Relating Hypergraph Parameters of Generalized Power Graphs

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Abstract Graph parameters like the chromatic number, independence number, clique number and many others alongside with their corresponding adjacency matrix have been broadly studied and extended to hypergraphs classes. A generalized power graph G_s^k of a graph G is a k-uniform hypergraph constructed by blowing up each vertex of G into a s-set of vertices and then adding $k-2s$ vertices of degree one to each edge, where $k \geq 2s$. A natural question is whether there exists any relation between structural parameters and spectral parameters of G_s^k with the respective parameters of the original graph G. In this paper we positively answer this question and investigate the parameters behavior.

Keywords Hypergraph · Generalized power graph · Strong chromatic number · Adjacency matrix of hypergraph · Spectral parameters

1 Introduction

A *hypergraph* $H = (V, E)$ is given by a vertex set V and a set $E = \{e : e \subseteq V\}$, whose elements are called (hyper) edges. A *graph* $G = (V, E)$ is a hypergraph such that $|e| \leq 2$ for every $e \in E$.

Different aspects of a graph like clique number, vertex or edge coloring, matching, connectivity, have been widely studied in many areas and can be generalized to hypergraph theory, for example hypergraph coloring and strong hypergraph coloring, weak and strong vertex connectivity [4, 9]. In [1], the authors stated that strong hypergraph coloring captures many previously studied graph coloring properties. These different ways of expanding a graph parameter have attracted the attention of researchers: [9] studied the difference between weak and strong vertex connectivity; and [2, 7, 11] exclusively focused their work on a single parameter.

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Spectral graph theory is another area that can be extended to hypergraphs. The goal of spectral graph theory is to study eigenvalues and eigenvectors of matrices associated with graphs finding information of structural properties of these graphs. Many graph matrices are studied in spectral graph theory which can also be extended to hypergraphs in different ways. The study of hypergraph matrices started in the 1990s with a generalization of the graph adjacency matrix [10], and new matrices are still being defined. In 2019, [3], a similar but different adjacency matrix of a hypergraph is defined, allowing a generalization of important spectral graph theory results to hypergraphs. Another approach to spectral hypergraph theory was given in 2012, [8], when it is proposed the study of hypergraphs through tensors.

This work aims to investigate the relation between hypergraph structural parameters and spectral parameters of a class of uniform hypergraphs, called generalized power graph, that was first considered in [15]. Recently, this class was studied by considering its tensor spectra [13–15].

Since these hypergraphs are constructed from a base graph, we discuss four main topics: the relation between hypergraph parameters with their respective graph parameters; the behavior of distinct variations of generalized graph parameters on this hypergraph class; the relation between the adjacency matrix of this hypergraph with matrices of the base graph; and new relations of hypergraph parameters and the adjacency matrix eigenvalues.

2 Preliminaries

A *hypergraph* $H = (V, E)$ is *k-uniform* if $|e| = k$ for every edge $e \in E(H)$. A *simple graph* $G = (V, E)$ is a 2-uniform hypergraph. In this work we consider only *simple hypergraphs*, i.e. it contains no loops (edges with $|e| = 1$) and no repeated edges. A *null hypergraph* contains no vertices (or no edges) and a hypergraph with only one vertex is called *trivial*. Two vertices in a hypergraph are *adjacent* if there is an edge which contains both vertices, and the *degree of a vertex* $v \in V$ is $d(v) = |\{e : v \in e\}|$, the number of edges that contain v.

A *path* P in a hypergraph H is a vertex-edge alternating sequence: $P = v_0, e_1, v_1, e_2, \ldots, v_{r-1}, e_r, v_r$ such that $v_0, v_1, \ldots, v_r$ are distinct vertices; $e_1, e_2, \ldots, e_r$ are distinct edges; and $v_{i-1}, v_i \in e_i$, $i = 1, 2, \ldots, r$. The *length* of a path P is the number of distinct edges. A hypergraph is *connected* if for any pair of vertices, there is a path which connects these vertices; it is *not connected* otherwise.

Let G be a graph and $s \geq 1$ an integer. The *s-extension* G_s of G is a $2s$-uniform hypergraph obtained from G by replacing each vertex $v_i \in V$ by a set $S_{v_i} = \{v_{i1}, \ldots, v_{is}\}$, where $S_{v_i} \cap S_{v_j} = \emptyset$ for every $v_i \neq v_j$. These s new vertices are called *copies* of v_i. More precisely, $V(G_S) = \{v_{11}, \ldots, v_{1s}, \ldots, v_{n1}, \ldots, v_{ns}\}$ and $E(G_s) = \left\{S_{v_i} \cup S_{v_j} : \left\{v_i, v_j\right\} \in E\right\}$. Note that $|V(G_s)| = s \cdot |V(G)|$ and $|E(G_s)| = |E(G)|$.

