Improved Bounds on the Span of $L(1, 2)$-edge Labeling of Some Infinite Regular Grids

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Abstract For two given non-negative integers h and k, an $L(h, k)$-edge labeling of a graph G is the assignment of labels $\{0, 1, \cdots, n\}$ to the edges so that two edges having a common vertex are labeled with difference at least h and two edges not having any common vertex but having a common edge connecting them are labeled with difference at least k. The span $\lambda'_{h,k}(G)$ is the minimum n such that G admits an $L(h, k)$-edge labeling. Here our main focus is on finding $\lambda'_{h,k}(G)$ for $L(1, 2)$-edge labeling of infinite regular hexagonal (T_3), square (T_4) and triangular (T_6) grids. It was known that $7 \leq \lambda'_{h,k}(T_3) \leq 8$, $10 \leq \lambda'_{h,k}(T_4) \leq 11$ and $16 \leq \lambda'_{h,k}(T_6) \leq 20$. Here we have shown that $\lambda'_{h,k}(T_3) \leq 7$, $\lambda'_{h,k}(T_4) \geq 11$ and $\lambda'_{h,k}(T_6) \geq 19$.

Keywords $L(1, 2)$-edge labelling · Bounds · Minimum span · Infinite regular grids

1 Introduction

Channel assignment problem (CAP) is one of the fundamental problems in wireless communication where frequency channels are assigned to transmitters such that interference can not occur. The objective of the CAP is to minimize the span of frequency spectrum. In 1980, Hale [6] first formulated the CAP as a classical vertex coloring problem. Later on, in 1988 Roberts [9] introduced $L(h, k)$-vertex labeling as defined below:

Definition 1 For two non-negative integers h and k, an $L(h, k)$-vertex labeling of a graph $G(V, E)$ is a function $\mathbf{f} : V \rightarrow \{0, 1, \cdots, n\}, \forall v \in V$ such that $|\mathbf{f}(u) - \mathbf{f}(v)| \geq h$ when $d(u, v) = 1$ and $|\mathbf{f}(u) - \mathbf{f}(v)| \geq k$ when $d(u, v) = 2$. For two vertices u and v, the distance, $d(u, v)$ is k' if at least k' edges are required to connect u and v.

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The *span* $\lambda_{h,k}(G)$ of $L(h,k)$-vertex labeling is the minimum n such that G admits an $L(h,k)$-vertex labeling. In 1992 Griggs and Yeh [5] extended the concept of $L(h,k)$ labeling by introducing $L(k_1, k_2, \cdots, k_l)$-vertex labeling with separation $\{k_1, k_2, \cdots, k_l\}$ for $\{1, 2, \cdots, l\}$ distant vertices and their main focus was on $L(h,k)$-vertex labeling for a special case $h = 2$, $k = 1$. In 2007, Griggs and Jin [4] studied $L(h,k)$-edge labeling, which can be formally defined as:

Definition 2 For two non-negative integers h and k, an $L(h,k)$-edge labeling of a graph $G(V,E)$ is a function $\mathbf{f}' : E \rightarrow \{0, 1, \cdots, n\}, \forall e \in E$ such that $|\mathbf{f}'(e_1) - \mathbf{f}'(e_2)| \geq h$ when $d(e_1, e_2) = 1$ and $|\mathbf{f}'(e_1) - \mathbf{f}'(e_2)| \geq k$ when $d(e_1, e_2) = 2$. Here, for any two edges e_1 and e_2, the distance $d(e_1, e_1)$ is k' if at least $(k' - 1)$ edges are required to connect e_1 and e_2.

Like $L(h,k)$-vertex labeling, the *span* $\lambda'_{h,k}(G)$ of $L(h,k)$-edge labeling is the minimum n such that G admits an $L(h,k)$-edge labeling. In 2011, Calamoneri did a rigorous survey [1] on both vertex and edge labeling problems. Authors in [2, 3, 7, 8] have studied $L(h,k)$-edge labeling of regular infinite hexagonal (T_3), square (T_4) and triangular (T_6) grids for the special case of $h = 1$ and $k = 2$. They obtained some upper and lower bounds on $\lambda'_{1,2}(G)$ for T_3, T_4 and T_6 with a gap between them. In this paper, we improve some of these gaps.

Given a graph $G(V,E)$, its *line graph* $L(G)(V', E')$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ have an edge if and only if their corresponding edges share a common vertex in G. It is well-known that if G is d-regular then $L(G)$ is $2(d-1)$-regular. Figure 1 shows T_3, T_4, $L(T_3)$, $L(T_4)$ and T_6. It is also well-known that edge labeling of G is equivalent to vertex labeling of $L(G)$. In our approach, instead of $L(1,2)$-edge labeling of T_3 and T_4, we use $L(1,2)$-vertex labeling of $L(T_3)$ and $L(T_4)$. Note that $L(T_6)$ is 10-regular. Because of this high degree, we consider $L(1,2)$-edge labeling of T_6 directly. Our results on $\lambda'_{1,2}(G)$ for T_3, T_4 and T_6 are stated in Table 1. In this table, $a - b$ represents that $a \leq \lambda'_{1,2}(G) \leq b$. Here, we use the term 'coloring' and 'labeling' interchangeably.

