On Superperfection of Edge Intersection Graphs of Paths

Hervé Kerivin and Annegret Wagler

Abstract The routing and spectrum assignment problem in flexgrid elastic optical networks can be modeled in two phases: a selection of paths in the network and an interval coloring problem in the edge intersection graph of these paths. The interval chromatic number equals the smallest size of a spectrum such that a proper interval coloring is possible, the weighted clique number is a natural lower bound. Graphs where both parameters coincide for all possible non-negative integral weights are called superperfect. We examine the question which minimal non-superperfect graphs can occur in the edge intersection graphs of paths in different underlying networks. We show that for any possible network (even if it is restricted to a path) the resulting edge intersection graphs are not necessarily superperfect and discuss some consequences.

Keywords Routing and spectrum assignment problem · Edge intersection graph of paths · Interval coloring · Superperfection

1 Introduction

Flexgrid elastic optical networks constitute a new generation of optical networks in response to the sustained growth of data traffic volumes and demands in communication networks. In optical networks, light is used as communication medium between sender and receiver nodes, and the frequency spectrum of an optical fiber is divided into narrow frequency slots of fixed spectrum width. Any sequence of consecutive slots can form a channel that can be switched in the network to create a lightpath (i.e., an optical connection represented by a route and a channel). The *routing and spectrum assignment (RSA) problem* consists of establishing the lightpaths for a set of end-to-end traffic demands, that is, finding a

H. Kerivin · A. Wagler (✉)
LIMOS (UMR 6158 CNRS), Université Clermont Auvergne, Clermont-Ferrand, France
e-mail: herve.kerivin@uca.fr; annegret.wagler@uca.fr

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route and assigning an interval of consecutive frequency slots for each demand such that the intervals of lightpaths using a same edge in the network are disjoint, see e.g. [17]. Thereby, the following constraints need to be respected when dealing with the RSA problem:

1. *spectrum continuity*: the frequency slots allocated to a demand remain the same on all the edges of a route;
2. *spectrum contiguity*: the frequency slots allocated to a demand must be contiguous;
3. *non-overlapping spectrum*: a frequency slot can be allocated to at most one demand.

The RSA problem has started to receive a lot of attention over the last few years. It has been shown to be NP-hard [3, 18]. In fact, if for each demand the route is already known, the RSA problem reduces to the so-called *spectrum assignment (SA) problem* and only consists of determining the demands' channels. Even the SA problem has been shown to be NP-hard on paths [16].

More formally, for the RSA problem, we are given an optical network G and a set $\mathscr{D}$ of end-to-end traffic demands where each demand is specified by a pair u, v of distinct nodes in G and the number d_{uv} of required frequency slots. The routing part of the RSA problem consists of selecting a route through G from u to v, i.e. a (u, v)-path P_{uv} in G, for each such traffic demand. The spectrum assignment can then be interpreted as an *interval coloring* of the *edge intersection graph* $I(\mathscr{P})$ of the set $\mathscr{P}$ of selected paths:

- Each path $P_{uv} \in \mathscr{P}$ becomes a node of $I(\mathscr{P})$ and two nodes are joined by an edge if the corresponding paths in G are in conflict as they share an edge (notice that we do not care whether they share nodes).
- Any interval coloring in this graph $I(\mathscr{P})$ weighted with the demands d_{uv} correctly solves the spectrum assignment: we assign a frequency interval of d_{uv} consecutive frequency slots (*spectrum contiguity*) to every node of $I(\mathscr{P})$ (and, thus, to every path $P_{uv} \in \mathscr{P}$ (*spectrum continuity*)) in such a way that the intervals of adjacent nodes are disjoint (*non-overlapping spectrum*).

