Total Chromatic Sum for Trees

Ewa Kubicka, Grzegorz Kubicki, Michał Małafiejski, and Krzysztof M. Ocetkiewicz

Abstract The total chromatic sum of a graph is the minimum sum of colors (natural numbers) taken over all proper colorings of vertices and edges of a graph. We provide infinite families of trees for which the minimum number of colors to achieve the total chromatic sum is equal to the total chromatic number. We construct infinite families of trees for which these numbers are not equal, disproving the conjecture from 2012.

Keywords Total colorings · Sum of colors · Trees

1 Introduction

Consider a proper coloring ϕ of vertices of a graph G using natural numbers; i.e. $\phi : V(G) \to N$ and $\phi(u) \neq \phi(v)$ whenever uv is an edge of G. The *chromatic sum* of G, denoted $\Sigma(G)$, is the minimum sum $\sum_{v \in V(G)} \phi(v)$ taken over all proper colorings ϕ of G. A coloring is *optimal* if the sum of colors equals $\Sigma(G)$.

This idea was introduced by Kubicka [4] in 1989, and since then much more work has been done with calculating the chromatic sums of graphs, generating algorithms to find chromatic sums and optimal colorings, and calculating the complexity of finding chromatic sums of graphs in certain families. Erdös et al. [2] constructed infinite families of graphs for which the minimum number of colors necessary to get an optimal coloring of G was larger than $\chi(G)$. This graph parameter, the minimum

E. Kubicka (✉) · G. Kubicki
University of Louisville, Louisville, KY, USA
e-mail: ewa@louisville.edu; gkubicki@louisville.edu

M. Małafiejski · K. M. Ocetkiewicz
Gdansk University of Technology, Gdansk, Poland
e-mail: michal@animima.org; krzysztof.ocetkiewicz@eti.pg.edu.pl

C. Gentile et al. (eds.), *Graphs and Combinatorial Optimization: from Theory to Applications*, AIRO Springer Series 5,
https://doi.org/10.1007/978-3-030-63072-0_3

number of colors necessary for an optimal coloring, is called the *strength* of G, and denoted by $\sigma(G)$ [3]. In [2], it is shown that even trees can have arbitrarily high strength, even though their chromatic number is 2. In fact, Erdös et al. [2] found for every $k \geq 3$ the smallest tree of strength k. In [3], Jiang and West also constructed trees with strength k but not of minimum order but of minimum maximum degree, $\Delta = 2k - 2$.

We say that a graph G is *strong* if $\chi(G) < \sigma(G)$. The smallest strong graph is the tree on eight vertices given in Fig. 1.

These color-sum concepts can be applied to edge coloring as well. In an analogous way, one can define the *edge chromatic sum* of a graph, its edge strength σ', and ask the question of whether or not $\chi' = \sigma'$. In 1997, Mitchem et al. [7] proved that every graph has a proper edge coloring with minimum sum that uses only Δ or $\Delta + 1$ colors. This implies that the only way for a graph to have $\chi' < \sigma'$ is to have both $\chi' = \Delta$ and $\sigma' = \Delta + 1$. We say that a graph G with this property, namely $\chi'(G) < \sigma'(G)$, is *E-strong*. The smallest known E-strong graph M is presented in Fig. 2.