For a graph $G = (V, E)$ and an integer $k \geq 2$, the *k-expansion* G^k of G (also called the k^{th} *power graph of* G) is a k-uniform hypergraph obtained from G by

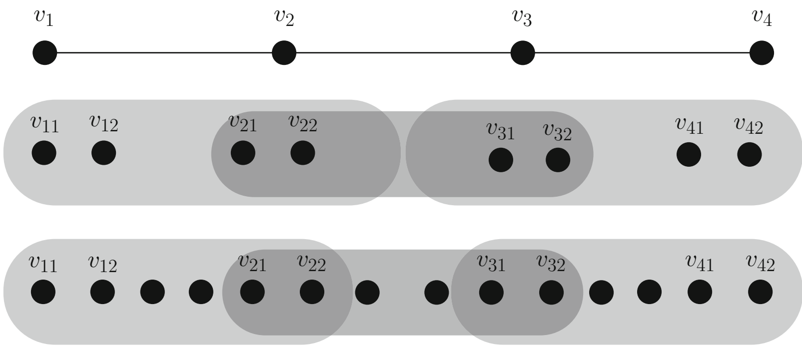

Fig. 1 Graph $G = P_4$ an its respective G_2 and G_2^6

adding $k-2$ new vertices of degree one to each edge of G. Note that $|V(G^k)| = |V(G)| + (k-2) \cdot |E(G)|$ and $|E(G^k)| = |E(G)|$.

Let $s \geq 1$ and $k \geq 2s$ be two integers and consider a graph G. The *generalized power graph* G_s^k is the k-uniform hypergraph $(G_s)^k$, obtained by adding $k-2s$ new vertices to each edge of G_s. These $(k-2s) \cdot |E(G)|$ new vertices of degree one are called *additional vertices* of G_s^k. Note that $|V(G_s^k)| = s \cdot |V(G)| + (k-2s) \cdot |E(G)|$ and $|E(G_s^k)| = |E(G)|$. See an example in Fig. 1.

Let G be a simple graph with n vertices. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ symmetric matrix with entries $a_{ij} = 1$ if there is an edge joining vertices v_i and v_j; and $a_{ij} = 0$ otherwise. The *degree matrix* of G, denoted by $D(G)$, is the $n \times n$ diagonal matrix defined as $D(G) = Diag(d(v_1), \ldots, d(v_n))$ where $d(v_i)$ is the degree of the vertex v_i. The *signless Laplacian matrix* for G, denoted by $Q(G)$, is the $n \times n$ symmetric matrix given by $Q(G) = D(G) + A(G)$. We denote the eigenvalues of $A(G)$ as $\lambda_1(G) \geq \ldots \geq \lambda_n(G)$ and the eigenvalues of $Q(G)$ as $q_1(G) \geq \ldots \geq q_n(G)$.

Let H be a hypergraph with n vertices. The *adjacency matrix* of H, denoted by $A(H)$ is the $n \times n$ symmetric matrix with entries $a_{ij} = |\left\{e \in E(H) : v_i, v_j \in e\right\}|$. We also denote the eigenvalues of $A(H)$ as $\lambda_1(H) \geq \ldots \geq \lambda_n(H)$.

Note that all previously defined matrices are real and symmetric, so they are Hermitian (a square matrix that is equal to its own conjugate transpose).

Now, we recall some matrix theory results that we use latter. Let X be a $m \times n$ matrix and let Y be a $p \times q$ matrix. The *kronecker product* $X \otimes Y$ is the $mp \times nq$ matrix:

$$X \otimes Y = \begin{bmatrix} x_{11}Y & \ldots & x_{1n}Y \\ \vdots & \ddots & \vdots \\ x_{m1}Y & \ldots & x_{mn}Y \end{bmatrix}.$$

Theorem 1 ([16]) *Let X be a $n \times n$ matrix and Y a $m \times m$ matrix. If $x_1 \geq \ldots \geq x_n$ are the eigenvalues of X and $y_1 \geq \ldots \geq y_m$ the eigenvalues of Y, then the nm eigenvalues of $X \otimes Y$ are: $x_1y_1, \ldots, x_1y_m, x_2y_1, \ldots, x_2y_m, \ldots, x_ny_1, \ldots, x_ny_m$.*

The next theorem, by Weyl [12], is a well known inequality that gives lower and upper bounds for the eigenvalues of a matrix sum.

Theorem 2 ([12]) *Let X and Y be square $n \times n$ Hermitian matrices with eigenvalues $x_1 \geq \ldots \geq x_n$ and $y_1 \geq \ldots \geq y_n$ respectively. If the eigenvalues of the sum $Z = X + Y$ are $z_1 \geq \ldots \geq z_n$, then $x_k + y_n \leq z_k \leq x_k + y_1$.*

A hypergraph version of the Wilf's theorem was established in [3] stating a relation between the chromatic number and the largest eigenvalue of its adjacency matrix. This generalization can be restricted to uniform hypergraphs as follows:

Theorem 3 ([3]) *Let H be a k-uniform hypergraph, then $\chi_S(H) \leq 1 + \lambda_1(H)$.*

3 Structural Parameters

Graph parameters can be extended to hypergraphs and most of them in more than one way. In this section we investigate how these parameters behave on the class G_s^k and their relation with the respective parameters of the original graph G.

