Chapter 1
Poisson and Symplectic Structures, Hamiltonian Action, Momentum and Reduction

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1.1 Introduction

These lectures have been delivered by the first author in the Summer School "*Differential Geometry, Differential Equations, and Mathematical Physics*" at Wisła, Poland from 19th to 29th of August 2019. The second author took the notes of these lectures.

As the title suggests, the material covered here includes the Poisson and symplectic structures (Poisson manifolds, Poisson bi-vectors, and Poisson brackets), group actions and orbits (infinitesimal action, stabilizers, and adjoint representations), moment maps, Poisson and Hamiltonian actions. Finally, the phase space reduction is also discussed. The Poisson structures are a particular instance of Jacobi structures introduced by A. Lichnerowicz back in 1977 [7]. Several capital contributions to this field were made by A. Weinstein, see e.g. [13].

The text below does not pretend to provide any new scientific results. However, we believe that this point of view and exposition will be of some interest to our readers. As other general (and excellent) references on this topic include:

- R. Abraham and J.E. Marsden. *Foundations of Mechanics*, second edn., Addison–Wesley Publishing Company, Redwood City, CA, 1987 [1]

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- V.I. Arnold. *Mathematical methods of classical mechanics*, second edn., Springer, New York, 1997 [2]
- A.M. Vinogradov and B.A. Kupershmidt. *The structures of Hamiltonian mechanics*, Russ. Math. Surv., **32**(4), 177–243, 1977 [12]

We can mention also another set of recent lecture notes on the symplectic and contact geometries [11]. We mention also the classical review of this topic [14].

Our presentation remains at quite elementary level of exposition. We restricted deliberately ourselves to the presentation of basic notions and the state of the art as it was in 1990–2000. Nevertheless, we hope that motivated students will be inspired to find more advanced and modern material which is inevitably based on these elementary notes.

In the following text each Section corresponds to a separate lecture. This text is organized as follows. The Poisson and symplectic structures are presented in Sect. 1.2. The group actions and orbits are introduced in Sect. 1.3. The moment map, Poisson and Hamiltonian actions are described in Sect. 1.4. Finally, the manuscript is ended with the description of the phase space reduction in Sect. 1.5. The very last Sect. 1.6 is a brief (and essentially incomplete) introduction to Poisson–Lie structures and some related notions. An excellent account of the last topic can be found in the survey paper by Y. Kossmann-Schwarzbach (1997) [6].

1.2 Poisson and Symplectic Structures

Hamiltonian systems are usually introduced in the context of the symplectic geometry [5, 10]. However, the use of Poisson geometry emphasizes the Lie algebra structure, which underlies the Hamiltonian mechanics.

1.2.1 Poisson Manifolds

Let $\mathbb{M}$ be a smooth manifold with a bracket

$$\{-,\,-\}:\; C^{\infty}(\mathbb{M})\times C^{\infty}(\mathbb{M})\;\longmapsto\; C^{\infty}(\mathbb{M}),$$

which verifies the following properties:

- Bi-linearity $\{-,\,-\}$ is real-bilinear.
- Anti-symmetry $\{F,\,G\} = -\{G,\,F\}$.
- Jacobi identity $\{\{F,\,G\},\,H\} + \{\{G,\,H\},\,F\} + \{\{H,\,F\},\,G\} = 0$.
- Leibniz identity $\{FG,\,H\} = F\{G,\,H\} + \{F,\,H\}G$.

Then, the bracket $\{-,\,-\}$ is a Poisson bracket and the pair $\bigl(\mathbb{M};\,\{-,\,-\}\bigr)$ will be called a Poisson manifold. A Poisson algebra is defined as the following pair

$\big(C^\infty(\mathbb{M};\mathbb{R});\{-,-\}\big)$. Thanks to the first three properties of the Poisson bracket, it is not difficult to see that $\big(C^\infty(\mathbb{M};\mathbb{R});\{-,-\}\big)$ is also a Lie algebra. The last property of the Poisson bracket (i.e. the Leibniz identity) implies that it is also a *derivative* in each of its arguments.

Let $\big(\mathbb{M};\{-,-\}\big)$ be a Poisson manifold and $H \in C^\infty(\mathbb{M};\mathbb{R})$, then there exists a unique vector field $\mathcal{X}_H$ such that

$$\mathcal{X}_H(G) = \{G, H\}, \qquad \forall G \in C^\infty(\mathbb{M};\mathbb{R}).$$

The vector field $\mathcal{X}_H$ is called the Hamiltonian vector field with respect to the Poisson structure with H being the Hamiltonian function. Let $\mathfrak{X}(\mathbb{M})$ denote the space of all vector fields on $\mathbb{M}$. Then, the just constructed mapping $C^\infty(\mathbb{M};\mathbb{R}) \longrightarrow \mathfrak{X}(\mathbb{M})$ is a Lie algebra morphism, i.e. $\mathcal{X}_{\{F,G\}} = [\mathcal{X}_F, \mathcal{X}_G]$.

Definition 1.1 A Casimir function on a Poisson manifold $\big(\mathbb{M};\{-,-\}\big)$ is a function $F \in C^\infty(\mathbb{M};\mathbb{R})$ such that for all $G \in C^\infty(\mathbb{M};\mathbb{R})$ one has

$$\{F, G\} = 0, \qquad \forall G \in C^\infty(\mathbb{M};\mathbb{R}).$$

1.2.2 Poisson Bi-vector

If $\big(\mathbb{M};\{-,-\}\big)$ is a Poisson manifold, then there exists a contravariant anti-symmetric two-tensor $\pi \in \Lambda^2(\mathbb{TM})$ or equivalently

$$\pi : \mathbb{T}^*\mathbb{M} \times \mathbb{T}^*\mathbb{M} \longrightarrow \mathbb{R}$$

such that

$$\langle \pi, \mathrm{d}F \wedge \mathrm{d}G\rangle(z) = \pi(z)\big(\mathrm{d}F(z); \mathrm{d}G(z)\big) = \{F, G\}(z).$$

In local coordinates $(z_1, z_2, \ldots, z_n)$ we have the following expression for the Poisson bracket:

$$\{F, G\} = \sum_{i,j} \pi^{ij} \frac{\partial F}{\partial z_i} \frac{\partial G}{\partial z_j},$$

where $\pi^{ij} \stackrel{\text{def}}{:=} \{z_i, z_j\}$ are called the elements of the *structure matrix* or the Poisson bi-vector of the underlying Poisson structure. For the vector field we have the corresponding expression in coordinates:

$$\mathcal{X}_H = \sum_{i,j} \pi^{ij} \frac{\partial H}{\partial z_i} \frac{\partial}{\partial z_j}, \qquad \text{or} \qquad \mathcal{X}_H^j = \sum_i \pi^{ij} \frac{\partial H}{\partial z_i}.$$

1.2.2.1 Hamilton Map and Jacobi Identity

Let $\pi = (\pi^{ij})$ be a Poisson bi-vector on $\mathbb{M}$, then there exists a $C^\infty(\mathbb{M};\ \mathbb{R})-$linear map $\pi^\sharp :\ \mathbb{T}^*\mathbb{M} \longmapsto \mathbb{T}\mathbb{M}$ given by

$$\pi^\sharp(\alpha) \lrcorner \beta = \{\pi, \alpha \wedge \beta\}(z) = \pi(z)\bigl(\alpha(z), \beta(z)\bigr),$$

where $\lrcorner$ denotes the usual interior product or the substitution of a vector field into a form.

If $\alpha = \mathrm{d}f$ for some $f \in C^\infty(\mathbb{M};\ \mathbb{R})$, then $\pi^\sharp(\mathrm{d}H) = \mathcal{X}_H$. It is not difficult to see how the map $\pi^\sharp$ acts on the basis elements of co-vectors:

$$\pi^\sharp(\mathrm{d}z_i) = \bigl\{z_i, z_j\bigr\}\frac{\partial}{\partial z_j}.$$

Finally, we also have the following Jacobi identity:

$$\pi^{il}\frac{\partial \pi^{jk}}{\partial z_l} + \pi^{jl}\frac{\partial \pi^{ki}}{\partial z_l} + \pi^{kl}\frac{\partial \pi^{ij}}{\partial z_l} = 0. \tag{1.1}$$

1.2.3 Symplectic Structures on Manifolds

Definition 1.2 A *symplectic form* on a real manifold $\mathbb{M}$ is a non-degenerate closed $2-$form $\omega \in \Omega^2(\mathbb{M}) \stackrel{\text{def}}{:=} \Lambda^2(\mathbb{T}^*\mathbb{M})$. Such a manifold is called a *symplectic manifold* and it is denoted by a couple $(\mathbb{M};\ \omega)$.

Let $\bigl(\mathbb{M};\ \{-, -\}\bigr)$ be a Poisson manifold with non-degenerate Poisson structure bi-vector (π^{ij}) and the Hamiltonian isomorphism $\pi^\sharp$ such that $\pi^\sharp(\alpha) = \mathcal{X}$. Then, there is the inverse map $(\pi^\sharp)^{-1} :\ \mathbb{T}\mathbb{M} \longrightarrow \mathbb{T}^*\mathbb{M}$ is defined by the following relation:

$$\mathcal{Y} \lrcorner (\pi^\sharp)^{-1}(\mathcal{X}) = \alpha(\mathcal{Y}).$$

Moreover, the inverse operator $(\pi^\sharp)^{-1}$ defines a $2-$form ω_π as follows:

$$\omega_\pi(\mathcal{X}, \mathcal{Y}) = \Bigl\langle (\pi^\sharp)^{-1}(\mathcal{X}), \mathcal{Y}\Bigr\rangle.$$

1.2.3.1 Darboux Theorem and Hamiltonian Vector Fields

The following Lemma describes some important properties of the just defined form ω_π along with the underlying manifold $\mathbb{M}$:

Lemma 1.1 *The real manifold* $\mathbb{M}$ *always has an even dimension. The form* ω_π *is a symplectic* $2-$*form on* $\mathbb{M}$. *The Jacobi identity* (1.1) *is equivalent to* $\mathrm{d}\,\omega_\pi = 0$.