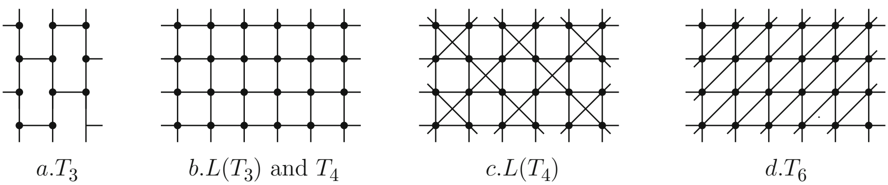


Fig. 1 T_3, T_4, their line graphs and T_6

Table 1 The main results

	T_3		T_4		T_6	
	Known	Ours	Known	Ours	Known	Ours
$\lambda'_{1,2}(G)$	7-8 [7]	7-7	10-11 [7]	11-11	16-20 [2]	19-20

2 Results

2.1 Hexagonal Grid

Let us consider the induced subgraph G_S of $L(T_3)$ as shown in Fig. 2a, where all vertices are at mutual distance at most three. It is clear that $\lambda_{1,2}(L(T_3)) \geq \lambda_{1,2}(G_S)$. A color can be reused at a pair of vertices at mutual distance three apart in G_S. But we observe that if any color is reused at distance three in G_S, then there exists a color which remains unused in G_S. Thus there is no such benefit of reusing a color over using all different colors in G_S. This motivates us to consider reusing a color at distance four only keeping all colors distinct at G_S. We show in Theorem 1 that such a coloring of $L(T_3)$ exists which uses colors from 0 to 7 only.

Theorem 1 $\lambda'_{1,2}(T_3) = 7$.

Proof Consider the coloring function g of vertices $v = (x, y)$ as $g(v)_{(x,y)} = (x + 5y)\ mod\ 8$. Here coordinates (x, y) of a vertex v can be computed from the origin $O(0, 0)$ as shown in Fig. 2b. The minimum and maximum color used here are 0 and 7 respectively. It can also be verified that g satisfies the $L(1, 2)$-vertex labeling requirements of $L(T_3)$. Hence $\lambda_{1,2}(L(T_3)) \leq 7$. It has been shown in [7] that $\lambda_{1,2}(L(T_3)) \geq 7$. Hence $\lambda'_{1,2}(T_3) = \lambda_{1,2}(L(T_3)) = 7$. In Fig. 2b, an $L(1, 2)$-vertex labeling of $L(T_3)$ has been shown. □

It is evident that $\lambda'_{1,2,1}(T_3) \geq \lambda'_{1,2}(T_3) = 7$. In the coloring function g stated above, observe that no vertices at distance three have the same color in $L(T_3)$. Hence

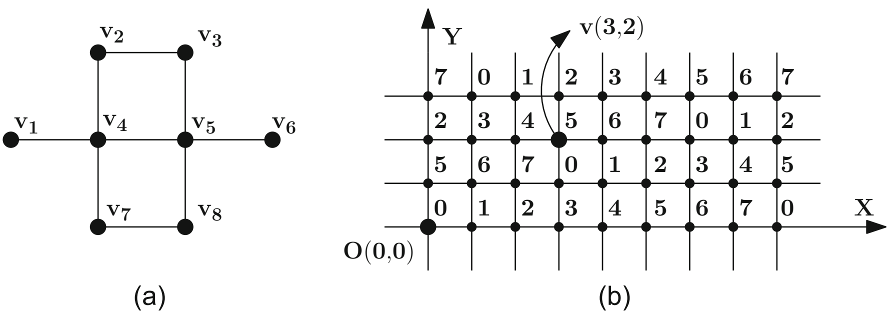


Fig. 2 (**a**) Sub graph G_s of $L(T_3)$. (**b**) A feasible $L(1, 2)$-labeling of $L(T_3)$

g also satisfies the $L(1, 2, 1)$-edge labeling requirements for T_3. So, $\lambda'_{1,2,1}(T_3) \leq 7$. Hence we have the following result.

Corollary 1 $\lambda'_{1,2,1}(T_3) = 7$.

2.2 Square Grid

Let us consider the induced subgraph G of $L(T_4)$ as shown in Fig. 3 where all vertices are at mutual distance at most three. Let $S_1 = \{a, b\}$, $S_2 = \{k, l\}$, $S_3 = \{c, g\}$, $S_4 = \{f, j\}$ and $S_5 = \{d, e, h, i\}$.

Definition 3 The set of vertices in S_5 are termed as **central vertices** in G.

Definition 4 The set of vertices in $S_1 \cup S_2 \cup S_3 \cup S_4$ are termed as **peripheral vertices** in G.

Now we have the following observations in G. Here the color of vertex a is denoted by $\mathbf{f}(a)$.

Observation 1 *If colors of vertices of G are all distinct then* $\lambda_{1,2}(G) \geq 11$.

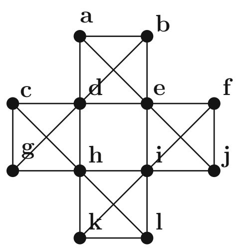
Proof As G has 12 vertices, if all of them get distinct colors then $\lambda_{1,2}(G) \geq 11$. □

Observation 2 *No color can be used thrice in G. Colors used at the central vertices in S_5 cannot be reused in G. Colors used at the peripheral vertices in S_1 can be reused only at the peripheral vertices in S_2. Similarly, colors used at the peripheral vertices in S_3 can be reused only at the peripheral vertices in S_4.*

Proof No three vertices are mutually distant three apart. Hence no color can be used thrice in G. For any central vertex in S_5 there does not exist any vertex in G which is distance three apart from it. So colors used in the central vertices in S_5 cannot be reused in G. For all peripheral vertices in $S_1 \cup S_2$, $d(x, y) = 3$ only when $x \in S_1$ and $y \in S_2$. Hence color used at peripheral vertex in S_1 can only be reused in S_2. Similarly, color used at peripheral vertex in S_3 can only be reused in S_4. □

