On the Asymptotics of Capelli Polynomials

Francesca Saviella Benanti and Angela Valenti

To Antonio Giambruno on the occasion of his 70th birthday

Abstract We present old and new results about Capelli polynomials, $\mathbb{Z}_2$-graded Capelli polynomials, Capelli polynomials with involution and their asymptotics.

Let $Cap_m = \sum_{\sigma \in S_m} (\mathrm{sgn}\sigma) t_{\sigma(1)} x_1 t_{\sigma(2)} \cdots t_{\sigma(m-1)} x_{m-1} t_{\sigma(m)}$ be the m-th Capelli polynomial of rank m. In the ordinary case (see Giambruno and Zaicev, Israel J Math 135:125–145, 2003) it was proved the asymptotic equality between the codimensions of the T-ideal generated by the Capelli polynomial Cap_{k^2+1} and the codimensions of the matrix algebra $M_k(F)$. In (Benanti, Algebr Represent Theory 18:221–233, 2015) this result was extended to superalgebras proving that the $\mathbb{Z}_2$-graded codimensions of the T_2-ideal generated by the $\mathbb{Z}_2$-graded Capelli polynomials Cap^0_{M+1} and Cap^1_{L+1} for some fixed M, L, are asymptotically equal to the $\mathbb{Z}_2$-graded codimensions of a simple finite dimensional superalgebra. Recently, the authors proved that the $*$-codimensions of a $*$-simple finite dimensional algebra are asymptotically equal to the $*$-codimensions of the T-$*$-ideal generated by the $*$-Capelli polynomials Cap^+_{M+1} and Cap^-_{L+1}, for some fixed natural numbers M and L.

Keywords Algebras with involution · Capelli polynomials · Codimension · Growth · Superalgebras

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1 Introduction

From Kemer's theory (see [35]), the polynomial identities of the matrix algebra $M_k(F)$ over a field F of characteristic zero are among the most intriguing topics in the PI-theory. There are a lot of open problems and conjectures concerning the bases of polynomial identities of $M_k(F)$, the minimal degree of identities which do not follow from the standard polynomial, the numerical invariants of polynomial identities, etc. Similar problems are also to consider for matrix algebras with additional structure as $\mathbb{Z}_2$-gradings, group gradings or involution. The Capelli polynomial plays a central role in the combinatorial PI-theory and in particular in the study of polynomial identities of the matrix algebra $M_k(F)$ in fact it was determinated a precise relation between the growth of the corresponding T-ideal and the growth of the T-ideal of the matrix algebra. Moreover the Capelli polynomials characterize the algebras having the cocharacter contained in a given strip (see [41]). Let us recall that, for any positive integer m, the m-th Capelli polynomial is the element of the free algebra $F\langle X\rangle$ defined as

$$Cap_m = Cap_m(t_1, \ldots, t_m; x_1, \ldots, x_{m-1}) =$$

$$= \sum_{\sigma \in S_m} (\text{sgn}\,\sigma) t_{\sigma(1)} x_1 t_{\sigma(2)} \cdots t_{\sigma(m-1)} x_{m-1} t_{\sigma(m)}$$

where S_m is the symmetric group on $\{1, \ldots, m\}$. It is an alternating polynomial and every polynomial which is alternating on $t_1, \ldots, t_m$ can be written as a linear combination of Capelli polynomials obtained by specializing the x_i's. These polynomials were first introduced by Razmyslov (see [39]) in his construction of central polynomials for $k \times k$ matrices. It is easy to show that if A is a finite dimensional algebra A and dim $A = m - 1$ then A satisfies Cap_m. Moreover, any finitely generated PI-algebra A satisfies Cap_m for some m (see, for example, Theorem 2.2 in [35]). Then the matrix algebra $M_k(F)$ satisfies Cap_{k^2+1} and $k^2 + 1$ is actually the minimal degree of a Capelli polynomial satisfied by $M_k(F)$.

The main purpose of this paper is to present a survey on old and new results concerning the Capelli polynomials. In particular, in Sect. 2 we recall the results about the T-ideal generated by the m-th Capelli polynomial Cap_m and in Sect. 3 the results concerning the T_2-ideal generated by the $\mathbb{Z}_2$-graded Capelli polynomials Cap^0_{M+1} and Cap^1_{L+1}. We show their relations with the T-ideal of the polynomial identities of $M_k(F)$ and, respectively, with the T_2-ideals of the $\mathbb{Z}_2$-graded identities of the simple finite dimensional superalgebra $M_k(F)$, $M_{k,l}(F)$ and $M_s(F \oplus tF)$. In Sect. 4 we present the recent results obtained by the authors about the study of the $*$-codimensions of the T-$*$-ideal generated by the $*$-Capelli polynomials Cap^+_{M+1} and Cap^-_{L+1}. These results has been announced in a complete version at the preprint server of Cornell University (https://arxiv.org/pdf/1911.04193.pdf) and has been submitted elsewhere.

2 Ordinary Case

Let F be a field of characteristic zero and let $F\langle X\rangle = F\langle x_1, x_2, \ldots\rangle$ be the free associative algebra on a countable set X over F. Recall that an ideal I of $F\langle X\rangle$ is a T-ideal if it is invariant under all endomorphisms of $F\langle X\rangle$. Let A be an associative algebra over F, then an element $f = f(x_1, \ldots, x_n) \in F\langle X\rangle$ is a polynomial identity for A if $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in A$. If f is a polynomial identity for A we usually write $f \equiv 0$ in A. Let $Id(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ in A}\}$ be the ideal of polynomial identities of A. When A satisfies a non trivial identity (i.e. $Id(A) \neq (0)$), we say that A is a PI- algebra. The connection between T-ideals of $F\langle X\rangle$ and PI-algebras is well understood: for any F-algebra A, $Id(A)$ is a T-ideal of $F\langle X\rangle$ and every T- ideal I of $F\langle X\rangle$ is the ideal of identities of some F-algebra A. For $I = Id(A)$ we denote by $\mathrm{var}(I) = \mathrm{var}(A)$ the variety of all associative algebras having the elements of I as polynomial identities. The language of varieties is effective for investigations of PI-algebras.

An important class of T-ideals is given by the so-called verbally prime T-ideals. They were introduced by Kemer (see [35]) in his solution of the Specht problem as basic blocks for the study of arbitrary T-ideals. Recall that a T-ideal $I \subseteq F\langle X\rangle$ is verbally prime if for any T-ideals I_1, I_2 such $I_1 I_2 \subseteq I$ we must have $I_1 \subseteq I$ or $I_2 \subseteq I$. A PI-algebra A is called verbally prime if its T-ideal of identities $I = Id(A)$ is verbally prime. Also, the corresponding variety of associative algebras $\mathrm{var}(A)$ is called verbally prime. By the structure theory of T-ideals developed by Kemer (see [35]) and his classification of verbally prime T-ideals in characteristic zero, the study of an arbitrary T-ideal can be reduced to the study of the T-ideals of identities of the following verbally prime algebras

$$F,\ F\langle X\rangle,\ M_k(F),\ M_k(G),\ M_{k,l}(G)$$

where G is the infinite dimensional Grassmann algebra, $M_k(F)$, $M_k(G)$ are the algebras of $k \times k$ matrices over F and G, respectively, and

$$M_{k,l}(G) = \begin{array}{c} \\ k \\ l \end{array} \!\!\begin{array}{c} \begin{array}{cc} k & l \end{array} \\ \begin{pmatrix} G_0 & G_1 \\ G_1 & G_0 \end{pmatrix} \end{array}.$$

Recall that G is the algebra generated by a countable set $\{e_1, e_2, \ldots\}$ subject to the conditions $e_i e_j = -e_j e_i$ for all $i, j = 1, 2, \ldots$, and $G = G_0 \oplus G_1$ is the natural $\mathbb{Z}_2$-grading on G, where G_0 and G_1 are the spaces generated by all monomials in the generators e_i's of even and odd length, respectively.