Let $\mathbf{d} \in \mathbb{Z}_+^{|\mathscr{D}|}$ be the vector whose entries d_{uv} are the slot requirements associated with the demands between pairs u, v of nodes in $\mathscr{D}$. The *interval chromatic number* $\chi_I(I(\mathscr{P}), \mathbf{d})$ is the minimum spectrum width such that $I(\mathscr{P})$ weighted with the vector $\mathbf{d}$ of traffic demands d_{uv} for each path P_{uv} has a proper interval coloring. Given G and $\mathscr{D}$, the minimum spectrum width of any solution of the RSA problem, thus, equals

$$\chi_I(G, \mathscr{D}) = \min\{\chi_I(I(\mathscr{P}), \mathbf{d}) : \mathscr{P} \text{ possible routing of demands } \mathscr{D} \text{ in } G\}.$$

For each routing $\mathscr{P}$, the *weighted clique number* $\omega(I(\mathscr{P}), \mathbf{d})$, also taking the traffic demands d_{uv} as weights, equals the weight of a heaviest clique in $I(\mathscr{P})$ and is a natural lower bound for $\chi_I(I(\mathscr{P}), \mathbf{d})$ (as clearly the intervals of all nodes in a clique in $I(\mathscr{P})$ have to be disjoint by construction of $I(\mathscr{P})$). However, it is not always possible to find a solution with this lower bound as spectrum width, as weighted clique number and interval chromatic number are not always equal.

Graphs where weighted clique number and interval chromatic number coincide for all possible non-negative integral weights are called *superperfect*.

A graph is *perfect* if and only if this holds for every (0, 1)-weighting **d** of its nodes. According to a characterization achieved by Chudnovsky et al. [4], perfect graphs are precisely the graphs without chordless cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, or their complements, the *odd antiholes* $\overline{C}_{2k+1}$ (the complement $\overline{G}$ has the same nodes as G, but two nodes are adjacent in $\overline{G}$ if and only if they are non-adjacent in G).

In particular, every superperfect graph is perfect.

On the other hand, comparability graphs form a subclass of superperfect graphs. A graph $G = (V, E)$ is *comparability* if and only if there exists a partial order $\mathscr{O}$ on $V \times V$ such that $uv \in E$ if and only if u and v are comparable w.r.t. $\mathscr{O}$. Hoffman [12] proved that every comparability graph is superperfect. Gallai [6] characterized comparability graphs by giving a complete list of minimal non-comparability graphs, that are

- odd holes C_{2k+1} for $k \geq 2$ and antiholes $\overline{C}_n$ for $n \geq 6$,
- the graphs J_k and J'_k for $k \geq 2$ and the graphs J''_k for $k \geq 3$ (see Fig. 1),
- the complements of D_k for $k \geq 2$ and of E_k, F_k for $k \geq 1$ (see Fig. 2),
- the complements of $A_1, \ldots, A_{10}$ (see Fig. 3).

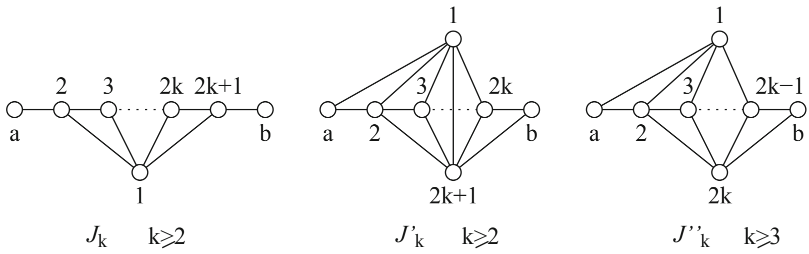

Fig. 1 Minimal non-comparability graphs: J_k, J'_k for $k \geq 2$ and J''_k for $k \geq 3$

