Edge Tree Spanners

Fernanda Couto, Luís Cunha, and Daniel Posner

Abstract A tree t-spanner of a graph G is a spanning tree T of G in which any two adjacent vertices of G have distance at most t in T. The line graph $L(G)$ of a graph G is the intersection graph of the edges of G. We define the edge tree t-spanner of a graph G as a spanning tree T of $L(G)$ in which any two edges that share an endpoint in G have distance at most t in T. Although determining if G has a tree 3-spanner is an open problem for more than 20 years, we settle that deciding if a graph G has an edge tree 3-spanner is polynomial-time solvable. As a consequence, we present polynomial time algorithms for the edge tree t-spanner problem for several graph classes such as trees, join of graphs, split graphs, P_4-tidy, and (1, 2)-graphs. Moreover, we establish that deciding whether a graph G has an edge tree 8-spanner is NP-complete, even if G is bipartite.

Keywords Tree t-spanner · Edge tree t-spanner · Polynomial time algorithms · NP-completeness · Line graphs · Graph classes

1 Introduction

The problem of looking for a spanning tree with constraints on the vertices' or edges' distances is a combinatorial challenge with many applications and approaches [1, 11]. A *tree t-spanner* of a graph G is a spanning tree T of G in which any two adjacent vertices of G have distance at most t in T. A graph G having a tree t-spanner is called a *t-admissible* graph. The smallest t for which a graph G is t-admissible is the *stretch index of* G and is denoted by $\sigma_T(G)$ (or

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simply $\sigma(G)$). The *t-admissibility* problem aims to decide whether a given graph G has $\sigma(G) \leq t$. The problem of determining the tree stretch index, i.e. *the minimum stretch spanning tree problem* (MSST) has been studied by establishing bounds on $\sigma(G)$ or developing the computational complexity of the decision version of MSST for several graph classes [2–4]. Cai and Corneil [2] proved that t-admissibility is NP-complete, for $t \geq 4$, whereas 2-admissible graphs can be recognized in polynomial-time. However, the characterization of 3-admissible graphs is still an open problem.

The characterization for 2-admissible graphs [2], stated in Theorem 1, deals with triconnected components of a connected graph, defined as any maximal subgraph that does not contain two vertices whose removal disconnects the graph (the authors also consider K_2 and K_3 as triconnected components). A *nonseparable* graph is a graph without a *cut vertex*, i.e., a vertex whose removal disconnects the graph. A *star* with $n+1$ vertices is the complete bipartite graph $K_{1,n}$. A *v-centered star* is a star centered on v, that is a universal vertex. Similarly, a bi-star is a graph such that there is an edge uv and every edge of E shares an endpoint with uv. Hence, uv is a *universal edge* of the bi-star. A *uv-centered bi-star* is a bi-star centered on a universal edge uv.

Theorem 1 ([2]) *A nonseparable graph G is 2-admissible if and only if G contains a spanning tree T such that for each triconnected component H of G, $T \cap H$ is a spanning star of H.*

Given a graph G, its *line graph* $L(G)$ is obtained as follows: $V(L(G)) = E(G)$; $E(L(G)) = \{\{uv, uw\} | uv, uw \in E(G)\}$. I.e., each edge of G is a vertex of $L(G)$ and if two edges share an endpoint, then their corresponding vertices are adjacent in $L(G)$. The *distance between two edges* e_1 and e_2 of G, for $e_1, e_2 \in E(G)$ is the distance between their corresponding vertices in $L(G)$.

We define the *edge tree t-spanner* of a graph G as a spanning tree T of $L(G)$ such that, for any two adjacent edges of G, their distance is at most t in T. Therefore, an edge tree t-spanner of G is a tree t-spanner of $L(G)$.

A graph G that has an edge tree t-spanner is called *edge t-admissible*. The smallest t for which G is an edge t-admissible graph is the *edge stretch index of* G, and is denoted by $\sigma'_T(G)$ (or simply $\sigma'(G)$). The *edge t-admissibility* problem aims to decide whether a given graph G has $\sigma'(G) \leq t$. Figure 1 depicts the relation between the edge tree spanner of a graph and the tree spanner of its line graph.

An immediate consequence of MSST is that the property of being t-admissible graph is not hereditary, i.e., if G is t-admissible then there may exist a subgraph H of G that is not t-admissible. Indeed, the addition of a universal vertex u to any t-admissible graph results in a 2-admissible graph by a u-centered star.

On the other hand, regarding the edge tree t-spanner, in Sect. 3 we prove that being an edge 3-admissible graph is a hereditary property, and based on that, we are able to decide whether G is edge 3-admissible in polynomial time. Moreover, in Sect. 4 we determine polynomial time algorithms to obtain the edge stretch index for some edge 4-admissible and edge 5-admissible classes, such as split graphs, join graphs, P_4-tidy graphs and $(1, 2)$-graphs. In Sect. 5, we prove that