Proof Left to the reader as an exercise. □

Theorem 1.1 (G. Darboux) *Let* $(\mathbb{M};\, \omega)$ *be a symplectic manifold. There exists a local coordinate system* $(q_1,\, q_2,\, \dots,\, q_n,\, p_1,\, p_2,\, \dots,\, p_n) \stackrel{\text{def}}{=:} (\mathbf{q},\, \mathbf{p})$ *such that* $\omega = \sum_{i=1}^{n} \mathrm{d}\,q_i \wedge \mathrm{d}\,p_i$. *Such coordinates are called* canonical *or Darboux coordinates.*

Lemma 1.2 *Let* $(\mathbb{M};\, \omega)$ *be a symplectic manifold and* $H \in C^\infty(\mathbb{M};\, \mathbb{R})$ *the Hamiltonian function. Then, there is a unique vector field* $\mathcal{X}_H$ *(i.e. the Hamiltonian vector field associated with the Hamiltonian* H*) on* $\mathbb{M}$ *such that*

$$\mathcal{X}_H \lrcorner\, \omega = \mathrm{d}H\,.$$

The Hamiltonian vector field $\mathcal{X}_H$ *can be written in the canonical coordinates* $(\mathbf{q},\, \mathbf{p})$ *on* $\mathbb{M}$ *as*

$$\mathcal{X}_H = \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right).$$

The Poisson bracket in these coordinates looks like

$$\{F\,,\, G\} = \mathcal{X}_F(G) = \sum_{i,\,j} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right).$$

Proof Left to the reader as an exercise. □

1.2.3.2 Example

Let $\mathbb{S}^2 \stackrel{\text{def}}{:=} \left\{ (x,\, y,\, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \right\}$ be the $2-$sphere, which can be naturally injected in $\mathbb{R}^3$: $\imath\,:\, \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$. The $2-$form $\bar{\omega} \in \Lambda^2(\mathbb{R}^3)$ is given by

$$\bar{\omega} = x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y$$

and $\omega \in \Lambda^2(\mathbb{S}^2)$ is defined as $\omega = \imath^*(\bar{\omega})$.

Lemma 1.3 *The form* ω *gives a symplectic structure on* $\mathbb{S}^2$, *i.e.* $\mathrm{d}\,\omega = 0$ *and this* $2-$*form is non-degenerate on* $\mathbb{S}^2$.

Proof First of all, we observe that the closedness of the $2-$form ω is a straightforward conclusion in view of

$$\mathrm{d}\omega = \mathrm{d}\big(\iota^*(\bar{\omega})\big) = \iota^*\big(\mathrm{d}\bar{\omega}\big) = 0,$$

since $\Lambda^3(\mathbb{S}^2) = 0$. To check that it is non-degenerate, we make a choice of the following charts:

$$\begin{aligned} \phi_N : \mathbb{S}^2 \setminus \{N\} &\longrightarrow \mathbb{R}^2, \\ (x, y, z) &\longmapsto \Big(\frac{x}{1-z}, \frac{y}{1-z}\Big), \end{aligned}$$

$$\begin{aligned} \phi_S : \mathbb{S}^2 \setminus \{S\} &\longrightarrow \mathbb{R}^2, \\ (x, y, z) &\longmapsto \Big(\frac{x}{1+z}, \frac{y}{1+z}\Big). \end{aligned}$$

It is not difficult to see that their inverses are given by the following maps:

$$\phi_N^{-1} : (u, v) \longmapsto \Big(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\Big),$$

$$\phi_S^{-1} : (u, v) \longmapsto \Big(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, -\frac{u^2+v^2-1}{1+u^2+v^2}\Big).$$

In both coordinate charts (u, v) induced by $\phi_{N,S}$ we obtain

$$\begin{aligned} \iota^*(\bar{\omega}) = (x\circ\iota)\,\mathrm{d}(y\circ\iota)\wedge\mathrm{d}(z\circ\iota) + (y\circ\iota)\,\mathrm{d}(z\circ\iota)\wedge\mathrm{d}(x\circ\iota) + (z\circ\iota)\,\mathrm{d}(x\circ\iota)\wedge\mathrm{d}(y\circ\iota) \\ = -\frac{4}{1+u^2+v^2}\,\mathrm{d}u\wedge\mathrm{d}v \neq 0. \end{aligned}$$

1.2.4 Co-tangent Bundle: Liouville Form

Let $\mathbb{M}$ be a smooth $n-$dimensional manifold and $\varpi : \mathbb{T}^*\mathbb{M} \longrightarrow \mathbb{M}$ is the projection map whose differential map is $T_\varpi : \mathbb{T}\mathbb{T}^*\mathbb{M} \longrightarrow \mathbb{T}\,\mathbb{M}$.

Definition 1.3 A differential $1-$form ρ on $\mathbb{T}^*\mathbb{M}$, which is defined by $\sigma_\rho : \mathbb{T}^*\mathbb{M} \longrightarrow \mathbb{T}^*\mathbb{T}^*\mathbb{M}$ as follows:

$$\big\langle \sigma_\rho, \mathcal{X}\big\rangle = \big\langle \rho, T_{\varpi_\rho}(\mathcal{X})\big\rangle, \qquad \mathfrak{X} \in T_{\varpi_\rho}(\mathbb{T}^*\mathbb{M})$$

is called the Liouville (or *action*) form.

If $(\mathbf{q}, \mathbf{p})$ is a local coordinate system on $\mathbb{T}^*\mathbb{M}$, then the form ρ can be written as

$$\rho = \mathbf{p}\,\mathrm{d}\,\mathbf{q} = \sum_{i=1}^{n} p_i\,\mathrm{d}\,q^i\,.$$

The 2−form $\omega \in \Omega^2(\mathbb{T}^*\mathbb{M})$, $\omega = \mathrm{d}\rho = \mathrm{d}\mathbf{p} \wedge \mathrm{d}\,\mathbf{q} = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}\,q^i$ is the *canonical* symplectic form.

1.2.5 Non-Symplectic Poisson Structures

Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra and $\mathfrak{g}^*$ be its dual (as a vector space). If we suppose that $\dim \mathfrak{g} = n$, then $\mathfrak{g}^*$ is isomorphic as a real smooth manifold to $\mathbb{R}^n$.

Theorem 1.2 *There exists a (non-symplectic) Poisson structure on* $\mathfrak{g}^*$.

1.2.6 Poisson Brackets on Dual of a Lie Algebra

A bracket can be defined on $C^\infty(\mathfrak{g}^*)$ by the identification:

$$\mathfrak{g} \simeq \mathfrak{g}^{**} \equiv C^\infty_{\mathrm{lin}}(\mathfrak{g}^*) \subset C^\infty(\mathfrak{g}^*)\,, \qquad \mathcal{X} \longmapsto F_{\mathcal{X}}$$

$$F_{\mathcal{X}}(\xi) = \langle \xi\,, \mathcal{X} \rangle = \xi(\mathcal{X})\,.$$

One has $\left\{ F_{\mathcal{X}}\,,\, F_{\mathcal{Y}} \right\} = F_{[\mathcal{X},\mathcal{Y}]}$. Let $\{e_i\}_{i=1}^{n}$ be a base of $\mathfrak{g}$ and $\left\{ C_{ij}^{k} \right\}$ be structure constants of the Lie algebra $\mathfrak{g}$, i.e.

$$[\,e_i\,,\, e_j\,] = \sum_{k=1}^{n} C_{ij}^{k}\, e_k\,.$$

Let $\{F_i\}_{i=1}^{n}$ be a dual basis of $\mathfrak{g}^*$ and $\{\mathcal{X}_i\}_{i=1}^{n}$ be the coordinate system of $\mathfrak{g}^*$, i.e.

$$\xi = \mathcal{X}_i(\xi) F_i\,, \qquad \mathcal{X}_i(F_j) = \delta_{ij}\,, \qquad \mathcal{X}_i = F_{e_i}\,.$$

Now, it is not difficult to see that

$$\{\mathcal{X}_i\,,\, \mathcal{X}_j\} = F_{[\,e_i\,,\,e_j\,]} = \sum_k C_{ij}^{k}\, F_{e_k} = \sum_k C_{ij}^{k}\, \mathcal{X}_k\,.$$

Finally, we can write the coordinate expression of the Poisson bracket on the dual of a Lie algebra:

$$\{ F , G \} = \sum_{i,j,k} C_{ij}^{k} \frac{\partial F}{\partial \mathcal{X}_i} \frac{\partial G}{\partial \mathcal{X}_j} \mathcal{X}_k .$$

1.2.6.1 Definition via Gradient Operator

We define the gradient operator

$$\nabla : C^{\infty}(\mathfrak{g}^*) \longrightarrow C^{\infty}(\mathfrak{g}^*)$$

as follows:

$$\langle \eta , \nabla F(\xi) \rangle \stackrel{\text{def}}{:=} \frac{\mathrm{d}}{\mathrm{d}t} F(\xi + t\eta)\Big|_{t=0}, \qquad \forall F \in C^{\infty}(\mathfrak{g}^*) .$$

In coordinates one simply has

$$\nabla F = \frac{\partial F}{\partial \mathcal{X}_i} e_i .$$

Using the gradient operator, the bracket is defined as

$$\{ F , G \}(\xi) \stackrel{\text{def}}{:=} \langle \xi , [\nabla F(\xi) , \nabla G(\xi)] \rangle$$

and in coordinates we obtain

$$\left\{ \mathcal{X}_i , \mathcal{X}_j \right\}(F_k) = \left\langle F_k , [\nabla \mathcal{X}_i (F_k) , \nabla \mathcal{X}_j (F_k)] \right\rangle = \left\langle F_k , [e_i , e_j] \right\rangle = C_{ij}^{k} .$$

1.2.6.2 Definition via Canonical Structure on $\mathbb{T}^*(G)$

Let $\mathfrak{g}$ be a Lie algebra, then there exists a unique (connected and simply connected up to an isomorphism) Lie group G such that $\mathbb{T}_e G \simeq \mathfrak{g}$. Let $L_g \in \mathbf{Aut}(G)$ be the translation by g, i.e. $\forall h \in G$:

$$\begin{aligned} L_g : G &\xrightarrow{\simeq} G \\ h &\longmapsto g h . \end{aligned}$$

