Chapter 19
Researching Modelling by Mathematics Teacher Educators: Shifting the Focus onto Teaching Practices

Francisco Rojas, Helena Montenegro, Manuel Goizueta, and Salomé Martínez

19.1 Introduction

In the last 10 years, there have been considerable efforts to improve teacher education in Chile. For this reason, multiple public policies have been implemented to regulate teacher education, such as the introduction of new pedagogical and disciplinary standards for pre-service teacher education programmes and the creation of nationwide diagnostic tests for measuring the knowledge of pre-service teachers during their final year of study. These policies have led to the implementation of curricular changes in teacher education programmes, strengthening disciplinary and methodological aspects over general pedagogy (Mineduc, 2011). Currently, according to a study focused on characterising pre-service primary teacher education programmes in Chile, most of the students take at least four mathematics courses (Mineduc, 2016), while as reported by Varas et al. (2008), in 2008, over 80% of prospective teachers were required to take no more than two. Despite the implementation of these measures, major challenges remain, especially in mathematics. For instance, concerning learning opportunities for pre-service primary mathematics teachers, Rojas (2017) notes that these students receive more

F. Rojas (✉)
School of Education – Pontificia Universidad Católica de Chile, Macul, Santiago de Chile, Chile
e-mail: frojass@uc.cl

H. Montenegro · S. Martínez
Center for Mathematical Modeling, UMI 2807 CNRS – Universidad de Chile, Santiago de Chile, Chile

M. Goizueta
Institute of Mathematics – Pontificia Universidad Católica de Valparaíso, Cerro Barón – Valparaíso, Chile

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_19

theoretical information (isolated mathematical concepts) than practical knowledge (strategies for teaching mathematics), regardless of the disciplinary contents studied.

Teacher educators are a key agent to consider when attempting to improve learning processes, because they are involved in curriculum design, its implementation and research into pre-service teacher education (Furlong, Barton, Miles, Whiting, & Whitty, 2000), although this key actor has been seldom discussed and researched in Chile (Cisternas, 2011; Montenegro, 2016). There is evidence that teacher educators' working conditions in Chile are not consistent with the importance of their role in preparing future teachers. Indeed, most educators have hourly contracts and lack professional development opportunities (Mineduc, 2016; Radovic, Peñafiel, San Martin, Bustos, & Martinez, 2018).

In this chapter, we conceptualise mathematics teacher educators as agents tasked with helping pre-service teachers improve their skills and facilitating the teaching of mathematics (Jaworski, 2008; Zaslavsky, 2009). Rojas and Deulofeu (2015) suggest two essential tasks for mathematics teacher educators: first, offering pre-service teachers the chance to learn the discipline in the same way as their students are expected to learn it (Chapman, 2008); second, promoting activities in university classrooms which allow pre-service teachers to learn how to teach mathematics (Watson & Mason, 2007), establishing a strong theory-practice link (Gellert, 2005).

The role as a model that mathematics teacher educators adopt when teaching how to teach becomes hugely relevant since it is a mechanism that can contribute to strengthening pre-service teacher education (Lunenberg, Korthagen, & Swennen, 2007). To understand the relevance of this role, the following sections introduce the concept of modelling and two methodological challenges to studying it. Firstly, we show the complexity and pertinence of including students in the modelling process and thus the need to devise ways to research modelling that take into account both mathematics teacher educators and prospective teachers. Second, we argue the need to understand modelling as a situated practice and thus the need to account for the complexity of the context in which it takes place. The discussion of both methodological challenges is supported by findings of specific research experiences with the purpose of exemplifying modelling as a relational and situated teaching practice.

19.2 Concept of Modelling

Every time a mathematics teacher educator teaches, he or she is enacting a way of thinking mathematically as well as a way of thinking about the teaching of mathematics, either intentionally or unintentionally. Therefore, mathematics teacher educators do not only organise and support the learning of their students; also, through their teaching, they model the practices that students learn (Korthagen, Loughran, & Lunenberg, 2005). Thus, the teaching process in which mathematics teacher educators engage appears to be as influential as the knowledge imparted. Russell (1997) famously summarised this view with the expression "how I teach IS the message," suggesting that teacher educators' teaching practices may be more relevant than the content of the said message when it comes to teaching how to teach.

Even though the implicit modelling of pedagogical reasoning conducted by mathematics teacher educators constitutes the first chance of showing best practices to prospective teachers, it does not necessarily generate substantial learning because it fails to identify such practices as an object of learning (Lunenberg et al., 2007). Thus, it is necessary to expand the concept of modelling teaching practices. Lunenberg et al. (2007) define modelling as a practice that involves intentionally deploying certain behaviours in one's teaching to promote the professional learning of prospective teachers. These authors have identified four types of modelling: implicit, explicit, transferred (facilitating the translation to the prospective teachers' practices) and connected (linking exemplary behaviour with theory). These types vary depending on their degree of explicitness, the connections made between theory and practice and the prospective teacher's role in the process.