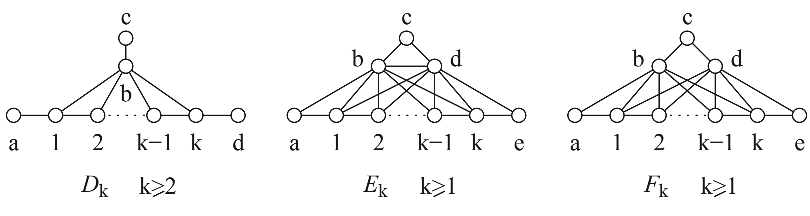

Fig. 2 Minimal non-comparability graphs: the complements of D_k, E_k, F_k

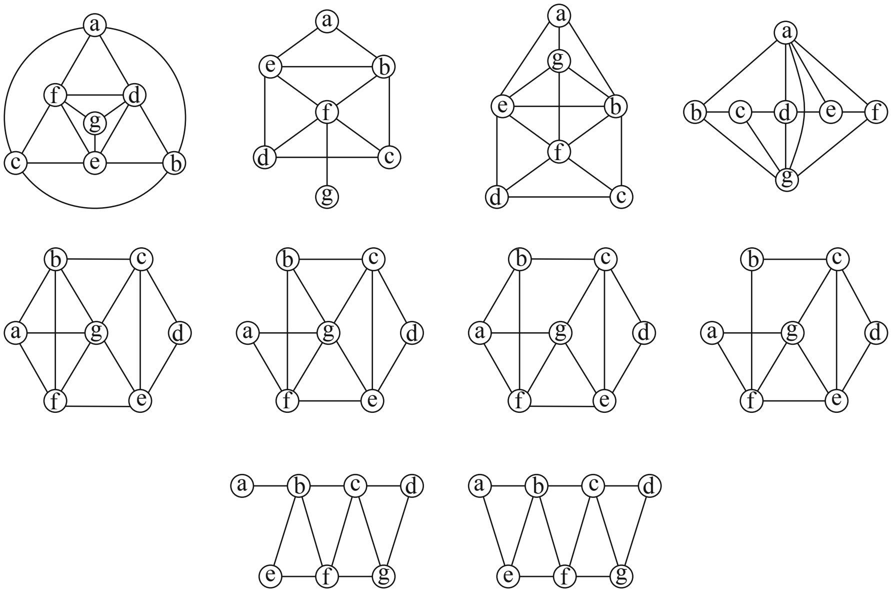

Fig. 3 Minimal non-comparability graphs: the graphs $\overline{A}_1, \ldots, \overline{A}_{10}$

As comparability graphs form a subclass of superperfect graphs, we have that every non-superperfect graph is in particular non-comparability, which raises the question which minimal non-comparability graphs are also minimal non-superperfect. Clearly, odd holes and odd antiholes are minimal non-superperfect (as they are minimal non-perfect). It has been shown by Golumbic [7] that $\overline{A}_1$, $\overline{D}_2$, $\overline{E}_1$, $\overline{E}_2$ and J_2 are non-superperfect, but that there are also superperfect non-comparability graphs such as e.g. even antiholes $\overline{C}_{2k}$ for $k \geq 3$.

Furthermore, Andreae showed in [1], that the graphs J''_k for $k \geq 3$ and the complements of $A_3, \ldots, A_{10}$ are superperfect, but that the graphs J_k for $k \geq 2$ and J'_k for $k \geq 3$ as well as the complements of D_k for $k \geq 2$ and of E_k, F_k for $k \geq 1$ are non-superperfect.

Note that Andreae wrongly determined $\overline{A}_2$ as superperfect which is, in fact, not the case (see Fig. 4 for a weight vector $\mathbf{d}$ and an optimal interval coloring showing that $\omega(\overline{A}_2, \mathbf{d}) = 5 < 6 = \chi_I(\overline{A}_2, \mathbf{d})$ holds). Moreover, Andreae wrongly determined J'_2 as non-superperfect which is, in fact, not the case:

Lemma 1 *J'_2 is a superperfect graph.*

Hence, all the previous results together imply the following:

Corollary 1 *The following minimal non-comparability graphs are also minimal non-superperfect:*

- *$\overline{A}_1$ and $\overline{A}_2$,*
- *odd holes C_{2k+1} and odd antiholes $\overline{C}_{2k+1}$ for $k \geq 2$,*

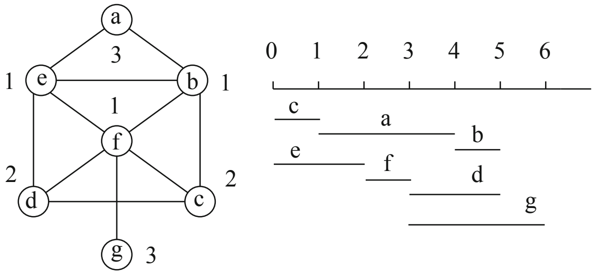

Fig. 4 The graph $\overline{A}_2$ together with node weights $\mathbf{d}$ and an optimal interval coloring showing $\omega(\overline{A}_2, \mathbf{d}) = 5 < 6 = \chi_I(\overline{A}_2, \mathbf{d})$

- *the graphs* J_k *for* $k \geq 2$ *and* J'_k *for* $k \geq 3$ *as well as*
- *the complements of* D_k *for* $k \geq 2$ *and of* E_k, F_k *for* $k \geq 1$.

