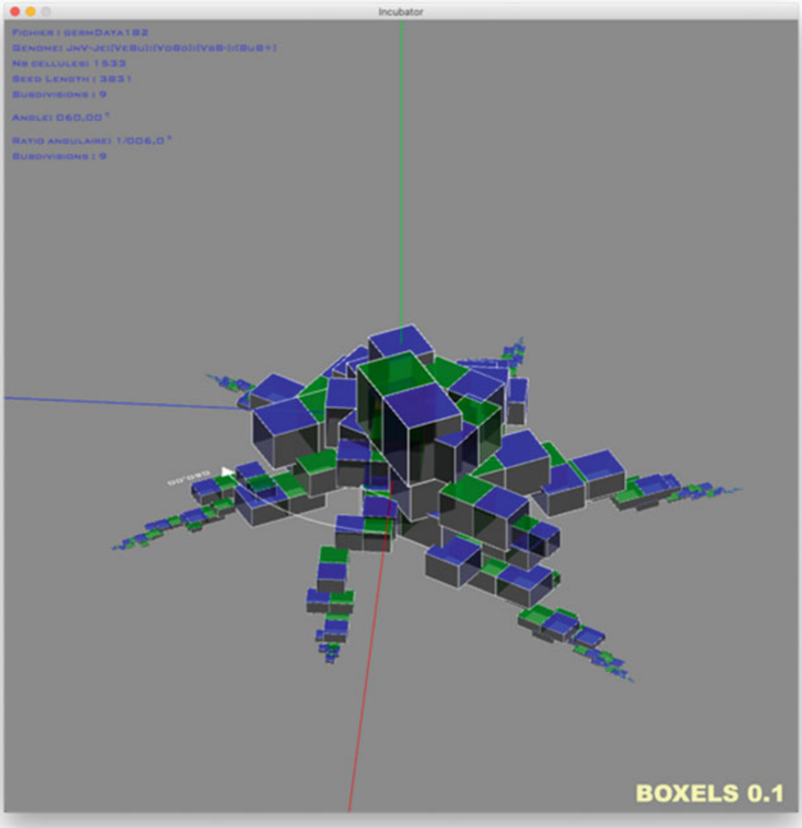

Fig. 4 Combining a GA approach with a L-System generator (2020)

adaptation to an unfamiliar environment simpler when one adopts star-shaped or symmetrical forms? This is an interesting avenue for architecture. Ernst Haeckel's plates artfully illustrate the impressive beauty of the biological world, a beauty that expresses itself in our eyes but which has no other reason in the end than to offer species the tools to survive and procreate.

4 Conclusion

Recalling Nature with efficient generative paradigms seems to be relevant to discriminate the exponential spread-out of possible solutions of un-controlled growth approaches. However, the drawback of such processes is in its unpredictability or its poor response to domains where it is hard or impossible to define a computational fitness function. Interactive Genetic Algorithms (IGA) or Aesthetic Selection uses human evaluation for the fitness function, typically when the form of fitness function is not known, such as visual appearance or aesthetics evaluation. It is so possible to use well-established mechanisms that have been experimented by nature for billions of years and that have produced—needless to say—workable results in many areas. Well implemented in today's 3D tools, some inspired organic formalisms are now used in many fields: although they certainly do not deploy the same functional complexity as their living counterparts, they are extremely gifted at optimizing multi-criteria problems, supervising monitoring operations or assisting in operational decision-making. As said before, it is no longer necessary to go through the tree of possibilities in its entirety, we will be able to make drastic shortcuts in the production of optimal solutions considering a set of constraints placed at the beginning and according to a time span more compatible with the duration of our own existence.

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Design of Individualized Digital Activities Fostering Strategic Planning in Linear Algebra

Agnese Ilaria Telloni

Abstract In this paper we present the design of the IPSE (Individualized Planned Strategy Environment), an online environment aimed at fostering the strategic planning competence for solving problems in linear algebra. The IPSE is composed by connected digital activities that guide university students in designing a plan and executing its phases by means of procedural steps theoretically justified. A peculiar feature of the IPSE is the individualization of teaching/learning, pursued by specific feedback provided within the activities and methodological choices left to the student. Moreover, we report the outcomes of a pilot study carried on with first year engineering students: relying on the notion of *aletheic component of rational behavior* [10], we classify some different expressions of it arisen in written problem solving processes by students who worked with the IPSE.

Keywords Strategic planning · Individualization · Digital tasks · Linear algebra · Components of rational behavior · Aletheic component of rational behavior

1 Strategic Planning and Linear Algebra

Linear algebra is one of the first courses attended by beginners enrolled in scientific subjects. It is a highly demanding field of knowledge, and it requires students to master an advanced mathematical thinking and an increasing level of abstraction. Freshmen often feel disoriented when taking part to the first lessons about vector spaces. This disorientation is due to the perception of a big discrepancy between mathematics previously learned and linear algebra, either from the content point of view and by the methodology and the language used [1, 2].

As widely recognized [3–5], linear algebra and geometry are characterized by an intertwinement between conceptual and procedural knowledge; as a consequence, the full understanding of the subject can be achieved only when these two dimensions are integrated each other and are perceived, according to Sfard [6], as two sides of

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the same coin. When facing linear algebra, students have to coordinate different semiotic registers, namely the geometric one, the symbolic one and the abstract one [2], that correspond to complementary viewpoints. Moreover, students should be able to integrate a local perspective, allowing to use mathematical objects like vectors or homomorphisms and to make operations between them, with a global perspective, according to which they are elements of a certain algebraic structure. As a further peculiarity, linear algebra envisages a multiplicity of methods to solve a problem, since the involved concepts can be regarded in many ways (for example, a matrix is an object itself, but also the representation of other objects like a homomorphism, or a scalar product), and the same computation takes a lot of information (for example, the rank of a matrix A is equal to the maximum order of a not null minor of A, but also to the dimension of the vector space spanned by the columns/rows of A).

With respect to the mentioned difficulties, sometimes students try to take a shortcut and avoid the complexity of the subject, so they learn procedures for solving standard exercises, without control on the meanings involved [1, 7]; in other cases, they display productive uses of computational thinking and procedural flexibility, not only to apply algorithms, but also to justify and interpret them [3, 8].

According to these aspects, it is especially important, in linear algebra and beyond, that students develop the competence of strategically planning the solving process of a problem, as well as the attitude of connecting the available knowledge with the aims to be reached; moreover, the plan is more likely to be effective the more it connects the desired goal with past experiences and skills at disposal of the individual.

In order to combine the need of nurturing the strategically planning competence and a didactic attention to the individual learning needs, attitudes and preferences—particularly important for freshmen, often feeling a sense of massification at the beginning of their university career [9] —we designed and implemented a digital environment where students are supported in problem solving about topics of linear algebra: we named it IPSE (Individualized Planned Strategy Environment). The IPSE consists in a set of connected digital activities aimed at guiding university students in solving a problem by designing a plan and implementing its steps, theoretically justified, by means of suitable procedures.

In a recent study [10] we described the rationale of the IPSE and presented the outcomes of a pilot study carried on with engineering freshmen of university of Salerno. In particular, we analyzed the evolution of students who interacted with the IPSE and their use of components of rational behavior, according to the notion introduced by Habermas [11] and adapted to the activity of proving in mathematics [12]. One of the main results of that study is the arising of a new component of rational behavior, which we called *aletheic* for its dynamic feature recalling the action of removing a veil.

In this paper we review the structure of the IPSE, focusing on some design choices and the related implementation details; moreover, we report and discuss the different ways in which the aletheic component of rational behavior emerged in productions of students who worked with the IPSE.

2 Theoretical Framework

In this section we recall two elements which guided the design of the IPSE, as well as the analysis of the students' productions: they are the individualization of teaching/learning and the construct of rational behaviour by Habermas, adapted to mathematics education and extended by means of the aletheic component.

Individualization is here intended as the differentiation of learning paths in order to enable students to reach common objectives [13]; it is considered as a key teaching/learning feature for the success of the educational process. Unfortunately, it is very hard to be pursued at university level, because of some logistic conditions of the context, at least in Italy, like the deep differences between the students' backgrounds and the high number of students per course. In this respect a crucial contribution for increasing the individualization of teaching/learning at university level would come from supporting traditional courses with e-learning resources [14–18]. The integration of technology within the educational processes may bring advantages to both students and teachers. Students can access the resources when they prefer accordingly to their own learning rhythms, they are stimulated by different learning channels, they also can receive response-specific feedback that scaffolds their work and supports self-assessment; as a consequence, they have the possibility of becoming progressively more and more autonomous and responsible of their own learning. From the teacher's viewpoint, a careful use of technology allows to keep track of the students' work and to follow their evolution, so as to propose them individualized remedial or advanced activities; moreover, both the formative and summative assessment can be simplified.