It is well known that in characteristic zero every T-ideal is completely determined by its multilinear elements. Hence, if P_n is the space of multilinear polynomials of degree n in the variables $x_1, \ldots, x_n$, the relatively free algebra $F\langle X\rangle/Id(A)$ is determined by the sequence of subspaces $\{P_n/(P_n \cap Id(A))\}_{n\geq 1}$.

The integer $c_n(A) = \dim P_n/(P_n \cap Id(A))$ is called the n-th codimension of A and gives a quantitative estimate of the polynomial identities satisfied by A.

Thus to each T-ideal $I = Id(A)$ one can associate the numerical sequence of codimensions $\{c_n(I)\}_{n\geq 1} = \{c_n(A)\}_{n\geq 1}$ of I, or A, that plays an important role in the study of $Id(A)$. It is well known that A is a PI-algebra if and only if $c_n(A) < n!$ for some $n \geq 1$. Regev in [40] showed that if A is an associative PI-algebra, then $c_n(A)$ is exponentially bounded i.e., there exist constants α, β which depend on A such that $c_n(A) \leq \alpha\beta^n$ for any $n \geq 1$ (see also [36] and [42] for the best known estimates). Giambruno and Zaicev improved this result and, in [23] and [24], proved that for a PI-algebra A

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and is an integer; $\exp(A)$ is called the PI-exponent of the algebra A. For the verbally prime algebras we have (see [14, 43, 44] and [24])

$$\exp(M_k(F)) = k^2,\ \exp(M_k(G)) = 2k^2,\ \exp(M_{k,l}(G)) = (k+l)^2.$$

In [43] Regev obtained the precise asymptotic behavior of the codimensions of the verbally prime algebra $M_k(F)$. It turns out that

$$c_n(M_k(F)) \simeq C(\frac{1}{n})^{(k^2-1)/2}k^{2n},$$

where C is a certain constant explicitly computed. For the other verbally prime algebras $M_k(G)$, $M_{k,l}(G)$ there are only some partially results (see [14] and [16]). More precisely,

$$c_n(M_{k,l}(G)) \simeq an^g\alpha^n,\ c_n(M_k(G)) \simeq bn^h\beta^n,$$

with $\alpha = (k+l)^2$, $g = -\frac{1}{2}(k^2+l^2-1)$, $\beta = 2k^2$, $h = -\frac{1}{2}(k^2-1)$, and a and b are undetermined constants. It turns out that it is in general a very hard problem to determine the precise asymptotic behavior of such sequences.

In [29] and in [10] it was found a relation among the asymptotics of codimensions of the verbally prime T-ideals and the T-ideals generated by Capelli polynomials or Amitsur's Capelli-type polynomials.

Now, if $f \in F\langle X\rangle$ we denote by $\langle f\rangle_T$ the T-ideal generated by f. Also for $V \subset F\langle X\rangle$ we write $\langle V\rangle_T$ to indicate the T-ideal generated by V. Let C_m be the set of 2^m polynomials obtained from the m-th Capelli polynomial Cap_m by deleting any subset of variables x_i (by evaluating the variables x_i to 1 in all possible ways) and let $\langle C_m\rangle_T$ denotes the T-ideal generated by C_m. If $U_m = \text{var}(C_m)$ is the variety corresponding to $\langle C_m\rangle_T$ then $\exp(C_m) = \exp(U_m)$. In case $m = k^2$, it follows from [43] that

$$\exp(C_{k^2+1}) = k^2 = \exp(M_k(F)).$$

Mishchenko, Regev and Zaicev in [37] computed the $\exp(C_m)$, for an arbitrary m, and in particular they proved (see also [30, Theorem 9.1.5])

Theorem 1 ([37, Theorem])

(1) $m-3 \le \exp(C_{m+1}) \le m$.
(2) $\exp(C_{m+1}) = max\{a_1, a_2, a_3, a_4\}$ *where* $a_j = max\{d_1^2 + \cdots + d_j^2 \mid d_1, \ldots, d_j \in \mathbb{Z}, d_1, \ldots, d_j > 0, d_1^2 + \cdots + d_j^2 + j \le m+1\}$.
(3) $\exp(C_{m+1}) \le m \Leftrightarrow m = q^2$, for some q.

The proof applies, in an essential way, the classical Lagrange's four square theorem.

In [29] Giambruno and Zaicev proved that the codimensions of U_{k^2+1} are asymptotically equal to the codimensions of the verbally prime algebra $M_k(F)$

Theorem 2 ([29, Theorem 3, Corollary 4]) *Let* $m = k^2$. *Then* $\mathrm{var}(C_{m+1}) = \mathrm{var}(M_k(F) \oplus B)$ *for some finite dimensional algebra* B *such that* $\exp(B) < k^2$. *In particular*

$$c_n(C_{k^2+1}) \simeq c_n(M_k(F)).$$

This result has been extended to the others verbally prime algebras by the so called Amitsur's Capelli-type polynomials.

Let L and M be two natural numbers, let $\hat{n} = (L+1)(M+1)$ and let μ be a partition of $\hat{n}$ with associated rectangular Young diagram, $\mu = ((L+1)^{M+1}) \vdash \hat{n}$. In [6] the following polynomials, denoted Amitsur's Capelli-type polynomials, were introduced

$$e^*_{M,L} = e^*_{M,L}(t_1, \ldots, t_{\hat{n}}; x_1, \ldots, x_{\hat{n}-1}) = \sum_{\sigma \in S_{\hat{n}}} \chi_\mu(\sigma) t_{\sigma(1)} x_1 t_{\sigma(2)} \cdots x_{\hat{n}-1} t_{\sigma(\hat{n})},$$

where $\chi_\mu(\sigma)$ is the value of the irreducible character χ_μ corresponding to the partition $\mu \vdash \hat{n}$ on the permutation σ. We note that for $L = 0$ we have $\mu = (1^{\hat{n}})$ and $e^*_{M,L} = Cap_{\hat{n}}$ is the $\hat{n}$-th Capelli polynomial. The Amitsur's Capelli-type polynomials generalize the Capelli polynomials in the sense that the Capelli polynomials characterize the algebras having the cocharacter contained in a given strip (see [41]) and the Amitsur's polynomials characterize the algebras having a cocharacter contained in a given hook (see [6, Theorem B]).