Note that we have $\omega(G, \mathbf{1}) < \chi_I(G, \mathbf{1})$ with $\mathbf{1} = (1, \ldots, 1)$ if G is an odd hole or an odd antihole (as they are not perfect), whereas the other minimal non-comparability non-superperfect graphs are perfect and, thus, $\omega(G, \mathbf{d}) < \chi_I(G, \mathbf{d})$ is attained for some $\mathbf{d} \neq \mathbf{1}$ (see Fig. 4).

We examine, for different underlying networks G, the question whether or not there is a solution of the RSA problem with

$$\omega(G, \mathscr{D}) = \min\{\omega(I(\mathscr{P}), \mathbf{d}) : \mathscr{P} \text{ possible routing of demands } \mathscr{D} \text{ in } G\}$$

as spectrum width which depends on the occurrence of (minimal) non-superperfect graphs in the edge intersection graphs $I(\mathscr{P})$.

Note that for some networks G, the edge intersection graphs form well-studied graph classes: if G is a path (resp. tree, resp. cycle), then $I(\mathscr{P})$ is an *interval graph* (resp. *EPT graph*, resp. *circular-arc graph*). However, if G is a sufficiently large grid, then it is known by Golumbic et al. [9] that $I(\mathscr{P})$ can be *any* graph. Modern optical networks do not fall in any of these classes, but are 2-connected, sparse planar graphs with small maximum degree with a grid-like structure.

We first study the cases when the underlying network G is a path, a tree or a cycle (see Sects. 2–4). We recall results on interval graphs, EPT graphs and circular-arc graphs from [5, 8, 14] and then discuss which minimal non-comparability non-superperfect graphs can occur. In addition, we exhibit new examples of minimal non-superperfect graphs within these classes.

All of these non-superperfect graphs are inherited for the case when G is an optical network, and we give also representations as edge intersection graphs for the remaining minimal non-comparability non-superperfect graphs. In view of the result on edge intersection graphs of paths in a sufficiently large grid [9], we expect that any further minimal non-superperfect graph has such a representation and give some further new examples of such graphs.

To find new examples, we make use of the complete list of minimal non-comparability graphs found by Gallai [6] and the fact that any candidate for a new minimal non-superperfect graph can neither be imperfect nor a comparability graph. Thus, among the graphs with n nodes, the candidates of new minimal non-superperfect graphs are all graphs that are

- perfect (i.e. do not contain odd holes or odd antiholes),
- do not contain any minimal non-superperfect graph with $\leq n$ nodes,
- contain a minimal non-comparability superperfect graph with $< n$ nodes.

We close with some concluding remarks and open problems.

2 If the Network Is a Path

If the underlying optical network is a path P, then there exists exactly one (u, v)-path P_{uv} in P for every traffic demand between a pair u, v of nodes. Hence, if P is a path, then $\mathscr{P}$ and $I(\mathscr{P})$ are uniquely determined for any set of end-to-end traffic demands, and the RSA problem reduces to the spectrum assignment part. The edge intersection graph $I(\mathscr{P})$ of the (unique) routing $\mathscr{P}$ of the demands is an *interval graph* (i.e. the intersection graph of intervals in a line, here represented as subpaths of a path).

Interval graphs are known to be perfect by Berge [2]. In order to examine which minimal non-comparability non-superperfect graphs are interval graphs, we rely on a characterization of minimal non-interval graphs from [14].

A graph is *triangulated* if it does not have holes C_k with $k \geq 4$ as induced subgraph. Interval graphs are triangulated [11] hence all holes are in particular minimal non-interval graphs.

Theorem 1 *If $\mathscr{P}$ is a set of paths in a* path*, then $I(\mathscr{P})$ is an* interval graph *and can contain the graphs J_k for all $k \geq 2$, J'_k for all $k \geq 3$ and $\overline{E}_2$, but none of the other minimal non-comparability non-superperfect graphs.*

This implies that edge intersection graphs of paths in a path are not necessarily superperfect.