The other reference of our theoretical background is the Habermas' construct of rational behavior [11]. The Habermas' model envisages three components of the rational behavior: an *epistemic component,* concerning the knowledge and its inner connections, a *teleological component,* concerning the actions performed towards a goal and a *communicative component*, concerning the choices of means of communication to reach an agreement with the target community. This model has been adapted to the mathematical activity of proving theorems [12] as follows: the *epistemic* component concerns the validation of statements based on shared premises and right ways of reasoning; the *teleological component* is related to strategic choices to achieve an aim, as a particular case of problem solving; the *communicative* component regards the effective sharing of steps of reasoning.

In Albano and Telloni [10] we analysed students' problem solving processes in terms of components of rational behaviour. As a remarkable result, a further component of rational behaviour arose from the students' productions: we called it *aletheic* from the Greek word *aletheia*, meaning "truth" as "not hidden", according to the etymology. In the quoted study, we gave the following description of the aletheic component of rational behavior: "This component is characterized by a dynamic integration between what is available from the performed calculations and the final aim; it manifests in a continuous search for a meta-control on the evidences following from the implemented procedures and for a strategic use of them with respect to a specific goal. […] The new component of rationality puts the epistemic and the teleological ones together, moving continuously back and forth from the available knowledge […] to the set goal, and interpreting the former in light of the latter.”

The *aletheic component* of rational behaviour seems to be very strongly linked with the competence of strategically planning, since it acts to realize a trade-off between what is at disposal (theoretical knowledge, specific available data, …) and what should be achieved.

3 Materials and Methodology

We used the software Geogebra for implementing five different IPSEs, focused on various topics in linear algebra:

- IPSE 1—basis of a subspace spanned by a set of vectors;
- IPSE 2—Cartesian equations of a vector subspace in $\mathbb{R}^3$;
- IPSE 3—Cartesian equations of a vector subspace in $\mathbb{R}^4$;
- IPSE 4—kernel and image of a homomorphism;
- IPSE 5—linear systems.

The IPSEs have been submitted to engineering freshmen of university of Salerno attending the face-to-face course “Geometria, Algebra e Logica”. The course was taught during the spring semester of the academic year 2018-2019 for an amount of 48 h of lectures and exercises sessions; the themes treated concern logic, linear algebra, geometry of the Euclidean plane and space.

Along the course, the IPSEs have been delivered to students through the university Moodle platform, so as they can access them in their own pace and without time constraints. Through the same platform some open-ended problems have been periodically handed out to students, with the request of solving them, justifying the reasoning steps and uploading the solving processes on the platform as photos or text documents up to four days from the publication. The timeline and the topics of the IPSEs and of the problems (indicated with a P) are shown in Fig. 1.

For the present study, in order to analyse the individual student’s evolution in dependence on his/her work with the IPSEs, we considered data of two kinds: a) the number of accesses he/she made to the IPSE resources; b) his/her solving processes to the open-ended problems proposed along the course.

In particular, we selected the students who accessed at least four times to the different IPSEs and uploaded solving processes of at least two of the five submitted problems, one of which among the last two problems (about homomorphisms and conics). This methodological choice was due to the will of evaluating a possible evolution in the students’ behaviour and of correlating it with their interaction with the IPSEs. For these students we analysed the whole set of their productions in terms of components of rational behaviour. In this study we focus on the aletheic component of rational behaviour, displayed by the students’ comments suggesting their back and forth moving between the information at disposal and the goal to be reached.

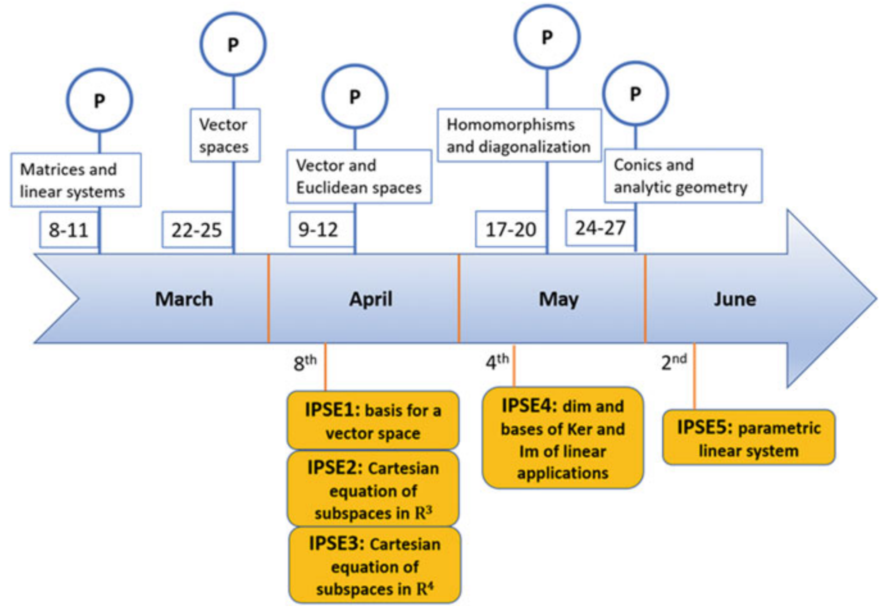

Fig. 1 The timeline of the distribution of the IPSEs and of the problems

4 Design and Implementation Details of the IPSE

In this section the essential elements of the design of the IPSE are explained, together with some implementation details.

4.1 The Structure of the IPSE

The IPSE consists in a route of digital activities designed to assist students in solving a problem by identifying and following a suitable plan: in detail, the plan envisages *sub-problems* to be solved by means of e*lementary step*s, theoretically justified, that can be performed in many ways through *procedural steps.* For these notions we provide the following operational definitions, being aware that they cannot be unambiguous and depend strictly on the designer's choice and on the supposed level of knowledge of learners the IPSE is addressed to. We call *sub-problems* intermediate targets, the achievement of which allows to solve the initial problem; the *elementary steps* are goals, that can be regarded also as procedures, needed for solving a sub-problem; the *procedural steps* are algorithms producing as outputs the goals indicated by the elementary steps.

The structure of the IPSE is shown in Fig. 2: the learning path begins with the activity A1 and each subsequent activity becomes accessible when the previous

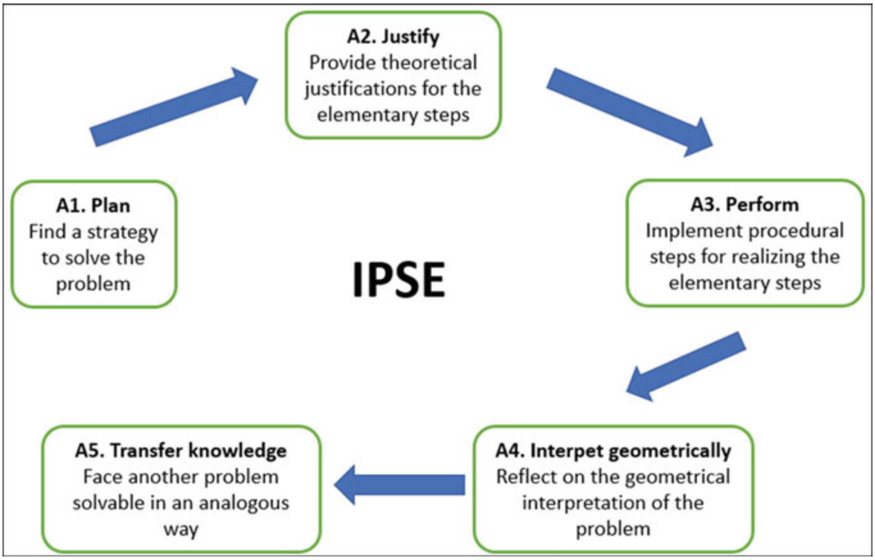

Fig. 2 The structure of the IPSE

one has been correctly performed. Within each activity, when the student submits his/her answer, an immediate feedback is provided; in case of wrong answer, it is a facilitative and response-specific feedback [19], mainly in the form of a guiding question or remark, aimed at bringing the student to think about the given answer and to find him-/herself the mistake made. According to the levels of feedback discussed by Hattie and Timperley [20], the IPSE provides feedback about the task, about the process and about self-regulation. Let's look the activities of the IPSE in detail:

A1. **Plan**. The first activity is focused on the identification of a plan to solve the problem. An ordered list of sub-problems and a not-ordered list of elementary steps are provided; the student is firstly required to tidy up the elementary steps and then to associate each sub-problem to the elementary steps it corresponds to.

A2. **Justify**. This activity concerns the theoretical justifications to the elementary steps. The ordered lists of the sub-problems and of the elementary steps are provided; the student is required to link each elementary step with the theoretical result justifying it.

A3. **Perform**. This activity guides the student in the implementation of the elementary steps. The student has at disposal the ordered lists of sub-problems and of elementary steps; he/she is required to choose subsequent procedural steps, according to the different available methods. Some of the proposed procedural steps are correct and some are wrong; the wrong procedures are chosen on the base of the common misconceptions on the topics at stake.

A4. **Interpret geometrically**. In this activity a geometrical interpretation of the initial problem is provided; the student can explore it by dragging and rotating the graph and by viewing separately the mathematical objects. This is a meta-level activity [1], aimed at favoring the understanding of the connections between the algebraic, geometric and abstract registers of linear algebra.