Fig. 1 The smallest strong tree

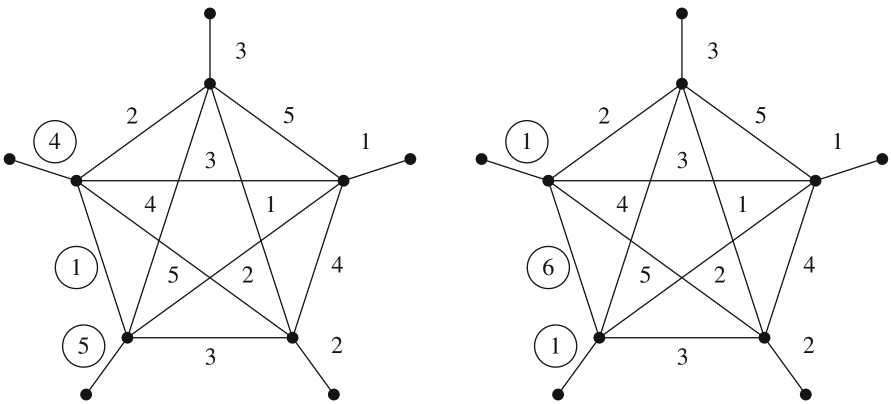

Fig. 2 E-strong graph M with $\Delta = 5$, $\chi' = 5$, and $\sigma' = 6$. On the left side with five colors used, the sum of the colors is 45. If we introduce a sixth color and change the colors of the edges whose color labels are circled, we obtain a coloring with sum 43

2 Total Chromatic Sum and Total Strength

A *total coloring* ϕ of G is an assignment of natural numbers to vertices and edges of a graph. A total coloring of G is *proper* if no pair of adjacent or incident elements (vertices or edges) is assigned the same color. A total-k-coloring ϕ of G is a proper total coloring that uses k colors. The *total chromatic number* $\chi''(G)$ of a graph G is the smallest number k for which G has a total-k-coloring. The famous Total Coloring Conjecture stating that

$$\chi''(G) \leq \Delta(G) + 2$$

for every graph G, where $\Delta(G)$ is the maximum degree of G, was posed independently by Vizing [8, 9] and Behzad [1].

The total chromatic sum of a graph is defined in a similar way to the chromatic sum. The *total chromatic sum* of G, denoted $\Sigma''(G)$, is the minimum sum $\sum_{x \in V(G) \cup E(G)} \phi(x)$ taken over all proper total colorings ϕ of G. A total coloring is *optimal* if the sum of colors of vertices and edges of G equals $\Sigma''(G)$. The minimum number of colors necessary for an optimal total coloring is called the *T- strength* of G, and is denoted by $\sigma''(G)$. We say that a graph G is *T-strong* if $\sigma''(G) > \chi''(G)$. The total chromatic sum and the related parameters were introduced by Leidner [6] in his Ph.D. dissertation.

The total chromatic sum was determined for many families of graphs. In [6] and [5], several infinite families of T-strong graphs, it means graphs for which we need more colors for optimal total coloring than the total chromatic number, were constructed. Each graph from the following list is T-strong:

1. Cycles of length $3n$, $n \geq 2$, with one chord joining vertices at distance congruent to 3 along the cycle. Those graphs have $\Delta = 3$, $\chi'' = 4$, and $\sigma'' = 5$.
2. Cycles of length $3n$, $n \geq 2$, with two independent chords with proper distances along the cycle. For those graphs also $\Delta = 3$, $\chi'' = 4$, and $\sigma'' = 5$.
3. Graphs G obtained from M, the graph from Fig. 2, by attaching a copy of K_{2k+1} to each vertex. Here $\Delta(G) = 2k + 5$, $\chi''(G) = 2k + 6$, and $\sigma''(G) = 2k + 7$.

The smallest T-strong graph is a 6-cycle with a diametral chord, or equivalently the graph $P_2 \times P_3$. Two colorings, the first using $\chi''(G)$ colors and the second that is optimal and uses one more color are depicted in Fig. 3. Leidner [6] verified by an exhaustive computer search that $P_2 \times P_3$ is the smallest T-strong graph and the only one of order smaller than 9.

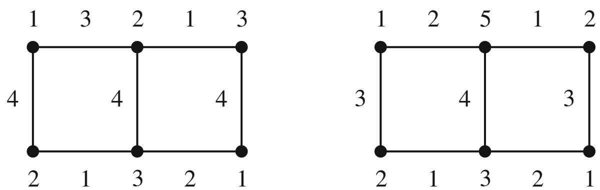

Fig. 3 The grid graph $P_2 \times P_3$ total-colored in two ways

In the next sections, we show new results for the total chromatic sum of trees and the existence of T- strong trees. Two conjectures about the total strength of a graphs were stated in [5].

