Oscillations and Periodic Solutions in a Two-Dimensional Differential Delay Model

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Abstract A class of two-dimensional differential systems with delay and overall negative feedback is considered. Sufficient conditions for the existence of periodic solutions are established. The instability of the unique equilibrium together with the one-sided boundedness of one of the two nonlinearities lead to the existence of periodic solutions. Systems of this type appear in various applications in engineering and natural sciences, in particular in mathematical biology and physiology as models of circadian rhythm generator and glucose-insulin regulation models in humans.

Keywords Delay differential equations · Slow oscillations · Periodic solutions · Ejective fixed point theory

1 Introduction

This paper deals with the problem of existence of slowly oscillating periodic solutions for a system of two-dimensional differential delay equations of the form

$$\begin{aligned} x'(t) &= -\alpha x(t) + f(x(t), y(t), x(t-\tau), y(t-\sigma)) \\ y'(t) &= -\beta y(t) + g(x(t), y(t), x(t-\tau), y(t-\sigma)) \end{aligned} \tag{1}$$

where nonlinearities f and g are continuous real-valued functions, decay rates α, β are positive, and delays τ, σ are non-negative with $\tau + \sigma := d > 0$.

Systems of type (1) appear in various applications, including physics and laser optics, physiology and mathematical biology, economics and life sciences among others. In particular, they naturally appear in physiology and mathematical biology

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[5, 9, 11, 14, 18, 25, 33, 35], where they serve as models of enzyme production [13, 27], of an intracellular circadian rhythm generator [30], or as glucose-insulin regulation models in humans [4, 26]. An extensive description of various applications can be found in e.g. [10, 11, 31, 35].

Sufficient conditions for the existence of periodic solutions in system (1) are established in this paper. The nonlinearities f and g are further assumed to satisfy either positive or negative feedback condition with the overall negative feedback in the system. The instability of the unique equilibrium together with a one-sided boundedness of either f or g lead to the existence of periodic solutions. The analysis of system (1) uses some of the results established for multi-dimensional systems and higher order differential delay equations [6, 20, 21].

The standard technique employed in the proof of existence of periodic solutions of autonomous equations and systems is the *Ejective Fixed Point Theory*. For details of the theory see respective chapters of monographs [8, 17]. It goes back to the classic works by Wright [34] and Jones [22–24]. Wright proved that the delay differential equation $x'(t) = -\alpha x(t-1)[1 + x(t)]$ has no slowly oscillating solutions converging to zero as $t \to \infty$ when $\alpha > \frac{\pi}{2}$ (for appropriately chosen initial functions). This fact together with adapted versions of the ejective fixed point theorem by Browder [7] allowed Jones to show that the latter equation and its analogues possess slowly oscillating periodic solutions. In the years following since then the approach was further theoretically developed and formalized to the level that allowed one to prove the existence of periodic solutions to various classes of delay differential equations and systems. The basics of the ejective fixed point techniques can be found in monographs [8, 17]; some examples of application of this theory to derive periodic solutions can be given by results in papers [1, 2, 6, 15, 16, 28, 32].

In this paper we provide a general outline of establishing the existence of slowly oscillating periodic solutions for system (1); detailed mathematical exposition with complete proofs will be given in a forthcoming work. We follow the established theory of the ejective fixed point theorem as described in monographs [8, 17], with recent supplementary results obtained in [21] for systems of delay equations, as well as some further results from [6, 20]. A significant difference for the developments in this work is that our system (1) cannot be reduces to the form used in [6, 21], where the original system is transformed to the form with a single delay present in only one equation. This difference is essential and requires modified considerations and approaches. The modified elements of our approach are the selection of an appropriate cone in the phase space of system (1), deriving the relevant properties of solutions, and construction of a nonlinear map on the cone. The ejectivity is derived from establishing linear lower bounds on the functionals which are specific for this two-dimensional system. Together with the one-sided boundedness of either f or g this implies the compactness of the nonlinear map on a set of slowly oscillating initial functions, leading in turn to the existence of periodic solutions to system (1).