For example, explicit modelling is conducted through the teacher educator's critical reflection on his or her practice (Boyd, 2014), which involves the use of teaching strategies that make explicit the decision-making process involved in the planning and implementation of his or her teaching, such as thinking aloud, co-teaching and meta-commentary (Swennen, Lunenberg, & Korthagen, 2008). On the other hand, transferred modelling is aimed at helping the prospective teacher see how the practices modelled can be applied to various teaching situations. Boyd (2014) points out that this type of modelling should make it possible for prospective teachers to reconstruct a teaching situation through learning activities that enable them to compare and analyse the teacher educator's teaching practices with their own. Finally, in connected modelling, the teacher educator links theory and practice whenever he or she treats his or her teaching as an object of reflection with his or her students. By connecting his or her teaching decisions with theories of learning, research evidence or even public policies, the teacher educator is expanding his or her modelling and sending prospective teachers a message: to perform well in a professional capacity, linking theory and practice is crucial.

Much of the research on modelling indicates that it can help prospective teachers learn based on their teacher educators' perspectives and teachings (Loughran & Berry, 2005). Prospective teachers would learn to teach more effectively if teacher educators shared and made explicit the pedagogical reasoning that supports their teaching, explaining the kinds of pedagogical decision that underpins their instructional practices (Bullock, 2009; Loughran, 2006). To do this, teaching must be intentional and congruent to connect prospective teachers' learning with teacher educators' teaching (Swennen et al., 2008), making clear the pedagogical rationality of the latter (Rojas & Deulofeu, 2015). However, the research on this topic has been focused on the teacher educator, leaving in the background critical aspects for the understanding of modelling. For instance, research has not taken into account that the modelling enacted by the teacher educator has an interactive nature and therefore is directly related to the prospective teacher. Furthermore, elements of academic communities and school classroom contexts take part in and mediate the instructional practice of the teacher educator (Goizueta, Montenegro, Rojas, & González, 2017).

Addressing these concerns, research on modelling presupposes new methodological challenges. As a way to advance in this discussion, in the following section, we link this new perspective with findings of studies conducted by the authors of this chapter to make sense of the ideas mentioned above.

19.3 Methodological Challenges in the Study of Modelling

This section will discuss two methodological challenges in the research of modelling practices enacted by mathematics teacher educators: the necessity of including the prospective teachers' perspective and the complexities of the educational context where future teachers will work.

19.3.1 Modelling as a Two-Sided Practice

Previous research on modelling has mainly focused on how teacher educators model (i.e. what teacher educators do) and on teacher educators' explicit claims about teaching (i.e. what teacher educators say). By contrast, there are few studies of what teacher educators model (i.e. techniques, values, dispositions, educational principles) and how prospective teachers interact with the contents of such modelling. The focus on teacher educators' performance neglects or even invisibilises prospective teachers as the necessary counterpart of the teacher educator's educational aims and actions (Boyd, 2014; Goizueta et al., 2017). We argue that the role of prospective teachers must be recognised and taken into account when modelling is used as a means of teaching how to teach.

To help prospective teachers identify the modelling practices enacted, it is essential that mathematics teacher educators ponder some crucial questions: what do prospective teachers look at in the teaching practices enacted by mathematics teacher educators? What instructional practices and teaching knowledge do prospective teachers incorporate into their pedagogical practices? Why do they make those particular choices? What impact do mathematics teacher educators have in these processes? These questions have in common that only prospective teachers can answer them. In other words, for achieving better understanding of these issues, we need to research the modelling practices enacted by mathematics teacher educators taking into account the prospective teachers' perspective. Hence, it seems reasonable to suggest that a first methodological approach to extend the research on modelling is to inquire what prospective teachers experience and think when the mathematics teacher educator is teaching.

To illustrate this methodological approach, we share two studies that consider the perspective of prospective teachers. The first study, conducted by Martínez (2017), focuses on the perceptions of prospective primary school teachers regarding the implementation of learning units for teaching mathematics. In the second study, Rojas and Montenegro (2018) explore how prospective secondary school mathematics teachers perceive a set of instructional practices enacted by their mathematics teacher educators according to the degree to which the latter make their pedagogical reasoning explicit.

Concerning the first study, Martínez (2017) leads a research and development project aimed at developing a system for supporting mathematics teaching in pre-service primary teacher education. In this project, learning units for teacher education are sequences of lessons around a mathematical topic and include mathematical tasks for teaching and supporting resources for mathematics teacher educators. A multidisciplinary team developed the learning units following an elaboration-testing-adjusting design cycle.

During 2017, four learning units were developed focused on topics selected for their high impact in initial teacher education. Two units deal with numbers. The first of these concerns addition and subtraction problems, covering the classification of these problems according to the actions involved and the place of the unknown (Lewin, López, Martínez, Rojas, & Zannoco, 2010). The second unit on numbers addresses representing addition and subtraction problems, which seeks to identify concrete and pictorial representations of these problems and discuss their pertinence (Veloo & Parmijt, 2017). In addition, two geometry units were developed. The first of these deals with definition of perimeter, which addresses the process of constructing a definition of the contour of a shape and problem-solving involving perimeters (Lu, Weng, & Tuo, 2013). The second geometry unit addresses variations of area and perimeter, which deals with the relationship between area and perimeter when changing geometric shapes (D'Amore & Fandiño Pinilla, 2006; Ma, 2010). In January 2018, the units reviewed were tested by mathematics teacher educators from the development team in two different short courses included in a summer programme for pre-service primary teachers.