Conjecture 1 For every graph G, $\chi"(G) \leq \sigma"(G) \leq \chi"(G) + 1$.

Conjecture 2 No tree is T-strong.

In this paper, we prove that no tree requires more than $\Delta + 2$ colors to achieve its total chromatic sum, which proves Conjecture 1 for trees. We construct a polynomial time algorithm to determine the total chromatic sum and the total chromatic strength of a tree. We prove that all trees with no adjacent vertices of maximum degree have the total chromatic number equal to the total chromatic strength. Finally, we disprove Conjecture 2 providing infinite families of T-strong trees.

3 Upper Bound on Total Strength of Trees

Let $G = (V, E)$ be a tree. We define a distance between elements of G.

- If $u, v \in V$, then $d(u, v)$ is the number of edges on the $u - v$ path.
- If $e, f \in E$, then $d(e, f)$ is the number of vertices on the $e - f$ path.
- If $e = uv \in E$ and $w \in V$, then $d(e, w) = \frac{1}{2} + min\{d(u, w), d(v, w)\}$.

It is easy to check that d is a metric on $V \cup E$.

It is well known that every tree is either **unicentral** or **bicentral**, i.e. has the center consisting of one vertex or two adjacent vertices, respectively. For the purpose of this note we call the **strong center** of G:

– the central vertex u if G if unicentral,
– the edge $e = uv$ if G is bicentral with the center $cent(G) = \{u, v\}$.

Theorem 1 *No tree requires $\Delta + 3$ or more colors for an optimal total coloring.*

Proof Let $\Delta = \Delta(G)$. We will show that there is an optimal coloring of G without a color $c = \Delta + 3$. For larger colors, the proof is similar but simpler. Suppose, to the contrary, that the color $c = \Delta + 3$ must occur in any optimal coloring of G. Among all such colorings select a coloring φ in which the color c occurs as far away from the strong center of G as possible. Suppose first that color $c = \Delta + 3$ occurs on the edge $e = xy$, x closer to the center than y (if $\{e\}$ is not a strong center of G). Let $E(x)$ and $E(y)$ denote the set of edges incident to x and y, respectively. Define two sets of elements of G, $L = \{x\} \cup E(x) - \{e\}$ and $R = \{y\} \cup E(y) - \{e\}$. Notice that $|L|, |R| \leq \Delta$; also all elements of L (and of R) must be colored with different colors. If $\varphi(R) \subset \varphi(L)$, then because $|\varphi(L)| \leq \Delta$, we can find a color $c_1 \in \{1, 2, \ldots, \Delta + 1\}$ that was not used on elements from L and from R. Recolor e with c_1 obtaining a total coloring with a smaller sum. If $\varphi(R) \setminus \varphi(L) \neq \emptyset$, say $c_1 \in \varphi(R) \setminus \varphi(L)$, and $c_1 = \varphi(\alpha)$, then recolor the element α by c and the edge e by c_1. This produces the total coloring of G with the same sum but with color

$c = \Delta + 3$ father away from the strong center of G; a contradiction. Notice that the same argument can be used if the color on e is $\Delta + 2$ (not $\Delta + 3$). This observation will be used in the proof of the next theorem.

