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Kim Beswick *Editors*

The Learning and Development of Mathematics Teacher Educators

International Perspectives and Challenges

Springer

Research in Mathematics Education

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Editors
Merrilyn Goos
School of Education
University of Limerick
Limerick, Ireland

Kim Beswick
School of Education
University of New South Wales
Sydney, NSW, Australia

ISSN 2570-4729 ISSN 2570-4737 (electronic)
Research in Mathematics Education
ISBN 978-3-030-62407-1 ISBN 978-3-030-62408-8 (eBook)
https://doi.org/10.1007/978-3-030-62408-8

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Acknowledgments

We would like to thank Jinfa Cai for encouraging us to pursue publication of this volume. The book grew from a Working Session that we convened at the Joint Meeting of the International Group for the Psychology of Mathematics Education (PME38) and the North American Chapter of the Psychology of Mathematics Education (PME-NA36) in Vancouver, Canada, in 2014. We thank the participants in that Working Session who have contributed chapters to this book, along with others who accepted our invitation to share their research and practice as mathematics teacher educators. We are grateful to those chapter authors who additionally acted as reviewers of other chapters in the book, and we especially wish to thank the following external reviewers who provided critical and constructive feedback:

Anne Bennison
Karin Brodie
Olivia Fitzmaurice
Michael Fried
Vince Geiger
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Contents

Part II Learning and Developing as a Mathematics Teacher Educator

Contributors

Ferdinando Arzarello Department of Mathematics “G. Peano”, University of Turin, Turin, Italy

Dionysia Bakogianni National and Kapodistrian University of Athens, Athens, Greece

Kim Beswick School of Education, University of New South Wales, Sydney, NSW, Australia

Alistair Bissell Mathematics Education and Industry (MEI), London, UK

Irene Biza University of East Anglia, Norwich, UK

Julian Brown University of Bristol, Bristol, UK

Laurinda Brown University of Bristol, Bristol, UK

Katharine Burn Department of Education, University of Oxford, Oxford, UK

Edwina Butler School of Mathematics and Physical Sciences, Faculty of Science, University of Newcastle, Callaghan, NSW, Australia

Jinfa Cai Department of Mathematical Sciences, University of Delaware, Newark, DE, USA

Olive Chapman Werklund School of Education, University of Calgary, Calgary, AB, Canada

Alf Coles University of Bristol, Bristol, UK

Luis Carlos Contreras Universidad de Huelva, Huelva, Spain

Ann Downton Monash University, Clayton, VIC, Australia

Dinazar Escudero-Ávila Benemérita Universidad de Puebla, Puebla, Mexico

Jen Fiddaman Bayards Hill Primary School, Oxford, UK

Manuel Goizueta Institute of Mathematics – Pontificia Universidad Católica de Valparaíso, Cerro Barón – Valparaíso, Chile

Merrilyn Goos School of Education, University of Limerick, Limerick, Ireland

The University of Queensland, Brisbane, QLD, Australia

Yvonne Grimeland Norwegian University of Science and Technology (NTNU, Norway), Trondheim, Norway

Tracy Helliwell University of Bristol, Bristol, UK

Dave Hewitt Loughborough University, Loughborough, UK

Jenni Ingram Department of Education, University of Oxford, Oxford, UK

Barbara Jaworski Mathematics Education Centre, Loughborough University, Epinal Way, Loughborough, Leicestershire, UK

Lindsay Keazer Sacred Heart University, Department of Teacher Education, Orange, CT, USA

Patrick M. Kimani Glendale Community College, Glendale, AZ, USA

Roza Leikin Department of Mathematics Education, RANGE Center (Research and Advancement of Giftedness and Excellence), Faculty of Education, University of Haifa, Haifa, Israel

Mariana Levin Department of Mathematics, Western Michigan University, Kalamazoo, MI, USA

Sharyn Livy Monash University, Clayton, VIC, Australia

Nicole Maher School of Education, University of Tasmania, Newnham Campus, Launceston, TAS, Australia

Angeliki Mali University of Groningen, Faculty of Science and Engineering, Institute for Science Education and Communication, Groningen, Groningen, The Netherlands

Ofer Marmur Mathematics Education Unit, Department of Mathematics, Faculty of Science, The University of Auckland, Auckland, New Zealand

Margaret Marshman School of Education, University of the Sunshine Coast, Sippy Downs, QLD, Australia

Salomé Martínez Center for Mathematical Modeling, UMI 2807 CNRS – Universidad de Chile, Santiago de Chile, Chile

Joanna O. Masingila Syracuse University, Syracuse, NY, USA

Helena Montenegro Center for Mathematical Modeling, UMI 2807 CNRS – Universidad de Chile, Santiago de Chile, Chile

Miguel Montes Universidad de Huelva, Huelva, Spain

Tracey Muir School of Education, University of Tasmania, Newnham Campus, Launceston, TAS, Australia

Carol Murphy School of Education, University of Tasmania, Newnham Campus, Launceston, TAS, Australia

Kathleen Nolan University of Regina, Faculty of Education, Regina, SK, Canada

Greg Oates School of Education, University of Tasmania, Newnham Campus, Launceston, TAS, Australia

Dana Olanoff Widener University, Chester, PA, USA

Judy-anne Osborn School of Mathematics and Physical Sciences, Faculty of Science, University of Newcastle, Callaghan, NSW, Australia

Colin Penfold Education Development Trust, Reading, UK

Georgia Petropoulou National and Kapodistrian University of Athens, Athens, Greece

Despina Potari National and Kapodistrian University of Athens, Athens, Greece

Elena Prieto School of Education, Faculty of Education and Arts, University of Newcastle, Callaghan, NSW, Australia

Giorgos Psycharis National and Kapodistrian University of Athens, Athens, Greece

Robyn Reaburn School of Education, University of Tasmania, Newnham Campus, Launceston, TAS, Australia

Francisco Rojas School of Education – Pontificia Universidad Católica de Chile, Macul, Santiago de Chile, Chile

Toby Rome Mathematics Education and Industry (MEI), London, UK

Charalambos Sakonidis Democritus University of Thrace, Komotini, Greece

Svein Arne Sikko Norwegian University of Science and Technology (NTNU, Norway), Trondheim, Norway

Department of Teacher Education, Faculty of Social and Educational Sciences, Norwegian University of Science and Technology (NTNU), Trondheim, Norway

Vasiliki Spiliotopoulou The School of Pedagogical and Technological Education (ASPETE), Heraklion, Greece

Eugenia Taranto Department of Mathematics and Computer Science, University of Catania, Catania, Italy

Clare Tope University of Winchester, Winchester, UK

Chrissavgi Triantafillou National and Kapodistrian University of Athens, Athens, Greece

Laura R. Van Zoest Department of Mathematics, Western Michigan University, Kalamazoo, MI, USA

Yingkang Wu School of Mathematical Sciences, East China Normal University, Shanghai, China

Rina Zazkis Faculty of Education, Simon Fraser University, Burnaby, BC, Canada

Editor and Author Biographies

Editors

Merrilyn Goos is Professor of STEM Education at the University of Limerick, Ireland. Previously she was Professor and Head of the School of Education at The University of Queensland, Australia. She is a former secondary school mathematics teacher who became a university-based mathematics teacher educator. Her research has investigated students' mathematical thinking, the impact of digital technologies on mathematics learning and teaching, numeracy across the curriculum, and the professional learning of mathematics teachers and mathematics teacher educators. She has served as Editor-in-Chief of *Educational Studies of Mathematics*, President of the Mathematics Education Research Group of Australasia, and Vice-President of the International Commission on Mathematical Instruction.

Kim Beswick is a Professor of Mathematics Education at the University of New South Wales, Sydney, Australia, where she is also Head of the School of Education. Her research interests centre on the beliefs, knowledge and professional development of mathematics teachers. She is particularly interested in how teachers' beliefs and knowledge relate to their expectations of and aspirations for their students. She is currently an Associate Editor of the *Journal for Mathematics Teacher Education* and convenor of the 15th International Congress on Mathematical Education to be held in 2024. She has received the research award of the Mathematics Education Research Group of Australasia and a prestigious Australian Research Council Future Fellowship.

Authors

Ferdinando Arzarello is Professor Emeritus in Mathematics Education at the University of Turin, Italy. He is author of more than 175 scientific publications on Mathematics Education and Foundation of Mathematics. He has been President of ERME from 2009 to 2013, and President of ICMI from 2013 to 2016; currently he is a member of the ICMI Executive Committee and a corresponding member of the Academy of Sciences of Turin. On behalf of the Italian Mathematical Union, he has worked with a Commission of the Italian Ministry of Education at the preparation of the Italian Mathematics Curriculum for preuniversity schools.

Dionysia Bakogianni is Doctor of Mathematics in Mathematics Education at the National and Kapodistrian University of Athens. Her research interest is on the development of statistics teaching and learning. Particularly, she focuses on teaching resources aimed at developing statistical reasoning and thinking in secondary education. In her dissertation she uses the lens of sociocultural theory for teaching and learning to study the development of teachers' practices as they collaborate to inquire into the teaching and learning of statistics. Her analytical tools are drawn mainly from qualitative research approaches. She has also explored further theoretical and analytical constructs through her experience in research programs and other research collaboration.

Alistair Bissell works for Mathematics in Education and Industry (MEI), UK, as Deputy Coordinator of the Teaching Advanced Mathematics (TAM) course. He started teaching mathematics in 2006, working in UK secondary schools (students aged 11–18 years) as a teacher, second in department and head of department. Alistair is studying for a doctorate in Mathematics Education and has been a member of the editorial board for *Mathematics Teaching*, the journal of the Association of Teachers of Mathematics. He is particularly interested in the nature of learning of mathematics teachers and the relationship between lesson observation debrief conversations and teacher learning.

Irene Biza is Associate Professor of Mathematics Education at the School of Education and Lifelong Learning at the University of East Anglia, UK. Irene has an MSc and a PhD in Mathematics Education and a BSc in Mathematics. She worked for several years as mathematics teacher and development officer in Greece before being appointed to a Lectureship in Mathematics Education at Loughborough University, UK. Irene's research focuses on the mathematical learning at the post-compulsory level, mathematics teachers' discourses and communities of practice in university mathematics. She is the leader of the MathTASK, a collaborative research and development programme in the UK, Brazil and Greece on mathematics teachers' pedagogical and mathematical discourses.

Julian Brown is a Lecturer in Education at the University of Bristol, UK, where he teaches on undergraduate and postgraduate programmes, including initial mathematics teacher education. Having moved into teaching with a background as a research physicist and software developer, Julian spent fourteen years as a mathematics teacher in schools in England with 11–18-year-olds. He has written for and presented at mathematics education research conferences in the UK and internationally. His current research interests include mathematics teacher awarenesses and the relevance of mathematics curricula in the light of local and global sustainability issues.

Laurinda Brown now retired, was a Reader in Mathematics Teacher Education at the University of Bristol, UK, and continues to supervise her doctoral students to completion. Throughout her academic career, she has enjoyed writing collaboratively and editing, particularly international mathematics education journals (past editor of *For the Learning of Mathematics*; special issues of *Journal of Mathematics Teacher Education* and *Educational Studies in Mathematics*; and currently a senior editor of the *International Journal of Science and Mathematics Education*. She was one of the editors for the Brill book, *Storytelling: Global Reflections on Narrative* (2019).