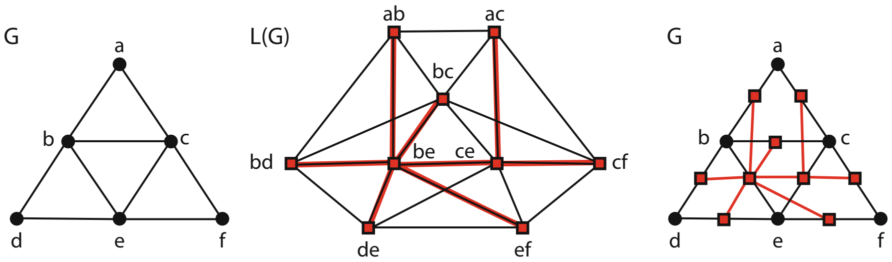

Fig. 1 A graph G, a tree 3-spanner of $L(G)$ in red, and G with the related edge 3-spanner in red

edge 8-admissibility is NP-complete for $(2, 0)$-graphs, i.e. bipartite graphs. In Sect. 6, we present concluding remarks. Next (Sect. 2), we relate admissibility and edge admissibility problems, presenting immediate consequences and preliminary results.

2 Admissibility Versus Edge Admissibility for Graph Classes

Since induced cycles in a graph G correspond to cycles of the same length in $L(G)$, we have that $\sigma'(C_n) = \sigma(C_n) = n - 1$. Although cycle graphs satisfy $\sigma' = \sigma$, for several other classes the stretch index is different of the edge stretch index.

For instance, trees are 1-admissible and the unique edge 1-admissible graphs are the ones such that their line graphs are trees. Since line graphs are claw-free, then path graphs are the unique edge 1-admissible graphs. In Proposition 1 we determine the edge stretch index of trees.

Proposition 1 *Let G be a tree. If G is a path graph then $\sigma'(G) = 1$, otherwise $\sigma'(G) = 2$.*

Proof Note that if G is a path, then $L(G)$ is a path and $\sigma'(G) = 1$. For any other tree there is a vertex of degree at least 3, implying a complete subgraph of length at least 3 in $L(G)$. Each internal node u of G correspond to a maximal complete subgraph of $L(G)$ of size $d_G(u)$ and two of such maximal complete subgraphs share at most a vertex in $L(G)$. Hence, any triconnected component of $L(G)$ is a complete subgraph and satisfies Theorem 1. □

Since the study of edge tree spanners is equivalent to the study of tree spanners of line graphs, and deciding whether a graph is 2-admissible is polynomial-time solvable, Theorem 1 implies Corollary 1.

Corollary 1 *Edge 2-admissibility is polynomial-time solvable.*

The edge stretch index of cycle graphs and complete graphs are useful to characterize edge 3-admissible graphs, as discussed in Sect. 3.

Complete graphs are 2-admissible, however their line graphs are not. In order to prove that $\sigma'(K_n) = 4$, from Lemma 1 we have that $\sigma'(K_5) \leq 4$, and it is possible to prove that K_5 is not edge 3-admissible, as highlighted below.

To prove that K_5 is not edge 3-admissible, one can verify by a case analysis that it is not possible obtain a spanning tree T such that $T \cap L(K_5)$ has at least 3 internal nodes. Clearly, $T \cap L(K_5)$ cannot have more than 3 internal nodes, because otherwise the edge factor of such a tree would be at least 4. Moreover, it is not possible obtain a spanning tree T such that $T \cap L(K_5)$ is a bi-star or it is a tree with three internal nodes whose leaves at distance 4 in T are not adjacent in $L(K_5)$.

In Sect. 3 we prove that being edge 3-admissible is a hereditary property for induced subgraphs (Lemma 2), then Corollary 3 states that $\sigma'(K_n) = 4$, for $n \geq 5$.

A graph G has a *distance two dominating edge* uv if every edge of $E(G)$ has a vertex in $N[u] \cup N[v]$ as one of its endpoints, where $N[x]$ is the *closed neighborhood of* x, i.e. $N[x] = N(x) \cup \{x\}$. Moreover, G has two adjacent distance two dominating edges uv and vw if every edge of $E(G)$ has a vertex in $N[u] \cup N[v] \cup N[w]$ as one of its endpoints.

Lemma 1 *A graph G with a distance two dominating edge uv has $\sigma'(G) \leq 4$.*

Proof Since G has a distance two dominating edge uv, there is a spanning tree with diameter at most four of $L(G)$ with the vertex uv as its root, the vertices $\{ux \mid ux \in E(G)\} \cup \{vy \mid vy \in E(G)\}$ adjacent to uv, and the remaining vertices of $L(G)$ adjacent to some vertex in $\{ux \mid ux \in E(G)\} \cup \{vy \mid vy \in E(G)\}$. □

Figure 2 depicts graphs with distance two dominating edges and their edge tree 4-spanners, as the proof of Lemma 1. A graph is *split* if its vertex set can be partitioned into a stable set and a clique. The *join* between two graphs G_1 and G_2 results in the graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Several graph classes can be constructed by join and complement of join operations, i.e. *union* operations. Cographs are the P_4-free graphs, i.e. graphs without a P_4 as an induced subgraph, and G is a cograph iff it has the following recursive definition: (i) G is a K_1; (ii) G is a join of cographs; (iii) G is a union of cographs. A generalization of cographs are the graphs with few P_4's, such as P_4-sparse and P_4-tidy [7].