It remains to consider the case when in the coloring φ of G, the element with color $c = \Delta + 3$ is a vertex v. Of course, v is not a leaf since any leaf can be colored with a color from $\{1, 2, 3\}$. Let u be a neighbor of v closer to the strong center of G than v, or the other vertex of the strong center if the center of G is $\{u, v\}$. Assume that $\varphi(u) = c_1$ and $\varphi(uv) = c_2$. Since $deg(v) \leq \Delta$, there are two colors, say c_3 and c_4 from $\{1, 2, \ldots, \Delta + 2\}$ that are not present on the edges incident to v (for illustration see Fig. 4). If $c_1 \in \varphi(E(v) - \{e\})$, then both colors c_3 and c_4 must be used on neighbors of v, say $c_3 = \varphi(x)$, $c_4 = \varphi(y)$, otherwise we could recolor v with c_3 or c_4, obtaining a smaller sum of colors. Therefore, there is a color $c_5 (c_5 \neq c_1)$ on an edge f incident to v that is not used on the neighbors of v. We can interchange colors c and c_5 on v and f obtaining a coloring in which c is father away from the strong center of G; a contradiction. Similar argument works if $c_1 \notin \varphi(E(v) - \{e\})$. Without loss of generality, we might assume that $c_1 = c_3$. Then color c_4 must occur on a neighbor of v, say $c_4 = \varphi(x)$; otherwise we could recolor v with c_4. Now, one of the colors from $\varphi(E(v) - \{e\})$ is not used on any neighbor of v, say this color is c_5 and it occurs on the edge f, $c_5 = \varphi(f)$. Similarly, we can interchange colors c and c_5 on v and f obtaining a coloring with the color c farther away from the strong center of G; a contradiction. □

It is well known that, with the exception of K_2, $\chi''(G) = \Delta + 1$ for every tree G with maximum degree Δ. Therefore, as the corollary of Theorem 1, for trees we have the following two possibilities:

1. $\chi''(G) = \Delta + 1 = \sigma''(G)$, which means that G is not T-strong, or
2. $\chi''(G) = \Delta + 1$ and $\sigma''(G) = \Delta + 2$, if a tree G is T-strong.

This observation verifies Conjecture 1 for trees.

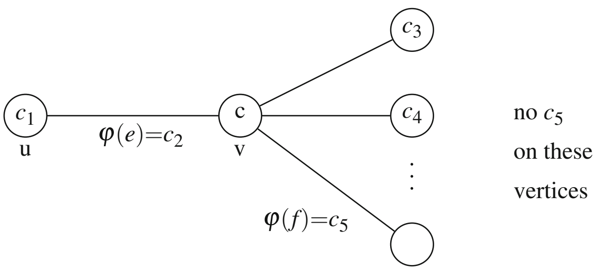

Fig. 4 Illustration of the proof of Theorem 1

4 Total Strength of Trees with No Adjacent Vertices of Maximum Degree

abel3:sec:4

A tree G is called $\Delta\Delta$**-free** if G has no adjacent vertices of maximum degree.

Theorem 2 *No $\Delta\Delta$-free tree is T-strong.*

Proof Assume, to the contrary, that there exists a T-strong tree. Such a tree must have $\Delta \geq 3$. Among all such trees consider those of minimum order. For each tree of this property, select an optimal total coloring φ (the coloring with the sum of colors equal to $\sum''(G)$) in which the color c ($c = \Delta + 2$) occurs as far away from the strong center of G as possible. Consider a particular tree G with such coloring φ. By the observation in the first part of the proof of Theorem 1, this color cannot occur on any edge of G.

Thus, one can assume that in the coloring $\varphi(G)$, the element colored with c is some vertex, say v. If $deg(v) \leq \Delta - 1$, then we get a contradiction using a similar argument as in the proof of Theorem 1, where numbers Δ and $\Delta + 3$ denoting the degree of v and the largest color c, respectively, are now replaced by $\Delta - 1$ and $\Delta + 2$. It remains to show that there is no $\Delta\Delta$-free tree with the color $\Delta + 2$ occurring on the vertex v with $deg(v) = \Delta$. So assume that $\varphi(v) = \Delta + 2$, the neighbor of v "toward the center of G" is u (notice that $deg(u) \leq \Delta - 1$ because G is $\Delta\Delta$-free), $\varphi(u) = c_1$, and $\varphi(uv) = c_2$.