Katharine Burn is Associate Professor of Education and Director of the Oxford Education Deanery in the Department of Education at the University of Oxford, UK. Her research interests include teachers' research engagement, including initial teacher education, professional learning and collaborative research between schools and academics. She is also Deputy President of the Historical Association and co-editor of the professional journal *Teaching History*. Her work has been published in the *Journal of Education for Teaching,* the *Journal of Education Policy,* the *Oxford Review of Education* and the *Curriculum Journal*, amongst others.

Edwina Butler is a teacher who has taught science in primary and secondary schools, as well as middle school mathematics. She has taught in both city schools and remote locations. She has written articles and produced and conducted teacher professional development workshops for Science teachers and been the organiser of a series of conferences for mathematics teachers. Her recent work has been on project teams at the University of Newcastle, Australia, with teacher educators with mathematical expertise, striving to improve mathematics education. She possesses a Bachelor of Applied Science, Graduate Diploma in Education and a Master's in Educational Leadership.

Jinfa Cai is the Kathleen and David Hollowell Professor of Mathematics Education at the University of Delaware, USA, and a Fellow of the American Educational Research Association. He serves as the Editor for the *Journal for Research in Mathematics Education*. In 2017, he received the University of Delaware's College of Arts and Sciences' Outstanding Scholarship Award, historical for a mathematics education researcher to receive such a prestigious award. In 2018, he received the Webber Award, honouring his significant contribution to the State of Delaware's mathematics education. He has served as Program Director at the National Science Foundation. He is the Editor of the 2017 *Compendium for Research in Mathematics Education* (by NCTM).

Olive Chapman is professor of mathematics education at the Werklund School of Education, University of Calgary, Canada. She served as editor-in-chief of the *Journal of Mathematics Teacher Education* and is a member of the Board of Directors of the National Council of Teachers of Mathematics. She is the series editor for the second edition of the four-volume *International Handbook of Mathematics Teacher Education*. She teaches mathematics education courses to undergraduate and graduate students. Her research interests and publications include prospective and practising mathematics teachers' thinking (beliefs, conceptions, practical knowledge), knowledge for teaching mathematics, and learning or professional development; inquiry-based mathematics pedagogy; mathematical problem solving and modelling; and authentic mathematics assessment tasks.

Alf Coles Associate Professor of Mathematics Education at the University of Bristol, UK, was a teacher of mathematics in schools for 17 years before moving into teacher education in 2010. One of his research strands focuses on engaging with teachers of mathematics and their learning, recently researching how mathematics education can respond to global and societal crises; what do they mean for the learning of mathematics and the education of teachers of mathematics? Alf is author of the book *Engaging in Mathematics in the Classroom* and was co-editor of *Teaching Secondary Mathematics as if the Planet Matters*, both published by Routledge.

Luis Carlos Contreras is Full Professor of Mathematics Education at the Integrated Didactics Department in the University of Huelva, Spain, since 1993. He earned his PhD in 1998. His research has been focused on teacher training, especially in mathematics teacher's specialized knowledge. In addition, his research focuses on teacher educators' knowledge, teachers' problem-solving skills and the role of exemplification in teacher knowledge.

Ann Downton is a Lecturer in Mathematics Education at Monash University, Australia. Her work is focused on the learning and teaching of mathematics in the early years, in particular, the experiences that support young children's development of multiplicative thinking and geometric reasoning. In relation to this, she also researches the role of the teacher and pedagogical practices used within the classroom. Her current major research project is focused on the implementation of sequences of challenging tasks in the early years.

Dinazar Escudero-Ávila is a Mexican mathematics educator who studied for a degree in Physics and Mathematics with a specialty in Mathematics Education from the National Polytechnic Institute, and a Master's degree in Mathematics Education in the Center for Research and Advanced Studies. She earned her PhD at the University of Huelva in Spain (2011–2015), focused on the creation of categories and subdomains for the Mathematics Teacher's Specialised Knowledge (MTSK), an analytical model created to identify and characterise the knowledge of mathematics teachers. She is a member of the Iberoamerican Network MTSK.

Jen Fiddaman is the Newly Qualified Teacher Mentor at Bayards Hill Primary School in Oxford, UK. She previously held the role of Primary Mathematics Achievement Lead there whilst also teaching Mathematics in the partner secondary school, Cheney School. Her current role focuses on staff development through various aspects of teaching and learning, with a focus on newly qualified teachers and trainees. She completed her MSc in Learning and Teaching at the University of Oxford in 2017.

Manuel Goizueta is associate professor at the Mathematics Institute of the Pontificia Universidad Católica de Valparaíso, Chile. His main research focus is on the role of argumentation both as a means to science and mathematics education, and as part of scientific and mathematical literacy. He has led and collaborated on a number of projects on argumentation and mathematics education from socio-cultural perspectives. He is also interested in ethnographic and cross-cultural approaches to mathematics education in multicultural environments. As a teacher educator, Manuel has been part of research groups interested in the teacher educator's knowledge and its development within teacher education programs.

Yvonne Grimeland is an Associate Professor in Mathematics Education at the Norwegian University of Science and Technology. She has a PhD (2014) in pure mathematics, more specifically within the field of representation theory of algebras. Since 2014 she has been working as a mathematics teacher educator. Her main areas of research interest are mathematics teacher education and early years mathematics.

Tracy Helliwell is a Senior Lecturer in Mathematics Education at the University of Bristol, UK. She was a teacher of mathematics in secondary schools (students aged 11–18 years) from 2002–2015 before moving to work primarily on the initial mathematics teacher education course at university. Tracy writes and presents her research in mathematics education both nationally and internationally. Her current area of interest is the use of narrative research methods within the area of mathematics teacher education, particularly in exploring the practices and expertise of mathematics teacher educators.

Dave Hewitt taught in secondary schools for 11 years and gained a national reputation as a mathematics teacher before moving into teacher education. In 1994, he gained his PhD based on the idea of the economic use of time and effort in the teaching and learning of mathematics. After working many years at the University of Birmingham, UK, Dave moved to Loughborough University to start a new mathematics teacher training course as well as work with master's and PhD students. His research interests include the role of awareness in teaching and learning, with particular interest in algebra and early number.

Jenni Ingram is Associate Professor of Mathematics Education in the Department of Education at the University of Oxford, UK. Her research interests include mathematics classroom interaction, mathematics teacher education and international comparative studies. Her work has been published in *Research in Mathematics Education*, *For the Learning of Mathematics, Cambridge Journal of Education* and the *Oxford Review of Education*, amongst others. She is also on the editorial board of *Research in Mathematics Education*.

Barbara Jaworski is Professor of Mathematics Education at Loughborough University, UK, and Doctor Honoris Causa at the University of Agder, Norway. Her career spans mathematics teaching at secondary and first-year university levels, teacher education at secondary level and teaching and supervision of doctoral students. Her research has mainly investigated the development of mathematics teaching through research and partnerships between teachers and researchers, at both school and university levels. She has been Chief Editor of the *Journal of Mathematics Teacher Education*, and President of the European Society of Mathematics Education and the International Group for the Psychology of Mathematics Education.

Lindsay Keazer is Assistant Professor of Teacher Education - Mathematics at Sacred Heart University (CT, USA) where she teaches undergraduate courses in mathematics and mathematics methods to prospective teachers. Her research focuses on mathematics teacher education, examining how prospective and practising teachers develop culturally relevant pedagogies and pedagogies for student-centred practices such as reasoning and sense making.

Patrick M. Kimani teaches at Glendale Community College, Arizona, USA. His research interests are in teacher learning and multiplicative reasoning. Patrick currently serves as a faculty research associate for AI@CC (NSF #1561436) studying the relationship between community college algebra instruction and student success. He has been engaged in the professional development of mathematics teachers and college faculty through various funded projects.

Roza Leikin is a Professor of Mathematics Education and Education of Gifted at the Faculty of Education, University of Haifa, Israel. Her research focuses on mathematical creativity and ability, mathematics teachers' professional potential, mathematically challenging tasks and integration of neuro-cognitive research in mathematics education. Dr Leikin is the Head of the Department of Mathematics Education and the Director of the RANGE Center – the Interdisciplinary Center for the Research and Advancement of Giftedness. During 2012–2017 she served as the

President of MCG – the International Group for Mathematical Creativity and Giftedness. She published more than 150 papers in peer-reviewed research journals, books and refereed conference proceedings.

Mariana Levin is an Assistant Professor of Mathematics Education in the Department of Mathematics at Western Michigan University, USA. Her primary research interests centre on the role of knowledge in mathematical sense-making processes: for students, teachers and teacher educators. Her recent work explores undergraduate students' experience of the critical transition to proof-intensive upper-level courses. She is particularly interested in supporting the development of productive disciplinary identities for pre-service mathematics teachers, both at the elementary and secondary level.

Sharyn Livy is a Senior Lecturer in mathematics education in the Faculty of Education, Monash University, Australia. Her research interests are closely linked to teaching in supporting primary school teachers to extend their knowledge for mathematics teaching, including ways to improve pre-service teachers' university experiences. Other research interests include implementation of sequences of challenging tasks, geometric reasoning and engaging children with mathematics through picture story books. Sharyn is currently working on an Australian Research Council (ARC) Linkage project with colleagues to provide mathematics leaders and teachers with strategies and tools to improve student learning in the early years.

Nicole Maher is a Lecturer in Primary Mathematics Education at the University of Tasmania, Australia. She was a teacher of secondary mathematics (years 7–12) before making the transition to teacher education and pursuing her long-term interest in what it means to teach mathematics well. To this end, Nicole's key research interests involve the exploration of pedagogical content knowledge (PCK) at the senior secondary level and investigating this knowledge from the perspectives of students as well as teachers.

Angeliki Mali is an Assistant Professor in mathematics education in the Institute for Science Education and Communication at the University of Groningen, the Netherlands. Angeliki previously worked as a Postdoctoral Research Fellow in mathematics education in the School of Education at the University of Michigan, USA. She has an MSc and a PhD in mathematics education and a BSc in mathematics. Angeliki's research is on the teaching practice of university mathematics and its relation to students' making of mathematical meaning. She supervises Bachelor's, Master's and PhD students and teaches mathematics courses as well as mathematics education courses at undergraduate and postgraduate levels.

Ofer Marmur is a lecturer/assistant professor of mathematics education in the Department of Mathematics at the University of Auckland, New Zealand. His research focuses on the interrelationship between the cognitive, affective, and social dimensions of student learning, specifically in the context of undergraduate mathematics and teacher education. He was a postdoctoral research fellow in the Faculty of Education at Simon Fraser University, Canada, after having completed his doctoral studies in the Faculty of Education in Science and Technology at the Technion – Israel Institute of Technology.

Margaret Marshman is a mathematics educator at the University of the Sunshine Coast, Australia. She has taught all levels of mathematics education from early years, primary and secondary and is passionate about mathematics teacher education. Margaret's research focuses on people's beliefs about mathematics, its teaching and learning, including mathematics teacher educators and how their beliefs influence mathematics pre-service teachers. She also researches how to support pre-service teachers to engage their students with real-world problems, helping others to visualise and reason in three dimensions and statistical thinking.

Salomé Martínez is a Professor of Universidad de Chile, and holds a PhD in Mathematics from the University of Minnesota. She is the director of the Laboratory of Mathematical Education at the Center for Mathematical Modeling. She has led several research and development projects related to the development of resources and instructional models for the teaching of mathematics in pre-service teacher education. Regarding teacher professional development, she also has led the development and nationwide implementation of a technology-based program for primary and secondary school teachers, constructed through a long-term partnership with the Chilean Ministry of Education.