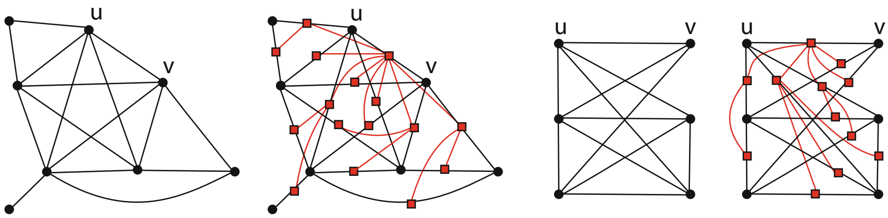

Fig. 2 A split graph and a join graph with their edge tree 4-spanners

A graph is P_4-*sparse* if for each set of 5 vertices, there is at most one induced P_4. A graph is P_4-*tidy* if for each induced P_4 of G, say P, there is at most one vertex $v \in V(G) \setminus V(P)$ such that $V(P) \cup \{v\}$ induces at most two P_4's in G. P_4-tidy generalizes P_4-sparse graphs, and G is a P_4-tidy graph iff it has the following recursive definition: (i) G is P_5, C_5, $\overline{P_5}$, or K_1; (ii) G is a join of P_4-tidy graphs; (iii) G is a union of P_4-tidy graphs; (iv) G is a spider; (v) G is an almost spider. A graph is a *spider* graph if its vertex set can be partitioned into $\mathscr{S}$, $\mathscr{K}$ and $\mathscr{R}$ such that (i) $\mathscr{K}$ is a clique ($\mathscr{K}$ is called *body*), $\mathscr{S}$ is a stable set and $|\mathscr{S}| = |\mathscr{K}| \geq 2$; (ii) each vertex of $\mathscr{R}$ ($\mathscr{R}$ is called *head*) is adjacent to all vertices of $\mathscr{K}$ and is non-adjacent to any vertex of $\mathscr{S}$; (iii) There is a bijection $f : \mathscr{S} \mapsto \mathscr{K}$ such that, for all $x \in \mathscr{S}$, either $N(x) = \{f(x)\}$, or $N(x) = \mathscr{K} - \{f(x)\}$. A graph is an *almost-spider* graph if it can be constructed from a spider graph $G = (\mathscr{S}, \mathscr{K}, \mathscr{R})$ by adding a vertex v' which is either a false twin of v or a true twin of v, such that $v \in \mathscr{S} \cup \mathscr{K}$ [10].

Split graphs, join graphs and P_4-tidy graphs are 3-admissible [3, 4]. Corollary 2 follows from Lemma 1 and: for split graphs, any clique's edge is distance two dominating; for join graphs between G_1 and G_2, any uv such that $u \in V(G_1)$ and $v \in V(G_2)$ is distance two dominating; for P_4-tidy graphs, any edge between the head and the body is distance two dominating.

Corollary 2 *Split graphs, join graphs and* P_4*-tidy graphs are edge* 4*-admissible.*

Since 3-admissibility is still open and t-admissibility is NP-complete, for $t \geq 4$, we are interested to establish the computational complexity of determining the edge stretch index. In Sect. 3, we prove that edge 3-admissibility is polynomial-time solvable, and as an immediate consequence, we are able to determine in polynomial time the edge stretch index for any edge 4-admissible graph, such as split graphs, join graphs and P_4-tidy graphs (Corollary 6).

3 Edge 3-Admissibility Is Polynomial-Time Solvable

Lemma 2 *Edge* 3*-admissibility is a hereditary property for induced subgraphs.*

Proof Assume that there is an edge 3-admissible graph G with an induced subgraph H such that H is not edge 3-admissible. W.l.o.g. let G' be an induced subgraph of G such that: $|V(G')| = |V(H)| + 1$, $u \in V(G') \cap V(H)$; G' is edge 3-admissible; H is edge k-admissible for $k \geq 4$; T' is an edge tree 3-spanner of G'; and T is an edge k-tree spanner of H with $k \geq 4$. In any edge tree k-spanner T of H there is a path P with $k + 1$ vertices using edges of T and an edge of G' not in T between the two endpoints of this path (see Fig. 3a that considers $k = 5$). Since G' is edge 3-admissible, the addition of the vertex u must remove a part of that path P from T. For the sake of contradiction, assume T'' is a tree that contains at least three internal nodes among the edges incident to u. Since these edges have u as endpoint, then the leaves that are at distance 4 in T'' correspond to adjacent edges in G', a contradiction. Therefore, the edges incident to u must be a bi-star in T' (see Fig. 3b).

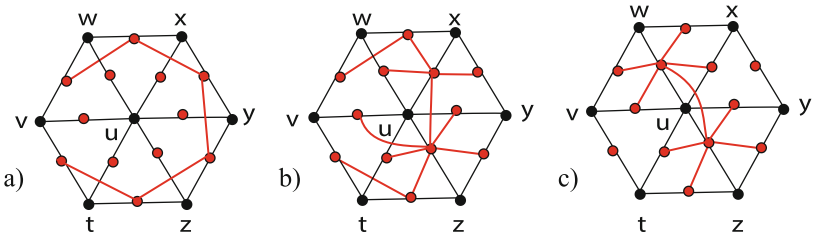

Fig. 3 (**a**) $V(H) = \{v, w, x, y, z, t\}$ and a path P in red. (**b**) In red a bi-star satisfying Case 1. (**c**) In red a bi-star satisfying Case 2