Let H be a k-uniform hypergraph. A set $U \subseteq V(H)$ is a *clique* if every subset of U with k elements is an edge of H. The *clique number* is $\omega(H) = max\{|U| : U \subseteq V(H)$ is a clique$\}$.

Proposition 1 *Given a graph G with at least one edge, $s \geq 1$ and $k \geq 2s$ (except the case where $s = 1$ and $k = 2$, i.e. $G_s^k = G$), we have that $\omega(G_s^k) = k$. Moreover, every clique in G_s^k is composed by the k vertices of an edge.*

Proof First, observe that the intersection between two edges of G_s^k is formed by a set of s vertices or is empty. Choose any set of $k + 1$ vertices of G_s^k and suppose it is a clique. This means that there exist two edges in G_s^k which share $k - 1$ common vertices. This is a contradiction since $k \geq 2s,\ s \neq 1$ and $k \neq 2$. Clearly any set of k vertices of an edge is a clique. □

A *matching* of a hypergraph $H = (V, E)$ is a set $M \subset E$ of pairwise disjoint hyperedges of H. The *matching number* $\nu(H)$ is the cardinality of a maximum matching.

Proposition 2 *If G is a graph with $s \geq 1$ and $k \geq 2s$, then $\nu(G_s^k) = \nu(G)$.*

A *perfect matching* of a hypergraph H is a matching M such that each vertex in $V(H)$ is covered by exactly one edge in M. It is easy to see that for $s \geq 1,\ G_s$ has a perfect matching if and only if G has a perfect matching.

Proposition 3 *Let G be a graph that is not the union of disjoint edges. For $s \geq 1$ and $k > 2s$ the hypergraph G_s^k does not have a perfect matching.*

Proof Since $k > 2s$, each edge of G_s^k have $k - 2s$ vertices of degree one. The only way that all those vertices are covered by a matching M is when $M = E(G_s^k)$, and that can happen only when G is the union of disjoint edges. □

Given a hypergraph $H = (V, E)$ we construct new hypergraphs by deleting vertices in the following ways. The *strong vertex deletion* of a vertex $v \in V$ creates the hypergraph $H' = (V', E')$ where $V' = V - v$ and $E' = \{e \in E : v \notin e\}$. That is, the strong deletion of v removes v from the vertex set and removes all edges that contain v from the hypergraph. For any subset X of V, we use $H -_{(S)} X$ to denote the hypergraph formed by strongly deleting all the vertices of X from H. A vertex $v \in V$ is called a *strong cut vertex* of H if $H -_{(S)} v$ has more connected components than H, and a set $X \subseteq V$ is called a *strong vertex cut* of H if $H -_{(S)} X$ is disconnected. We define the *strong vertex connectivity* of H, denoted $\kappa_S(H)$ as follows: if H has at least one strong vertex cut, then $\kappa_S(H)$ is the cardinality of a minimum strong vertex cut of H; otherwise, $\kappa_S(H) = |V| - 1$. By convention, the strong vertex connectivity of a null or trivial hypergraph is 1. Observe that $\kappa_S(H) \leq \delta(H)$.

Proposition 4 *Given a connected graph G, $s \geq 1$ and $k \geq 2s$ integers such that $G_s^k \neq G$ then $\kappa_S(G_s^k) = 1$.*

Proof If $k > 2s$ removing a vertex that is originally from G_s disconnects G_s^k, since its deletion removes at least one edge and hence the $k - 2s$ additional vertices of this edge become isolated. Similarly, if $k = 2s$ then $s > 1$ and $G_s^k = G_s$. Removing any vertex leaves the $s - 1$ vertices that are its copies isolated. □

The *weak vertex deletion* of a vertex $v \in V$ creates the hypergraph $H' = (V', E')$ where $V' = V - v$ and $E' = \{e - \{v\} : e \in E\}$. That is, the weak deletion of v removes v from the vertex set, and all occurrences of v from the edges of the hypergraph H. For any subset X of V, we use $H -_{(W)} X$ to denote the hypergraph formed by weakly deleting all the vertices of X from H. Since we are only considering simple hypergraphs, we remove edges with only one vertex. A vertex $v \in V$ is called a *weak cut vertex* of H if $H -_{(W)} v$ has more connected components than H, and a set $X \subseteq V$ is called a *weak vertex cut* of H if $H -_{(W)} X$ is disconnected. We define the *weak vertex connectivity* of H, denote $\kappa_W(H)$ as follows: if H has at least one weak vertex cut, then $\kappa_W(H)$ is the cardinality of a minimum weak vertex cut of H; otherwise, $\kappa_W(H) = |V| - 1$. By convention, the weak vertex connectivity of a null or trivial hypergraph is 1.