A5. **Transfer knowledge**. In this activity a problem different from the first one is proposed; some remarks induce the student in understanding that it can be solved by applying the knowledge just grasped. This activity is of meta-level kind; it aims at supporting students in developing a unifying and structural vision of linear algebra, by means of a trans-object level of thinking [1].

Within the IPSE, the individualization of teaching and learning is pursued at different levels. The IPSE stimulates students according to different learning channels and registers (symbolic, verbal language, graphical). Each activity requires students to make choices, according to their own skills and preferences; moreover, the program adapts itself to the single student's behavior: it evaluates as correct those answers that are consistent with the choices he/she did and the computations he/she performed. Another feature of individualization is the response-specific and facilitative feedback, designed so that it has impact on the individual's difficulties and learning needs.

In the rest of the section, we go through the IPSE4; because of space limitations, we describe the implementation details of activities A1 and A4 included in it.

4.2 *Examples of Activities of the IPSE4*

The problem of the IPSE4 is "*given a homomorphism* φ*, find the dimension and a basis for* Ker(φ) *and* Im(φ)" (Fig. 3). We identified the following sequence of sub-problems allowing to solve the problem: (1) determine $k = \dim(\mathrm{Ker}(\varphi))$; (2) find a base B for Ker(φ); (3) determine $h = \dim(\mathrm{Im}(\varphi))$; (4) find a base H for Im(φ). Corresponding to them, we planned the elementary steps shown in the yellow boxes on the right-side of Fig. 3, suitable theoretical results justifying them and procedures allowing the achievement of the elementary steps.

When the student accesses the activity A1 (Plan), he/she sees the screenshot in Fig. 3, where the ordered list of sub-problems is provided and he/she is required to tidy up the elementary steps by dragging the yellow boxes to the list with blue bullets at the center of the screen. If a wrong answer is submitted, a facilitative feedback appears, which can address technical issues (*Be careful! You did not insert some blocks in the list, or they are not well centered*) or conceptual issues (*How can you find* dim(Ker(φ))*?What is the common characteristic of the vectors in* Ker(φ)?). The feedback messages are designed so that they address one mistake at a time, starting from the possible technical troubles to the conceptual ones, according to a bottom-up scheme (for example, if two blocks are ordered in a wrong way in the list of elementary steps, a message appears, addressing only the mistake about the first wrong block with respect to the correctly ordered list). When the student submits the correct answer, it is furtherly required to associate the sub-problem to the elementary steps it corresponds to. Also in this case a facilitative, response-specific

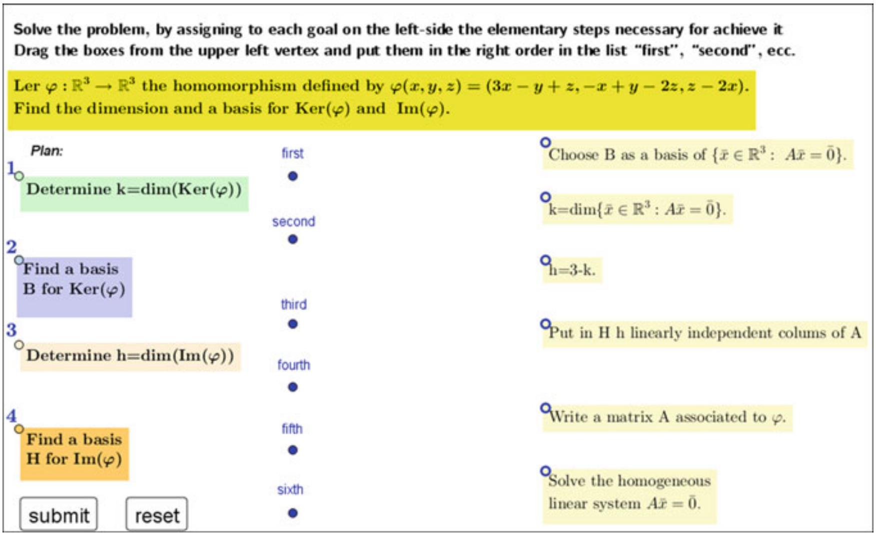

Fig. 3 The opening screenshot from the activity A1 (Plan) of the IPSE4

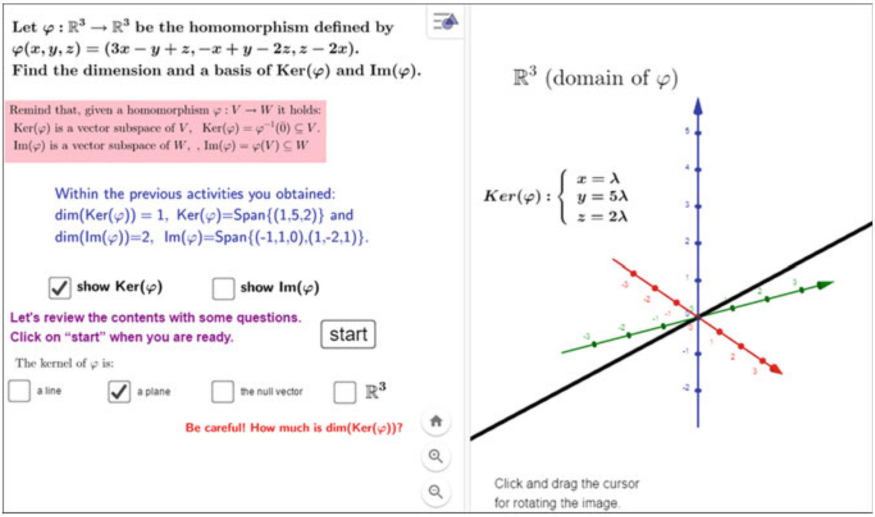

Fig. 4 A screenshot from the activity A4 (Interpret geometrically) of the IPSE4

and bottom-up feedback is provided; it addresses the task, the process of the task and the self-regulation.

In the activity A4 (Interpret geometrically) the screen is subdivided in two parts (Fig. 4). On the left-side, from the top to the bottom, there are: the text of the problem; a remind about the definition of Ker(φ) and Im(φ) and their property of being vector subspaces; a summary of the results yet obtained in the previous activities of the IPSE4; two check boxes “show Ker(φ)” and “show Im(φ)”; a “start” button with an invitation of clicking it for facing some review questions. On the right-side there is a 3D Cartesian system that can be dragged and rotated to improve the visualization.

When the student clicks on the “start” button, the first one of a sequence of multiple-choice questions about the homomorphism φ appears; when it is correctly answered, the second one appears and so on. If the student submits a wrong answer, as shown in Fig. 4, a guiding question appears, either addressing the already obtained results (*Are you sure? How much is* dim(Ker(φ))?), or inviting the student to check the consistency between his/her own assertions.

The two check boxes on the left-side of the screen are always available and allow viewing the geometrical representations of Ker(φ) and Im(φ), so as the student can use them either as a guide for correctly answer the questions, or as a confirmation of the given answer, or as a further resource at the end of the whole activity. The check boxes can be activated one at a time and have been designed so as favoring the linking between the algebraic and the geometric representations of the Ker(φ) and Im(φ), when φ is an endomorphism of $\mathbb{R}^3$.

5 Outcomes of the Study and Analysis

In this section we consider the solving processes of the open-ended problems submitted along the course by students who interacted with the IPSEs; in particular, we focus on the emergence of what we called aletheic component of rational behaviour in their productions [10]. In our previous study we focused on the students’ evolution for what concerns the capability of strategically planning for solving a problem, according to their use of the epistemic, teleological and communicative Habermas’ components of rational behaviour. In the present study we extend and deepen the analysis of the aletheic component of rational behavior, trying to give, as a new result, a possible classification of its manifestations (Table 1).

The aletheic component of rationality emerged when students relyed on an *a posteriori checking of the outcomes of computational procedures*. Typically in this case students performed computations and finally compared the obtained results

Table 1 Different manifestations of the aletheic component of rational behavior

Manifestation of the aletheic component	Typical sentence
A posteriori checking	“It was expected that…”
Prefiguring results	“It can occur that…”
Avoiding useless computations	
– at the beginning of the problem solving process	“I know a priori that…”
– after some computations	“I already conclude that…”

with theoretical facts; in other cases they verified the appropriateness of the found solution. An example of this manifestation of the aletheic component is the transcript of S1, who was asked to study the diagonalizability of an endomorphism associated to a parametric matrix A. S1 found that A is symmetric for a certain value of the parameter, for which he performed suitable calculations and obtained that A has three real eigenvalues. Finally, S1 commented "*It was expected that all the eigenvalues would be real for the Spectral Theorem*", so displaying an attempt of crossing the results of calculations with other information at disposal, by means of a regressive movement.