We next briefly discuss which further minimal non-superperfect graphs can be interval graphs. Recall that all of them have to contain a minimal non-comparability superperfect graph as proper induced subgraph. We observe that any further minimal non-superperfect interval graph can contain

- no even antihole $\overline{C}_{2k}$ for $k \geq 3$ (as they all contain a C_4 induced by $1, 2, 4, 5$),
- none of the graphs J''_k for all $k \geq 3$ (as they all contain a C_4 induced by $1, 2, 2k, 2k-1$),
- none of the graphs $\overline{A}_3, \dots \overline{A}_8$ (as they all contain a C_4, see Fig. 3),

but only $\overline{A}_9$, $\overline{A}_{10}$ and J'_2. However, there is no example of a minimal non-superperfect interval graph containing $\overline{A}_9$, $\overline{A}_{10}$ or J'_2 known yet.

3 If the Network Is a Tree

If the underlying network G is a tree, then there exists also exactly one (u, v)-path P_{uv} in G for every traffic demand between a pair u, v of nodes. Hence, if G is a tree, then $\mathscr{P}$ and $I(\mathscr{P})$ are uniquely determined for any set $\mathscr{D}$ of end-to-end traffic demands, and the RSA problem again reduces to the spectrum assignment part. The resulting edge intersection graph $I(\mathscr{P})$ belongs to the class of EPT graphs studied in [8]. We recall results from [8] on holes in EPT graphs and examine which minimal non-superperfect graphs can occur in such graphs.

It is known from [8] that EPT graphs are not necessarily perfect as they can contain odd holes. More precisely, Golumbic and Jamison showed the following:

Theorem 2 (Golumbic and Jamison [8]) *If the edge intersection graph $I(\mathscr{P})$ of a collection $\mathscr{P}$ of paths in a tree T contains a hole C_k with $k \geq 4$, then T contains a star $K_{1,k}$ with nodes $b, a_1, \ldots, a_k$ and there are k paths $P_1, \ldots, P_k$ in $\mathscr{P}$ such that P_i precisely contains the edges ba_i and ba_{i+1} of this star (where indices are taken modulo k).*

Figure 5 illustrates the case of $C_5 = I(\mathscr{P})$. From the above result, Golumbic and Jamison deduced the possible adjacencies of a hole which further implies that several graphs cannot occur as induced subgraphs of EPT graphs, including the complement of the P_6 and the two graphs G_1 and G_2 shown in Fig. 6.

That $\overline{P}_6$ is a non-EPT graph shows particularly that no antihole $\overline{C}_k$ for $k \geq 7$ can occur in such graphs. This implies:

Theorem 3 (Golumbic and Jamison [8]) *An EPT graph is perfect if and only if it does not contain an odd hole.*

With view on Theorem 2, this is clearly the case when the underlying tree has maximum degree 4, as noted in [8].

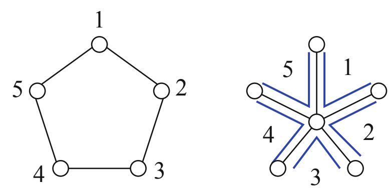

Fig. 5 The odd hole $C_5 = I(\mathscr{P})$ with $\mathscr{P}$ in a star

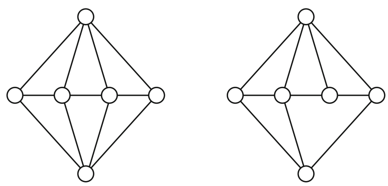
Fig. 6 The non-EPT graphs G_1 and G_2

Based on the above results, we further examine which minimal non-comparability non-superperfect graphs can occur in edge intersection graphs of paths in a tree:

Theorem 4 *If $\mathscr{P}$ is a set of paths in a* tree*, then the EPT graph $I(\mathscr{P})$ can contain $\overline{A}_1$, $\overline{A}_2$ and*

- *odd holes C_{2k+1} for $k \geq 2$, but no odd antiholes $\overline{C}_{2k+1}$ for $k \geq 3$,*
- *the graphs J_k for all $k \geq 2$ and J'_k for all $k \geq 3$,*
- *$\overline{D}_2$, $\overline{D}_3$, $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$, $\overline{F}_1$, $\overline{F}_2$, $\overline{F}_3$, but none of $\overline{D}_k$, $\overline{E}_k$, $\overline{F}_k$ for $k \geq 4$.*

This implies that perfect EPT graphs are not necessarily superperfect.