Proposition 5 *Given a connected graph G that is not the complete graph and an integer $s \geq 1$, then $\kappa_W(G_s) = s.\kappa(G)$.*

Proof Note that by the construction of G_s, we have that if $X \subset V(G_s)$ is a weak vertex cut of G_s then $X = S_{v_1} \cup S_{v_2} \ldots \cup S_{v_r}$ and $\{v_1, \ldots, v_r\} \subseteq V(G)$ is a vertex cut of G. Now, let $\{v_1, \ldots, v_r\}$ be a minimum vertex cut of G. So, $S_{v_1} \cup S_{v_2} \ldots \cup S_{v_r}$ is a minimum weak vertex cut in G_s with $s.\kappa(G)$ elements, otherwise $v_1, \ldots v_r$ would not be a minimum vertex cut of G, a contradiction. □

Proposition 6 *Let G be a connected graph that is not the complete graph, $s \geq 1$ and $k > 2s$ be two integers. Then:*

(i) If $\kappa_W(G_s) = s$, then $\kappa_W(G_s^k) = s$.
(ii) If $\kappa_W(G_s) \geq 2s$, then $\kappa_W(G_s^k) = 2s$.

Proof First we observe that a vertex cut of G_s is a vertex cut of G_s^k. Also, after the k-expansion G_s^k of G_s, the only new minimum vertex cut is the one where we isolate the additional $k - 2s$ new vertices of an edge by removing the $2s$ already existing vertices (since the new vertices of G_s^k make no difference in a vertex cut). Hence:

(i) if X is a minimum weak vertex cut of G_s with less than $2s$ elements, then it is a minimum weak vertex cut of G_s^k.
(ii) if $\kappa_W(G_s) \geq 2s$, a minimum weak vertex cut of G_s has more than $2s$ elements. For each edge of G_s^k, the set of the $2s$ vertices that came from G^s is a minimum weak vertex cut of G_s^k since their removal leaves the additional $k-2s$ remaining vertices isolated. □

Next result follows from the fact that if $s = 1$ then $G_s = G$ and $G_s^k = G^k$.

Corollary 1 *Let G be a connected graph. For any $k > 2$ we have that:*

(i) if $\kappa(G) = 1$, then $\kappa_W(G^k) = 1$;
(ii) if $\kappa(G) \geq 2$, then $\kappa_W(G^k) = 2$.

We observe from the previous results that the difference between weak and strong vertex connectivity of hypergraphs can be arbitrarily large, since $\kappa_S(G_s^k) = 1$ and $\kappa_W(G_s^k) \geq s$, with s as large as desired. Finally, we also remark that the inequality $\kappa_W(H) \leq \delta(H)$ is not valid: if G is a connected graph with $\kappa(G) \geq 2$, we have for $k > 2s$ that $\delta(G_s^k) = 1 < 2s = \kappa_W(G_s^k)$.

The *distance* $d(v, u)$ between two vertices v and u is the minimum length of a path that connects v and u. The *diameter* $d(H)$ of H is defined by $d(H) = max\,\{d(v, u) : v, u \in V\}$. It is easy to see that given a graph G and $s \geq 1$, then $d(G_s) = d(G)$. But this is not always true for the k-expansion.

Proposition 7 $d(G) \leq d(G_s^k) \leq d(G) + 2$, *for any graph G, $s \geq 1$ and $k \geq 2s$, .*

Proof Suppose $d(G_s) = r$ and $P = v_1, e_1, v_2, e_2, \ldots, v_r, e_r, v_{r+1}$ be a maximum path of G_s. If $k > 2s$, we add $k - 2s$ vertices on each edge to obtain G_s^k. After that, if there is an additional vertex u such that $\{u, v_1\}$ belongs to an edge $e \neq e_1$ and another additional vertex w such that $\{w, v_{r+1}\}$ belongs to an edge $f \neq e_r$, the path $P = u, e, v_1, e_1, v_2, e_2, \ldots, v_r, e_r, v_{r+1}, f, w$ have length $d(G) + 2$. Moreover, $d(G_s^k) = d(G) + 2$ since we have at most 2 additional vertices on a path and the path must start and end on them, otherwise we would have to repeat edges. □

A *hypergraph coloring* is an assigning of colors $\{1, 2, \ldots, c\}$ to each vertex of $V(H)$ in such a way that each edge contains at least two vertices of distinct colors. A coloring using at most c colors is called a c-coloring. The *chromatic number* $\chi(H)$ of a hypergraph H is the least integer c such that H has a c-coloring.

It is easy to see that given a graph G we have that $\chi(G^k) = \chi(G_s) = \chi(G_s^k) = 2$ (except when $s = 1$ and $k = 2$). Another type of coloring, that is also a generalization of graph coloring, is the *strong hypergraph coloring*: is an assigning of colors $\{1, 2, \ldots, c\}$ to each vertex of $V(H)$ in such a way that every vertex of an edge has distinct colors. The *strong chromatic number* $\chi_S(H)$ of a hypergraph H is the least integer c such that H has a strongly c-coloring. Given a hypergraph H, note that:

1. $\chi_S(H) \geq |e|$ for every $e \in E(H)$;
2. $\chi(H) \leq \chi_S(H)$, since a strong hypergraph coloring is also a hypergraph coloring;
3. $\omega(H) \leq \chi_S(H)$ (similarly to graphs);
4. for the class G_s^k, the inequality $\omega(H) \leq \chi(H)$ is not valid, since $\chi(G_s^k) = 2$ but we can have edges (cliques) arbitrarily large.