2 Preliminaries

2.1 Assumptions and Basics

Throughout the paper we make the following assumptions:

(A1) (*Continuity*) Functions $f(u, v, w, z), g(u, v, w, z)$ are continuous, $f, g \in C(\mathbf{R}^4, \mathbf{R})$, the decay coefficients are positive, $\alpha > 0, \beta > 0$, and the delays τ, σ are nonnegative with $\tau + \sigma = d > 0$;

(A2) (*Differentiability at zero*) Partial derivatives $f_u, f_v, f_w, f_z, g_u, g_v, g_w, g_z$ are continuous in a neighborhood of zero, $(u, v, w, z) \in [-\delta, \delta]^4$ for some $\delta > 0$;

(A3) (*Overall negative feedback*) Function f satisfies the positive feedback condition

$$f(u, v, w, z) \cdot z > 0 \qquad \forall (u, v, w, z) \in \mathbf{R}^4, z \neq 0, \qquad (pf)$$

while function g satisfies the negative feedback condition

$$g(u, v, w, z) \cdot w < 0 \qquad \forall (u, v, w, z) \in \mathbf{R}^4, w \neq 0; \qquad (nf)$$

(A4) (*One sided boundedness*) Either nonlinearity f or nonlinearity g is one-sided bounded:

$$f(u, v, w, z) \leq M > 0 \quad \text{or} \quad f(u, v, w, z) \geq -M < 0 \quad \forall (u, v, w, z) \in \mathbf{R}^4; \quad (bd)$$

(similar inequalities for g when it is one-sided bounded).

For some of the considerations of the paper a higher smoothness of the partial derivatives in $(A2)$ may be required (see e.g. Theorem 1, $(O1)$). Therefore, we shall also use the following enhancement of assumption $(A2)$:

(A2*) Partial derivatives $f_u, f_v, f_w, f_z, g_u, g_v, g_w, g_z$ are of C^1-class in a neighborhood of zero, $(u, v, w, z) \in [-\delta, \delta]^4$ for some $\delta > 0$.

Given initial data for each of the two components $x = \varphi(s) \in C([-\tau, 0], \mathbf{R}) := X_1$, $y = \psi(s) \in C([-\sigma, 0], \mathbf{R}) := X_2$ one has to solve a sequence of ordinary differential systems in order to derive the corresponding solution to system (1) for $t \geq 0$. Therefore, the natural phase space for system (1) is the direct product of two Banach spaces $X = C([-\tau, 0], \mathbf{R}) \times C([-\sigma, 0], \mathbf{R}) := X_1 \times X_2$. We shall assume that for every initial function $(\varphi, \psi) \in X$ there exists unique corresponding solution $(x(t), y(t))$ to systems (1) defined for all $t \geq 0$. Such conditions for the existence and uniqueness can be e.g. the uniform Lipschitz continuity of functions $f(u, v, w, z), g(u, v, w, z)$ in the first two variables u, v. The solution (x, y) is then constructed for $t \geq 0$ by the standard step method [3, 8, 17].

Note that due to symmetry considerations one can make the assumption in (A3) that the nonlinearity f satisfy the negative feedback condition (nf) while the nonlinearity g satisfies the positive feedback condition (pf). One can also assume, that

$\sigma \leq \tau$ is satisfied. The feedback assumptions (pf) and (nf) of (A3) imply that $(x, y) \equiv (0, 0)$ is the unique equilibrium of system (1).

System (1) includes a general cyclic system with the overall negative feedback [6, 21] as a partial case of $N = 2$, when $f(u, v, w, z) = F(z)$ and $g(u, v, w, z) = G(w)$ and functions F, G satisfy the positive and negative feedback conditions respectively in dimension one: $x \cdot F(x) > 0$, $x \cdot G(x) < 0$, $\forall x \neq 0$. An example of such systems in applications is e.g. the mathematical model for the intracellular circadian rhythm generator [30]. When $\alpha = \beta$ and $\tau = \sigma$ a partial case of system (1) was studied in papers [2, 32]. Some second order delay differential equations can also be viewed as a partial case of system (1) [1].

