Space Kinematics and Projective Differential Geometry over the Ring of Dual Numbers

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Abstract. We study an isomorphism between the group of rigid body displacements and the group of dual quaternions modulo the dual number multiplicative group from the viewpoint of differential geometry in a projective space over the dual numbers. Some seemingly weird phenomena in this space have lucid kinematic interpretations. An example is the existence of non-straight curves with a continuum of osculating tangents which correspond to motions in a cylinder group with osculating vertical Darboux motions. We also look at the set of osculating conics of a curve in projective space, suggest geometrically meaningful examples and briefly discuss and illustrate their corresponding motions.

1 Introduction

The eight-dimensional real algebra $\mathbb{DH}$ of dual quaternions provides a well-known model for the group SE(3) of rigid body displacements. Dual quaternions with non-zero real norm represent elements of SE(3) and are uniquely determined up to real scalar multiples. In the projectivization $\mathbb{P}(\mathbb{DH}) \cong \mathbb{P}^7(\mathbb{R})$ they correspond to points of the Study quadric $\mathcal{S}$ minus an exceptional subspace E of dimension three.

The Study quadric model provides a rich geometric and algebraic environment for investigating questions of space kinematics. However, its "curved" nature poses serious problems in numerous applications. One way of getting around this is to consider dual quaternions modulo multiplication by *dual numbers* instead of just real numbers. The locus of the ensuing geometry is then not the set $\mathcal{S} \setminus E \subset \mathbb{P}^7(\mathbb{R})$ but the projective space $\mathbb{P}^3(\mathbb{D})$ of dimension three over the dual numbers (minus a low dimensional subset.) It provides now a *linear* model of space kinematics which is certainly a big advantage. However, it also comes with some rather counter-intuitive properties: The connecting straight line of two points is no longer unique and there exist curves with an osculating tangent in any of their points.

What seems rather strange from a traditional geometric viewpoint becomes much more natural in a kinematic interpretation where straight lines in $\mathbb{P}^3(\mathbb{D})$ correspond to vertical Darboux motions. Two poses may be interpolated by

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an infinity of vertical Darboux motions [5] and motions in cylinder groups, for example helical motions, admit osculating Darboux motions at any instance. We demonstrate and illustrate this in Sect. 3.

In Sect. 4 we present some preliminary results on osculating conics/motions. Generically, there exists a four dimensional set of osculating conics in every curve point. Among them we find the well-known Bennett motions but we also suggest another type of osculating conic with geometric significance. Its construction is based on the construction of osculating circles in elliptic geometry.

2 Preliminaries

A dual number is an element of the factor ring $\mathbb{D} := \mathbb{R}[\varepsilon]/\langle\varepsilon^2\rangle$. It is uniquely represented by a linear polynomial $a+\varepsilon b$ in the indeterminate ε with coefficients a, $b \in \mathbb{R}$. Sum and product of two dual numbers as implied by this definition are

$$(a+\varepsilon b)+(c+\varepsilon d)=a+c+\varepsilon(b+d), \quad (a+\varepsilon b)(c+\varepsilon d)=ac+\varepsilon(ad+bc).$$

Multiplication obeys the rule $\varepsilon^2 = 0$. The multiplicative inverse of $a+\varepsilon b$ exists if $a \neq 0$ and is then given by $(a+\varepsilon b)^{-1} = a^{-1} - \varepsilon b a^{-2}$. We denote the set of invertible dual numbers by $\mathbb{D}^\times$.

2.1 Projective Geometry over Dual Numbers

Similar to the common projective geometry over the real or complex numbers, we can study projective geometry over the dual numbers. We focus on the projective space $\mathbb{P}^3(\mathbb{D})$ of dimension three over the dual numbers as this will be the relevant case for doing rigid body kinematics. The elements of $\mathbb{P}^3(\mathbb{D})$ are equivalence classes of elements of $\mathbb{D}^4\setminus\{0\}$ where two vectors x and y are considered equivalent if there exists an invertible dual number $a+\varepsilon b$ such that $(a+\varepsilon b)x = y$. We denote equivalence classes by square brackets as $[x]$ where $x \in \mathbb{D}^4$ or as $[x_0, x_1, x_2, x_3]$ where x_0, x_1, x_2, $x_3 \in \mathbb{D}$.

