Chapter 5
Pedagogical Tasks Toward Extending Mathematical Knowledge: Notes on the Work of Teacher Educators

Rina Zazkis and Ofer Marmur

5.1 Introduction

The work of mathematics teacher educators is multifaceted and relies on extensive knowledge in various related areas. This includes knowledge in mathematics, pedagogy, didactics, school curriculum, assessment methods, education research literature, research methodologies, and different theories of learning and teaching (Jaworski, 2008).

Our focus in this chapter is on teacher educators' work in professional development courses for secondary school mathematics teachers. On the one hand, our explicit goal in these courses is to extend the teachers' *mathematical* knowledge. On the other hand, as mathematics teacher educators, we are expected by teachers to address pedagogical issues associated with the *teaching* of mathematics. Consequently, to balance existing expectations of teachers with our goal of extending and strengthening their mathematical knowledge, we must engage the teachers in a delicate interplay between mathematics and pedagogy. We design tasks of pedagogical nature,

The original version of this chapter was revised due to some errors (listed below) in the text at page numbers 98 and 99.
• On pages 98 and 99, there are incorrect "+" symbols (with a circle around them), where this should have been an approximation symbol.
• Also on pages 98 and 99, there are two instances of wrong indentations.
The correction to this chapter is available at https://doi.org/10.1007/978-3-030-62408-8_23

R. Zazkis (✉)
Faculty of Education, Simon Fraser University, Burnaby, BC, Canada
e-mail: zazkis@sfu.ca

O. Marmur
Mathematics Education Unit, Department of Mathematics, Faculty of Science, The University of Auckland, Auckland, New Zealand

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_5

which are additionally aimed at extending the teachers' understanding of the underlying mathematics (e.g. Biza, Nardi, & Zachariades, 2007, 2018; Peng, 2007).

In order to advance teachers' mathematics, it is essential to first gain insight into the mathematical knowledge they possess, as teaching mathematics undoubtedly relies on prior knowledge of learners. School teachers usually have a relatively good grasp of their students' knowledge based on their prior work with the students and the curriculum sequence they follow. Teacher educators do not have such a privilege. This is particularly evident in postbaccalaureate teacher education programmes, to which students come from various undergraduate experiences, often educated in different decades and different countries. How is it possible to get a sense of where a group is in terms of their mathematical maturity and sophistication? How can a teacher educator be able to "scan" the group's knowledge of a mathematical topic in order to plan for, or adjust, subsequent instruction?

We address these questions by providing two illustrative examples: the first attends to the concept of a function, and the second deals with the concept of irrational exponents. In each case, we:

(a) Describe a task of a pedagogical nature that provides teacher educators with a window into strengths and weaknesses in teachers' knowledge of a particular mathematical topic.
(b) Provide a brief overview of the main themes that emerged from teachers' responses to the task, and exemplify several responses that informed our design of follow-up instructional activities.
(c) Illustrate the follow-up instructional engagements.

In the following section, we present the idea of script-writing, which served as the guiding principle for the design of our pedagogical tasks. We then discuss the *usage-goal framework* (Liljedahl, Chernoff, & Zazkis, 2007) that we employ to illustrate the symbiosis of mathematics and pedagogy in our approach. Subsequently we introduce the two examples as described above.

5.2 Script-Writing in Mathematics Education

Script-writing is a valuable pedagogical strategy and an innovative research tool that was adopted and developed in the context of mathematics teacher education. While script-writing is novel in mathematics education research, its roots trace to the Socratic dialogue and to the style of Lakatos' (1976) evocative *Proofs and Refutations* in which a fictional interaction between a teacher and students investigates mathematical claims.

Initially, script-writing was introduced in mathematics teacher education as a *lesson play*, a task in which participants script interaction between an imaginary teacher-character and student-character(s) (Zazkis, Liljedahl, & Sinclair, 2009; Zazkis, Sinclair, & Liljedahl, 2013). Juxtaposed to a classical lesson plan describing merely content and activities, the lesson play reveals how a teaching-learning interaction unfolds. In later research, the idea of a lesson play was extended to an activity of writing an imaginary dialogue that is not necessarily restricted to a lesson, referred to as script-writing. When used in teacher education, script-writing is a tool related to "approximations of practice" (Grossman, Hammerness, & McDonald, 2009), which "include opportunities to rehearse and enact discrete components of complex practice in settings of reduced complexity" (p. 283), and is advocated as an essential part of teacher preparation.

Script-writing is used as both an instructional tool for the advancement of pedagogy and mathematics and a research tool for data collection. It has been implemented in recent research (e.g. Brown, 2018; Koichu & Zazkis, 2013; Zazkis & Kontorovich, 2016; Zazkis & Zazkis, 2014) where participants had to identify problematic issues in the presented topics and subsequently clarify these by designing a scripted dialogue. The affordances of script-writing were detailed for script-writers (students, prospective teachers, and teachers), teacher educators, and researchers. In particular, for teachers, writing a script is an opportunity to examine a personal response to a situation, explore erroneous or incomplete approaches of a student, revisit and possibly enhance personal understanding of the mathematics involved, and enrich the repertoire of potential responses to be used in future real teaching. For teacher educators, the scripts provide a lens on planned pedagogical approaches that can be consequently highlighted and discussed in working with teachers. For researchers, the scripts form a rich data source that can be examined from various perspectives and provide a lens for examining images of teaching and insights into the script-writers' understanding of mathematics (e.g. Zazkis et al., 2013).

In this chapter, we highlight an additional use and benefit of script-writing tasks: they provide teacher educators with insight into the mathematical knowledge of their students, which can subsequently be used for planning follow-up instructional activities aimed at extending and strengthening this knowledge.

5.3 The Usage-Goal Framework

As mathematics educators, we draw on our knowledge of mathematics and pedagogy to design tasks aimed at advancing both the mathematical and pedagogical understanding of the teachers we teach. However, the interweaving of mathematics and pedagogy in professional development courses might create a complex educational setting in which the mathematical and pedagogical ideas become entangled and, consequently, harder to be discerned. The *usage-goal framework* suggested by Liljedahl et al. (2007) illustrates a way of examining the use of tasks in teacher education while attending to both mathematical and pedagogical perspectives. More specifically, the framework is organised in four cells of a 2 × 2 array (presented in Fig. 5.1), where the content of the cells should be read as "the use of **x** to promote understanding of **Y**", ***x*** and ***Y*** being either mathematics or pedagogy (e.g. mP is read as "the use of mathematics (m) to promote understanding of Pedagogy (P)"). The suggested array, according to Liljedahl et al. (2007), serves to disaggregate "our knowledge of mathematics and use of pedagogy [as teacher

USAGE		GOALS	
		Mathematics (M)	Pedagogy (P)
	mathematics (m)	**mM**	**mP**
	pedagogy (p)	**pM**	**pP**

Fig. 5.1 Goals and usage grid

educators] from the mathematical and pedagogical understandings we wish to instill within our students [teachers or prospective teachers]" (p. 240).

As we demonstrate in what follows, our scripting tasks are situated within the lower row of the grid (pM and pP). In short, these tasks present a prompt for a beginning of a dialogue between imaginary students and their teacher and ask the participating teachers to continue the dialogue in a way they find fit. These prompts typically attend to a problematic issue, a potential error or misconception, or an unexpected student question. This requires the participants to consider both the pedagogical issues of how to explain the topic and the mathematical knowledge involved. Accordingly, the scripting tasks utilise a pedagogical perspective to promote both pedagogy and mathematics.

The follow-up classroom activities in a teacher education course, however, are predominantly situated within the upper row of the grid (mM and mP). These activities utilise those mathematical themes that require further attention and deepening of understanding according to the tasks. Accordingly, the mathematics involved is used as a basis for the promotion of related mathematical knowledge, as well as pedagogical considerations specific to the topic.

5.4 Context for the Examples

The two examples described below took place in the context of a professional development course for practicing secondary school mathematics teachers. In both cases, the course participants were given a scripting task in the form of a prompt that was the beginning of a dialogue between a teacher and students and were asked to continue it according to their mathematical and pedagogical understanding (Part A). In addition to writing a script that extends the dialogue, the participants were also asked to explain their choice of the presented instructional approach (Part B). Furthermore, they were asked to explain their personal understanding of the mathematics involved in the task and note whether their potential explanation to a "mathematically sophisticated colleague" differed from what they chose to include in the scripted conversation with students (Part C).