Joanna O. Masingila is Dean of the School of Education at Syracuse University, USA, and a Professor of mathematics and mathematics education. Her research interests include teacher development, teacher educator development, mathematical knowledge for teaching teachers, and students' out-of-school mathematics practice. She is or has been a principal investigator or co-principal investigator on over $7 million in grants. Masingila has published widely in journals and books, and has developed a number of multimedia case studies examining issues in teaching practice. Masingila is active in policy and advocacy work for teacher and leader education in New York and nationally.

Helena Montenegro is an educational psychologist and holds a PhD in education. She is working as a postdoctoral researcher at the Faculty of Education in the Pontificia Universidad Católica de Chile. Her research topics are in the field of teacher education and the challenges of learning about teaching for both teacher educators and student teachers. She is methodologically experienced in mixed-method research, phenomenography and discourse analysis. She has been involved in several research projects related to the work of teacher educators and publishes widely on that topic. She is currently working on a project about teaching and learning experiences in primary teacher education.

Miguel Montes is teacher of Mathematics Education at the Integrated Didactics Department in the University of Huelva, Spain, since 2011. He earned his PhD in 2015. His research has been focused on teacher education, especially in relation to teachers' knowledge. He also researches teacher educators' knowledge and teachers' problem posing skills.

Tracey Muir is an Associate Professor in Mathematics Education at the University of Tasmania, Australia. Her research interests include effective teaching for numeracy, problem solving in mathematics, the use of ICT in mathematics and parental involvement in mathematics education. Her recent research includes flipped learning in mathematics, personalised learning and online engagement in higher education. Her teaching expertise has been recognised through a number of institutional and national teaching awards. She regularly conducts professional learning for teachers and has delivered workshops, presentations and keynote addresses at national and international conferences.

Carol Murphy is a Senior Lecturer in Mathematics Education at the University of Tasmania, Australia. She teaches mathematical knowledge and inclusive practices to pre-service primary teachers. Her research interests have included studies on pre-service teachers' PCK and on children's mathematical thinking in calculation strategies. Recent studies have focused on young children's dialogue in collaborative group work in the UK, the use of ICT in primary mathematics classrooms in New Zealand and inquiry-based learning in STEM with middle school students in Qatar. Carol has engaged in professional development workshops in New Zealand and in Tasmania on the use of classroom talk in mathematics.

Kathleen Nolan is Professor in Mathematics Education at the University of Regina, Canada, where she teaches undergraduate and graduate courses in mathematics curriculum, qualitative research, and contemporary issues in education. Her research focuses on mathematics teacher education, exploring issues of teacher identity, and the regulatory practices of schooling, learning and knowing, with Bourdieu's social field theory and self-study methodology featured prominently in her work. Kathleen's research has support from the *Social Sciences and Humanities Research Council of Canada*.

Greg Oates is a Senior Lecturer in Mathematics Education at the University of Tasmania, Australia, where he teaches pre-service teachers of primary and secondary mathematics. Greg is a cheerleader for mathematics and has taught mathematics at all levels from middle through to senior school and undergraduate calculus, linear algebra and mathematics education. His research interests include the integration of technology into mathematics curricula; the use of technology to promote collaborative learning in mathematics; and professional development for teachers, with a specific focus on pedagogical content knowledge (PCK) and the use of videos as a focus for lesson study.

Dana Olanoff is an Associate Professor of Mathematics and Mathematics Education in the Department of Mathematics at Widener University, USA. She teaches mathematics content courses for prospective elementary and middle school teachers. Her research interests include Mathematical Knowledge for Teaching Teachers (MKTT), prospective teachers' knowledge of fractions, and designing, implementing, and modifying high-quality mathematics tasks for prospective teachers.

Judy-anne Osborn is a researcher in Mathematics and Education in the School of Mathematical and Physical Sciences at the University of Newcastle, Australia. She is an executive member of the Computer-Assisted Research Mathematics and its Applications (CARMA) Priority Research Centre at Newcastle, within which she is a member of the Discrete Mathematics Research Group, and leads the Educational Research Group. She loves to explore mathematics and to teach. She also enjoys writing, and has written about mathematics for a general audience in *The Conversation*. In the context of Mathematics Education, she is particularly interested in inclusive approaches and equity.

Colin Penfold is Principal Education Consultant at the Education Development Trust, UK. His interests lie in education system reform including the professional development of teachers in a range of countries and settings, including Brunei, Rwanda, Lebanon, UK and Ghana. He completed his MSc in Teacher Education at the University of Oxford in 2016.

Georgia Petropoulou has MSc and PhD in Mathematics Education and BSc in Mathematics. Georgia is a secondary mathematics teacher in collaboration with both the Greek University of Athens and the Greek Institute of Educational Policy as a mathematics teacher educator. She has been involved in mathematics curriculum development in Greece for grades 9–12. Georgia's research focuses on university teaching practices that meet student learning needs.

Despina Potari is Professor in the Mathematics Department of the National and Kapodistrian University of Athens. She obtained her first degree in Mathematics from the same university and a PhD in Mathematics Education from the University of Edinburgh. She has been visiting professor at several universities abroad and currently at Linnaeus University in Sweden. Her research interest is on the development of mathematics teaching and learning and teacher development. Her current focus is on macro and micro aspects of mathematics teaching and on mathematics teacher collaboration. Since 2017 she is chief editor of the *Journal of Mathematics Teacher Education*.

Elena Prieto holds a Bachelor degree in Mathematics and a PhD in Theoretical Computer Science. From 2005, she has worked extensively in Mathematics education, including several Australia-wide research projects. She is currently engaged in several projects focused on the use of technology for the learning of mathematics and mathematics teacher training and professional development. She is a member of the University of Newcastle Teachers and Teaching Research Centre where she leads the STEM Education Group.

Giorgos Psycharis is Assistant Professor of Mathematics Education at the Department of Mathematics, National and Kapodistrian University of Athens. His fields of interest include the design of learning environments for mathematics with the use of digital technologies, students' construction of mathematical meanings with the use of digital tools, connecting mathematics teaching to authentic situations, and mathematics teacher education (levels of prospective and practising teachers as well as teacher educators). He has participated in more than twelve STEM-related national and European projects and he has more than 50 publications in peered-reviewed journals, books and refereed conference proceedings.

Robyn Reaburn is a lecturer in Mathematics Education at the University of Tasmania, Australia. She completed her doctorate in statistics education. She has taught in high schools, in Technical and Further Education, at the Australian Maritime College, and at the University of Tasmania. Before this she worked as a biomedical scientist specialising in haematology. Her current research interests include tertiary students' understanding of statistical concepts, statistics lecturers' knowledge of content and pedagogy, the use of p-values in research, mathematics in home schooling, and academics' self-efficacy in teaching students with diverse needs and backgrounds.

Francisco Rojas is assistant professor at the Faculty of Education of the Pontificia Universidad Católica de Chile and holds a PhD in Mathematics Education from Universitat Autònoma de Barcelona. His research areas are focused on pre-service teacher education in mathematics, mainly for primary levels, regarding the mathematical knowledge of the pre-service teacher at different moments of their training, and on mathematics teacher educator knowledge and instructional practices in the university classroom. He has participated in research and development projects related to producing standards of knowledge for prospective teachers and standardized instruments to measure mathematical knowledge at the first year of instruction.

Toby Rome is a National Coordinator for the Advanced Mathematics Support Programme, which is managed by Mathematics in Education and Industry (MEI) in the UK. His main responsibilities relate to supporting teachers with their continuing professional development, with a particular focus on A-level mechanics. Prior to joining MEI, Toby taught mathematics for 16 years at a large 11–18 comprehensive school near Bristol, where he specialised in sixth-form education (16–18-year-olds) and served on the School Leadership Team as an Assistant Headteacher.

Charalambos Sakonidis is a Professor in Mathematics Education at Democritus University of Thrace, Greece. He is a mathematician and a mathematics educator involved first in research on pupils' learning of algebra, the role of representational systems in students' mathematical thinking and modelling in computational contexts for educational purposes. Since the 1990s his research focus gradually moved to socio-cultural aspects of mathematics teaching and learning, especially of the mathematical meaning being constructed within the classroom, as well as on mathematics teachers' professional development. Sakonidis participated in various national and European projects related to his research interests and his published works appear in refereed international as well as national journals, conference proceedings and books.

Svein Arne Sikko is an Associate Professor in Mathematics Education at Norwegian University of Science and Technology. He has a doctoral degree in pure mathematics within the field of representation theory of algebras. Since 1998, he has been working in mathematics teacher education. His main areas of research interest are mathematics teacher education, teacher professional development, lesson design studies, inquiry-based learning and issues related to the transition between educational levels.

Vasiliki Spiliotopoulou is Professor of Research Methods in Science, Mathematics and Technology Education at the Department of Education of the School of Pedagogical and Technological Education (ASPETE), Greece. She has long experience as a teacher educator with secondary teachers in ASPETE and prospective primary teachers in the University of Patras. She has participated in different European research projects (CHEMINC, STEDE, MASCIL, INCLUSEME), while she has published more than 100 papers in international journals, book volumes and conference proceedings, and her scientific work has been acknowledged though more than 500 citations.

Eugenia Taranto obtained her PhD in Pure and Applied Mathematics at the University of Turin and the Polytechnic of Turin, Italy. She has currently a post-doc position at the University of Catania, where she is also a lecturer of Mathematics Education in a graduate-level course. Her research fields include: MOOCs (Massive Open Online Courses), technologies to mediate the learning of mathematics (dynamic geometry systems, MathCityMap, learning videos, ...) and the role of the teacher as the focus of class activities. She has authored papers and chapters in various prestigious journals and books.

Clare Tope is a senior lecturer in the Department of Teacher Development at the University of Winchester, UK. She is the PGCE Primary Programme Leader and teaches mathematics on a range of undergraduate and postgraduate programmes working with both students in the initial stage of their teacher education and practising teachers. She completed her MSc in Teacher Education at the University of Oxford in 2016.

Chrissavgi Triantafillou is Assistant Professor in Mathematics Education in the Mathematics Department of University of Athens, Greece. Her BA is in Mathematics and her PhD in Mathematics Education. Her doctoral dissertation was on studying mathematical practices in a workplace setting. She was involved in PD activities with in-service teachers in the context of Mascil. Her current research interests include how teacher educators could support Mathematics and Science teachers and how workplace tasks could be used and implemented in mathematics classrooms. She has four publications in international journals and more than 25 papers published in conference proceedings.

Laura R. Van Zoest is a Professor of Mathematics Education in the Department of Mathematics at Western Michigan University, USA. She focuses specifically on the process of becoming an effective mathematics teacher and ways university coursework and professional development can accelerate that process. Her research and teacher development practice are intertwined and have included investigating the effect of reform curriculum materials on teacher development, the use of practice-based materials in university methods courses, and the cultivation of productive norms in teacher education. Her current work focuses on identifying mathematical opportunities in student thinking (MOSTs) and supporting teachers in capitalizing on MOSTs.

Yingkang Wu is an associate professor at East China Normal University, Shanghai. She obtained her PhD in mathematics education from the National Institute of Education, Nanyang Technological University of Singapore. Dr Wu has taught mathematics pedagogical courses for pre-service secondary teachers, and also taught calculus for college students. Her research interests include mathematics teacher education, mathematics classroom teaching, mathematics assessment and teaching and learning of school statistics.