We next briefly discuss which further minimal non-superperfect graphs can be EPT graphs. Recall that all of them have to be perfect and have to contain a minimal non-comparability superperfect graph as proper induced subgraph. Among the minimal non-comparability superperfect graphs, the following are EPT graphs: J'_2 and

- $\overline{C}_6$ but no even antihole $\overline{C}_{2k}$ for $k \geq 4$ (as they all contain $\overline{P}_6$) by Golumbic and Jamison [8],
- none of the graphs J''_k for all $k \geq 3$ (as they all contain G_1 induced by the nodes $1, 2, 3, 4, 5, 2k$),
- the graphs $\overline{A}_3, \ldots \overline{A}_6$, $\overline{A}_8, \ldots \overline{A}_{10}$ (but not $\overline{A}_7$ as it has a G_2).

Hence, any minimal non-superperfect EPT graph not being minimal non-comparability has to contain one of $\overline{C}_6$, $\overline{A}_3, \ldots \overline{A}_6$, $\overline{A}_8, \ldots \overline{A}_{10}$ or J'_2 as proper induced subgraph. Figure 7 shows one example containing $\overline{A}_{10}$: it is non-superperfect (due to the indicated weight vector **d** causing a gap between weighted clique and interval chromatic number), it is minimal (as it does not have a non-comparability subgraph different from $\overline{A}_{10}$), it is an EPT graph (see the according path representation). However, note that the graph is not an interval graph (as it contains a C_4 induced by a, e, f, h).

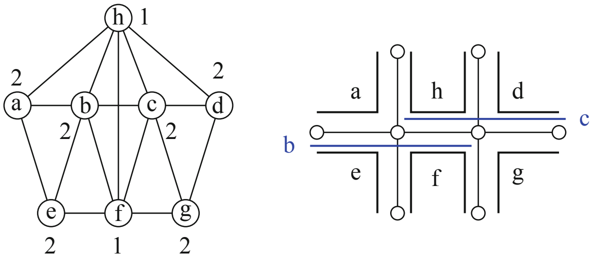

Fig. 7 A minimal non-superperfect EPT graph containing $\overline{A}_{10}$

4 If the Network Is a Cycle

If the underlying optical network is a cycle C, then there exist exactly two (u, v)-paths P_{uv} in C for every traffic demand between a pair u, v of nodes. Hence, if C is a cycle, then the number of possible routings $\mathscr{P}$ (and their edge intersection graphs $I(\mathscr{P})$) is exponential in the number $|\mathscr{D}|$ of end-to-end traffic demands, namely $2^{|\mathscr{D}|}$.

Moreover, the edge intersection graphs of paths in a cycle are clearly *circular-arc graphs* (that are the intersection graphs of arcs in a cycle, here represented as paths in a hole C_n). It is well-known that circular-arc graphs are not necessarily perfect as they can contain both odd holes and odd antiholes, see e.g. [5] and Fig. 8 for illustration.

In order to address the question which of the studied perfect minimal non-comparability, non-superperfect graphs can occur in circular-arc graphs, we either present according path collections for the affirmative cases or exhibit a minimal non-circular-arc graph otherwise. For that, we first show the following:

Lemma 2 *$\overline{E}_3$ is a minimal non-circular-arc graph.*

Making use of the above facts, we can prove:

Theorem 5 *If $\mathscr{P}$ is a set of paths in a* cycle*, then the* circular-arc graph *$I(\mathscr{P})$ can contain $\overline{A}_1$ but not $\overline{A}_2$,*

- *all odd holes C_{2k+1} and odd antiholes $\overline{C}_{2k+1}$ for $k \geq 2$,*
- *the graphs J_k for all $k \geq 2$ and J'_k for all $k \geq 3$,*
- *$\overline{D}_2$, $\overline{D}_3$, $\overline{D}_4$, but not the graphs $\overline{D}_k$ for $k \geq 5$,*
- *$\overline{E}_1$ and $\overline{E}_2$, but not the graphs $\overline{E}_k$ for $k \geq 3$,*
- *$\overline{F}_2$, but not $\overline{F}_1$ neither the graphs $\overline{F}_k$ for $k \geq 3$.*

We next discuss which further minimal non-superperfect graphs can be circular-arc graphs. For that, we first show the following:

Lemma 3 *J''_3 is a minimal non-circular-arc graph.*

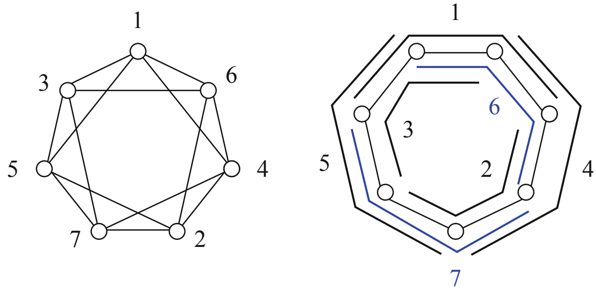

Fig. 8 The odd antihole $\overline{C}_7 = I(\mathscr{P})$ with $\mathscr{P}$ in a cycle

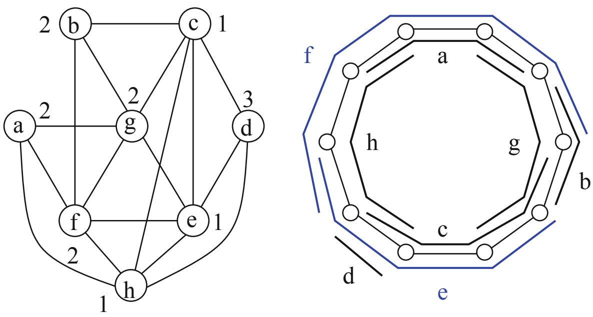

Fig. 9 A minimal non-superperfect circular-arc graph containing $\overline{A}_6$

Remark 1 Note that $\overline{E}_3$ and J_3'' are, to the best of our knowledge, *new* examples of minimal non-circular-arc graphs (see e.g. the results on circular-arc graphs surveyed in [5]).

Recall that every further minimal non-superperfect graph has to be perfect and has to contain a minimal non-comparability superperfect proper induced subgraph. Among the perfect minimal non-comparability superperfect graphs, the following are circular-arc graphs: J_2' but

- no even antihole $\overline{C}_{2k}$ for $k \geq 3$ ("folklore"),
- neither J_3'' (by Lemma 3) nor the graphs J_k'' for all $k \geq 4$ (as they all contain the well-known minimal non-circular-arc graph $K_{2,3}$ induced by the nodes $1, 2, 4, 6, 2k$),
- all of the graphs $\overline{A}_3, \dots \overline{A}_{10}$.

Hence, any minimal non-superperfect circular-arc graph not being minimal non-comparability has to contain one of $\overline{A}_3, \dots \overline{A}_{10}$ or J_2' as proper induced subgraph. Figure 9 shows one example containing $\overline{A}_6$: it is non-superperfect (due to the indicated weight vector **d** causing a gap between weighted clique and interval chromatic number), it is minimal (as it does not have a non-comparability subgraph different from $\overline{A}_6$), it is a circular-arc graph (see the according path representation). However, note that the graph is not an interval graph (as $\overline{A}_6$ is not).

5 The General Case

Modern optical networks have clearly not a tree-like structure neither are just cycles due to survivability aspects concerning node or edge failures in the network G, see e.g. [13]. Instead, today's optical networks are 2-connected, sparse planar graphs with small maximum degree and have more a grid-like structure, see as example Fig. 10 showing the Telefónica network of Spain taken from [15].

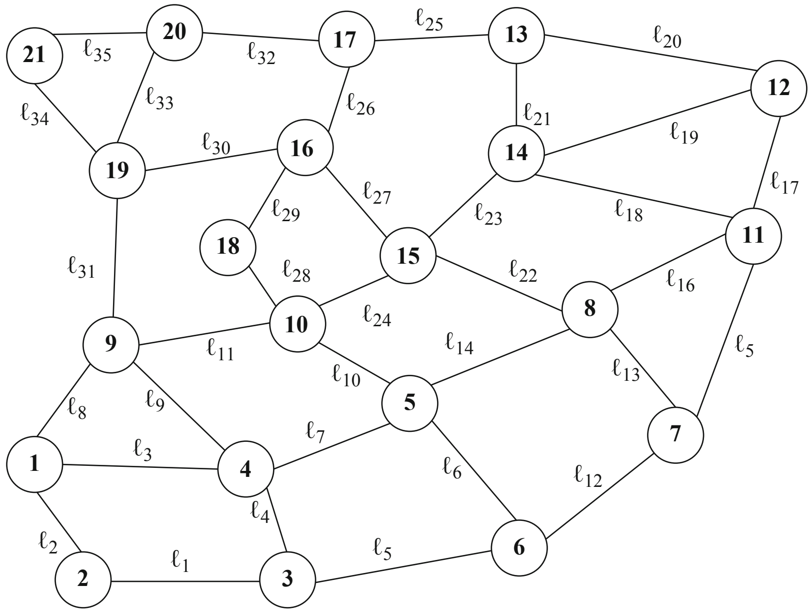

Fig. 10 The Telefónica network of Spain from [15]