Some students displayed the use of the aletheic component of rational behavior by *prefiguring results of computations and their interpretations*, so activating forms of anticipatory thought [21]. Students generally proceeded by reducing the given problem to another one or by decomposing it through a "*divide et impera*" approach (in tune with the philosophy of the IPSE), often shifting from a semiotic register to another one. Example of this aspect are the transcripts by S2 and S3. The former was asked to establish whether a vector v belongs to a subspace S; before making any calculations, he wrote "*if v would belong to S, then I would have that rk(B v) = dimS*", where B was the matrix having as columns the components of vectors of a basis for S. The latter was required to determine the mutual position of two planes (given in Cartesian equations) in the Euclidean space; he said that the problem could be solved by comparing the ranks of the incomplete and the complete matrices associated to the linear system of the equations of the planes and listed all the possible cases, giving the related geometrical interpretation. In this case a sort of forth moving can be highlighted, which reveals an attempt of metacognitive control on the spectrum of the possibilities that can happen.

Some students *avoided pleonastic computations*, becoming aware that the information at disposal in a certain moment were sufficient to give the required answers. This understanding arose in different moments: in some cases at the beginning of the problem solving process students noticed that provided data, connections between available facts, possibly together with theoretical results allowed them to give the required answer without implementing procedures; in other cases, after some calculations, they realized that it was not necessary to conclude them, since the desired information had already arisen. An example of this behavior is that of S4, who was required to determine the dimension of the kernel of an endomorphism the eigenspaces of which he had already calculated; at the beginning of his problem solving process he wrote "*I know* a priori *that* dim(KerL) = *1, since it is the geometric multiplicity of the eigenvalue 0 of L*", so avoiding any calculation. Examples of stopping calculations at an intermediate step are the behaviors of S5 and S6. S5 was required to say if a point belongs to a line in $\mathbb{R}^3$, so she set up the substitution of the point coordinates within the equations of the line, but stopped the computation when she saw that one of the equations was not satisfied; she wrote "*I already conclude that P does not belong to the line r*" and chose not to solve the other equations. In an analogous way, S6, having been required to establish if a vector v belongs to the orthogonal complement of a vector subspace W, wrote a linear system where the scalar products of v with all the vectors of a basis for W were set equal to zero; she did not conclude the computations, since he noticed that in an equation the scalar product has all positive addends, hence it cannot be 0. He commented that "*since the first equality does not hold, it has not sense verifying the other ones; I conclude* $v \notin W^{\perp}$". In this case the arising of the aletheic component of rationality reveals a continued back and forth moving and a strategic attitude by the students, who aim to reach their own goals as quickly as possible.

6 Conclusive Discussion

This study discussed the design and the implementation of the digital environment IPSE, aimed at supporting the students' development of the strategic planning competence. Productions of engineering freshmen who worked with the IPSE have been analyzed with specific focus on their use of the aletheic component of rationality [10]. As a result, we identified three categories of manifestations of the aletheic component, each one involving a dynamic approach to problem solving and displaying an interplay between theoretical knowledge, provided data and goals. These findings link with the concepts of mathematical *flexibility* [1, 22] and *versatility* [23].

According to our interpretation, the students' evolution in using the aletheic component of rationality is strongly linked with their interaction with the IPSEs. In fact, the aletheic component arose in students' productions later than their work within the IPSEs and mirror the learning experience and the steps envisaged by the IPSE. In particular, students' activities and productions reflected the change of register addressed by the meta-level activities (A4 and A5) of the IPSE; moreover, their a priori analysis of the cases that can occur, as well as the a posteriori control of the appropriateness of the found solutions correspond to the focus of the IPSE on the consistency between calculations and the meanings of mathematical concepts.

Further theoretical research would be necessary to fully comprehend the epistemology of the aletheic component of rational behavior and possible implications for teaching and learning; moreover, educational experiments on large and heterogeneous samples could allow a correlation between the manifestations of the aletheic component of the rational behavior and mathematical contexts or activities the students are engaged in.

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Measures

Daniela Velichová

Abstract Paper brings few reflections on attempts to confront mathematical and artistic approach to measuring various aspects of creativity, imagination and our understanding of aesthetic values hidden in the produced fine art and mathematical artefacts. Phenomena that are carriers of these subtle values enable comparison of the concepts of measures and measurement strategies in maths and art. Short revision of the surface measure definitions and application within the two environments is presented, together with continuous efforts of finding their common features and differences on some examples of abstract visual art and its geometric relation to measure theory in mathematics.

Keywords Mathematics · Geometry · Art · Measures

1 Parallels of Mathematics and Art

Mathematics and art are two highly imaginative intellectual domains of human activities with no strict boundaries dividing sphere of competencies for one or another. In both, such outstanding abilities of human brain as abstraction, imagination, comparison, analysis and synthesis of pieces of knowledge and perceptions, generalization, intuition, resourcefulness, aesthetic feelings or creativity are essential and obviously used and developed. Ability to convey a message on experienced and purposely investigated events in the form of intellectually elaborated judgments and emotionally touched up pieces of art is a unique ability of mankind. This enables us to express, exchange, share and reclaim outcomes of mental activities of individuals for the advantage of the whole community. Methods of these two distinct branches of systematic intellectual activities of mankind are different, and their working approaches differ. But the level of abstraction, timeless validity and recentness of achieved results and articulated values of the conveyed messages of these abstract disciplines are comparably important pieces of information about perception and understanding of

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our world, in which we are doomed to live, work and create. Products of these seemingly antithetical domains represent evidence of the level of development of human society, as they reflect the society latest achievements in the never-ending efforts for finding and perception of truth, searching for it and understanding it as a backbone of "knowing & feeling" undoubtedly present in harmony in both mathematics and art.

Mathematical theorem is essentially an exactly formulated truth, which is generalization of a certain precisely repeating phenomenon. It is a symbolic description of observed behavioural feature of an object, presented in abstract form, and explaining simultaneously also the circumstances under which the studied phenomena could appear. These circumstances are assumptions of the preposition. Mathematical hypothesis is a symbolically coded message, often understandable only to experts, and subdued to strict laws of logics. Thus, conveyed truth must be exactly verified by means of logical rules. It can be well comprehensible also to a layman, when the main idea of hypothesis is explained and illustrated, and its message can be visualised on a particular example, a geometric/graphical model preferably. Geometric interpretation helps to show the human face of mathematics. It serves as platform for science and art matching, as space for overlapping of artistic and mathematical abstraction. Anyhow, their common intersection is not only in the usage of geometric forms in the fine arts. Into a much greater extent it can be found on the abstract level, in searching for a description of natural relations, principles and rules of our Universe.

Artist is also expressing and describing observed reality and truth by creating the masterpiece of whatever type—painting, musical piece, sculpture, film, or theatre performance. Developing own specific artistic approach, he/she presents these observations in abstract form, symbolically, although "in his/her way". Reflecting own observations in mind and intimate emotional world, personalizing them, they are receiving a remarkable emotional mettle and generate thus the most remarkable power of created piece of art.

Fine art and mathematical abstract worlds are two parallels dealing with understanding of our world and its laws, each from its own point of view and by its own means of expressions. Maths and Art represent two "on the edge" attitudes of describing the world's truth. They seem to be the most imaginative, but in spite if this common feature, also the most descent achievements of human mind. Anyhow, their naturally different character in attitude and work processes is more unifying then separating them, as these two worlds are meeting on a higher abstract level, not clearly apparent on the first sight to laymen. Mathematical preposition and fine art work are two different representations of the observed reality, two abstractions appearing seemingly on the opposite poles of human mind, two different illustrations of truth and effort to comprise it and present in an abstract way. These two representations of "world truths" are available to complement each other perfectly.

In the following section we give some examples on how the two disciplines differ and what are the limits of their convergence. Various illustrations of pieces of art will be presented in comparison to mathematical definitions and symbolic formulas representing the concept of measurement, as the background for assessing understanding, truthfulness and inner value.

2 Measures in Mathematics

Finding a measure of any kind of unknown, we get the feeling that we have reached the first level of understanding the matter.

Measuring is one of the pretty natural human activities, when attempting to evaluate whatever we might come across, objects we are meeting, activities we are performing and their consequences and results, our relations and their manifestation, but at the first place the space in which we are obliged to live or work. Finding a measure of unknown objects or feelings, we get the impression that we understand the matter. Concept of measure is a well-defined phenomenon in mathematics, and measure theory represents a complex field of mathematical theory on a highly abstract level, with verified geometric models illustrating on particular examples and calculations concept of different measures as length, area, or volume of geometric figures. What exactly is a measure in mathematics?

Let X be a non-empty set and M a non-empty system of its subsets. M is called σ-algebra, if it is closed with respect to all complements and countable unions, i.e.

$$A \in \mathrm{M} \Rightarrow X \backslash A \in M, \quad A_n \in M \Rightarrow \bigcup_{n=1}^{n} A_n \in M.$$

If X is a topological space, then elements of σ-algebra B generated by system of all opened sets in X are called Borel sets.

Let M be a σ-algebra on a set X. Function $\mu : M \to [0, +\infty]$ is called (non-negative) measure, if it is σ-additive, i.e.

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

for any arbitrary system of mutually disjoint sets, while $\{A_n\}_{n=1}^{\infty} \subset M$ and $\mu(\emptyset) = 0$. Elements of M are called measurable sets, while triple (X, M, μ) is called space with a measure.