In spite of its formal similarity with $\mathbb{P}^3(\mathbb{R})$ or $\mathbb{P}^3(\mathbb{C})$, the space $\mathbb{P}^3(\mathbb{D})$ exhibits some rather unusual properties. Let us consider the connecting straight line of two points $[a]$ and $[b]$. Already for its definition we have two choices. It can be considered as point set

$$\{[\alpha a+\beta b] \mid (\alpha,\beta)\in\mathbb{F}^2, (\alpha,\beta)\neq(0,0)\} \tag{1}$$

where $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{D}$, respectively. We will reserve the word *straight line* for the case $\mathbb{F}=\mathbb{R}$. There are two reasons for this preference: Firstly, it seems to be the common notion in projective geometry over rings. Secondly, a straight line in this sense has real dimension one (while it has real dimension two otherwise). With regard to kinematics, this means that a straight line describes a more common one-parametric motion.

A first, possibly surprising, geometric property of $\mathbb{P}^3(\mathbb{D})$ refers to the connecting straight lines of two points. In contrast to geometry over the real numbers, it is no longer unique.

Proposition 1. *Any two different points* $[c]$ *and* $[d] \in \mathbb{P}^3(\mathbb{D})$ *with invertible* c *and* d *have infinitely many connecting straight lines. The real dimension of the set of all connecting straight lines is two.*

Proof. We may parameterize any straight line connecting the given points by (1) where $a = \gamma c$, $b = \delta d$ and γ, $\delta \in \mathbb{D}^\times$. This gives four real parameters—the coefficients of γ and δ. But multiplying c and d simultaneously with the same invertible dual number yields identical lines. Thus, only two essential real parameters remain. □

2.2 Space Kinematics

A *quaternion* is an element of the algebra $\mathbb{H}$ generated by basis elements 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ with generating relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ over the real numbers. A *dual* quaternion q is an element of the algebra $\mathbb{DH}$ with the same basis elements and generating relations but over the *dual numbers.* Thus, we may write $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ with q_0, q_1, q_2, $q_3 \in \mathbb{D}$ or, separating primal and dual parts, $q = p + \varepsilon d$ where $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ and $d = d_0 + d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ are elements of $\mathbb{H}$.

The conjugate dual quaternion is $q^\star = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k} = p^\star + \varepsilon d^\star$, the dual quaternion norm is $qq^\star$. In terms of (coefficients of) p and d it may be written as $qq^\star = pp^\star + \varepsilon(pd^\star + dp^\star) = p_0^2 + p_1^2 + p_2^2 + p_3^2 + 2\varepsilon(p_0d_0 + p_1d_1 + p_2d_2 + p_3d_3)$. It is thus a dual number. The unit norm conditions reads as

$$pp^\star = 1, \quad pd^\star + dp^\star = 0.$$

Because the norm is multiplicative, the unit dual quaternions form a multiplicative group $\mathbb{DH}_0^\times$. We embed $\mathbb{R}^3$ into $\mathbb{DH}$ via $(x_1, x_2, x_3) \hookrightarrow 1 + \varepsilon(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})$ and define the action of $q = p + \varepsilon d \in \mathbb{DH}_0^\times$ on points of $\mathbb{R}^3$ via

$$1 + \varepsilon(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}) \mapsto 1 + \varepsilon(y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}) = (p - \varepsilon d)x(p^\star + \varepsilon d^\star). \quad (2)$$

This action provides us with an isomorphism between the groups $\mathbb{DH}_0^\times$ and SE(3), the group of rigid body displacements.