5.5 Example 1: Functions, Not Just Linear

The concept of a function is fundamental in mathematics, and it has been repeatedly regarded in the education literature as a central concept in the school curriculum (e.g. Ayalon, Watson, & Lerman, 2017; Dreyfus & Eisenberg, 1983; Hitt, 1998; Paz & Leron, 2009). Rather recently, Dubinsky and Wilson (2013) have conducted a longitudinal literature review, covering over 50 years of research on student learning of this concept. However, the examination of the vast amount of mathematics education literature related to the understanding of functions reveals that there has been relatively little research performed specifically in relation to teachers' and prospective teachers' understanding of the concept. The following example provides a glimpse into this issue.

5.5.1 The Scripting Task: Functions

Figure 5.2 presents a prompt for a scripting task alongside a particular table of values. In this task, the participants were invited to explore an imaginary student question, whether there are functions other than $y = 3x$ that satisfy the same table of values. The task was designed to address several known misconceptions regarding the function concept (elaborated below), which are attributed in the literature to either secondary school students, undergraduate students, or prospective teachers. Accordingly, the pedagogical task could promote either mathematical growth, in case the teachers' existing understanding of the function concept was challenged (pM), or pedagogical growth, in case the teachers decided to address possible student mistakes that might arise (pP), or both.

The above task addresses two issues described in mathematics education research. Firstly, the task attends to the phenomenon of linear functions as "overpowering" prototypical examples, both for undergraduate students (e.g. Dreyfus &

Teacher:	Consider the following table of values. What function can this describe?
Alex:	y = 3x
Teacher:	And why do you say so?
Alex:	Because you see numbers on the right are 3 times numbers on the left
Jamie:	I agree with Alex, but is this the only way?
Teacher:	…

x	y
1	3
2	6
3	9
4	12
5	
6	

Fig. 5.2 A prompt for the table of values scripting task

Eisenberg, 1983) and secondary school students (e.g. Markovits, Eylon, & Bruckheimer, 1986; Schwarz & Hershkowitz, 1999). For example, Markovits et al. (1986) reported that half of the participating ninth-grade students in their study claimed that there is only one given function that passes through two given points and that this function is a straight line. In the current presented task, the table of values contains four points that satisfy the line $y = 3x$, which in comparison to the case presented by Markovits et al., where there were only two given points, further "strengthens" the idea of the line as the only available option. Secondly, the task addresses the issue of teachers' potential lack of understanding of the arbitrary nature of how a function may be defined (e.g. Even, 1990). More specifically, Thomas (2003) reported on teachers' need for an algebraic formula to describe a function that has a tabular representation, as is the case with the current "table of values" task.

Through Jamie's question in the task, "but is this the only way?", we expected the consideration of various suitable functions that constitute an example space (Watson & Mason, 2005) applicable to the task, which in turn could shed light on the teachers' understanding of the function concept.

5.5.2 Snapshots from the Scripts: Functions

Our analysis identified the main themes that emerged from the scripts and examined the structure of the exhibited example spaces of functions that were given by the participants. We distinguished between examples used in Part A, which could have been purposefully restricted in the scripts based on pedagogical and instructional considerations, and the examples mentioned in Part B or Part C, which pointed to the participants' own understanding of the task. In the context of the current chapter, however, we do not provide a detailed analysis of participants' responses. Rather, we highlight several themes that emerged, which served as a motivation for developing follow-up instructional activities.

5.5.2.1 On the Notion of Function

The following excerpt from Taylor's[1] script exemplifies several features evident in the participants' perception of the function concept:

Teacher: Excellent question Jamie, what's your instinct, are there other ways?
Jamie: Well I don't know, I guess there could be, but how could we tell?
Teacher: Why don't we start by plotting these points. And by we I mean you.

[Students plot the points]

Teacher: Good, so how would it look if we used Alex's function?

[1] All participant names are pseudonyms.

Jamie:	It would have a straight line through all the points.
Teacher:	Yes, but how else can we connect these points?
Jamie:	I suppose we could do a zig zag line.
Teacher:	Sure, that would work. But we want this to be a function, so what rule do we need to follow?
Jamie:	The vertical line test.
Teacher:	Which is the easy way of remembering what?
Jamie:	Each output can only have 1 input.
Teacher:	Correct, so how can we connect these points then?
Jamie:	Any way we want as long as we don't break the vertical line test.

In this excerpt, the teacher's request to plot the points explicitly leads to the consideration of a graphical representation. The question "but how else can we connect these points?" leads students to explore alternative options to the straight line. However, all examples in the script explicitly or implicitly regarded the domain as the set of all real numbers.

In Taylor's script, we note the reference to the "vertical line test" as an implicit working definition of a function. We further note that the teacher-character agrees with a student's incorrect definition, "Each output can only have 1 input". This could be either a misconception or lack of attention on the part of the script-writer.

The features of this script – infinite and unbounded domain, continuous function ("connected points"), vertical line test, and lack of a correct definition – were typical in this group of participants. This led us to design an activity that focused on different definitions of a function and a historical evolution of the contemporary definition.

5.5.2.2 Polynomial Expressions

While some script-writers expressed difficulty in considering examples other than the linear function, a possibility to fit a polynomial function to the given table of values was featured in several scripts. This is illustrated in the following excerpt from Logan's script:

Teacher:	Well in all of these cases we have assumed something subtle. If we filled the table of values what would we get for the remaining y entries?
Alex:	15 and 18
Teacher:	Does it have to be those values? What if I put 16 and 23?
Jamie:	… Can you do that?
Teacher:	Why not? The points could be modeling anything! There is nothing there that says it has to be a line.
Jamie:	Can we find an equation for that though?
Teacher:	Certainly, but I need to talk about degrees of freedom. In our table of values we could make up 6 values of y and therefore we have 6 degrees of freedom. Simple enough?
Jamie:	Mhmm.
Teacher:	So we need to find a polynomial with at least 6 degrees of freedom to describe it, that is a polynomial with at least 6 terms.
Alex:	So a 5^{th} order polynomial?

Teacher: Exactly Alex, we could find a polynomial of the form $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ that fits the table of values.

Jamie: But how can we ever assume that any patterns we see in a table of values continues?

Teacher: An excellent question, short answer is we don't. When we make these equations we are assuming that the trend we observe will continue. When making this assumption we need to look for reasons to explain the trend and then ask if we expect those factors to stay the same. Maybe the data was showing the population of a species but at x = 5 more food is introduced or a predator is removed and the species can grow at a faster rate.

While general solutions are usually considered in mathematics as more valuable than specific ones, Zazkis and Leikin (2007) noted that often general examples point to an individual's inability to generate a specific one. In this case, the presented example of a polynomial function points to Logan's awareness of a possibility of a polynomial function, while it may also indicate Logan's difficulty in producing an explicit formula for the polynomial.

While Logan noted the existence of a polynomial function, Corey provided such a function "out of the blue" and left it for the students to verify that it is consistent with the entries in the table of values.

Jamie: It's kind of obvious that it's y = 3x. What are we learning here?

Alex: I guess it's making us think outside the box a little, but yeah, our other answers are kind of lousy.

[…]

Teacher: Then let me give you an extension. Check out this function $y = x^4 - 10x^3 + 35x^2 - 47x + 24$

Alex: Where did you get that from?

Teacher: You tell me.

Jamie: Ug. This works doesn't it? That is so annoying! How did you get this?

[…]

Alex: But it's not a line!

Jamie: Who cares? It's a function. And I guess it takes going to the power of four to hit all four points.

Teacher: I'll leave you to it. Figure out how to derive that equation! I didn't just pull it out of thin air.

In his commentary in Part C, Corey added that the polynomial was generated by a computer program, using matrices to solve systems of linear equations. He felt, however, that this material was inappropriate for his students. Corey wrote: "[t]he level of math needed to determine the final function is beyond what I consider high school level math. After being given the function the answer can be easily revealed, but it still is not easy".

We agree with Corey that generating polynomials from a manipulation of matrices may be beyond high school student capabilities. However, the scripts of Corey and Logan informed a follow-up instructional activity aimed at extending the teachers' knowledge of mathematics. In the next section, we present and discuss an instructional exploration on how such polynomials can be found and how this approach could be utilised to extend teachers' connections between tertiary and school mathematics.