Let $E^*_{M,L}$ denote the set of $2^{\hat{n}-1}$ polynomials obtained from $e^*_{M,L}$ by evaluating the variables x_i to 1 in all possible ways. Also we denote by $\Gamma_{M,L} = \langle E^*_{M,L} \rangle_T$ the T-ideal generated by $E^*_{M,L}$. Moreover we write $\mathcal{V}_{M,L} = \mathrm{var}(E^*_{M,L}) = \mathrm{var}(\Gamma_{M,L})$, $c_n(E^*_{M,L}) = c_n(\Gamma_{M,L})$ and $\exp(E^*_{M,L}) = \exp(\Gamma_{M,L})$. The following relations between the exponent of the Capelli-type polynomials and the exponent of the verbally prime algebras are well known (see [15])

$$\exp(E^*_{k^2,k^2}) = 2k^2 = \exp(M_k(G)), \quad \exp(E^*_{k^2+l^2,2kl}) = (k+l)^2 = \exp(M_{k,l}(G)).$$

In [15] (see also [30]) Berele and Regev, by using the generalized-six-square theorem [17], proved that

Theorem 3 ([15, Proposition 4.4.]) *Let $l \leq k$. Then $k+l-3 \leq \exp(E^*_{k,l}) \leq k+l$.*

Finally, in [10] it was shown the following asymptotical equalities

Theorem 4 ([10, Theorem 5]) *Let $k, l \in \mathbb{N}$. Then $\mathrm{var}(E^*_{k^2+l^2,2kl}) = \mathrm{var}(M_{k,l}(G) \oplus G(D'))$, where D' is a finite dimensional superalgebra such that $exp(D') < (k+l)^2$. In particular*

$$c_n(E^*_{k^2+l^2,2kl}) \simeq c_n(M_{k,l}(G)).$$

Theorem 5 ([10, Theorem 10]) *Let $k \in \mathbb{N}$, $k > 0$. Then $\mathrm{var}(E^*_{k^2,k^2}) = \mathrm{var}(M_k(G) \oplus G(D'))$, where D' is a finite dimensional superalgebra such that $\exp(D') < 2k^2$. In particular*

$$c_n(E^*_{k^2,k^2}) \simeq c_n(M_k(G)).$$

3 $\mathbb{Z}_2$-Graded Case

Recall that an algebra A is a superalgebra (or $\mathbb{Z}_2$-graded algebra) with grading $(A^{(0)}, A^{(1)})$ if $A = A^{(0)} \oplus A^{(1)}$, where $A^{(0)}, A^{(1)}$ are subspaces of A satisfying:

$$A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)} \text{ and } A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}.$$

The elements of $A^{(0)}$ and of $A^{(1)}$ are called homogeneous of degree zero (or even elements) and of degree one (or odd elements), respectively. If we write $X = Y \cup Z$ as the disjoint union of two countable sets, then the free associative algebra $F\langle X\rangle = F\langle Y \cup Z\rangle = \mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ has a natural structure of free superalgebra with grading $(\mathcal{F}^{(0)}, \mathcal{F}^{(1)})$, where $\mathcal{F}^{(0)}$ is the subspace generated by the monomials of even degree with respect to Z and $\mathcal{F}^{(1)}$ is the subspace generated by the monomials having odd degree in Z.

Recall that an element $f(y_1, \ldots, y_n, z_1, \ldots, z_m)$ of $F\langle Y \cup Z\rangle$ is a $\mathbb{Z}_2$-graded identity or a superidentity for A if $f(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$, for all $a_1, \ldots, a_n \in A^{(0)}$ and $b_1, \ldots, b_m \in A^{(1)}$. The set $Id^{sup}(A)$ of all $\mathbb{Z}_2$-graded identities of A is a T_2-ideal of $F\langle Y \cup Z\rangle$ i.e., an ideal invariant under all endomorphisms of $F\langle Y \cup Z\rangle$ preserving the grading. Moreover, every T_2-ideal Γ of $F\langle Y \cup Z\rangle$ is the ideal of $\mathbb{Z}_2$-graded identities of some superalgebra $A = A^{(0)} \oplus A^{(1)}$, $\Gamma = Id^{sup}(A)$. For $\Gamma = Id^{sup}(A)$ a T_2-ideal of $F\langle Y \cup Z\rangle$, we denote by supvar(Γ) or supvar(A) the supervariety of superalgebras having the elements of Γ as $\mathbb{Z}_2$-graded identities.

As it was shown by Kemer (see [34, 35]), superalgebras and their $\mathbb{Z}_2$-graded identities play a basic role in the study of the structure of varieties of associative algebras over a field of characteristic zero. More precisely Kemer showed that any variety is generated by the Grassmann envelope of a suitable finite dimensional superalgebra (see Theorem 3.7.8 [30]) and moreover he established that an associative variety is a prime variety if and only if it is generated by the Grassmann envelope of a simple finite dimensional superalgebra.

Recall that, if F is an algebraically closed field of characteristic zero, then a simple finite dimensional superalgebra over F is isomorphic to one of the following algebras (see [30, 35]):

1. $M_k(F)$ with trivial grading $(M_k(F), 0)$;
2. $M_{k,l}(F)$ with grading $\left(\begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}, \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix}\right)$, where F_{11}, F_{12}, F_{21}, F_{22} are $k \times k$, $k \times l$, $l \times k$ and $l \times l$ matrices respectively, $k \geq 1$ and $l \geq 1$;
3. $M_s(F \oplus tF)$ with grading $(M_s(F), tM_s(F))$, where $t^2 = 1$.

Thus an important problem in the theory of PI-algebras is to describe the T_2-ideals of $\mathbb{Z}_2$-graded identities of finite dimensional simple superalgebra: $Id^{sup}(M_k(F))$, $Id^{sup}(M_{k,l}(F))$, $Id^{sup}(M_s(F \oplus tF))$.

In case char$F = 0$, it is well known that $Id^{sup}(A)$ is completely determined by its multilinear polynomials and an approach to the description of the $\mathbb{Z}_2$-graded identities of A is based on the study of the $\mathbb{Z}_2$-graded codimensions sequence of this superalgebra. If P_n^{sup} denotes the space of multilinear polynomials of degree n in the variables $y_1, z_1, \ldots, y_n, z_n$ (i.e., y_i or z_i appears in each monomial at degree 1), then the sequence of spaces $\{P_n^{sup} \cap Id^{sup}(A)\}_{n \geq 1}$ determines $Id^{sup}(A)$ and

$$c_n^{sup}(A) = \dim_F \left(\frac{P_n^{sup}}{P_n^{sup} \cap Id^{sup}(A)} \right)$$

is called the n-th $\mathbb{Z}_2$-graded codimension of A. The asymptotic behaviour of the $\mathbb{Z}_2$-graded codimensions plays an important role in the PI-theory of superalgebras. In 1985, Giambruno e Regev (see [22]) proved that the sequence $\{c_n^{sup}(A)\}_{n \geq 1}$ is exponentially bounded if and only if A satisfies an ordinary polynomial identity. In [12] it was proved that if A is a finitely generated superalgebra satisfying a polynomial identity, then $\lim_{n \to \infty} \sqrt[n]{c_n^{sup}(A)}$ exists and is a non negative integer. It is called superexponent (or $\mathbb{Z}_2$-exponent) of A and it is denoted by

$$\text{supexp}(A) = \lim_{n \to \infty} \sqrt[n]{c_n^{sup}(A)}.$$

We remark that in [21] the existence of the G-exponent has been proved when G is a group of prime order and, in general, in [2, 31] and [1] for an arbitrary PI-algebras graded by a finite abelian group G.

