b-Generalized Skew Derivations on Multilinear Polynomials in Prime Rings

Vincenzo De Filippis, Giovanni Scudo, and Feng Wei

Abstract Let R be a prime ring of characteristic different from 2, Q_r be its right Martindale quotient ring and C be its extended centroid. In this paper we define b-generalized skew derivations of prime rings. Then we describe all possible forms of two b-generalized skew derivations F and G satisfying the condition $F(x)x - xG(x) = 0$, for all $x \in S$, where S is the set of the evaluations of a multilinear polynomial $f(x_1, \ldots, x_n)$ over C with n non-commuting variables. Several potential research topics related to our current work are also presented.

Keywords Prime rings · Generalized skew derivations · Multilinear polynomials

1 Introduction

In this paper, unless otherwise mentioned, R always denotes a prime ring with center $Z(R)$. We denote the *right Martindale quotient ring* of R by Q_r. The center of Q_r is denoted by C, which is called *extended centroid* of R. We refer the reader to the book [4] for more details.

An additive mapping $d\colon R \longrightarrow R$ is said to be a *derivation* of R if

$$d(xy) = d(x)y + xd(y)$$

for all $x, y \in R$. An additive mapping $F\colon R \longrightarrow R$ is called a *generalized derivation* of R if there exists a derivation d of R such that

$$F(xy) = F(x)y + xd(y)$$

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for all $x, y \in R$. The derivation d is uniquely determined by F, which is called an *associated derivation* of F.

The definition of generalized skew derivation is a unified notion of skew derivation and generalized derivation, which are considered as classical additive mappings of non-commutative algebras, have been investigated by many people from various views, see [1, 9, 11–14, 16, 24, 25, 28, 29, 39, 42, 45]. Let R be an associative ring and α be an automorphism of R. An additive mapping $d\colon R \longrightarrow R$ is said to be a *skew derivation* of R if

$$d(xy) = d(x)y + \alpha(x)d(y)$$

for all $x, y \in R$. The automorphisms α is called an *associated automorphism* of d. An additive mapping $F\colon R \longrightarrow R$ is called a *generalized skew derivation* of R if there exists a skew derivation d of R with associated automorphism α such that

$$F(xy) = F(x)y + \alpha(x)d(y)$$

for all $x, y \in R$. In this case, d is called an *associated skew derivation* of F and α is called an *associated automorphism* of F. It was Chang who first introduced this notion and initiated the study of generalized skew derivations of (semi-)prime rings in [10]. Therein, he described the identity of the form $h(x) = af(x) + g(x)b$, where f, g and h are the so-called generalized (α, β)-derivations of a prime ring R, a and b are some fixed noncentral elements of R.

It is worth pointing out that many research papers are devoting to studying the additive mappings in the interfaces between algebra and operator algebra. In [7], Brešar and Villena investigate the automatic continuity of skew derivations on Banach algebras and gave the skew derivation version of noncommutative Singer-Wermer conjecture on Banach algebras. Various technical generalizations of derivations on (semi-)prime rings are used to discuss the range inclusion problems of generalized derivations on noncommutative Banach algebras, see [5, 8, 27, 46, 47]. More recently, Eremita et al determine the structure of generalized skew derivations implemented by elementary operators [30]. Liu and his students characterize a (generalized-)skew derivation F of Banach algebras so that the values of F on a left ideal are nilpotent [41, 43]. Qi and Hou in [45] study generalized skew derivations on nest algebras determined by acting on zero products.

Brešar in [6] gives a description of additive mappings which are commuting on a prime ring R. More precisely, he proves that if F is an additive mapping of R into itself which is centralizing on R and if either R has a characteristic different from 2 or F is commuting on R, then F is of the form $F(x) = \lambda x + \zeta(x)$, where λ is an element of the extended centroid C of R and ζ is an additive mapping of R into C. Moreover, the general situation when two additive mappings F and G of the ring R satisfy $F(x)x - xG(x) \in Z(R)$ for all x in a subset S of R is considered. In particular, it is showed that if $0 \neq F$ and G are both derivations of R and S is a nonzero left ideal of R, then R is commutative. Many researchers successfully extended this result concerning derivations, by replacing S with other subsets of R or replacing F and G with other types of additive mappings. In [49], Wong characterizes derivations F and G of R such that $F(x)x - xG(x) \in Z(R)$, for all $x \in S$, where S is the set of all the evaluations (in a non-zero ideal of R) of a non-central multilinear polynomial over C. Later, Lee and Shiue in [36] extend Wong's result to derivations acting on arbitrary polynomials. Then, in [40], Liu generalizes the theorem of Wong to one-sided ideals. More recently, Chen in [15] extends Lee and Shiue's result to generalized derivations.

In a recent paper [34], Koşan and Lee propose the following new definition. Let $d\colon R \longrightarrow Q_r$ be an additive mapping and $b \in Q_r$. An additive map $F\colon R \longrightarrow Q_r$ is called a *left b-generalized derivation*, with associated mapping d, if $F(xy) = F(x)y + bxd(y)$, for all $x, y \in R$. In the present paper this mapping F will be called *b-generalized derivation* with associated pair (b, d). Clearly, any generalized derivation with associated derivation d is a b-generalized derivation with associated pair $(1, d)$.

In view of this idea, we now give the following:

Definition 1 Let $b \in Q_r$, $d : R \longrightarrow Q_r$ an additive mapping and α be an automorphism of R. An additive mapping $F : R \longrightarrow Q_r$ is called a *b-generalized skew derivation* of R, with associated term (b, α, d) if

$$F(xy) = F(x)y + b\alpha(x)d(y)$$

for all $x, y \in R$.

According to the above definition, we can conclude that general results about b-generalized skew derivations may give useful and powerful corollaries about derivations, generalized derivations, skew derivations and generalized skew derivations.

The main goal of the present paper is to prove the following theorem. It characterizes b-generalized skew derivations which are commuting on multilinear polynomials in prime rings:

Theorem 1 *Let R be a prime ring of characteristic different from 2, Q_r be its right Martindale quotient ring and C be its extended centroid, $\alpha \in Aut(R)$, d and δ skew derivations of R with associated automorphism α, such that both d and δ are commuting with α. Suppose that F, G are b-generalized skew derivations of R, with associated terms (b, α, d) and (p, α, δ), respectively. Let $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over C with n non-commuting variables. If*

$$F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)G(f(r_1, \ldots, r_n)) = 0 \qquad (1)$$

for all $r_1, \ldots, r_n \in R$, then one of the following statements holds:

1. *there exists $a' \in Q_r$ such that, $F(x) = xa'$ and $G(x) = a'x$ for all $x \in R$;*
2. *$f(x_1, \ldots, x_n)^2$ is central-valued on R and there exist $a', b' \in Q_r$ such that $F(x) = a'x + xb'$, $G(x) = b'x + xa'$, for all $x \in R$.*

Let us recall some results which will be useful in the sequel.

Note 1 Let R be a prime ring, then the following statements hold:

1. Every generalized derivation of R can be uniquely extended to Q_r [35, Theorem 3].
2. Any automorphism of R can be uniquely extended to Q_r [19, Fact 2].
3. Every generalized skew derivation of R can be uniquely extended to Q_r [10, Lemma 2].

Lemma 1 *Let R be a prime ring, $\alpha \in \mathrm{Aut}(R)$, $0 \neq b \in Q_r$, $d\colon R \longrightarrow R$ be an additive mapping of R and F be the b-generalized skew derivation of R with associated term (b, α, d). Then d is a skew derivation of R with associated automorphism α.*

Proof See [26, Lemma 3.2].

Lemma 2 *Let R be a prime ring, $\alpha \in Aut(R)$, $b \in Q_r$, $d\colon R \longrightarrow R$ be an additive mapping of R and F be the b-generalized skew derivation of R with associated term (b, α, d). Then F can be uniquely extended to Q_r and assumes the form $F(x) = ax + bd(x)$, where $a \in Q_r$.*

Proof See [26, Lemma 3.3].

2 Some Results on Differential Identities with Automorphisms

In order to proceed with our proofs, we need to recall some well-known results on skew derivations and automorphisms involved in generalized polynomial identities for prime rings.

Let us denote by $\mathrm{SDer}(Q_r)$ the set of all skew-derivations of Q_r. By a *skew-derivation word* we mean an additive mapping Δ of the form $\Delta = d_1 d_1 \dots d_m$, where $d_i \in \mathrm{SDer}(Q_r)$. A *skew-differential polynomial* is a generalized polynomial with coefficients in Q_r of the form $\Phi(\Delta_j(x_i))$ involving noncommutative indeterminates x_i on which the skew derivation words Δ_j act as unary operations. The skew-differential polynomial $\Phi(\Delta_j(x_i))$ is said to be a *skew-differential identity* on a subset T of Q_r if it vanishes on any assignment of values from T to its indeterminates x_i.

Let R be a prime ring, $\mathrm{SD}_{\mathrm{int}}$ be the C-subspace of $\mathrm{SDer}(Q_r)$ consisting of all inner skew-derivations of Q_r, and let d and δ be two non-zero skew-derivations of Q_r. The following results follow as special cases from results in [18–21, 33].

Note 2 Let d and δ be skew derivations on R, associated with the same automorphism α of R. Assume that d and δ are $C-$linearly independent modulo $\mathrm{SD}_{\mathrm{int}}$.
If d and δ are commuting with the automorphism α and $\Phi(\Delta_j(x_i))$ is a skew-differential identity on R, where Δ_j are skew-derivations words from the set $\{d, \delta\}$,

then $\Phi(y_{ji})$ is a generalized polynomial identity of R, where y_{ji} are distinct indeterminates (see [33, Theorem 6.5.9]).