Two focus groups were conducted (Flick, 2002) to assess the implementation of these four learning units, with pre-service teachers who took part in each of the two courses. These focus groups sought to examine in more detail the implementation and experiences associated with the numbers and geometry units. The discussion was guided by a set of questions aimed at evaluating the activities designed, as well as making explicit the teaching practices adopted by the mathematics teacher educators when implementing the learning units. The focus groups were also used to explore how the students perceived their teacher educators' modelling role. Both were recorded and transcribed in full for subsequent analyses. The transcripts were examined using thematic content analysis (Bardin, 2002).

Concerning the participants' perceptions regarding mathematics teacher educators' role as models, the prospective teachers in both groups pointed out that they learned not only the content imparted but also from the practices of the mathematics teacher educators. That is, they learned from the modelling in which they implicitly engaged, as the following extract shows:

> Also, not only... at least in my case, I learned from what we were taught about mathematics and also from the teacher herself (Numbers FG).

Specifically, the prospective teachers who took the geometry course pointed out that the mathematics teacher educator was able to anticipate students' questions, which he used to make it easier for them to learn the content:

> Yes, I feel that he [the mathematics teacher educator] anticipated our heuristics, so to speak, like our ways of thinking or tackling an exercise. He took our answers into account and knew how to use our ways of reasoning to construct the content of the course. I think that was quite admirable (Geometry FG).

For their part, the prospective teachers who took the numbers course stressed the mathematics teacher educator's ability to organise learning according to their mistakes. In other words, prospective teachers had a positive opinion of how the mathematics teacher educator managed the classroom climate to encourage them to share their answers without fearing criticism. This perception is observed in the following extract:

> In contrast, with her [the mathematics teacher educator], if I made a mistake it was the opposite, it was a good thing. Because I know she is going to clarify it for me, she is going to make it clear. I know that when she explained something… I think sometimes you can also learn from your mistakes, and you should… but I was not afraid of making mistakes (Numbers FG).

Lastly, prospective teachers in both groups mentioned that these pedagogical practices constituted another type of learning that they think will be essential in their own work as teachers in the future. For instance, a prospective teacher said the following:

> I wish I could do the same later, with the children, so they would not be afraid to make mistakes. Because sometimes children make a mistake one day and they do not want to work anymore (Numbers FG).

As these excerpts show, prospective teachers were able to see mathematics teacher educators as teaching models from whom they learned some teaching practices that they would like to implement in school classrooms. In addition, they considered that this type of learning was positive for their professional education. The above observations are especially relevant considering that all the teaching practices identified by the participants were enacted through implicit modelling. As a consequence, the professional role model that the mathematics teacher educator enacts while teaching has an impact on learning outcomes beyond the explicit pedagogical and disciplinary content at stake, and such effects relate to what prospective teachers notice about such role models. Precisely as Russell (1997) suggested, there is "a message" about teaching in teaching itself and thus the criticality of the teaching model's role enacted by the teacher educator.

Nevertheless, according to Loughran and Berry (2005), when prospective teachers learn about teaching, what is evident for teacher educators might not be so for their students. Thus, for modelling to be an effective teaching and learning tool, explicit attention must be intentionally directed to particular features of teaching practices, to make implicit content about teaching available and to address possible differing interpretations. However, despite the efforts made by the teacher educator to make explicit his or her pedagogical reasoning and thus to justify his or her teaching practices, prospective teachers still might not perceive what has been

Table 19.1 Descriptions of instructional practices

Mathematical task (MT)	Interaction (INT)	Consideration (CON)
Generation of mathematical reasoning opportunities through the design and application of mathematical tasks	Generation of teacher-student interactions to promote mathematical reasoning	Observation and consideration of students' actions, behaviours, responses and mathematical output

modelled by the teacher educator, or these practices might be perceived differently from what the latter intends.

In this regard, the second study developed by Rojas and Montenegro (2018) explores how prospective teachers perceive the modelling enacted by their mathematics teacher educators, specifically related to their instructional practices. These practices were defined using several standardised protocols for the observation of mathematics lessons (e.g. Boston, Bostic, Lesseig, & Sherman, 2015; Hill et al., 2008) and grouped into three categories (Rojas & Chandía, 2015) (Table 19.1).

These categories were transformed into a Likert-type questionnaire (Rojas & Chandía, 2015), in which prospective secondary school mathematics teachers were asked which modelling type – implicit, explicit, transferred or connected modelling (Lunenberg et al. (2007) – they identified in the mathematics teaching practices of their mathematics teacher educators.

Rojas and Montenegro (2018) analysed the results of the previous questionnaire applied to a subsample of 61 prospective teachers taking mathematics teaching methods courses at eight Chilean universities. Two-stage cluster analysis was conducted to characterise their perceptions of the instructional practices modelled by mathematics teacher educators. This approach made it possible to group together continuous and categorical variables and to form groups with a high degree of internal homogeneity and high heterogeneity with respect to each other (Hair, Anderson, Tatham, & Black, 1998).