2.2 Linearization and Characteristic Equation

The linearized system about the equilibrium $(x, y) = (0, 0)$ is given by

$$\begin{aligned} x'(t) &= -\alpha x(t) + a_1 y(t - \sigma) \\ y'(t) &= -\beta y(t) - a_2 x(t - \tau), \end{aligned} \tag{2}$$

where $a_1 = f_z(0, 0, 0, 0) > 0$ and $a_2 = -g_w(0, 0, 0, 0) > 0$. Note that all other partial derivatives of f and g at $(0, 0, 0, 0)$ are zero due to both positive and negative feedback assumptions (nf) and (pf). The characteristic equation of the linear system (2) is found when one seeks its solutions in the exponential form $(x, y) = (x_0, y_0) \exp\{\lambda t\}$; it has the following form:

$$(\lambda + \alpha)(\lambda + \beta) + a \exp\{-d\lambda\} = 0, \tag{3}$$

where $a = a_1 a_2 > 0$ and $d = \tau + \sigma > 0$.

The transcendental equation (3) is extensively studied in several publications; we will adopt and use in this paper corresponding results from papers [1, 6].

The linear system (2) and its characteristic equation (3) determine several important properties of the original nonlinear system (1). In particular, when (3) has no real eigenvalues then all solutions to systems (1) and (2) oscillate. When (3) has a pair of complex conjugate eigenvalues with positive real part then the zero solution for both systems is unstable. See [6] for exact statements and more details.

2.3 Oscillation

We are interested in the oscillatory behavior of all solutions of system (1). Recall that a scalar continuous function $u(t)$ is said to be oscillatory on a semi-axis $[t_0, \infty)$ if there is an increasing sequence of values $t_n \to \infty$ such that $u(t_n) \cdot u(t_{n+1}) < 0$.

We shall call a solution (x, y) to the system to be oscillatory if both components $x(t)$ and $y(t)$ oscillate on the semi-axis $[t_0, \infty)$. Note that an assumption about the oscillatory behavior of one component of system (1) implies that the other component is oscillatory as well (see [6], subsection "Oscillation").

Sufficient conditions for the oscillation of all solutions to system (1) are given by the following statement.

Theorem 1 *Suppose that at least one of the following two conditions is satisfied:*

(O1) *The characteristic equation (3) has no real solutions (while (A2*) holds);*
(O2) $ad > \max\{\alpha, \beta\}$.

Then all solutions to system (1) oscillate about the equilibrium $(x, y) = (0, 0)$.

Part $(O1)$ of the theorem can be derived from an analogue of Theorem 1 ([6], p. 17) when one assumes the additional smoothness properties (A2*). Part $(O2)$ can be established along the same lines as the proof of Theorem 1 in [19] (with no principal changes). Additional oscillation criteria for delay differential equations and systems can be found in e.g. [12].

An important particular type of the oscillatory behavior is the so-called *slow oscillation*. It is associated with the size of a delay in a particular equation/system. With regard to system (1) we shall call either one of the components x or y to be slowly oscillating on a semi-axis $[t_0, \infty)$ if the distance between its any two zeros there is greater that the overall delay $d = \tau + \sigma$ in the system.

The slow oscillation is present and typical in scalar equations and systems of type (1) with the overall negative feedback. This property allows one to define an associated cone K of initial functions in the phase space X, and follow corresponding slowly oscillating solutions in forward time until they enter the cone again at some point. This return point defines a nonlinear invariant map F on cone K which fixed points correspond to periodic solutions of the original system (1). Typically the zero element $(0, 0)$ is a part of the cone, however, it produces the trivial fixed point for the nonlinear map F, as it results in constant equilibrium solution $(x, y) \equiv (0, 0)$ to system (1). One needs a second fixed point of F in order to derive a nontrivial periodic solution to the system. This is achieved by establishing the ejectivity of the trivial fixed point under the non-linear map F on the cone K.