Observation 3 *If* $\mathbf{f}(x) = \mathbf{f}(y) = \mathbf{c}$ *where* $x \in S_1$ *and* $y \in S_2$ *then either* $\mathbf{c} \pm 1$ *is to be used in* $(S_1 \cup S_2) \setminus \{x, y\}$ *or it should remain unused in G. Similarly, if* $\mathbf{f}(x) = \mathbf{f}(y) = \mathbf{c}$ *where* $x \in S_3$ *and* $y \in S_4$ *then either* $\mathbf{c} \pm 1$ *is to be used in* $(S_3 \cup S_4) \setminus \{x, y\}$ *or it should remain unused in G.*

Fig. 3 Sub graph G of $L(T_4)$



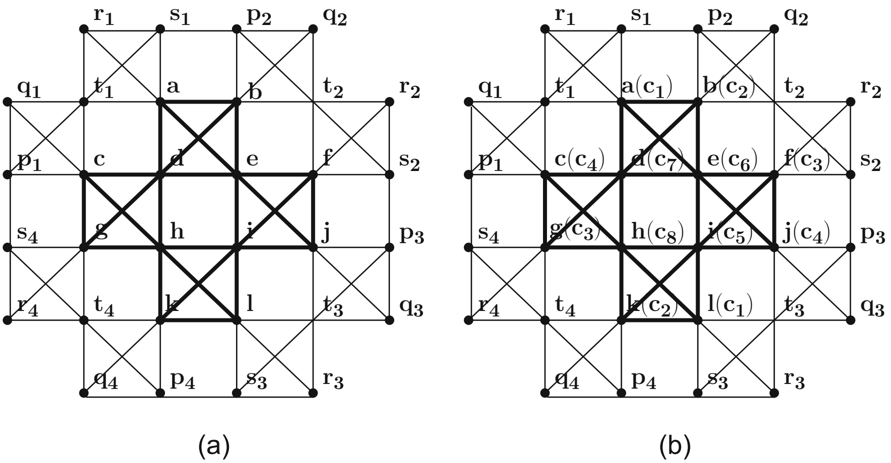


Fig. 4 (**a**) The subgraph G_1. (**b**) A feasible $L(1, 2)$-labeling of G

Proof Note that for all vertices $z \in V(G) \setminus (S_1 \cup S_2)$, either $d(z, x) = 2$ or $d(z, y) = 2$, where $x \in S_1$ and $y \in S_2$. Hence $\mathbf{c} \pm 1$ cannot be used in $V(G) \setminus (S_1 \cup S_2)$. So $\mathbf{c} \pm 1$ can only be used in $(S_1 \cup S_2) \setminus \{x, y\}$ or it should remain unused in G. Similarly, if $\mathbf{f}(x) = \mathbf{f}(y) = \mathbf{c}$, where $x \in S_3$ and $y \in S_4$, then $\mathbf{c} \pm 1$ can only be used in $(S_3 \cup S_4) \setminus \{x, y\}$ or it should remain unused in G. □

Observation 4 *Let* $\mathbf{f}(x) = \mathbf{f}(y) = \mathbf{c}$ *where* $x \in S_1$ *and* $y \in S_2$. *If* $|\mathbf{f}(x) - \mathbf{f}(x')| \geq 2$, *where* $x' \in S_1 \setminus \{x\}$, *then one of* $\mathbf{c} \pm 1$ *must remain unused in* G. *Similarly if* $|\mathbf{f}(y) - \mathbf{f}(y')| \geq 2$, *where* $y' \in S_2 \setminus \{y\}$, *then one of* $\mathbf{c} \pm 1$ *must remain unused in* G. *Similar facts hold when* $x \in S_3$, $x' \in S_3 \setminus \{x\}$, $y \in S_4$ *and* $y' \in S_4 \setminus \{y\}$.

Proof Since $|\mathbf{f}(x) - \mathbf{f}(x')| \geq 2$, $\mathbf{f}(x') \neq \mathbf{c} \pm 1$. Hence from Observation 3, one of $\mathbf{c} \pm 1$ must remain unused in G. □

If no color is reused in G, then $\lambda_{1,2}(G) \geq 11$ from Observation 1. To make $\lambda_{1,2}(G) < 11$, at least one color must be reused in G. From Observation 2, there are at most 4 distinct pairs of peripheral vertices in G where a pair can have the same color. Now consider the subgraph G_1 of $L(T_4)$ as shown in Fig. 4a. Note that G_1 consists of 5 subgraphs G', G'_1, G'_2, G'_3 and G'_4 which all are isomorphic to G having central vertices $\{d, h, i, e\}$, $\{t_1, c, d, a\}$, $\{b, e, f, t_2\}$, $\{i, l, t_3, j\}$ and $\{g, t_4, k, h\}$ respectively. Based on the span requirements of coloring G_1, we derive the following theorem.

Theorem 2 $\lambda_{1,2}(L(T_4)) \geq \lambda_{1,2}(G_1) \geq 11$.

Proof

Case 1 When at most one pair of peripheral vertices use the same color in any sub graph of $L(T_4)$ isomorphic to G.