A slight modification of (2) extends the action to points $[x_0, x_1, x_2, x_3]$ in the projective extension $\mathbb{P}^3(\mathbb{R})$ of $\mathbb{R}^3$:

$$[x_0 + \varepsilon(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})] \mapsto [y_0 + \varepsilon(y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k})] = [(p - \varepsilon d)x(p^\star + \varepsilon d^\star)]. \quad (3)$$

This gives an isomorphism between the dual quaternions of non-zero real norm modulo $\mathbb{R}^\times$, the real multiplicative group, and SE(3). The unit norm condition of (2) is replaced by the condition that the norm of q be real but non-zero:

$$pd^\star + dp^\star = 0, \quad pp^\star \neq 0. \quad (4)$$

This means that $[q] = [p + \varepsilon d]$ is a point of the quadric given by the quadratic form $pd^\star + dp^\star$—the so-called *Study quadric* $\mathcal{S}$—minus the *null cone* $\mathcal{N}$ given by the singular quadratic form $pp^\star$. The only real points of $\mathcal{N}$ are those of its three-dimensional vertex space E. We call it the *exceptional generator.*

A crucial observation for this article is that even the real norm requirement (4) can be abandoned: As long as $qq^\star$ is invertible, (3) will describe a valid action on $\mathbb{P}^3(\mathbb{R})$ and provide a *homomorphism* from the group $\mathbb{DH}^\times$ of invertible dual quaternions modulo $\mathbb{R}^\times$ to SE(3) or even *isomorphism* between $\mathbb{DH}^\times/\mathbb{D}^\times$ and SE(3).

Proposition 2. *The groups* $\mathbb{DH}^\times/\mathbb{D}^\times$ *and* SE(3) *are isomorphic via the action* (3)*.*

Proof. It is easy to see that $\mathbb{DH}^\times$ is homomorphic to SE(3) via (3). In order to see that $\mathbb{DH}^\times/\mathbb{D}^\times$ is isomorphic, we have to show that dual multiples yield the same action and that identical action implies a dual factor.

Using the notation $q_\varepsilon := p - \varepsilon d$ for the ε-conjugate of $q = p + \varepsilon d$ we can write the right-hand side of (3) as $q_\varepsilon x q^\star$. Multiplying q with a dual number a yields $(aq)_\varepsilon x a q^\star = a_\varepsilon q_\varepsilon x a q^\star = (a_\varepsilon a) q_\varepsilon x q^\star$. Because $a_\varepsilon a$ equals the primal part of a squared, this does not change the action on $\mathbb{P}^3(\mathbb{R})$. Existence of a dual factor from identical action follows from equal dimension of $\mathbb{DH}^\times/\mathbb{D}^\times$ and SE(3) and the fact that these groups have only one connected component.

Since all elements of $\mathbb{DH}^\times/\mathbb{D}^\times$ are points of $\mathbb{P}^3(\mathbb{D})$, it is natural to study space kinematics via the projective geometry of $\mathbb{P}^3(\mathbb{D})$. This point of view is not new. It played a role in [2] or [3]. From an old paper by C. Segre [6] we even infer that probably already E. Study and his disciples were aware of these connections in the first decades of the 20th century.

2.3 Straight Lines

Via the action (3), a curve in $\mathbb{P}^3(\mathbb{D})$ corresponds to a one-parametric rigid body motion. In particular, polynomial curves yield motions with polynomial trajectories in homogeneous coordinates, that is, *rational motions.* The simplest example of such motions comes from straight lines in $\mathbb{P}^3(\mathbb{D})$ which correspond to *vertical Darboux motions* [3,4]. A vertical Darboux motion is the composition of a unit speed rotation about a fixed axis with a harmonic oscillation along the axis such that one full rotation corresponds to one oscillation period. Its trajectories are bounded rational curves of degree two (ellipses). Rotations and translations are considered as special cases of vertical Darboux motions with zero or infinite oscillation amplitude, respectively.

We illustrate a vertical Darboux motion in Fig. 1. This figure also helps us explain a generally useful concept: Motions obtained as composition of rotation around an axis and translation along the same axis have trajectories on a right circular cylinder. Any curve γ on such a cylinder can be used to completely specify the motion by adding a Cartesian frame consisting of cylinder normal, cylinder generator and horizontal cylinder tangent. Instead of the curve on the cylinder, we may equally well consider its image when developing the cylinder