Rina Zazkis is a Professor of Mathematics Education at the Faculty of Education and an associate member in the Department of Mathematics at the Simon Fraser University, Canada. Her research is in the area of undergraduate mathematics education, with a general focus on mathematical content knowledge of teachers and the ways in which this knowledge is acquired and modified. She currently serves as a co-editor of the *Journal of Mathematical Behavio*r. In her career she supervised numerous graduate students and received an award for excellence in graduate supervision. She was appointed as Tier 1 Canada Research Chair, a prestigious recognition of excellence in research and research training.

Chapter 1
Introduction: The Learning and Development of Mathematics Teacher Educators

Merrilyn Goos and Kim Beswick

1.1 Rationale

Research in mathematics teacher education as a distinctive field of inquiry has grown substantially over the past 20 years, as evidenced by the establishment of an international journal (*Journal of Mathematics Teacher Education*), the commissioning of the 15th ICMI Study on the professional education and development of teachers of mathematics (Ball & Even, 2008), and publication of the first and second editions of the *International Handbook of Mathematics Teacher Education* (Chapman, 2020; Wood, 2008). Within this field there is emerging interest in how mathematics teacher educators (MTEs) themselves learn and develop. An early contribution was the fourth volume of the *International Handbook of Mathematics Teacher Education* (Jaworski & Wood, 2008) in which Even (2008) commented that neglect of the education of mathematics teacher educators, by comparison to that of mathematics teachers (MTs), mirrors earlier research in mathematics education that focused more on students' learning than on teachers' learning. At that time, the processes by which mathematics teacher educators learn, and the forms of knowledge they require for effective practice, had not been systematically investigated (Llinares & Krainer, 2006).

However, researchers in mathematics education are increasingly investigating the development of MTE expertise and associated issues. As evidence of growing interest in this area, we can point to discussion groups that we have co-convened

M. Goos (✉)
School of Education, University of Limerick, Limerick, Ireland

The University of Queensland, Brisbane, QLD, Australia
e-mail: merrilyn.goos@ul.ie

K. Beswick
School of Education, University of New South Wales, Sydney, NSW, Australia

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_1

with colleagues at PME conferences and at ICME-12 in 2012 (Beswick, Chapman, Goos, & Zaslavsky, 2015; Beswick, Goos, & Chapman, 2014; Goos, Chapman, Brown, & Novotna, 2010, 2011, 2012), a special issue of the *Journal of Mathematics Teacher Education* (Volume 21(5), published in 2018), a new *International Handbook* volume on mathematics teacher education (Beswick & Chapman, 2020), and an ICME-14 Topic Study Group on the knowledge and practices of mathematics teacher educators. This book, which draws on the latest research and thinking in the field, is therefore timely to stimulate future development and directions.

1.2 Who Is a Mathematics Teacher Educator?

Preparing a book on the topic of the learning and development of MTEs raises the question of *who is a mathematics teacher educator*? We can tease this question further apart by asking:

- Where are MTEs located?
- What kinds of teacher do they work with?
- What do they teach?
- What qualifications and experience are needed or expected, and how is this acquired?
- Do the answers to these questions vary from one country to another in light of the different policies, practices, and norms that guide teacher preparation and development?

We know that many people contribute to the professional formation of mathematics teachers. They include university academics, from the disciplines of mathematics and mathematics education, who teach in prospective or practising teacher education programmes or who engage in research with teachers; practising teachers who supervise and mentor prospective teachers during their school placement; officers of local or national education authorities who are involved in professional development programmes; and private providers of educational consultancy services. All of these types of MTEs appear in the chapters of this book, highlighting the diversity of contexts in which MTEs work and develop.

1.3 Structure of the Book

The book is organised in three main sections related to the themes discussed below.

1.3.1 Theme 1: The Nature of Mathematics Teacher Educator Expertise

One element of this theme includes research that investigates the application or transfer of the various knowledge types proposed for mathematics teachers to the knowledge needed by mathematics teacher educators. Much of this research that concerns mathematics teachers has focused on the roles of content and pedagogical knowledge and their interaction. For example, the concept of pedagogical content knowledge (PCK) introduced by Shulman (1987) has received considerable attention in the mathematics education community and has been elaborated by Ball and colleagues (e.g. Ball, Thames, & Phelps, 2008; Hill, Sleep, Lewis, & Ball, 2007) in their extensive and influential work on mathematical knowledge for teaching (MKT). Much research in the MTE domain has portrayed the knowledge of mathematics teacher educators as an extension of mathematics teachers' knowledge that encompasses that knowledge in the same way as mathematics teachers' knowledge encompasses the knowledge that they intend their students to develop (e.g. Beswick, 2020; Zaslavsky & Leikin, 2004). In terms of the knowledge types proposed by Shulman (1987) and Ball and colleagues, MTEs require a kind of meta-knowledge which could be described as *knowledge for teaching knowledge for teaching mathematics*.

Chick, Pham, and Baker (2006) proposed a framework for analysing PCK in which they listed categories of mathematics teachers' knowledge that are clearly PCK because the content and pedagogy are inseparable (e.g. analysing student thinking); content knowledge in a pedagogical context (e.g. deconstructing content); or pedagogical knowledge in a content context (e.g. maintaining student engagement). Chick and Beswick (2018) then evaluated the usefulness of this framework for analysing mathematics teacher educators' PCK as a meta-PCK. Their analysis also demonstrated how the MTE's PCK develops during practice and highlighted the role of contingency (Rowland, Huckstep, & Thwaites, 2005) in MTE impromptu decision-making during teaching sessions.

As well as comparing the PCK for teaching mathematics with the PCK for teaching prospective teachers of mathematics, this theme also encompasses aspects of MTE knowledge that may be unique to them or overlapping with mathematics teachers' knowledge. For example, in addition to having mathematical knowledge, MTEs need to be able to empathise with prospective primary teachers' difficulties with mathematics (Zazkis & Zazkis, 2011) and be aware of psychological characteristics of the mathematics teachers they teach (Leikin, 2020). Other researchers are exploring how MTEs develop their own understanding of mathematical content knowledge as it influences their work with prospective teachers (e.g. Li & Castro Superfine, 2016). MTEs also need to know how new teaching practices are learned and the pitfalls associated with promoting this learning. For example, Bergsten and Grevholm (2008) discussed knowledge of how to design teacher education activities, especially activities that connect prospective teachers' learning in the university and practicum contexts.

Beliefs are sometimes considered to be a contentious form of knowledge (Beswick, 2007). Mathematics teacher beliefs have been extensively researched, but the beliefs of MTEs have received little attention in studies published to date although Chick and Beswick (2018) included aspects of MTEs' beliefs in their knowledge framework and Appova (2020) argued for inclusion of Schoenfeld's (2010) broader notion of orientations in her model of MTE knowledge. MTE beliefs about teaching and learning are likely to be influenced by theoretical studies and research (Bergsten & Grevholm, 2008) and to vary among different categories of MTEs (Goodchild, 2020; Goos, 2020).

1.3.1.1 Questions Addressed by Theme 1 Chapters

The six chapters in this section concern questions about differences between the knowledge required of MTEs and teachers of school mathematics and between MTEs of differing backgrounds and working in differing contexts. The role of beliefs in relation to MTE knowledge and their influence on practice is also addressed.

Question 1: To what extent are the various knowledge types for mathematics teachers applicable/transferable to mathematics teacher educators?

Five of the six chapters in this theme address the relationship between mathematics teacher knowledge and MTE knowledge. The conceptualisations of teacher knowledge upon which they draw include those of Shulman (1986, 1987), Ball and colleagues (e.g. Ball et al., 2008), and Rowland and colleagues (e.g. Rowland, Turner, Thwaites, & Huckstep, 2009). The ways in which these conceptualisations of teacher knowledge are applied to MTE knowledge include a relatively direct translation (Muir, Livy, and Downton), a broader more generic reconceptualisation (Escudero-Ávila, Montes, and Contreras), and a further development of an existing reconceptualisation of a single aspect of a model for teacher knowledge (Mali, Petropoulou, Biza, and Hewitt). Two of the chapters make less direct links with mathematics teacher knowledge. Leikin, for example, builds from the notion of students' mathematical potential and the related idea of mathematics teachers' professional potential, to discuss the knowledge needed by MTEs to facilitate the development of that potential, whereas Zazkis and Marmur consider MTE knowledge in terms of usage-goal theory (Liljedahl, Chernoff, & Zazkis, 2007). Common to all, albeit implicitly in the case of Zazkis and Marmur, is a hierarchical view of MTE knowledge in relation to mathematics teacher knowledge. In addition to the range of conceptualisations of teacher knowledge drawn upon is the diversity of MTEs to which they are applied. These include practising teachers and mathematics education researchers working with practising or prospective teachers (primary or secondary) and research mathematicians working with university students that may include prospective teachers.

Escudero-Ávila et al. structure their consideration of MTE knowledge around Shulman's knowledge types, specifically CK and PCK. They conceptualise MTE knowledge as a meta-version of MT knowledge distinguished by the nature of the content. For MTEs, they argue, content knowledge can be considered according to the three categories proposed by Ponte (2012) to describe the content of teacher education. These are professional knowledge (encompassing the mathematics CK and PCK required by mathematics teachers), knowledge of mathematics teaching skills and practices, and knowledge of mathematics teachers' professional identities. Escudero-Ávila et al. describe PCK for MTEs as comprising knowledge of professional development for mathematics teachers, knowledge of the content of teacher education programmes, and knowledge of the standards for teacher education programmes.

Muir et al. examined the transferability of Rowland's (2013) Knowledge Quartet, which emerged from observations of prospective primary school teachers, to MTEs' work with prospective primary school teachers. They were able to identify aspects of each category of the Knowledge Quartet in the two vignettes of MTEs' practice that they considered.

A particular aspect of mathematical knowledge for teaching (Ball et al., 2008) that has served as a point of departure for researchers considering MTE knowledge has been Horizon Content Knowledge. Mali et al. take this line of work further by building upon reconceptualisations of the mathematical horizon for teachers by Zazkis and Mamolo (2011) and Figueiras, Ribeiro, Carrillo, Fernandez, and Deulofeu (2011). For Zazkis and Mamolo, the mathematical horizon was the point at which university-level mathematical content meets the mathematics content of school curricula. Figueiras et al. (2011) added to this the need for mathematics teachers and MTEs to reflect upon connections among mathematical concepts. For Mali et al. the horizon unifies university and school mathematics and encompasses aspects of both mathematical and teaching practice along with reflection on both content and practice.

Rather than focussing on MTE knowledge, Zazkis and Marmur focus on the work of MTEs, specifically the need to uncover the existing mathematical knowledge of the mathematics teachers with whom they work and then to extend that knowledge. Implicit in their account is the relatively deeper knowledge required of MTEs that includes the ability to connect sophisticated mathematics knowledge with school mathematics.

Leikin describes the dual hierarchical relation of mathematics student knowledge and mathematics teacher knowledge and mathematics teacher knowledge and MTE knowledge. Rather than starting with a particular conceptualisation of teacher knowledge, Leikin begins with the notion of students' mathematical potential to consider the nature of challenging content for mathematics teachers and, therefore, mathematics teachers' professional potential. She argues that both form part of MTEs' knowledge and skills.

Question 2: How does the knowledge needed by mathematics teacher educators differ from that required by mathematics teachers? How does it differ from that needed by mathematicians who teach undergraduate courses to prospective teachers?