Now, if $f \in F\langle Y \cup Z \rangle$ we denote by $\langle f \rangle_{T_2}$ the T_2-ideal generated by f. Also for a set of polynomials $V \subset F\langle Y \cup Z \rangle$ we write $\langle V \rangle_{T_2}$ to indicate the T_2-ideal generated by V. Let denote by $Cap_m[Y, X] = Cap_m(y_1, \ldots, y_m; x_1, \ldots, x_{m-1})$

and $Cap_m[Z, X] = Cap_m(z_1, \dots, z_m; x_1, \dots, x_{m-1})$ the m-th $\mathbb{Z}_2$-graded Capelli polynomial in the alternating variables of homogeneous degree zero $y_1, \dots, y_m$ and of homogeneous degree one $z_1, \dots, z_m$, respectively. Then Cap_m^0 indicates the set of 2^{m-1} polynomials obtained from $Cap_m[Y, X]$ by deleting any subset of variables x_i (by evaluating the variables x_i to 1 in all possible way). Similarly, we define by Cap_m^1 the set of 2^{m-1} polynomials obtained from $Cap_m[Z, X]$ by deleting any subset of variables x_i.

If L and M are two natural numbers, let $\Gamma_{M+1,L+1}$ be the T_2-ideal generated by the polynomials Cap^0_{M+1}, Cap^1_{L+1}, $\Gamma_{M+1,L+1} = \langle Cap^0_{M+1}, Cap^1_{L+1}\rangle_{T_2}$. We also write $\mathcal{U}^{sup}_{M+1,L+1} = \text{supvar}(\Gamma_{M+1,L+1})$.

In [8] it was calculated the $\text{supexp}(\mathcal{U}^{\text{sup}}_{M+1,L+1})$. We recall the following

Definition 1 (see [8]) Let M and L be fixed. Then, for any integers $s, t \geq 0$, $r \geq 1$ such that $r - 1 = r_0 + r_1$ for some non-negative integers r_0, r_1, we define the set

$$\overline{A}_{r,s,t;r_0,r_1} = \{a_1, \dots, a_r, k_1, l_1, \dots, k_s, l_s, b_1, \dots, b_t \in \mathbb{Z}^+ \mid$$

$$a_1^2 + \cdots + a_r^2 + (k_1^2 + l_1^2) + \cdots (k_s^2 + l_s^2) + b_1^2 + \cdots + b_t^2 + r_0 + s + t \leq M,$$

$$\text{and } 2k_1l_1 + \cdots + 2k_sl_s + b_1^2 + \cdots + b_t^2 + r_1 + s + t \leq L\}.$$

Also, given integers $s, t \geq 0$ $(r = 0)$, we define the set

$$\widetilde{A}_{s,t} = \{k_1, l_1, \dots, k_s, l_s, b_1, \dots, b_t \in \mathbb{Z}^+ \mid$$

$$(k_1^2 + l_1^2) + \cdots (k_s^2 + l_s^2) + b_1^2 + \cdots + b_t^2 + s + t \leq M + 1,$$

$$\text{and } 2k_1l_1 + \cdots + 2k_sl_s + b_1^2 + \cdots + b_t^2 + s + t \leq L + 1\}.$$

Moreover, let

$$\overline{a}_{r,s,t;r_0,r_1} = \max_{a_i,k_i,l_i,b_i \in \overline{A}_{r,s,t;r_0,r_1}} \{a_1^2 + \cdots + a_r^2 + (k_1 + l_1)^2 + \cdots (k_s + l_s)^2 + 2b_1^2 + \cdots + 2b_t^2\}$$

and

$$\widetilde{a}_{s,t} = \max_{k_i,l_i,b_i \in \widetilde{A}_{s,t}} \{(k_1 + l_1)^2 + \cdots (k_s + l_s)^2 + 2b_1^2 + \cdots + 2b_t^2\},$$

then we define

$$a_0 = \max\{\overline{a}_{r,s,t;r_0,r_1}, \widetilde{a}_{s,t} \mid r + s + t \leq 11\}.$$

Theorem 6 ([8, Theorem 4]) *If $M \geq L \geq 0$, then*

(1) $\text{supexp}(\mathcal{U}^{\text{sup}}_{M+1,L+1}) = a_0$;
(2) $(M + L) - 10 \leq \text{supexp}(\mathcal{U}^{\text{sup}}_{M+1,L+1}) \leq (M + L)$.

This result was inspirated by the ordinary case. Moreover, we should mention that in the previous theorem an essential tool is the generalized-six-square theorem proved in [17] (see also Appendix A of [30]).

The following relations between the superexponent of the $\mathbb{Z}_2$-graded Capelli polynomials and the superexponent of the simple finite dimensional superalgebras are well known (see [8, 12, 28])

$$\mathrm{supexp}(\mathcal{U}^{\mathrm{sup}}_{k^2+1,1}) = k^2 = supexp(M_k(F))$$

$$\mathrm{supexp}(\mathcal{U}^{\mathrm{sup}}_{k^2+l^2+1,2kl+1}) = (k+l)^2 = supexp(M_{k,l}(F))$$

$$\mathrm{supexp}(\mathcal{U}^{\mathrm{sup}}_{s^2+1,s^2+1}) = 2s^2 = supexp(M_s(F \oplus tF)).$$

In [9] it was found a close relation among the asymptotics of $\mathcal{U}^{sup}_{k^2+l^2+1,2kl+1}$ and $M_{k,l}(F)$ and the asymptotics of $\mathcal{U}^{sup}_{s^2+1,s^2+1}$ and $M_s(F \oplus tF)$. More precisely it was showed that

Theorem 7 ([9, Theorem 9]) *Let $M = k^2 + l^2$ and $L = 2kl$ with $k, l \in \mathbb{N}$, $k > l > 0$. Then $\mathcal{U}^{sup}_{M+1,L+1} = supvar(\Gamma_{M+1,L+1}) = supvar(M_{k,l}(F) \oplus D')$, where D' is a finite dimensional superalgebra such that $supexp(D') < M + L$. In particular*

$$c_n^{sup}(\Gamma_{M+1,L+1}) \simeq c_n^{sup}(M_{k,l}(F)).$$

Theorem 8 ([9, Theorem 14]) *Let $M = L = s^2$ with $s \in \mathbb{N}$, $s > 0$. Then $\mathcal{U}^{sup}_{M+1,L+1} = supvar(\Gamma_{M+1,L+1}) = supvar(M_s(F \oplus tF) \oplus D'')$, where D'' is a finite dimensional superalgebra such that $supexp(D'') < M + L$. In particular*

$$c_n^{sup}(\Gamma_{M+1,L+1}) \simeq c_n^{sup}(M_s(F \oplus tF)).$$

In [29] Giambruno and Zaicev proved that $c_n^{sup}(\Gamma_{k^2+1,1}) \simeq c_n^{sup}(M_k(F))$.