Case 1 If $c_1 \notin \varphi(E(v) - \{e\})$ and $\varphi(N(v)) - u) = \varphi(E(v) - \{e\})$ (the same palette of colors is used on $\Delta - 1$ edges and $\Delta - 1$ neighbors of v away from the strong center), then there must be a color among them, say c_3 on the edge f that is not used on $E(u)$. We can modify φ by coloring v by c_2, f by $\Delta + 2$, and e by c_3, obtaining a coloring with the same sum but with color $\Delta + 2$ father away from the strong center of G; a contradiction.

Case 2 If $c_1 \notin \varphi(E(v) - \{e\})$ and $\varphi(N(v)) - u) \neq \varphi(E(v) - \{e\})$, then there is a color $c_3 \in \varphi(E(v) - \{e\})$ that is not present on any neighbor of v, suppose that c_3 occurs on an edge f. By interchanging colors c_3 and $\Delta + 2$ on f and v, f will receive color $\Delta + 2$ which is father away from the strong center of G; a contradiction.

Case 3 If $c_1 \in \varphi(E(v) - \{e\})$, say $\varphi(f) = c_1$, then there is a color $c_3 (c_3 \neq c_1, c_3 \neq c_2)$ that is not present on $E(v) - e$ but is present on some neighbor of v, say x.

Case 3a If no other neighbor of v has color c_3, we swap colors of v and x obtaining the coloring with color $\Delta + 2$ farther away from the strong center of G (Fig. 5).

Case 3b If there is another neighbor of v with color c_3, then one of the $\Delta - 2$ colors from the edges $E(v) - e - f$ is not present on the neighbors of v; say color c_4 occurring on an edge h. Modify the coloring φ by swapping colors of v and h; the color $\Delta + 2$ will be on h that is farther away from the strong center of G; a contradiction (Fig. 6).

□

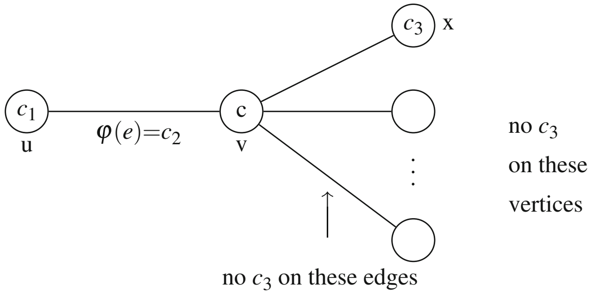

Fig. 5 Illustration of Case 3a

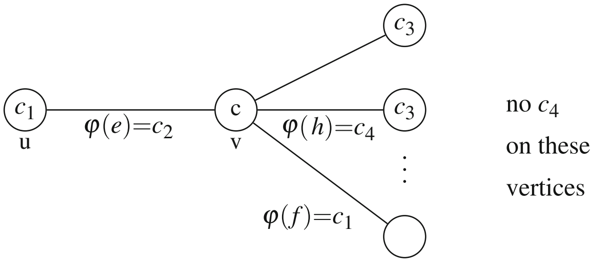

Fig. 6 Illustration of Case 3b

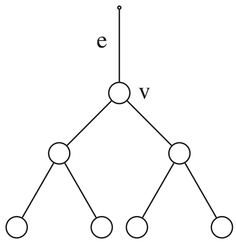

Fig. 7 An example of a tree fragment

The only case in which we cannot "push" color $\Delta + 2$ away from the strong center is for a tree with two adjacent vertices u and v both of degree Δ such that the pallets of colors on $E(u) - e$, $E(v) - e$, and $N(v) - u$ are identical.

5 Polynomial Time Algorithm for Trees

In this section we propose an $O(n\Delta^4)$ algorithm for finding the total chromatic sum, the minimum sum total coloring and the total strength of an arbitrary tree.