In particular, we have

Note 3 In [22] Chuang and Lee investigate polynomial identities with a single skew derivation. They prove that if $\Phi(x_i, D(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and D is an outer skew derivation of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates. Furthermore, they observe [22, Theorem 1] that in the case $\Phi(x_i, D(x_i), \alpha(x_i))$ is a generalized polynomial identity for a prime ring R, D is an outer skew derivation of R and α is an outer automorphism of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i, z_i)$, where x_i, y_i, and z_i are distinct indeterminates.

Note 4 If d and δ are $C-$linearly dependent modulo $\mathrm{SD}_{\mathrm{int}}$, then there exist $\lambda, \mu \in C$, $a \in Q_r$ and $\alpha \in \mathrm{Aut}(Q_r)$ such that $\lambda d(x) + \mu\delta(x) = ax - \alpha(x)a$ for all $x \in R$.

Note 5 By Chuang and Lee [22] we can state the following result. If d is a non-zero skew-derivation of R and

$$\Phi\Big(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n)\Big)$$

is a skew-differential polynomial identity of R, then one of the following statements holds:

1. either $d \in \mathrm{SD}_{\mathrm{int}}$;
2. or R satisfies the generalized polynomial identity

$$\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

Note 6 Let R be a prime ring and I be a two-sided ideal of R. Then I, R, and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r (see [18]). Furthermore, I, R, and Q_r satisfy the same generalized polynomial identities with automorphisms (see [20, Theorem 1]).

Note 7 Let R be a prime ring, $\mathrm{Inn}(Q_r)$ be the C-subspace of $\mathrm{Aut}(Q_r)$ consisting of all inner automorphisms of Q_r and let α and β be two non-trivial automorphisms of Q_r.
α and β are called *mutually outer* if $\alpha\beta^{-1}$ is not an inner automorphism of Q_r.
If α and β are mutually outer automorphisms of Q_r and $\Phi(x_i, \alpha(x_i), \beta(x_i))$ is an automorphic identity for R, then by Kharchenko [32, Theorem 4] we know that $\Phi(x_i, y_i, z_i)$ is a generalized polynomial identity for R, where x_i, y_i, z_i are distinct indeterminates.

Note 8 Let R be a prime ring, $\alpha, \beta \in \mathrm{Aut}(Q_r)$ and $d : R \to R$ be a skew derivation, associated with the automorphism α. If there exist $0 \neq \theta \in C$, $0 \neq \eta \in C$ and $u, b \in Q_r$ such that

$$d(x) = \theta\Big(ux - \alpha(x)u\Big) + \eta\Big(bx - \beta(x)b\Big), \quad \forall x \in R \tag{2}$$

then d is an inner skew derivation of R. More precisely, either $b = 0$ or $\alpha = \beta$.

Proof Starting from relation (2) we have

$$d(xy) = \theta\Big(uxy - \alpha(x)\alpha(y)u\Big) + \eta\Big(bxy - \beta(x)\beta(y)b\Big), \quad \forall x, y \in R. \tag{3}$$

On the other hand,

$$\begin{aligned} &d(xy) = d(x)y + \alpha(x)d(y) = \\ &\theta\Big(ux - \alpha(x)u\Big)y + \eta\Big(bx - \beta(x)b\Big)y + \\ &\alpha(x)\theta\Big(uy - \alpha(y)u\Big) + \alpha(x)\eta\Big(by - \beta(y)b\Big). \end{aligned} \tag{4}$$

Comparision of (3) with (4) leads to

$$\eta\Big(\beta(x)\beta(y)b - \beta(x)by + \alpha(x)by - \alpha(x)\beta(y)b\Big) = 0, \quad \forall x, y \in R. \tag{5}$$

Suppose first that α and β are mutually outer, in the sense of Note 7. Therefore, by (5) and since $\eta \neq 0$, it follows that

$$y_1y_2b - y_1by + x_1by - x_1y_2b = 0, \quad \forall x, y, x_1, y_1, y_2 \in R. \tag{6}$$

In particular, for $y_2 = x_1 = 0$ we get $y_1by = 0$, for any $y, y_1 \in R$ and, by the primeness of R, it follows $b = 0$, as required.
Now we assume that α and β are not mutually outer, that is there exists an invertible element $q \in Q_r$ such that $\alpha\beta^{-1}(x) = qxq^{-1}$, for any $x \in R$. Replacing x by $\beta(x)$, it follows easily that $\alpha(x) = q\beta(x)q^{-1}$. Hence by (5)

$$\beta(x)\beta(y)b - \beta(x)by + q\beta(x)q^{-1}by - q\beta(x)q^{-1}\beta(y)b = 0, \quad \forall x, y \in R$$

that is

$$\Big(q\beta(x)q^{-1} - \beta(x)\Big)\Big(\beta(y)b - by\Big) = 0, \quad \forall x, y \in R. \tag{7}$$

Now replace y by yz in (7), then

$$\left(q\beta(x)q^{-1} - \beta(x)\right)\left(\beta(y)\beta(z)b - byz\right) = 0, \quad \forall x, y, z \in R \tag{8}$$

and using (7) in (8) it follows

$$\left(q\beta(x)q^{-1} - \beta(x)\right)\beta(y)\left(\beta(z)b - bz\right) = 0, \quad \forall x, y, z \in R. \tag{9}$$

By the primeness of R, one has that either $\beta(z)b - bz = 0$, for any $z \in R$, or $q\beta(x)q^{-1} - \beta(x) = 0$, for any $x \in R$. In the first case $d(x) = \theta\left(ux - \alpha(x)u\right)$ and we are done. In the latter case, for any $x \in R$ we get $\beta(x) = q\beta(x)q^{-1} = \alpha(x)$ and we are done again.

Note 9 Assuming that $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C and d is a skew derivation of R, associated with the automorphism α, we denote

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \gamma_\sigma x_{\sigma(1)} \cdot x_{\sigma(2)} \cdots x_{\sigma(n)}, \quad \gamma_\sigma \in C.$$

Let $f^d(x_1, \ldots, x_n)$ be the polynomial originated from $f(x_1, \ldots, x_n)$ by replacing each coefficient γ_σ with $d(\gamma_\sigma)$. Thus

$$d\left(\gamma_\sigma \cdot x_{\sigma(1)} \cdot x_{\sigma(2)} \cdots x_{\sigma(n)}\right) = d(\gamma_\sigma)x_{\sigma(1)} \cdot x_{\sigma(2)} \cdots x_{\sigma(n)} +$$

$$+\alpha(\gamma_\sigma)\sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdot x_{\sigma(2)} \cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)} \cdots x_{\sigma(n)}$$

and

$$d(f(x_1, \ldots, x_n)) = f^d(x_1, \ldots, x_n) +$$

$$+\sum_{\sigma \in S_n} \alpha(\gamma_\sigma)\sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdot x_{\sigma(2)} \cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)} \cdots x_{\sigma(n)}.$$

3 Commuting Generalized Derivations and Commuting Generalized Skew Derivations

Here we would like also to collect some results in literature concerning commuting generalized derivations and commuting generalized skew derivations. This section will be useful in the sequel in order to conclude the proof of our main results.

Proposition 1 ([2, Lemma 3]) *Let R be a prime ring, Q_r be its right Martindale quotient ring and C be its extended centroid, $f(x_1,\ldots,x_n)$ be a multilinear polynomial over C, which is not central-valued on R. Suppose there exist $a,b,c,q \in Q_r$ such that*

$$\begin{aligned}\Big(af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b\Big)f(r_1,\ldots,r_n)\\ -f(r_1,\ldots,r_n)\Big(cf(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)q\Big)=0\end{aligned} \tag{10}$$

for all $r_1,\ldots,r_n \in R$. Then one of the following statements holds:

1. *$a,q \in C$, $q-a=b-c=\alpha \in C$;*
2. *$f(x_1,\ldots,x_n)^2$ is central-valued on R and there exists $\alpha \in C$ such that $q-a=b-c=\alpha$;*
3. *$char(R)=2$ and R satisfies S_4.*

Corollary 1 *Let R be a prime ring and $f(x_1,\ldots,x_n)$ be a multilinear polynomial over C with n non-commuting variables. Let $a,b \in R$ be such that*

$$af(r_1,\ldots,r_n)^2+f(r_1,\ldots,r_n)bf(r_1,\ldots,r_n)=0$$

for all $r_1,\ldots,r_n \in R$. If $f(x_1,\ldots,x_n)$ is not central valued on R, then either $a=-b \in C$, or $char(R)=2$ and R satisfies S_4.