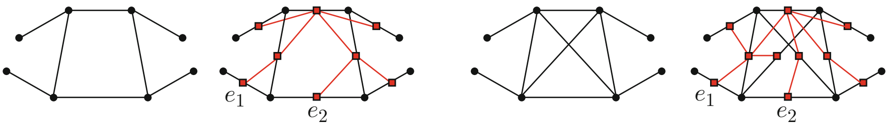

Fig. 4 C_4 and K_4 whose vertices have degree at least 3 and 4 in G, resp. Note that $d_T(e_1, e_2) = 4$

W.l.o.g. assume that u is adjacent to all vertices of G related to the path P of T. The edges of the bi-stars cover at most four vertices of P. We have two cases: Case 1: the bi-star connects consecutive vertices of P. In this case it does not reduce the distance between the vertices of P in T' (e.g. see Fig. 3b, the distance between vw and vt is 5 in T') and T' is not an edge tree 3-spanner, a contradiction; Case 2: the bi-star connects non-consecutive vertices of P. In this case it does reduce the distance between vertices of P, however, the vertex xy between this non consecutive vertex of P is connected to leaves of the two centers of the bi-star in $L(G)$, which implies that T' is not edge 3-admissible, a contradiction (Fig. 3c). □

Corollary 3 *Any complete graph K_n has $\sigma'(K_n) = 4$, for $n \geq 5$.*

Proof Since $\sigma'(K_5) = 4$ (Sect. 2) and for $n \geq 5$, K_n has a K_5 as an induced subgraph, then, by Lemma 2, we have that K_n are not edge 3-admissible, for $n \geq 5$. Furthermore, complete graphs have a distance two dominating edge, hence by Lemma 1, $\sigma'(K_n) \leq 4$, for $n \geq 5$, and the result follows. □

Line graphs of K_n are complement of *Kneser graph* $KG_{n,2}$ [8], then $\sigma(\overline{KG_{n,2}}) = 4$.

Note that C_k and K_k, for $k \geq 5$ are not subgraphs of edge 3-admissible graphs. See Fig. 4 for examples of C_4 and K_4 where all vertices have degree at least 3 and 4 in G, resp. Suppose H is an induced C_4 (or K_4) in G. In $L(G[H])$ there must be a path through all $L(C_4)$'s vertices (or through four $L(K_4)$'s vertices) and one more vertex corresponding to an edge that does not belong to the C_4 (to the K_4) in H. Hence, it implies that $\sigma'(H) \geq 4$, and Corollary 4 follows.

Corollary 4 *Let G be an edge 3-admissible graph. If $X \in \{C_4, K_4\}$ is an induced subgraph of G, then there is a vertex $v \in V(X)$ such that $N_G(v) \subseteq V(X)$.*

By Corollary 4, any edge 3-admissible graph has vertices of degree 2 and 3 in each induced C_4's and K_4, resp. Hence, Construction 2 presents a way to break C_4's and K_4's into P_5's and K_3's, resp., in order to present a stronger necessary condition in Lemma 4.

Construction 2 *Let G be a graph that satisfies: G does not have induced C_k nor K_k, for $k \geq 5$, as induced subgraphs; for each induced C_4 there is a vertex of degree two in G; and for each induced K_4 there is a vertex of degree three in G. We construct a graph H from G as follows:*

1. *each induced $C_4 = a, b, c, d, a$, for $d_G(a) = 2$, is transformed into a $P_5 = a, b, c, d, a'$ by adding a new vertex a' and the edge da', and removing the edge da;*
2. *each induced $K_4 = \{a, b, c, d\}$, for $d_G(a) = 3$, is transformed into three complete graphs K_3 by adding a new vertex a' and: removing edge ba; adding edges ba' and ca'.*

Lemma 3 *A graph G is edge 3-admissible if and only if the graph H from Construction 2 is edge 3-admissible.*

Proof If G is edge 3-admissible, then all edges of an edge tree 3-spanner of G are used to obtain a spanning tree of H and we do not increase the edge stretch index from G to H, because, by construction, we are not increasing a maximum path between any two adjacent vertices of G in H. If H is edge 3-admissible, then all edges of an edge tree 3-spanner of H are used for a spanning tree of G and, since we are identifying vertices that belong only to C_4's or K_4's in G, such identification does not affect cycles that give the edge tree 3-spanner of H and does not increase such index of G by the used edges of H. □

A *k-tree* is a graph obtained from a K_{k+1} by repeatedly adding vertices in such a way that each added vertex v has exactly k neighbors defining a clique of size $k+1$. A *partial k-tree* is a subgraph of a k-tree [9].

Lemma 4 *Let G be an edge 3-admissible graph. If H is the graph obtained from G in Construction 2, then H is a chordal partial 2-tree graph.*

Proof If G is edge 3-admissible with $X \in \{C_4, K_4\}$ as an induced subgraph, then, by Corollary 4, X must have at least one vertex a such that $N(a) \subseteq X$. Based on that, in Construction 2 we obtain a graph without C_4's nor K_4's. Since, by Lemma 3, the transformed graph H from an edge 3-admissible graph G is also edge 3-admissible, we have that the length of any clique is at most 3 and it does not have C_k, for $k \geq 4$. Since chordal graphs with maximum clique of length 3 are partial 2-tree [9], we have that H is a chordal partial 2-tree graph. □

By Lemma 4, edge tree 3-spanner graphs are formed by 2-trees where either an edge or a vertex connects two 2-trees. Hence, for the former case such edge is a bridge and for the later case it is a cut vertex of the graph. Lemmas 5 and 6 present conditions that force spanning trees correspond to edge 3-admissible graphs.