2.4 Ejective Fixed Point Theorem

For the objectives of this paper we are adapting more general definitions and considerations of the ejective fixed point theory from [8, 17] to the partial case of two-dimensional system (1). The Banach space $X = X_1 \times X_2 = C([-\tau, 0], \mathbf{R}) \times C([-\sigma, 0], \mathbf{R})$ is the phase space of system (1). The norm $|| \cdot ||_X$ is defined as the maximum of the two norms for the Banach spaces X_1 and X_2; each one of the latter is defined as the supremum norm on the sets of continuous functions on initial intervals $[-\tau, 0]$ and $[-\sigma, 0]$, respectively.

Let U be a subset of X, $F : U \mapsto X$ be a mapping on U, and $x_* \in U$ be a fixed point of F. The fixed point x_* is called *ejective* if there exists its open neighborhood $G \subset X$ such that for every $x \in G \cap U$, $x \neq x_*$, there is an integer $m = m(x)$ such that $F^m(x) \notin G \cap U$.

The following statement is taken from [17] (Theorem 2.1, Sect. 11.2); its original version is given in paper [28].

Theorem 2 *Suppose $\mathscr{K}$ is closed, bounded, convex infinite-dimensional set in X, map $\mathscr{F} : \mathscr{K} \setminus \{x_*\} \to \mathscr{F}$ is completely continuous, and x_* is an ejective fixed point of $\mathscr{F}$. Then there is a fixed point of $\mathscr{F}$ in $\mathscr{K} \setminus \{x_*\}$.*

In applications of this theorem to delay differential equations the set U is usually a set of initial functions which give rise to slowly oscillating solutions (cone $\mathscr{K}$ mentioned above). The fixed point x_* of the map $\mathscr{F}$ is a trivial fixed point generated by a constant solution of a differential delay system. The other fixed point from $\mathscr{K} \setminus \{x_*\}$ generates a non-constant periodic solution.

3 Main Results

The following theorem is the main result of this paper

Theorem 3 *Suppose that the assumptions $(A1)$–$(A4)$ are satisfied and the characteristic equation (3) has a pair of complex conjugate solutions with positive real part. Then delay differential system (1) has a nontrivial slowly oscillating periodic solution.*

The principal components of the proof are the construction of a cone $\mathscr{K} \subset X$ of initial functions, building of a nonlinear invariant map $\mathscr{F}$ on $\mathscr{K}$ as an appropriate shift along the corresponding solutions, showing the complete continuity of $\mathscr{F}$, and establishing the ejectivity of the zero fixed point of $\mathscr{F}$. These are outlined in the following subsections.

The ejectivity is proved under the assumption that the characteristic equation (3) has a leading pair of complex conjugate solutions $\lambda = \gamma \pm \omega i$ with the positive real part $\gamma > 0$ and the imaginary part satisfying $0 < \omega < \pi/(\tau + \sigma)$. The existence of such leading eigenlavue also implies the oscillatory behavior of all solutions of system (1).

3.1 Invariant Cone, Slow Oscillation, and Nonlinear Mapping

Consider the following cone $\mathscr{K}$ of initial functions:

$$\mathscr{K} = \{(\varphi, \psi) \in X \mid \varphi(s) \geq 0, \varphi(s)\exp\{\alpha s\} \uparrow, s \in [-\tau, 0];$$
$$\psi(s) \geq 0, \psi(s)\exp\{\beta s\} \uparrow, s \in [-\sigma, 0]\}.$$

$\mathscr{K}$ is a closed convex set that includes the zero element $(\varphi, \psi) = (0, 0)$. The latter generates the trivial solution $(x, y) \equiv (0, 0),\ \forall t \geq 0$. When an initial function is not the zero element, $(\varphi, \psi) \not\equiv (0, 0)$, then the corresponding solution $(x(t), y(t))$ is not the trivial zero solution at any $t \geq 0$.