Lemma 3 ([2, Lemma 1]) *Let R be a prime ring and $f(x_1,\ldots,x_n)$ be a polynomial over C with n non-commuting variables. Let $a,b \in R$ be such that $af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b=0$ for all $r_1,\ldots,r_n \in R$. If $f(x_1,\ldots,x_n)$ is not a polynomial identity for R, then either $a=-b \in C$, or $f(x_1,\ldots,x_n)$ is central-valued on R and $a+b=0$, unless $char(R)=2$ and $R \subseteq M_2(C)$, the 2×2 matrix ring over C.*

Corollary 2 *Let R be a prime ring of characteristic different from 2 and $f(x_1,\ldots,x_n)$ be a polynomial over C with n non-commuting variables. Let $a \in R$ be such that $f(r_1,\ldots,r_n)a=0$ (or $af(r_1,\ldots,r_n)=0$) for all $r_1,\ldots,r_n \in R$. If $f(x_1,\ldots,x_n)$ is not a polynomial identity for R, then $a=0$.*

Theorem 2 ([2, Theorem 1]) *Let R be a prime ring, Q_r be its right Martindale quotient ring and C be its extended centroid, I a non-zero two-sided ideal of R, F and G non-zero generalized derivations of R. Suppose that $f(x_1,\ldots,x_n)$ is a* *non-central multilinear polynomial over* C *such that*

$$F(f(r_1,\dots,r_n))f(r_1,\dots,r_n) - f(r_1,\dots,r_n)G(f(r_1,\dots,r_n)) = 0$$

for all $r_1,\dots,r_n \in I$, *then one of the following statements holds:*

1. *there exists* $a \in Q_r$ *such that,* $F(x) = xa$ *and* $G(x) = ax$ *for all* $x \in R$;
2. $f(x_1,\dots,x_n)^2$ *is central-valued on* R *and there exist* $a, b \in Q_r$ *such that* $F(x) = ax + xb$, $G(x) = bx + xa$, *for all* $x \in R$;
3. $char(R) = 2$ *and* R *satisfies* S_4, *the standard identity of degree 4.*

4 Some Remarks on Matrix Algebras

Let us state some well-known facts concerning the case when $R = M_m(K)$ is the algebra of $m \times m$ matrices over a field K. Note that the set $f(R) = \{f(r_1,\dots,r_n)|r_1,\dots,r_n \in R\}$ is invariant under the action of all inner automorphisms of R. Let us write $r = (r_1,\dots,r_n) \in R \times R \times \dots \times R = R^n$. Then for any inner automorphism φ of $M_m(K)$, we get that $\underline{r} = (\varphi(r_1),\dots,\varphi(r_n)) \in R^n$ and $\varphi(f(r)) = f(\underline{r}) \in f(R)$. As usual, we denote the matrix unit having 1 in (i, j)-entry and zero elsewhere by e_{ij}.

Let us recall some results from [37]. Let T be a ring with 1 and let $e_{ij} \in M_m(T)$ be the matrix unit having 1 in (i, j)-entry and zero elsewhere. For a sequence $u = (A_1,\dots,A_n)$ in $M_m(T)$, the value of u is defined to be the product $|u| = A_1A_2\cdots A_n$ and u is nonvanishing if $|u| \neq 0$. For a permutation σ of $\{1, 2,\cdots, n\}$, we write $u^\sigma = (A_{\sigma(1)},\dots,A_{\sigma(n)})$. We call u *simple* if it is of the form $u = (a_1e_{i_1j_1},\cdots, a_ne_{i_nj_n})$, where $a_i \in T$. A simple sequence u is called *even* if for some σ, $|u^\sigma| = be_{ii} \neq 0$, and *odd* if for some σ, $|u^\sigma| = be_{ij} \neq 0$, where $i \neq j$. In [37] it is proved that:

Note 10 Let T be a K-algebra with 1 and let $R = M_m(T)$, $m \geq 2$. Suppose that $g(x_1,\dots,x_n)$ is a multilinear polynomial over K such that $g(u) = 0$ for all odd simple sequences u. Then $g(x_1,\dots,x_n)$ is central-valued on R.

Note 11 Let T be a K-algebra with 1 and let $R = M_m(T)$, $m \geq 2$. Suppose that $g(x_1,\dots,x_n)$ is a multilinear polynomial over K. Let $u = (A_1,\dots,A_n)$ be a simple sequence from R.

1. If u is even, then $g(u)$ is a diagonal matrix.
2. If u is odd, then $g(u) = ae_{pq}$ for some $a \in T$ and $p \neq q$.

We also notice that:

Note 12 Since $f(x_1,\dots,x_n)$ is not central-valued on R, then by Note 10 there exists an odd simple sequence $r = (r_1,\dots,r_n)$ from R such that $f(r) = f(r_1,\dots,r_n) \neq 0$. By Note 11, $f(r) = \beta e_{pq}$, where $0 \neq \beta \in C$ and $p \neq q$. Since $f(x_1,\dots,x_n)$ is a multilinear polynomial and C is a field, we may assume that $\beta = 1$. Now, for distinct i, j, let $\sigma \in S_n$ be such that $\sigma(p) = i$ and $\sigma(q) = j$, and let ψ be the automorphism of R defined by $\psi(\sum_{s,t} \xi_{st} e_{st}) = \sum_{s,t} \xi_{st} e_{\sigma(s)\sigma(t)}$. Then $f(\psi(r)) = f(\psi(r_1), \dots, \psi(r_n)) = \psi(f(r)) = \beta e_{ij} = e_{ij}$.

Note 13 By Note 11 and [37, Lemma 9], since $f(x_1, \dots, x_n)$ is not central-valued on R, then there exists a sequence of matrices $r_1, \dots, r_n \in R$ such that $f(r_1, \dots, r_n) = \sum_i \alpha_i e_{ii} = D$ is a non-central diagonal matrix, for $\alpha_i \in C$. Suppose $r \neq s$ such that $\alpha_r \neq \alpha_s$. For all $l \neq m$, let $\psi \in \mathrm{Aut}_C(R)$ defined by $\psi(x) = \psi(\sum_{ij} \alpha_{ij} e_{ij}) = \sum_{ij} \alpha_{ij} e_{\sigma(i)\sigma(j)}$, where σ is a permutation in the symmetric group of n elements, such that $\sigma(r) = l$ and $\sigma(s) = m$. Thus $\psi(D)$ is an element of $f(R)$ and it is a diagonal matrix with (l, l) and (m, m) entries distinct.

Note 14 ([23, Lemma 1.5]) Let H be an infinite field and $n \geq 2$. If $A_1, \dots, A_k$ are not scalar matrices in $M_m(\mathcal{H})$ then there exists some invertible matrix $P \in M_m(H)$ such that each matrix $PA_1P^{-1}, \dots, PA_kP^{-1}$ has all non-zero entries.

5 Commuting Inner b-Generalized Skew Derivations

The present section is devoted to the proof of a reduced version of Theorem 1. More precisely, we prove the Theorem in the case α, β are automorphisms of R and F, G are inner b-generalized skew derivations of R respectively defined as follows:

$$F(x) = ax + b\alpha(x)c, \qquad G(x) = ux + p\beta(x)w$$

for all $x \in R$ and suitable fixed $a, b, c, u, p, w \in Q_r$.
We would like to remark that in this section F and G have not necessarily the same associated automorphism.

We start with the following case:

Lemma 4 *Let $R = M_m(C)$, $m \geq 2$ and let C be infinite. Suppose that F, G are inner b-generalized skew derivations of R respectively defined as follows:*

$$F(x) = ax + bqxq^{-1}c, \qquad G(x) = ux + pvxv^{-1}w$$

for all $x \in R$ and suitable fixed $a, b, c, u, p, w, q, v \in Q_r$, with invertible elements q, v of Q_r. Let $f(x_1, \dots, x_n)$ be a non-central multilinear polynomial over C with n non-commuting variables. If

$$F(f(r_1, \dots, r_n))f(r_1, \dots, r_n) - f(r_1, \dots, r_n)G(f(r_1, \dots, r_n)) = 0 \tag{11}$$

for all $r_1, \dots, r_n \in R$, then the following statements hold simultaneously:

1. *either $bq \in Z(R)$ or $q^{-1}c \in Z(R)$.*
2. *either $pv \in Z(R)$ or $v^{-1}w \in Z(R)$.*

Proof We assume that $bq \notin Z(R)$ and $q^{-1}c \notin Z(R)$, that is both $q^{-1}c$ and bq are not scalar matrices, and prove that a contradiction follows. By Note 14, there exists some invertible matrix $P \in M_m(C)$ such that each matrix $PbqP^{-1}$, $P(q^{-1}c)P^{-1}$ has all non-zero entries. Denote by $\varphi(x) = PxP^{-1}$ the inner automorphism induced by P. Say $\varphi(bq) = \sum_{hl} q_{hl}e_{hl}$ and $\varphi(q^{-1}c) = \sum_{hl} c_{hl}e_{hl}$ for $0 \neq q_{hl}, 0 \neq c_{hl} \in C$. Without loss of generality, we may replace bq and $q^{-1}c$ with $\varphi(bq)$ and $\varphi(q^{-1}c)$, respectively. Hence, for $f(r_1, \ldots, r_n) = \lambda e_{ij} \neq 0$ in (11), we get that the (j, j)-entry in (11) is

$$q_{ji}c_{ji} = 0,$$

which is a contradiction.

Assume now that $pv \notin Z(R)$ and $v^{-1}w \notin Z(R)$, that is both $v^{-1}w$ and pv are not scalar matrices, and prove that a contradiction follows. As above, there exists $\chi(x) = QxQ^{-1}$ the inner automorphism induced by $Q \in R$, such that $\chi(pv) = \sum_{hl} p_{hl}e_{hl}$ and $\chi(v^{-1}w) = \sum_{hl} w_{hl}e_{hl}$ for $0 \neq p_{hl}, 0 \neq w_{hl} \in C$. Moreover we replace pv and $v^{-1}w$ with $\chi(pv)$ and $\chi(v^{-1}w)$, respectively. Hence, again for $f(r_1, \ldots, r_n) = \lambda e_{ij} \neq 0$ in (20), we observe that the (i, i)-entry in (11) is

$$p_{ji}w_{ji} = 0,$$

which is also a contradiction.

Lemma 5 *Let $R = M_m(C)$, $m \geq 2$ and let $char(C) \neq 2$. Suppose that F, G are inner b-generalized skew derivations of R respectively defined as follows:*

$$F(x) = ax + bqxq^{-1}c, \qquad G(x) = ux + pvxv^{-1}w$$

for all $x \in R$ and suitable fixed $a, b, c, u, p, w, q, v \in Q_r$, with invertible elements q, v of Q_r. Let $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over C with n non-commuting variables. If

$$F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)G(f(r_1, \ldots, r_n)) = 0$$

for all $r_1, \ldots, r_n \in R$, then one of the following assertions holds:

1. *$bq \in Z(R)$ and $pv \in Z(R)$;*
2. *$bq \in Z(R)$ and $v^{-1}w \in Z(R)$;*
3. *$q^{-1}c \in Z(R)$ and $pv \in Z(R)$;*
4. *$q^{-1}c \in Z(R)$ and $v^{-1}w \in Z(R)$.*

Proof If one assumes that C is infinite, the conclusion follows from Lemma 4.