Lemma 5 *Given an edge* 3*-admissible graph* G *and two* 2*-trees* A_1 *and* A_2 *connected by a bridge* uv*, such that* $|V(A_i)| > 3$ *for* $i \in \{1, 2\}$*, then for any edge* 3*-spanner* T*,* uv *is a pendant vertex in* $T[A_1 \cup \{u, v\}]$*, i.e.* $d_{T[A_1 \cup \{u,v\}]}(uv) = 1$.

Proof Assume $u \in A_1, u, x, y$ is a triangle and $v \in A_2$. Suppose $d_{T[A_1 \cup \{u,v\}]}(uv) \geq 2$, hence xy must be adjacent to either ux or to uy in T. W.l.o.g., let xy be adjacent to uy, then, there is an edge wx in A_1 which implies the distance between wx and xy to be equal to 4 by a path through uv, a contradiction. □

Each bridge forces a unique way to obtain an edge tree 3-spanner of G. Hence, by Lemma 5, assume G is 2-*edge connected*, i.e. there is not a bridge in G. Otherwise, we consider each connected component separately after the bridges removal of G.

Now, consider the case that G has a cut vertex. Let a *windmill graph* $Wd(3, n)$ be the graph constructed for $n \geq 2$ by identifying n copies of K_3 at a universal vertex. Since an edge 3-admissible graph is partial 2-tree, we have that if there is a cut vertex u in G, then $G[N_G[u]]$ contains a windmill graph $Wd(3, d)$, for $2 \leq d \leq \frac{d_G(u)}{2}$. Let a *diamond graph* be a K_4 minus an edge. Each K_3 of a windmill centered in u has two vertices of degree 2, or it has a cut vertex of G distinct of u, or it belongs to a diamond graph of G.

Lemma 6 *Let* G *be* 2-edge connected *graph with a cut vertex* u *and edge* 3*-admissible. If the associated windmill graph* $Wd(3, n)$ *centered in* u *satisfies* $n \geq 3$*, then* u *belongs to at most* 2 *diamonds in* G.

Proof Assume that u is center of the windmill graph $Wd(3, 3)$ and it belongs to 3 diamonds D_1, D_2 and D_3 in G. We prove that G is not edge 3-admissible, and then it implies that if G is edge 3-admissible, then u does not belong to more than 3 diamonds for every $n \geq 3$, either, because the hereditary property proved in Lemma 2.

Note that $L(H)$, for $H = Wd(3, 3) \cup D_1 \cup D_2 \cup D_3$, is composed by a K_6 and the addition of three other subgraphs, named B_1, B_2 and B_3, constructed by a join between a vertex and a C_4. Moreover, each edge of a perfect matching of the K_6, $\{e_1, e_2, e_3\}$, is identified to an edge of B_1, B_2 and B_3 that belongs to the C_4s, resp. Suppose that $L(H)$ is 3-admissible, hence for any tree 3-spanner T of $L(H)$ we have that $T \cap L(H)$ is a fl-centered bi-star, for f and l being any two K_6's vertices. Since any vertex of the K_6 belongs to exactly one of the other three subgraphs added to it, i.e. each K_6's vertex belongs to either B_1, B_2 or B_3, then at least two adjacent vertices of $L(H)$ are adjacent to leaves of the fl-centered bi-star, implying $\sigma'(H) = 4$. □

If there is a vertex u that belongs to $Wd(3, 2)$ then there are two solutions in $T \cap Wd(3, 2)$, less than isomorphism. Consider a $Wd(3, 2)$ such that $V(Wd(3, 2)) = \{u, v, w, v', w'\}$ such that u, v, w and u, v', w' induce K_3's. Note that an edge tree 3-spanner $T \cap Wd(3, 2)$ can be formed as follows: Case 1: $\{uv, uw\}$, $\{uv, vw\}$, $\{uv, uv'\}$, $\{uv', uw'\}$, $\{uv', v'w'\}$; Case 2: $\{uv, uw\}$, $\{uv, vw\}$, $\{uv, uv'\}$, $\{uv, uw'\}$, $\{uv', v'w'\}$. Any other edge tree spanner of $Wd(3, 2)$ is not edge tree 3-spanner.

Although a $Wd(3,2)$ graph centered in u may have two spanning trees, if each triangle also belongs to a diamond, let D_1 and D_2 be such diamonds with vertices $V(D_1) = \{u, v, w, x\}$ and $V(D_1) = \{u, v', w', x'\}$, then the previous Case 1 is the unique edge tree 3-spanner for $T \cap Wd(3,2)$, less than isomorphism.

Furthermore, let $H = Wd(3,2) \cup D_1$ be formed by a $Wd(3,2)$ centered in u with vertices $V(Wd(3,2)) = \{u, v, w, v', w'\}$ such that vw belongs to the diamond D_1 with vertices $V(D_1) = \{v, w, s, t\}$, then we have that H is not edge 3-admissible, which can be verified by conditions above and a simple case analyses.