Let G be a tree, and let $v \in V(G)$ and $e \in E(G)$, $e = vu$. We define a **fragment tree** or an **f-tree** with respect to v and e, to be a component of $G \backslash e$ containing v, together with an attached edge e joining v and u, but without the vertex u (see Fig. 7). We denote this fragment tree by $Q(v, e)$. Formally, $Q(v, e)$ is not a graph. By $p(v)$ we mean the vertex u.

Let $F = Q(v, e)$ be an f-tree of some tree. By the **root** of F, denoted by $r(F)$, we mean the vertex v, and by the **extent** of F, denoted by $x(F)$, we mean the edge e. Obviously, $Q(r(F), x(F)) = F$. By $\hat{F}$ we mean F with attached $p(r(F))$. By a total coloring of an f-tree F we mean a partial total coloring of $\hat{F}$, with colors assigned to $V(\hat{F})\backslash\{p(r(F))\} \cup E(\hat{F})$. By $\Delta(F)$ we mean $\Delta(\hat{F})$.

Let F be an f-tree of some tree G and let $c \geq \Delta(F) + 1$ be an integer (an upper bound for the number of colors). The **cost matrix** C_F^c of the f-tree F is a square $c \times c$ matrix whose (p, q)-entry, for $p \neq q$, denotes the sum of colors of an optimal total coloring of F using at most c colors where the root $r(F)$ has color p and the extend $x(F)$ has assigned color q. The diagonal entries of C_F^c are undefined, since in a proper total coloring we must have $p \neq q$.

If $F = Q(v, e)$ and the neighbors of v are $v_1, v_2, \ldots, v_k$ adjacent to v by the edges $e_1, e_2, \ldots, e_k$ (see Fig. 8), then knowing the cost matrices $C_{F_i}^c$ for the k fragments $F_i = Q(v_i, e_i)$, $1 \leq i \leq k$, we can evaluate the cost matrix for F. Namely, the (p, q)-entry of C_F^c is the minimum of the sums $p + q + C_{F_1}^c[p_1, q_1] + \ldots + C_{F_k}^c[p_k, q_k]$ taken over all colors $p_i, q_i \in \{1, \ldots, c\}$ such that for each $i, 1 \leq i \leq k$, $p_i \neq q_i$ and $p_i \neq p$ and all colors in the set $\{p, q, q_1, \ldots, q_k\}$ are different.

If G is a tree rooted at v and the neighbors of v are $v_1, v_2, \ldots, v_k$, then in a similar manner we can evaluate the cost vector C_G^c for G knowing cost matrices for all k fragments $F_i = Q(v_i, e_i)$, $1 \leq i \leq k$, where $e_i = vv_i$. The p-entry $C_G^c[p]$ of this vector equals the minimum of the sums $p + C_{F_1}^c[p_1, q_1] + \ldots + C_{F_k}^c[p_k, q_k]$ taken over all colors $p_i, q_i \in \{1, \ldots, c\}$ such that for each i, $1 \leq i \leq k$, $p_i \neq q_i$ and $p_i \neq p$ and all colors in the set $\{p, q_1, \ldots, q_k\}$ are different.

Theorem 3 *There is an algorithm of complexity $O(n\Delta^4)$ for finding the minimum total chromatic sum of a tree in the class of trees of order n and the degree bounded by Δ.*