If no color is reused in G', then $\lambda_{1,2}(G') \geq 11$ from Observation 1. We now consider the case when exactly one pair reuse a color in G'. Without loss of generality, consider $\mathbf{f}(a) = \mathbf{f}(l) = c_1$. From Observation 3, $c_1 \pm 1$ can only be put in $\{b, k\}$. Let $\mathbf{f}(k) = c_1 - 1$ and $\mathbf{f}(b) = c_1 + 1$. We assume that $c_1 - 1$ is the minimum color. Let us consider $\mathbf{f}(d) = c_1 + n$ where $n \in \mathbb{N}$ and $n \geq 2$. From Observation 4, $x \in \{c_1, c_1 + n\}$ can be reused in G_2' only if one of $x \pm 1$ remains unused in G_2'. In either case, $\lambda_{1,2}(G_2') \geq 11$. So x cannot be reused in G_2'. Since $\mathbf{f}(a) = \mathbf{f}(l) = c_1$, $c_1 - 1$ can only be put in $\{r_2, s_2\}$ as vertex b is already colored and for all other vertices $z \in V(G_2') \setminus \{r_2, s_2\}$, either $d(z, a) = 2$ or $d(z, l) = 2$. Without loss of generality, let $\mathbf{f}(r_2) = c_1 - 1$. In that case, $c_1 + n \pm 1$ can only be put in $\{e, s_2\}$. Without loss of generality, let $\mathbf{f}(e) = c_1 + n - 1$ and $\mathbf{f}(s_2) = c_1 + n + 1$. Since $\mathbf{f}(a) = \mathbf{f}(l) = c_1$, $\mathbf{f}(i) \neq c_1 \pm 1$ and hence $|\mathbf{f}(l) - \mathbf{f}(i)| \geq 2$. Now if $|\mathbf{f}(d) - \mathbf{f}(c)| \geq 2$, then from Observation 4, one of $\mathbf{f}(c) \pm 1$, $\mathbf{f}(d) \pm 1$ and $\mathbf{f}(i) \pm 1$ remains unused in G_4' if $\mathbf{f}(c)$ or $\mathbf{f}(d)$ or $\mathbf{f}(i)$ is reused in G_4' respectively. In either case, this implies $\lambda_{1,2}(G_4') \geq 11$. So $|\mathbf{f}(d) - \mathbf{f}(c)| = 1$ and $\mathbf{f}(c) = c_1 + n + 1$. There are 5 more vertices $\{g, h, i, j, f\}$ in G' which are to be colored with 5 distinct colors. Hence at least color $c_1 + n + 6$ must be used. Observe that if $\mathbf{f}(f) = c_1 + n + 2$ then $|\mathbf{f}(e) - \mathbf{f}(f)| = 3$ and $|\mathbf{f}(k) - \mathbf{f}(h)| \geq 3$ implying $\lambda_{1,2}(G_3') \geq 11$ from Observation 4. As $d(s_2, i) = d(s_2, j) = 2$ and $\mathbf{f}(s_2) = c_1 + n + 1$, we get $\mathbf{f}(i) \neq c_1 + n + 2$ and $\mathbf{f}(j) \neq c_1 + n + 2$. Therefore, either $\mathbf{f}(g) = c_1 + n + 2$ or $\mathbf{f}(h) = c_1 + n + 2$. So, $\mathbf{f}(p_4) \neq c_1 + n + 1$ and $\mathbf{f}(q_4) \neq c_1 + n + 1$. In that case, $\mathbf{f}(p_4)$ and $\mathbf{f}(q_4)$ must be in $\{c_1 + n, c_1 + n - 1\}$ if color $c_1 + n$ is to be reused in G_4', otherwise, $\lambda_{1,2}(G_1) \geq 11$. As c_1 cannot be reused in G_4', either $\mathbf{f}(r_4) = c_1 + 1$ or $\mathbf{f}(s_4) = c_1 + 1$. Let $\mathbf{f}(r_4) = c_1 + 1$. When $n = 2$, $c_1 + n - 1 = c_1 + 1$ and when $n = 3$, $c_1 + n - 1 = c_1 + 2$. As $d(p_4, l) = d(p_4, r_4) = d(q_4, l) = d(q_4, r_4) = 2$, $\mathbf{f}(p_4), \mathbf{f}(q_4) \notin \{c_1 + 1, c_1 + 2\}$. So, $n \geq 4$ and hence $c_1 + n + 6 \geq c_1 + 10$. So at least 12 color are required in G_1 including $c_1 - 1$ and $c_1 + 10$. Hence $\lambda_{1,2}(G_2) \geq 11$.

Case 2 There exists at least one subgraph of $L(T_4)$ isomorphic to G where two pairs of peripheral vertices use a color each.

There are two different ways of reusing two colors in G'.

Case 2.1 First consider the case when $\mathbf{f}(a) = \mathbf{f}(l) = c_1$ and $\mathbf{f}(c) = \mathbf{f}(j) = c_2$. From Observation 3, $c_1 \pm 1$ and $c_2 \pm 1$ must be used in $\{b, k\}$ and $\{g, f\}$ respectively. From Observation 2, c_1 can only be reused in $\{r_2, s_2\}$ in G_2'. But $\mathbf{f}(r_2) \neq c_1$ and $\mathbf{f}(s_2) \neq c_1$ as $|\mathbf{f}(b) - c_1| = 1$ and $d(b, r_2) = d(b, s_2) = 2$. Again, from Observation 2, c_2 can only be reused in $\{p_2, q_2\}$. But $\mathbf{f}(p_2) \neq c_2$ and $\mathbf{f}(q_2) \neq c_2$ as $|\mathbf{f}(f) - c_2| = 1$ and $d(f, p_2) = d(f, q_2) = 2$. From Observation 3, if $\mathbf{f}(i)$ is to be reused in G_2', then $|\mathbf{f}(i) - c_2| = 1$. But $\mathbf{f}(i) \neq c_2 \pm 1$ as $d(c, i) = 2$ and $\mathbf{f}(c) = c_2$. If $\mathbf{f}(d)$ is to be reused in G_2', then $|\mathbf{f}(d) - c_1| = 1$. But $\mathbf{f}(d) \neq c_1 \pm 1$ as $d(d, l) = 2$ and $\mathbf{f}(l) = c_1$. Therefore, no color can be reused in G_2' and hence $\lambda_{1,2}(G_1) \geq 11$.