Now let E be an infinite field which is an extension of the field C and let $\overline{R} = M_t(E) \cong R \otimes_C E$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned}&\Psi(x_1,\ldots,x_n)=\\&\Big(af(x_1,\ldots,x_n)+bqf(x_1,\ldots,x_n)q^{-1}c\Big)f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)\Big(uf(x_1,\ldots,x_n)+pvf(x_1,\ldots,x_n)v^{-1}w\Big),\end{aligned}\tag{12}$$

which is a generalized polynomial identity for R. Moreover, it is multi-homogeneous of multi-degree $(2,\ldots,2)$ in the indeterminates $x_1,\ldots,x_n$. Hence the complete linearization of $\Psi(x_1,\ldots,x_n)$ is a multilinear generalized polynomial $\Theta(x_1,\ldots,x_n,y_1,\ldots,y_n)$. Moreover,

$$\Theta(x_1,\ldots,x_n,x_1,\ldots,x_n)=2^n\Psi(x_1,\ldots,x_n).$$

Clearly, the multilinear polynomial $\Theta(x_1,\ldots,x_n,y_1,\ldots,y_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $\mathrm{char}(C)\neq 2$, we obtain $\Psi(r_1,\ldots,r_n)=0$ for all $r_1,\ldots,r_n\in\overline{R}$, and the conclusion follows from Lemma 4.

Lemma 6 *Assume that*

$$\begin{aligned}&\Psi(x_1,\ldots,x_n)=\\&\Big(af(x_1,\ldots,x_n)+bqf(x_1,\ldots,x_n)q^{-1}c\Big)f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)\Big(uf(x_1,\ldots,x_n)+pvf(x_1,\ldots,x_n)v^{-1}w\Big)\end{aligned}\tag{13}$$

is a generalized polynomial identity for R. If R does not satisfy any non-trivial generalized polynomial identity, then one of the following holds:

1. *$bq\in C$ and $p=0$;*
2. *$bq\in C$ and $v^{-1}w\in C$;*
3. *$q^{-1}c\in C$ and $p=0$;*
4. *$q^{-1}c\in C$ and $v^{-1}w\in C$;*
5. *$a=u\in C$, $q^{-1}c\in C$, $pv\in C$, $bc=0$ and $pw=0$.*

Proof We firstly assume that $a\notin C$.
If $\{a,bq,1\}$ is linearly C-independent and since $\Psi(x_1,\ldots,x_n)$ is a trivial generalized polynomial identity for R, then the component $af(x_1,\ldots,x_n)^2$ is also a trivial generalized identity for R, implying the contradiction $a=0$. Hence we assume there exist $\alpha,\gamma\in C$, such that $bq=\alpha a+\gamma$. In this case (13) reduces to

$$\begin{aligned}&af(x_1,\ldots,x_n)^2+(\alpha a+\gamma)f(x_1,\ldots,x_n)q^{-1}cf(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)uf(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)pvf(x_1,\ldots,x_n)v^{-1}w.\end{aligned}\tag{14}$$

Since $\{1, a\}$ is linearly C-independent and (14) is a trivial generalized polynomial identity for R, then the components

$$af(x_1, \ldots, x_n)(1 + \alpha q^{-1}c) \tag{15}$$

and

$$\begin{aligned}&\gamma f(x_1, \ldots, x_n)q^{-1}cf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)uf(x_1, \ldots, x_n)\\&-f(x_1, \ldots, x_n)pvf(x_1, \ldots, x_n)v^{-1}w\end{aligned} \tag{16}$$

are also trivial generalized polynomial identities for R. By (15), we get $q^{-1}c \in C$. Thus, in the case $v^{-1}w \in C$ we are done. Here we assume that $v^{-1}w \notin C$, that is $\{1, v^{-1}w\}$ is linearly C-independent. Therefore, by (16) it follows that R satisfies $f(x_1, \ldots, x_n)pvf(x_1, \ldots, x_n)v^{-1}w$, which implies $pv = 0$, that is $p = 0$ (since v is invertible).

Assume now both $a \in C$ and $bq \in C$. Hence (13) reduces to

$$\begin{aligned}&f(x_1, \ldots, x_n)(a + bc)f(x_1, \ldots, x_n)\\&-f(x_1, \ldots, x_n)uf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)pvf(x_1, \ldots, x_n)v^{-1}w.\end{aligned} \tag{17}$$

Also in this case, if $v^{-1}w \in C$ we are done.

Assume that $\{1, v^{-1}w\}$ is linearly C-independent. Starting from (17) one has that the component $f(x_1, \ldots, x_n)pvf(x_1, \ldots, x_n)v^{-1}w$ must be a trivial generalized polynomial identity for R. This gives that $pv = 0$, that is $p = 0$.
Finally, we consider the case $a \in C$ and $bq \notin C$. Thus, by (13) we have that

$$\begin{aligned}&bqf(x_1, \ldots, x_n)q^{-1}cf(x_1, \ldots, x_n)\\&+f(x_1, \ldots, x_n)(a - u)f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)pvf(x_1, \ldots, x_n)v^{-1}w\end{aligned} \tag{18}$$

is a trivial generalized polynomial identity for R. Since $bq \notin C$ and by (18), it follows that $bqf(x_1, \ldots, x_n)q^{-1}cf(x_1, \ldots, x_n)$ is also a trivial generalized polynomial identity for R, implying $q^{-1}c \in C$ and $bc = 0$. As above, if $v^{-1}w \in C$ we are done. On the other hand, if $v^{-1}w \notin C$ and again by (18), one has that $f(x_1, \ldots, x_n)pvf(x_1, \ldots, x_n)v^{-1}w$ is a trivial generalized polynomial identity for R. This means that $pv \in C$ and $pw = 0$. In light of what has just been said and by (18), R satisfies

$$f(x_1, \ldots, x_n)(a - u)f(x_1, \ldots, x_n) \tag{19}$$

that is $a = u$.

Remark 1 We would like to remark that any conclusion of the previous Lemma implies that F and G are generalized derivations of R. Hence, in view of Theorem 2, the statement of Lemma 6 can be written as follows: there exists $a' \in Q_r$ such that $F(x) = xa'$ and $G(x) = a'x$ for all $x \in R$.

Proposition 2 *Let R be a prime ring of characteristic different from 2, Q_r be its right Martindale quotient ring and C be its extended centroid. Suppose that F, G are inner b-generalized skew derivations of R respectively defined as follows:*

$$F(x) = ax + bqxq^{-1}c, \qquad G(x) = ux + pvxv^{-1}w$$

for all $x \in R$ and suitable fixed $a, b, c, u, p, w, q, v \in Q_r$, with invertible elements q, v of Q_r. Let $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over C with n non-commuting variables. If

$$F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)G(f(r_1, \ldots, r_n)) = 0$$

for all $r_1, \ldots, r_n \in R$, then one of the following statements holds:

1. *$bq \in Z(R)$ and $pv \in C$;*
2. *$bq \in Z(R)$ and $v^{-1}w \in C$;*
3. *$q^{-1}c \in Z(R)$ and $pv \in C$;*
4. *$q^{-1}c \in Z(R)$ and $v^{-1}w \in C$.*

In other words, F and G are generalized derivations of R and one of the following statements holds:

1. *there exists $a' \in Q_r$ such that, $F(x) = xa'$ and $G(x) = a'x$ for all $x \in R$;*
2. *$f(x_1, \ldots, x_n)^2$ is central-valued on R and there exist $a', b' \in Q_r$ such that $F(x) = a'x + xb'$, $G(x) = b'x + xa'$, for all $x \in R$.*

Proof If R does not satisfy any non-trivial generalized polynomial identity, then the conclusion follows from Lemma 6. Therefore we may assume that

$$\begin{aligned} &\Psi(x_1, \ldots, x_n) = \\ &\Big(af(x_1, \ldots, x_n) + bqf(x_1, \ldots, x_n)q^{-1}c\Big)f(x_1, \ldots, x_n) \\ &-f(x_1, \ldots, x_n)\Big(uf(x_1, \ldots, x_n) + pvf(x_1, \ldots, x_n)v^{-1}w\Big) \end{aligned} \tag{20}$$

is a non-trivial generalized polynomial identity for R.

By Chuang [18] it follows that $\Psi(x_1, \ldots, x_n)$ is a non-trivial generalized polynomial identity for Q_r. By the well-known Martindale's theorem of [44], Q_r is a primitive ring having nonzero socle with the field C as its associated division ring. By Jacobson [31, Page 75] Q_r is isomorphic to a dense subring of the ring of linear transformations of a vector space V over C, containing nonzero linear transformations of finite rank. Assume first that $\dim_C V = k \geq 2$ is a finite positive integer, then $Q \cong M_k(C)$ and the conclusion follows from Lemma 5.

Let us now consider the case of $\dim_C V = \infty$. As in [48, Lemma 2], the set $f(R) = \{f(r_1, \ldots, r_n)|r_i \in R\}$ is dense on R. By the fact that $\Psi(r_1, \ldots, r_n) = 0$ is a generalized polynomial identity of R, we know that R satisfies

$$\left(ax + bqxq^{-1}c\right)x - x\left(ux + pvxv^{-1}w\right). \tag{21}$$

Recall that if an element $r \in R$ centralizes the non-zero ideal $H = \mathrm{soc}(RC)$, then $r \in C$.