This process yielded four groups[1], two of which show consistency in the type of perceived modelling (clusters 3 and 4) and two with greater divergence (clusters 1 and 2), according to different categories of teaching practices. In cluster 1, prospective teachers tended to recognise that mathematics teacher educators based their practices on a theoretical knowledge that informs their pedagogical decisions, specifically in categories of practice regarding mathematical tasks and consideration of students' productions. In contrast, in cluster 2, prospective teachers perceived that the mathematics teacher educator implicitly guides them through various mathematics teaching practices. In this group, prospective teachers were unable to identify a specific modelling practice for actions related to mathematical tasks (they tended to choose "Does not apply" in these cases). Cluster 3 comprises prospective teachers who perceived that their mathematics teacher educators explicitly support their actions as models of teaching practice, in all three categories. However, this does not necessarily mean that these explicit explanations have a theoretical basis or

[1] The analysis of the χ^2 test revealed statistically significant differences in the distribution within the groups ($\chi^2 = 28.685$, df = 12, $p < 0.004$).

Table 19.2 Type of modelling practices by cluster and category

	Dimensions of instructional practices		
	MT	INT	CON
Cluster 1	Connected	Transferred	Connected
Cluster 2	Does not apply	Implicit	Implicit
Cluster 3	Explicit	Explicit	Explicit
Cluster 4	Transferred	Transferred	Transferred

are linked to experiences in mathematics classrooms. Finally, prospective teachers in cluster 4 perceived that their mathematics teacher educators are permanently transferring to the school classroom that which is studied in the university classroom. As in cluster 3, the students in this group also considered that their teacher educators connect their practices in the three categories with the school context, but do not necessarily make decisions based on their knowledge of public theory. The following table summarises the type of modelling perceived in each dimension for the four clustered student groups (Table 19.2).

Since each cluster is composed of students from different universities, this analysis shows that the perception of prospective teachers is heterogeneous within the same university classroom. Regarding the type of instructional practice that the mathematics teacher educator promotes more explicitly, the analysis shows that those related to the consideration of student productions (CON) are those that the prospective teacher most easily discriminates. This result suggests that mathematics teacher educators can make explicit to various degrees the pedagogical reasoning that supports their teaching decisions, which prospective teachers are unable to see clearly. These results concerning prospective teachers' perceptions of the type of modelling employed by mathematics teacher educators highlight the importance of harmonising prospective teachers' learning and mathematics teacher educators' teaching (Swennen et al., 2008).

Although the prospective teachers' perspective can inform the modelling enacted by the mathematics teacher educators, there is still a question about why one type or another is perceived, besides knowing what kind of instructional practice they see most clearly. The answers to these questions should not tend to seek a homogenisation of the perception of modelling by prospective teachers. Heterogeneity tells us about the level of involvement and evolution of pedagogical thinking that prospective teachers have. Even so, and at a theoretical level, it is desirable that the interrelation between the mathematics teacher educator and his or her students tends to project and perceive, respectively, a modelling closer to what we define as connected practice.

19.3.2 Modelling as a Situated Practice

Various authors highlight the relevance of researching teaching and learning processes from a situated perspective, taking into account the context and how it shapes both individuals and teaching practices (Borko, 2004). From a sociocultural perspective, researching teaching and learning implies taking into account not only teacher educators' and prospective teachers' views on the setting in which they interact but also the broader social and school contexts in which the latter (will) teach. In this regard, the classroom setting and sociocultural context are fundamental to facilitate a more comprehensive and connected understanding of teaching and learning experiences in the pre-service teacher education classroom (Marton & Tsui, 2004).

Research on modelling teaching practices from a situated perspective presupposes methodological challenges for approaching the complexities associated with learning to teach from a relational perspective. On the one hand, we need to consider teaching and learning from the positioning of those engaged in teaching and learning processes. That is, the focus should be on studying both mathematics teacher educators' and prospective teachers' conceptions of teaching and learning and the perceptions of their educational settings with the purposes of enhancing the learning experience of school students. On the other hand, the situated character of teaching requires an understanding that prospective teachers learn to teach in a particular educational context (teacher education classroom), but in the future, their teaching will take place in a different educational context (school classroom). As Boyd (2014) suggests, learning to teach implies "becoming within a transitional process of boundary-crossing" (p. 53). This idea illustrates the challenging task that prospective teachers face in developing their teaching practice and professional identity inside teacher education programmes, a different workplace setting compared to the school system. Loughran and Berry (2005) state that for many teacher educators, this dual setting is an ever-present feature of their teaching context.

Hence, it is possible to argue that the value of research on modelling from a situated perspective depends on integrating descriptions of those elements in an interrelated way, giving a whole and complex picture of the educational phenomenon. In this regard, phenomenographic research inquires into how teachers and students in naturalistic teaching contexts approach their teaching and learning processes (Marton & Tsui, 2004). Phenomenography studies how an experience can vary by identifying the qualitatively different ways in which a phenomenon is experienced, perceived or conceptualised. The results of this variation are systematised using categories of description that are hierarchically organised to create an outcome space (Bowden & Walsh, 2000).