5.5.3 Follow-Up Activities: Functions

In the current section, we present two follow-up activities that were designed in response to the participants' scripts: (1) classifying different definitions of functions for the deepening of the conceptual understanding of the function concept and (2) seeking to "create" a polynomial function that fits the original table of values. The rationale behind the activities, their mathematical details, and the resulting classroom events are elaborated below.

5.5.3.1 Function Definition

The scripts revealed that the participants mostly considered functions as continuous, written as a single formula, defined on an infinite and unbounded domain, and continue the given pattern. More specifically, the examples that the teachers used lacked any referral to the *arbitrary* nature of functions, that is, that there is no need for regularity or a representative expression for the correspondence, nor are there rules for what the sets of the domain and codomain must be (Even, 1990; Steele, Hillen, & Smith, 2013). As explained by Even (1990), and in line with the scripts, the rejection of the arbitrary characteristic of functions may be the result of "a prototypical judgement whether an instance is a function, combined with a limited concept image" (p. 528). Accordingly, we decided to initiate a follow-up classroom activity that "goes back to the basics" and focuses on the function definition.

Initially, the participants were asked to provide a definition of a function in writing. While "function" is a familiar and frequently used term both in school mathematics and in undergraduate studies, only one half of the participating teachers succeeded in providing an accurate definition. The other half provided definitions that either did not allude to the necessity of a *unique* corresponding value to each value in the domain or, to the contrary, required *different* image values for different domain values (i.e. defined an injective function instead of a function).

After providing a personal definition, the participants were given a list of 17 definitions and were asked to classify them according to criteria of their choice. The classification itself was a subordinate goal (Hewitt, 1996) in order to invoke a thorough consideration of the details and features of each definition. Out of the given definitions, 11 definitions were taken from different textbooks and web-based sources, and the remaining 6 "definitions" were in fact incorrect, chosen to highlight frequent misconceptions (such as the aforementioned confusion between the definition of a "function" and an "injective function").

As a group, the participants succeeded in rejecting the incorrect or incomplete definitions. The subsequent conversation focused on two chosen clusters of definitions. The first cluster included set-based definitions, exemplified by Definition S below (based on Usiskin, Peressini, Marchisotto, & Stanley, 2003, p. 70), in which the sets consisting the domain and codomain were explicitly addressed:

> Definition S: The Cartesian product of two sets A and B, denoted A × B, is the set of all ordered pairs (a, b) such that a ∈ A and b ∈ B. For any sets A and B, a function f from A to B, f : A → B, is a subset f of the Cartesian product A × B such that every a ∈ A appears once and only once as a first element of an ordered pair (a, b) in f.

This continued into a discussion on the arbitrary choice of the two sets. More specifically, an important realisation that did not appear in any of the scripts was that the table of values in Fig. 5.2 already provides a function without any need for domain expansion: the set of ordered pairs S = {(1,3), (2,6), (3,9), (4,12)} given in the table determines a function with domain {1,2,3,4} and codomain {3,6,9,12}.

The second cluster consisted of the following two definitions, H1 and H2, where the reader may recognise their historical significance:

> Definition H1: A function of a variable quantity is an analytic expression composed in any manner from that variable quantity and numbers or constant quantities.

> Definition H2: y is a a function of a variable x, defined on an interval $a < x < b$, if for every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established.

Definition H1 belongs to Euler and dates to the year 1748 (see Euler, 1988, for a translated version of his original book *Introductio in Analysin Infinitorum*). As explained by Kleiner (1993), this early version of a definition of a function refers to a single analytic expression with an unrestricted domain (and corresponds to what we nowadays call “elementary functions”). Definition H2 belongs to Dirichlet from the 1820s and is considered as the basis of the current contemporary definition of a function (Kleiner, 1993). While this definition does not yet permit arbitrary sets as domain and codomain, it is nonetheless freed from the requirement of an analytic expression or a corresponding curve to define a function (e.g. the famous Dirichlet function, which is defined separately for rational and irrational numbers).

In addition to the meta-mathematical dimension of exposing the teachers to the developing nature of mathematical definitions (e.g. Kjeldsen & Blomhøj, 2012; Kjeldsen & Petersen, 2014), these historical definitions were discussed in order to highlight the arbitrary nature of the correspondence defining a particular function. In this regard, Euler’s eighteenth-century definition was presented to illustrate the participants’ own possible misconception of the function definition. As articulated by Even (1990), “[w]e cannot accept a situation where secondary teachers at the end of the 20th century have a limited concept of function, similar to the one from the 18th century” (p. 530). Juxtaposed with Euler’s definition, the difference that emerged when alluding to Dirichlet’s definition was that “it is *irrelevant* in what way this correspondence is established” (Definition H2). This led the class to the additional realisation that the function defined by the set of ordered pairs T = {(1, 3), (2, 6), (3, 9), (4, 12), (7.5, π), ($-\sqrt{2}$, 103.54)} is another possible solution to the task, in which not only are the domain and codomain sets arbitrarily chosen, but so is the correspondence itself. The latter example helped the participants realise that there are infinitely many arbitrary functions that fit the original table of values presented in Fig. 5.2.

While a set-based definition of a function is usually beyond school mathematics, we believe that exposure to this contemporary convention in disciplinary mathematics adds an important dimension to teachers' mathematical knowledge (mM, in the usage-goal terminology). Furthermore, we suggest that the engagement with a mathematical task that explores different function definitions may additionally affect teachers' future pedagogical approaches (mP). Even while attending to functions prescribed by the school curriculum, the raised awareness to the topic would (hopefully) result in teachers introducing students to rigorous definitions and highlighting the arbitrary nature of the function concept.

5.5.3.2 Fitting Polynomials

As became evident from the scripts, some teachers considered the option of polynomials when dealing with the mathematical aspects of the task yet struggled to come up with concrete formulas for these. Accordingly, an additional classroom activity focused on generating a nonlinear polynomial function consistent with the originally given table of values (see Fig. 5.2). Corey shared with the class his polynomial function $y = x^4 - 10x^3 + 35x^2 - 47x + 24$ for verification. While it is easily confirmed, both numerically and graphically (see Fig. 5.3), that this function indeed conforms to the table of values in Fig. 5.2, questions arose as to how this function could be generated.

To address this question, instead of producing a function that intersects the line $y = 3x$ at exactly four different points, the teachers were asked to create a function that intersects the line $y = 0$ (the x-axis) at exactly four distinct points. While the teachers easily generated a function that has zeros at 1, 2, 3, and 4 – $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$ – it was nonetheless a conceptual leap to combine it with the function $g(x) = 3x$ suggested by the table of values to generate a polynomial function $h(x) = f(x) + g(x)$ (see Fig. 5.4). However, this initial example naturally led to additional examples of the form $h_k(x) = kf(x) + g(x)$. Of note, for k = 1, $h(x)$ is simplified to $x^4 - 10x^3 + 35\,x^2 - 47x + 24$, which is the function that a computer program generated for Corey.

The creation of the above algebraic examples was accompanied with computer-generated graphs, providing yet additional visual evidence that the points from the table of values satisfied the generated functions (see Fig. 5.4). While, mathematically, such a confirmation was unnecessary, we suggest that this satisfied an aesthetic dimension of the teachers' mathematical experience, using various values for k.