Beyond considering MTE knowledge as a kind of a meta-knowledge in relation to that of mathematics teachers, authors of chapters in this section differentiate the knowledge requirements of MTEs and mathematics teachers in terms of the greater depth and width of MTE knowledge (Escudero-Ávila et al., Mali et al., Muir et al., Zazkis & Marmur) and the greater density of connections that MTEs need to have between mathematical ideas and between these and aspects of teachers' identities and practices (Escudero-Ávila et al.). MTEs need to hold the knowledge that mathematics teachers need in such a way that they can see it from a higher vantage point (Escudero-Ávila et al.). Part of this perspective is the notion of meta-mathematical ideas discussed by both Escudero-Ávila et al. and Mali et al. and considered important to the knowledge of MTEs. For Escudero-Ávila et al., these include the meanings of such things as theorems and definitions and the essence of proofs, whereas Mali et al., whose focus is on research mathematicians as MTEs, emphasise MTEs' awareness of their own mathematical practices and the implications of these for their teaching. They identify meta-commenting, that is, being explicit about such things as the reasons for which a mathematical idea is important or how particular mathematical practices work, as particularly helpful.

Escudero-Ávila et al. provide a detailed comparison of differences between the knowledge needed by MTEs and mathematics teachers that illustrates its necessarily more holistic nature. For example, although teachers need to know about theories of teaching and learning and to develop their own, MTEs additionally need to know how to empower teachers to do this, to become reflective practitioners able to learn from their practice (a capacity identified by Zazkis and Marmur as common to both MTEs and mathematics teachers). In addition to analogues between the knowledge of the two groups, Escudero-Ávila et al. highlight three unique components of MTE PCK. These are knowledge of professional development of mathematics teachers, the contexts (national, institutional, and programme) in which they engage in teacher education, and standards and evaluation methods for their programmes.

The distinctions made by Escudero-Ávila et al. are, in the main, presented as applying to MTEs generally, but they also draw brief distinctions between three types of MTEs: professional mathematicians, denoted research mathematicians by Mali et al. and Muir et al.; didacticians, otherwise known as mathematics education researchers; and school teachers engaged in mathematics teacher education. Principal differences centre on the professional mathematicians' deep knowledge of mathematics, on the mathematics education researchers' dual focus on the transformation of knowledge for prospective teachers and their need to learn how to make mathematics accessible to school students, and on the unique positioning of school teachers in MTE roles to connect theory and practice (Escudero-Ávila et al.).

Mali et al. focus on aspects of research mathematicians' teaching of undergraduate students, likely to include some prospective teachers that support their students' understanding of advanced mathematical ideas and thereby foster their mathematical horizons. They provide a categorisation of extending practices that relate to these MTEs' own mathematical research and argue that each of the four categories – drawing on examples, connecting mathematical areas, visualising, and simplifying – is applicable to school mathematics teaching. Indeed, each of the practices operates at the levels of research mathematics, undergraduate mathematics, (teacher education – in countries where this level exists), and school mathematics (Mali et al.). It is noteworthy that these practices are likely to be enacted by different types of MTEs as well as school mathematics teachers and, hence, they provide a kind of unifying conceptualisation of the practices and knowledge of these groups (Mali et al.).

In the study on which Muir et al. report, a primary school teacher and a mathematics education researcher co-taught prospective primary school teachers. This context afforded a unique opportunity to consider differences in the knowledge, conceptualised according to the KQ (Rowland, 2013), that each of these MTEs brought to the work. Among their findings was that the mathematics teacher was less inclined than the mathematics education researcher to act on contingent moments. Whereas Escudero-Ávila et al. suggest school teacher MTEs can bridge the gap between theory and practice, Muir et al. locate knowledge of the theoretical underpinnings of mathematics teaching practice with the mathematics education researcher. Nevertheless, the prospective teachers valued the school teacher MTE's current classroom experience. Unique also to the knowledge requirements of researcher MTEs is the knowledge that enables them to respond to prospective teachers' misconceptions (Muir et al.). Mathematics teachers, but not other MTEs, Muir et al. contend, need detailed understanding of children's mathematics learning and the ability to assess and report on that learning, including to the parents.

Question 3: How do the beliefs of mathematics teacher educators (whether mathematics educators or mathematicians) inform their knowledge and practice? How do prospective teachers reconcile any differences in the beliefs communicated by the mathematics educators and mathematicians who teach them?

Two chapters, those by Escudero-Ávila et al. and Marshman, explicitly address the beliefs of MTEs. The beliefs of different categories of MTEs are the focus of Marshman's chapter. She adopts Beswick's (2005) definition of beliefs as "anything that an individual regards as true" (p. 39), thereby encompassing knowledge, whereas Escudero-Ávila et al. do not explicitly define the construct. Nevertheless, their conception of beliefs is linked with the notion of professional identity in relation to which Wenger (1998) is cited, and so it may be that they have in mind a more sociocultural than psychological construct when they refer to beliefs. In fact, Escudero-Ávila et al. include in Ponte's third category of teacher education content, professional identity, everything affective including beliefs. They propose that MTEs need to know about the influences, including beliefs, on the development of mathematics teachers' identities and how they can foster that development. They thus position beliefs as both part of and an influence on teachers' professional identity. In addition, they discuss the beliefs of both mathematics teachers and MTEs although their focus is on the former and its implications for MTEs' knowledge. In particular, MTEs need to be aware of beliefs commonly held by prospective teachers and have the knowledge required to make these beliefs available for reflection (Escudero-Ávila et al.).

Escudero-Ávila et al. distinguish mathematicians in the role of MTE from others in that role in terms of their tendency to have Platonist beliefs about the discipline, and to attach greater importance than do other MTEs to mathematical content knowledge for MTs. Similarly to the way that Marshman describes the influence of teachers' beliefs on their students, Escudero-Ávila et al. argue that MTEs' practice is influenced by their beliefs about what prospective mathematics teachers need to learn; how they can best assist MTs to learn; how they can collaborate as they do this; and how to evaluate MTs' learning/development. What MTEs believe mathematics teachers should learn influences their choice of content and "the general orientation of the education" they provide (Escudero-Ávila et al.). Escudero-Ávila et al. illustrate the influence of MTEs' beliefs by stating some of their own in relation to programme design, specifically the need for input from diverse MTEs and the need for discussions of "both pedagogical and mathematical topics". They also express a belief in the need for specific education for MTEs that, nevertheless, respects their autonomy.

Marshman reports on a study of the beliefs about the nature of mathematics, mathematics teaching, and mathematics learning of MTEs that included both research mathematicians and mathematics education researchers. They divided the participating MTEs in two ways in order to consider differences among various groups. First considering their qualifications resulted in four groups: those without a PhD, those with a PhD in mathematics, those with a PhD in mathematics education, and those whose PhD was in mathematics but who also had a Graduate Diploma in Education. Second, they divided them according to teaching responsibilities: those who taught only mathematics or statistics, those who taught only mathematics pedagogy, and those who taught both mathematics or statistics content and pedagogy. There were differences in mean responses to several items concerning mathematics teaching among the groups defined by qualifications but not among the groups of MTEs defined by teaching responsibilities. There were no differences among either set of groups in relation to the nature of mathematics.

Marshman's interviews with three MTEs – a pure mathematician, a mathematics education researcher, and an applied mathematician – revealed interesting differences between the ways in which these MTEs connected their beliefs about mathematics with their teaching practice. One described having been significantly influenced by collaborating with another MTE with a different disciplinary background to develop a course for prospective teachers. Although there were few differences among mean survey responses for prospective teachers and the various categories of MTEs, prospective teachers indicated in interviews that they were aware of the inconsistencies between the way they were being taught mathematics at university and the ways that they were being taught to teach it to school students, and they linked these differences to what they perceived to be mathematician MTEs' beliefs. Marshman drew a parallel between prospective teachers managing the differences they experienced between MTEs and their likely encounters with colleagues with differing beliefs when they became practising teachers. She recommended greater dialogue among MTEs with a view to increasing the coherence of prospective teachers' programmes.

1.3.2 Theme 2: Learning and Developing as a Mathematics Teacher Educator

Theoretical approaches found in early studies of mathematics teacher educator development were largely based on constructivist views of teaching and learning, in particular the notion of reflective practice as a means of establishing relationships between activity and consequences to explain how human beings advance their thinking. For example, Tzur (2001) and Krainer (2008) provided reflective self-studies of their own developmental trajectories, tracing their experiences as mathematics learners, mathematics teachers, mathematics teacher educators, and mentors of fellow mathematics teacher educators to identify critical events and experiences that advanced their professional knowledge and practice. More recently Chapman, Kastberg, Suazo-Flores, Cox, and Ward (2020) illustrated the potential of reflecting together to generate insight into MTEs' own practices in ways that are also more broadly applicable. Reflection was also the tool used in meta-studies where mathematics teacher educators analysed their own learning as part of a larger teacher professional development project (e.g. Diezmann, Fox, de Vries, Siemon, & Norris, 2007; Even, 2008; Zaslavsky & Leikin, 2004).

Some researchers have represented MTEs' learning as a lifelong process of growth through practice. For example, Zaslavsky and Leikin (2004) presented a three-layered hierarchical model of learning, where each successive layer contains the knowledge of mathematics learners, mathematics teachers, and mathematics teacher educators, respectively. A recursive relationship exists between the layers as each form of knowledge operates and reflects on knowledge in the layer beneath. There is also space for a fourth layer representing the knowledge of educators of MTEs. Tzur's (2001) self-reflective analysis of his own growth as a MTE is an example of how an individual moves through these four layers of learning mathematics, learning to teach mathematics, learning to teach mathematics teachers, and learning to mentor fellow MTEs.

Mathematics teacher educators are also well positioned to learn from their research with teachers. This learning has often been left unacknowledged and unarticulated (Jaworski, 2001), but there is beginning to be greater attention paid to this avenue for MTE learning (e.g. Chen, Lin, & Yang, 2018). Chapman (2008) suggested that an explicit goal of mathematics teacher educators' research of their practice should be self-understanding and professional development. Reports of such studies, therefore, need to include how the teacher educator-researchers reflected, what practical knowledge they acquired, and how this knowledge impacted or is likely to impact their future behaviour in working with their students. This will allow such research to contribute to greater theoretical understanding about mathematics teacher educator learning and to the improvement of practice.

Rather than appealing to cognitive or constructivist theories that treat learning as an internal mental process (as in the studies mentioned above), many mathematics education researchers have begun to draw on sociocultural theories in proposing that teachers' learning is better understood as increasing participation in socially organised practices that develop their professional identities (Lerman, 2001). Goos (2020) has identified two lines of sociocultural inquiry that might help organise the field of research into MTE learning and development: a *change* perspective influenced by Vygotsky's (1978) advocacy for a genetic or developmental method and a *practice* perspective informed by Wenger's (1998) ideas about situated learning in communities of practice.

Exemplifying the sociocultural *change* perspective, Valsiner (1997) extended Vygotsky's (1978) conceptualisation of the zone of proximal development (ZPD) to account for the changing relationship between individuals and their environments. He proposed the existence of two additional zones, the zone of free movement (ZFM) and zone of promoted action (ZPA). The ZFM structures an individual's access to different areas of the environment, the availability of different objects within an accessible area, and the ways the individual is permitted or enabled to act with accessible objects in accessible areas. The ZPA comprises activities, objects, or areas in the environment in respect of which the individual's actions are promoted. The ZFM and ZPA are dynamic and inter-related, forming a ZFM/ZPA complex that is constantly being re-organised through interaction between people in the learning environment. Valsiner's theoretical ideas have been taken up by researchers interested in studying the learning and development of mathematics teachers (e.g. Bansilal, 2011; Hussain, Monaghan, & Threlfall, 2013) and mathematics teacher educators (Goos & Bennison, 2019).