Hence we may assume there exist $r_1, r_2, r_3, r_4 \in H = \mathrm{soc}(RC)$ such that:

1. either $[bq, r_1] \neq 0$ or $[pv, r_1] \neq 0$;
2. either $[bq, r_2] \neq 0$ or $[v^{-1}w, r_2] \neq 0$
3. either $[q^{-1}c, r_3] \neq 0$ or $[pv, r_3] \neq 0$
4. either $[q^{-1}c, r_4] \neq 0$ or $[v^{-1}w, r_4] \neq 0$

and prove that a number of contradictions follows.

By Litoff's Theorem [31, Page 90] there exists $e^2 = e \in H$ such that

- $r_1, r_2, r_3, r_4 \in eRe$;
- $ar_1, r_1a, ar_2, r_2a, ar_3, r_3a, ar_4, r_4a \in eRe$;
- $br_1, r_1b, br_2, r_2b, br_3, r_3b, br_4, r_4b \in eRe$;
- $cr_1, r_1c, cr_2, r_2c, cr_3, r_3c, cr_4, r_4c \in eRe$;
- $qr_1, r_1q, qr_2, r_2q, qr_3, r_3q, qr_4, r_4q \in eRe$;
- $ur_1, r_1u, ur_2, r_2u, ur_3, r_3u, ur_4, r_4u \in eRe$;
- $pr_1, r_1p, pr_2, r_2p, pr_3, r_3p, pr_4, r_4p \in eRe$;
- $vr_1, r_1v, vr_2, r_2v, vr_3, r_3v, vr_4, r_4v \in eRe$;
- $wr_1, r_1w, wr_2, r_2w, wr_3, r_3w, wr_4, r_4w \in eRe$;
- $pvr_1, r_1pv, pvr_2, r_2pv, pvr_3, r_3pv, pvr_4, r_4pv \in eRe$;
- $bqr_1, r_1bq, bqr_2, r_2bq, bqr_3, r_3bq, bqr_4, r_4bq \in eRe$;
- $q^{-1}cr_1, r_1q^{-1}c, q^{-1}cr_2, r_2q^{-1}c, q^{-1}cr_3, r_3q^{-1}c, q^{-1}cr_4, r_4q^{-1}c \in eRe$;
- $v^{-1}wr_1, r_1v^{-1}w, v^{-1}wr_2, r_2v^{-1}w, v^{-1}wr_3, r_3v^{-1}w, v^{-1}wr_4, r_4v^{-1}w \in eRe$,

where $eRe \cong M_m(C)$, the matrix ring over the extended centroid C. Note that eRe satisfies (21). By the above Lemma 5, we have that one of the following assertions holds:

1. $ebqe \in C$ and $epve \in C$, which contradicts with the choice of $r_1 \in H$;
2. $ebqe \in C$ and $ev^{-1}we \in C$, which contradicts with the choice of $r_2 \in H$;
3. $eq^{-1}ce \in C$ and $epve \in C$, which contradicts with the choice of $r_3 \in H$;
4. $eq^{-1}ce \in C$ and $ev^{-1}we \in C$, which contradicts with the choice of $r_4 \in H$.

As an easy consequence of Proposition 2 we also obtain a reduced version of Theorem 1 for the case both F and G are inner b-generalized derivations of R:

Proposition 3 *Let R be a prime ring of characteristic different from 2, Q_r be its right Martindale quotient ring and C be its extended centroid. Suppose that F, G are inner b-generalized derivations of R respectively defined as follows:*

$$F(x) = ax + bxc, \qquad G(x) = px + qxv$$

for all $x \in R$ and suitable fixed $a, b, c, p, q, v \in Q_r$. Let $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over C with n non-commuting variables. If

$$F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)G(f(r_1, \ldots, r_n)) = 0$$

for all $r_1, \ldots, r_n \in R$, then one of the following holds:

1. *there exists $a' \in Q_r$ such that, $F(x) = xa'$ and $G(x) = a'x$ for all $x \in R$;*
2. *$f(x_1, \ldots, x_n)^2$ is central-valued on R and there exist $a', b' \in Q_r$ such that $F(x) = a'x + xb'$, $G(x) = b'x + xa'$, for all $x \in R$.*

We are now ready to prove the more general result of this section.
We permit the following facts:

Note 15 Let R be a non-commutative prime ring, $a, b \in R$ such that $axb \in Z(R)$, for all $x \in R$. Then either $a = 0$ or $b = 0$.

Proof We assume that $a \neq 0$ and $b \neq 0$. For any $x \in R$ and by our assumption, both $a(xb) \in Z(R)$ and $a(xb)b \in Z(R)$. Thus we have that either $b \in Z(R)$ or $axb = 0$ for all $x \in R$. In the first case it follows that $aR \subseteq Z(R)$, which contradicts with the non-commutativity of R. In the latter case, by the primeness of R, we have the required conclusion.

Note 16 Let R be a non-commutative prime ring, $a, b \in R$, $f(x_1, \ldots, x_n)$ a polynomial over C, which is not central valued on R. If $af(r_1, \ldots, r_n)b \in Z(R)$, for all $r_1, \ldots, r_n \in R$, then either $a = 0$ or $b = 0$.

Proof Let S be the additive subgroup of R generated by $\{f(y_1, \ldots, y_n) : y_i \in R\}$. Since $f(y_1, \ldots, y_n)$ is not central and $char(R) \neq 2$, it is well known that S contains a non-central Lie ideal L of R (see [17]). Moreover, since L is not central then there exists a non-central ideal I of R such that $[I, R] \subseteq L$. Therefore $a[i, r]b \in Z(R)$, for any $i \in I$, $r \in R$. Since I and Q_r satisfy the same generalized identities it follows that $a[x, y]b \in C$ for any $x, y \in Q_r$. In this situation we may apply the main result in [3] and one of the following holds: either $a = 0$ or $b = 0$ or Q_r is a central simple algebra of dimension at most 4 over C. Moreover, since Q_r is not commutative, then Q_r contains some non-trivial idempotent elements $e = e^2$. In this last case, by the main hypothesis, one has $a[e, x(1-e)]b \in C$, that is $aex(1-e)b \in C$, for all $x \in Q_r$. By Note 15, either $ae = 0$ or $(1 - e)b = 0$.
If $ae = 0$ and by $a[y, ex]b \in C$, we get $ayexb \in C$, for any $x, y \in Q_r$. Thus, using Note 15 and since $e \neq 0$, it follows that either $a = 0$ or $b = 0$, as required.

On the other hand, if $(1-e)b = 0$ and by $a[x, y(1-e)]b \in C$, we have that $ay(1-e)xb \in C$, for any $x, y \in Q_r$. Once again by Note 15 and since $e \neq 1$, we get their required conclusion.

Theorem 3 *Let R be a prime ring of characteristic different from 2, Q_r be its right Martindale quotient ring and C be its extended centroid. Suppose that F, G are inner b-generalized skew derivations of R respectively defined as follows:*

$$F(x) = ax + b\alpha(x)c, \qquad G(x) = ux + p\beta(x)w$$

for all $x \in R$ and suitable fixed $a, b, c, u, p, w \in Q_r$, and $\alpha, \beta \in \mathrm{Aut}(Q_r)$. Let $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over C with n non-commuting variables. If

$$F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)G(f(r_1, \ldots, r_n)) = 0 \qquad (22)$$

for all $r_1, \ldots, r_n \in R$, then one of the following statements holds:

1. *$\alpha = \beta = id$, where id denotes the identical mapping on Q_r;*
2. *$\alpha = id$ and there exists an invertible element $v \in Q_r$ such that $\beta(x) = vxv^{-1}$, for all $x \in R$;*
3. *$\beta = id$ and there exists an invertible element $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$, for all $x \in R$;*
4. *$\beta = id$ and $b = 0$;*
5. *$\beta = id$ and $c = 0$;*
6. *$\alpha = id$ and $p = 0$;*
7. *$\alpha = id$ and $w = 0$;*
8. *there exists an invertible element $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$, for all $x \in R$, and either $p = 0$ or $w = 0$;*
9. *there exists an invertible element $v \in Q_r$ such that $\beta(x) = vxv^{-1}$, for all $x \in R$, and either $b = 0$ or $c = 0$;*
10. *$b = p = 0$;*
11. *$b = w = 0$;*
12. *$c = p = 0$;*
13. *$c = w = 0$;*
14. *there exist invertible elements $q, v \in Q_r$ such that $\alpha(x) = qxq^{-1}$ and $\beta(x) = vxv^{-1}$, for all $x \in R$.*

In other words one of the following occurs:

- *F and G are ordinary generalized derivations of R.*
- *F and G are inner b-generalized derivations;*
- *F and G are inner b-generalized skew derivations of R, associated with inner automorphisms;*

In any case, respectively in light of Propositions 1, 3 and 2, we have that one of the following statements holds:

1. *there exists $a' \in Q_r$ such that, $F(x) = xa'$ and $G(x) = a'x$ for all $x \in R$;*
2. *$f(x_1, \ldots, x_n)^2$ is central-valued on R and there exist $a', b' \in Q_r$ such that $F(x) = a'x + xb'$, $G(x) = b'x + xa'$, for all $x \in R$.*

Proof On the contrary, we assume that the following hold simultaneously:

- either $\alpha \neq id$ or $\beta \neq id$;
- either $\alpha \neq id$ or β is not an inner automorphism on Q_r;
- either $\beta \neq id$ or α is not an inner automorphism on Q_r;
- either $\alpha \neq id$ or $b \neq 0$;
- either $\alpha \neq id$ or $c \neq 0$;
- either $\beta \neq id$ or $p \neq 0$;
- either $\beta \neq id$ or $w \neq 0$;
- either α is not inner, or both $p \neq 0$ and $w \neq 0$;
- either β is not inner, or both $b \neq 0$ and $c \neq 0$;
- either $b \neq 0$ or $p \neq 0$;
- either $b \neq 0$ or $w \neq 0$;
- either $c \neq 0$ or $p \neq 0$;
- either $c \neq 0$ or $w \neq 0$;
- at least one among α and β is not an inner automorphism of R.