Another polynomial function, $s(x)$, that appeared in Eric's script was presented for classroom consideration, with the (undeclared) goal of making connections between tertiary and secondary school mathematics:

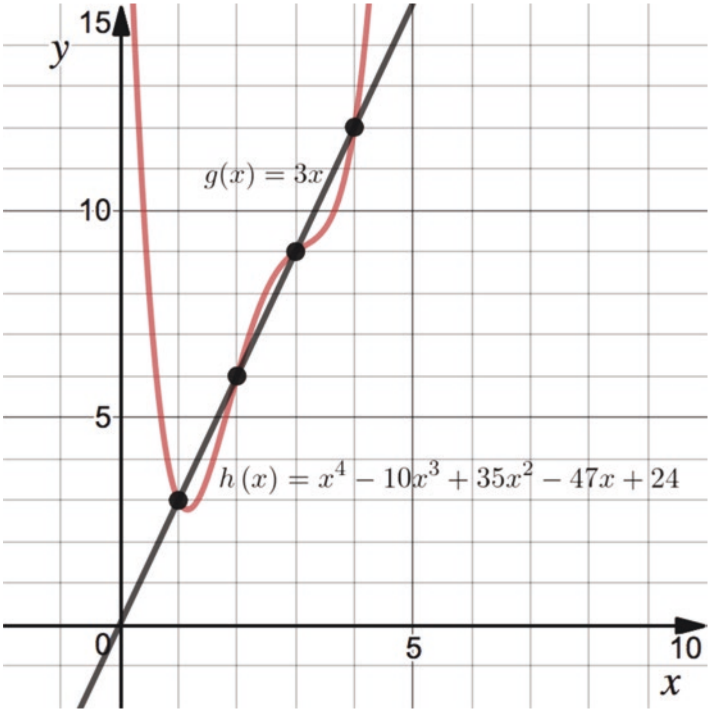

Fig. 5.3 A polynomial that passes through the points (1, 3), (2.6), (3, 9), and (4, 12)

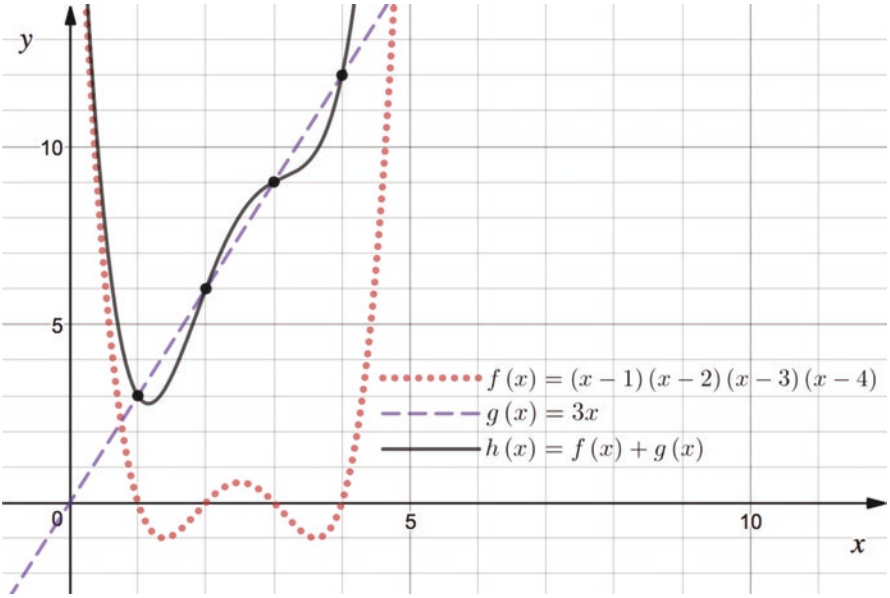

Fig. 5.4 Generating the polynomial $h(x)$ as the sum of $f(x)$ and $g(x)$

$$s(x) = -\frac{1}{3}(x-2)(x-3)(x-4) + 3(x-1)(x-3)(x-4)$$
$$-\frac{9}{2}(x-1)(x-2)(x-4)$$
$$+2(x-1)(x-2)(x-3)$$

It was presented as a student's example, and the teachers were asked to consider the correctness of the example. Having examined the function and focusing on the factors in each component, the participants expressed a belief that "it should work". However, a close examination of the graph of $s(x)$ (see Fig. 5.5) revealed that one of the required points was in fact not on the graph.

In order to "fit" the point (1, 3) to the graph, the participants suggested replacing $\left(-\frac{1}{3}\right)$ with $\left(-\frac{1}{2}\right)$, suspecting that Eric's function was not copied properly. At that point, different participants expressed appreciation for Eric's idea of easily producing a suitable polynomial by "controlling" the values of the function at x=1, 2, 3, and 4. However, graphing the "corrected" function $q(x)$

$$q(x) = -\frac{1}{2}(x-2)(x-3)(x-4) + 3(x-1)(x-3)(x-4)$$
$$-\frac{9}{2}(x-1)(x-2)(x-4)$$
$$+2(x-1)(x-2)(x-3)$$

led to an unexpected result. The participants realised that $q(x)$ can be simplified to $3x$ and presents, as the participants referred to it, a "linear function in disguise".

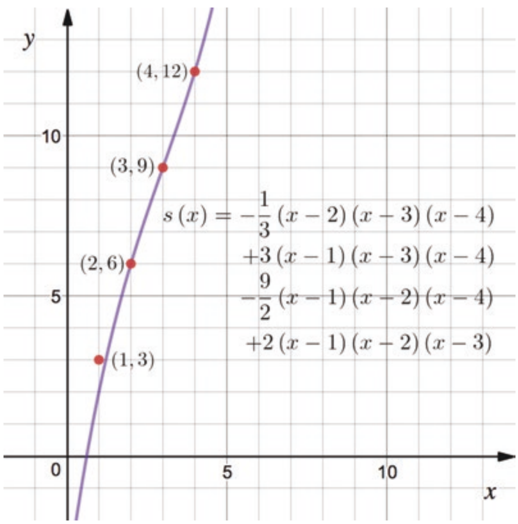

Fig. 5.5 Eric's function

This realisation led to a follow-up question of whether (and, if so, how) it is possible to generate a polynomial function of the third degree that corresponds to the table of values in Fig. 5.2. After some consideration, Logan suggested that this would be impossible, as "third degree functions go up-down-up" (this was accompanied with a wave hand gesture), so they would not reach the fourth point on the line. While this was a reasonable explanation, a more rigorous one was sought in inviting the participants to consider possible zeros of a cubic function. This led to recalling the fundamental theorem of algebra and to acknowledging that the existence of three complex roots means that there are at most three real roots for any cubic function. Therefore, in analogy with the previous discussion, cubic functions can be generated to pass through any three points on the same line, but not four.

The above discussion of Eric's example from his script clearly demonstrates how participants' scripts can be utilised in the work of a teacher educator. In this case, a pedagogical task of considering the correctness of a student's solution was used toward the goal of extending the teachers' mathematics. The activity guided the teachers toward an explicit connection between undergraduate and secondary school mathematics, a goal considered by many researchers as highly valuable (e.g. Wasserman, 2016; Watson & Harel, 2013). Furthermore, it demonstrated the utility of advanced mathematical knowledge as a tool to instantly recognise student mistakes (mP): in this case, through the realisation that there is no need to check for calculation errors, as there could be no cubic function that intersects a line in four different points.

5.6 Example 2: Irrational Exponents, Not Just with a Calculator

The idea of irrational exponents lies at the juxtaposition of two mathematical concepts: irrational numbers and exponentiation. Both these concepts have been recognised in the literature as conceptually challenging for secondary school students and teachers alike (e.g. Confrey & Smith, 1995; Davis, 2009; Fischbein, Jehiam, & Cohen, 1995; Kidron, 2018; Sirotic & Zazkis, 2007a; Weber, 2002). However, we have not found any research that focuses on the combination of the two – that is, learners' understanding of *irrational exponents*. The following case presents a step toward this goal.

5.6.1 The Scripting Task: Irrational Exponents

Figure 5.6 presents a prompt for a scripting task on the topic of irrational exponents. In the task, the teachers were encouraged to consider the mathematical meaning of irrational exponents, as well as how this concept could be explained to a group of secondary school students.

Your class learned how rational exponents are interpreted and practiced simplifying expressions that involved rational exponents.