Then, we define

$$\lambda_g(h) \stackrel{\text{def}}{:=} \left(\mathbb{T}L_g(h)\right)^* : \mathbb{T}^*_{gh}(G) \longrightarrow \mathbb{T}^*_h(G) ,$$

which gives rise to the isomorphism $\lambda_g \; : \; \mathbb{T}^*(G) \longrightarrow \mathbb{T}^*(G)$ with the inverse λ_g^{-1}. Define a map

$$\begin{aligned} \lambda \, : \, G \times \mathfrak{g}^* &\longrightarrow \mathbb{T}^* G \\ (g,\, \xi) &\longmapsto \lambda_g^{-1}(g)\,(\xi)\,, \end{aligned}$$

which is a diffeomorphism in the following commutative diagram:

$$\begin{array}{ccc} G \times \mathfrak{g}^* & \xrightarrow{\quad \lambda \quad} & \mathbb{T}^* G \\ & {\scriptstyle \mathsf{pr}_1} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & G & \end{array}$$

The co-tangent bundle $\mathbb{T}^* G \simeq G \times \mathfrak{g}^*$ is a trivial vector bundle with the fiber $\mathfrak{g}^*$. The Liouville form $\rho \in \Omega^1(\mathbb{T}^* G)$ defines by the section $\sigma_\rho \, : \, \mathbb{T}^* G \longrightarrow \mathbb{T}^*\mathbb{T}^* G$ similar to Sect. 1.2.4:

$$\left\langle \sigma_\rho(\xi)\,,\, \mathcal{X} \right\rangle = \left\langle \rho\,,\, \mathbb{T}_{\pi_\xi}(\mathcal{X}) \right\rangle, \qquad \mathcal{X} \in \mathbb{T}_{\pi_\xi}(\mathbb{T}^* G)\,, \qquad \xi \in \mathbb{T}^*_h G\,.$$

Let $g \in G$, $\xi \in \mathbb{T}^*_{gh} G$, $\mathcal{X} \in \mathbb{T}_{\pi_{\lambda_g(\xi)}}(\mathbb{T}^* G)$, then we have

$$\begin{aligned} \left\langle \sigma_\rho \circ \lambda_g(\xi)\,,\, \mathcal{X} \right\rangle &= \left\langle \lambda_g(\xi)\,,\, \mathbb{T}_{\pi_{\lambda_g(\xi)}} \right\rangle = \left\langle (\mathbb{T}_h L_g)^* \xi\,,\, \mathbb{T}_{\lambda_{g(\xi)}} \right\rangle = \\ &\left\langle \xi\,,\, (\mathbb{T}_h L_g)\mathbb{T}_{\lambda_g(\xi)}\mathcal{X} \right\rangle = \left\langle \xi\,,\, \mathbb{T}_{\lambda_g(\xi)}(L_g \circ \pi)\mathcal{X} \right\rangle. \end{aligned}$$

We can make two observations:

- $(L_g \circ \pi)\,(h,\, \xi) = gh\,,$
- $\pi \circ \lambda_{g^{-1}} = gh\,.$

Henceforth, $L_g \circ \pi \equiv \pi \circ \lambda_{g^{-1}}$. The Liouville form becomes

$$\begin{aligned} &\left\langle \sigma_\rho \circ \lambda_g(\xi)\,,\, \mathcal{X} \right\rangle = \\ &\left\langle \xi\,,\, \mathbb{T}_{\lambda_g(\xi)}(\pi \circ \lambda_{g^{-1}})\mathcal{X} \right\rangle = \left\langle \xi\,,\, \mathbb{T}_\xi \pi \circ \mathbb{T}_{\lambda_g(\xi)\lambda_{g^{-1}}(\mathcal{X})} \right\rangle = \\ &\left\langle \rho(\xi)\,,\, \mathbb{T}_{\lambda_g(\xi)}\lambda_{g^{-1}}(\xi)\mathcal{X} \right\rangle = \left\langle \lambda^*_{g^{-1}} \circ \rho(\xi)\,,\, \mathcal{X} \right\rangle. \end{aligned}$$

We have $\omega = d\rho$ as the canonical symplectic form on $\mathbb{T}^* G$. We observe also that

- $\sigma_\rho \circ \lambda_g = \lambda^*_{g^{-1}}(\sigma_\rho)\,,$
- $\omega \circ \lambda_g = \lambda^*_{g^{-1}}(\omega)\,.$

We recall that a Poisson bracket $\{F, G\}$ for $F, G \in C^\infty(\mathbb{T}^*G)$ can be defined via $\{F, G\} \stackrel{\text{def}}{:=} \mathcal{X}_F(G)$, where $\mathcal{X}_F$ is the unique vector field on $\mathbb{T}^*G$ such that $\mathcal{X}_F \lrcorner\, \omega = \mathrm{d}F$.

Lemma 1.4 *Let $g \in G$ and $F, G \in C^\infty(\mathbb{T}^*G)$, then*

$$\{F \circ \lambda_g, G \circ \lambda_g\} = \{F, G\} \circ \lambda_g .$$

Proof Left to the reader as an exercise. □

Let $C^\infty(\mathbb{T}^*G)^G$ denote a subspace of stable or invariant functions with respect to the mapping λ_g, i.e.

$$C^\infty(\mathbb{T}^*G)^G \stackrel{\text{def}}{:=} \{F \in C^\infty(\mathbb{T}^*G) \mid F \circ \lambda_g = F\}, \qquad g \in G .$$

Lemma 1.4 shows that the set $C^\infty(\mathbb{T}^*G)^G$ is closed with respect to the Poisson bracket.

Let the linear mapping Φ be defined as

$$\Phi : C^\infty(\mathfrak{g}^*) \longrightarrow C^\infty(\mathbb{T}^*G)$$
$$F \longmapsto F \circ \mathbf{pr}_2 ,$$

where $\mathbf{pr}_2 : \mathbb{T}^*G \equiv G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ is the canonical projection on the second argument. Let $\iota : \mathfrak{g}^* \longrightarrow \mathbb{T}^*G$ be the canonical embedding. Then,

$$\Psi : C^\infty(\mathbb{T}^*G)^G \longrightarrow \mathfrak{g}^*$$
$$F \longmapsto F \circ \iota$$

is linear and inverse to Φ, i.e. $\Psi \equiv \Phi^{-1}$.

Lemma 1.5 *The bracket* $\{F, G\} \stackrel{\text{def}}{:=} \Phi^{-1}\big(\{\Phi(F), \Phi(G)\}\big)$ *is a Poisson bracket on $C^\infty(\mathfrak{g}^*)$ coinciding with two previous definitions.*

Proof Left to the reader as an exercise. □

1.3 Group Actions and Orbits

Let G be a Lie group and $\mathbb{M}$ is a smooth manifold.

Definition 1.4 A *left action* of G on $\mathbb{M}$ is a smooth map $\mu : G \times \mathbb{M} \longrightarrow \mathbb{M}$ such that

- $\mu(e, m) = m, \forall m \in \mathbb{M}$,

- $\mu\big(g, \mu(h, m)\big) = \mu(gh, m), \forall g, h \in G$ and $\forall m \in \mathbb{M}$.

Definition 1.5 A *right action* of G on $\mathbb{M}$ is a smooth map $\rho : \mathbb{M} \times G \longrightarrow \mathbb{M}$ such that

- $\rho(m, e) = m, \forall m \in \mathbb{M}$,
- $\rho\big(\rho(m, h), g\big) = \rho(hg, m), \forall g, h \in G$ and $\forall m \in \mathbb{M}$.

Left and right actions of G and $\mathbb{M}$ are in one-to-one correspondence by the following relation:

$$\rho(m, g^{-1}) = \mu(g, m).$$

From now on we shall denote the left Lie group action $\mu(g, m)$ simply by $g \cdot m$. We can define several important action types:

Effective or Faithful	$\forall g \in G, g \neq e \implies \exists m \in \mathbb{M}$ such that $g \cdot m \neq m$.
Free	If g is a group element and $\exists m \in \mathbb{M}$ such that $g \cdot m = m$ (that is, if g has at least one fixed point), $\implies g = e$. Note that a free action on a non-empty M is faithful.
Transitive	If $\forall m, n \in \mathbb{M}, \exists g \in G$ such that $g \cdot m = n$. In this case the smooth manifold $\mathbb{M}$ is called homogeneous.

Important examples of group actions include:

Example 1.1 Example G acts on itself by left multiplication:

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto gh. \end{aligned}$$

This action is effective and transitive. Indeed, $g \cdot h = h \implies g = e$ and if $g \cdot m = n \implies g = n \cdot m^{-1}$.

Example 1.2 Example G acts on itself by conjugation:

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto g \cdot h \cdot g^{-1}. \end{aligned}$$

Generally, this action is not free, transitive, or effective.

Example 1.3 Example $\mathbf{GL}_n(\mathbb{R})$ acts on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ by matrix multiplication on the left:

$$\begin{aligned} \mathbf{GL}_n(\mathbb{R}) \times \mathbb{R}^n \setminus \{\mathbf{0}\} &\longrightarrow \mathbb{R}^n \setminus \{\mathbf{0}\} \\ (A, x) &\longmapsto Ax. \end{aligned}$$

This is an example of an effective and transitive action.

1.3.1 Stabilizers and Orbits

Let G be a Lie group which acts on a smooth manifold $\mathbb{M}$. The *orbit* of a point $m \in \mathbb{M}$ is

$$G \cdot m \stackrel{\text{def}}{:=} \{ g \in G \mid g \cdot m \} \subseteq \mathbb{M}.$$

A *stabilizer* of a point $m \in \mathbb{M}$ is

$$G_m \stackrel{\text{def}}{:=} \{ g \in G \mid g \cdot m = m \} \subseteq G.$$

Proposition 1.1 *The stabilizer G_m is a closed sub-group of G and $G_{g \cdot m} = g \cdot G_m \cdot g^{-1}, \forall g \in G$.*

Proof Left to the reader as an exercise. □

We mention here two technical theorems regarding the orbits and stabilizers:

Theorem 1.3 *Let G be a Lie group which acts on a smooth manifold $\mathbb{M}$ and $m \in \mathbb{M}$. There is a manifold structure on the orbit $G \cdot m$ such that the map*

$$\begin{aligned} G &\longrightarrow G \cdot m \\ g &\longmapsto g \cdot m \end{aligned}$$

is a submersion and the embedding $\imath : G \cdot m \hookrightarrow \mathbb{M}$ is an immersion.