Montenegro (2018) is currently conducting a phenomenographic research project aimed at understanding how the notions of modelling held by mathematics teacher educators influence their teaching practices. As part of this study, phenomenographic interviews (Trigwell, 2000) have been held with a sample of 12 mathematics teacher educators teaching disciplinary and pedagogical courses in three programmes for pre-service teachers. The results of the preliminary analyses reveal findings that are interesting to examine. Four categories of descriptions, structured by complexity, emerged from the analysis. Table 19.3 reports the name of the categories of descriptions and representative quotations from the interviews.

Table 19.3 Categories of description of modelling

	Mathematics teacher educators model
A	Pedagogical activities that can be replicated in school classrooms "The use of the body is also relevant, especially in geometry. I don't know, angles, parallel lines, you can show all that using your arms. I said to them "everyone, show me an obtuse angle with your arms, an acute angle", that sort of thing... and I also said to them explicitly that it is good for them to do that with their students" (MTE1).
B	Pedagogical interactions to be conducted with students "There is also the emotional aspect... in my opinion, if there is no emotion, there is no learning. So I become emotionally involved with students, I mean, I tell them that they can do it, that they can generate changes. That they can change mathematics teaching" (MTE9).
C	Teaching connected with school classrooms "I try to model, with a theoretical basis, a way of thinking about designs and their objectives that is not unique… it is like thinking aloud about what I want to achieve in the classroom regarding a mathematical objective" (MTE6).
D	Teaching practices consistent with the context where they are carried out "Because otherwise there is no consistency, how can I… so if I am not a model, I can just babble about how I think students should learn mathematics. But if I am not [a model], students will not have a point of reference to observe how you can do those things that the teacher says you can do. So, I think discourse and practice must coincide" (MTE2).

Regarding the first category, mathematics teacher educators point out that they model pedagogical activities that prospective teachers will be able to replicate when they become teachers. In the second category of description, mathematics teacher educators model pedagogical interactions that can facilitate learning in the classroom, a process in which it is fundamental to establish an appropriate bond with students. Regarding the third category of description, mathematics teacher educators conceive modelling as a teaching practice linked to the school classroom. Here, mathematics teacher educators model a type of teaching that is aimed towards the mathematics taught in schools predicting the most frequent errors and difficulties observed in school students. Finally, the fourth category of description views modelling as the use of a consistent set of teaching practices that allow prospective teachers to experience mathematical learning and replicate it with students in the school system. In this category of description, mathematics teacher educators are interested in modelling teaching practices consistent with the theoretical model that they ascribe to since they regard this as essential for learning how to teach mathematics.

These results support the view that mathematics teacher educators have different notions of their role as models, which vary regarding the position that they adopt and their awareness of the effect that they can have on their prospective teachers' learning. When mathematics teacher educators see modelling as a practice with a focus on performing pedagogical activities and interactions with prospective teachers, they attempt to recreate the complexity of the school classroom inside teacher education programmes. In contrast, mathematics teacher educators who regard modelling as a teaching practice linked to the school classroom and supported by a corresponding modelling approach not only connect their teaching to the university classroom but also invite prospective teachers to reflect on the school classroom where they will work in the future. In other words, learning to teach is viewed as a complex phenomenon that can be only understood if it is discussed and pondered considering the context where it will take place (Boyd, 2014; Loughran, 2006).

19.4 The Next Step in Researching Modelling with a Focus on Teaching Practices Inside the Classroom

In this chapter, we have discussed some methodological challenges associated with research on modelling practices of teacher educators when they teach about teaching mathematics. These challenges are related to how to incorporate the prospective teachers and the critical role they play in the practice of modelling, as well as to understand that modelling is a practice situated in a university context but, at the same time, directed to the school context. Based on our experience in studying mathematics teacher educators' conceptions and teaching practices, we have shared some interesting findings as a way to contribute to this discussion. However, those studies focused on the pedagogical discourse and on perceptions, ideas and teaching practices of both prospective teachers and mathematics teacher educators. That is, they do not explore what the experience of modelling inside teacher education classrooms is like.

To move forward in research on modelling with a focus on displayed teaching practices, including prospective teachers and the educational context where it is materialised, it is fundamental to take into account new issues and methodological challenges in this field of research. For instance, it is necessary to think about the content of what is modelled, particularly in how the mathematics teacher educator makes visible the disciplinary reasoning that underpins his or her mathematical knowledge for teaching. Despite the acknowledgement of disciplinary differences in teaching and learning and how to learn to teach, previous research on modelling has mainly focused on general aspects and has not taken into account specificities and nuances associated with the disciplinary content at stake (see, e.g. Boyd, 2014; Lunenberg et al., 2007; Loughran & Berry, 2005). We claim that it is necessary to consider that, besides general pedagogical principles, values and knowledge, mathematics teacher educators deploy specific mathematics knowledge for teaching (Ball, Thames, & Phelps, 2008) associated with discipline-specific teaching practices. Thus, modelling should also be understood and researched with its discipline-specific features. In the same vein, we consider it essential to research how this disciplinary reasoning is perceived by prospective teachers regarding the possible improvement of their learning to teach mathematics as well as how prospective teachers might transfer this disciplinary reasoning to students when they become teachers.