Robin:	So when there is a fraction in an exponent you rewrite it as a root.
Teacher:	Indeed
Robin:	But what if there is a root in the exponent, do we make it a fraction?
Teacher:	What do you mean?
Robin:	So we learned that $x^{\frac{m}{k}} = \sqrt[k]{x^m}$, like $8^{\frac{2}{3}} = \sqrt[3]{8^2} = 4$. But what about $x^{\sqrt{k}}$, like $4^{\sqrt{2}}$, what would this mean?
Teacher:	This is a very interesting question, let us look at this together …

Fig. 5.6 A prompt for the irrational exponents scripting task

The task was designed to expand the teachers' mathematical knowledge of irrational exponents and strengthen their knowledge of rational exponents. We suspected that while the meaning of rational exponents would be familiar to them, this might not be the case for irrational exponents. This assumption was based on the following two considerations: (a) secondary mathematics textbooks typically introduce the idea of exponentiation as "repeated multiplication" and obtain the graphs of exponential functions without raising the issue of continuity or irrational exponents (Confrey, 1991; Davis, 2009; Davis, 2014); (b) previous research has pointed toward prospective secondary school teachers' reliance on calculators, rather than conceptual understanding, when dealing with irrational numbers (e.g. Zazkis & Sirotic, 2010) – a tendency we suspected to be even stronger in the case of two "irrationality layers" (i.e. the irrational number $4^{\sqrt{2}}$ that contains an irrational exponent $\sqrt{2}$). Accordingly, the formulation of the task encouraged the teachers to first make sense of irrational exponents, whether independently and/or by utilising external mathematical sources, and subsequently consider how to present the underlying ideas to students.

5.6.2 *Snapshots from the Scripts: Irrational Exponents*

Most of the script-writers presented the idea of irrational exponents to a group of imaginary students in an adequate manner. This included the creation of a sequence a_n that converges to $\sqrt{2}$ and subsequently considering the sequence 4^{a_n} to approximate $4^{\sqrt{2}}$ (we note that the terminology used in the scripts differed from the one presented here; some of the related nuances are elaborated below). Nonetheless, responses to Part C of the task revealed that the notion of irrational exponents was new to the teachers. Most of the teachers explicitly admitted they had never considered the meaning of irrational exponents before and were pleased with the opportunity to do so, as exemplified by the following excerpt by one of the teachers:

> I have taught Powers and Exponents to students through grade 8 to 10 a number of times so far, but strangely enough, the question such as 'what if the exponents are irrational numbers' has never crossed my mind, nor was ever asked by any of my students. […] As such, I sincerely appreciated this assignment for it guided me, or better yet, intrigued me to spend almost ridiculous amount of hours to think about the issues linked to this topic.

In the following sections, we focus on two prominent themes that emerged in the scripts, which led to follow-up instructional activities aimed at addressing the teachers' conceptions and difficulties related to the topic.

5.6.2.1 Irrationals Can Only Be Approximated

We acknowledged in the scripts a certain conceptual difficulty regarding the teachers' perception of irrational numbers. More specifically, most of the teachers claimed, both in the script and Part C of the task (referring to their personal mathematical understanding), that the value of $4^{\sqrt{2}}$ can be *approximated only* and that an accurate value either does not exist or cannot be described. Consider, for example, the following excerpt from Eden's script:

Teacher: So things are much easier with an integer or a fraction in the exponent. How about a decimal in the exponent, such as $4^{1.2}$?

Robin: That is easy. We can re-write 1.2 as a fraction, then we have a fractional exponent.

Teacher: Excellent! So what makes $\sqrt{2}$ different? Why is it difficult to interpret it?

Robin: I guess if we could write $\sqrt{2}$ as a decimal, then turn it into a fraction, then we would be able to work with it. But its decimal expansion is infinitely long…

Chris: And there is no repeating pattern. I guess what makes $\sqrt{2}$ different is that it is an irrational number!

Teacher: Exactly. We would not be able to find an exact answer. Now, if I draw a number line and ask you to put $4^{\sqrt{2}}$ on it, where would you put it?

Chris: Since we can have a decimal in the exponent, and $\sqrt{2}$ is about 1.4, we can say that $4^{\sqrt{2}}$ is about $4^{1.4}$ which is about 6.96. So close to 7?

Robin: If we use a more accurate approximation of $\sqrt{2}$, we can get a more accurate approximation of $4^{\sqrt{2}}$, right? If we use $\sqrt{2} \approx 1.414$, we have 7.101.

Teacher: Excellent! As you use better approximation using $4^{1.4}$, then $4^{1.41}$, then $4^{1.414}$ which you can all make sense of as fractional exponents, or rational exponents, your estimated value is getting closer and closer to the real value of $4^{\sqrt{2}}$.

$$4^{1.4} \approx 6.964404$$

$$4^{1.41} \approx 7.061623$$

$$4^{1.414} \approx 7.100891$$

$$4^{1.4142} \approx 7.102860$$

And so on… So our best attempt to interpret an irrational exponent is to write it as adding a sequence of rational exponents.

While the above script correctly describes the process of approximating $4^{\sqrt{2}}$, which is at the basis of how to define irrational exponents, it nonetheless treats irrational numbers as numbers that can only be approximated. This is exemplified by the teacher-character's claim "we would not be able to find an exact answer" in regard to $\sqrt{2}$ and having solely a "best attempt to interpret an irrational exponent" in regard to $4^{\sqrt{2}}$. This approach was further supported by Eden's answer to Part C in which she claimed: "there are no exact answers, just approximations". Similar claims were found in the tasks of the other participants, such as the following assertion by Leslie in Part C: "this discussion leads the student that the value of the irrational power … cannot be stated 'definitively'. We can only give the bounds of the region where we would find the value". Accordingly, one of the goals of the follow-up activity was to demonstrate that irrational numbers could be perceived not only as approximations but also as exact values on the number line.

5.6.2.2 Attempting to Make Sense of Irrational Exponents with the Use of Graphs

One prominent method the teachers used to cope with the issue of "what $4^{\sqrt{2}}$ really means" was by providing additional explanations that used graphical representations, mostly of $y = 4^x$ or $y = x^{\sqrt{2}}$. As is illustrated by the following excerpt from Leslie's script:

Robin: I had lots of decimals in the answer when I used decimals instead $\sqrt{2}$. It is interesting that your calculator can answer $4^{\sqrt{2}}$; it doesn't give an error. That means $4^{\sqrt{2}}$ exists; now we need to figure out what it means.

Jo: When you were playing with decimal values, were you doing things like calculating $4^{1.4142}$?

Robin: Yes, I started with $4^{1.41}$, and kept increasing the number of decimal places that I used.

Jo: You realise that you were just using the ideas from today's class? $1.41 = 141/100$, which means you were finding the 100$^{\text{th}}$ root of 4 to the exponent 141, or $4^{\frac{141}{100}} = \sqrt[100]{4^{141}}$.

You know $\sqrt{2}$ is irrational, its decimal form will go on forever and you can't write it as a fraction. So, we can't understand $4^{\sqrt{2}}$ using roots. We are going to need another way of explaining irrational exponents.

[…]

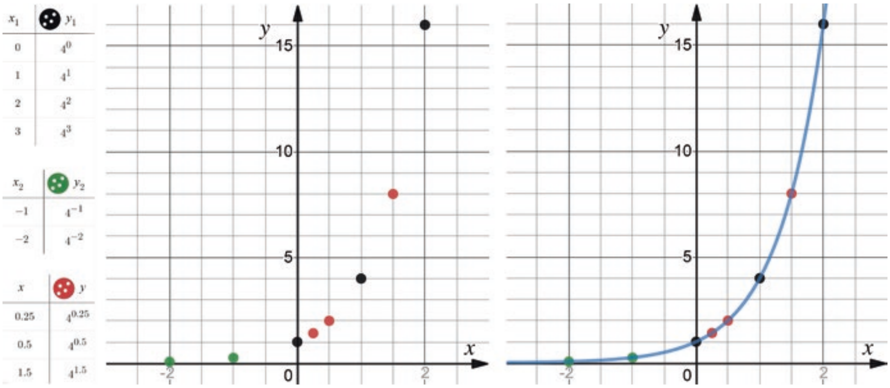

Fig. 5.7 The graphs shown in Leslie's script

Mel: (peer tutor) Do you folks need some help? (Jo recaps the discussion so far.) What an interesting idea! Let's pull up Desmos and look at a graph of the powers of four. Maybe that will help.

(Mel enters first table: whole number exponents.)

Robin: You should include negative exponents (Mel adds another table)

Jo: What about exponents that are decimals or fractions? (Mel adds another table)

[see Fig. 5.7]

Robin: The graph is sort of a line, with a weird bit at the bottom.

Mel: We can find the equation for this graph; what do you think it is?

Jo: Our "y" values are all 4 to the power of something, $y = 4^x$?

Mel: Let's see what that looks like.

[see Fig. 5.7]

Robin: All the points fit on the graph, you found the equation, Jo! We haven't seen graphs like that before, what is it?