Case 2.2 Consider the case when $\mathbf{f}(a) = \mathbf{f}(l) = c_1$ and $\mathbf{f}(b) = \mathbf{f}(k) = c_2$. Without loss of generality, assume $c_2 > c_1$. From Observation 3, $c_1 \pm 1$ and $c_2 \pm 1$ must be used in $\{b, k\}$ and $\{a, l\}$ respectively. Even if we set $c_2 = c_1 + 1$, at least one of $c_1 - 1$ and $c_2 + 1$ must remain unused in G'. So the 8 vertices in $V(G') \setminus (\{a, l\} \cup \{b, k\})$ must get 8 distinct colors other than c_1 and c_2. So, $\lambda_{1,2}(G') \geq 10$. Note that $\lambda_{1,2}(G') = 10$ only if $c_2 = c_1 + 1$, c_1 is minimum color ($c_1 - 1$ does not exists) or c_2 is maximum color ($c_2 + 1$ does not exists). If both c_1 and c_2 are non-extreme color, then $\lambda_{1,2}(G') \geq 11$ and we are done. So, we consider $c_1 = 0$, $c_2 = c_1 + 1 = 1$ and $c_2 + 1 = 2$ as unused in G'. In that case, $\mathbf{f}(d) = x \geq 3$ and hence $|\mathbf{f}(d) - \mathbf{f}(a)| \geq 3$. From Observation 4, if x is reused in G'_2, then one of $x \pm 1$ cannot be used in G'_2. If only x is reused in G'_2, then $\lambda_{1,2}(G'_2) \geq 11$. If x and one of $\{\mathbf{f}(i), \mathbf{f}(j)\}$ are reused in G'_2, then from Case 2.1 above, $\lambda_{1,2}(G_1) \geq 11$. If x and both of $\{\mathbf{f}(i), \mathbf{f}(j)\}$ are reused in G'_2, from Case 3 below, we will see that $\lambda_{1,2}(G_1) \geq 11$. So, to keep $\lambda_{1,2}(G_1) < 11$, x should not be reused in G'_2. In that case, $x - 1$ must be used at one of $\{c, g, h, e\}$ in G'. Now arguing similarly as stated in case **1**, we can conclude that $x + 7$ must be used in G'_1 or G'_2. If $x = 3$, then $x - 1 = 2$ must be used in G' which is a contradiction, as 2 must remain unused in G'. Hence $x \geq 4$ implying $x + 7 = 11$. Hence $\lambda_{1,2}(G_1) \geq 11$.

Case 3 The exists at least one sub graph of $L(T_4)$ isomorphic to G where three pairs of peripheral vertices use a color each.

Without loss of generality, let us consider $\mathbf{f}(a) = \mathbf{f}(l) = c_1$, $\mathbf{f}(b) = \mathbf{f}(k) = c_2$ and $\mathbf{f}(c) = \mathbf{f}(j) = c_3$. From Observation 3, $c_1 \pm 1$ and $c_2 \pm 1$ must be used in $\{b, k\}$ and $\{a, l\}$ respectively. It can be observed that $\lambda_{1,2}(G') = 9$ only if $|c_1 - c_2| = 1$, $|c_3 - \mathbf{f}(g)| = 1$, $|c_3 - \mathbf{f}(f)| = 1$ and any one of $\{c_1, c_2\}$ is one extreme color. Without loss of generality consider $\mathbf{f}(g) = c_3 + 1$, $\mathbf{f}(f) = c_3 - 1$, c_1 is minimum color and $c_2 = c_1 + 1$. From Observation 2, c_3 can only be reused in $\{p_2, q_2\}$. But $\mathbf{f}(p_2) \neq c_3$ and $\mathbf{f}(q_2) \neq c_3$ as $\mathbf{f}(f) = c_3 - 1$ and $d(f, p_2) = d(f, q_2) = 2$. From Observation 3, if $\mathbf{f}(i)$ is to be reused in G'_2, then $|\mathbf{f}(i) - c_3| = 1$. But $\mathbf{f}(i) \neq c_3 \pm 1$ as $d(c, i) = 2$ and $\mathbf{f}(c) = c_3$. From Observation 2, c_1 can only be reused in $\{r_2, s_2\}$. But $\mathbf{f}(r_2) \neq c_1$ and $\mathbf{f}(s_2) \neq c_1$ as $\mathbf{f}(b) = c_2 = c_1 + 1$ and $d(b, r_2) = d(b, s_2) = 2$. Now arguing similarly as stated in case **2.2** above, we can conclude that $c_2 + 1$ must remain unused in G'. So, $(c_1 - \mathbf{f}(d)) \geq 3$. Now from Observation 4, if $\mathbf{f}(d)$ is reused in G'_2 then any one of $\mathbf{f}(d) \pm 1$ must remain unused in G'_2. Thus in G'_2, only $\mathbf{f}(d)$ can be reused by keeping one of $\mathbf{f}(d) \pm 1$ as unused. Hence $\lambda_{1,2}(G_1) \geq 11$. If we consider $\lambda_{1,2}(G') = 10$, the same result can be obtained by considering the corresponding G'_i, $1 \leq i \leq 4$.

Case 4 The exists at least one subgraph of $L(T_4)$ isomorphic to G where all four pairs of peripheral vertices use a color each.