The significance of modelling disciplinary reasoning is in line with our results. For example, Montenegro's findings highlight that mathematics teacher educators position themselves differently in terms of what they model. This positioning oscillates from teaching pedagogical activities that can be replicated in school classrooms (focus on mathematical procedures) to teaching styles connected with classrooms in schools (focus on school mathematical knowledge). Martinez's research shows that prospective teachers tend to pay more attention to pedagogical approaches to teaching. Particularly, when asked about the mathematics teacher educator's teaching practices, prospective teachers highlighted general pedagogical resources, such as learning from mistakes, anticipating answers and approaches and cultivating confidence among students. Nevertheless, they also highlighted the criticality of unpacking and making certain mathematics-related elements of teaching explicit for prospective teachers to notice and reflect about them. In the same way, Rojas and Montenegro's findings make evident that prospective teachers have difficulty in identifying the mathematical reasoning related to the specific mathematical task. In other words, prospective teachers do not recognise teaching practices specific for learning to teach mathematics as part of what mathematics teacher educators enact as a role model when they are teaching.

Furthermore, another methodological challenge that we consider crucial in research on modelling from this new approach is to explore it in a holistic and contextualised way. To strengthen approaches to research on modelling with a focus on teaching practices materialised within the classroom, we should move towards two new developments and levels of complexity. Firstly, it is required to inquire how prospective teachers engage with the modelling enacted by the mathematics teacher educator. To analyse the interactions between teachers and students, we need new methods and instruments for identifying patterns of interactions, widely developed for the school classroom but scarce in the teacher education programme contexts. Second, it is fundamental to think of teacher education programmes as a community of practice (Wenger, 1998) in which researchers, mathematics teacher educators and prospective teachers reflect on the complex task of learning to teach mathematics and how the modelling enacted by mathematics teacher educators might contribute to improving the learning experience when they work as future teachers.

Concerning the first new development, Lunenberg et al. (2007) argue that teacher educators have difficulties becoming aware of their role as models and the influence of their teaching practices and their pedagogical choices on prospective teachers' learning about teaching. Therefore, a challenge for mathematics teacher educators is to pay attention to what is being taught and to how it is taught, taking into account the need for congruency between the pedagogical theories they introduce and the teaching practices they enact (Swennen et al., 2008). Similarly, the shift in focus to enacted teaching practices turns such practices, pedagogical reasoning and rationale behind them into objects of collective conscious reflection. Such space for reflection constitutes an opportunity for mathematics teacher educators and prospective teachers to develop professional scrutiny and critique and to explicitly connect professional practice to the knowledge basis behind it (Loughran & Berry, 2005).

Unpacking teaching and learning activities in the classroom might also constitute an opportunity to discern different elements and aspects of the mathematics teacher's professional knowledge and thus an opportunity to introduce, make accessible and connect with actual practice the knowledge basis and theory behind it. Nonetheless, some mathematics teacher educators tend to overlook prospective teachers as the other side of the coin in the process of learning to teach because they do not realise that prospective teachers are learning both content and teaching strategies. Thus, modelling might be an effective way to connect and bridge different elements of the prospective teachers' and the mathematics teacher educator's professional knowledge. This approach might raise awareness about how the prospective teachers' and the mathematics teacher educators' professional practices are related, how they develop together and how modelling might contribute to such development. Because modelling promotes a reflective stance towards teaching and learning practices in the classroom, it might help prospective teachers to understand teaching mathematics as a layered activity related to both content knowledge and knowledge for teaching such content.

Regarding the second new development, the perspective of teaching education programmes as communities of practice reflects that teaching is a relational and interactive activity. However, at the same time, it generates a new methodological challenge: that teaching, being a relational practice, should include research by mathematics teacher educators and their prospective teachers, together. To advance in this matter, we propose that modelling is better understood as a collaborative practice in which both mathematics teacher educators and prospective teachers participate and for which explicitness about what is worth paying attention to, reflecting on and learning is needed and negotiated between the participants.

In this regard, collectively reflecting on teaching and learning mathematics through modelling is thus an opportunity for prospective teachers and their mathematics teacher educators to learn about how to teach mathematics. From this perspective, mathematics teacher educators can be seen both as facilitators of learning and as learners themselves (Zaslavsky, 2009), so that they and prospective teachers can be regarded as learning in two interrelated communities of practice improving each other's professional learning (Jaworski, 2008; Wenger, 1998). These communities of practice might allow the emergence of opportunities to learn that are not likely to materialise otherwise (Loughran & Berry, 2005).

Additionally, this collaborative learning between mathematics teacher educators and prospective teachers broadly contributes to the professional development of both. It allows for the explanation of the different roles and tasks carried out in the profession of school mathematics teaching (Jaworski, 2008) and to develop a sense of belonging in teacher education programmes (Loughran, 2006). A collaborative approach also allows to make explicit the tacit knowledge of teaching when it is verbalised and discussed with others (Loughran & Berry, 2005) and to consolidate a language which can be analysed with other teachers, mathematics teacher educators and researchers in the field (Ball et al., 2008; Lunenberg et al., 2007). As a result, we must start to see teacher education programmes as learning communities, not only for prospective teachers but also for the mathematics teacher educators who are part of them. It is the responsibility of both the programme and mathematics teacher educators to ensure that challenges associated with entry into this new educational context are discussed and scrutinised (Jaworski, 2008; Loughran, 2006).