Let $A \subset E_n$, and let $S = \{I_{1,} \ldots, I_m\}$ be a system of closed intervals, in which no two different intervals have common interior points. Then we define $|A| = 0$, if set A does not contain interval, and if it contains some interval, then we define

$$|A|_* = \sup \sum_{k=1}^{m} |I_k|, \quad \bigcup_{k=1}^{m} I_k \subset A, \quad |A|^* = \inf \sum_{k=1}^{m} |I_k|, \quad A \subset \bigcup_{k=1}^{m} I_k.$$

For any bounded set A the following relation is true: $|A|_* \le |A|^*$.

Number $|A|_*$ is the inner Jordan measure of the set A (Fig. 1), number $|A|^*$ is the outer Jordan measure of the set A (Fig. 2). Bounded set A is measurable in the Jordan sense, if $|A|_* = |A|^* = |A|$.

Number $|A| \neq 0$ is called the Jordan measure of set, it is represented as

$|A| = \int_A \chi_A(x)dx = \int \int_A \ldots \int 1.dx_1 \ldots dx_m$, where $\chi_A(x)$ is a characteristic function such, that $\chi_A(x) = 1$ for $x \in A$, $\chi_A(x) = 0$, $x \in E_n - A$.

If (X, ρ) is a metric space and p is a positive number, then for any $A \subset X$ holds.

$$\mu_p^*(A) = \sup_{\varepsilon>0} \inf \left\{ \sum_{n=1}^{\infty} (d(A_n))^p;\ A = \bigcup_{n=1}^{\infty} A_n, d(A_n) < \varepsilon, n = 1, 2, \ldots \right\}.$$

Outer measure μ_p^* is called the Hausdorff p-dimensional measure (Fig. 3) determining fractal dimension (Fig. 4).

Measure is a certain limit that can be reached in-between two different measurements based on similar, anyhow not completely equal principles. Limit, in which inner and outer measures meet and overlap finding thus an agreement reflected in the real value, must not exist necessarily in all cases.

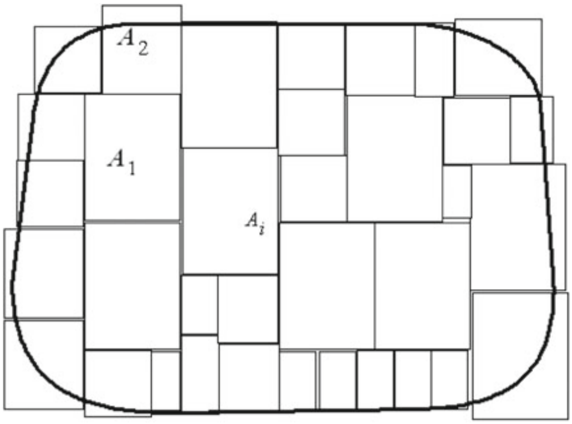

Fig. 1 Inner Jordan measure $|A|_*$ of set A

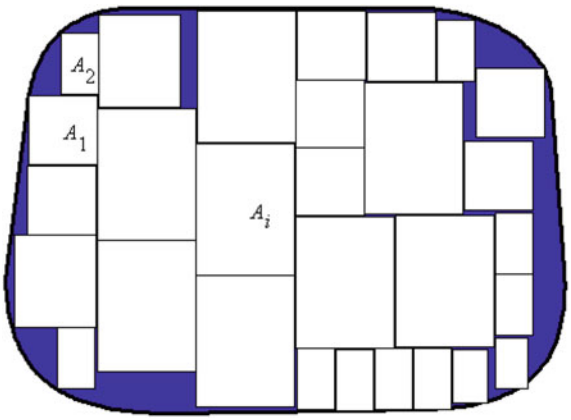

Fig. 2 Outer Jordan measure $|A|^*$ of set A

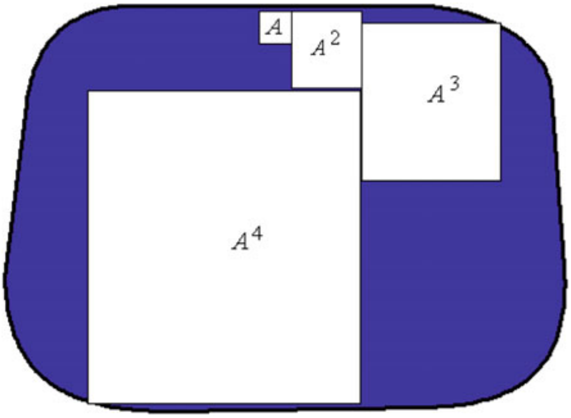

Fig. 3 Hausdorff *p*-dimensional measure of set *A*

Fig. 4 Fractal

Measure of an empty space is zero. Just the measurement units, objects filling the space, determine the non-zero measure value. Consequently, these objects themselves, items representing our expectations, define the final measure value. The preconditions defined as assumptions influence a sort of the measure quality or type. Mathematics states these clearly by determining initial conditions related to the specific desired type of measure, which are generally agreed and accepted and they serve as certain guide for related measurements.

Let (X, M, μ) be a space with measure and let simultaneously X be a topological space. Measure μ is called

- Borel measure, if M contains all Borel sets, so $B \subset M$
- regular, if it is a Borel set and for all $A \in M$ holds
  $\mu(A) = \inf\{\mu(U);\ A \subset U,\ U \text{ opened}\}$
- or
  $\mu(A) = \sup\{\mu(K);\ K \subset A,\ K \text{ compact}\}$
- (non-negative) Radon, if it is regular and for any compact $K \subset X$ holds $\mu(K) < \infty$
- complete, if for any $A \in M$ holds: if $\mu(A) = 0$ and $B \subset A$, then $B \in M$
- finite (or bounded), if $\mu(X) < \infty$
- σ-finite, if there exists $A_n \in M$ such, that $\mu(A_n) < \infty$ and $X = \bigcup_{n=1}^{\infty} A_n$.

Several other measures can be defined, such as arithmetic, Dirac, Carathéodory or Lebesgue-Stieltjes measures, we can speak about regular, complete, finite, σ-finite, arithmetic, and many other measures, which differ in the measuring approach and the measure quality for which they are applied, details see in [5], [1].

3 Measures in Art

Do similar laws exist also in art, or is the imagination space of an artist completely unlimited? Are there certain obstacles, defined by authors themselves, imposed maybe intuitively due to culture development of mankind, that are influencing creative work on the masterpiece, in the sense of its pre-defined inner desirable measure? Are the different measures reflected in different artistic styles that are approaching different aims and looking for different qualities?

Painters might apply a principle similar to mathematical definition of a measure, when filling the empty two-dimensional space represented by a piece of canvas, in order to change it into a painting. According to his/her own strategy he/she starts to distribute separate objects in there, creating thus a unique composition reflecting his/her own inner criteria posed on a surface, i.e. painting measure. Authors emotional world and creative potential influence quality of his/her measurements, it has a strong impact on the whole composition, as it determines the inner measure of the created piece of art. Outer measure of a painting is perceived by the observers. It comes out of the 2D space of the painting and it is independent on the viewer in certain sense. Although this outer measure can be generally perceived differently on the individual level, it is defined and influenced by well-formed rules of our culture heritage applicable in painting, or sculpture and design, which are developing together with the mankind knowledge, science and culture in the historical context, as mentioned also in [6].

It is very interesting and inspiring to find out, how a fine artist reacts to an abstract formalized mathematical proposition from the measure theory coded in mathematical symbolism, and how mathematician could understand his artistic illustration of the abstract mathematical imaginations in the fine art shaping the intuitive vision of artist. We still know only very little about general connections between different abilities of human brain to perceive pieces of art and to understand mathematical formulas. Is the first one not just an illustrative emotional visualization and model of the latter one? Are they interfering and cooperating, or neglecting each other passing in parallels? How can we make these two abstract abilities to respect or influence each other, and utilize thus the synergetic effect of imaginative human abilities resulting from this cooperation? Do there exist certain general rules, which human beings apply when measuring space by dividing it into smaller parts that actually live in there, and populating it make the space alive, being the carriers of interrelation with the exterior viewer? Are all these measurement processes intuitive, inborn or natural, or are they only side effects of our material obsession and desire to evaluate all our possessions?

Some of the questioned topics can be studied in the illustrations presenting artwork of the Slovak acknowledged women painters of the twentieth century. The first presented painting "Country" by Ester Šimerová Martinčeková (Fig. 5) is an example of 2D picture with the 3D exterior measure that is visualising the 3D scene—snow-covered Slovak mountainous countryside. Chosen technique of the painting by shades of blue and grey and simple division of the canvas area into several parts that determine the picture interior measure create in the synergic effect its 3D dimensional expression forming the outer measure of this oil painting on canvas.

Art-piece "Mother" by Svetlana Ilavská (Fig. 6) presents the unlimited power of the mother love. Mother, more dead than alive, carries the ill child on her back, doing her best to rescue him. The external dimension of this picture dwells in the depicted indescribable emotions—despair, belief and hope. Colour contrasts of the mother and child silhouettes masterfully scratched by just few differently coloured domains in the two-dimensional space of the canvas, produce an emotionally rich atmosphere and radiate this sensation out of the painting as its exterior dimension.