Let us consider $\mathbf{f}(a) = \mathbf{f}(l) = c_1$, $\mathbf{f}(b) = \mathbf{f}(k) = c_2$, $\mathbf{f}(g) = \mathbf{f}(f) = c_3$ and $\mathbf{f}(c) = \mathbf{f}(j) = c_4$. From Observation 3, $c_1 \pm 1$, $c_2 \pm 1$, $c_3 \pm 1$ and $c_4 \pm 1$ must be used in $\{b, k\}$, $\{a, l\}$, $\{c, j\}$ and $\{g, f\}$ respectively. It can be observed that $\lambda_{1,2}(G') = 9$ only if $|c_1 - c_2| = 1$, $|c_3 - c_4| = 1$, one of $\{c_1, c_2\}$ is an extreme color and one of $\{c_3, c_4\}$ is the other extreme color. Without loss of generality, consider $c_1 = 0$, $c_4 = 9$, $c_2 = c_1 + 1 = 1$ and $c_3 = c_4 - 1 = 8$. So $c_2 + 1 = 2$ and $c_3 - 1 = 7$ are two distinct unused colors. Without loss of generality, consider $c_8 = c_2 + 2$, $c_5 = c_8 + 1$, $c_6 = c_5 + 1$ and $c_7 = c_6 + 1$. Since $|c_3 - c_4| = 1$ and $d(g, p_4) = d(g, q_4) = 2$, we get $\mathbf{f}(p_4) \neq c_4$ and $\mathbf{f}(q_4) \neq c_4$. Similarly, $\mathbf{f}(r_4) \neq c_1$ and $\mathbf{f}(s_4) \neq c_1$. From

Observation 2, c_5 can only be reused at $\{s_4, r_4\}$ in G'_4 but $\mathbf{f}(s_4) \neq c_5$ and $\mathbf{f}(r_4) \neq c_5$ as $d(h, s_4) = d(h, r_4) = 2$ and $\mathbf{f}(h) = c_8 = c_5 - 1$. Therefore, only c_7 can be reused in $\{p_4, q_4\}$. From Observation 4, one of $c_7 \pm 1$ must remain unused in G'_4 as $(c_4 - c_7) = 3$. Hence $\lambda_{1,2}(G_1) \geq 11$. For other assignment of central vertices and for the case when $\lambda_{1,2}(G') = 10$, we can obtain the same result by considering the corresponding G'_i, $1 \leq i \leq 4$.

□

2.3 Triangular Grid

Here we first define some notations.

For any vertex u, the set of vertices which are adjacent to u is called $N(u)$. Let us define $N(S) = \{\cup_{u \in S} N(u) : u \in S\}$. Let v be any vertex in T_6. Consider the subgraph $G_v(V, E)$ of T_6 centering v as shown in Fig. 5, where $V = N(v) \cup N(N(v))$ and E is set of all the edges which are incident to u where $u \in N(v)$. Observe that in G_v, for any two edges e_1 and e_2, $d(e_1, e_2) \leq 3$. Now we define the following three sets of edges S_1, S_2 and S_3:

S_1: Edges of G_v incident to v.

S_2: Edges of G_v whose both end points incident to e_1 and e_2 where $e_1,\ e_2 \in S_1$.

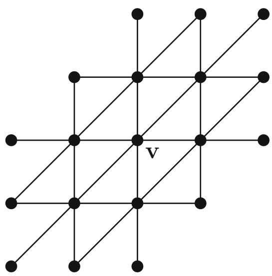
S_3: $E \setminus (S_1 \cup S_2)$.

Consider the 6-cycle, H_v formed with the edges of S_2 in G_v. We say e and e_1 as a pair of *opposite edges* in H_v if and only if $d(e, e_1) = 3$. This implies that the same color can be used at a pair of opposite edges in $L(1, 2)$-edge labeling. An edge $e(v, w)$ *covers* the set of edges E' if for every $e' \in E'$, $d(e, e') \leq 2$. This implies that a color used at e cannot be used at any edge $e' \in E'$ in $L(1, 2)$-edge labeling. Now we have the following lemmas.

Lemma 1 *If c be a color used to color an edge e in S_1, then c cannot be used in $E \setminus e$.*

Proof Since e is incident to v, for any other edge $e_1 \in E$, $d(e, e_1) \leq 2$. Hence $f'(e_1) \neq c$ for $L(1, 2)$-edge labeling, where $f'(e_1)$ denotes the color of e_1. □

Fig. 5 A subgraph G_v of T_6



Lemma 2 *If c be a color used to color an edge in S_1, then $c+1$ and $c-1$ both can be used at most once in G_v.*

Proof Let e be an edge in S_1 such that $f'(e) = c$. Since e is incident to v, for any other edge $e_1 \in E$, $d(e, e_1) \leq 2$. Let $S_e = \{e_1 : d(e, e_1) = 1\}$. For $L(1,2)$-edge labeling, $c+1$ can only be used in an edge e_1 in S_e. It can be noted that for any two edges $e_1, e_2 \in S_e$, $d(e_1, e_2) \leq 2$. Hence $c+1$ can be used at most once. Proof for $c-1$ can be done in similar manner. □

Lemma 3 *If c be a color used to color an edge e in S_2, then c can be used at most one edge in $E \setminus e$ in G_v.*

Proof Note that c cannot be used at any edge in S_1. Here c can be used at the *opposite edge* e_1 of e in S_2 or at an edge e_2 in S_3, which is adjacent to e_1. When c is used at e and e_1, then c cannot be used again in G_v as e and e_1 together cover all the edges of G_v. When c is used at e and e_2, c cannot be used again in G_v as e and e_2 together also cover all the edges of G_v. □

Lemma 4 *If c be a color used to color an edge e in S_2, then $c+1$ and $c-1$ both can be used at most twice in G_v.*

Proof Suppose e_1 be an edge colored with $c+1$. If e_1 is not adjacent to e then $d(e_1, e) = 3$. From statement of Lemma 3, it follows that there does not exist two edges along with e in G_v which are mutually distance 3 apart, otherwise c would have been used for three times. Hence $c+1$ can be used at most once.