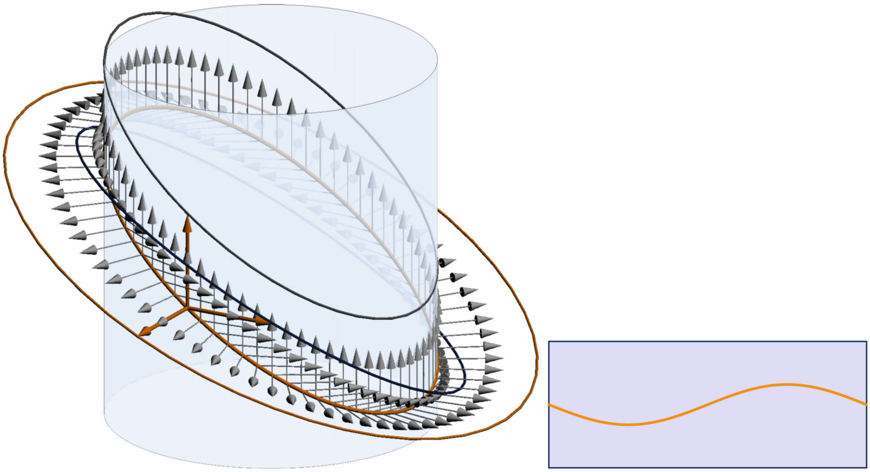

Fig. 1. Vertical Darboux motion with some elliptic trajectories, right circular cylinder, and development

surface. In case of a vertical Darboux motion, γ is an ellipse. Its development is a suitable positioned sine curve which is scaled in direction of the developed cylinder generators in order to adapt to the oscillation's amplitude.

3 Osculating Lines

In this section we demonstrate that a helical motion and a vertical Darboux motion can have second order contact at any parameter value. Since vertical Darboux motions correspond to straight lines in $\mathbb{P}^3(\mathbb{D})$ this amounts to saying that a curve corresponding to a helical motions has an osculating tangent at any point. This is a remarkable difference to classical differential geometry over the real numbers where this property characterizes straight lines.

A rotation about axis $\mathbf{k}$ with rotation angle ω is given by the dual quaternion $r = \cos\frac{\omega}{2} + \sin\frac{\omega}{2}\mathbf{k}$, a translation with oriented distance δ in direction of $\mathbf{k}$ is given by $t = 1 - \frac{1}{2}\varepsilon\delta\mathbf{k}$. Thus, a helical motion h with pitch p and a Darboux motion d are obtained by substituting $p\omega$ and $p\sin\omega$, respectively, for δ in the product rt:

$$
\begin{aligned}
h &= \cos\left(\tfrac{\omega}{2}\right) + \sin\left(\tfrac{\omega}{2}\right)\mathbf{k} + \frac{p}{2}\omega\varepsilon\left(\sin\left(\tfrac{\omega}{2}\right) - \cos\left(\tfrac{\omega}{2}\right)\mathbf{k}\right),\\
d &= \cos\left(\tfrac{\omega}{2}\right) + \sin\left(\tfrac{\omega}{2}\right)\mathbf{k} + \frac{p}{2}\sin\omega\varepsilon\left(\sin\left(\tfrac{\omega}{2}\right) - \cos\left(\tfrac{\omega}{2}\right)\mathbf{k}\right)
\end{aligned}
\tag{5}
$$

With this, we compute

$$
\frac{\mathrm{d}h}{\mathrm{d}\omega}(0) = \frac{\mathrm{d}d}{\mathrm{d}\omega}(0) = \frac{1}{2}\mathbf{k} - \frac{1}{2}p\varepsilon\mathbf{k} \quad \text{and} \quad \frac{\mathrm{d}^2h}{\mathrm{d}\omega^2}(0) = \frac{\mathrm{d}^2d}{\mathrm{d}\omega^2}(0) = -\frac{1}{4} + \frac{1}{2}p\varepsilon,
$$