We do not consider $\chi(G) = \chi_S(G_s^k) = 1$, since G has at least one edge. The following results establish relations between $\chi(G)$, $\chi_S(G_s)$ and $\chi_S(G_s^k)$.

Proposition 8 *If G is a graph and $s \geq 1$ is an integer, then $\chi_S(G_s) \leq s.\chi(G)$.*

Proof Let $\chi(G) = c$, we obtain a sc-strong coloring of G_s as follows: if $v \in V(G)$ has color $c(v) \in \{1, \ldots, c\}$ then, in G_s, assign colors $\{1 + (c(v) - 1)s, 2 + (c(v) - 1)s, \ldots, s + (c(v) - 1)s\}$ to S_v. □

Note that this bound is tight in the sense that the equality holds for any s-extension of the complete graph and does not hold for the 2-extension of C_5.

Proposition 9 *Let $s \geq 1$, $k > 2s$ be two integers and let G be a graph. We have that:*

(i) if $\chi_S(G_s) < k$ then $\chi_S(G_s^k) = k$;
(ii) if $\chi_S(G_s) \geq k$ then $\chi_S(G_s^k) = \chi_S(G_s)$.

Proof

(i) Let $\chi_S(G_s) = c < k$, we obtain a k-strong coloring of G_s^k as follows: we color the vertices of G_s^k that came from G_s with the same c-colors used in G_s. Hence, for each edge of G_s^k, we already used $2s$ colors from the set $\{1, 2, .., k\}$, $k > 2s$. Again, for each edge, we color the $k - 2s$ new additional vertices with the remaining $k - 2s$ distinct colors. This k-strong coloring of G_s^k is minimum, since k is the size of each edge of G_s^k.

(ii) Let $\chi_S(G_s) = c \geq k$ and consider a c-strong coloring of G_s. We color the vertices of G_s^k that came from G_s with the same c-colors used in G_s. For each edge, we color the $k - 2s$ additional vertices with any $k - 2s$ distinct colors from $\{1, 2, .., c\}$ different from the $2s$ colors already used in the vertices that came from G_s (since $c \geq k \geq 2s$ such colors exist). Suppose that it is possible to use less than c-colors in G_s^k. This implies that we can color all the vertices of G_s^k that came from G_s with less than c-colors and hence G_s with less than c colors, a contradiction. □

Corollary 2 *Let G be a graph and $k \geq 2$ an integer. Thus:*

(i) if $\chi(G) < k$ then $\chi_S(G^k) = k$;
(ii) if $\chi(G) \geq k$ then $\chi_S(G^k) = \chi(G)$.

A set $U \subseteq V$ is a *strong independent set* if no two vertices of U are adjacent. The *strong independence number* is $\alpha'(H) = max\{|U| : U \subseteq V(H)$ is a strong independent set of H $\}$. Let G be a graph and $s \geq 1$. From the construction of G_s we have that $\alpha'(G_s) = \alpha(G)$.

Proposition 10 *If G is a graph, $s \geq 1$ and $k > 2s$, then $\alpha'(G_s^k) = |E(G)|$.*

Proof Since $k > 2s$, every edge of G_s^k has at least one additional vertex. A set formed by choosing, for each edge, one of these additional vertices is a strong independent set of size $|E(G_s^k)| = |E(G)|$. This set is maximum since $\alpha'(H) \leq |E(H)|$, for any hypergraph H. □

Another generalization of a graph independent set is as follows: a set $U \subseteq V$ is an independent set if no edge of H is contained in U. As before, the *independence number* is $\alpha(H) = max\{|U| : U \subseteq V(H)$ is an independent set of H $\}$. Observe that if U is a strong independent set of a hypergraph H then U is also an independent set of H, since if U contains no two adjacent vertices then U does not contain an edge of H. So we have that $\alpha'(H) \leq \alpha(H)$.