4 Involution Case

Let $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ denote the free associative algebra with involution $*$ generated by the countable set of variables $X = \{x_1, x_1^*, x_2, x_2^*, \ldots\}$ over a field F of characteristic zero. Let $(A, *)$ be an algebra with involution $*$ over F, recall that an element $f(x_1, x_1^*, \cdots, x_n, x_n^*)$ of $F\langle X, *\rangle$ is a $*$-polynomial identity (or $*$-identity) for A if $f(a_1, a_1^*, \cdots, a_n, a_n^*) = 0$, for all $a_1, \ldots, a_n \in A$. We denote by $Id^*(A)$ the set of all $*$-polynomial identities satisfied by A. $Id^*(A)$ is a T-$*$-ideal of $F\langle X, *\rangle$ i.e., an ideal invariant under all endomorphisms of $F\langle X, *\rangle$ commuting with the involution of the free algebra. Similar to the case of ordinary identities any T-$*$-ideal Γ of $F\langle X, *\rangle$ is the ideal of $*$-identities of some algebra A

with involution $*$, $\Gamma = Id^*(A)$. For $\Gamma = Id^*(A)$ we denote by $\mathrm{var}^*(\Gamma) = \mathrm{var}^*(A)$ the variety of $*$-algebras having the elements of Γ as $*$-identities.

It is well known that in characteristic zero $Id^*(A)$ is completely determinated by the multilinear $*$-polynomials it contains. To the T-$*$-ideal $\Gamma = Id^*(A)$ one associates a numerical sequence called the sequence of $*$-codimensions $c_n^*(\Gamma) = c_n^*(A)$ which is the main tool for the quantitative investigation of the $*$-polynomial identities of A. Recall that $c_n^*(A)$, $n = 1, 2, \ldots$, is the dimension of the space of multilinear polynomial in n-th variables in the corresponding relatively free algebra with involution of countable rank. Thus, if we denote by P_n^* the space of all multilinear polynomials of degree n in $x_1, x_1^*, \cdots, x_n, x_n^*$ then

$$c_n^*(A) = \dim P_n^*(A) = \dim \frac{P_n^*}{P_n^* \cap Id^*(A)}.$$

It is clear that the ordinary free associative algebra $F\langle X\rangle$ (without involution) can be considered as a subalgebra of $F\langle X, *\rangle$ and, in particular, an ordinary polynomial identity (without involution) can be considered as an identity with involution. Hence if A is a $*$-algebra, then $Id(A) \subseteq Id^*(A)$. Moreover, a celebrated theorem of Amitsur ([4, 5], see also [30]) states that if an algebra with involution satisfies a $*$-polynomial identity then it satisfies an ordinary polynomial identity. At the light of this result in [22] it was proved that, as in the ordinary case, if A satisfies a non trivial $*$-polynomial identity then $c_n^*(A)$ is exponentially bounded, i.e. there exist constants a and b such that $c_n^*(A) \leq ab^n$, for all $n \geq 1$. Later (see [7]) an explicit exponential bound for $c_n^*(A)$ was exhibited and in [28] a characterization of finite dimensional algebras with involution whose sequence of $*$-codimensions is polynomial bounded was given. This result was extended to non-finite dimensional algebras (see [27]) and $*$-varieties with almost polynomial growth were classified in [26] and [38]. The asymptotic behavior of the $*$-codimensions was determined in [13] in case of matrices with involution.

Recently (see [33]), for any algebra with involution, it was studied the exponential behavior of $c_n^*(A)$, and it was showed that the $*$-exponent of A

$$\exp^*(A) = \lim_{n\to\infty} \sqrt[n]{c_n^*(A)}$$

exists and is a non negative integer. It should be mentioned that the existence of the $*$-exponent was proved in [25] for finite dimensional algebra with involution.

An interesting problem in the theory of PI-algebras with involution $*$ is to describe the T-$*$-ideals of $*$-polynomial identities of $*$-simple finite dimensional algebras. Recall that, if F is an algebraically closed field of characteristic zero, then, up to isomorphisms, all finite dimensional $*$-simple are the following ones (see [30, 45]):

- $(M_k(F), t)$ the algebra of $k \times k$ matrices with the transpose involution;
- $(M_{2m}(F), s)$ the algebra of $2m \times 2m$ matrices with the symplectic involution;

- $(M_h(F) \oplus M_h(F)^{op}, exc)$ the direct sum of the algebra of $h \times h$ matrices and the opposite algebra with the exchange involution.

Let G be the infinite dimensional Grassmann algebra over F. G is generated by the elements $e_1, e_2, \ldots$ subject to the following condition $e_i e_j = -e_j e_i$, for all $i, j \geq 1$. Recall that G has a natural Z_2-grading $G = G_0 \oplus G_1$ where G_0 (resp. G_1) is the span of the monomials in the e_i's of even length (resp. odd length). If $B = B_0 \oplus B_1$ is a superalgebra, then the Grassmann envelope of B is defined as $G(B) = (G_0 \otimes B_0) \oplus (G_1 \otimes B_1)$. The relevance of $G(A)$ relies in a result of Kemer ([35, Theorem 2.3]) stating that if B is any PI-algebra, then its T-ideal of polynomial identities coincides with the T-ideal of identities of the Grassmann envelope of a suitable finite dimensional superalgebra. This result has been extended to algebras with involution in fact in [3] it was proved that, if A is a PI-algebra with involution over a field F of characteristic zero, then there exists a finite dimensional superalgebra with superinvolution B such that $Id^*(A) = Id^*(G(B))$.

Recall that a superinvolution $*$ of B is a linear map of B of order two such that $(ab)^* = (-1)^{|a||b|} b^* a^*$, for any homogeneous elements $a, b \in B$, where $|a|$ denotes the homogeneous degree of a. It is well known that in this case $B_0^* \subseteq B_0, B_1^* \subseteq B_1$ and we decompose $B = B_0^+ \bigoplus B_0^- \bigoplus B_1^+ \bigoplus B_1^-$. We can define a superinvolution $*$ on G by requiring that $e_i^* = -e_i$, for any $i \geq 1$. Then it is easily checked that $G_0 = G^+$ and $G_1 = G^-$. Now, if B is a superalgebra one can perform its Grassmann envelope $G(B)$ and in [3] it was shown that if B has a superinvolution $*$ we can regard $G(B)$ as an algebra with involution by setting $(g \otimes a)^* = g^* \otimes a^*$, for homogeneous elements $g \in G, a \in B$. By making use of the previous theorem, in [33] it was proved the existence of the $*$-exponent of a PI-algebra with involution A and also an explicit way of computing $\exp^*(A)$ was given. The $*$-exponent is computed as follows: if B is a finite dimensional algebra with superinvolution over F, then by Giambruno et al. [32] we write $B = \bar{B} + J$ where $\bar{B}$ is a maximal semisimple superalgebra with induced superinvolution and $J = J(B) = J^*$. Also we can write $\bar{B} = B_1 \oplus \cdots \oplus B_k$, where $B_1, \cdots, B_k$ are simple superalgebras with induced superinvolution. We say that a subalgebra $B_{i_1} \oplus \cdots \oplus B_{i_t}$, where $B_{i_1}, \ldots, B_{i_t}$ are distinct simple components, is admissible if for some permutation $(l_1, \ldots, l_t)$ of $(i_1, \ldots, i_t)$ we have that $B_{l_1} J B_{l_2} J \cdots J B_{l_t} \neq 0$. Moreover if $B_{i_1} \oplus \cdots \oplus B_{i_t}$ is an admissible subalgebra of B then $B' = B_{i_1} \oplus \cdots \oplus B_{i_t} + J$ is called a reduced algebra. In [33] it was proved that $\exp^*(A) = \exp^*(G(B)) = d$ where d is the maximal dimension of an admissible subalgebra of B. It follows immediately that if A is a $*$-simple algebra then $exp^*(A) = \dim_F A$. If $\mathcal{V} = \mathrm{var}^*(A)$ is the variety of $*$-algebras generated by A we write $Id^*(\mathcal{V}) = Id^*(A)$, $c_n^*(\mathcal{V}) = c_n^*(A)$ and $\exp^*(\mathcal{V}) = \exp^*(A)$.