Lemma 1 *Suppose one of the two conditions of Theorem 1 is satisfied. Assume an initial function* $\mathscr{K} \ni (\varphi, \psi) \neq (0, 0)$ *is given, and let* $(x(t), y(t)),\ t \geq 0$, *be the corresponding solution to system (1). Then each component* $x(t)$ *and* $y(t)$ *is slowly oscillating with the following properties holding:*

(i) *The component* $x(t)$ *has an infinite sequence* $\{\xi_n\},\ n \in \mathbf{N}$, *of simple zeros such that* $\xi_{n+1} - \xi_n > d$ *and* $x(t) < 0\ \forall t \in (\xi_{2n-1}, \xi_{2n}),\ x(t) > 0\ \forall t \in (\xi_{2n}, \xi_{2n+1})$*;*
(ii) *The component* $y(t)$ *has an infinite sequence* $\{\eta_n\},\ n \in \mathbf{N}$, *of simple zeros such that* $\eta_{n+1} - \eta_n > d$ *and* $y(t) < 0\ \forall t \in (\eta_{2n-1}, \eta_{2n}),\ y(t) > 0\ \forall t \in (\eta_{2n}, \eta_{2n+1})$*;*
(iii) *Between any two zeros* ξ_n *and* ξ_{n+1} *of the component* $x(t)$ *there is exactly one zero* η_{n+1} *of the component* $y(t)$. *Likewise, between any two zeros* η_n *and* η_{n+1} *of the component* $y(t)$ *there is exactly one zero* ξ_n *of the component* $x(t)$*;*
(iv) *Moreover, there is additional separation between zeros* $\{\xi_n\}$ *and* $\{\eta_n\}$ *so that the following inequalities are satisfied:*

$$\xi_n - \eta_n > \sigma \quad \text{and} \quad \eta_{n+1} - \xi_n > \tau, \quad n \in \mathbf{N}.$$

Note that Lemma 1 also holds under the assumption that the characteristic equation (3) has an eigenvalue with the positive real part. Such conditions are known explicitly (see e.g. [1] for the case $d = 1$), and can be shown to be more restrictive than those of Theorem 1.

We indicate main components of a general outline of the proof of Lemma 1.

First, each component of the solution (x, y) can be represented by an equivalent integral equation as follows. The first component $x(t)$ of the solution (x, y) of system (1) satisfies the following integral equation

$$x(t) = x_0 \exp\{-\alpha(t - t_0)\} + \int_{t_0}^{t} \exp\{\alpha(s - t)\} f(x(s), y(s), x(s - \tau), y(s - \sigma))\, ds, \quad (4)$$

for $t \geq t_0$, where $x_0 = x(t_0)$. Likewise, the second component $y(t)$ satisfies the integral equation

$$y(t) = y_0 \exp\{-\beta(t - t_0)\} + \int_{t_0}^{t} \exp\{\beta(s - t)\} g(x(s), y(s), x(s - \tau), y(s - \sigma))\, ds, \quad (5)$$

for $t \geq t_0$ with $y(t_0) = y_0$. Second, due to the positive and negative feedback assumptions on the nonlinearities f and g as given in assumption (A3), the increasing and decreasing nature of the expressions $x(t) \exp\{\alpha t\}$ and $y(t) \exp\{\beta t\}$ can be deduced from the representations

$$\frac{d}{dt}\left[x(t)\cdot\exp\{\alpha t\}\right]=\exp\{\alpha t\}\, f(x(t),y(t),x(t-\tau),y(t-\sigma)) \tag{6}$$

and

$$\frac{d}{dt}\left[y(t)\cdot\exp\{\beta t\}\right]=\exp\{\beta t\}\, g(x(t),y(t),x(t-\tau),y(t-\sigma)), \tag{7}$$

when $y(t-\sigma)$ and $x(t-\tau)$ are of definite sign respectively.

It follows from Eq. (5) that the component y is the fist one to change the sign at some point $\eta_1 \geq 0$; moreover, $y(t)$ in decreasing on $[0,\eta_1]$ (we make a generic assumption $x(0)>0$, $y(0)>0$; other possibilities when $x(0)=0$ or $y(0)=0$ are analogous and eventually reduced to this one). In view of the integral equation (4) the component x remains positive on the interval $(0,\eta_1+\sigma]$, while the component y is negative on the interval $(\eta_1,\eta_1+\sigma]$. The component $x(t)$ is decreasing for $t\geq\eta_1$; there exists its first simple zero at $t=\xi_1>\eta_1+\sigma$. The component y remains negative on the interval $(\eta_1,\xi_1]$. Equations (4) and (6) next show that the component $x(t)$ is negative on $(\xi_1,\xi_1+\sigma]$ and $x(t)\cdot\exp\{\alpha t\}$ is decreasing there. Due to the integral equation (5) the component $y(t)$ is negative in interval $[\xi_1,\xi_1+\tau]$. In view of (7) the component y is increasing in some right neighborhood $[\xi_1+\tau,\eta_2]$ of $\xi_1+\tau$ where η_2 is its second simple zero. The component x remains negative on the interval $[\xi_1,\eta_2]$.