Theorem 1.4 *The Lie algebra $\mathfrak{g}_m$ of the stabilizer G_m for a point $m \in \mathbb{M}$ coincides with $\ker \mathbb{T}_e \Phi$, where the mapping Φ is defined as*

$$\begin{aligned} \Phi : G &\longrightarrow \mathbb{M} \\ g &\longmapsto g \cdot m. \end{aligned}$$

1.3.2 Infinitesimal Action

Let $\mu : G \times \mathbb{M} \longrightarrow \mathbb{M}$ be a Lie group action on $\mathbb{M}$ and $\mathfrak{g} = \mathsf{Lie}(G)$ be its Lie algebra.

Definition 1.6 Let $\mathcal{X} \in \mathfrak{g}$ and $\phi : \mathbb{R} \longrightarrow G$ its exponential flow, i.e. $\phi(t) = \exp(t\,\mathcal{X})$. Then, there exists the unique vector field $\mathcal{X}_{\mathbb{M}} \in \mathfrak{X}(\mathbb{M})$ with the flow $\phi_m : \mathbb{R} \longrightarrow \mathbb{M}$ defined by $\phi_m(t) = \phi(t) \cdot m$. The vector field $\mathcal{X}_{\mathbb{M}}$ is defined by

$$\mathcal{X}_{\mathbb{M}}(m)(f) \stackrel{\text{def}}{:=} \frac{\mathrm{d}}{\mathrm{d}t} \bigl(f \circ \phi(t) \cdot m \bigr) \Bigr|_{t=0}.$$

The mapping $\mu_* : \mathfrak{g} \longrightarrow \mathfrak{X}(\mathbb{M})$ is called the *infinitesimal action* of $\mathfrak{g}$ on $\mathbb{M}$.

Proposition 1.2 *The mapping $\mu_* : \mathfrak{g} \longrightarrow \mathfrak{X}(\mathbb{M})$ is a Lie algebra (anti-)homomorphism (and it is in particular a linear mapping):*

$$\mu_*\big([\mathcal{X}, \mathcal{Y}]\big) = -[\mu_*(\mathcal{X}), \mu_*(\mathcal{Y})].$$

Proof Left to the reader as an exercise. □

Remark 1.1 One can see that $\mu_*(\mathcal{X})_m(f) = \mathcal{X}(f \circ \Phi_m)$. In other words, $\mu_*(\mathcal{X})_m = \mathbb{T}_e\Phi_m(\mathcal{X})$.

Proposition 1.3 *Let $m \in \mathbb{M}$, then*

$$\mathbb{T}_m G \cdot m = \{\mathcal{X} \in \mathfrak{g} \mid \mu_*(\mathcal{X})_m\}.$$

Proof Left to the reader as an exercise. □

The following difficult result is left without the proof:

Theorem 1.5 (R. Palais) *Let G be a simply connected Lie group with the Lie algebra $\mathfrak{g} = \textbf{Lie}(G)$ and $\mathbb{M}$ be a smooth compact manifold such that there exists a homomorphism of Lie algebras $\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(\mathbb{M})$. Then, there is a unique action $\mu : G \times \mathbb{M} \longrightarrow \mathbb{M}$ such that $\mu_* = \rho$.*

Proposition 1.4 *Let $\mu : G \times \mathbb{M} \longrightarrow \mathbb{M}$ be an action of G on a smooth manifold $\mathbb{M}$ and $m \in \mathbb{M}$. Then, the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mu_*} & \mathfrak{X}(\mathbb{M}) \\ \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\mu_m]{} & \mathbb{M} \end{array}$$

or, in other words:

$$\forall \mathcal{X} \in \mathfrak{g} : \qquad \mu_m(\mathrm{e}^{t\mathcal{X}}) = \mathrm{e}^{t\,\mu_*(\mathcal{X})_m}.$$

1.3.3 Lie Group and Lie Algebra Representations

Let G be a Lie group, $\mathfrak{g} = \textbf{Lie}(G)$ be its Lie algebra and V be a real vector space.

Definition 1.7 A *representation* of the Lie group G in the vector space V is a homomorphism of Lie groups (i.e. a smooth group morphism) $\varphi : G \longrightarrow \mathbf{GL}(V)$.

Definition 1.8 A representation of the Lie algebra $\mathfrak{g}$ in the vector space V is a Lie algebra homomorphism $\phi : \mathfrak{g} \longrightarrow \mathbf{End}\,(V)$.

Here $\mathbf{End}\,(V)$ is enabled with the Lie algebra structure given by the endomorphisms commutator:

$$\forall A,\ B \in \mathbf{End}\,(V) \qquad [\,A\,,\ B\,] \stackrel{\text{def}}{:=} A \cdot B - B \cdot A\,.$$

If $\varphi : G \longrightarrow \mathbf{GL}\,(V)$ is a Lie group representation, then

$$\phi \stackrel{\text{def}}{:=} \mathbb{T}_e\varphi : \mathbb{T}_e G = \mathfrak{g} \longrightarrow \mathbb{T}_{\mathrm{id}}\bigl(\mathbf{GL}\,(V)\bigr) = \mathbf{End}\,(V)$$

is a representation of the Lie algebra $\mathfrak{g}$.

1.3.3.1 Adjoint Representations

Let $g \in G$, $V = \mathfrak{g}$, then the composition $L_g \circ R_{g^{-1}} : G \longrightarrow G$ induces a linear mapping $\mathbb{T}_e\,(\,L_g \circ R_{g^{-1}}\,) \stackrel{\text{def}}{:=} \mathbf{Ad}\,(\,g\,) : \mathfrak{g} \longrightarrow \mathfrak{g}$ and, hence, a group morphism $\mathbf{Ad} : G \longrightarrow \mathbf{GL}\,(\,\mathfrak{g}\,)$. Then, the following Lemma holds:

Lemma 1.6 *The group morphism* $\boldsymbol{Ad}$ *is a smooth map which gives a representation of G in $\mathfrak{g}$, which is called the* adjoint *Lie group representation.*

Proof Left to the reader as an exercise. □

Let $\mathbf{ad} \stackrel{\text{def}}{:=} \mathbb{T}_e\,(\,\mathbf{Ad}\,) : \mathfrak{g} \longrightarrow \mathbf{End}(\,\mathfrak{g}\,)$. Then, $\mathbf{ad}$ is also called the *adjoint* Lie group representation.

Lemma 1.7

$$\boldsymbol{ad}(\,\mathcal{X}\,)(\,\mathcal{Y}\,) = [\,\mathcal{X}\,,\ \mathcal{Y}\,], \qquad \forall \mathcal{X},\ \mathcal{Y} \in \mathfrak{g}\,.$$

Proof Left to the reader as an exercise. □

1.3.3.2 Co-Adjoint Representations

Let $g \in G$, $V \in \mathfrak{g}^*$ and $f^* \in \mathbf{End}\,(\,\mathfrak{g}^*\,)$ is defined by $f^*\,(\xi) \stackrel{\text{def}}{:=} \xi \circ f$ for any element $f \in \mathbf{End}\,(\,\mathfrak{g}\,)$. Then, we can write down the following

Definition 1.9 The following smooth map

$$\begin{aligned} \mathbf{Ad}^* : G &\longrightarrow \mathbf{GL}\,(\,\mathfrak{g}^*\,)\,, \\ g &\longmapsto \mathbf{Ad}\,(\,g^{-1}\,)^*\,, \end{aligned}$$

which gives a representation of G in $\mathfrak{g}^*$ is called the co-adjoint Lie group representation.

The last definition makes sense because $\mathbf{Ad}^* = F \circ \mathbf{Ad}$ and $F(f) = f^*$, where the map $F : \mathbf{End}(V) \longrightarrow \mathbf{End}(V^*)$. Similarly, we can also define

$$\begin{aligned} \mathbf{ad}^* : \mathfrak{g} &\longrightarrow \mathbf{End}(\mathfrak{g}^*), \\ \mathcal{X} &\longmapsto -\mathbf{ad}^*(\mathcal{X}), \end{aligned}$$

where

$$\mathbf{ad}^*(\mathcal{X})(\xi)(\mathcal{Y}) = -\langle \xi, [\mathcal{X}, \mathcal{Y}] \rangle .$$

Then, $\mathbf{ad}^*$ is also called the co-adjoint Lie algebra representation in $\mathfrak{g}^*$.

Lemma 1.8

$$[\boldsymbol{ad}^*(\mathcal{X}), \boldsymbol{ad}^*(\mathcal{Y})] = \boldsymbol{ad}^*([\mathcal{X}, \mathcal{Y}]), \qquad \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{g}, \quad \forall \xi \in \mathfrak{g}^* .$$

Proof Left to the reader as an exercise. □

Proposition 1.5 *The co-adjoint representation* $\boldsymbol{Ad}^*$ *of a Lie group G gives rise to a co-adjoint left action of G on* $\mathfrak{g}^*$*:*

$$\begin{aligned} G \times \mathfrak{g}^* &\longrightarrow \mathfrak{g}^*, \\ (g, \xi) &\longmapsto \boldsymbol{Ad}^*(\xi) . \end{aligned}$$

Let $\mathcal{X} \in \mathfrak{g}$ and $F_{\mathcal{X}} \in C^\infty(\mathfrak{g}^*)$ be the evaluation function defined by $F_{\mathcal{X}}(\xi) \overset{\text{def}}{:=} \xi(\mathcal{X})$. Then, the following Propositions hold:

Proposition 1.6

$$\frac{\mathrm{d}}{\mathrm{d}t} F\left(\boldsymbol{Ad}^*_{\exp(t\mathcal{X})}(\xi)\right)\Big|_{t=0} = \{F, F_{\mathcal{X}}\}(\xi) .$$

Proposition 1.7 *A function* $F \in C^\infty(\mathfrak{g}^*)$ *is a Casimir function for the Lie–Poisson structure on* $\mathfrak{g}^*$ *if and only if*

$$\{F, F_{\mathcal{X}}\} = 0, \qquad \forall \mathcal{X} \in \mathfrak{g} .$$

Finally, we can state without the proof the following important

Theorem 1.6 (Lie–Berezin–Kirillov–Kostant–Souriau)

$$\boldsymbol{Cas}\left(C^\infty(\mathfrak{g}^*)\right) = C^\infty(\mathfrak{g}^*)^{\boldsymbol{Ad}^*_G},$$

where

$$C^{\infty}(\mathfrak{g}^*)^{\boldsymbol{Ad}^*_G} \stackrel{\text{def}}{:=} \Big\{ F \in C^{\infty}(\mathfrak{g}^*) \,\Big|\, F \circ \boldsymbol{Ad}^*_g = F, \quad \forall g \in G \Big\}.$$