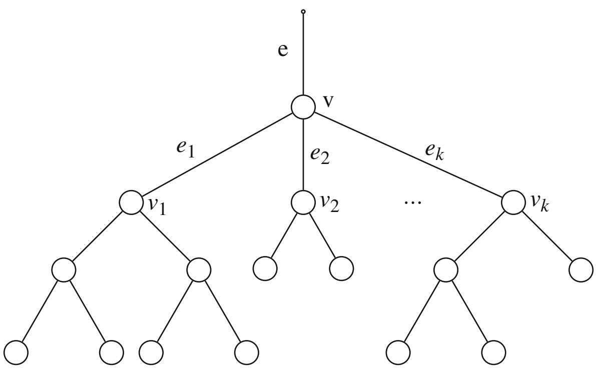

Fig. 8 Decomposing a tree fragment into k smaller fragments

Proof Let G be a tree of order n and maximum degree $\Delta(G) = \Delta$. Select any vertex of G as a root; call it r. Direct all edges of G toward the root. Sort the vertices in a topological order in accordance to the orientation of edges. Select $c = \Delta + 2$, since by Theorem 1 no tree needs more than $\Delta + 2$ colors for a optimal total coloring. Initialize the algorithm by assigning the cost matrix for any fragment F consisting of a leaf with its pendant edge $C_F^c(p, q) = p + q$.

If v is not a root and not a leaf, and has $v_1, v_2, \ldots, v_k$ as predecessors, compute cost matrix C_F^c for the f-fragment $Q(v, e)$, where e is the only edge of v directed towards r. If $v = r$ is the root of G, compute the cost vector C_G^c. Total chromatic sum of G equals

$$\Sigma''(G) = \min_{1 \le p \le \Delta+2} \{C_G^c[p]\}.$$

It is not difficult to see that the complexity of this algorithm is $O(n\Delta^4)$. □

By running this algorithm with $c = \Delta(G) + 1$ and $c = \Delta(G) + 2$, we can determine the total strength $\sigma''(G)$ of a tree G. If the algorithm returns the same costs for both upper bounds for c, then $\sigma''(G) = \Delta(G) + 1$. If the cost for $c = \Delta(G) + 2$ is smaller than for $c = \Delta(G) + 1$, then $\sigma''(G) = \Delta(G) + 2$ and G is T-strong.

6 Existence of T-strong Trees

From the proof of Theorem 2, we can get some information about an optimal total coloring of any T-strong tree G. The structure of such a tree and an optimal total coloring of G must be as follows:

1. G must have two adjacent vertices, say u and v, both of degree $\Delta(G)$.
2. Color $\Delta + 2$ must occur at one of those vertices, say v.
3. The three palettes of colors occurring on vertices adjacent to v (not counting u), the edges incident to v (not counting uv), and the edges incident to u (without uv as well) must be identical.

Using this observation. we were able to construct a T-strong tree. The smallest in the family of all subcubic trees ($\Delta \le 3$) is the tree T_{50}, of order 50, depicted in Fig. 9. Our algorithm verified that $\sigma"(T_{50}) = 5$. We used the vertex marked black as the root for running our algorithm. Notice that its neighbor $r(L)$ is the root of a subtree isomorphic to the other half of T_{50}.