Fig. 5 Ester Šimerová Martinčeková: Country

Fig. 6 Svetlana Ilavská: Mother

Fig. 7 Lýdia Jergušová-Vydarená: Antimass

The third illustration in Fig. 7. "Antimass" by Lýdia Jergušová-Vydarená [4], is the artist visualisation of her own understanding of physical and mathematical basic concepts. The famous Einstein equation $E = mc^2$, representing that matter can be turned into energy, and energy into matter, is presented in connection to the exploration of antimass, black matter and invisible energy hidden as the black holes of the Universe predicted by Dirac and studied by Anderson. Artist´s imagination is expressed in the slightly adapted form of the equation with the doubled right side, expressing thus her belief in the existence of the antimass and its equivalence with the mass. Both are regarded just as two different forms of energy that are available in the Universe. The antimass is threatening and it makes us feeling unsafe, as something unknown, mysterious, and perhaps dangerous. Sharp frame coming out from the mild background represents antimass as the outer measure of this piece of art which is earnestly and intrusively approaching observer, making him/her feeling precariously. Inner measure in warm pleasant colours acts reassuringly and it calms our worry. Together they balance the story conveyed by author in a common limit that defines the real value of the aesthetic measure of this piece of art.

Examples in Fig. 8 and Fig. 9 present work of a young Slovak artist Žofia Dubová [3], who precisely takes into consideration inner and outer measure of her pieces of art. She is openly stressing the role of visualised matter with inner measure surrounded by the border area forming the frame, which itself is a part of the image as its outer measure. Thus, observer is invited to perceive the artist message in the context of a framed object, realising both inner and outer measure described explicitly by author in the masterpiece. Experiencing both in the context of the articulated author´s message, gives observer a chance to find the actual value measured as the limit balancing the inner and outer part of the piece of art. This approach can be identified precisely with the main idea of an object measure as defined in mathematics, where both inner and outer measures are given explicitly, e.g. in Jordan measure

Fig. 8 Žofia Dubová: Shadows (on the left), The falling being (on the right)

Fig. 9 Žofia Dubová: Over thee mountain

definition. Žofia Dubová was the winner of the Slovak young artists contents Maľba 2017 with her unusually interesting piece of art "Over the mountain".

Another chapter of geometric feelings in art can be seen in art using entire geometric objects filling the canvas space in various colour and position variations. Some of these pieces remind us on special mathematical phenomena, as e.g. Fibonacci series, or spirals, or simply rolling a circle inside or outside the other one. Here one can include also Op-art or geometric abstraction, which is a non-figurative art born after the year 1910 as a branch of abstract art based on usage of geometric forms placed in non-illusory space and combined into non-objective compositions (Fig. 10, 11, 12). How can be measured the artistic value of these pieces of art?

It is a pursuit of elegance that captures essence and gives us a precise insight on relations. Xah Lee

Fig. 10 Dada: Circles, inspired by [2]

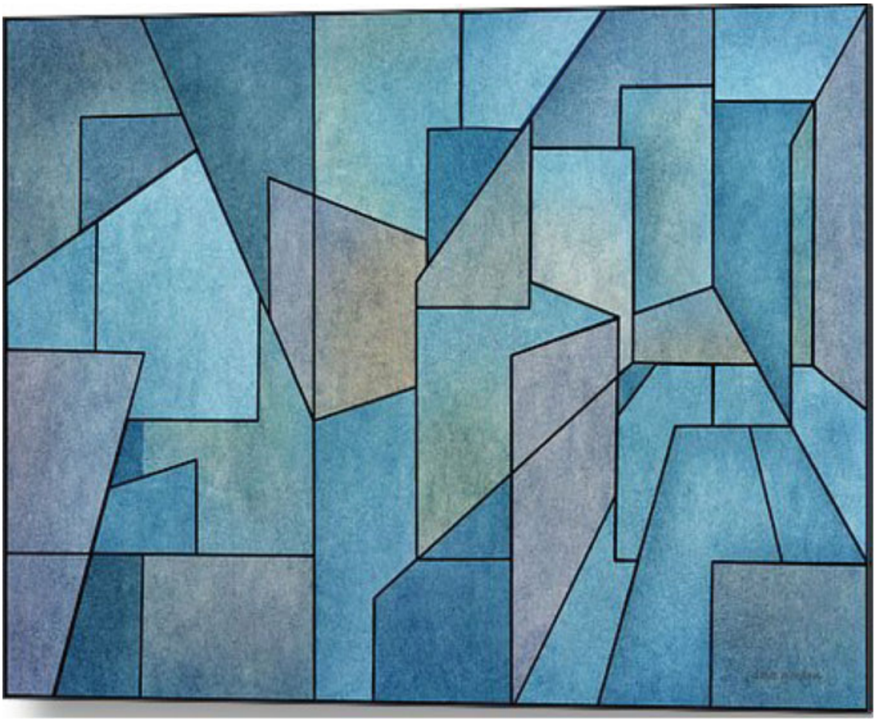

Fig. 11 David Gordon: Geometric abstraction

Fig. 12 Exhibition of the Slovak artistic association **VEKTORYart**, see details in [7]

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Patterns

Mária Ždímalová

Abstract A pattern is a regularity in the world. In human—made design, or in abstract ideas. The elements of a pattern repeat in a predictable manner. A geometric pattern is a kind of pattern formed of geometric shapes and typically repeated like a wallpaper design. Nature provides examples of many kinds of patterns, including symmetries, trees, and other structures with fractal dimension, spirals, meanders, waves, foams, tillings, cracks and stripes. Patterns are simply a repetition of more than one design element working in concert with each other. A seamless pattern working is one where every element within a design (no matter how often it is repeated) combines to form a whole. Designers base most patterns on colors, textures and shapes, rather than words. Patterns in architecture is the idea of capturing architectural design ideas as archetypal and reusable descriptions. The patterns serve as an aid to design cities and buildings. This contribution discuss the philosophy of patterns and their applications in mathematics, geometry, architecture and design. Finally, we discuss how to use patterns for aggregations and tessellation. We discuss how we can use weighted Voronoi diagram for patterns or vice versa and we consider as well weighted Voronoi tessellation and Voronoi patterns.

1 Introduction

Any of the senses may directly observe patterns. Conversely, abstract patterns in science, mathematics, or language may be observable only by analysis. Direct observation in practice means seeing visual patterns, which are widespread in nature and in art [1, 2, 5, 6]. Visual patterns in nature are often chaotic, never exactly repeating and often involve fractals. Natural patterns include spirals, meanders, waves, foams, tillings, cracks, and those created by symmetries of rotation and reflection. Patterns have of rotation and reflection. Patterns have an underlying mathematical structure; indeed, mathematics can be seen as the search for regularities, and

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the output of any function is a mathematical pattern. Similarly in the sciences, theories explain and predict regularities in the world. In art and architecture, decorations or visual motifs may be combined and repeated to form patterns designed to have a chosen effect on the viewer. In computer science, a software design pattern is a known solution to a class of problems in programming. In fashion, the pattern is a template used to create any number of similar garments. Nature provides examples of many kinds of pattern, including symmetries, trees and other structures With a fractal dimension, spirals, waves, foams, tilings, cracks, and stripes [6, 7].

1.1 What Is a Pattern?

The itsy-bitsy spider went up the water spout. Down came the rain and washed the spider out. Out came the sun and dried up all the rain, and the itsy-bitsy spider went up the spout again.' The itsy-bitsy spider song is an example of a pattern. A **pattern** [1, 2, 5] is a series or sequence that repeats. The itsy-bitsy spider climbed the water spout, and then did the same thing again after the weather cleared up. You can observe patterns—things like colors, shapes, actions, or other sequences that repeat—everywhere. Think about words or melodies in songs, lines and curves on buildings, or even in the grocery store where boxes and jars of various items are lined up. But, one of the most common places to find patterns is in math. **Math patterns** are sequences that repeat according to a rule or rules. In math, a **rule** is a set way to calculate or solve a problem.

Number Patterns: One common type of math pattern is a number pattern. Number patterns are a sequence of numbers that are ordered based upon a rule. There are many ways to figure out the rule, such as: using a number line to see the distance between the numbers or what they have in common, look at the last one or two digits or the first digit to see if they repeat in a special manner, look at the numbers and see if there is a pattern, like taking each number and multiplying by 3 for instance, considering about common number patterns, like counting by 2s, 5s, or 10s, and/or finding the difference between the numbers. It is important to remember that a number pattern can have more than one solution and a combination of rules. If this is the case, try to think of the simplest rule possible, like adding 1 or multiplying by 2 with a difference of 3. There are different types of number patterns in Mathematic like: Arithmetic Sequence, Geometric Sequence, Square Numbers, Cube Numbers, Triangular Numbers, Fibonacci Numbers [1, 2, 5, 6, 9].