Mel: It is the graph of an *exponential function*. We are just learning about this function in Pre-Calculus 12.

The need for additional ways to approach the topic is understandable when considering that this topic was new to the teachers (as they expressed themselves). However, the presented graphical approach and implied continuity avoid the issue of attending to an explicit definition of irrational exponents. Accordingly, in the follow-up activity, we wished to respond to the teachers' need for a graphical approach to address the idea of irrational exponents.

5.6.3 Follow-Up Activities: Irrational Exponents

In the current section, we focus on two follow-up activities that were designed to address the participants' tasks: (1) finding the exact location of irrational numbers on the number line and (2) graphing functions of the form $y = x^{\frac{m}{n}} \left(m, n \in \mathbb{N}\right)$ to deepen the learners' understanding of both rational and irrational exponents.

5.6.3.1 Finding Irrational Numbers on the Number Line

This activity was designed as a response to claims found in the scripts that irrational numbers can only be approximated. By demonstrating that $\sqrt{2}$ and other irrational numbers can in fact be placed in an accurate manner on the number line, we hoped to challenge the participants' aforementioned view. Sirotic and Zazkis (2007b) claimed that by placing irrational numbers on the number line, learners' understanding of irrational numbers can be improved in two ways. Firstly, it strengthens the distinction between an irrational number and its rational approximation; and secondly, it shifts learners' attention away from the conceptually challenging never-ending limit process associated with the decimal representation. As Sirotic and Zazkis (2007b) stated, "the geometric representation of irrational number may well turn out to be a very powerful and indispensable teaching tool for encapsulating a process into an object, especially in the case where the learner is on the verge of the reification stage in the development of the concept of irrationality" (p. 488).

At the onset of the activity, a number line was drawn on the board, and the participants were asked to place $\sqrt{2}$ on it. It took a few minutes of thought until one of the participants found and shared a solution: build an equilateral right-angle triangle of sides equal to one on the number line (where one of the sides rests on the number

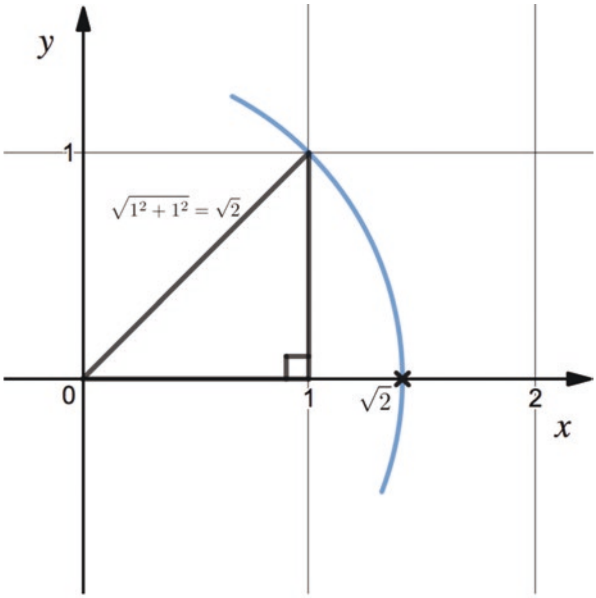

Fig. 5.8 Placing $\sqrt{2}$ on the number line

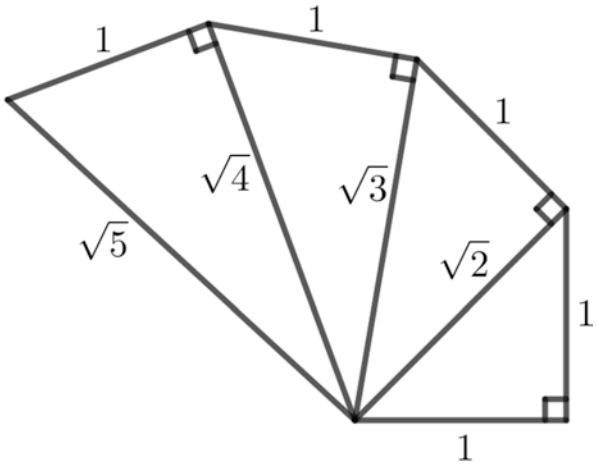

Fig. 5.9 Roots of natural numbers via successive triangles

line), and rotate the hypotenuse of length $\sqrt{2}$ until it is contained in the number line (see Fig. 5.8). This solution evoked enthusiastic and surprised responses in class and was received with clapping of hands. It is interesting to note that even though the solution based on the Pythagorean theorem was simple and clearly in the teachers' repertoire of knowledge, they did not immediately think of connecting this to the location of irrational numbers on the line.

Subsequent to finding $\sqrt{2}$, the participants were asked to find $\sqrt{3}, \sqrt{5}$, etc. on the number line. For $\sqrt{3}$, the immediate suggestions were to build a right-angle triangle in which one side equals $\sqrt{2}$ (as already found in the previous step) and the other equals 1. This leads to a hypotenuse of length $\sqrt{3}$, which again can be rotated until it sits on the number line. Similar logic was subsequently applied to find the location of other radicals, each step utilising the previous as a side in an appropriate right-angle triangle. One of the participants suggested that this could be visually done in a "manifold" of successive right-angle triangles (see Fig. 5.9). The general conclusion that in mathematics we do not always have to work with approximation of irrational numbers only was followed by exploratory participant questions, such as "how do we find π accurately?" and "what is the difference between transcendental and algebraic numbers and is there a relation between these properties and whether the number can be placed on the line?"

These questions demonstrate that while the original task regarded irrational exponents, responding to issues that arise may actively engage the participants and raise their interest in the topic. In this sense, the mathematical activity managed to promote the teachers' mathematical understanding and trigger their mathematical curiosity (mM). Furthermore, we suggest that the mathematical awareness of different representations for irrational numbers, as well as the distinction between irrational numbers and their approximations, may in turn raise the teachers' pedagogical attention to this issue when they discuss the topic with their students (mP).

5.6.3.2 Graphing Rational Exponents

As explained earlier, the scripts revealed a need for graphical ways to approach the issue of irrational exponents, manifested by using the graphs of $y = 4^x$ and $y = x^{\sqrt{2}}$ to explain what irrational exponents are. In the follow-up activity, we chose to focus on the latter, that is, on a possible method of how the graph of $y = x^{\sqrt{2}}$ could be created. More specifically, we planned to use a sequence of graphs of rational exponents (in the form of $y = x^{\frac{m}{n}}$, where $m, n \in N$) that (pointwise) converge to a graph of an irrational exponent (in this case, $y = x^{\sqrt{2}}$). The focus on $y = x^{\sqrt{2}}$ (over $y = 4^x$) was decided based on two learning affordances this option entailed. Firstly, this enabled us to emphasise the *same* approach to irrational exponents that the scriptwriters had already presented in their tasks (i.e. a sequence of rational exponents that converges to an irrational exponent), only this time via a graphical, rather than numerical, representation. Secondly, the focus on $y = x^{\sqrt{2}}$ was additionally planned to be used as a subordinate goal (Hewitt, 1996) in order to deepen the teachers' understanding of rational exponents and explore properties of their graphs.

In the classroom discussion, we invited the teachers to first consider graphs of rational exponents that are "close" to $\sqrt{2}$ and utilise computer software (Desmos) to do so. The teachers began with the estimation $1.4 < \sqrt{2} < 1.5$ and accordingly graphed the functions $f(x) = x^{1.4}$ and $g(x) = x^{1.5}$. Figure 5.10 shows the resulting graphs. While the teachers correctly predicted that the graph of $y = x^{\sqrt{2}}$ is located "in between" the graphs of $f(x) = x^{1.4}$ and $g(x) = x^{1.5}$, seeing these graphs seemed to have evoked reactions of puzzlement and surprise in class, and some responses of "What???" and "Wow!" were heard. The revelation that was surprising to the participants was that one of the functions was drawn for all $x \in R$, whereas the other function only for all $x > 0$.