Denote by $\mathcal{O}_{\xi} \stackrel{\text{def}}{:=} G \cdot \xi \subseteq \mathfrak{g}^*$ a co-adjoint orbit of a co-vector $\xi \in \mathfrak{g}^*$. Recall that these orbits are manifolds such that $\mathcal{O}_{\xi}$ admits a submersion $\phi : G \longrightarrow \mathcal{O}_{\xi}$ and an immersion $\imath : \mathcal{O}_{\xi} \hookrightarrow \mathfrak{g}^*$. Define a $\mathfrak{g}-$valued $1-$form ω on G as

$$\omega_g(\mathcal{X}) = \mathbb{T}_g L_{g^{-1}}(\mathcal{X}) \in \mathfrak{g}.$$

Then, $L_h^*(\omega) = \omega$. In other words, the $1-$form ω is $G-$invariant. Here we take $g,\, h \in G$ and $\mathcal{X} \in \mathfrak{g}$. By Maurer–Cartan formula we have that

$$\mathrm{d}\omega = -\frac{1}{2}[\omega,\, \omega].$$

Let us define also the $1-$form $\omega_{\xi} \in \Lambda^1(G)$ by

$$\omega_{\xi}(g)(\mathcal{X}) \stackrel{\text{def}}{:=} \big\langle \xi,\, \omega_g(\mathcal{X}) \big\rangle.$$

Then, the following result can be shown:

Theorem 1.7 (Kirillov–Kostant–Souriau) *There exists a unique $2-$form $\Omega_{\xi} \in \Lambda^2(\mathcal{O}_{\xi})$ such that $\phi^*(\Omega_{\xi}) = \mathrm{d}\omega_{\xi}$. This form is symplectic on the co-adjoint orbit $\mathcal{O}_{\xi}$.*

1.4 Moment Map, Poisson and Hamiltonian Actions

1.4.1 Introductory Motivation

Let $\mathbb{R}^3$ be a basic configuration space with coordinate or position vectors $\mathbf{r} = (q_1,\, q_2,\, q_3)$ and velocity vectors:

$$\dot{\mathbf{r}} = (\dot{q}_1,\, \dot{q}_2,\, \dot{q}_3) \stackrel{\text{def}}{=:} \mathbf{p} \stackrel{\text{def}}{:=} (p_1,\, p_2,\, p_3).$$

$$E_{\mathrm{T}} \stackrel{\text{def}}{:=} \frac{\langle \dot{\mathbf{r}},\, \dot{\mathbf{r}} \rangle}{2} + U(\mathbf{r})$$

and the equation of motion is $\ddot{\mathbf{r}} = -\nabla_{\mathbf{r}} U(\mathbf{r})$. The total mechanical energy is conserved, i.e. $\dfrac{\mathrm{d}E_{\mathrm{T}}}{\mathrm{d}t} \equiv 0$. The angular momentum is also constant along a trajectory. It implies that

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = -\mathbf{r} \times \nabla_{\mathbf{r}} U = \mathbf{0},$$

which is equivalent to say that $\mathbf{r} = \lambda \nabla_{\mathbf{r}} U$ for some $\lambda \in \mathbb{R}$.

Let $\mathfrak{so}(3)$ be the Lie algebra of skew-symmetric 3×3 matrices with real entries. This is a three-dimensional vector space with the basis $\{X_1, X_2, X_3\}$ given by three following matrices:

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The Lie brackets in the Lie algebra $\mathfrak{so}(3)$ are given by

$$[X_i, X_j] = X_k, \qquad (i, j, k) = (1, 2, 3),$$

with all circular permutations. The Killing form $\kappa(-, -)$ defined as

$$\begin{aligned} \kappa : \mathfrak{so}(3) \times \mathfrak{so}(3) &\longrightarrow \mathbb{R} \\ (X, Y) &\longmapsto \mathrm{tr}(X Y) \end{aligned}$$

is symmetric, bi-linear and non-degenerate. Here $\mathrm{tr}(-)$ is the trace form of a square matrix. This form identifies $\mathfrak{so}(3)$ and $\mathfrak{so}(3)^*$ by the interior product rule $X \lrcorner \kappa$.

The Lie–Poisson structure on $\mathfrak{so}(3)^*$ in the coordinates (x_1, x_2, x_3) on $\mathfrak{so}(3)$ can be expressed as

$$\{F, G\}(x_1, x_2, x_3) = \sum_{i, j, k=1}^{3} c_{ij}^{k} \left(\frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} - \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial x_i} \right) x_k .$$

Here c_{ij}^k is the structure constant tensor of the Lie algebra $\mathfrak{so}(3)$.

The angular momentum $\mathbf{L}$ s defined as a map

$$\begin{aligned} \mathbb{T}^*\mathbb{R}^3 \simeq \mathbb{R}^6 &\longrightarrow \mathfrak{so}(3)^* \\ (\mathbf{q}, \mathbf{p}) &\longmapsto \mathbf{q} \times \mathbf{p} = \sum_{i, j, k} (q_i p_j - p_i q_j) X_k . \end{aligned}$$

The angular momentum map $\mathbf{L} : \mathbb{T}^*\mathbb{R}^3 \longrightarrow \mathfrak{so}(3)^*$ is a Poisson morphism.

Definition 1.10 Let $\mu : G \times \mathbb{M} \longrightarrow \mathbb{M}$ be a Lie group action on a Poisson manifold $\big(\mathbb{M}; \{-, -\}\big)$. This action is called a Poisson action if the map

$$\mu_g^* : C^\infty(\mathbb{M}, \mathbb{R}) \longrightarrow C^\infty(\mathbb{M}, \mathbb{R})$$

defined by $\mu_g^*(F)(m) \stackrel{\text{def}}{:=} F\bigl(\mu_g(m)\bigr)$ satisfies the following condition:

$$\mu_g^*\bigl(\{F, G\}\bigr)(m) = \Bigl\{\mu_g^*(F), \mu_g^*(G)\Bigr\}(m), \qquad \forall F, G \in C^\infty(\mathbb{M}, \mathbb{R}).$$

Let a Poisson structure $(\mathbb{M}, \pi)$ be symplectic.[1] In this case this Poisson action can be called the Hamiltonian action.

1.4.2 Momentum Map

Definition 1.11 Let $\mathfrak{g}$ be a Lie algebra and $\bigl(\mathbb{M}, \{-, -\}\bigr)$ be a Poisson manifold. A *momentum map* is a Poisson morphism $\mu : \mathbb{M} \longrightarrow \mathfrak{g}^*$. In other words, it is a smooth map μ such that for $\forall\, F, G \in C^\infty(\mathfrak{g}^*)$:

$$\mu^*\bigl(\{F, G\}_{\mathfrak{g}^*}\bigr) = \bigl\{\mu^*(F), \mu^*(G)\bigr\}_{\mathbb{M}}.$$

Let $\bar{\lambda} : \mathfrak{g} \longrightarrow C^\infty(\mathbb{M}, \mathbb{R})$ be a smooth linear map. Then, there is a unique map $\lambda : \mathbb{M} \longrightarrow \mathfrak{g}^*$ defined by $\bar{\lambda}$:

$$\{\lambda(m), \mathcal{X}\} = \bar{\lambda}(\mathcal{X})(m), \qquad \forall m \in \mathbb{M}, \quad \forall \mathcal{X} \in \mathfrak{g}.$$

Proposition 1.8 *Let $\bigl(\mathbb{M}, \{-, -\}\bigr)$ be a Poisson manifold and $\mu : \mathbb{M} \longrightarrow \mathfrak{g}^*$ is a smooth map. Then, μ is a momentum map if and only if the associated map $\bar{\mu} : \mathfrak{g} \longrightarrow C^\infty(\mathbb{M}, \mathbb{R})$ is a Lie algebra homomorphism:*

$$\bar{\mu}\bigl([\mathcal{X}, \mathcal{Y}]\bigr) = \{\bar{\mu}(\mathcal{X}), \bar{\mu}(\mathcal{Y})\}_{\mathbb{M}}, \qquad \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{g}.$$

Recall that the map $\chi : C^\infty(\mathbb{M}, \mathbb{R}) \longrightarrow \mathfrak{X}(\mathbb{M})$ such that

$$\chi(F) = \mathcal{X}_F = \{F, -\}$$

is a Lie algebra morphism. We take the composition $\Theta \stackrel{\text{def}}{:=} \chi \circ \bar{\mu} : \mathfrak{g} \longrightarrow \mathfrak{X}(\mathbb{M})$, where $\bar{\mu} : \mathfrak{g} \longrightarrow C^\infty(\mathbb{M}, \mathbb{R})$ and $\mathfrak{g} = \mathsf{Lie}(G)$ with a simply connected Lie group G. For compact manifolds $\mathbb{M}$, Theorem 1.5 ensures the existence of an action $\lambda : G \times \mathbb{M} \longrightarrow \mathbb{M}$ with $\lambda_* = -\Theta$.