Hence, we have presented necessary conditions of a 2-edge connected graph G satisfying Construction 2 to be edge 3-admissible when it has a cut vertex.

Now, consider G a biconnected graph. Theorem 2 characterizes such graphs. The *diameter* of a graph G is the greatest distance between any pair of vertices, and is denoted by $D(G)$.

Theorem 2 *Given G a biconnected graph with $D(G) \leq 3$. We have that $\sigma'(G) \leq 3$ if and only if either there is distance two dominating edge $e_1 = uv$ or for any edges $e_1 = uv$, $e_2 = uw$, and $e_3 \notin N(u) \cup N(v) \cup N(w)$, e_3 is adjacent to edges only of $N(v)$ (or equivalently, only of $N(w)$).*

Proof If G has a dominating edge, for $D(G) \leq 3$, then $\sigma'(G) \leq 3$ by a uv centered bi-star. Or, if any edge is not dominated by e_1 but it is adjacent to edges only of $N(v)$, then in the solution spanning tree such vertex is adjacent to a leaf of v and it does not turn $\sigma'(G) \geq 4$ because it is not adjacent to leaves of u. Assume that G is edge 3-admissible, there is not a distance two dominating edge and there is an edge e_3, such that $e_3 \notin N(u) \cup N(v) \cup N(w)$ that is adjacent to edges of $N(v)$ and $N(w)$. In this case e_3 is connected to leaves of the two centers of the bi-star in $L(G)$, which implies that T' is not edge 3-admissible, a contradiction. □

Note that Theorem 2 gives another argument on the lower bound of Corollary 3, since a K_n does not satisfy conditions of Theorem 2.

Corollary 5 *Edge 3-admissibility is polynomial-time solvable.*

4 Edge Stretch Index for Split and Generalized Split Graphs

Since $\sigma'(G) \leq 4$ for graphs with a distance two dominating edge (Theorem 1), the polynomial time algorithm for edge 3-admissible of Corollary 5 also works for these graphs and their subclasses, such as split graphs, join graphs and P_4-tidy graphs. I.e., we know whether these graphs have $\sigma'(G) = 2$, $\sigma'(G) = 3$ or $\sigma'(G) = 4$.

Corollary 6 *Edge t-admissibility is polynomial-time solvable for split graphs, join graphs and P_4-tidy graphs.*

As presented in Corollary 6, we are able to determine the edge stretch index for split graphs. Split graphs can be generalized as the (k, ℓ)-graphs, which are the

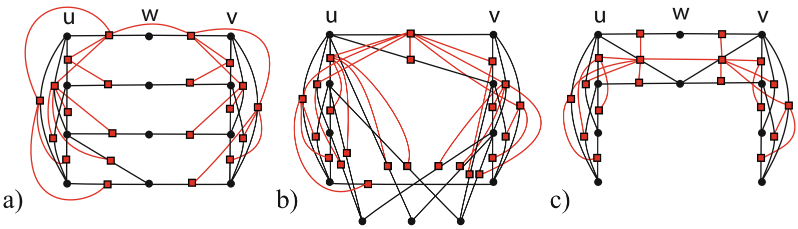

Fig. 5 Cases of (1, 2)-graphs and the corresponding edge tree spanners. (**a**) an edge 5-admissible graph. (**b**) and (**c**) are edge 4-admissible graphs

graphs that the vertex set can be partitioned into k stable sets and ℓ cliques. The (k, ℓ)-graphs are also denoted as the generalized split graphs [5].

In [4], the dichotomy *P versus* NP-complete on deciding the stretch index for (k, ℓ)-graphs was partially classified. One of the open problems regarding MSST is to establish the computational complexity for (1, 2)-graphs. Next, we prove that the edge stretch index for (1, 2)-graphs can be determined in polynomial time.

We denote a (1, 2)-graph as a graph $G = (V, E)$ where V is partitioned into $V = \mathscr{K}_1 \cup \mathscr{K}_2 \cup S$, such that each $\mathscr{K}_i$ induces a clique and S is a stable set.

Lemma 7 *If G is a* (1, 2)*-graph, then G is edge* 5*-admissible.*

Proof Since G is connected, there is a path between a vertex $u \in \mathscr{K}_1$ and $v \in \mathscr{K}_2$ by an edge uv or by a $P_3 = u, w, v$. Figure 5 depicts the cases of (1, 2)-graphs and their edge 5-tree spanners. In Fig. 5a there is an induced C_6 by two vertices of each clique and two vertices of S, implying a non-edge in any tree, hence $\sigma'(G) \leq 5$. □

Theorem 3 *A* (1, 2)*-graph* $G = (\mathscr{K}_1 \cup \mathscr{K}_2 \cup S, E)$ *has* $\sigma'(G) \leq 4$ *if and only if G has a distance two dominating edge or two adjacent distance two dominating edges that are adjacent to at least one edge of each pair of edges incident to a vertex of S such that one endpoint of an edge of this pair is in* $\mathscr{K}_1$ *and another one in* $\mathscr{K}_2$*.*

Proof From Lemma 1, if G has a distance two dominating edge, then G is edge 4-admissible. Moreover, if G has two distance two dominating edges e_1 and e_2 adjacent to at least one edge of each pair of edges incident to a vertex of S such that one endpoint of an edge of this pair is in $\mathscr{K}_1$ and an endpoint of the other edge is in $\mathscr{K}_2$, one obtain an edge tree 4-spanner T of G by selecting any spanning tree of $L(G)$ that maximizes the degrees of these two distance two dominating edges in T.