When e_1 is adjacent to e, e_2 can be colored with $c+1$ if e_2 is at distance 3 apart from both e_1 and e. Again from the statement of Lemma 3, it follows that there does not exist two edges along with e in G_v which are mutually distance 3 apart, otherwise c would have been used for three times. So, $c+1$ can be used at most twice, one in one of the edges adjacent to e and other in one of the edges which are at distance 3 apart from e. Proof for $c-1$ can be done in similar manner. □

Lemma 5 *If c be a color used to color an edge e in S_3, then c can be used at most twice in $E \setminus e$.*

Proof It follows from Fig. 5 that exactly one end point of e is incident to a vertex in H_v. Note that for any walk through H_v, every third vertex is distance 2 apart. So edges incident to those vertices are distance 3 apart. Since the order of H_v is 6, there can be at most $6/2 = 3$ vertices which are mutually distance 2 apart. Hence c can be used thrice. □

Lemma 6 *If c be a color used to color an edge e in S_3, then $c+1$ and $c-1$ both can be used at most thrice in G_v.*

Proof We know that $c+1$ can be used at an edge adjacent to e. From Lemma 5 it is clear that c can be used at most thrice. So, $c+1$ can also be used at most thrice, where each such edge is adjacent to one of the three edges colored with c. It can be proved similarly for $c-1$. □

Lemma 7

i. To color the edges of S_1, at least 6 colors are required.
ii. To color the edges of S_2, at least 3 colors are required.
iii. To color the edges of S_3, at least 6 colors are required.

Proof

i. From Lemma 1, every edge of S_1 has an unique color. As there are 6 edges in S_1, 6 distinct colors are required here.
ii. In S_2, there are 3 pairs of *opposite edges*. Each pair of opposite edges requires at least one unique color. So at least 3 colors are required.
iii. A color can be used thrice in S_3 by Lemma 5. In S_3, there are 18 edges. So, at least 6 colors are required.

□

Theorem 3 *For any optimal labeling of G_v, 6 consecutive colors including either the minimum color or the maximum color must be used in S_1.*

Proof It is clear from Lemma 7.i that S_1 needs at least 6 colors to color its edges. From Lemma 2, note that if c be a color used in an edge of S_1 then both $c + 1$ and $c - 1$ can be used at most once in G_v. Whereas a color can be used twice in S_2 and thrice in S_3. Thus our aim should be to minimize the number of colors which can be used only once in G_v. This implies that consecutive colors should be used in S_1 for optimal coloring. If the minimum color (min) or the maximum color (max) is used in S_1 then further benefit can be achieve as $min - 1$ or $max + 1$ does not exist. Therefore, optimal span can be achieved only when the colors of S_1 are consecutive including either min or max. □

Theorem 4 *For any optimal labeling of G_v, 3 colors like $\{c, c + 2, c + 4\}$ have to be used twice each in S_2.*

Proof Let c be a color used in S_2. From Lemma 3, observe that c can be used at most twice in G_v. Also, no matter how many times c is used in S_2, it follows from Lemma 4 that both $c + 1$ and $c - 1$ can be used at most twice in G_v. Let $C_{S_2} = \{c, c + 1, c - 1 | \ \forall c \ used \ at \ S_2\}$. Note that a color can be used at most thrice in G_v. So our goal is to minimize $|C_{S_2}|$, where $|C_{S_2}|$ is the cardinality of set C_{S_2}. Observe that minimum 3 colors are required and maximum 6 colors can be used to color S_2. If 3 colors $\{c, c + 2, c + 4\}$ are used then $|C_{S_2}| \geq 6$, assuming one of them is an extreme color. If 6 consecutive colors are used then $|C_{S_2}| \geq 7$, assuming one of them is an extreme color. One can follow that in all the other cases $|C_{S_2}| > 7$. So for optimal coloring of G_v, 3 colors such as $\{c, c + 2, c + 4\}$ have to be used twice each in S_2. □

Lemma 8 *If three consecutive colors c, $c + 1$, $c + 2$ are used thrice each in S_3 then neither $c - 1$ nor $c + 3$ can be used in S_3.*

Proof Observe that there are exactly 2 sets of three alternating vertices in H_v where a color can be used thrice at edges incident to any set of alternating vertices. If $c - 1$

would have been used in S_3 then either it was used at an edge adjacent to the edges colored with c or at an edge distance 3 apart from the edge colored with c. Now observe that c and $c - 1$ are used at two edges of S_3 which form a triangle with one edge of S_2. Suppose c, $c - 1$ be the colors used at those two edges $e, e_1 \in S_3$ respectively, where e is incident to u and e_1 is incident to w where $uw \in S_2$. Note that c is used thrice in S_3. Then c must be reused at an edge incident to x, and $xw \in S_2$. So c and $c - 1$ are used at two edges at distance 2 apart, which violets the condition of $L(1, 2)$-edge labeling. Hence $c - 1$ cannot be used in G_v. Similarly it can be shown that $c+3$ can also not be used in G_v. This implies that no 4 consecutive colors can be used thrice each in G_v. □

Lemma 9 *If all colors in* $\{c, c + 2, c + 4\}$ *are used twice each in* S_2 *then at least* 6 *colors are required which all are either higher than* $c + 4$ *or lower than* c.

Proof From Theorem 4, it follows that $|C_{S_2}| = 6$ for optimal coloring of G_v. So $c+1, c+3$ and one of $c+5$ and $c-1$ must be used in S_3. Without loss of generality, assume that colors $\{c + 1, c + 3, c + 5\}$ are used in S_3. Using Lemma 4 it can be verified that colors $c + 1$, $c + 3$ and $c + 5$ can be used at most twice in S_3. That means using these three colors at most 6 edges can be colored in S_3. So 12 edges remain uncolored till now. By Lemma 6 a color can be used thrice in S_3. Again, it follows from Lemma 8 that no 4 consecutive colors can be used thrice each in S_3. Hence the maximum color used in S_3 will be at least $(c + 5) + 5$. Similarly it can be shown that minimum color used in S_3 will be at most $(c - 1) - 5$ for the case when $\{c - 1, c + 1, c + 3\}$ are used at S_3. □

Theorem 5 $\lambda'_{1,2}(G_v) \geq 17$.