Concept of patterns will help us to learn the basic number patterns and table patterns. Animals such as all cows, all lions, all dogs and all other animals have dissimilar features. All mangoes have similar features and shapes. Leaves of the same tree have similar pattern of shape. The doors and windows of a house have similar patterns. We may search for similar patterns in mathematics also. The study of patterns helps students to observe relationships and find continuous connections, to deduce generalizations and predictions. In everyday life we observe dress patterns, carpet patterns, wall-paper patterns, sari-design patterns as well as number pattern in arithmetic. How could Whitney describe this visual pattern? What type of a visual pattern is it?

A **visual pattern** [1, 6, 7] is a sequence of pictures or geometric objects that have been created based on some rule. There are two categories of visual patterns that will be taken in consideration in the present paper.

a. Some visual patterns are made up of a sequence of pictures or geometric objects that repeat over and over. These types of patterns are called **repeating patterns**. The sequence of pictures that repeat is called the **pattern unit**.
b. Some visual patterns are based on a picture or geometric object that keeps increasing or decreasing in size in a specific way. These types of patterns have a **pattern rule** that help to describe how the picture or object is growing or shrinking, star, triangle, star, square, square, triangle, star, triangle. First notice that this is a repeating pattern. The images haven't been displayed, but they have been described. The images are being repeated as opposed to increasing/decreasing in size. Next, look carefully to see how the shapes have been ordered. After "triangle, star, triangle, star" is "square, square". After "square, square" the "triangle, star, …" sequence seems to repeat. It appears that the same set of six shapes will repeat over and over. The pattern unit is "triangle, star, triangle, star, square, square". Now that you have the pattern unit, you can extend the pattern. After the "triangle, star, triangle" at the end of the pattern will be another star, then two squares, and then back to the beginning of the pattern unit with "triangle, star, triangle, star, square, square…".

Symmetry: [2, 6, 7] Snowflake sixfold symmetry: Symmetry is widespread in living things. Animals that move usually have bilateral or mirror symmetry as this favours movement. Plants often have radial or rotational symmetry, as do many flowers, as well as animals which are largely static as adults, such as sea anemones. Fivefold symmetry is found in the echinoderms, including starfish, sea urchins, and sea lilies. Among non-living things, snowflakes have striking sixfold symmetry: each flake is unique, its structure recording the varying conditions during its crystallisation similarly on each of its six arms. Crystals have a highly specific set of possible crystal symmetries; they can be cubic or octahedral, but cannot have fivefold symmetry (unlike quasicrystals).

Spirals: Spiral patterns are found in the body plans of animals including molluscs such as the nautilus, and in the phyllotaxis of many plants, both of leaves spiralling around stems, and in the multiple spirals found in flowerheads such as the sunflower and fruit structures like the pineapple.

Chaos, turbulence, meanders and complexity: Chaos theory predicts that while the laws of physics are deterministic, there are events and patterns in nature that never exactly repeat because extremely small differences in starting conditions can lead to widely differing outcomes. The patterns in nature tend to be static due to dissipation on the emergence process, but when there is interplay between injection of energy and dissipation there can arise a complex dynamic. Many natural patterns are shaped by this complexity, including vortex streets, other effects of turbulent flow such as meanders in rivers or nonlinear interaction of the system.

Dune tipple [5, 6]: Waves are disturbances that carry energy as they move. Mechanical waves propagate through a medium—air or water, making it oscillate as they pass by. Wind waves are surface waves that create the chaotic patterns of the sea. As they pass over sand, such waves create patterns of ripples; similarly, as the wind passes over sand, it creates patterns of dunes. Foams obey Plateau's laws, which require films to be smooth and continuous, and to have a constant average curvature. Foam and bubble patterns occur widely in nature, for example in radiolarians, sponge spicules, and the skeletons of silicoflagellates and sea urchins. Cracks form in materials to relieve stress: with 120 degree joints in elastic materials, but at 90 degrees in inelastic materials. Thus the pattern of cracks indicates whether the material is elastic or not. Cracking patterns are widespread in nature, for example in rocks, mud, tree bark and the glazes of old paintings and ceramics.

Spots, stripes: Alan Turing, and later the mathematical biologist James D. Murray and other scientists, described a mechanism that spontaneously creates spotted or striped patterns, for example in the skin of mammals or the plumage of birds: a reaction–diffusion system involving two counter-acting chemical mechanisms, one that activates and one that inhibits a development, such as of dark pigment in the skin. These spatiotemporal patterns slowly drift, the animals' appearance changing imperceptibly as Turing predicted [2, 6, 7].

In **sewing and fashion design** [1, 5, 6], a pattern is the template from which the parts of a garment are traced onto fabric before being cut out and assembled. Patterns are usually made of paper, and are sometimes made of sturdier materials like paperboard or cardboard if they need to be more robust to withstand repeated use. The process of making or cutting patterns is sometimes condensed to the one-word—Pattern making, but it can also be written pattern(-)making or pattern cutting. Pattern (home sewing) or block pattern (industrial production) is a custom-fitted, basic pattern from which patterns for many different styles can be developed. The process of changing the size of a finished pattern is called grading. Several companies, like Butterick and Simplicity, specialize in selling pre-graded patterns directly to consumers who will sew the patterns at home. Commercial clothing manufacturers make their own patterns in-house as part of their design and production process, usually employing at least one specialized patternmaker. In bespoke clothing, slopers and patterns must be developed for each client, while for commercial production, patterns will be made to fit several standard body sizes.

Pattern in **architecture** [6–9] is the idea of capturing architectural design ideas as archetypal and reusable descriptions. The term "pattern" in this context is usually attributed to Christopher Alexander, an Austrian born American architect. The patterns serve as an aid to design cities and buildings. The concept of having collections of "patterns", or typical samples as such, is much older. One can think of these collections as forming a pattern language, whereas the elements of this language may be combined, governed by certain rules. This may be distinct from common use of pattern books, which are collections of architectural plans which may be copied in new works. Patterns may be collected together into a pattern language that addresses a particular domain. A large body of patterns was published by Alexander and his collaborators as **A Pattern Language** [9]. The patterns in that book were intended to enable communities to construct and modify their own homes, workplaces, towns and cities. Other than Alexander's own projects, few building projects have tried to use Alexander's patterns. Those that have done so have met a mixed response from other architects, builders, architectural critics, and users. Alexander has come to believe that patterns themselves are not enough, and that one needs a "morphogenetic" understanding of the formation of the built environment. He has published his ideas in the four-volume work *The Nature of Order*. While the pattern language idea has so far had limited impact on the building industry, it has had a profound influence on many workers in the information technology industry [9].

A design pattern [7, 8] is the re-usable form of a solution to a design problem. The idea was introduced by the architect Christopher Alexander and has been adapted for various other disciplines, notably software engineering. An organized collection of design patterns that relate to a particular field is called a pattern language. This language gives a common terminology for discussing the situations designers are faced with.

The elements of this language are entities called patterns. Each pattern describes a problem that occurs over and over again in our environment, and then describes the core of the solution to that problem, in such a way that you can use this solution a million times over, without ever doing it the same way twice—Christopher Alexander.. Documenting a pattern requires explaining why a particular situation causes problems, and how the components of the pattern relate to each other to give the solution. Christopher Alexander describes common design problems as arising from "conflicting forces"—such as the conflict between wanting a room to be sunny and wanting it not to overheat on summer afternoons. A pattern would not tell the designer how many windows to put in the room; instead, it would propose a set of values to guide the designer toward a decision that is best for their particular application. Alexander, for example, suggests that enough windows should be included to direct light all around the room. He considers this a good solution because he believes it increases the enjoyment of the room by its occupants. Other authors might come to different conclusions, if they place higher value on heating costs, or material costs. These values, used by the pattern's author to determine which solution is "best", must also be documented within the pattern. Pattern documentation should also explain when it is applicable. Since two houses may be very different from one another, a design pattern for houses must be broad enough to apply to both of them, but not so vague that it doesn't help the designer make decisions. The range of situations in which a pattern can be used is called its context. Some examples might be "all houses", "all two-story houses", or "all places where people spend time". For instance, in Christopher Alexander's work, bus stops and waiting rooms in a surgery center are both within the context for the pattern "A PLACE TO WAIT" [1, 7, 8].

Pattern Language by Christopher Alexander is renowned for providing simple, conveniently formatted, humanist solutions to complex design problems ranging in scale from urban planning through to interior design [9]. Thus, while *A Pattern Language* is widely referenced in architectural scholarship, most of these references simply acknowledge its existence and fail to engage with its content. Furthermore, the literature that does critically engage with Alexander's theory, challenging its ideas and assumptions [9], is often difficult to find, and the criticisms are diverse and complex. The intent of this paper is to facilitate a deeper understanding of these criticisms and the relationships between them. The 28 criticisms identified in past research are organised hierarchically in this paper into three tiers representing those associated with the: (i) conceptualisation, (ii) development and documentation and, (iii) implementation and outcomes of Alexander's theory. Despite the influence and impact of Alexander's second theory, it has been rejected or ignored by many architects, and many academic references to the theory simply acknowledge its existence rather than attempting to engage with its ideas.