Research on MTE learning as a sociocultural *practice* invokes Wenger's (1998) community of practice concept, for example, when novice and experienced MTEs within a disciplinary community reflect on their practice both collectively and individually (Masingila, Olanoff, & Kimani, 2018). Goos and Bennison (2018) have also identified the learning opportunities brought about by boundary crossing between disciplinary communities of mathematicians and mathematics educators who are engaged in teacher education.

1.3.2.1 Questions Addressed by Theme 2 Chapters

The ten chapters in this section address questions about transition and identity formation as a MTE, contexts for MTE learning, conditions and contexts for learning that differ across cultures, and theories of MTE learning.

Question 1: How is the transition to developing as a MTE made? What changes in this process? What is gained or lost in the transition?

Three chapters explore the transition to developing as a MTE from the perspectives of being a graduate student, a mathematics teacher in school, or a research mathematician. Olanoff, Masingila, and Kimani discuss how they formed a community of practice comprising an expert and two novice MTEs who worked together on a mathematics content course for prospective elementary teachers in the USA. This was a mentored teaching experience that contributed to the award of a Certificate in University Teaching for the two novice MTEs who were graduate students of the university. In this community of practice, the MTEs engaged with each other as reflective practitioners who designed, enacted, and modified teacher education tasks and observed each other teaching the mathematics content course. Their joint enterprise was to develop mathematics knowledge for teaching teachers.

The chapter by Bissell, Brown, Helliwell, and Rome provides a narrative analysis of Bissell's transition from teaching mathematics in school to working with practising mathematics teachers in professional development settings in the UK. The case study of his transition from expert teacher to novice MTE is further illuminated by two of the other chapter authors (Helliwell and Rome), whose own transition narratives resonated with the *straplines* in Bissell's narrative – that is, the taglines he identified that captured important elements of his transition experience. The nuanced accounts of transition in this chapter bring to the fore notable similarities and differences between being a mathematics teacher and a mathematics teacher educator.

Sikko and Grimeland investigate a different kind of transition, from research mathematician to MTE for prospective primary school teachers in Norway. While the first author experienced this process of change 20 years previously, the chapter focuses on the more recent transition of the second author. Educational reforms in Norway have led to increased recruitment of MTEs to university positions, and the shortage of well-prepared teacher educators with PhDs in mathematics education has resulted in research mathematicians filling many of these posts. Sikko and Grimeland analyse two key challenges in this transition: developing a pedagogy for teaching prospective teachers and becoming research active in mathematics education.

Question 2: In what contexts do MTEs learn? How does the context influence what and how we learn? What is the difference between learning in structured and spontaneous contexts?

The majority of chapters in this section discuss MTE learning in the authors' professional contexts – reflecting on their practice as teacher educators or their research with teachers. In many cases, reflection on practice involves collegial expert-novice partnerships, such as those described by Sikko and Grimeland and by Olanoff et al. The chapter by Van Zoest and Levin, working in the USA, presents a further example of learning in the context of the authors' initial teacher education practice, which they refer to as Artifact-Enhanced Collegial Inquiry (ACI). This structured process supported joint inquiry between an experienced (Van Zoest) and a novice (Levin) MTE in order to articulate what is involved in teaching a mathematics methods course for prospective middle school teachers. As many of these chapters demonstrate, it is rare for MTEs to experience any formal preparation for their role. In contrast, the chapter by Ingram, Burn, Fiddaman, Penfold, and Tope analyses the learning of three MTEs who were undertaking a research-based Master's programme in a UK university where the focus was on their work as teacher educators. This chapter is concerned with the affordances for learning and change offered by the different contexts in which the MTEs worked – either as a school teacher, a university academic, or an educational consultant to a group of private schools in the Middle East.

In several chapters, the authors discuss what they learned as MTEs from their research with teachers. Nolan and Keazer present a research conversation reflecting on the dilemmas and tensions they experience in teaching courses on culturally relevant pedagogy in two Canadian universities. Bakogianni et al. report on similar professional challenges faced by their large team of MTEs and mathematics teachers in Greece as they collaboratively planned, enacted, and reflected on mathematics lessons that exploited workplace situations. Osborn, Prieto, and Butler draw attention to the diversity of MTE roles in Australian universities and analyse how a group of MTEs from different disciplinary backgrounds (mathematics, statistics, computer science, science, secondary school teaching, and tertiary education) found a collective identity in the context of a large research project that deliberately fostered interdisciplinary collaboration in initial teacher education.

Question 3: What role does cultural context play in MTE learning and development? How do conditions and contexts for this learning differ across cultures? How do these differences influence the practice of MTEs?

Teacher education is strongly grounded in local educational and cultural contexts, and cross-cultural comparisons can reveal interesting differences, for example, in the educational qualifications and amount of school teaching experience expected of teacher educators. Many of the chapters in this section allude to such cultural variations and their impact on MTE practice, such as in Sikko and Grimeland's account of the Norwegian cultural context and the changing expectations of the qualifications and experience that MTEs need to possess. Rather than treating the cultural context as the backdrop to an analysis of MTE learning and development, Wu and Cai bring culture to the foreground in comparing the developmental trajectories of different types of MTEs in China. Their chapter describes the unique teaching research system that supports practising teacher professional development in China and identifies challenges faced by university-based MTEs in their teaching work possibly arising from a lack of prior school teaching experience.

Question 4: What theories of learning might be useful for understanding how MTEs learn and develop?

The chapters in this section draw on a range of theories that conceptualise MTE learning as a process of collegial reflection. Thus, while reflection in and on practice is considered to be important, this process is typically supported by socially organised practices rather than only by individual introspection. There are examples of MTE learning from a sociocultural *practice* perspective, typically involving communities of practice that include experienced and novice MTEs (e.g. Olanoff et al.; Van Zoest & Levin) or MTEs from different professional or disciplinary backgrounds (e.g. Bakogianni et al.; Osborn et al.). Some chapters invoke the concept of boundary crossing between communities of practice to analyse MTE learning (e.g. Bakogianni et al.; Osborn et al.; Sikko & Grimeland). Ingram et al. illustrate the sociocultural *change* perspective on MTE learning, drawing firstly on Clarke and Hollingsworth's (2002) interconnected model of teacher professional growth and secondly on Goos's (2013) adaptation of zone theory to examine how this growth changes the context in which it occurs. Two chapters examine MTE learning from an enactivist perspective (Bissell et at.; Brown et al.). While enactivism is grounded in introspection and the elucidation of first-person experience, these chapters make methodological use of an empathic "second person" as an expert observer who is a partner in the process of MTEs developing multiple layers of awareness. This approach to educating awareness is consistent with the ways in which these MTEs work with prospective teachers in their initial teacher education courses.

1.3.3 Theme 3: Methodological Challenges in Researching Mathematics Teacher Educator Expertise, Learning, and Development

The learning and development of mathematics teacher educators is naturally of interest to MTEs themselves, and it is, therefore, MTEs who have conducted what little research has been reported. This situation raises obvious questions about the capacity of researchers to examine their own and colleagues' practice in valid and ethical ways (Burns, 2000). Consequently, much existing research is self-reflective (e.g. Alderton, 2008; Schuck, 2002). Chick (2011) was aware of this fact and illustrated it in her description of the dilemmas inherent in the choices that mathematics teacher educators, including herself, routinely make about the content, tasks, and emphases that they include in their courses. Different choices made by different mathematics educators relate to both the individual's particular beliefs about mathematics teaching and to the differing constraints imposed by such things as course structures and prospective teachers' prior knowledge and experience in different contexts (Chick, 2011). These varied beliefs and contexts also influence the stances that mathematics teacher educators take as they reflect on and observe their own and colleagues' practice with the resulting danger that findings might be too personally and contextually specific to be generalisable. Schuck (2002) stressed the importance of involving a critical friend in the interpretation of the findings of self-study in order to avoid simply confirming one's existing beliefs, and Chapman et al. (2020) illustrated the value of collective reflection on practice.

Zaslavsky and Leikin (2004) introduced the role of mathematics teacher educator to describe a person responsible for the development of mathematics teacher educators. This work has led to programmes being designed to educate MTEs (e.g. Abboud, Robert, & Rogalski, 2020; Thornton, Beaumont, Lewis, & Penfold, 2020). This introduces a new "layer" that could be seen as analogous to mathematics teachers researching their students and mathematics teacher educators researching mathematics teachers. MTE educators could thus be the appropriate people to research mathematics teacher educators. In reality, however, those who take the role of MTE educators often are also mathematics teacher educators and hence, as was the case in the study reported by Zaslavsky and Leikin (2004), likely to be involved in the milieus that they are researching as well as personally engaged with the same issues with which their research subjects (mathematics teacher educators) are grappling.

The MTE educator is essentially taking the role of a participant observer in accordance with a well-established research methodology (Burns, 2000).

Another possible way of ameliorating the introspective tendency of participant observer or self-study research on mathematics teacher educators' expertise, learning, and development is the use of research teams. Lovin et al. (2012) used this approach among mathematics teacher educators who had shared experiences of a doctoral programme at the same university before becoming mathematics teacher educators at various institutions. Arguably the greater the diversity of such teams, the better. It is evident from Beswick and Callingham (2011) that diversity within the research team in terms of backgrounds, work contexts, beliefs, and knowledge militates against personal biases exerting undue dominance and highlights the role that individuals' beliefs play in their design and interpretation of items used to assess the knowledge of mathematics teacher educators as well as prospective teachers.

1.3.3.1 Questions Addressed by Theme 3 Chapters

The three chapters in this section address questions concerning dilemmas and opportunities associated with researching ourselves as mathematics teacher educators and methodologies that might be effective in building an evidence base for successful mathematics teacher education. We should also point out that the other chapters in this book, while more closely associated with Theme 1 on the nature of MTE expertise and Theme 2 on MTE learning and development, offer additional insights into methodological challenges in this emerging field of research.

Question 1: What are the dilemmas and opportunities associated with researching ourselves?

In their survey of research in mathematics teacher education conducted from 1999 to 2003, Adler, Ball, Krainer, Lin, and Novotna (2005) highlighted four main claims about the state of the field at that time. One of these claims was that most teacher education research is conducted by teacher educators studying the teachers with whom they are working. Adler concluded that MTEs need to find ways of taking a sceptical stance towards our work, for example, by inviting "external eyes" to scrutinise what we are doing, and to develop strong theoretical languages to create a distance between ourselves and what we are looking at. As we have indicated earlier, the same conclusion would apply to research in which the object of study is mathematics teacher educators ourselves.

Both of these approaches suggested by Adler are evident in the chapter by Oates, Muir, Murphy, Reaburn, and Maher. These Australian MTEs investigated how they brought their professional knowledge and beliefs to bear on their decision-making processes in the context of initial teacher education course design and review. Their search for suitable theoretical frameworks led them to activity theory and professional capital in order to make sense of, and yet distance themselves from, the tensions they experienced in balancing individual agency in course design with the need for alignment between the separate units they taught. These MTEs also invited a former colleague to conduct a focus group interview with the course team, both as a means of distancing themselves from what they were looking at and to supplement the data they had gathered from audio-recorded team meetings and document analysis.