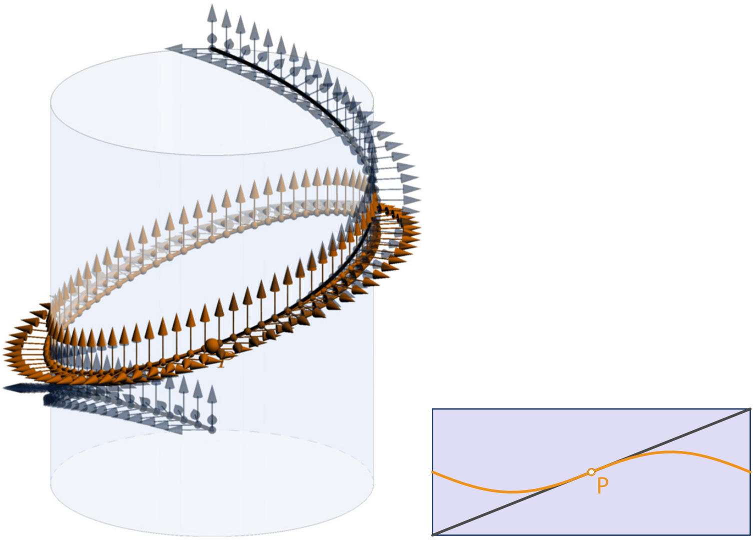

Fig. 2. Geometric interpretation of osculating lines

while

$$\frac{\mathrm{d}^3 h}{\mathrm{d}\omega^3}(0) = -\frac{1}{8}\mathbf{k} + \frac{3}{8}p\varepsilon\mathbf{k} \neq -\frac{1}{8}\mathbf{k} + \frac{7}{8}p\varepsilon\mathbf{k} = \frac{\mathrm{d}^3 d}{\mathrm{d}\omega^3}(0).$$

Thus, the motions (5) have second order contact at $\omega = 0$. Since this parameter value has no particular meaning for a helical motion, we may state that *for any instance of a helical motion there exists a vertical Darboux motion with second order contact.*

Let us also verify that d is actually a straight line in $\mathbb{P}^3(\mathbb{D})$ by multiplying its parametric representation (5) with a suitable dual number valued function. Indeed, we have

$$(1 + p\varepsilon\cos^2(\tfrac{\omega}{2}))d = \cos(\tfrac{\omega}{2})(1 + p\varepsilon) + \sin(\tfrac{\omega}{2})\mathbf{k}$$

which is a parametric representation of the straight line spanned by $1 + p\varepsilon$ and $\mathbf{k}$. Summarizing, we can thus state

Theorem 1. *At any instance in time any helical motion, viewed as a curve in kinematic space $\mathbb{P}^3(\mathbb{D})$, has second order contact with a straight line. Yet, it is not a straight line itself.*

This seemingly strange behavior allows a clear geometric interpretation that also gives additional insight. Figure 2 displays helical motion and osculating Darboux motion via the cylinder model we discussed earlier. In the development, the helical motion corresponds to a straight line while the Darboux motion is a sine curve with this line as inflection tangent. Obviously, it is possible to determine uniquely a suitable sine function in every point. It gives rise to the unique osculating vertical Darboux motion in a point of the helical motion.

Helical motions are not the only curves in $\mathbb{P}^3(\mathbb{D})$ susceptible to second order approximation by straight lines in every point. An arbitrary motion in the cylinder group C corresponds to a curve in the development. There, an osculating sine function can be drawn in any sufficiently smooth point and gives rise to an osculating vertical Darboux motion. The possibility to do so is a direct consequence of the following lemma: Among all candidate sine functions there exist one with prescribed slope and curvature.

Lemma 1. *Given two real numbers k, $\varkappa \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that some point on the graph of the function $\varphi \mapsto a \sin\varphi$ has slope k and curvature $\varkappa$.*

Proof. The subgraphs corresponding to parameter intervals $[i\frac{\pi}{2}, (i+1)\frac{\pi}{2}]$ for $i \in \{0, 1, 2, 3\}$ are congruent and, up to respective signs, have points of equal slope and curvature. Thus, we may restrict to the case $i = 0$, $k \geq 0$, $\varkappa \leq 0$ and search for $a > 0$. Slope and curvature are given by

$$k = a\cos\varphi \quad \text{and} \quad \varkappa = -\frac{a\sin\varphi}{(1 + a^2\cos^2\varphi)^{3/4}}$$

Because of $\varphi \in [0, \frac{\pi}{2}]$ we may substitute c for $\cos\varphi$ and $\sqrt{1-c^2}$ for $\sin\varphi$. With this, $a = k/c$ and the formula for $\varkappa$ reduces to an even quartic equation for a with discriminant $4\varkappa^4(1+k^2)^3 \geq 0$. Thus, solutions for a in $\mathbb{R}$ do exist. Because of our assumptions on φ, k and $\varkappa$, it must necessarily be positive. □