Proposition 11 *If G is a graph and $s \geq 1$, then $\alpha(G_s) \geq (s-1) \cdot |V(G)| + \alpha(G)$.*

Proof Let $V(G) = \{v_1, \ldots, v_n\}$ and $V(G_s) = S_{v_1} \cup \ldots \cup S_{v_n}$. We obtain an independent set with $(s-1) \cdot n$ elements by choosing $s-1$ vertices of S_{v_i}, for each $i = 1, \ldots, n$. Now, adding a maximum stable set of G to the previous set produces a stable set of G_s with $(s-1) \cdot n + \alpha(G)$ vertices. □

Proposition 12 *If G be a graph, $s \geq 1$ and $k \geq 2s$, then $\alpha(G_s^k) \geq (s-1) \cdot |V(G)| + \alpha(G) + (k-2s) \cdot |E(G)|$.*

Proof By the construction of G_s^k and Proposition 11, a stable set of G_s is also a stable set of G_s^k with $(s-1) \cdot |V(G)| + \alpha(G)$ vertices. Adding to this stable set every $k-2s$ additional vertices of each edge of G_s^k produces a stable set with $(s-1) \cdot |V(G)| + \alpha(G) + (k-2) \cdot |E(G)|$ elements. □

Corollary 3 *Let G be a graph and $k \geq 2$, then $\alpha(G^k) \geq \alpha(G) + (k-2) \cdot |E(G)|$.*

4 Spectral Parameters

In this section we investigate spectral properties of hypergraphs and establish relations with structural parameters. The following result relates the adjacency matrix of G_s with the matrices $A(G)$ and $Q(G)$.

Proposition 13 *Let G be a graph with n vertices and $s > 1$. The adjacency matrix $A(G_s)$ is given on $s \times s$ blocks of size $n \times n$ by:*

$$A(G_s) = \begin{bmatrix} A(G) & Q(G) & Q(G) & \dots & Q(G) \\ Q(G) & A(G) & Q(G) & \dots & Q(G) \\ Q(G) & Q(G) & A(G) & \dots & Q(G) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q(G) & Q(G) & Q(G) & \dots & A(G) \end{bmatrix} = (J_s \otimes Q(G)) + (I_s \otimes -D(G)),$$

where J_s is the $s \times s$ matrix with 1 on all entries and I_s is the $s \times s$ identity matrix.

Proof First we show that $A(G_s)$ can be written in blocks like above. Let G be a graph on n vertices, then $|V(G_s)| = sn$. So, we order the vertices of the matrix as follows: $V(G_s) = \{v_{11},\ v_{21}, \dots,\ v_{n1},\ v_{12},\ v_{22}, \dots,\ v_{n2},\ v_{13},\ v_{23}, \dots,\ v_{n3}, \dots,$ $v_{1s},\ v_{2s}, \dots,\ v_{sn}\}$, where $S_{v_1} = \{v_{11}, v_{12}, \dots, v_{1s}\}$, $S_{v_2} = \{v_{21}, v_{22}, \dots, v_{2s}\}$, $\dots$, $S_{v_n} = \{v_{n1}, v_{n2}, \dots, v_{ns}\}$. We suppose that the vertices $v_{11}, v_{21}, \dots, v_{n1}$ are the vertices that come from G. So the $n \times n$ block formed by these is $A(G)$, since two vertices that are not copies from each other, share an edge in G_s if and only if they share an edge in G. Hence, we can see that all the diagonal blocks, formed by the vertices $\{v_{1i}, v_{2i}, \dots, v_{ni}\} \times \{v_{1i}, v_{2i}, \dots, v_{ni}\}$, $i = 1, \dots, s$, also correspond to $A(G)$.

For the other blocks we observe that, for every $i \neq j$, the blocks formed by $\{v_{1i}, v_{2i}, \dots, v_{ni}\} \times \left\{v_{1j}, v_{2j}, \dots, v_{nj}\right\}$ are always the same, since the vertices are copies from one another.

The block where $i = 1$ and $j = 2$ have the following structure: the vertices v_{11} and v_{12} are copies so they are in the same edges; and the number of edges they belong is exactly $d_G(v_1)$. So, their entry is equal $d_G(v_1)$, the degree of v_1 in G. The same works for the entries $v_{21} \times v_{22},\ v_{31} \times v_{32}, \dots, v_{n1} \times v_{n2}$. So, the diagonal of the block is made of the degrees in G. The entries that are not in the diagonal, for example, the entry $v_{11} \times v22$ is the same entry as $v_{11} \times v_{21}$, since v_{22} is a copy of the vertex v_{21}. So, these blocks are equal $D(G) + A(G) = Q(G)$. □

Proposition 14 *Let G be a graph on n vertices, $s > 1$ an integer and $d_1, \dots, d_n$ the vertices degree of G. Then $-d_1, \dots, -d_n$ are eigenvalues of $A(G_s)$. Moreover, each $-d_i$ has multiplicity at least $s - 1$.*

Proof Consider the vector $(-1, 0, \dots, 0|, 1, 0, \dots, 0|, 0, \dots, 0|, \dots, |0, \dots, 0) \in R^{sn}$, formed of s "blocks" with n entries each (ie, $|-1, 0, \dots, 0|$ has n entries, $|1, 0, \dots, 0|$ has n entries, $|0, \dots, 0|$ has n entries). This vector is an eigenvector of