The reduced algebras are basic elements of any $*$-variety in fact we have the following (see [11])

Theorem 9 *Let $\mathcal{V}$ be a proper variety of $*$-algebras. Then there exists a finite number of reduced superalgebras with superinvolution $B_1, \ldots, B_t$ and a finite* *dimensional superalgebra with superinvolution D such that*

$$\mathcal{V} = \mathrm{var}(G(B_1) \oplus \cdots \oplus G(B_t) \oplus G(D))$$

with $\exp^*(\mathcal{V}) = \exp^*(G(B_1)) = \cdots = \exp^* G((B_t))$ *and* $\exp^*(G(D)) < \exp^*(\mathcal{V})$.

In terms of $*$-codimensions we obtain

Corollary 1 *Let $\mathcal{V} = \mathrm{var}^*(A)$ be a proper variety of $*$-algebras. Then there exists a finite number of reduced superalgebras with superinvolution $B_1, \ldots, B_t$ and a finite dimensional superalgebra with superinvolution D such that*

$$c_n^*(A) \simeq c_n^*(G(B_1) \oplus \cdots \oplus G(B_t)).$$

If A is a finite dimensional $*$-algebra, then we have the following

Corollary 2 *Let A be a finite dimensional $*$-algebra. Then there exists a finite number of reduced $*$-algebras $B_1, \ldots, B_t$ and a finite dimensional $*$-algebra D such that*

$$\mathrm{var}^*(A) = \mathrm{var}^*(B_1 \oplus \cdots \oplus B_t \oplus D)$$

$$c_n^*(A) \simeq c_n^*(B_1 \oplus \cdots \oplus B_t)$$

and

$$\exp^*(A) = \exp^*(B_1) = \cdots = \exp^*(B_t), \;\; \exp^*(D) < \exp^*(A).$$

4.1 $*$-Capelli Polynomials and the $*$-Algebra $UT^*(A_1, \ldots, A_n)$

In this paragraph we shall recall the relation among the asymptotics of the $*$-codimensions of the $*$-simple finite dimensional algebras and the T-$*$-ideals generated by the $*$-Capelli polynomials recently proved by the authors. If $(A, *)$ is any algebra with involution $*$, let $A^+ = \{a \in A \mid a^* = a\}$ and $A^- = \{a \in A \mid a^* = -a\}$ denote the subspaces of symmetric and skew elements of A, respectively. Since $\mathrm{char} F = 0$, we can regard the free associative algebra with involution $F\langle X, *\rangle$ as generated by symmetric and skew variables. In particular, for $i = 1, 2, \ldots$, we let $y_i = x_i + x_i^*$ and $z_i = x_i - x_i^*$, then we write $X = Y \cup Z$ as the disjoint union of the set Y of symmetric variables and the set Z of skew variables and $F\langle X, *\rangle = F\langle Y \cup Z\rangle$. Hence a polynomial $f = f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in F\langle Y \cup Z\rangle$ is a $*$-polynomial identity of A if and only if $f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0$ for all $a_i \in A^+$, $b_i \in A^-$. Let $Cap_m^*[Y, X] = Cap_m(y_1, \ldots, y_m; x_1, \ldots, x_{m-1})$ denote the m-th $*$-Capelli polynomial in the alternating symmetric variables $y_1, \ldots, y_m$

and let $Cap_m^*[Z, X] = Cap_m(z_1, \dots, z_m; x_1, \dots, x_{m-1})$ be the m-th $*$-Capelli polynomial in the skew variables $z_1, \dots, z_m$. Then we denote by Cap_m^+ the set of 2^{m-1} polynomials obtained from $Cap_m^*[Y, X]$ by deleting any subset of variables x_i (by evaluating the variables x_i to 1 in all possible way). Similarly, we define by Cap_m^- the set of 2^{m-1} polynomials obtained from $Cap_m^*[Z, X]$ by deleting any subset of variables x_i. If L and M are two natural numbers, we denote by $\Gamma_{M+1,L+1}^* = \langle Cap_{M+1}^+, Cap_{L+1}^- \rangle$ the T-$*$-ideal generated by the polynomials Cap_{M+1}^+, Cap_{L+1}^-. We also write $\mathcal{U}_{M+1,L+1}^* = \mathrm{var}^*(\Gamma_{M+1,L+1})$ for the $*$-variety generated by $\Gamma_{M+1,L+1}^*$.

The following results give us a characterization of the $*$-varieties satisfying a Capelli identity. The proof of the next result follows closely the proof given in [30, Theorem 11.4.3]

Theorem 10 *Let $\mathcal{V}$ be a variety of $*$-algebras. If $\mathcal{V}$ satisfies the Capelli identity of some rank then $\mathcal{V} = \mathrm{var}^*(A)$, for some finitely generated $*$-algebra A.*

Let M, L be two natural numbers. Let $A = A^+ \oplus A^-$ be a generating $*$-algebra of $\mathcal{U}_{M+1,L+1}^*$. It is easy to show that A satisfies a Capelli identity. Hence by the previous theorem, we may assume that A is a finitely generated $*$-algebra. Moreover by Sviridova [46, Theorem 1] we may consider A as a finite dimensional $*$-algebra. Since any polynomial alternating on $M + 1$ symmetric variables vanishes in A (see [30, Proposition 1.5.5]), we get that $\dim A^+ \leq M$. Similarly we get that $\dim A^- \leq L$ and $\exp^*(A) \leq \dim A \leq M + L$. Thus we have the following

Lemma 1 $\exp^*(\mathcal{U}_{M+1,L+1}^*) \leq M + L$.

Now, we recall the construction of the $*$-algebra $UT^*(A_1, \dots, A_n)$ given in Section 2 of [18]. Let $A_1, \dots, A_n$ be a n-tuple of finite dimensional $*$-simple algebras, then $A_i = (M_{d_i}(F), \mu_i)$, where μ_i is the transpose or the symplectic involution, or $A_i = (M_{d_i}(F) \oplus M_{d_i}(F)^{op}, exc)$, where exc is the exchange involution.