Proposition 1.9 *If G is connected, then $\lambda_* = -\Theta$ gives a Poisson morphism $\lambda_g^* : C^\infty(\mathbb{M}, \mathbb{R}) \longrightarrow C^\infty(\mathbb{M}, \mathbb{R})$ for $\forall\, g \in G$ and $\forall\, u, v \in C^\infty(\mathbb{M}, \mathbb{R})$:*

[1] Here we mean that the bi-vector π is non-degenerate, i.e. π is invertible when it is seen as a banal matrix.

$$\left\{ \lambda_g^* (u) , \lambda_g^* (v) \right\} = \lambda_g^*\big(\{ u , v \} \big) .$$

Proposition 1.10 *Let $\mathbb{M}$ be a compact manifold and G is connected and simply connected. Then, the action λ is $G-$equivariant:*

$$\begin{array}{ccc} \mathbb{M} & \xrightarrow{\ \mu\ } & \mathfrak{g}^* \\ {\scriptstyle \lambda_g}\downarrow & & \downarrow{\scriptstyle Ad^*_{\mathfrak{g}}} \\ \mathbb{M} & \xrightarrow[\ \mu\ ]{} & \mathfrak{g}^* \end{array}$$

1.4.3 Moment and Hamiltonian Actions

Let $\big(\mathbb{M}, \omega \big)$ be a symplectic manifold and the corresponding Poisson brackets are defined by a pair of Hamiltonian vector fields:

$$\{ u , v \} = \mathcal{X}_u (v) = \omega\big(\mathcal{X}_v, \mathcal{X}_u \big) , \qquad \mathcal{X}_u \lrcorner\, \omega = \mathrm{d}u .$$

Lemma 1.9 *If $H^1 (\mathbb{M}, \mathbb{R}) = 0$ and $\mathcal{X} \in \mathfrak{X} (\mathbb{M})$ "infinitesimally" preserves the symplectic form ω, i.e. $\mathcal{L}_{\mathcal{X}} (\omega) = 0$, then there exists a unique $u \in C^\infty (\mathbb{M}, \mathbb{R})$ such that $\mathcal{X} = \mathcal{X}_u$.*

Here we should remark that the function u is uniquely defined only modulo a locally constant function on M (which is usually identified with an element of $H^0 (\mathbb{M}, \mathbb{R})$.

Lemma 1.10 *Let $\lambda : G \times \mathbb{M} \longrightarrow \mathbb{M}$ be an action of a Lie group G on a symplectic manifold $\big(\mathbb{M}, \omega \big)$. The action λ is a Poisson (more precisely, in this case we may call it a Hamiltonian) action if and only if $\lambda_g^* = \omega$.*

Proposition 1.11 *Let $\lambda : G \times \mathbb{M} \longrightarrow \mathbb{M}$ be a Hamiltonian action on a symplectic manifold $\big(\mathbb{M}, \omega \big)$ and $\lambda_* : \mathfrak{g} \longrightarrow \mathfrak{X} (M)$ is the corresponding Lie algebra homomorphism. Then, $\forall\, \mathcal{X} \in \mathfrak{g}$:*

$$\mathcal{L}_{\lambda_* (\mathcal{X})} (\omega) = 0 .$$

Definition 1.12 Let $\lambda : G \times \mathbb{M} \longrightarrow \mathbb{M}$ be an action of G on $\mathbb{M}$ and $\big(\mathbb{T}^*\mathbb{M}, \Omega \big)$ is the co-tangent bundle with the canonical symplectic form $\Omega = \mathrm{d}\rho$, where ρ is the Liouville $1-$form. This action can be lifted to an action $\Lambda : G \times \mathbb{T}^*\mathbb{M} \longrightarrow \mathbb{T}^*\mathbb{M}$ defined by

$$\Lambda (g, \xi_m) \overset{\text{def}}{:=} (\mathbb{T}_{g \cdot m}\, \lambda^*_{g^{-1}}) (\xi_m) .$$

Theorem 1.8 *The action Λ is Hamiltonian and the induced momentum map $\mu_{\Lambda}\,:\,\mathbb{T}^*\mathbb{M}\;\longrightarrow\;\mathfrak{g}^*$ is defined by*

$$\{\,\mu_{\Lambda}\,(\,\xi_{m}\,)\,,\;\mathcal{X}\,\}\;=\;\{\,\xi_{m}\,,\;\mathbb{T}_{e}\,\lambda_{m}\,(\,\mathcal{X}\,)\,\}\,.$$

1.4.3.1 Examples

Example 1.4 Lifting of the left $G-$action *on* G to $\mathbb{T}^*G$:

$$\begin{aligned}\lambda\,:\;G\times G\;&\longrightarrow\;G\,,\\(\,g,\,h\,)\;&\longmapsto\;g\cdot h\,.\end{aligned}$$

Then, we obtain the required lifting:

$$\begin{aligned}\Lambda\,:\;G\times\mathbb{T}^*G\;\simeq\;G\times\mathfrak{g}^*\;&\longrightarrow\;\mathbb{T}^*G\;\simeq\;G\times\mathfrak{g}^*\,,\\\big(\,g,\,(\,h,\,\xi\,)\big)\;&\longmapsto\;(\,g\cdot h,\,\xi\,)\,.\end{aligned}$$

The associated momentum can be also easily computed:

$$\mu\,(\,\xi_{h}\,)\;=\;-\,(\,\mathbb{T}_{e}R_{h}\,)^{*}(\,\xi_{h}\,)\,,\qquad\mu\,(\,h,\,\xi\,)\;=\;\mathbf{Ad}_{h}^{*}\,(\,\xi\,)\,.$$

Similarly, we can consider lifting of the right $G-$action *on* G to $\mathbb{T}^*G$:

$$\begin{aligned}\lambda\,:\;G\times G\;&\longrightarrow\;G\,,\\(\,g,\,h\,)\;&\longmapsto\;h\cdot g^{-1}\,.\end{aligned}$$

Then, we obtain the required lifting:

$$\begin{aligned}\Lambda\,:\;G\times\mathbb{T}^*G\;\simeq\;G\times\mathfrak{g}^*\;&\longrightarrow\;\mathbb{T}^*G\;\simeq\;G\times\mathfrak{g}^*\,,\\\big(\,g,\,(\,h,\,\xi\,)\big)\;&\longmapsto\;(\,h\cdot g^{-1},\,\mathbf{Ad}_{g}^{*}\,\xi\,)\,.\end{aligned}$$

The associated momentum can be also easily computed:

$$\mu\,(\,\xi_{h}\,)\;=\;-\,(\,\mathbb{T}_{e}L_{h}\,)^{*}(\,\xi_{h}\,)\,,\qquad\mu\,(\,h,\,\xi\,)\;=\;-\,\xi\,.$$

Example 1.5 Let us consider also the action of $\mathbb{S}^1$ on $\mathbb{C}$:

$$\mathbb{C}\;\simeq\;\mathbb{R}^2\;\simeq\;\mathbb{T}^*\mathbb{R}^1\,,\qquad\Omega\;=\;\mathrm{d}q\wedge\mathrm{d}p\,.$$

The action is given by

$$\begin{aligned} \lambda \, : \, \mathbb{S}^1 \times \mathbb{C} &\longrightarrow \mathbb{C} \,, \\ \left(e^{i\theta}, \, z \right) &\longmapsto e^{i\theta} z \,, \end{aligned}$$

for some $\theta \in [0, \, 2\pi[$. Above i is the complex imaginary unit, i.e. $i^2 = -1$. Then, one can easily obtain the expression for $\lambda_* \, : \, \mathbb{T}_e\mathbb{S}^1 \longrightarrow \mathbb{C}$:

$$\lambda_* \left(\frac{d}{d\theta} \right) (q, \, p) = \mathbb{T}_e \left(\lambda_{q,\, p} \right) \left(\frac{d}{d\theta} \right) = -p \, \frac{\partial}{\partial q} + q \, \frac{\partial}{\partial p} \,.$$

The interior product with the symplectic form can be also easily obtained:

$$\lambda_* \left(\frac{d}{d\theta} \right) \lrcorner \, \Omega = -\left(p dp + q dq \right) = -\frac{1}{2} \, d\left(p^2 + q^2 \right) .$$

The momentum map is given by

$$\mu \left(z \right) = \mu \left(q + i p \right) = \frac{p^2 + q^2}{2} \,.$$

The last construction can be easily generalized to $\mathbb{C}^n$:

$$\begin{aligned} \lambda \, : \, \mathbb{S}^1 \times \mathbb{C}^n &\longrightarrow \mathbb{C}^n \,, \\ \left(e^{i\theta}, \, (z_1, \, z_2, \ldots, \, z_n) \right) &\longmapsto \left(e^{i\theta} z_1, \, e^{i\theta} z_2, \ldots, \, e^{i\theta} z_n \right) . \end{aligned}$$

Then, the associated momentum map is given by

$$\mu \left(z_1, \, z_2, \ldots, \, z_n \right) = \sum_{i=1}^{n} | \, z_i \, |^2 \,.$$

Example 1.6 Let us consider the action of $\mathbb{S}^1$ on $\mathbb{S}^2$. The manifold $\mathbb{S}^2$ is equipped with local coordinates $(z, \, \phi)$ and $\Omega = dz \wedge d\phi$. The action of $\mathbb{S}^1$ is given by rotation in $z-$planes:

$$\begin{aligned} \lambda \, : \, \mathbb{S}^1 \times \mathbb{S}^2 &\longrightarrow \mathbb{S}^2 \,, \\ \left(e^{i\theta}, \, (z, \, \phi) \right) &\longmapsto \left(z, \, \phi + \theta \right) . \end{aligned}$$

It is not difficult to see that

$$\lambda_* \left(\frac{d}{d\theta} \right) (z, \, \phi) = \frac{\partial}{\partial \phi} \,, \qquad \lambda_* \left(\frac{d}{d\theta} \right) \lrcorner \, \Omega = dz \,.$$

Finally, the momentum map is

$$\mu(z, \phi) = z .$$

Example 1.7 We consider now the action of $\mathbb{S}^1$ on the torus $\mathbb{T}^2 \stackrel{\text{def}}{:=} \mathbb{S}^1 \times \mathbb{S}^1$. The torus $\mathbb{T}^2$ is equipped with local coordinates (ϕ_1, ϕ_2) and the symplectic form is $\Omega = \mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2$. The action is defined as

$$\begin{aligned} \lambda : \mathbb{S}^1 \times \mathbb{T}^2 &\longrightarrow \mathbb{T}^2 , \\ \bigl(\mathrm{e}^{\mathrm{i}\theta}, (\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2})\bigr) &\longmapsto \bigl(\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}(\phi_1 + \theta)}\bigr) . \end{aligned}$$

Then, we have

$$\lambda_* \Bigl(\frac{\mathrm{d}}{\mathrm{d}\theta}\Bigr)(\Omega) = \mathrm{d}\phi_1 \wedge \mathrm{d}(\theta + \phi_2) = \Omega$$

and

$$\lambda_* \Bigl(\frac{\mathrm{d}}{\mathrm{d}\theta}\Bigr) \lrcorner\, \Omega = -\,\mathrm{d}\phi_1 .$$

Since the coordinate function ϕ_1 is defined only locally, the momentum map μ and the morphism $\bar{\mu}$ do not exist.