Conversely, for the sake of contradiction assume that G does not have such distance two dominating edges and T is an edge tree 4-spanner of G. Since G is connected, there is a vertex of S adjacent to both $\mathscr{K}_1$ and $\mathscr{K}_2$ and we can select these two edges of S to be two distance two dominating edges of G. Therefore, for all distance two dominating edges e_1 and e_2 of G we have two edges e_i and e_f incident to a vertex of S such that these edges are both not adjacent to e_1 and e_2. Therefore, in the best case scenario these two edges are adjacent to edges e'_1 and e'_2

adjacent to e_1 and e_2. However, we have a path in T e_i e'_1 e_1 e_2 e'_2 e_f with these two edges e_i and e_f sharing an endpoint, which implies that T is not an edge 4-tree spanner of G. □

Corollary 7 *Edge t-admissibility is polynomial-time solvable for* $(1, 2)$*-graphs.*

5 Edge 8-Admissibility Is NP-Complete for Bipartite Graphs

Next, we present a polynomial time transformation from 3-SAT [6] to edge 8-admissibility for $(2, 0)$-graphs, i.e. bipartite graphs.

Construction 3 *Given an instance $I = (U, C)$ of 3-SAT we construct a graph G as follows. We add a P_2 with labels x and x' to G. For each variable $u \in U$ we add a C_8 to G with three consecutive vertices labeled as u, m_u, and $\overline{u}$ and the other five consecutive vertices labeled as u_1 to u_5. For each u_i, $i = 1, \ldots, 5$, u and $\overline{u}$ we add a pendant vertex. For each variable $u \in U$ we add the edge xm_u to G. For each clause $c_1 = (u, v, w) \in C$, we add two vertices vertex c_1 and c'_1 to G and the edges $c_1c'_1$, c_1u, c_1v, and c_1w. For each variable $u \in U$ we add a P_4 to G with endpoints labeled p_{u1} and p_{u4} and the edges $p_{u1}x$ and $p_{u4}m_u$.*

Figure 6 depicts an example of a graph obtained from a 3-SAT instance.

The key idea of the proof of Theorem 4 is that, for each variable $u \in U$, we have exactly one edge in the edge tree 8-spanner T which is near to x and u or $\overline{u}$. We relate this proximity to a true assignment of that literal. Next, we require that at least one edge incident to each clause to be connected to a true literal. Otherwise, if they are all false literals, we end up with two of the edges incident to that clause being vertices of $L(G)$ with distance at least 9 in T.

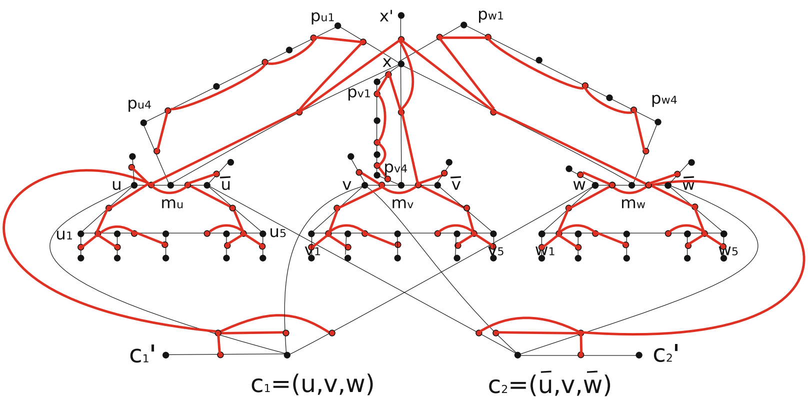

Fig. 6 Graph obtained from Construction 3 on the instance $I = (\{u, v, w\}, \{(u, v, w), (\overline{u}, v, \overline{w}\})$ and an edge tree 8-spanner of it in red

Theorem 4 *Edge* 8*-admissibility is* NP*-complete for bipartite graphs.*

Proof By construction, G is bipartite. Moreover, not only the problem is in NP, but also the size of the graph G, obtained from Construction 3 on an instance $I = (U, C)$ of 3-SAT, is polynomially bounded by the size of I. We prove that G is edge 8-admissible if and only if there is a truth assignment to I. Consider a truth assignment of $I = (U, C)$. We obtain an edge tree 8-spanner T of G as follows (see Fig. 6).