By our assumption R satisfies the following generalized polynomial

$$\begin{aligned}&\Big(af(x_1,\ldots,x_n) + b\alpha\big(f(x_1,\ldots,x_n)\big)c\Big)f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)\Big(uf(x_1,\ldots,x_n) + p\beta\big(f(x_1,\ldots,x_n)\big)w\Big).\end{aligned} \tag{23}$$

In view of the Note 6, Q_r satisfies (23).

In case $\alpha = id$, then β is not inner. Thus, by (23), Q_r satisfies the generalized polynomial

$$\begin{aligned}&\Big(af(x_1,\ldots,x_n) + b(f(x_1,\ldots,x_n)c\Big)f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)\Big(uf(x_1,\ldots,x_n) + pf^{\beta}(y_1,\ldots,y_n)w\Big).\end{aligned} \tag{24}$$

In particular, $pf^{\beta}(y_1,\ldots,y_n)w$ is a generalized polynomial identity for Q_r. It is easy to see that $pXw = 0$, for any $X \in S$, the additive subgroup of Q_r generated by $\{f^{\beta}(y_1,\ldots,y_n) : y_i \in Q_r\}$. Since $f^{\beta}(y_1,\ldots,y_n)$ is not central and $char(Q_r) \neq 2$, it is well known that S must contain a non-central Lie ideal L. This implies $pLw = (0)$ and, by the primeness of Q_r we get the contradiction that either $p = 0$ or $w = 0$.

Similarly, if we assume that $\beta = id$, then we obtain the contradiction that either $b = 0$ or $c = 0$.

Thus we may suppose both $\alpha \neq id$ and $\beta \neq id$. In what follows we denote $f^{\alpha}(x_1,\ldots,x_n) = \alpha\Big(f(x_1,\ldots,x_n)\Big)$.

If α and β are mutually outer, then by (23), Q_r satisfies

$$\Big(af(x_1,\ldots,x_n) + bf^{\alpha}(y_1,\ldots,y_n)c\Big)f(x_1,\ldots,x_n) \\ -f(x_1,\ldots,x_n)\Big(uf(x_1,\ldots,x_n) + pf^{\beta}(z_1,\ldots,z_n)w\Big). \tag{25}$$

In particular, Q_r satisfies both

$$bf^{\alpha}(y_1,\ldots,y_n)cf(x_1,\ldots,x_n)$$

and

$$f(x_1,\ldots,x_n)pf^{\beta}(z_1,\ldots,z_n)w.$$

Applying twice Corollary 2 to both last relations yields that either $b = 0$ or $c = 0$ and simultaneously either $p = 0$ or $w = 0$, which is a contradiction.

Assume finally that α and β are not mutually outer, then exists an invertible element $q \in Q_r$ such that $\alpha\beta^{-1}(x) = qxq^{-1}$, for any $x \in R$. Therefore $\alpha(x) = q\beta(x)q^{-1}$ and by (23) it follows that Q_r satisfies

$$\Big(af(x_1,\ldots,x_n) + bq\beta\big(f(x_1,\ldots,x_n)\big)q^{-1}c\Big)f(x_1,\ldots,x_n) \\ -f(x_1,\ldots,x_n)\Big(uf(x_1,\ldots,x_n) + p\beta\big(f(x_1,\ldots,x_n)\big)w\Big). \tag{26}$$

If β is an inner automorphism of Q_r, then the required conclusion follows from Proposition 2. On the other hand, if β is outer, then, by (26) we have that Q_r satisfies

$$\Big(af(x_1,\ldots,x_n) + bqf^{\beta}(y_1,\ldots,y_n)q^{-1}c\Big)f(x_1,\ldots,x_n) \\ -f(x_1,\ldots,x_n)\Big(uf(x_1,\ldots,x_n) + pf^{\beta}(y_1,\ldots,y_n)w\Big) \tag{27}$$

and in particular

$$bqf^{\beta}(y_1,\ldots,y_n)q^{-1}cf(x_1,\ldots,x_n) - f(x_1,\ldots,x_n)pf^{\beta}(y_1,\ldots,y_n)w \tag{28}$$

is a generalized polynomial identity for Q_r. Since $f(x_1,\ldots,x_n)$ is not central valued and in light of Lemma 3, one has that $bqf^{\beta}(y_1,\ldots,y_n)q^{-1}c =$

$pf^{\beta}(y_1, \ldots, y_n)w \in C$ for any $y_1, \ldots, y_n \in Q_r$. Hence Note 16 implies that the following hold simultaneously:

- either $b = 0$ or $c = 0$;
- either $p = 0$ or $w = 0$

and in any case we get a contradiction.

6 Commuting *b*-Generalized Derivations on Multilinear Polynomials

In this section we provide a proof of Theorem 1 in the case both F and G are arbitrary b-generalized derivations (not necessarily inner) and prove the following:

Theorem 4 *Let R be a prime ring of characteristic different from 2, Q_r be its right Martindale quotient ring and C be its extended centroid, F and G non-zero b-generalized derivations of R. Suppose that $f(x_1, \ldots, x_n)$ is a non-central multilinear polynomial over C such that $F(f(X))f(X) - f(X)G(f(X)) = 0$, for all $X = (x_1, \ldots, x_n) \in R^n$, then one of the following statements holds:*

1. *there exists $u \in Q_r$ such that, $F(x) = xu$ and $G(x) = ux$ for all $x \in R$;*
2. *$f(x_1, \ldots, x_n)^2$ is central-valued on R and there exist $a, b \in Q_r$ such that $F(x) = ax + xb$, $G(x) = bx + xa$, for all $x \in R$.*

Hence F and G are generalized derivations of R.

Proof As mentioned in the Introduction, we can write $F(x) = ax + bd(x)$, $G(x) = px + q\delta(x)$ for all $x \in R$, where $a, b, p, q \in Q_r$ and d, δ are derivations of R. In light of Proposition 3, we may assume that:

- At least one among d and δ is not an inner derivation of R;
- At least one among b and q is not zero;
- If d is an inner derivation of R, then $\delta \neq 0$ and $q \neq 0$;
- If δ is an inner derivation of R then $d \neq 0$ and $b \neq 0$.

We will prove that, under these assumptions, a number of contradiction follows.

Assume first that d and δ are both non-zero derivations and linearly C-independent modulo Q_r-inner derivations. Since Q_r satisfies

$$\Big(af(x_1, \ldots, x_n) + bf^d(x_1, \ldots, x_n) + b\sum_{i=1}^n f(x_1, \ldots, d(x_i), \ldots, x_n)\Big)f(x_1, \ldots, x_n)$$
$$-f(x_1, \ldots, x_n)\Big(pf(x_1, \ldots, x_n) + qf^{\delta}(x_1, \ldots, x_n) + q\sum_{i=1}^n f(x_1, \ldots, \delta(x_i), \ldots, x_n)\Big) \quad (29)$$

and by Kharchenko [32], we arrive at that Q_r satisfies

$$\Big(af(x_1,\ldots,x_n)+bf^d(x_1,\ldots,x_n)+b\sum_{i=1}^n f(x_1,\ldots,y_i,\ldots,x_n)\Big)f(x_1,\ldots,x_n)$$
$$-f(x_1,\ldots,x_n)\Big(pf(x_1,\ldots,x_n)+qf^\delta(x_1,\ldots,x_n)+q\sum_{i=1}^n f(x_1,\ldots,z_i,\ldots,x_n)\Big). \tag{30}$$

In particular, Q_r satisfies the blended components

$$bf(y_1,x_2,\ldots,x_n)\cdot f(x_1,\ldots,x_n)$$

and

$$f(x_1,\ldots,x_n)\cdot q\cdot f(y_1,x_2,\ldots,x_n),$$

which imply the contradiction $b=0$ (by Corollary 2) and $q=0$ (by Corollary 1).

Assume now that d and δ are both non-zero derivations and C-dependent modulo Q_r-inner derivations. Without loss of generality, we assume that $\delta=\lambda d+ad_w$, that is $\delta(x)=\lambda d(x)+[w,x]$, for suitable $0\neq\lambda\in C$ and $w\in Q_r$. Moreover, in light of the previous remarks, d is not an inner derivation of R. By the hypothesis we have that

$$\Big(af(x_1,\ldots,x_n)+bf^d(x_1,\ldots,x_n)+b\sum_{i=1}^n f(x_1,\ldots,d(x_i),\ldots,x_n)\Big)f(x_1,\ldots,x_n)$$
$$-f(x_1,\ldots,x_n)\Big(pf(x_1,\ldots,x_n)+\lambda qf^d(x_1,\ldots,x_n)+$$
$$+\lambda q\sum_{i=1}^n f(x_1,\ldots,d(x_i),\ldots,x_n)+q[w,f(x_1,\ldots,x_n)]\Big) \tag{31}$$

is a differential polynomial identity for Q_r, and again by Kharchenko [32] it follows that Q_r satisfies

$$\Big(af(x_1,\ldots,x_n)+bf^d(x_1,\ldots,x_n)+b\sum_{i=1}^n f(x_1,\ldots,y_i,\ldots,x_n)\Big)f(x_1,\ldots,x_n)$$
$$-f(x_1,\ldots,x_n)\Big(pf(x_1,\ldots,x_n)+\lambda qf^d(x_1,\ldots,x_n)+$$
$$+\lambda q\sum_{i=1}^n f(x_1,\ldots,y_i,\ldots,x_n)+q[w,f(x_1,\ldots,x_n)]\Big). \tag{32}$$

In particular, Q_r satisfies the blended component

$$b\sum_i f(x_1,\ldots,y_i,\ldots,x_n)f(x_1,\ldots,x_n)-\lambda f(x_1,\ldots,x_n)q\sum_i f(x_1,\ldots,y_i,\ldots,x_n). \tag{33}$$

Let us choose $y_2 = y_3 = \ldots = y_n = 0$ and $y_1 = x_1$ in (33). This yields that Q_r satisfies

$$bf(x_1, \ldots, x_n)^2 - \lambda f(x_1, \ldots, x_n)qf(x_1, \ldots, x_n). \tag{34}$$

Moreover, for $z \notin C$ and $y_i = [z, x_i]$ for any $i = 1, \ldots, n$ in (33), we also have that

$$b[z, f(x_1, \ldots, x_n)]f(x_1, \ldots, x_n) - \lambda f(x_1, \ldots, x_n)q[z, f(x_1, \ldots, x_n)] \tag{35}$$

is a generalized polynomial identity for Q_r. Application of Proposition 1 to (34) implies that $b = \lambda q \in C$. Therefore, by (35) it follows that Q_r satisfies

$$b[z, f(x_1, \ldots, x_n)]_2.$$

Since $z \notin C$ and since neither $char(R) = 2$ nor $f(x_1, \ldots, x_n)$ is central-valued on R, by Liu [38] we get $b = 0$, and so also $q = 0$, which is a contradiction.