One can consider now the constructed solution (x,y) on the interval $[0,\eta_2]$ as a new initial function, an element of X. One concludes that the new initial functions $\varphi_1(s):=x(\eta_2+s)$, $s\in[-\tau,0]$ and $\psi_1(s):=y(\eta_2+s)$, $s\in[-\sigma,0]$ belong to the symmetric "negative set" $-\mathscr{K}$ consisting of initial functions (φ,ψ) such that $(-\varphi,-\phi)\in\mathscr{K}$. One can construct now the solution (x,y) for $t\geq 0$ in the very same way to conclude that there exists the second simple zero ξ_2 of the component x such that $\xi_2>\eta_2+\sigma$ and $x(t)>0, t\in[\eta_2+\sigma,\xi_2)$, $y(t)>0, t\in(\eta_2,\xi_2]$ and $y(t)\exp\{\beta t\}$ is increasing in $[\eta_2,\xi_2]$. Continuing further one shows that $x(t)>0, t\in(\xi_2,\xi_2+\sigma]$ and $x(t)\exp\{\alpha t\}$ is increasing there. At the same time $y(t)>0$ and $y(t)\exp\{\beta t\}$ is increasing on the interval $(\eta_2,\xi_2+\sigma]$.

One now defines a mapping $\mathscr{F}$ on $\mathscr{K}$ as follows

$$\forall(\varphi,\phi)\in\mathscr{K}:\ \mathscr{F}(\varphi,\phi)=(\varphi_1,\phi_1),$$

where $(\varphi,\phi)\neq(0,0)$ and $\varphi_1(s)=x(\xi_2+\tau+s)$, $s\in[-\tau,0]$ and $\phi_1(s)=y(\xi_2+\tau+s)$, $s\in[-\sigma,0]$. Due to the construction described in Lemma 1 one has that $(\varphi_1,\phi_1)\in\mathscr{K}$, thus showing that $\mathscr{F}$ maps $\mathscr{K}$ into itself. The one-sided boundedness of either f or g implies that the derivatives of both components x and y are bounded (after the second zeros ξ_2 and η_2). Therefore the map $\mathscr{F}$ is completely continuous, as the set $\mathscr{F}(\mathscr{K})$ of functions is bounded and uniformly continuous.

By the continuity of $\mathscr{F}$ the zero element $(\varphi,\psi)\equiv(0,0)\in\mathscr{K}$ is defined to be mapped into itself under $\mathscr{F}$, as the initial function $(\varphi,\psi)\equiv(0,0)$ results in the zero trivial solution of system (1). Any nonzero fixed point $\mathscr{K}\ni(\varphi_*,\psi_*)\neq(0,0)$ of map $\mathscr{F}$, $\mathscr{F}(\varphi_*,\psi_*)=(\varphi_*,\psi_*)$, gives rise to a nontrivial slowly oscillating periodic solution to system (1). The ejectivity property of the trivial fixed point (0, 0) guarantees the existence of such second fixed point (φ_*, ψ_*). Its general outline is given in the next sub-section.