We first wonder which minimal non-comparability non-superperfect graphs can occur in edge intersection graphs of paths in such networks G and can show:

Theorem 6 *All minimal non-comparability non-superperfect graphs can occur in edge intersection graphs $I(\mathscr{P})$ of sets $\mathscr{P}$ of paths in optical networks G.*

In addition, there are further minimal non-superperfect graphs in edge intersection graphs of paths in networks.

Figure 11 shows one example containing $\overline{A}_7$: it is non-superperfect (due to the indicated weight vector $\mathbf{d}$ causing a gap between weighted clique and interval chromatic number), it is minimal (as removing node g or h yields $\overline{A}_7$, and removing any other node results in a comparability graph), and it has a path representation in

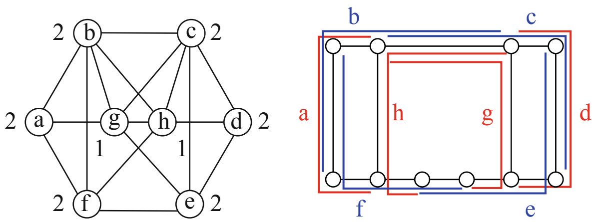

Fig. 11 A minimal non-superperfect graph containing $\overline{A}_7$ and a path representation in a sparse planar graph

a sparse planar graph. However, note that the graph is neither an EPT graph (as $\overline{A}_7$ is not), nor a circular-arc graph (as nodes a, e, f, g, h induce a $K_{2,3}$).

We expect that *all* minimal non-superperfect graphs can occur in edge intersection graphs of paths in networks, as soon as the networks G satisfy minimal survivability conditions concerning edge or node failures.

6 Concluding Remarks

From the fact that both, EPT graphs and circular-arc graphs, are not necessarily perfect, we notice that also edge intersection graphs of paths in networks are not necessarily perfect and, thus, also not necessarily superperfect. If we restrict the networks to paths, then $I(\mathscr{P})$ is an interval graph, but still not necessarily superperfect (as the minimal non-superperfect graphs J_k for all $k \geq 2$, J'_k for all $k \geq 3$ and $\overline{E}_1$ can occur). This is in accordance with the fact that the SA problem has been showed to be NP-hard on paths [16].

Hence, in all networks, it depends on the weights $\mathbf{d}$ induced by the traffic demands whether there is a gap between the weighted clique number $\omega(I(\mathscr{P}), \mathbf{d})$ and the interval chromatic number $\chi_I(I(\mathscr{P}), \mathbf{d})$. To determine the size of this gap, we propose to extend the concept of χ-binding functions introduced in [10] for usual coloring to interval coloring in weighted graphs, that is, to χ_I-binding functions f with

$$\chi_I(I(\mathscr{P}), \mathbf{d}) \leq f(\omega(I(\mathscr{P}), \mathbf{d}))$$

for edge intersection graphs $I(\mathscr{P})$ in a certain class of networks and all possible non-negative integral weights $\mathbf{d}$.

It is clearly of interest to study such χ_I-binding functions for different families of minimal non-superperfect graphs and to identify a hierarchy of graph classes between trees respectively cycles and sparse planar graphs resembling the structure of modern optical networks in terms of the gap between $\omega_I(I(\mathscr{P}), \mathbf{d})$ and $\chi_I(I(\mathscr{P}), \mathbf{d})$.

Furthermore, in networks different from trees, the routing part of the RSA problem is crucial and raises the question whether it is possible to select the routes in $\mathscr{P}$ in such a way that neither non-superperfect subgraphs nor unnecessarily large weighted cliques occur in $I(\mathscr{P})$.

Finally, giving a complete list of minimal non-superperfect graphs is an open problem, so that our future work comprises to find more minimal non-superperfect graphs and to examine the here addressed questions for them.

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