Example 1.8 In this example we consider the action of $\mathbf{SU}(n)$ on $\mathbb{T}^*\bigl(\mathfrak{su}(n)\bigr)$. We remind that $\mathbf{SU}(n)$ is the Lie group of special unitary matrices with complex coefficients:

$$\mathbf{SU}(n) \stackrel{\text{def}}{:=} \bigl\{\mathbb{A} \in \mathrm{Mat}_n(\mathbb{C}) \bigm| \mathbb{A}\mathbb{A}^* = \mathbb{I},\ \det(\mathbb{A}) = 1\bigr\},$$

where $\mathbb{I}$ is the identity matrix and $\mathbb{A}^*$ is the conjugate (or Hermitian) transpose of $\mathbb{A}$. The corresponding Lie algebra is defined as

$$\mathsf{\textbf{Lie}}\bigl(\mathbf{SU}(n)\bigr) = \mathfrak{su}(n) \stackrel{\text{def}}{:=} \{\mathbb{A} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathbb{A}^* = -\mathbb{A},\ \mathrm{tr}(\mathbb{A}) = 0\}.$$

The Lie algebra $\mathfrak{su}(n)$ is an example of a semi-simple Lie algebra with a Killing form $\kappa(X, Y) = 2n\,\mathrm{tr}(XY)$ and $\mathfrak{su}(n) \simeq \mathfrak{su}(n)^*$. The action is defined as

$$\begin{aligned} \lambda : \mathbf{SU}(n) \times \mathbb{T}^*\bigl(\mathfrak{su}(n)\bigr) &\longrightarrow \mathbb{T}^*\bigl(\mathfrak{su}(n)\bigr) , \\ \bigl(g, (X, L)\bigr) &\longmapsto \bigl(g X g^{-1}, g L g^{-1}\bigr) . \end{aligned}$$

Then, $\Omega = \mathrm{tr}(\mathrm{d}X \wedge \mathrm{d}L)$ and the momentum map is given by

$$\mu(X, L) = [X, L] .$$

1.5 Reduction of the Phase Space

Let $(\mathbb{M}, \omega)$ be a symplectic manifold and $\lambda : G \times \mathbb{M} \longrightarrow \mathbb{M}$ is a Hamiltonian action, i.e.

$$\lambda_g^*(\omega) = \omega, \qquad \forall g \in G.$$

We justify the terminology by the following observation:

Lemma 1.11 *Assume that there exists a momentum map $\mu : \mathbb{M} \longrightarrow \mathfrak{g}^*$, one necessarily obtains that*

$$\lambda^*(\mathcal{Y}) = -\mathcal{X}_{\bar{\mu}(\mathcal{Y})}, \qquad \forall \mathcal{Y} \in \mathfrak{g}.$$

Definition 1.13 An element $c \in \mathfrak{g}^*$ is called a *regular* if $\mathbb{M}_c \stackrel{\text{def}}{:=} \mu^{-1}(c)$ is a sub-manifold in $\mathbb{M}$ and if

$$\ker(\mathbb{T}_m \mu) = \mathbb{T}_m \mathbb{M}_c, \qquad \forall c \in \mathbb{M}_c.$$

Lemma 1.12 *Let $G_c \stackrel{\text{def}}{:=} \left\{ g \in G \,\middle|\, \boldsymbol{Ad}_g^*(c) = c \right\}$. If G is connected and simply connected, then $\forall m \in \mathbb{M}_c$ and $\forall g \in G_c$:*

$$g \cdot m \in \mathbb{M}_c.$$

Proof Left to the reader as an exercise. □

Remark 1.2 In the case when c is a regular element and G is connected and simply connected, the action of G on $\mathbb{M}$ induces an action of the Lie sub-group $G_c \subseteq G$ on the sub-manifold $\mathbb{M}_c \subseteq \mathbb{M}$.

1.5.1 The Main Results

Theorem 1.9 *If G is connected and simply connected and, in addition:*

- *c is a regular element;*
- *G_c is compact;*
- *G_c acts on $\mathbb{M}_c$ by free and transitive action.*

Then, there exists a natural smooth structure on $\mathbb{M}_c / G_c$ such that the mapping $\pi_c : \mathbb{M}_c \longrightarrow \mathbb{M}_c / G_c$ is a submersion.

Remark 1.3 The quotient space $\mathbb{M}_c / G_c$ is called in this case the *reduced phase space*.

Theorem 1.10 (Marsden–Weinstein) *If G is connected and simply connected and, in addition:*

- *c is a regular element;*
- *G_c is compact;*
- *G_c acts on $\mathbb{M}_c$ by free and transitive action.*

Then, there exists a unique symplectic $2-$form ω_c on $\mathbb{M}_c / G_c$ such that

$$\pi_c^*(\omega_c) = \iota_c^*(\omega),$$

where $\pi_c : \mathbb{M}_c \longrightarrow \mathbb{M}_c / G_c$ is the canonical submersion and $\iota_c : \mathbb{M}_c \hookrightarrow \mathbb{M}$ is the canonical embedding.

The proof of this Theorem is based on the following

Lemma 1.13 *Let $m \in \mathbb{M}_c$. Then, $\mathbb{T}_m \mathbb{M}_c = (\mathbb{T}_m G \cdot m)^{\perp}$. In other words,*

$$\mathbb{T}_m \mathbb{M}_c = \{ \mathcal{X} \in \mathbb{T}_m \mathbb{M} \mid \omega_m(\mathcal{X}, \mathcal{Y}) = 0, \forall \mathcal{Y} \in \mathbb{T}_m G \cdot m \}.$$

Proof Left to the reader as an exercise. □

Remark 1.4 Observe that $\mathbb{T}_m \mathbb{M}_c \bigcap \mathbb{T}_m G \cdot m \neq \varnothing$. More precisely, $\mathbb{T}_m \mathbb{M}_c \bigcap \mathbb{T}_m G \cdot m = \mathbb{T}_m G_c \cdot m$.

Corollary 1.1 *Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2 \in \mathbb{T}_m \mathbb{M}_c \subseteq \mathbb{T}_m \mathbb{M}$ such that*

$$\mathbb{T}_m \pi_c(\mathcal{X}_1) = \mathbb{T}_m \pi_c(\mathcal{X}_2),$$
$$\mathbb{T}_m \pi_c(\mathcal{Y}_1) = \mathbb{T}_m \pi_c(\mathcal{Y}_2),$$

then, $\omega_m(\mathcal{X}_1, \mathcal{Y}_1) = \omega_m(\mathcal{X}_2, \mathcal{Y}_2)$.

Lemma 1.14 *Let $m, n \in \mathbb{M}_c$ such that $\pi_c(m) = \pi_c(n)$ and $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2 \in \mathbb{T}_m \mathbb{M}_c \subseteq \mathbb{T}_n \mathbb{M}_c \subseteq \mathbb{T}_n \mathbb{M}$ such that*

$$\mathbb{T}_m \pi_c(\mathcal{X}_1) = \mathbb{T}_n \pi_c(\mathcal{X}_2),$$
$$\mathbb{T}_m \pi_c(\mathcal{Y}_1) = \mathbb{T}_n \pi_c(\mathcal{Y}_2),$$

then, $\omega_m(\mathcal{X}_1, \mathcal{Y}_1) = \omega_n(\mathcal{X}_2, \mathcal{Y}_2)$.

Proof Left to the reader as an exercise. □

Lemma 1.15 *Let c be a regular element and $\mathcal{O}_c \stackrel{\text{def}}{:=} G \cdot c$ be its co-adjoint orbit. Then, $\mu^{-1}(\mathcal{O}_c)$ is a sub-manifold in $\mathbb{M}$.*

Proof Left to the reader as an exercise. □

Theorem 1.11 *The mapping*

$$\phi : \mathcal{O}_c \longrightarrow \mathbb{M}_c / G_c ,$$
$$m \longmapsto \pi_c (g^{-1} m) ,$$

where $\mu (m) = \boldsymbol{Ad}^*_g (c)$ *is correctly defined, induces a diffeomorphism:*

$$\Phi : \pi \left(\mu^{-1} (\mathcal{O}_c) \right) \longrightarrow \mathbb{M}_c / G_c .$$

1.5.2 Example

In this Section we consider again the action of $\mathbb{S}^1$ on $\mathbb{C}^n$, which is defined as

$$\lambda : \mathbb{S}^1 \times \mathbb{C}^n \longrightarrow \mathbb{C}^n ,$$
$$\left(\mathrm{e}^{\mathrm{i}\theta}, \mathbf{q} + \mathrm{i}\mathbf{p} \right) \longmapsto \mathrm{e}^{\mathrm{i}\theta} \mathbf{q} + \mathrm{i}\, \mathrm{e}^{\mathrm{i}\theta} \mathbf{p} ,$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$. The momentum map is

$$\mu : \mathbb{C}^n \longrightarrow \mathbf{Lie} (\mathbb{S}^1) ,$$
$$\mathbf{q} + \mathrm{i}\,\mathbf{p} \longmapsto - \sum_{i=1}^{n} \frac{q_i^2 + p_i^2}{2} .$$

Then, $\mathbb{M}_c = \left\{ z \in \mathbb{C}^n \,\middle|\, \sum_{i=1}^{n} | z_i |^2 = 2c \right\} \simeq \mathbb{S}^n , c > 0$. It is also clear that $G_c \simeq \mathbb{S}^1$ and $\mathbb{S}^1$ is an Abelian group. Thus, we have

$$L_g R_{g^{-1}} = \mathbb{I} \qquad \Longrightarrow \qquad \mathbf{Ad}_g = \mathbf{Ad}^*_g = \mathbb{I} .$$

Henceforth,

$$\mathbb{M}_c / G_c = \mathbb{S}^n / \mathbb{S}^1 \simeq \mathbb{P}^{n-1} .$$

1.6 Poisson–Lie Groups

A Lie group G is called *Poisson–Lie group* if it is a Poisson manifold such that the multiplication $m : G \times G \longrightarrow G$ is a morphism of Poisson manifolds. Let $\mathfrak{g}$ be Lie algebra, $\mathfrak{g}^*$ be dual vector space to $\mathfrak{g}$.