3.2 Ejectivity

The ejectivity of mapping $\mathscr{F}$ is decided from certain properties of linear functionals constructed on the basis of the linearized system (2) (see [8, 17, 21] for details of general exposition about the functionals). The functionals are coming from projections on eigenspaces associated with particular eigenvalues; they are related to the Laplace transform of the linearized systems (see e.g. [1, 15, 21, 29, 34] for more details on specific cases). The functionals for system (2) turn out to be of the form:

$$L_1(x, y) = (\lambda + \beta)x(0) + (\lambda + \alpha)(\lambda + \beta)\int_{-\tau}^{0} \exp\{-\lambda s\}x(s)\,ds$$
$$+ a_1 \exp\{-\lambda\sigma\}y(0) + a_1(\lambda + \beta)\exp\{-\lambda\sigma\}\int_{-\sigma}^{0} \exp\{-\lambda s\}y(s)\,ds$$

and

$$L_2(x, y) = (\lambda + \alpha)y(0) + (\lambda + \alpha)(\lambda + \beta)\int_{-\sigma}^{0} \exp\{-\lambda s\}y(s)\,ds$$
$$- a_2 \exp\{-\lambda\tau\}x(0) - a_2(\lambda + \alpha)\exp\{-\lambda\tau\}\int_{-\tau}^{0} \exp\{-\lambda s\}x(s)\,ds.$$

The two functionals are derived by using the Laplace transform $\mathscr{L}_u(\lambda) := \int_0^\infty \exp\{-\lambda t\}u(t)\,dt$ on the components x and y of the solutions of system (2) when the latter is subject to the Laplace transformation. Functional L_1 comes out when $\mathscr{L}_x(\lambda)$ is excluded from the algebraic system, while L_2 appears when $\mathscr{L}_y(\lambda)$ is excluded (therefore, they are equivalent). The functionals are also used to represent the projection $\Pi(\lambda)$ on the eigenspace corresponding to an eigenvalue λ (see e.g. [8, 17, 20, 21] for more details, where the relationship between the Laplace transforms of solutions and the projection Π is described and studied).

The ejectivity follows from either one of the two inequalities

$$|L_1(x, y)| \geq c_1||(x, y)|| \quad \text{or} \quad |L_2(x, y)| \geq c_2||(x, y)||, \quad \forall (x, y) \in \mathscr{K},$$

which are in turn equivalent to the projection's boundedness away from zero

$$\sup\{|L_k(x, y)|, ||(x, y)|| = 1\} = l_k > 0, \; k = 1, 2,$$

when $\lambda = \gamma + \omega i$ is the leading solution of the characteristic equation with the positive real part $\gamma > 0$ and the imaginary part satisfying $0 < \omega < \pi/(\tau + \sigma)$ (see e.g. [17], Theorem 2.3 (ii), p. 337). The latter is proved by using a detailed analysis of functionals L_1 and L_2 under the assumption $||(x, y)|| = 1$. For example, when $|x| = 1$ then $x(0) \geq \exp\{-\alpha\tau\}$ is satisfied. One considers next the expression $L_1^* = L_1(x, y)\exp(\lambda\sigma)$ and estimates the lower bound of its imaginary part. The first term of L_1^* is estimated as $|\text{Im}\{(\lambda + \beta)\exp(\lambda\sigma)\}| \geq m_0$ for some $m_0 > 0$ independent of the particular choice of $x(s)$, $s \in [-\tau, 0]$ (when $\sigma < \tau$). The imaginary parts of the second and forth terms of L_1^* can be shown to be each positive (however, not uniformly bounded away from zero). The third term of L_1^* is pure real. Therefore, one deduces that $|L_1^*| \geq |\text{Im}(L_1^*)| \geq m_1 > 0$ is satisfied, implying also that $|L_1| \geq m_2 > 0$ is valid. The other consideration $|y| = 1$ is similar with the use of functional L_2.

4 Conclusion

We establish sufficient conditions for the existence of non-trivial slowly oscillating periodic solutions for a new class of two-dimensional differential delay systems. Those systems are more general than some of the previously studied models such as systems with cyclic overall negative feedback [6, 21], or a model of a circadian rhythm generator [30], or a second order non-linear differential delay equation [1]; they include those mentioned as partial cases. The proof of the existence of periodic solutions follows along the lines of standard techniques of the ejective fixed point theory [8, 17]. However, the specific considerations on major steps somewhat differ from those previously employed. These distinctions require new appropriate adjustments and further developments to several points of the established theory.

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