Corollary 1. *Any sufficiently smooth motion in the group generated by all rotations around a fixed axis and all translations in direction of this axis has an osculating Darboux motion in any of its points (at any instance).*

4 Osculating Conics

We now turn our attention to conic sections in $\mathbb{P}^3(\mathbb{D})$. We study them as rational curves of degree two. A parametric representation is simply a polynomial C of degree two in one indeterminate t that serves as a real parameter. We assume that C has no scalar polynomial factor of positive degree, as otherwise it would be a linear parametric representation in disguise, and also that the coefficients are independent, as otherwise it would be a quadratic parametrization of a straight line. A conic parameterizes a rational motion with trajectories of degree at most four.

In line with the general philosophy we should consider a polynomial C up to multiplication with a dual number valued function but in the context of this article it is be sufficient to consider only dual number multiples, that is, polynomial representations of minimal degree. In projective differential geometry over the real numbers, a generic smooth space curve admits a two parametric set of osculating conics in a generic point. In projective geometry over the dual numbers, a further degree of freedom is added:

For the case of interpolating conics for three finitely separated points $[c_0]$, $[c_1]$, $[c_2] \in \mathbb{P}^3(\mathbb{D})$, this is easy to see. An interpolating conic may be parameterized as $[c(t)]$ where

$$c(t) = c_0 + (c_1 - c_0 - c_2)t + c_2 t^2.$$

The points $[c_0]$, $[c_1]$, $[c_2]$ correspond to parameter values $t = 0$, $t = 1$, and $t = \infty$, respectively. Obviously, different dual number multiples of c_0, c_1 and c_2 yield different conics, unless the dual factor is the same for all three points. We may use this freedom to have the dual factor 1 for c_1 whence a general parametric representation for interpolating conics can be written as

$$c(t) = \gamma_0 c_0 + (c_1 - \gamma_0 c_0 - \gamma_2 c_2)t + \gamma_2 c_2 t^2 \tag{6}$$

where γ_0 and γ_2 are invertible dual numbers.

In view of Sect. 3 it is natural to ask for space curves that admit a conic with even higher order contact in every point. We will not pursue this question any further at this place. Instead, we present two examples of osculating conics in this set with a special meaning for space kinematics.

4.1 Bennett Motions

The *Bennett motion* is a well-known example of a quartic space motion whose kinematic image in the "classical" sense is a conic section on the Study quadric $\mathcal{S}$ and which is determined by three general finitely separated or infinitesimally neighboring points in the Study quadric. In fact, we may simply define it as a regular conic in the Study quadric that does not intersect the exceptional generator E. In our context, we can re-derive the motion from the following observation:

Lemma 2. *Given an invertible dual quaternion p there exists an invertible dual number a such that ap has real norm. The dual number a is determined up to a real multiple.*

Proof. Write $p = p' + \varepsilon p''$ and $a = a' + \varepsilon a''$ with quaternions p', p'' and real numbers a', a''. The dual part of the norm of ap then reads as $a'^2(p'p''^\star + p''p'^\star) + 2a'a''p'p'^\star$. Both, $p'p''^\star + p''p'^\star$ and $p'p'^\star$ are real numbers and the latter is different from zero (because p is invertible). We divide by a' (because we want to find an invertible dual number a) so that ultimately $a = a' + \varepsilon a''$ is determined, up to a real multiple, by one non-vanishing homegeneous linear equation. A solution with $a' = 0$ is not possible because p is invertible whence $p'p'^\star \neq 0$. □

Returning to (6), we may assume $[c_0]$, $[c_1]$, $[c_2] \in \mathcal{S}$ as otherwise we can multiply with suitable dual numbers by Lemma 2. Now we are still free to multiply c_0, c_1, and c_2 with real numbers and it is well-known (c. f. for example [1]) that this freedom is enough to ensure that $[c(t)]$ lies on the Study quadric $\mathcal{S}$.