To conclude, we believe that reconceptualising research on modelling from a more integrated, holistic perspective should take into account the complementary roles of mathematics teacher educators and prospective teachers and how they complement each other in the challenge of learning to teach mathematics. Furthermore, this new approach must consider modelling as having disciplinary specificities for teaching mathematics from a situated perspective. We hope that future studies will not only contribute to the improvement of this proposal but also generate and develop new strategies for enhancing the disciplinary and pedagogical development of mathematics teacher education.

Acknowledgements This work was carried out within the following research projects: VRI INICIO 23/2013 "Estudio de las Prácticas Instruccionales del Formador de Profesores de Matemática de Educación Media: Construcción de Indicadores de Coherencia" (*Study of the Instructional Practices of the Secondary Mathematics Teacher Educators: Construction of Coherence Indicators*) led by Francisco Rojas, FONDECYT POSTDOCTORADO 3170423 "Caracterización de las experiencias de enseñanza y aprendizaje en la formación docente inicial: un estudio exploratorio" (*Characterization of teaching and learning experiences in pre-service teacher education: an exploratory study*) led by Helena Montenegro, FONDEF ID16I10119 "Metodologías innovadoras para la formación inicial de profesores de educación básica en matemática: modelo basado en prácticas efectivas de aula y tecnologías de la información" (*Innovative methodologies for the pre-service training of elementary school teachers in mathematics: a model based on classroom practices and information technologies*) led by Salomé Martínez and CONICYT + PIA/Concurso de Apoyo a Centros Científicos y Tecnológicos de Excelencia con Financiamiento Basal AFB170001.

References

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching. What makes it special? *Journal of Teacher Education, 59*(5), 389–407.
Bardin, L. (2002). *Análisis de contenido [Content Analysis]*. Madrid, Spain: Akal Ediciones.
Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher, 33*(8), 3–15.
Boston, M., Bostic, J., Lesseig, K., & Sherman, M. (2015). A comparison of mathematics classroom observation protocols. *Mathematics Teacher Educator, 3*(2), 154–175.
Bowden, J., & Walsh, E. (2000). *Phenomenography*. Melbourne, Australia: RMIT University Press.
Boyd, P. (2014). Using 'modelling' to improve the coherence of initial teacher education. In P. Boyd, A. Szplit, & Z. Zbróg (Eds.), *Teacher educators and teachers as learners: International perspectives* (pp. 51–73). Kraków, Poland: Wydawnictwo Libron.
Bullock, S. M. (2009). Learning to think like a teacher educator: Making the substantive and syntactic structures of teaching explicit through self-study. *Teachers and Teaching: Theory and Practice, 15*(2), 291–304.
Chapman, O. (2008). Mathematics teacher educator's learning from research on their instructional practices. In B. Jaworski & T. Wood (Eds.), *International handbook of mathematics teacher education* (Vol. 4, pp. 115–134). Rotterdam, The Netherlands: Sense Publishers.
Cisternas, T. (2011). La investigación sobre formación docente en Chile. Territorios explorados e inexplorados [Research on teacher education in Chile: Known and unknown territories]. *Calidad en la Educación, 35*, 131–164.

D'Amore, B., & Fandiño Pinilla, M. I. (2006). Relationships between area and perimeter: Beliefs of teachers and students. *Mediterranean Journal for Research in Mathematics Education, 5*(2), 1–29.

Flick, U. (2002). *An introduction to qualitative research*. London: SAGE.

Furlong, J., Barton, L., Miles, S., Whiting, C., & Whitty, G. (2000). *Teacher education in transition*. Buckingham, UK: OUP.

Gellert, U. (2005). La formación docente entre lo teórico y lo práctico [Teacher education: between the theory and practice]. In I. M. Gómes-Chacón y E. Planchart (Eds.), *Educación Matemática y Formación de Profesores. Propuestas para Europa y Latinoamérica*. Bilbao, Spain: Universidad de Deusto.

Goizueta, M., Montenegro, H., Rojas, F., & González, M. P. (2017). Rethinking modeling in pre-service teacher education: Implications for teacher educators. In M. Juanjo, A. García-Valcárcel, F. García-Peñalvo, & M. Martín del Pozo (Eds.), *Search and research: Teacher education for contemporary contexts* (pp. 139–147). Salamanca, Spain: Ediciones Universidad de Salamanca.

Hair, J., Anderson, R., Tatham, R., & Black, W. (1998). *Multivariate data analysis* (5th ed.). Upper Saddle River, NJ: Prentice Hall.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., et al. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430–511.

Jaworski, B. (2008). Mathematics teacher educator learning and development. In B. Jaworski & T. Wood (Eds.), *International handbook of mathematics teacher education* (Vol. 4, pp. 1–13). Rotterdam, The Netherlands: Sense Publishers.

Korthagen, F., Loughran, J., & Lunenberg, M. (2005). Teaching teachers-studies into the expertise of teacher educators: An introduction to this theme issue. *Teaching and Teacher Education, 21*(2), 107–115.

Lewin, R., López, A., Martínez, S., Rojas, D., & Zannoco, P. (2010). *Números para futuros profesores de educación básica, REFIP [Numbers for prospective primary teachers, REFIP]*. Santiago, Chile: Ediciones SM.