We finally consider the case either $d = 0$ or $\delta = 0$. Without loss of generality, we may assume $\delta = 0$ (the case $d = 0$ is similar and we omit it for brevity). By our assumption it follows that Q_r satisfies

$$\Big(af(x_1, \ldots, x_n) + bf^d(x_1, \ldots, x_n) + b\textstyle\sum_{i=1}^n f(x_1, \ldots, d(x_i), \ldots, x_n)\Big)f(x_1, \ldots, x_n)$$
$$-f(x_1, \ldots, x_n)pf(x_1, \ldots, x_n). \tag{36}$$

Moreover, as above remarked, in this case d is not an inner derivation of R. In view of Kharchenko's theorem in [32], Q_r satisfies

$$\Big(af(x_1, \ldots, x_n) + bf^d(x_1, \ldots, x_n) + b\textstyle\sum_{i=1}^n f(x_1, \ldots, y_i, \ldots, x_n)\Big)f(x_1, \ldots, x_n)$$
$$-f(x_1, \ldots, x_n)pf(x_1, \ldots, x_n). \tag{37}$$

Therefore

$$bf(y_1, x_2, \ldots, x_n)f(x_1, \ldots, x_n) \tag{38}$$

is a generalized polynomial identity for Q_r, implying again the contradiction $b = 0$.

7 The Main Result

The last part of our paper is dedicated to the proof of Theorem 1 in its most general form. For sake of clearness and completeness, we recall our hypothesis.

We assume that R is a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid, $\alpha \in \text{Aut}(R)$, d and δ skew derivations of R associated with α, such that both d and δ are commuting with α. We suppose that F, G are b-generalized skew derivations of R, respectively associated with terms (b, α, d) and (p, β, δ). We may write $F(x) = ax + bd(x)$ and $G(x) = ux + p\delta(x)$, for all $x \in R$ and suitable $a, u \in Q_r$. We assume that $f(x_1, \ldots, x_n)$ is a non-central multilinear polynomial over C with n non-commuting variables, such that

$$F(f(r_1, \ldots, r_n))f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)G(f(r_1, \ldots, r_n)) = 0 \tag{39}$$

for all $r_1, \ldots, r_n \in R$, that is R satisfies

$$\begin{aligned} &\Big(af(x_1, \ldots, x_n) + bd(f(x_1, \ldots, x_n))\Big)f(x_1, \ldots, x_n) \\ &-f(x_1, \ldots, x_n)\Big(uf(x_1, \ldots, x_n) + p\delta(f(x_1, \ldots, x_n))\Big). \end{aligned} \tag{40}$$

Under these assumptions, we'll prove that one of the following statements holds:

1. $d = \delta = 0$;
2. $\alpha = id$;
3. there exist $b', c' \in Q_r$ such that $d(x) = b'x - \alpha(x)b'$ and $\delta(x) = c'x - \alpha(x)c'$, for all $x \in R$;
4. $b = p = 0$;
5. $b = 0$ and $\delta = 0$;
6. $p = 0$ and $d = 0$.

In other words, either F and G are generalized derivations of R, or F and G are b-generalized derivations of R, or F and G are inner b-generalized skew derivations of R. Therefore, respectively in light of Theorems 2, 4 and 3, we have that one of the following holds:

1. there exists $a' \in Q_r$ such that, $F(x) = xa'$ and $G(x) = a'x$ for all $x \in R$;
2. $f(x_1, \ldots, x_n)^2$ is central valued on R and there exist $a', b' \in Q_r$ such that $F(x) = a'x + xb'$, $G(x) = b'x + xa'$, for all $x \in R$.

Proof of Theorem 1 By (40) and Note 9,

$$\begin{aligned} &\Big(af(x_1, \ldots, x_n) + bf^d(x_1, \ldots, x_n) + \\ &b\textstyle\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdot x_{\sigma(2)} \cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)} \cdots x_{\sigma(n)}\Big)f(x_1, \ldots, x_n) \\ &-f(x_1, \ldots, x_n)\Big(uf(x_1, \ldots, x_n) + pf^\delta(x_1, \ldots, x_n) \\ &+p\textstyle\sum_{\sigma \in S_n} \beta(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdot x_{\sigma(2)} \cdots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)} \cdots x_{\sigma(n)}\Big) \end{aligned} \tag{41}$$

is a generalized identity for R.

On the contrary we assume that the following hold simultaneously:

- either $d \neq 0$ or $\delta \neq 0$;
- $\alpha \neq id$;
- at least one among d and δ is not an inner skew derivation of R;
- at least one among b and p is not zero;
- at least one among b or δ is not zero;
- at least one among p or d is not zero.

7.1 Let d and δ be $C-$Linearly Independent Modulo SD_{int}

In this case, in view of (41) we know that R satisfies the generalized polynomial

$$\begin{aligned}&\Big(af(x_1,\ldots,x_n)+bf^d(x_1,\ldots,x_n)+\\&b\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdot x_{\sigma(2)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)\Big(uf(x_1,\ldots,x_n)+pf^\delta(x_1,\ldots,x_n)\\&+p\textstyle\sum_{\sigma\in S_n}\beta(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdot x_{\sigma(2)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big).\end{aligned} \tag{42}$$

In particular, R satisfies any blended component

$$b\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{i=1}^{n}\alpha(x_{\sigma(1)}\cdot x_{\sigma(2)}\cdots x_{\sigma(i-1)})y_{\sigma(i)}x_{\sigma(i+1)}\cdots x_{\sigma(n)}\Big)f(x_1,\ldots,x_n). \tag{43}$$

In light of the Note 6, Q_r satisfies (43).

Suppose there exists an invertible element $q\in Q_r$ such that $\alpha(x)=qxq^{-1}$ for all $x\in Q_r$. Since $\alpha\neq id\in \mathrm{Aut}(R)$, we may assume $q\notin C$. Moreover, it is clear that $\alpha(\gamma_\sigma)=\gamma_\sigma$ for all coefficients involved in $f(x_1,\ldots,x_n)$. If we replace each $y_{\sigma(i)}$ with $qx_{\sigma(i)}$ in (43), then Q_r satisfies the generalized polynomial

$$b\Big(q\sum_{\sigma\in S_n}\gamma_\sigma x_{\sigma(1)}\cdot x_{\sigma(2)}\cdots x_{\sigma(i-1)}x_{\sigma(i)}x_{\sigma(i+1)}\cdots x_{\sigma(n)}\Big)f(x_1,\ldots,x_n).$$

That is $bqf(x_1,\ldots,x_n)^2=0$, which implies $bq=0$. Since q is invertible, we obtain that $b=0$.

Finally, assume that α is outer. By (43) it follows that Q_r satisfies the generalized polynomial

$$b\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{i=1}^{n}z_{\sigma(1)}\cdot z_{\sigma(2)}\cdots z_{\sigma(i-1)}y_{\sigma(i)}x_{\sigma(i+1)}\cdots x_{\sigma(n)}\Big)f(x_1,\ldots,x_n). \tag{44}$$

For any $i = 1, \ldots, n$, Q_r also satisfies the generalized polynomial

$$b\left(\sum_{\sigma \in S_{n-1}} \alpha(\gamma_\sigma) z_{\sigma(1)} \cdot z_{\sigma(2)} \cdots z_{\sigma(i-1)} \cdot z_{\sigma(i+1)} \cdots z_{\sigma(n)} \cdot y_i\right) f(x_1, \ldots, x_n). \quad (45)$$

Let us write

$$\sum_{\sigma \in S_{n-1}} \alpha(\gamma_\sigma) x_{\sigma(1)} \cdots x_{\sigma(j-i)} x_{\sigma(j+1)} \cdots x_{\sigma(n)} = t_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n),$$

where any t_j is a multilinear polynomial of degree $n-1$ and x_j never appears in any monomial of t_j. It follows from (45) that Q_r satisfies the generalized polynomial

$$bt_j(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) f(x_1, \ldots, x_n).$$

As a consequence of Lemma 3 and Corollary 2, either $b = 0$ or $t_j(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$ is a generalized polynomial identity for Q_r for all $j = 1, \ldots, n$. Moreover, we also denote $f^\alpha(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient γ_σ with $\alpha(\gamma_\sigma)$ and notice that $f^\alpha(r_1, \ldots, r_n) \neq 0$. Hence, in the case $t_j(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$ is a generalized polynomial identity for Q_r for all $j = 1, \ldots, n$, and since

$$f^\alpha(x_1, \ldots, x_n) = \sum_j x_j t_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n),$$

$f^\alpha(x_1, \ldots, x_n)$ is a generalized polynomial identity for Q_r, which is also a contradiction. Thus we conclude again that $b = 0$.
The previous argument shows that $b = 0$ in any case.
Moreover, by (42) it follows that Q_r satisfies

$$f(x_1, \ldots, x_n) p\left(\sum_{\sigma \in S_n} \beta(\gamma_\sigma) \sum_{i=1}^{n} \alpha(x_{\sigma(1)} \cdot x_{\sigma(2)} \cdots x_{\sigma(i-1)}) z_{\sigma(i)} x_{\sigma(i+1)} \cdots x_{\sigma(n)}\right).$$

By using the same above argument, one can show that $p = 0$, which is a contradiction. We omit the proof for brevity.