$A(G_s)$ associated to the eigenvalue $-d_1$. Indeed:

$$\begin{bmatrix} A(G) & Q(G) & Q(G) & \dots & Q(G) \\ Q(G) & A(G) & Q(G) & \dots & Q(G) \\ Q(G) & Q(G) & A(G) & \dots & Q(G) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q(G) & Q(G) & Q(G) & \dots & A(G) \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ -- \\ 1 \\ 0 \\ \vdots \\ 0 \\ -- \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} -1.0 + 1.d_1 \\ -a_{2,1} + a_{2,1} \\ \vdots \\ -a_{n,1} + a_{n,1} \\ -- \\ -1.d_1 + 1.0 \\ -a_{2,1} + a_{2,1} \\ \vdots \\ -a_{n,1} + a_{n,1} \\ -- \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} d_1 \\ 0 \\ \vdots \\ 0 \\ -- \\ -d_1 \\ 0 \\ \vdots \\ 0 \\ -- \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note that the vectors
$(-1, 0, \dots, 0|, 0, \dots, 0|, 1, 0, \dots, 0|, \dots, |0, \dots, 0)$,
$(-1, 0, \dots, 0|, 0, \dots, 0|, 0, \dots, 0|, 1, 0, \dots, 0|, \dots, |0, \dots, 0), \dots$,
$(-1, 0, \dots, 0|, 0, \dots, 0|, 0, \dots, 0|, \dots, |1, \dots, 0)$
are also eigenvectors of $A(G_s)$ associated to the eigenvalue $-d_1$. Since we have s blocks, the multiplicity of $-d_1$ is at least $s - 1$. Similarly to $-d_2$, starting with the eigenvector:
$(0, -1, \dots, 0|, 0, 1, \dots, 0|, 0, \dots, 0|, \dots, |0, \dots, 0)$
up to $-d_n$, when starting with the eigenvector:
$(0, 0, \dots, -1|, 0, 0, \dots, 1|, 0, \dots, 0|, \dots, |0, \dots, 0)$. □

Next result immediately follows from the previous proposition observing that if G is connected then every vertex degree is positive.

Corollary 4 *If G is a graph on n vertices and $s > 1$ an integer, then $A(G_s)$ has at least $n \cdot (s - 1)$ non positive eigenvalues. Moreover, if G is connected then $A(G_s)$ has at least $n \cdot (s-1)$ negative eigenvalues (hence, $A(G_s)$ has at most n non negative eigenvalues).*

Next proposition provides bounds for the greatest eigenvalue of $A(G_s)$.

Proposition 15 *If G be a graph with n vertices and $s > 1$ integer, then*

$$s.q_1(G) - \Delta(G) \leq \lambda_1(G_s) \leq s.q_1(G) - \delta(G).$$

Proof For the left inequality, we observe that is known that the largest eigenvalue of J_s is s. Thus, by Theorem 1, the largest eigenvalue of $J_s \otimes Q(G)$ is $s \cdot q_1(G)$. Also, the smallest eigenvalue of $-D(G)$ is $-\Delta(G)$. So, by Theorem 1, the smallest eigenvalue of $I_s \otimes -D(G)$ is $-\Delta(G)$. Since $A(G_s) = (J_s \otimes Q(G)) + (I_s \otimes -D(G))$, from Theorem 2, we have that $s \cdot q_1(G) - \Delta(G) \leq \lambda_1(G_s)$.

For the right inequality, we observe again that the largest eigenvalue of $J_s \otimes Q(G)$ is $s.q_1(G)$. Also, the largest eigenvalue of $-D(G)$ is $-\delta(G)$. So, by Theorem 1, the largest eigenvalue of $I_s \otimes -D(G)$ is $-\delta(G)$. Since $A(G_s) = (J_s \otimes Q(G)) + (I_s \otimes -D(G))$, we have from theorem 2 that $\lambda_1(G_s) \leq s.q_1(G) - \delta(G)$. □

We observe that the bound given by Proposition 15 is tight in the sense that the equality holds for any regular graph G and for any $s > 1$. In what follows we obtain some results relating structural and spectral parameters.

A well known spectral graph theory result is: given a connected graph G the number of distinct eigenvalues of $A(G)$ is at least $d(G) + 1$ (this is also true for the number of distinct eigenvalues of $Q(G)$). This result is still true on hypergraphs, and the proof is basically the same. In [5] this bound is proved for the signless Laplacian matrix of a hypergraphs. We prove this result for hypergraphs adjacency matrix but first we present the following lemma.