Add to T the edges: $\{x'x, xm_u \mid u \in U\}$; $\{xm_u, m_u u \mid u \in U$ and u is true$\}$ or

$\{xm_u, m_u\overline{u} \mid u \in U$ and $\overline{u}$ is true$\}$; $\{um_u, \overline{u}m_u \mid u \in U\}$; For each clause select a true literal and add to T: $\{c'c, uc \mid c$ is a clause with the selected true literal $u\}$;

$\{uc, um_u \mid c$ is a clause with the selected true literal $u\}$;

$\{\overline{u}c, \overline{u}m_u \mid c$ is a clause with the selected true literal $\overline{u}\}$;

$\{uc, vc \mid c$ is a clause with the selected trueliteral u and v is other literal of $c\}$;

For each variable $u \in U$ add to T the edges: $\{m_u p_{u_4}, p_{u_4} p_{u_3}\}$; $\{p_{u_4} p_{u_3}, p_{u_3} p_{u_2}\}$; $\{p_{u_3} p_{u_2}, p_{u_2} p_{u_1}\}$; $\{p_{u_2} p_{u_1}, p_{u_1} x\}$; $\{p_{u_1} x, xm_u\}$; $\{um_u, uu_1\}$; $\{\overline{u}m_u, \overline{u}u_5\}$; $\{uu_1, u_1u_2\}$; $\{u_3u_4, u_4u_5\}$; $\{u_4u_5, \overline{u}u_5\}$; and each pendant G is added to a solution tree as Fig. 6

Consider an edge tree 8-spanner T of G (resp. tree 8-spanner of $L(G)$), we present a truth assignment of $I = (U, C)$. First we claim that for each variable $u \in U$, there is exactly one of these two edges in T: $\{xm_u, um_u\}$ and $\{xm_u, \overline{u}m_u\}$. Assume that both edges are in T. There are in $L(G)$ two adjacent vertices $u_i u_{i+1}$ and $u_{i+1}u_{i+2}$ of the cycle C_9 of variable u with distance 9 in T, a contradiction. Now, assume that both edges are not in T. We consider two cases. If there are no edges $p_{u_4}m_u, um_u$ or $p_{u_4}m_u, \overline{u}m_u$, then there are in $L(G)$ two adjacent vertices $p_{u_4}m_u$ and um_u (or $\overline{u}m_u$) with distance at least 9 in T, since it is necessary to make a path passing through xx', a contradiction. Otherwise, there is an edge $p_{u_4}m_u, um_u$ or $p_{u_4}m_u, \overline{u}m_u$. In both cases, let $c_1 = (u, v, w)$ be a clause that contains u, there are in $L(G)$ two adjacent vertices c_1v, vv_1 that have distance at least 9 in T, a contradiction.

Hence, relate the edge $\{xm_u, um_u\}$ or $\{xm_u, m_u\overline{u}\}$ in T for each variable $u \in U$ to a true assignment to the literal u or $\overline{u}$. Assume that there is a clause with three false literals $c_3 = (x, y, z)$. No matter how we connect the vertices $c_3'c_3$, c_3x, c_3y and c_3z in T, two of them have distance at least 9 in T, a contradiction. Therefore, each clause has at least one true literal, and this is a truth assignment of I. □

Construction 3 can be adapted in order to prove that edge $2k$-admissibility is NP-complete, for $k \geq 5$. It can be obtained by subdividing the edge $m_u x$ and the cycles corresponding to each variable u.

6 Concluding Remarks

We have obtained the edge stretch index of some graph classes, or equivalently, the stretch index of line graphs, such as gridline graphs (line graphs of bipartite graphs); complement of Kneser graphs $KG_{n,2}$ (line graphs of complete graphs); and line graphs of (k, ℓ)-graphs. Although deciding the 3-admissibility is open for more than 20 years, we characterize the edge 3-admissible graphs in polynomial time, and we also prove that edge 8-admissibility is NP-complete, even for bipartite graphs. Hence, some open questions arise, such as determine the computational complexity of edge t-admissibility for $4 \leq t \leq 7$, and $t = 2k + 1$, $k \geq 4$.

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References

1. Bhatt, S., Chung, F., Leighton, T., Rosenberg, A.: Optimal simulations of tree machines. In: 27th Annual Symposium on Foundations of Computer Science, pp. 274–282. IEEE, Piscataway (1986)
2. Cai, L., Corneil, D.G.: Tree spanners. SIAM J. Discrete Math. **8**(3), 359–387 (1995)
3. Couto, F., Cunha, L.F.I.: Tree t-spanners of a graph: minimizing maximum distances efficiently. In: 12th COCOA, Lecture Notes in Computer Science, vol. 11346, pp. 46–61 (2018)
4. Couto, F., Cunha, L.F.I.: Hardness and efficiency on minimizing maximum distances for graphs with few P4's and (k, ℓ)-graphs. Electron. Notes Theor. Comput. Sci. **346**, 355–367 (2019)
5. Couto, F., Faria, L., Gravier, S., Klein, S.: Chordal-(2, 1) graph sandwich problem with boundary conditions. Electron. Notes Discrete Math. **69**, 277–284 (2018)
6. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman Co., New York (1979)
7. Giakoumakis, V., Roussel, F., Thuillier, H.: On P4-tidy graphs. Discr. Math. Theoretical Comput. Sci. **1**, 17–41 (1997)
8. Godsil, C., Royle, G.: Kneser graphs. In: Algebraic Graph Theory, pp. 135–161. Springer, New York (2001)
9. Heggernes, P.: Treewidth, partial k-trees, and chordal graphs. INF334-Advanced algorithmical techniques, Department of Informatics, University of Bergen (2005)
10. Jamison, B., Olariu, S.: P-components and the homogeneous decomposition of graphs. SIAM J. Discrete Math. **8**(3), 448–463 (1995)
11. Peleg, D., Ullman, J.D.: An optimal synchronizer for the hypercube. SIAM J. Comput. **18**(4), 740–747 (1989)