Arzarello and Taranto likewise acknowledge the dilemmas of "researching ourselves" as MTEs, which they address by drawing on a key feature of the Italian mathematics education tradition in the form of the researcher-teacher. This person belongs to both the community of university-based MTE researchers and the community of mathematics teachers and acts as a broker between communities who translates research knowledge into knowledge for teaching. The distinctive aspect of the study described by Arzarello and Taranto is that it created a massive open online course (MOOC) as a distance learning experience for Italian teachers and MTEs.

Question 2: What methodologies might be effective in building an evidence base for successful mathematics teacher education?

The chapters in this book offer many examples of methodologies as a counterpoint to the largely introspective tendencies of early research into teacher educator learning. In this section, for example, Oates et al. called on an external colleague to elicit the beliefs, motives, and actions of the MTE course team. The chapter by Rojas, Montenegro, Goizueta, and Martínez explicitly addresses methodological challenges in studying the teaching practices that MTEs model for prospective teachers. They pay attention to the pedagogical reasoning of MTEs and how this is shared with their students, and they identify different categories of pedagogical modelling and its impact on prospective teachers. In particular, they argue that MTE modelling is a two-way, interactive process, and so research needs to find out how prospective teachers interpret MTEs' practice. Rojas et al. also propose the use of phenomenographic interviews to uncover MTEs' understanding of the relationship between the practices they model in the university classroom and the practices of the school classroom where prospective teachers will work in future.

1.3.4 Commentary Chapters

We invited two eminent scholars in the field of MTE research to provide critical commentaries on the chapters addressing Themes 1, 2, and 3. Olive Chapman, reflecting on the chapters on MTE expertise, acknowledges that a category-based perspective on MTE knowledge and beliefs is reasonable at this early stage of development of the research field. However, she argues that this perspective is inadequate for capturing the complexity, contingency, and integrated nature of MTE knowledge and encourages further research to explore other ways of understanding and representing MTE expertise. Barbara Jaworski offers deeply personal insights into the key ideas expressed by the chapters on MTE learning and development and on methodological challenges. Her commentary chapter identifies personal reflections, narrative accounts, and voice as tools used by MTEs to advance their development.

Jaworski points to areas needing further research by posing questions directly to the reader, emphasising in particular how issues of identity are implicated in the question of who are "we" as MTEs?

1.4 Contributions to Advancing the Field

We are grateful to chapter authors for their contributions to developing this new field of research. Thanks to their insights, we believe that this book makes three significant contributions to advancing knowledge of the learning and development of mathematics teacher educators.

First, the book surveys an emerging field of inquiry in mathematics education, combining the work of established scholars with perspectives of newcomers to the field. There are still few journal articles or books on this topic but greater activity in conference proceedings suggesting that the field is reaching a more mature stage of development. Our book aims to influence this development.

Second, the book invites cross-cultural comparisons in becoming a MTE. International studies such as TEDS-M (Tatto et al., 2008) and the 15th ICMI Study (Tatto, Lerman, & Novotna, 2010) have greatly increased our understanding of the structure and content of initial teacher education programmes around the world, and the chapters in this book highlight similar issues in the development of MTEs in different countries.

Finally, the book examines the roles of both mathematics educators and mathematicians in preparing future teachers of mathematics, thus exposing any contrasting and complementary perspectives and examining possibilities for collaboration in course design and delivery. Previous research has highlighted differences between the epistemologies and values of these two communities (e.g. Goldin, 2003). Our book instead contributes to a search for common ground, as envisaged by Fried and Dreyfus (2014).

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Part I
The Nature of Mathematics Teacher Educator Expertise

Chapter 2
What Do Mathematics Teacher Educators Need to Know? Reflections Emerging from the Content of Mathematics Teacher Education

Dinazar Escudero-Ávila, Miguel Montes, and Luis Carlos Contreras

2.1 Introduction

Research into the work of those charged with training future teachers has found different ways of referring to this role, both at a general level, using terms such as *teacher trainer* (Palhares, Gomes, Carvalho, & Cebolo, 2009), *didactician* (Coles, 2014), *mentor* (Halai, 1998) and *facilitator*, and at the more subject specific level with expressions such as *mathematics educator* (Zavlavsky & Peled, 2007) and *mathematics teacher educator* (Beswick & Chapman, 2012, 2013; Jaworski & Huang, 2014). The choice of designation provides clues to the underlying conception of the role these agents play in the initiation to professional development and the configuration of the training they provide. In our view, the appellation *mathematics teacher educator* is a useful one as it encapsulates a multiplicity of approaches that teacher training can take, from the various roles the trainer might play to the different elements that can become the object of construction during these vital early stages of training. Mathematics teacher educator (MTE) is thus the term that we shall use in this chapter.

We make the assumption that decades of research about and with mathematics teachers should have an influence on what an MTE ought to know. Hence, the organisation of the chapter is structured around the content knowledge (the nature of which we will discuss in the course of the chapter) and pedagogical knowledge (associated with this content) which we consider essential for an MTE to be in possession of and which we believe is available to them as an interconnected whole. In

D. Escudero-Ávila
Benemérita Universidad de Puebla, Puebla, Mexico

M. Montes (✉) · L. C. Contreras
Universidad de Huelva, Huelva, Spain
e-mail: Miguel.montes@ddcc.uhu.es

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_2

the last section, we present some reflections about the different profiles of teacher educators, in relation with their knowledge.

2.2 Mathematics Teacher Educator Knowledge

It is widely acknowledged that there is a difference, both qualitative and quantitative, in the knowledge of those doing the teaching and those doing the learning. This has led some authors (e.g. Beswick & Chapman, 2012, 2013) to wonder whether models of mathematics teachers' knowledge can be applied to MTEs, allowing for inevitable differences. The focus of this chapter is not the comparison between MTEs and prospective teachers; rather we will use as a reference point the knowledge future mathematics teachers need to acquire in order to discuss what kind of knowledge MTEs should, therefore, have. One aspect that needs to be decided on from the outset is which model of teachers' knowledge to adopt as reference for defining the teacher education content, whether, for example, mathematical knowledge for teaching (MKT) (Ball, Thames, & Phelps, 2008) or mathematics teacher's specialized knowledge (MTSK) (Carrillo et al. 2018). However, we found it advantageous to situate this study in a previous theoretical approach, specifically the seminal work of Shulman (1986), and it is this model, with its two extensive domains of content knowledge and pedagogical content knowledge, which guided our research into the knowledge of MTEs, albeit that in our case the 'content' under investigation goes from being 'mathematics' to 'those elements which enable mathematics to be understood as the object of teaching and learning'.

In line with Ponte (2012), we divide teacher education content into three key areas: professional knowledge, practices and skills for the management of mathematics teaching and professional identity (in which we include all elements relating to the affective sphere). Different teacher education programmes place differing emphases on these components, so it should be borne in mind that the set of items of knowledge we consider here should not be taken as intending to be exhaustive nor even a list of minimum requirements; rather, it is one possible configuration of elements of utility to educate teachers. It should also be noted that though we deal with various elements of knowledge in separate sections, we regard them as interdependent and constituent of a single integrated whole of professional knowledge. Indeed, we have found that one of the basic characteristics of an MTE's knowledge lies in the sheer density of the connections which they are able to make between the diverse elements of knowledge relating to mathematics as the object of teaching and learning. And it is these connections that allow the MTE to create a suitable environment for prospective teachers to construct their own rich and integrated knowledge.

As mentioned above, among MTEs' knowledge, we also include elements from the affective domain. The MTE's conceptions and beliefs go beyond their view of mathematics as a science and object of teaching and learning. He or she has a personal vision of what future teachers should learn, and this has an impact on determining the content and general orientation of the education. Likewise, they will have their own ideas about how best to help their students on their journey to becoming teachers, that is, ideas about how people learn (what he or she understands should be learnt) and how best to collaborate in the process (which determines their personal model of mathematics teacher education). Finally, they will have some ideas about what, how and when to evaluate the development of the prospective teachers, which will without doubt have implications for those aspects mentioned above.

The chapter is divided into seven sections. The organisation of the chapter follows the prior ideas, presenting first some reflections about MTEs' content knowledge and later about MTEs' pedagogical content knowledge (PCK). Reflections about content knowledge are organised following Ponte's (2012) ideas about the teacher education content. Sections 1 and 2 concern MTEs' knowledge about the elements of knowledge to be built in mathematics teacher education, that is, mathematical knowledge and PCK, following Shulman (1986). Section 3 explores MTEs' knowledge about mathematics teaching practices and skills, while Section 4 examines MTEs' knowledge about professional identity. This is followed by three sections concerning MTEs' PCK. The three sections are knowledge of the features of the professional development of mathematics teachers, knowledge of teaching the content of initial mathematics teacher education programmes and knowledge of the standards of mathematics teacher education programmes. The structure of these three sections follows the classic focusing teaching, learning and syllabus also used in the characterisation teacher knowledge of Carrillo et al. (2018)[1] and by Kilpatrick and Spangler (2016) for teacher educators.

2.3 Mathematical Knowledge

In this section, we focus on the mathematical knowledge of MTEs. Although this is a more recent and less developed area of research than that of teachers' knowledge, one question which merits debate is the quantity, quality, robustness and depth of mathematical knowledge required by an MTE. Furthermore, as mentioned above, whatever the configuration of this mathematical knowledge might be, it is closely intertwined with other kinds of knowledge.

One way, we believe, of approaching the nature and characteristics of MTEs' knowledge is to focus on how this knowledge develops over time. Tzur (2001) explores the development of an MTE's knowledge, from the initial stage as novice

[1]This framework elaborates on Shulman's (1986) and Ball et al.'s (2008) conceptualisations of teacher knowledge, proposing an intrinsic approach towards specialisation (Scheiner, Montes, Godino, Carrillo, & Pino-Fan, 2019). Carrillo et al.'s (2018) framework proposes a structuration of content knowledge in three subdomains – knowledge of topics, knowledge of the structure of mathematics and knowledge of the practice of mathematics – and PCK in another three subdomains, knowledge of mathematics teaching, knowledge of the features of the learning of mathematics and knowledge of mathematics learning standards.

to the final stage as mentor, indicating the different areas of knowledge required for each intermediate stage. With respect to mathematical knowledge, there is a wealth of research describing very precisely the differences in terms of the knowledge and use of mathematics between, on the one hand, primary and secondary school children and, on the other, prospective teachers (Ball, et al., 2008; Jaworski, 2008; Lynch-Davis & Rider, 2006; Smith, 2005; Zopf, 2010), which in turn are suggestive of differences in knowledge between teachers and MTEs. The first difference is that teaching children is not the same as teaching adults. In the former case, the teacher supplies information not previously known to the learner; in the latter, the MTE is guiding the prospective teachers to mobilise information which they frequently already know (or least think they know as their knowledge often tends to be patchy) and confront it to a broader vision of mathematics education (Lloyd, 2006). In our view, prospective teachers not only learn mathematics but also restructure and reorganise their knowledge, transforming it into a specialised knowledge for teaching and learning, at the same time as they equip themselves in pedagogical terms.

In this respect, we share Zopf's (2010) view that mathematical knowledge, along with mathematical knowledge for teaching, are subsumed within the deeper and broader knowledge of mathematics for teaching teachers possessed by MTEs. In other words, MTEs' knowledge should encompass, not being limited to, the mathematical knowledge to be built by their students; it takes a panoramic and interrelated view, is fluid and intentional in nature and so emphasises connections (the scope and organisation of knowledge) and depth.