Let γ_d be the orthogonal involution defined on the matrix algebra $M_d(F)$ by putting, for all $a \in M_d(F)$, $a^{\gamma_d} = g^{-1}a^t g = ga^t g$, where a^t is the transposed of the matrix a and

$$g = \begin{pmatrix} 0 & \dots & 1 \\ & & \cdot \\ & \cdot & \\ \cdot & & \\ 1 & \dots & 0 \end{pmatrix}.$$

If $d = \sum_{i=1}^n \dim_F A_i$, then we can consider an embedding of $*$-algebras

$$\Delta : \bigoplus_{i=1}^n A_i \to (M_{2d}(F), \gamma_{2d})$$

defined by

$$(a_1, \dots, a_n) \to \begin{pmatrix} \bar{a}_1 & & & & & \\ & \ddots & & & & \\ & & \bar{a}_n & & & \\ & & & \bar{b}_n & & \\ & & & & \ddots & \\ & & & & & \bar{b}_1 \end{pmatrix}$$

where, if $a_i \in A_i = (M_{d_i}(F), \mu_i)$, then $\bar{a}_i = a_i$ and $\bar{b}_i = a_i^{\mu_i \gamma_{d_i}}$, and if $a_i = (\tilde{a}_i, \tilde{b}_i) \in A_i = (M_{d_i}(F) \oplus M_{d_i}(F)^{op}, exc)$, then $\bar{a}_i = \tilde{a}_i$ and $\bar{b}_i = \tilde{b}_i$. Let $D = D(A_1, \dots, A_n) \subseteq M_{2d}(F)$ be the $*$-algebra image of $\bigoplus_{i=1}^n A_i$ by Δ and let U be the subspace of $M_{2d}(F)$ so defined:

$$\begin{pmatrix} 0 & U_{12} & \cdots & U_{1t} & & & & \\ & \ddots & \ddots & \vdots & & & & \\ & & 0 & U_{t-1t} & & & & \\ & & & 0 & & & & \\ & & & & 0 & U_{tt-1} & \cdots & U_{t1} \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & 0 & U_{21} \\ & & & & & & & 0 \end{pmatrix}$$

where, for $1 \le i, j \le n$, $i \ne j$, U_{ij} denote the vector space of the rectangular matrices of dimensions $d_i \times d_j$. Let define (see section 2 of [18]) $UT^*(A_1, \dots, A_n) = D \oplus U \subseteq M_{2d}(F)$. It is easy to show that $UT^*(A_1, \dots, A_n)$ is a subalgebra with involution of $(M_{2d}(F), \gamma_{2d})$ in which the algebras A_i are embedded as $*$-algebras and whose $*$-exponent is given by

$$\exp^*(UT^*(A_1, \dots, A_n)) = \sum_{i=1}^{n} \dim_F A_i.$$

In [20] and [19] the link between the degrees of $*$-Capelli polynomials and the $*$-polynomial identities of $UT^*(A_1, \dots, A_n)$ was investigated. If we set $d^+ := \sum_{i=1}^n \dim_F A_i^+$ and $d^- := \sum_{i=1}^n \dim_F A_i^-$, then the following result applies

Lemma 2 *Let $R = UT^*(A_1, \dots, A_n)$. Then $Cap_M^*[Y, X]$ and $Cap_L^*[Z, X]$ are in $Id^*(R)$ if and only if $M \ge d^+ + n$ and $L \ge d^- + n$.*

4.2 Asymptotics for ∗-Capelli Polynomials

In this section we shall state our main results about the ∗-Capelli polynomials and their asymptotics (see [11]).

The following two key lemmas hold for any ∗-simple finite dimensional algebra.

Lemma 3 *Let $A = \bar{A} \oplus J$ where $\bar{A}$ is a ∗-simple finite dimensional algebra and $J = J(A)$ is its Jacobson radical. Then J can be decomposed into the direct sum of four $\bar{A}$-bimodules*

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$$

where, for $p, q \in \{0, 1\}$, J_{pq} is a left faithful module or a 0-left module according to $p = 1$, or $p = 0$, respectively. Similarly, J_{pq} is a right faithful module or a 0-right module according to $q = 1$ or $q = 0$, respectively. Moreover, for $p, q, i, l \in \{0, 1\}$, $J_{pq}J_{ql} \subseteq J_{pl}$, $J_{pq}J_{il} = 0$ for $q \neq i$ and there exists a finite dimensional nilpotent ∗-algebra N such that $J_{11} \cong \bar{A} \otimes_F N$ (isomorphism of $\bar{A}$-bimodules and of ∗-algebras).

Proof It follows from the proof of Lemma 2 in [29].

Lemma 4 *Let $\bar{A}$ be a ∗-simple finite dimensional algebra. Let $M = \dim_F \bar{A}^+$ and $L = \dim_F \bar{A}^-$. Then $\bar{A}$ does not satisfy $Cap^*_M[Y, X]$ and $Cap^*_L[Z, X]$.*

Proof The result follows immediately from [21, Lemma 3.1].

Lemma 5

(1) *Let $M_1 = k(k + 1)/2$ and $L_1 = k(k - 1)/2$ with $k \in \mathbb{N}$, $k > 0$ and let $J_{11} \cong M_k(F) \otimes_F N$, as in Lemma 3. If $\Gamma_{M_1+1,L_1+1} \subseteq Id^*(M_k(F) + J)$, then $J_{10} = J_{01} = (0)$ and N is commutative.*

(2) *Let $M_2 = m(2m - 1)$ and $L_2 = m(2m + 1)$ with $m \in \mathbb{N}$, $m > 0$ and let $J_{11} \cong M_{2m}(F) \otimes_F N$, as in Lemma 3. If $\Gamma_{M_2+1,L_2+1} \subseteq Id^*(M_{2m}(F) + J)$, then $J_{10} = J_{01} = (0)$ and N is commutative.*

(3) *Let $M_3 = L_3 = h^2$ with $h \in \mathbb{N}$, $h > 0$ and let $J_{11} \cong (M_h(F) \oplus M_h(F)^{op}) \otimes_F N$, as in Lemma 3. If $\Gamma_{M_3+1,L_3+1} \subseteq Id^*((M_h(F) \oplus M_h(F)^{op}) + J)$, then $J_{10} = J_{01} = (0)$ and N is commutative.*

Lemma 6

(1) *Let $M_1 = k(k + 1)/2$ and $L_1 = k(k - 1)/2$ with $k \in \mathbb{N}$, $k > 0$. Then*

$$\exp^*(\mathcal{U}^*_{M_1+1,L_1+1}) = M_1 + L_1 = k^2 = \exp^*((M_k(F), t)).$$

(2) *Let* $M_2 = m(2m-1)$ *and* $L_2 = m(2m+1)$ *with* $m \in \mathbb{N}$, $m > 0$. *Then*

$$\exp^*(\mathcal{U}^*_{M_2+1,L_2+1}) = M_2 + L_2 = 4m^2 = \exp^*((M_{2m}(F), s)).$$

(3) *Let* $M_3 = L_3 = h^2$ *with* $h \in \mathbb{N}$, $h > 0$. *Then*

$$\exp^*(\mathcal{U}^*_{M_3+1,L_3+1}) = M_3 + L_3 = 2h^2 = \exp^*((M_h(F) \oplus M_h(F)^{op}, exc)).$$

Proof (1) The exponent of $\mathcal{U}^*_{M_1+1,L_1+1}$ is equal to the exponent of some minimal variety lying in $\mathcal{U}^*_{M_1+1,L_1+1}$ (for the definition of minimal variety see [30]). Let $d^+ := \sum_{i=1}^n \dim_F A_i^+$ and $d^- := \sum_{i=1}^n \dim_F A_i^-$, then, by Di Vincenzo and Spinelli [20, Theorem 2.1] and Lemma 2, we have that

$$\exp^*(\mathcal{U}^*_{M_1+1,L_1+1}) = \max\{\exp^*(UT^*(A_1,\ldots,A_n)) \mid d^+ + n \le M_1+1 \text{ and } d^- + n \le L_1+1\}.$$

Then

$$\exp^*(\mathcal{U}^*_{M_1+1,L_1+1}) \ge M_1 + L_1 = k^2 = \exp^*(UT^*(M_k(F))).$$

Since by Lemma 1, $\exp^*(\mathcal{U}^*_{M_1+1,L_1+1}) \le M_1 + L_1$ then the proof is completed. (2), (3) The proof is the same of that of point (1).