Interaction design patterns are design patterns applied in the context human-computer interaction, describing common designs for graphical user interfaces [5, 6]. A design pattern is a formal way of documenting a solution to a common design problem. The idea was introduced by the architect Christopher Alexander for use in urban planning and building architecture and has been adapted for various other disciplines, including teaching and pedagogy, development organization and process, and software architecture and design. Thus, interaction design patterns are a way to describe solutions to common usability or accessibility problems in a specific context. They document interaction models that make it easier for users to understand an interface and accomplish their tasks. Some people say that doing the same thing over and over again will not get you results, but this lesson will teach you about mathematical patterns and show you how to prove those people wrong.

2 Applications of Pattern in Mitla, Čičmany and Tessellation

2.1 Patterns in Mitla

We can approach the rectangular structure [12], the many fringes containing geometric designs start to show their intricate details. In Spanish, they are known as "grecas" after the Greek symbol of eternal life that is repeated in so many decorative motives. There are three patterns in that building. Two more on another. Is this design different from the other? The grecas are everywhere. On the side of the structures. On the facades of a palace. Inside the priests' rooms. Mitla, the place of the dead or the place of rest, was always a site of religious importance. When Monte Alban declined, a significant amount of people moved in (and around) making Mitla the center of power for the Zapotecs residing on the Central Valleys of Oaxaca. There is no other archeological site in Mexico like this one. An intricate mosaic fretwork adorns the walls of two of the five groups remaining to this day. The geometric patterns are made from thousands of cut, polished stones that are fitted together without mortar. Therefore, none of the designs are repeated exactly anywhere in the complex. **Grecas Detail**: [12]. There arised a question: How many basic patterns are on the site?" "Impossible to tell," is the correct answer. Some designs have been lost because of many causes: vandalism, erosion, rain. However, some believe there are 13 patterns or a combination that is related to that number. It has to do with the 13 heavens of Zapotec legend. It makes sense since structures in archeological sites have features based on myths and/or astronomical knowledge. The number 13 is very important to religion people there referring to his Zapotec ascendants. The most beautiful building in Mitla is The Palace. The structure is an architectonical gem with its façade and four rooms (priests' living quarters) covered in "grecas." It only has one entrance which allowed for intimacy during rituals and other activities. Some parts of The Palace are open to the elements.

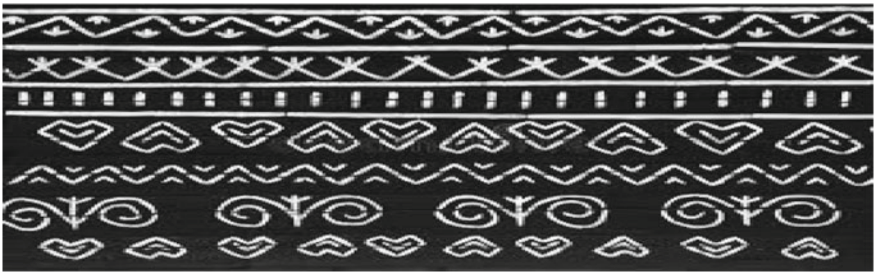
The most striking aspect of Mitla is the amazing geometric stepped-fret designs that adorn walls everywhere; both inside and outside, around doorways and lining entire walls. Their placement, particularly around the entrances to buildings, suggests they are designed to communicate to us some sort of meaning—such as who is allowed to enter the building, what will happen to those choose to enter, or what the purpose or significance of the building has. The mosaics are found on the internal and external walls of almost every building, including palaces, municipal buildings and within tombs. Because they feature on such a broad array of buildings and with only 6 or so different designs in total, it would seem their meaning must be fairly broad or very simple. However, the individual designs typically found together, normally stratified into horizontal bands using 3 different designs. This suggests they may have been creating compositions that bore new extended meanings, thus providing the potential for more than 100 permutations and a far more complex language. This isn't the only hidden complexity, within the designs. From a distance, it would seem obvious that they would have been produced using moulds to set the shapes in plaster which were then glued to the wall. However, up close it is easy to see the shapes are individually carved from rock and slotted together like a jigsaw. Even closer inspection shows that these individual pieces were slotted together so tightly that they didn't even require mortar. Most of the buildings are covered in plaster, so clearly they had the technology to make these patterns with moulds and chose the more complicated option deliberately. Even with today's technology, including laser guided machine cutting, the scale on which this decorative technique is used at Mitla would make it prohibitively expensive. As the builders at Mitla [13] had nothing more than chisels, hammers and man-power, it would have meant the cost of decorating each building would have outstripped the cost of building it by tenfold. So either there was a deeply religious or symbolic reason for using such a labour intensive technique, or it was used as a demonstration of wealth, power and opulence. It is most likely the latter, and this tells us that the ruling elite of Mitla enjoyed broadcasting their supremacy through this lavish ornamentation and that the city was visited by Royalty and High-Priests from far and wide whom the city wished to impress. With such opulence built into the masonry of the buildings, it is almost certain that inside the furnishings and ornaments would have been equally lavish, and this may be the real reason why the city was dismantled by the Catholic missionaries and colonialists. The gallery demonstrates how all the palatial and municipal buildings Mitla have step-fret designs around their entrances. The interlocking stonework at Mitla can either be described as fretwork or mosaic—but it doesn't fit precisely into either.

The designs are very similar to those used by the Romans, Greeks, Egyptians and Mesopotamian civilisations, which are known as step-fret or stepped-fret designs. At Mitla they are not simply Classic Greek Step-Fret Motif decorative though, and are thought to have religious significance—although the exact meaning of each shape is unkown. Most decipherments are based complex associations with serpents or waves. More informed research suggests the round shape depicts the world, the wind or clouds; the "S" style shape represents the serpent, whilst the wave style depicts reincarnation with the steps leading down to the underworld then looping through a cycle and back to the top of the next stairs. There is more to be uncovered of their meanings and undoubtedly numerics will play a role in their decipherment—for example, the "wave" has five steps which normally corresponds to the number of planets visible to the naked eye. Also, is it the raised or the sunken part of the image that is supposed to be meaningful? The most elegant answer is that individually they are sacred symbols, but when used together they signify people, families, or tribes [12, 13].

2.2 Patterns of Čičmany

The tiny village of Čičmany (population 204) looks just like it did hundreds of years ago [11] untouched by time. But this is no ordinary preservation area. The village is full of black timber houses, each one decorated with intricate traditional patterns in white lime paint, see Fig. 1. Shapes are based on geometrical patterns. Street after street is lined with folk art. The records of the village at Čičmany date back to the thirteenth century, covering the houses in lacy, geometric and stylized patterns is a much later tradition. It began around 200 years ago, when white lime would be used to help preserve damaged wood. The image of the bright lime pigment on the dark wood was striking, and people began elaborating on themes, eventually covering most of the timber-frame structures with remarkably uniform designs. In 1921 a fire raced through the village, and many structures had to be restored. With the help of the Slovakian government, funds were made available to keep Čičmany as it had been

Fig. 1 Patterns in Mitla, Mexico



Picture 2 Geometric Patterns of the Houses in Čičmany, Slovakia

for centuries. It was established in 1977 as the world's first folk architecture reserve, ensuring the protection of its buildings, and unique cultural heritage. Čičmany is in northwestern Slovakia, about a two-hour drive from Bratislava. It is possible to reach Čičmany by public transport. Visitors stranded in Čičmany can take advantage of the traditional houses turned into pensions [11] (Picture 2).

2.3 Tessalation Ad Voronoi Patterns

Tessellation [4, 5, 10] (or tiling) is a pattern of geometrical objects that covers the plane. The geometrical objects must leave no holes in the pattern and they must not overlap. It should be able to extend the pattern to infinity. It makes a tessellation by starting with one or several figures and then rotate it, translate or reflect them; or do a combination of transformations, in order to get a repeating pattern. If there is interest only want to use one regular polygon to make a tessellation, there are only three possible polygons to use: **triangle, square and hexagon**. We need also to mention different methods of clusterings [3] and its applications to tilings.

In mathematics, a **weighted Voronoi diagram** [10] in n dimensions is a special case of a Voronoi diagram. The Voronoi cells in a weighted Voronoi diagram are defined in terms of a distance function. The distance function may specify the usual Euclidean distance, or may be some other, special distance function. Usually, the distance function is a function of the generator points' weights. The multiplicatively weighted Voronoi diagram is defined when the distance between points is multiplied by positive weights. In the plane under the ordinary Euclidean distance, the multiplicatively weighted Voronoi diagram is also called circular Dirichlet tessellation and its edges are circular arc and straight line segments. A Voronoi cell may be non-convex, disconnected and may have holes. This diagram arises, e.g., as a model of crystal growth, where crystals from different points may grow with different speed. Since crystals may grow in empty space only and are continuous objects, a natural variation is the crystal Voronoi diagram, in which the cells are defined somewhat differently. The additively weighted Voronoi diagram is defined when positive weights are subtracted from the distances between points. In the plane under the ordinary Euclidean distance this diagram is also known as the hyperbolic Dirichlet tessellation and its edges are hyperbolic arc and straight line segments. The power diagram is defined when weights are subtracted from the squared Euclidean distance. It can also be defined using the power distance defined from a set of circles.

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