Theorem 4 *There is an infinite family of T-strong trees.*

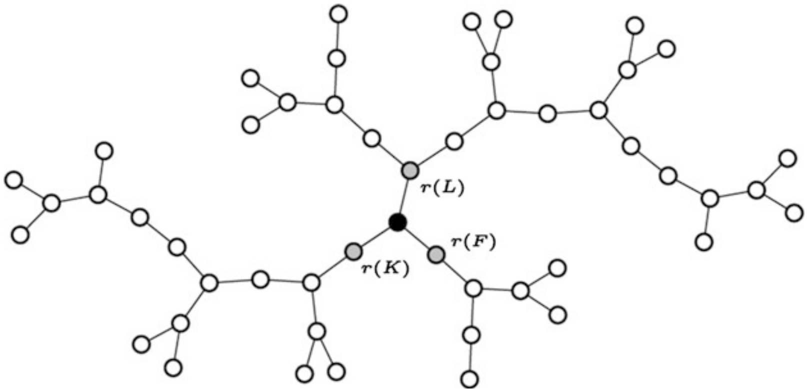

Fig. 9 The smallest subcubic T-strong tree T_{50}

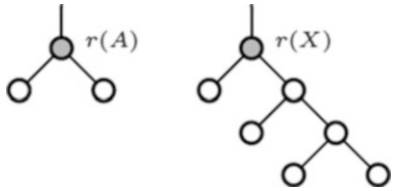

Fig. 10 Fragments A and X used for constructing larger T-strong trees

Proof Two f-trees A and X given in Fig. 10 have the following property. In any optimal coloring of A and X roots $r(A)$ and $r(X)$ must have colors 2, 3, or 4 and the extents must be colored with 1. Moreover, changing colors on those elements increases the cost of both f-trees by the same amount. This means that replacing a fragment A in any tree G by the fragment X does not change the coloring of the rest of G. Notice that X has four more vertices than A and X contains A as a subfragment. Starting with the T-strong tree T_{50}, we can replace any f-tree A in it by a copy of X obtaining a T-strong tree of order 54. Continuing these fragments' replacements, we can construct subcubic T-strong trees of arbitrarily large order. □

Our algorithm verified that the smallest T-strong tree with vertices of degree 1 and 3 only is the tree T_{122} depicted in Fig. 11. We also found a T-strong tree of order 266 with $\Delta = 4$. Both of these trees can generate infinite families of T-strong trees by similar fragment replacements.

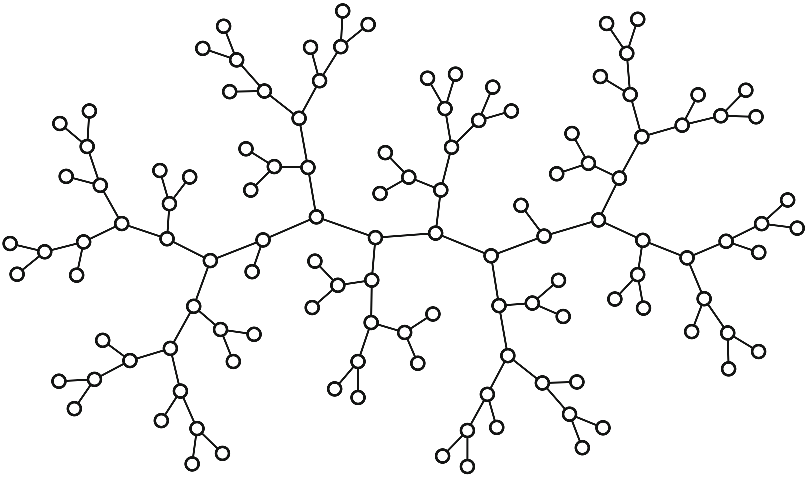

Fig. 11 T-strong tree T_{122} of order 122 with degree set $\{1, 3\}$

References

1. Behzad, M.: Graphs and their chromatic numbers. Ph.D. Thesis, Michigan State University, 69 pp. (1965)
2. Erdős, P., Kubicka, E., Schwenk, A.: Graphs that require many colors to achieve their chromatic sum. Congr. Numer. **71**, 17–28 (1990)
3. Jiang, T., West, D.B.: Coloring of trees with minimum sum of colors. J. Graph Theory **32**, 354–358 (1999)
4. Kubicka, E.: The chromatic sum and efficient tree algorithms. Ph.D. Thesis, Western Michigan University, 149 pp. (1989)
5. Kubicka, E., Kubicki, G., Leidner, M.: Total colorings of graphs with minimum sum of colors. Graphs Comb. **32**, 2515–2524 (2016)
6. Leidner, M.: The study of the total coloring of graphs. Ph.D. Thesis, University of Louisville, 101 pp. (2012)
7. Mitchem, J., Morris, P., Schmeichel, E.: On the cost chromatic number of outerplanar, planar, and line graphs. Discuss. Math. Graph Theory **2**, 229–241 (1997)
8. Vizing, V.G.: On an estimate of the chromatic class of a p-graph. Diskret. Anal. **3**, 29–39 (1964)
9. Vizing, V.G.: The chromatic class of a multigraph. Kibernetika (Kiev) **3**, 25–30 (1965)