7.2 Let d and δ be C−Linearly Dependent Modulo SD_{int}

We firstly assume that there exist $0 \neq \lambda \in C$, $0 \neq \mu \in C$, $c \in Q_r$ and $\gamma \in \mathrm{Aut}(R)$ such that $\lambda d(x) + \mu\delta(x) = cx - \gamma(x)c$ for all $x \in R$. Denote $\eta = -\mu^{-1}\lambda$ and $q = \mu^{-1}c$. Thus $\delta(x) = \eta d(x) + qx - \gamma(x)q$ for all $x \in R$. Therefore by (40), Q_r

satisfies the generalized polynomial

$$
\begin{aligned}
& af(x_1,\dots,x_n)^2 + bd\big(f(x_1,\dots,x_n)\big)f(x_1,\dots,x_n) \\
& -f(x_1,\dots,x_n)(u+pq)f(x_1,\dots,x_n) - \eta f(x_1,\dots,x_n)pd\big(f(x_1,\dots,x_n)\big) \\
& +f(x_1,\dots,x_n)p\gamma\big(f(x_1,\dots,x_n)\big)q.
\end{aligned} \tag{46}
$$

That is, Q_r satisfies the generalized polynomial

$$
\begin{aligned}
& af(x_1,\dots,x_n)^2 + bf^d(x_1,\dots,x_n)f(x_1,\dots,x_n) \\
& +b\Big(\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdot x_{\sigma(2)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\dots,x_n) \\
& -f(x_1,\dots,x_n)(u+pq)f(x_1,\dots,x_n) - \eta f(x_1,\dots,x_n)pf^d(x_1,\dots,x_n) \\
& -\eta f(x_1,\dots,x_n)p\Big(\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdot x_{\sigma(2)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big) \\
& +f(x_1,\dots,x_n)p\gamma\big(f(x_1,\dots,x_n)\big)q.
\end{aligned} \tag{47}
$$

In case d is outer, by (47) Q_r satisfies the generalized polynomial

$$
\begin{aligned}
& af(x_1,\dots,x_n)^2 + bf^d(x_1,\dots,x_n)f(x_1,\dots,x_n) \\
& +b\Big(\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdot x_{\sigma(2)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\dots,x_n) \\
& -f(x_1,\dots,x_n)(u+pq)f(x_1,\dots,x_n) - \eta f(x_1,\dots,x_n)pf^d(x_1,\dots,x_n) \\
& -\eta f(x_1,\dots,x_n)p\Big(\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdot x_{\sigma(2)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big) \\
& +f(x_1,\dots,x_n)p\gamma\big(f(x_1,\dots,x_n)\big)q.
\end{aligned} \tag{48}
$$

In particular,

$$
\begin{aligned}
& b\Big(\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\dots,x_n) \\
& -\eta f(x_1,\dots,x_n)p\Big(\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)
\end{aligned} \tag{49}
$$

is satisfied by R as well as Q_r (see Note 6 again).

Suppose there exists an invertible element $w \in Q_r$ such that $\alpha(x) = wxw^{-1}$ for all $x \in Q_r$. Since $\alpha \neq 1 \in \mathrm{Aut}(R)$, we may assume $w \notin C$. As above, we remark that $\alpha(\gamma_\sigma) = \gamma_\sigma$ for all coefficients involved in $f(x_1,\dots,x_n)$. Therefore, if we replace each $y_{\sigma(i)}$ with $wx_{\sigma(i)}$ in (49), we obtain that Q_r satisfies the generalized polynomial

$$\Big(bwf(x_1,\dots,x_n) - f(x_1,\dots,x_n)(\eta pw)\Big)f(x_1,\dots,x_n).$$

Applying again Corollary 1 yields $bw = \eta pw \in C$. In particular $b = \eta p$. Let us now replace each $y_{\sigma(i)}$ with $w[z, x_{\sigma(i)}]$ in (49), for some element $z \notin C$. Thus we obtain that Q_r satisfies the generalized polynomial

$$bw\Big[z, f(x_1, \ldots, x_n)\Big]_2.$$

Since $f(x_1, \ldots, x_n)$ is not central-valued on Q_r and $z \notin C$, we get the contradiction $b = p = 0$.

Finally, assume that α is outer. By (49) we know that Q_r satisfies the generalized polynomial

$$\begin{aligned}&b\Big(\textstyle\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{i=1}^{n} z_{\sigma(1)} \cdot z_{\sigma(2)} \cdots z_{\sigma(i-1)} y_{\sigma(i)} x_{\sigma(i+1)} \cdots x_{\sigma(n)} \Big) f(x_1, \ldots, x_n)\\ &-\eta f(x_1, \ldots, x_n) p\Big(\textstyle\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{i=1}^{n} z_{\sigma(1)} \cdot z_{\sigma(2)} \cdots z_{\sigma(i-1)} y_{\sigma(i)} x_{\sigma(i+1)} \cdots x_{\sigma(n)} \Big) f(x_1, \ldots, x_n)\end{aligned} \tag{50}$$

and, for any $i = 1, \ldots, n$, Q_r also satisfies the generalized polynomial

$$\begin{aligned}&b\Big(\textstyle\sum_{\sigma \in S_{n-1}} \alpha(\gamma_\sigma) z_{\sigma(1)} \cdot z_{\sigma(2)} \cdots z_{\sigma(i-1)} \cdot z_{\sigma(i+1)} \cdots z_{\sigma(n)} \cdot y_i \Big) f(x_1, \ldots, x_n)\\ &-\eta f(x_1, \ldots, x_n) p\Big(\textstyle\sum_{\sigma \in S_{n-1}} \alpha(\gamma_\sigma) z_{\sigma(1)} \cdot z_{\sigma(2)} \cdots z_{\sigma(i-1)} \cdot z_{\sigma(i+1)} \cdots z_{\sigma(n)} \cdot y_i \Big).\end{aligned} \tag{51}$$

As above, let us write

$$\sum_{\sigma \in S_{n-1}} \alpha(\gamma_\sigma) x_{\sigma(1)} \cdots x_{\sigma(j-i)} x_{\sigma(j+1)} \cdots x_{\sigma(n)} = t_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n),$$

where any t_j is a multilinear polynomial of degree $n-1$ and x_j never appears in any monomial of t_j. In view of (51), we get

$$\begin{aligned}&b\Big(t_j(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) y\Big) f(x_1, \ldots, x_n)\\ &-\eta f(x_1, \ldots, x_n) p\Big(t_j(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) y\Big).\end{aligned} \tag{52}$$

From Lemma 3 it follows that

$$bt_j(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) y = \eta p t_j(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) y \in C. \tag{53}$$

Suppose that $t_j(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$ is central-valued on Q_r for all $j = 1, \ldots, n$. Since

$$f^{\alpha}(x_1, \ldots, x_n) = \sum_j x_j t_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n),$$

it follows that $f^{\alpha}(x_1,\ldots,x_n)$ is a central-valued on Q_r, a contradiction. Therefore (53) forces $b=0$ and $\eta p=0$, which is again a contradiction.

Let us next start from (46) and consider the case when $d(x)=vx-\alpha(x)v$ for all $x\in R$ and for some fixed $v\in Q_r$. Hence, $\delta(x)=(\eta v+q)x-\alpha(x)\eta v-\gamma(x)q$. Therefore, by Note 8, F and G are simultaneously inner b-generalized skew derivations of R and, by Theorem 3 a number of contradictions follows.

We analyze now the last case. Let us start again from relation (40) and assume again that d and δ are C−linearly dependent modulo $\mathrm{SD}_{\mathrm{int}}$. That is $\lambda d(x)+\mu\delta(x)=cx-\gamma(x)c$ for all $x\in R$. Moreover, in view of the previous argument, we have to assume now $\lambda=0$. Thus $\delta(x)=qx-\gamma(x)q$ for all $x\in R$ and $q=\mu^{-1}c$. Therefore by (40), Q_r satisfies the generalized polynomial

$$\begin{aligned}&af(x_1,\ldots,x_n)^2+bd\big(f(x_1,\ldots,x_n)\big)f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)(u+pq)f(x_1,\ldots,x_n)\\&+f(x_1,\ldots,x_n)p\gamma\big(f(x_1,\ldots,x_n)\big)q.\end{aligned}\tag{54}$$

We finally observe that (54) is equivalent to (46) in case $\eta=0$. Therefore the same above argument completes our proof.

8 Some Open Problems

In the light of the motivation and contents of this article, we will propose several topics for future research in this field. More precisely, some informations about the structure of a prime ring R and the description of all possible forms of a b-generalized skew derivation F of R can be obtained if one of the following conditions is satisfied:

1. $F(x)^n=0$ for all $x\in L$, where n is a fixed positive integer and L is a noncommutative Lie ideal of R.
2. $F(x)^n\in Z(R)$ for all $x\in L$, where n is a fixed positive integer and L is a noncommutative Lie ideal of R.
3. $F(x)^n\in Z(R)$ for all $x\in I$, where n is a fixed positive integer and I is a non-zero one sided ideal of R.
4. $aF(x)^n=0$ for all $x\in I$, where n is a fixed positive integer, I is an ideal of R and a is a non-zero element of R.

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