Loughran, J. (2006). *Developing a pedagogy of teacher education. Understanding teaching and learning about teaching*. New York: Routledge.

Loughran, J., & Berry, A. (2005). Modelling by teacher educators. *Teaching and Teacher Education, 21*(2), 193–203.

Lu, P., Weng, K., & Tuo, Y. (2013). Perimeter in the primary school mathematics curriculum of Singapore. *Journal of the Association of Teachers of Mathematics, 236*, 27–30.

Lunenberg, M., Korthagen, F., & Swennen, A. (2007). The teacher educator as a role model. *Teaching and Teacher Education, 23*(5), 586–601.

Ma, L. (2010). *Knowing and teaching elementary mathematics. Teachers' understanding of fundamental mathematics in China and United States*. New York: Routledge.

Martínez, S. (2017). FONDEF ID16I10119. *Metodologías innovadoras para la formación inicial de profesores de educación básica en matemática: modelo basado en prácticas efectivas de aula y tecnologías de la información. [Innovative methodologies for the pre-service training of elementary school teachers in mathematics: a model based on classroom practices and information technologies] Internal report*.

Marton, F., & Tsui, A. (2004). *Classroom discourse and the space of learning*. New York: Routledge.

Mineduc. (2011). *Estándares Orientadores para Egresados de Carreras de Pedagogía en Educación Básica [Educational Standards for Primary Teacher Education Graduates]*. Santiago de Chile: Ministerio de Educación.

Mineduc. (2016). *Final Report. Identification of critical elements to strengthen the pre-service elementary teacher training of teachers in Mathematics*. Santiago de Chile: Ministerio de Educación.

Montenegro, H. (2016). The professional path to become a teacher educator: The experience of Chilean teacher educators. *Professional Development in Education, 42*(4), 527–546.
Montenegro, H. (2018). *Conceptions of modeling by Chilean mathematics teacher educators: A phenomenographic study*. Paper presented at First PME Regional Conference South America, Rancagua, Chile.
Radovic, D., Peñafiel, B., San Martin, A., Bustos, S., & Martinez, S. (2018). Enseñando matemáticas a futuros profesores en un sistema complejo: Formadores de profesores en universidades de Chile. [Teaching mathematics in a complex system: Teacher educators in universities in Chile]. *Estudios Pedagógicos, XLIV*(3), 101–116.
Rojas, F. (2017). Oportunidades para aprender a enseñar matemática en la formación inicial [Opportunities to learn to teach mathematics in pre-service teacher education]. En L. Pino-Fan, A. Poblete y V. Díaz (Eds.), *Perspectivas de la investigación sobre la formación de profesores de matemáticas en Chile* (pp. 45- 84). Osorno, Chile: Cuadernos de Sofía.
Rojas, F., & Chandía, E. (2015). Instructional coherence as perceived by prospective mathematics teachers: A case study in Chilean universities. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (CERME9, 4-8 February 2015) (pp. 2909-2915). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.
Rojas, F., & Deulofeu, J. (2015). El formador de profesores de matemática: un análisis de las percepciones de sus prácticas instruccionales desde la «tensión» estudiante-formador [Mathematics teacher educators: An analysis of the perceptions of their instructional practices based on the student-educator «tension»]. *Enseñanza de las Ciencias, 33*(1), 47–61.
Rojas, F., & Montenegro, H. (2018). Approaches to modeling by mathematics teacher educators: The perception of student teachers. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, p. 149). Umeå, Sweden: PME.
Russell, T. (1997). How I teach IS the message. In J. Loughran & T. Russell (Eds.), *Teaching about teaching: Purpose, passion and pedagogy in teacher education* (pp. 32–47). London/Washington, DC: Falmer Press.
Swennen, A., Lunenberg, M., & Korthagen, F. (2008). Preach what you teach! Teacher educators and congruent teaching. *Teachers and Teaching: Theory and Practice, 14*(5–6), 531–542.
Trigwell, K. (2000). A phenomenographic interview on phenomenography. In J. Bowden & E. Walsh (Eds.), *Phenomenography* (pp. 62–82). Melbourne, Australia: RMIT University Press.
Varas, L., Felmer, P., Gálvez, G., Lewin, R., Martínez, C., Navarro, S., et al. (2008). Oportunidades de preparación para enseñar matemática de futuros profesores de Educación General Básica en Chile [Opportunities for learning to teach mathematics in Chilean prospective primary teachers]. *Calidad en la Educación, 29*, 64–88.
Veloo, P., & Parmijt, S. (2017). Using representations to develop mathematical thinking. In B. Kaur & N. H. Lee (Eds.), *Empowering mathematics learners. Yearbook 2017* (pp. 249–269). Singapore: Association of Mathematics Educators.
Watson, A., & Mason, J. (2007). Taken-as-shared: A review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education, 10*, 205–215.
Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. Cambridge, UK: Cambridge University Press.
Zaslavsky, O. (2009). Mathematics educators' knowledge and development. In R. Evan & D. L. Ball (Eds.), *The professional education and development of teachers of mathematics. The 15th ICMI Study* (pp. 105–111). New York: Springer.