Proof By Theorem 3, 6 consecutive colors must be used to color the edges of S_1. Recall that, we assume the minimum color is used at S_1. Let c' be the maximum color used at S_1 and c'' be the minimum color used at S_2. From Theorem 4, 3 colors must be used to color the edges of S_2 and in that case by Lemma 1, $(c'' - c') \geq 2$. Note that, $c'+1$ and $c''-1$ can be used at most once and twice respectively. However a color can be used thrice in S_3. Therefore, it is beneficial if $c' + 1 = c'' - 1$. Now if $\{c, c + 2, c + 4\}$ are used at S_2 then $\{c - 2, c - 3, \cdots c - 7\}$ are used at S_1. Now if $\{c+1, c+3, c+5\}$ are used in S_3 then from Lemma 9, it follows that $c+10$ must be used at S_3. So, $\lambda'_{1,2}(G_v) \geq ((c+10)-(c-7)) = 17$. Similarly, if $\{c-1, c+1, c+3\}$ are used at S_3, $\lambda'_{1,2}(G_v) \geq ((c + 11) - (c - 6)) = 17$. Hence the proof. □

We assume that the minimum color is used in S_1. The maximum color can be used at most thrice in S_3 and at most twice in S_2. In all cases, there exists a vertex say v' in H_v such that color of any edge incident to v' is neither minimum nor maximum. Now we consider the subgraph $G_{v'}$ of T_6 centering v' and isomorphic to G_v.

Let min_1 and max_1 be the minimum and maximum colors used to color the edges of S'_1 in $G_{v'}$.

Lemma 10 *If $max_1 - min_1 \geq 7$, i.e., there exists at least two intermediate colors between min_1 and max_1 which are not used in S_1', then $\lambda'_{1,2}(G_{v'}) \geq 19$.*

Proof There must be at least two unused colors say, $\{c_1, c_2\}$ such that for each $c \in \{c_1, c_2\}$ either $c + 1$ or $c - 1$ is used in S_1'. From Lemma 2, it can be said that each of c_1, c_2, $min_1 - 1$ and $max_1 + 1$ can be used at most once in $G_{v'}$. Observe from discussion of Theorem 5, that for any optimal coloring of $G_{v'}$, each color must be used at least twice in $G_{v'} \setminus S_1'$. Note that at most 4 edges can be colored by c_1, c_2, $min_1 - 1$ and $max_1 + 1$. But for optimal coloring, c_1, c_2, $min_1 - 1$ and $max_1 + 1$ should have been colored at least 8 edges. For those four uncolored edges, at least two additional colors must be required as a color can be used at most thrice in $G_{v'}$. Hence the proof. □

Theorem 6 $\lambda'_{1,2}(T_6) \geq 19$.

Proof Assume that x be such a vertex which is not adjacent to edges colored with any of min or max in G_x. Let us consider G_x is not colored and u, w be two vertices of H_x in G_x. Let us define S_{x1} as the set of edges adjacent to x. Now we consider the following two cases.

- When $w \in N(u)$: u and w are connected by an edge e. Let $\{c_1, \cdots, c_6\}$ and $\{c_1', \cdots, c_6'\}$ be two sequences consisting of consecutive colors are used at the edges incident to u and w respectively. It is possible to assign consecutive colors at those edges when e is colored with either $c_6 = c_1'$ or $c_1 = c_6'$. Now observe two edges e' and e_1' of S_{x1} are already colored and those are not consecutive. Note that $|f'(e') - f'(e_1')| \geq 2$. If $|f'(e') - f'(e_1')| = 2$ then $f'(e')$ and $f'(e_1')$ is neither minimum nor maximum color used in u and w. Then any color of any other edge in S_{x1} is neither consecutive to $f'(e')$ nor $f'(e_1')$. So $max - min \geq 7$ where min and max be the minimum and maximum colors used to color the edges of S_{x1}. If $|f'(e') - f'(e_1')| > 2$, then also $max - min \geq 7$. Therefore from Lemma 10, at least 20 colors are required for G_x. Hence $\lambda'_{1,2}(T_6) \geq \lambda'_{1,2}(G_x) \geq 19$.
- When $w \notin N(u)$: Note that $x \in \{N(u) \cap N(w)\}$. Let two sequences $\{c_1, \cdots, c_6\}$ and $\{c_1', \cdots, c_6'\}$consisting of consecutive colors are used at the edges incident to u and w respectively. Let uv and wv are e' and e_1' respectively. If $f'(e')$ and $f'(e_1')$ are consecutive then either $f'(e') = c_6,\ f'(e_1') = c_1'$ or $f'(e') = c_1,\ f'(e_1') = c_6'$. Now observe that for any other edge e in S_{x1}, $|f'(e) - f'(e')| > 2$ implying $max - min \geq 7$ where min and max be the minimum and maximum colors used to color the edges of S_{x1}. If $f'(e')$ and $f'(e_1')$ are not consecutive then $|f'(e') - f'(e_1')| \geq 2$. If $|f'(e') - f'(e_1')| = 2$ then the intermediate color must be used at an edge $e \in S_{x1}$. There are still 4 edges remain uncolored. It can be checked that for any coloring of the rest of the graph, there exists a vertex y in $H_x \cup x$, for which $max - min \geq 7$ where min and max be the minimum and maximum colors used to color the edges incident to u. Hence from Lemma 10, at least 20 colors are required for G_x. Hence $\lambda'_{1,2}(T_6) \geq \lambda'_{1,2}(G_x) \geq 19$.

Hence the proof. □

3 Conclusions

In this article we improved some lower and upper bounds for infinite regular hexagonal, square and triangular grids using structural properties of those graphs. An interesting problem will be to improve or introduce new bounds on those graphs for other values of h and k. It would also be interesting to examine similar bounds for other infinite regular grids.

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