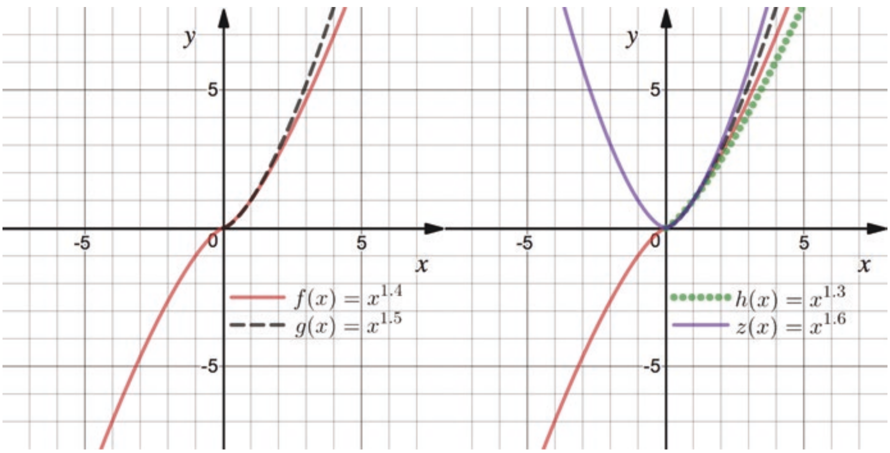

Fig. 5.10 Surprising behaviours in graphs of rational exponents

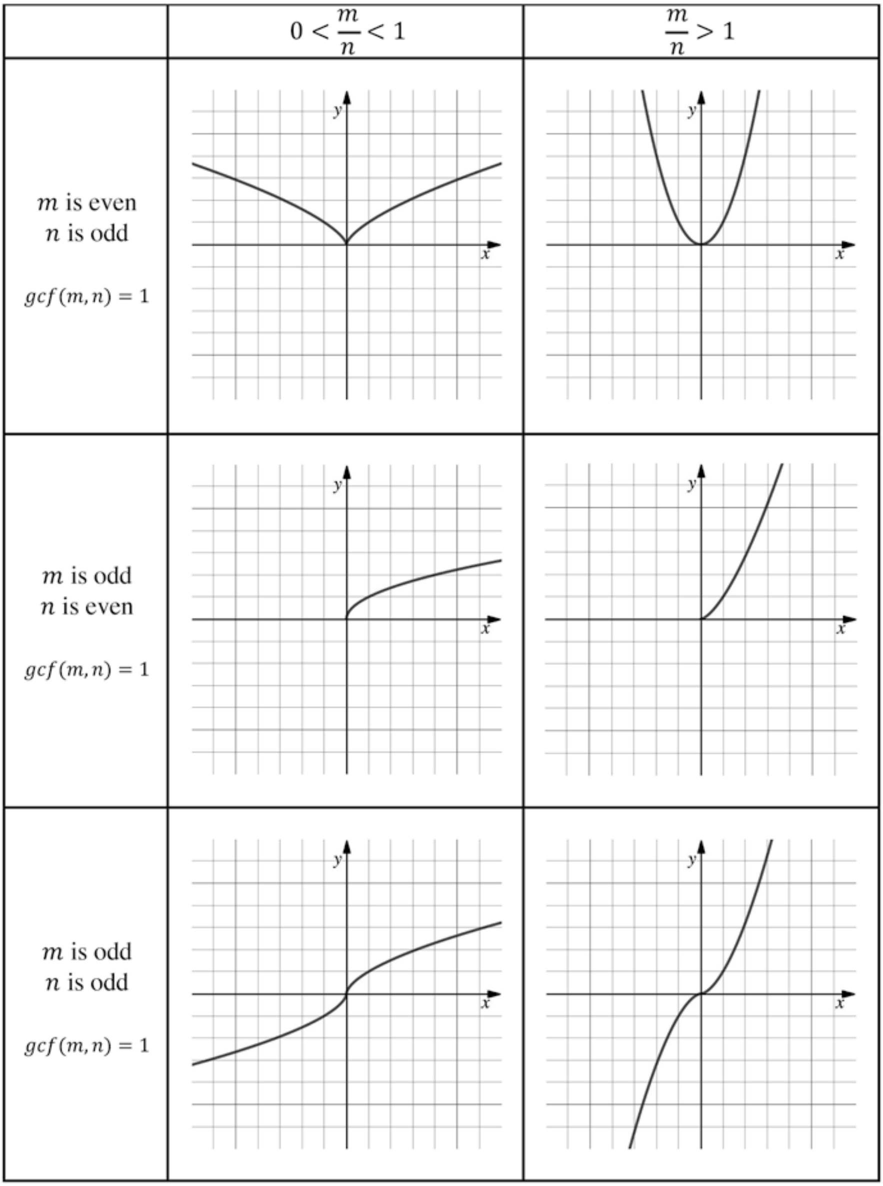

Fig. 5.11 The six different “shapes” of functions of the form $y = x^{\frac{m}{n}}$

Intrigued by the graphs, several participants asked to graph the functions of $h(x) = x^{1.3}$ and $z(x) = x^{1.6}$ as well (see Fig. 5.10). While $h(x)$ seemed similar to $g(x)$, the graph of $z(x)$, which according to the participants looked like a parabola, triggered additional surprise which invoked the need for a systematic investigation.

The participants subsequently worked in small groups on exploring the behaviour of the graphs, as well as other properties of the functions. Topics and conclusions that came up in the group activity and the following whole classroom discussion included the domain of the function as related to the evenness/oddness of the denominator, the convexity/concavity as related to the exponent being smaller or larger than 1, and functions that were each other's inverses (such as $y = x^{1.4} = x^{\frac{7}{5}}$ and $y = x^{\frac{5}{7}}$) and the graphic implication of reflection around $y = x$. In particular, the participants identified six possible shapes of a graph of the form $y = x^{\frac{m}{n}}$ and were able to predict the shape based on the given values of m and n (see Fig. 5.11). We note the participants' enthusiasm that portrayed itself in embodied gestures illustrating the different "shapes" of functions of the form $y = x^{\frac{m}{n}}$.

Subsequently, the discussion returned to irrational exponents and the underlying reason to define the domain of $y = x^{\sqrt{2}}$ as $\{x | x > 0\}$ as a common/joint domain to all approximating functions. Additionally, graphs of the form $y = x^{\frac{m}{n}}$ were plotted, where $\frac{m}{n}$ became "closer and closer" to $\sqrt{2}$, and were compared to the actual plotting of the graph of $y = x^{\sqrt{2}}$.

The mathematical activity regarding graphs of rational exponents served to advance the teachers' mathematical knowledge of both rational and irrational exponents (mM). This included observations made regarding properties of the functions, their geometric shapes, and reasoning behind their domain. Additionally, it raised the teachers' awareness to the possible limitations of using graphing computer software as a classroom pedagogical tool (mP). Whereas in the scripts the teachers used computer-generated graphs to explain irrational exponents, in the follow-up discussion, they became aware that these software tools cannot be designed without a pre-existing definition of irrational exponents. Such a pedagogical awareness of teachers is especially crucial when considering the strong technological dependence of their students.

5.7 Conclusion

The chapter provides a window into the work of mathematics teacher educators, which involves challenges in bringing together mathematics and pedagogy. In order to deal with this challenge, we utilised the usage-goal framework (Liljedahl et al., 2007), which highlights the interplay between mathematics and pedagogy involved in the tasks designed by teacher educators and accordingly can serve as a useful tool in the planning and refining of these tasks.

The usage-goal framework was originally illustrated by Liljedahl et al. (2007) through an account of an iterative process, where the *same* task went through a series of changes and adaptations, based on reflections on its implementation with *different* groups of teachers. The activities presented in this chapter, however, demonstrate a modified approach on how the usage-goal framework can be used in the work of teacher educators.

In our case, it served as a guide for a two-step instructional method, where the *same* group of teachers engaged with *different* tasks built upon each other, the design of which was informed by the usage-goal framework. In the first step, we made *usage* of scripting tasks – pedagogical tasks with the *goal* of promoting and revealing teachers' understanding of Mathematics and Pedagogy (pM and pP). The rationale was based on previous research, which illustrated that scripts generated by teachers provide a lens – for researchers and teacher educators alike – to examine teachers' mathematical knowledge and instructional choices. Extending this observation, we demonstrated that scripts can serve as a springboard for follow-up classroom activities aimed at strengthening teachers' mathematical knowledge. That is, we then made *usage* of mathematics in order to further promote the *goal* of Mathematics (mM), in particular by making connections between disciplinary and school mathematics. However, we believe these activities indirectly supported the goal of Pedagogy as well (mP), as the teachers' extended mathematical understanding of the related issues may well inform their future pedagogical approaches.