Now we are able to prove the main results.

Theorem 11 *Let* $M_1 = k(k+1)/2$ *and* $L_1 = k(k-1)/2$ *with* $k \in \mathbb{N}$, $k > 0$. *Then*

$$\mathcal{U}^*_{M_1+1,L_1+1} = \operatorname{var}^*(\Gamma^*_{M_1+1,L_1+1}) = \operatorname{var}^*(M_k(F) \oplus D'),$$

where D' *is a finite dimensional* $*$*-algebra such that* $\exp^*(D') < M_1 + L_1$. *In particular*

$$c_n^*(\Gamma^*_{M_1+1,L_1+1}) \simeq c_n^*(M_k(F)).$$

Sketch of the Proof By the previous Lemma we have that $\exp^*(\mathcal{U}^*_{M_1+1,L_1+1}) = M_1 + L_1$.

Let $A = A^+ \oplus A^-$ be a generating finite dimensional $*$-algebra of $\mathcal{U}^*_{M_1+1,L_1+1}$. By Corollary 2, there exist a finite number of reduced $*$-algebras $B_1, \ldots, B_s$ and a finite dimensional $*$-algebra D' such that $\mathcal{U}^*_{M_1+1,L_1+1} = \operatorname{var}^*(A) = \operatorname{var}^*(B_1 \oplus \cdots \oplus B_s \oplus D')$, with $\exp^*(B_1) = \cdots = \exp^*(B_s) = \exp^*(\mathcal{U}^*_{M_1+1,L_1+1}) = M_1 + L_1$ and $\exp^*(D') < \exp^*(\mathcal{U}^*_{M_1+1,L_1+1}) = M_1 + L_1$. Then, it is enough to analyze the structure of a finite dimensional reduced $*$-algebra R such that $\exp^*(R) = M_1 + L_1 = \exp^*(\mathcal{U}^*_{M_1+1,L_1+1})$ and $\Gamma^*_{M_1+1,L_1+1} \subseteq Id^*(R)$. Let write $R = R_1 \oplus \cdots \oplus R_q + J$, where $J = J(R)$, $R_1 J \cdots J R_q \neq 0$ and R_i is isomorphic to one of the following algebras :$(M_{k_i}(F), t)$ or $(M_{2m_i}(F), s)$ or $(M_{h_i}(F) \oplus M_{h_i}(F)^{op}, exc)$.

Let t_1 be the number of $*$-algebras R_i of the first type, t_2 the number of $*$-algebras R_i of the second type and t_3 the number of R_i of the third type, with $t_1 + t_2 + t_3 = q$. By [18, Theorem 4.5] and [18, Proposition 4.7] there exists a $*$-algebra $\overline{R}$ isomorphic to the $*$-algebra $UT^*(R_1, \ldots, R_q)$ such that $\exp^*(R) = \exp^*(\overline{R}) = \exp^*(UT^*(R_1, \ldots, R_q))$. Let observe that

$$k^2 = M_1 + L_1 = \exp^*(R) = \exp^*(\overline{R}) = \exp^*(UT^*(R_1, \ldots, R_q)) =$$

$$\dim_F R_1 + \cdots + \dim_F R_q = k_1^2 + \cdots + k_{t_1}^2 + (2m_1)^2 + \cdots + (2m_{t_2})^2 + 2h_1^2 + \cdots + 2h_{t_3}^2.$$

Let $d^{\pm} = \dim_F (R_1 \oplus \cdots \oplus R_q)^{\pm}$ then

$$d^+ + d^- = d = \dim_F (R_1 \oplus \cdots \oplus R_q) = \exp^*(\overline{R}) = M_1 + L_1.$$

By [20, Lemma 3.2] $\overline{R}$ does not satisfy the $*$-Capelli polynomials $Cap^*_{d^+ + q - 1}[Y; X]$ and $Cap^*_{d^- + q - 1}[Z; X]$, but $\overline{R}$ satisfies $Cap^*_{M_1+1}[Y; X]$ and $Cap^*_{L_1+1}[Z; X]$. Thus $d^+ + q - 1 \leq M_1$ and $d^- + q - 1 \leq L_1$. Hence $d^+ + d^- + 2q - 2 \leq M_1 + L_1$. Since $d^+ + d^- = M_1 + L_1$ we obtain that $q = t_1 + t_2 + t_3 = 1$. Since t_1, t_2 and t_3 are nonnegative integers by considering all possible cases we get that $t_1 = 1$ and $R \cong M_k(F) + J$. From Lemmas 3 and 5 we obtain

$$R \cong (M_k(F) + J_{11}) \oplus J_{00} \cong (M_k(F) \otimes N^{\sharp}) \oplus J_{00}$$

where $N^{\sharp}$ is the algebra obtained from N by adjoining a unit element.

Thus var*(R) $=$ $\mathrm{var}^*(M_k(F) \oplus J_{00})$ with J_{00} a finite dimensional nilpotent $*$-algebra. Hence, recalling the decomposition given above, we get

$$\mathcal{U}^*_{M_1+1, L_1+1} = \mathrm{var}^*(\Gamma_{M_1+1, L_1+1}) = \mathrm{var}^*(M_k(F) \oplus D'),$$

where D' is a finite dimensional $*$-algebra with $\exp^*(D') < M_1 + L_1$. Then

$$c_n^*(\Gamma_{M_1+1, L_1+1}) \simeq c_n^*(M_k(F))$$

and the theorem is proved.

In a similar way we can prove the next two theorems.

Theorem 12 *Let $M_2 = m(2m - 1)$ and $L_2 = m(2m + 1)$ with $m \in \mathbb{N}$, $m > 0$. Then*

$$\mathcal{U}^*_{M_2+1, L_2+1} = \mathrm{var}^*(\Gamma_{M_2+1, L_2+1}) = \mathrm{var}^*(M_{2m}(F) \oplus D''),$$

where D'' is a finite dimensional $*$-algebra such that $\exp^*(D'') < M_2 + L_2$. *In particular*

$$c_n^*(\Gamma_{M_2+1,L_2+1}) \simeq c_n^*(M_{2m}(F)).$$

Theorem 13 *Let $M_3 = L_3 = h^2$ with $h \in \mathbb{N}$, $h > 0$. Then*

$$\mathcal{U}^*_{M_3+1,L_3+1} = \mathrm{var}^*(\Gamma_{M_3+1,L_3+1}) = \mathrm{var}^*((M_h(F) \oplus M_h(F)^{op}) \oplus D'''),$$

where D''' is a finite dimensional $*$-algebra such that $\exp^*(D''') < M_3 + L_3$. *In particular*

$$c_n^*(\Gamma_{M_3+1,L_3+1}) \simeq c_n^*(M_h(F) \oplus M_h(F)^{op}).$$

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