Lemma 1 *Let H be a hypergraph and $A = A(H)$ its adjacency matrix. $(A^l)_{i,j} > 0$ if there is a path with length l connecting two distinct vertices i and j, and $(A^l)_{i,j} = 0$ otherwise (where $(A^l)_{i,j}$ denotes the entry i, j of $A(H)^l$).*

Proof The proof is by induction on l. If $l = 1$ the property clearly holds. Suppose the statement is true for $l \geq 1$ and now we check for $l + 1$. Note that $(A^{l+1})_{i,j} = \sum_{k=1}^{n}(A^l)_{i,k}(A)_{k,j}$. If there is a path with length $l + 1$ joining i and j then there must exist a path with length l joining i to a neighbor u of j. So $(A)_{u,j} = 1$ and by induction hypothesis $(A^l)_{i,u} > 0$. Therefore $(A^{l+1})_{i,j} > 0$. If there is no path with length $l + 1$ joining i and j then there does exist no path with length l joining i to any neighbor of j. So, if u is a neighbor of j we have that $(A^l)_{i,u} = 0$. When u is not a neighbor of j, we have that $(A)_{u,j} = 0$. Therefore $(A^{l+1})_{i,j} = 0$. □

Proposition 16 *If H is a connected hypergraph then $|\{$distinct eigenvalues of $A(H)\}| \geq d(H) + 1$.*

Proof Let $\lambda_1, \ldots, \lambda_t$ be all the distinct eigenvalues of $A = A(H)$. Then $(A - \lambda_1 I)\ldots(A - \lambda_t I) = 0$. So, we have that A^t is a linear combination of $A^{t-1}, \ldots, A, I$. Suppose by contradiction that $t \leq d(H)$. Hence there exist vertices i and j such that $d(i, j) = t$ and from our previous lemma, we have that $(A^t)_{i,j} > 0$. since there exists no path with length shorter than t joining i and j, $(A^{t-1})_{i,j} = 0, \ldots, (A)_{i,j} = 0, (I)_{i,j} = 0$. This is a contradiction, since $(A^t)_{i,j} = c_1(A^{t-1})_{i,j} + \ldots + c_{t-1}(A)_{i,j} + c_t(I)_{i,j}$. □

Previous proposition together with Proposition 7 result this simple corollary.

Corollary 5 *If G is connected then $|\left\{\text{distinct eigenvalues of } A(G_s^k)\right\}| \geq d(G)+1$.*

In other words, to find connected hypergraphs of the class G_s^k with few distinct adjacency eigenvalues, we have to look for graphs G with small diameter.

The next proposition gives us a different bound for $\chi_S(G_s^k)$, in terms of the largest eigenvalue of $Q(G)$ and the minimum degree of the graph G.

Proposition 17 *Given a graph G, $s > 1$ and $k \geq 2s$ we have that $\chi_S(G_s^k) = k$ or $\chi(G_s^k) \leq 1 + s.q_1(G) - \delta(G)$.*

Proof If $\chi_S(G_s^k) \neq k$ then by Proposition 9 $\chi_S(G_s^k) = \chi_S(G_s)$. Where by Theorem 3 and Proposition 15 we have: $\chi_S(G_s) \leq 1+\lambda_1(G_s) \leq 1+s.q_1(G)-\delta(G)$. □

A result from spectral graph theory states that if G is a graph, then $\alpha(G) \leq min\left\{\lambda(G)^-, \lambda(G)^+\right\}$, where $\lambda(G)^-$ is the number of non positive eigenvalues of $A(G)$ and $\lambda(G)^+$ is the number of non negative eigenvalues of $A(G)$". We show that this is not valid for the independence number of the class G_s.

Proposition 18 *If $s > 1$ and G is a connected graph on n vertices, then $\alpha(G_s) > min\left\{\lambda(G_s)^-, \lambda(G_s)^+\right\}$.*

Proof From Corollary 4, we have that $A(G_s)$ has at most n non negative eigenvalues. Hence, by Proposition 11: $\alpha(G_s) \geq (s-1)n + \alpha(G) > n \geq \lambda(G_s)^+ \geq min\left\{\lambda(G_s)^-, \lambda(G_s)^+\right\}$. □

Another result states that, for any graph G, $\frac{|V(G)|}{\alpha(G)} \leq \lambda_1(G) + 1$". This fact has not yet been generalized for hypergraphs and we prove its validity for connected hypergraphs in the class G_s.

Proposition 19 *If G is connected on n vertices and $s > 1$ then $\frac{|V(G_s)|}{\alpha(G_s)} \leq \lambda_1(G_s) + 1$.*

Proof By Proposition 11, we have that $\frac{|V(G_s)|}{\alpha(G_s)} = \frac{sn}{\alpha(G_s)} \leq \frac{sn}{(s-1)n+\alpha(G)} \leq \frac{sn}{(s-1)n} = \frac{s}{s-1}$. From Proposition 15, we have that $s \cdot q_1(G) - \Delta(G) \leq \lambda_1(G_s)$. Thus, it suffices to show that $\frac{s}{(s-1)} \leq s.q_1(G) - \Delta(G) + 1$ or, in other words, that $s \leq (s-1)(s.q_1(G) - \Delta(G) + 1)$. Since $s > 1$, if $s.q_1(G) - \Delta(G) + 1 \geq 2$ then the above inequality is valid, indeed: $s.q_1(G) - \Delta(G) + 1 \geq s(\Delta(G)+1) - \Delta(G) + 1 = (s-1)\Delta(G) + s + 1 \geq 2$. Where the first inequality holds because: [6] If G is a connected graph then $q_1(G) \geq \Delta(G) + 1$. □

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