Definition 1.14 We say that $\mathfrak{g}$ is a *Lie bi-algebra* if there is a Lie algebra structure $[\,-\,,\,-\,]_*$ on $\mathfrak{g}^*$ such that the map $\delta\,:\,\mathfrak{g}\longrightarrow\Lambda^2\,\mathfrak{g}$ (called the *co-bracket*), dual to the bracket $[\,-\,,\,-\,]_*\,:\,\Lambda^2\,\mathfrak{g}^*\longrightarrow g^*$ is a $1-$cocycle with respect to the adjoint action of $\mathfrak{g}$ on $\Lambda^2\,\mathfrak{g}$.

1.6.1 Modified Classical Yang–Baxter Equation

Let G be connected and a simply connected Lie group, and let $\mathfrak{g}$ be its Lie algebra. Then there is one-to-one correspondence between Poisson–Lie group structures on G and Lie bi-algebra structures on $\mathfrak{g}$.

As V. Drinfel'd showed [3], every structure on a semi-simple connected G has the following form:

$$\pi\,(g)\;=\;\Lambda^2\Big((\mathcal{L}_g)_*\Big)(\mathbf{r})\;-\;\Lambda^2\Big((\mathcal{R}_g)_*\Big)(\mathbf{r}),\qquad(1.2)$$

where $(\mathcal{L}_g)_*$ and $(\mathcal{R}_g)_*$ denote tangent maps of left and right translations by $g\in G$. The element $\mathbf{r}\in\Lambda^2\,\mathfrak{g}$ satisfies the following condition:

$$[\![\mathbf{r},\,\mathbf{r}]\!]\;\stackrel{\text{def}}{:=}\;[\,\mathbf{r}_{12}\,,\,\mathbf{r}_{13}\,]\;+\;[\,\mathbf{r}_{12}\,,\,\mathbf{r}_{23}\,]\;+\;[\,\mathbf{r}_{13}\,,\,\mathbf{r}_{23}\,]\;\in\;\Lambda^3\,\mathfrak{g},\qquad(1.3)$$

where the right-hand side is invariant under the adjoint action of $\mathfrak{g}$. The condition (1.3) is called a *modified Yang–Baxter equation* and the bracket

$$[\![-,\,-]\!]\;:\;\Lambda^2\,\mathfrak{g}\otimes\Lambda^2\,\mathfrak{g}\;\longrightarrow\;\Lambda^3\,\mathfrak{g}$$

is a so-called *Schouten–Nijenhuis bracket*. This is the natural graded (or *super-*) Lie algebra structure on the exterior algebra

$$\Lambda^\bullet\,\mathfrak{g}\;=\;\bigoplus_k\,\Lambda^k\,\mathfrak{g}\,.$$

Here $\mathbf{r}_{12}$, to give an example, denotes an element $\mathbf{r}_{12}\;=\;\mathbf{r}\otimes\mathbb{I}_3\;\in\;(\mathfrak{g}\otimes\mathbf{k})^{\otimes 3}$; $\mathbf{k}\in\{\,\mathbb{R},\,\mathbb{C}\,\}$ and $\mathbf{r}$ being usually called a *classical* $\mathbf{r}$*−matrix*.

The condition (1.3) ensures that the bracket $\{\,-\,,\,-\,\}_*$ on $\mathfrak{g}^*$ satisfies the Jacobi identity. The corresponding Lie bi-algebra structure is calculated in the obvious way. Namely, the co-bracket δ is given by

$$\delta\,(x)\;=\;\mathrm{d}_e\,\pi\,(x)\;=\;\mathcal{L}_{\bar{x}}\,\pi\,(e)\;=\;\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{r}_{(\mathrm{e}^{-tx})_*}\,\pi\,(\mathrm{e}^{tx})\Big|_{t=0}\;=\;\mathbf{ad}_x\,(\mathbf{r}),$$

where $\mathrm{d}_e\,\pi$ is the intrinsic derivative of a poly-vector field on G with $\pi\,(e)\;=\;0$, $\bar{x}$ is any vector field on G with $\bar{x}\,(e)\;=\;x$, and $\mathcal{L}_{\bar{x}}$ denotes the Lie derivative [8].

The Poisson structures of the form (1.2) are called *co-boundary* or $\mathbf{r}-$*matrix structures*. Since for a connected semi-simple or a compact Lie group G every $1-$cocycle is a co-boundary, one has the following

Proposition 1.12 . *The Poisson–Lie structures on a connected semi-simple or a compact Lie group G are in one-to-one correspondence with the solutions* $\mathbf{r} \in \Lambda^2 \mathfrak{g}$ *of the modified Yang–Baxter equation.*

1.6.2 Manin Triples

Let $\mathfrak{g}$ be a Lie bi-algebra. There is a unique Lie algebra structure on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ such that

1. $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie sub-algebras.
2. The symmetric bi-linear form on $\mathfrak{g} \oplus \mathfrak{g}^*$ given by the relation

$$\langle \mathcal{X} + \xi , \mathcal{Y} + \eta\mathcal{Y} \rangle = \langle \mathcal{X} , \eta \rangle + \langle \mathcal{Y} , \xi \rangle , \quad \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{g}, \quad \forall \xi, \eta \in \mathfrak{g}^*$$

is invariant.

This structure is given by

$$\{ \mathcal{X} , \xi \} = -\mathbf{ad}^*_{\mathcal{X}} (\xi) + \mathbf{ad}^*_{\xi} (\mathcal{X}) ,$$

for $\mathcal{X} \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, where $\mathbf{ad}^*$ is the co-adjoint action. This Lie algebra is denoted by $\mathfrak{g} \bowtie \mathfrak{g}^*$ and $(\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ is an example of a Manin triple. In general, a Manin triple is a decomposition of a Lie algebra $\mathfrak{g}$ with a non-degenerate invariant scalar product $\langle \ , \ \rangle$ into direct sum of isotropic with respect to $\langle \ , \ \rangle$ vector spaces, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ such that $\mathfrak{g}_\pm$ are Lie sub-algebras of $\mathfrak{g}$. It is well-known that there is one-to-one correspondence between Lie bi-algebras and Manin triples. These triples were introduced by V. Drinfel'd [4] and named after Yu. I. Manin.

1.6.3 Poisson–Lie Duality

Let G be a connected and simply connected Poisson–Lie group, $\mathfrak{g} = \mathbf{Lie} (G)$ its Lie algebra and $\left(\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^* \right)$ the Manin triple. By duality, $\left(\mathfrak{g}^* \bowtie \mathfrak{g}, \mathfrak{g}^*, \mathfrak{g} \right)$ is also a Manin triple. Then $\mathfrak{g}^*$ is a Lie bi-algebra. This enables us to consider a connected and simply connected Lie group G^* with a Poisson–Lie structure π^* and with the tangent Lie bi-algebra $\mathfrak{g}^*$. The Poisson–Lie group (G^*, π^*) is called the *Poisson–Lie dual* to (G, π).

1.6.4 Example of Non-Hamiltonian Action

Let G be a Poisson–Lie group with a multiplicative Poisson tensor π_g and $\mathbb{M}$ be a smooth Poisson manifold with a Poisson structure given by $\pi_{\mathbb{M}}$. Then, the product $G \times \mathbb{M}$ can be considered as a Poisson manifold with the direct sum structure $\tilde{\pi}$.

Proposition 1.13 *An action $\sigma : G \times \mathbb{M} \longrightarrow \mathbb{M}$ of a Poisson–Lie group G on a Poisson manifold $\mathbb{M}$ is a Poisson–Lie action if and only if*

$$\pi_{\mathbb{M}}(g \cdot m) = \Lambda^2\bigl((\sigma_g)_*\bigr)\bigl(\pi_{\mathbb{M}}(m)\bigr) + \Lambda^2\bigl((\sigma_m)_*\bigr)\bigl(\pi_G(g)\bigr).$$

Remark 1.5 One can consider any Lie group G as a Poisson–Lie group with $\pi_G \equiv 0$ then the action σ is a Poisson (action) if it gives a Poisson morphism $\pi_{\mathbb{M}}(g \cdot m) = (\sigma_g)_*\bigl(\pi_{\mathbb{M}}(m)\bigr)$.

Definition 1.15 The action $\sigma : G \times \mathbb{M} \longrightarrow \mathbb{M}$ is called a Poisson–Lie action if $\pi^* : C^\infty(\mathbb{M}) \longrightarrow C^\infty(G \times \mathbb{M})$ is a Poisson morphism:

$$\pi^*\bigl(\{F, H\}_{\pi_{\mathbb{M}}}\bigr) = \bigl\{\pi^*(F), \pi^*(H)\bigr\}_{\tilde{\pi}}.$$

Infinitesimally, a Poisson–Lie action of a Lie bi-algebra $\mathfrak{g}$ on a Poisson manifold $(\mathbb{M}, \{-, -\})$ is given by an action

$$\begin{aligned} \rho : \mathfrak{g} &\longrightarrow \mathfrak{X}(\mathbb{M}), \\ \mathcal{X} &\longmapsto V_{\mathcal{X}}, \end{aligned}$$

with $\mathcal{X} \in \mathfrak{g}$ such that

$$\begin{aligned} V_{\mathcal{X}}\{f, g\}(m) = \{V_{\mathcal{X}} f, g\}(m) &+ \{f, V_{\mathcal{X}} g\}(m) \\ &- \bigl\{\mathcal{X}, \bigl[\rho^* \mathrm{d} f(m), \rho^* \mathrm{d} g(m)\bigr]\bigr\}, \end{aligned}$$

where $\rho^* \mathrm{d} f(m) \in \mathfrak{g}^*$ and $\langle \mathcal{X}, \rho^* \mathrm{d} f(m)\rangle = V_{\mathcal{X}} f(m)$. In other words,

$$\langle \mathcal{X}, \bigl[\tilde{\rho}(\mathrm{d}F)(m), \tilde{\rho}(\mathrm{d}G)(m)\bigr]_*\rangle = \langle \mathrm{d}F, \mathrm{d}G\rangle\bigl(\rho(\mathcal{X})\bigr)(m)$$

define a *Lie algebroid* structure on $\mathbb{T}^*\mathbb{M}$.

Example 1.9 There are natural left and right actions of dual Poisson–Lie group G^* on G. These actions are called *left (right) dressing* transformations. The dressing transformations are not Hamiltonian as Semenov-Tian-Shansky proved but these actions are genuine Poisson–Lie actions [9].

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