We suggest that this idea of role alternation of mathematics/pedagogy between usage and goal is an effective instructional method for mathematics teacher educators, since it both acknowledges the gradual and continual process of knowledge construction by learners (in this case, through a series of activities with a common theme) and responds to the needs of the *particular* group of teachers the teacher educator encounters (in this case, by planning the follow-up activities in accord with themes that emerge from the teachers themselves).

As a concluding note, we noticed that our pedagogical engagement in designing and implementing the instructional activities described above contributed to our personal and more nuanced understanding of the corresponding mathematical ideas. We relate this observation to Leikin and Zazkis' (2010) notion of teachers *learning through teaching*, which includes teachers "making connections between different components of previous knowledge, achieving deeper awareness of what concepts entail, and enriching their personal repertoire of problems and solutions" (p. 8). In this regard, our experience described in this chapter shows that the notion of learning through teaching applies not only to mathematics teachers but also to mathematics teacher educators.

References

Ayalon, M., Watson, A., & Lerman, S. (2017). Students' conceptualisations of function revealed through definitions and examples. *Research in Mathematics Education, 19*(1), 1–19.

Biza, I., Nardi, E., & Zachariades, T. (2007). Using tasks to explore teacher knowledge in situation-specific contexts. *Journal of Mathematics Teacher Education, 10*, 301–309.

Biza, I., Nardi, E., & Zachariades, T. (2018). Competences of mathematics teachers in diagnosing teaching situations and offering feedback to students: Specificity, consistency and reification of pedagogical and mathematical discourses. In T. Leuders, J. Leuders, & K. Philipp (Eds.), *Diagnostic competence of mathematics teachers: Unpacking a complex construct in teacher education and teacher practice* (pp. 55–78). New York: Springer.

Brown, S. (2018). Difficult dialogs about degenerate cases: A proof script study. *Journal of Mathematical Behavior, 52*, 61–76.

Confrey, J. (1991). The concept of exponential functions: A student's perspective. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 124–159). New York: Springer.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66–86.

Davis, B. (2014). Toward a more power-full school mathematics. *For the Learning of Mathematics, 34*(1), 12–17.

Davis, J. D. (2009). Understanding the influence of two mathematics textbooks on prospective secondary teachers' knowledge. *Journal of Mathematics Teacher Education, 12*(5), 365–389.

Dreyfus, T., & Eisenberg, T. (1983). The function concept in college students: Linearity, smoothness and periodicity. *Focus on Learning Problems in Mathematics, 5*(3), 119–132.

Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. *Journal of Mathematical Behavior, 32*(1), 83–101.

Euler, L. (1988). *Introduction to analysis of the infinite: Book I*. New York: Springer Science+Business Media.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*(6), 521–544.

Fischbein, E., Jehiam, R., & Cohen, D. (1995). The concept of irrational numbers in high-school students and prospective teachers. *Educational Studies in Mathematics, 29*(1), 29–44.

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching, 15*(2), 273–289.

Hewitt, D. (1996). Mathematical fluency: The nature of practice and the role of subordination. *For the Learning of Mathematics, 16*(2), 28–35.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior, 17*(1), 123–134.

Jaworski, B. (2008). Mathematics teacher educator learning and development: An introduction. In B. Jaworski & T. Wood (Eds.), *The mathematics teacher educator as a developing professional* (pp. 1–13). Rotterdam, The Netherlands: Sense Publishers.

Kidron, I. (2018). Students' conceptions of irrational numbers. *International Journal of Research in Undergraduate Mathematics Education, 4*(1), 94–118.

Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation: History as a method for learning meta-discursive rules in mathematics. *Educational Studies in Mathematics, 80*(3), 327–349.

Kjeldsen, T. H., & Petersen, P. H. (2014). Bridging history of the concept of function with learning of mathematics: Students' meta-discursive rules, concept formation and historical awareness. *Science and Education, 23*(1), 29–45.

Kleiner, I. (1993). Functions: Historical and pedagogical aspects. *Science and Education, 2*(2), 183–209.

Koichu, B., & Zazkis, R. (2013). Decoding a proof of Fermat's Little Theorem via script writing. *Journal of Mathematical Behavior, 32*(3), 364–376.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge: Cambridge University Press.

Leikin, R., & Zazkis, R. (2010). Teachers' opportunities to learn mathematics through teaching. In R. Leikin & R. Zazkis (Eds.), *Learning through teaching mathematics* (pp. 3–21). Dordrecht, The Netherlands: Springer.

Liljedahl, P. G., Chernoff, E., & Zazkis, R. (2007). Interweaving mathematics and pedagogy in task design: A tale of one task. *Journal of Mathematics Teacher Education, 10*(4–6), 239–249.

Markovits, Z., Eylon, B. S., & Bruckheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics, 6*(2), 18–24.

Paz, T., & Leron, U. (2009). The slippery road from actions on objects to functions and variables. *Journal for Research in Mathematics Education, 40*(1), 18–39.

Peng, A. (2007). Knowledge growth of mathematics teachers during professional activity based on the task of lesson explaining. *Journal of Mathematics Teacher Education, 10*, 289–299.

Schwarz, B. B., & Hershkowitz, R. (1999). Prototypes: Brakes or levers in learning the function concept? The role of computer tools. *Journal for Research in Mathematics Education, 30*(4), 362–389.

Sirotic, N., & Zazkis, R. (2007a). Irrational numbers: The gap between formal and intuitive knowledge. *Educational Studies in Mathematics, 65*(1), 49–76.

Sirotic, N., & Zazkis, R. (2007b). Irrational numbers on the number line – Where are they? *International Journal of Mathematical Education in Science and Technology, 38*(4), 477–488.

Steele, M. D., Hillen, A. F., & Smith, M. S. (2013). Developing mathematical knowledge for teaching in a methods course: The case of function. *Journal of Mathematics Teacher Education, 16*(6), 451–482.

Thomas, M. (2003). *The role of representation in teacher understanding of function*. Paper presented at the 27th International Group for the Psychology of Mathematics Education conference held jointly with the 25th PME-NA conference, Honolulu, HI. Retrieved from https://files.eric.ed.gov/fulltext/ED501144.pdf

Usiskin, Z., Peressini, A. L., Marchisotto, E., & Stanley, D. (2003). *Mathematics for high school teachers: An advanced perspective*. Upper Saddle River, NJ: Pearson Education Inc.

Wasserman, N. H. (2016). Abstract algebra for algebra teaching: Influencing school mathematics instruction. *Canadian Journal of Science, Mathematics and Technology Education, 16*(1), 28–47.

Watson, A., & Harel, G. (2013). The role of teachers' knowledge of functions in their teaching: A conceptual approach with illustrations from two cases. *Canadian Journal of Science, Mathematics and Technology Education, 13*(2), 154–168.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Lawrence Erlbaum Associates.

Weber, K. (2002). *Students' understanding of exponential and logarithmic functions*. Paper presented at the second conference on the teaching of mathematics, Crete. Retrieved from https://files.eric.ed.gov/fulltext/ED477690.pdf.

Zazkis, R., & Kontorovich, I. (2016). A curious case of superscript (−1): Prospective secondary mathematics teachers explain. *Journal of Mathematical Behavior, 43*, 98–110.

Zazkis, R., & Leikin, R. (2007). Generating examples: From pedagogical tool to a research tool. *For the Learning of Mathematics, 27*(2), 15–21.

Zazkis, R., Liljedahl, P. G., & Sinclair, N. (2009). Lesson plays: Planning teaching versus teaching planning. *For the Learning of Mathematics, 29*(1), 40–47.

Zazkis, R., Sinclair, N., & Liljedahl, P. G. (2013). *Lesson play in mathematics education: A tool for research and professional development*. Dordrecht, The Netherlands: Springer.

Zazkis, R., & Sirotic, N. (2010). Representing and defining irrational numbers: Exposing the missing link. *CBMS Issues in Mathematics Education, 16*, 1–27.

Zazkis, R., & Zazkis, D. (2014). Script writing in the mathematics classroom: Imaginary conversations on the structure of numbers. *Research in Mathematics Education, 16*(1), 54–70.