Bennett motions are rational motions with entirely circular trajectories of degree four. They appear as coupler motions of *Bennett linkages*, that is, spatial four-bar linkages with exceptional mobility [1]. An example can be found later in Fig. 4.

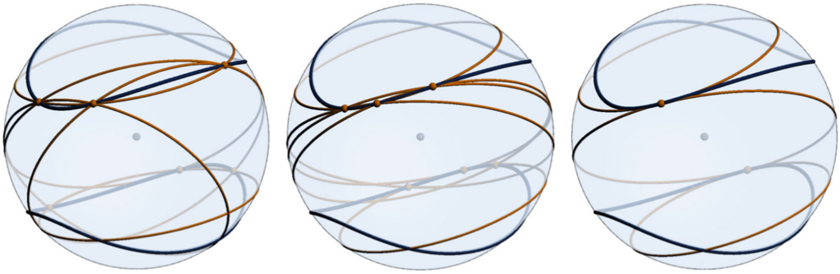

Fig. 3. Circles through three points (left and middle) and osculating circle in elliptic geometry (right)

4.2 Motions Based on Osculating Circles of Elliptic Geometry

An important object for the kinematic geometry in $\mathbb{P}^3(\mathbb{D})$ is the null cone $\mathcal{N}$. It consists of points represented by non-invertible dual quaternions—a property that does not change under coordinate changes and thus makes $\mathcal{N}$ a *geometric invariant*. With this in mind, it is thus natural to look for osculating conics in special position with respect to $\mathcal{N}$. For a general parametric representation of the shape (6) it is possible to determine the dual factors $\gamma_0, \gamma_2 \in \mathbb{D}$ in such a way that the conic parameterized by $[c(t)]$ is *tangent to* $\mathcal{N}$ *in two points.* In fact, if we only consider *real* factors, this amounts to determining a circle through three points in the real elliptic plane with absolute conic $\mathcal{N} \cap \varphi$ where φ is the conic's plane. For three finitely separated points this problem has four solutions as can be seen in the spherical model of elliptic geometry (Fig. 3). But this property

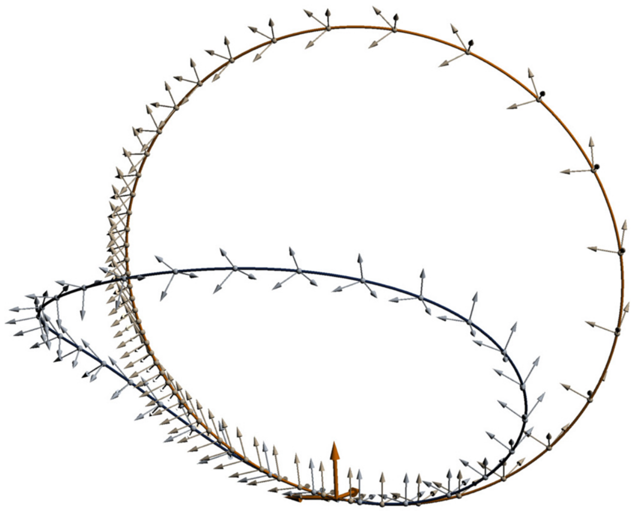

Fig. 4. A Bennet motion (orange) and an osculating null cone motion (blue)

does not translate to three infinitesimally neighboring points as in the limit three of the four circle converge to the curve tangent so that the osculating circle is unique. This is also visualized in Fig. 3.

In lack of a better name, we refer to the motions in question as *quadratic null cone motions.* The four-dimensional set of osculating conics contains a two-dimensional set of these motions. Their generic trajectories are rational of degree four, not circular in general but tangent to the plane at infinity in a pair of conjugate complex points and hence bounded (Fig. 4).

Figure 4 actually displays a null cone motion and a Bennett motion that osculate at one pose which is drawn a little larger.

5 Conclusion

We have related space kinematics to the geometry of the projective space $\mathbb{P}^3(\mathbb{D})$ over the ring of dual numbers. This interpretation seems well suited for kinematic visualization of certain differential geometric aspects and it also provides the proper mathematical framework for the systematic study of osculating motions. We presented results for ordinary and osculating tangents and some preliminary ideas about osculating conics that shall be deepened in the future.

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