From this perspective, the challenges facing an MTE can be characterised as the creation of opportunities to (re)construct and develop ideas of use to the teaching and learning of mathematics. Hence the most significant difference in terms of knowledge between teachers and MTEs is that MTEs need to be aware of and to understand a wide network of connections between both purely mathematical items and between mathematics and other professional elements, such as identity and practices, to a sufficient degree so as to be able to exploit any opportunity for constructing specialised knowledge emerging in the course of mathematics education and to a sufficient depth so as to be able to tackle the foundations of mathematical knowledge which a teacher should know.

The breadth and depth of MTEs' knowledge are the result of a process of growth in which mathematics progressively acquires increased complexity and is viewed from an increasingly holistic perspective in which the links between concepts become multiplied and more available to the MTE. Jaworski and Huang (2014) consider the transformation of mathematical knowledge in ways which enable the novices (students or prospective teachers) to develop their own versions of this knowledge. This process of transformation is articulated at two levels (from MTEs to teachers and from teachers to students), which interface with each other and allow interaction. The process can be seen as comprising continuous cycles whereby knowledge which is initially exclusively mathematical is transformed into an element of knowledge now perceived as an object of teaching and learning and ultimately becomes an element of professional knowledge.

It is clear that the complexity accruing to any item of mathematical knowledge in its progress through the cycle does not depend solely on mathematics, but on the knowledge, abilities and identity necessary to teach it. Zopf (2010) summarises this as knowing how to help unpack teachers' compressed mathematical knowledge and attempting to decompress it for the work of teaching children. For their part, Masingila, Olanoff and Kimani (2018) regard problem-solving as the ideal vehicle for this process, as it is through solving problems that one reasons, models, argues a case, sees the underlying structure and makes generalisations. Knowing how to help teachers reconstruct their own mathematical knowledge and decompress it so as to understand the connections between items (Ma, 1999), so that they conceptualise it as an object of teaching and learning, supposes that the MTE is capable of analysing the mathematical knowledge implicit in every situation. In like fashion, it implies that the MTE organises their knowledge in such a way that the trainees construct their own knowledge in such a way as to be able to carry out their own process of unpackaging (Masingila, et al., 2018).

A second area of knowledge which distinguishes MTEs from teachers concerns the importance that the former need to confer on syntactic aspects of mathematical knowledge (Rowland & Turner, 2009). If it is true that students tend not to see any need to justify certain mathematical relationships they regard as valid (Harel & Sowder, 2007), it is no less true that teachers do, over time, give more and more importance to such justifications (Douek, 2007). MTEs, on the other hand, need to recognise that substantive knowledge in itself is necessary but not sufficient, and as such they need to gain a full appreciation of meta-mathematical considerations such as understanding the essence of proofs, the meaning of theorems and definitions and the rigour of mathematical language (Leikin, Zazkis, & Meller, 2018), as well as being aware of the relevance of beliefs about the subject (Grossman, Wilson, & Shulman, 1989).

A third area marking out MTEs' knowledge is the manner in which it is organised (Leinhardt & Smith, 1985). Having a clear understanding of the structuring ideas underlying mathematics, the connections which serve to simplify or increase the complexity of an item (Montes, Ribeiro, Carrillo, & Kilpatrick, 2016) and cross-curricular connections is something which characterises a deep knowledge of content, and such networks of links need to be consolidated within the domains of MTE knowledge if he or she is to promote their construction on the part of the trainee teachers.

In order to explore these differences further, we will consider the organisation of mathematical knowledge and the use to which it is put by each actor (student, teacher, MTE). As Zopf (2010) points out, students learn mathematics for their own use, and the organisation of their mathematical knowledge is fragile. Teachers learn mathematical knowledge for teaching their students, to which end they unpack their knowledge of mathematical topics and transform it by making use of other elements of their knowledge in order that their students learn to make associations between different areas of their knowledge. To help teachers unpack their knowledge in such a way that it makes sense to their students, MTEs must contribute to the development, on the part of the prospective teachers, of this mathematical knowledge in relation to the other knowledge subdomains which they want the prospective teacher to construct. Hence, MTEs must bear in mind not only these subdomains but also the connections between them, which are not (just) connections between items of mathematical knowledge, but between subdomains of mathematics teachers' specialised knowledge.

2.4 Knowledge About Teachers' PCK

The fundamental contribution of Shulman (1986) was the recognition of a component of pedagogical knowledge intrinsic to the discipline and distinct from general pedagogical knowledge (Shulman, 1987). These two papers laid the groundwork for recognising the importance of taking into consideration pedagogical aspects of specific content, such that the characteristics specific to teaching and learning each were recognised. This very special kind of knowledge was named pedagogical content knowledge (PCK). The work triggered various studies into mathematics teachers' PCK, aimed at exploring its nature and demarking what it encompassed, so as to achieve a better understanding of the construct and its features (e.g. Depaepe, Verschaffel, & Kelchtermans, 2013).

Focusing on the MTE as the person responsible for managing and promoting the construction of teachers' professional knowledge, we consider it of fundamental importance that he or she takes account of PCK as an essential part of any teacher education programme and has an understanding of the different elements of which it is composed. In this regard, it makes sense to expect that an MTE is aware of the characteristics of PCK along with, should he or she be familiar with the research literature, the different conceptualisations and formulations that can be found in this regard.

Drawing on the organisation of PCK offered by Carrillo et al. (2018) and Kilpatrick and Spangler (2016), we consider that MTEs should be familiar with the following three areas (in each case, as appropriate to the level at which their prospective teachers will be working):

- Theories of teaching, teaching strategies and methodological resources for teaching mathematics. This should include such things as, for example, being aware of how GeoGebra can scaffold mathematical learning or knowing different applications of ICT to mathematics.
- Key features of learning mathematics. This includes theories of learning mathematics (e.g. van Hiele theory) and their application to each concept. Likewise, it could include knowledge about how special needs students learn that could offer recognition of diversity in modes of learning. It also includes the features of learning in identifiable cultural groups (e.g. drawing on the approach of ethnomathematics). It is also important that an MTE has an in-depth knowledge of the different kinds of strengths, weaknesses, errors and obstacles which can be revealed by different mathematical tasks (Kilpatrick and Spangler (2016).

– Learning standards. Clearly, knowing the relevant syllabus specifications is indispensable, but it might also include a personal view of how topics should be sequenced or the precise degree of conceptual or procedural development appropriate to the level in question. In like fashion, it would be advantageous for the MTE to be familiar with internationally recognised reference documents on standards (e.g. National Council of Teachers of Mathematics (NCTM) standards). An MTE should also be familiar with perspectives on school mathematics education (e.g. modern mathematics, mathematical competencies, problem-solving and so on).

It is to be expected that an MTE should not only have much wider knowledge than his or her students but also be able to take a higher standpoint (Kilpatrick, 2008), being aware of the connections between the different elements of the knowledge they aim for their students to construct. For example, an MTE might know about what Simon and Tzur (2009) call *hypothetical learning trajectories* for a given concept, involving elements from each of the three prior components, which would enable him or her, for example, to discuss with their students theoretical issues related to how the concept is learned or to structure a debate based around a video of a lesson on the concept. Likewise, drawing on their understanding of teachers' PCK, an MTE could lay the foundations for trainees to develop their own theories of teaching and learning mathematics, contributing from their own teaching experience and providing elements of formal theory while encouraging their reflection and independence. In this regard, what distinguishes the knowledge of an MTE from that of a teacher is that while the teacher needs to make associations between the items of knowledge within their grasp, the MTE needs to be aware of a wide variety of available associations that can be made and to promote their establishment.

2.5 Knowledge About Mathematics Teaching Practices and Skills

MTEs should also be familiar with how teachers use knowledge and how they focus their teaching practice. For example, in the NCTM *Principles to Actions*, eight effective mathematics teaching practices were proposed: establish mathematics goals to focus learning, implement tasks that promote reasoning and problem-solving, use and connect mathematical representations, facilitate meaningful mathematical discourse, pose purposeful questions, build procedural fluency from conceptual understanding, support productive struggle in learning mathematics and elicit and use evidence of student thinking (NCTM, 2014). Familiarity with these practices includes knowing in-depth the desirable mathematics teaching practices in the context he or she educates teachers; being aware of their (e.g. mathematical, didactical and historical) foundations, likely difficulties and limitations in their implementation; and the expected benefits for pupils' learning. From our perspective, the MTE should be very flexible about the teaching practices, assuming that 'teacher education must therefore be conceived of not as the experience and interpretation of a predetermined, prescribed pedagogic practice but rather as an ongoing, dialogically constructed entity involving two or more critically reflective interlocutors' (Kumaradivelu, 1999, p. 552). Therefore, an MTE should have the ability to discuss this issue with their students, gradually empowering them to become reflective practitioners (Schön, 1983).

A second key element intertwined with the mathematics teaching practices are mathematics teaching skills. These combine knowledge and practices such as professional noticing (Mason, 2002) and classroom preparation (Perks & Prestage, 1994). This combination requires both good mathematical knowledge and knowledge of classroom practices requiring students to make their thinking explicit. In addition, they should be able to foster the development of noticing skills, which requires different levels of awareness of teachers' professional abilities. At a first level, this would be an informal perception of what a teacher should learn in order to draw detailed inferences from particular learning scenarios. At an intermediate level, it would be knowing that prospective teachers need development in identifying and experience in interpreting what their students do and why, so as to be able to then make accurate decisions on how to respond (Sherin, Linsenmeier, & Van Es, 2009). At a deeper level, the MTE would recognise the importance of having developed noticing skills linked to specific concepts (e.g. Sánchez-Matamoros, Llinares, & Fernández, 2015).

Both elements are closely linked to the (future) teachers' 'mathematical work of teaching' (Ball, 2017). This concerns 'mathematical know-how' applied to the moment of teaching. It is dynamic, contingent and fluid, as happens when a teacher decides which student to invite to speak, based on their knowledge of the mathematical arguments and the kind of reasoning the student typically employs. To this end, MTEs' knowledge should encompass both awareness of how the different skills of the mathematical work of teaching are related and the relationship of these skills with different elements of knowledge or professional identity which are mobilised in putting them into practice. There are various sources from which this knowledge might be drawn, including the MTE's own teaching experience, discussions with other MTEs and related literature and research results.

2.6 Knowledge About Professional Identity

Professional identity (Wenger, 1998) is a part of the content of teacher education. When an MTE interacts with the prospective teachers he or she is educating, two professional identities come into play, that of the MTE and that of the prospective teacher. The identity of the MTE is not the focus of discussion of this chapter; rather we concentrate on the elements an MTE might/should know about the professional identity of prospective teachers.

Teacher educators have a crucial role contributing not only to the development of what teachers know or do but also to what teachers are and become, both individually and collectively (Wenger, 1998). They should therefore recognise the many different factors (e.g. beliefs, interaction with environment, attitudes, emotions) that influence the development of identity (van Putten, Stols, & Howie, 2014) and be aware of what they contribute to this complex and multifaceted process (Kelchtermans, 2009). This awareness would allow MTEs to foster the reflection of students, allowing them to assume their own positioning. Likewise, an MTE could be expected to know the different facets linked to the development of a professional identity, along with detailed aspects of these, so that they can contribute to its development.

One important facet of professional identity concerns the affective domain, which includes conceptions and beliefs. MTEs need to be aware of the prevalent conceptions and beliefs in their working environment, in respect of mathematics and mathematics teaching, so that they can help trainees become aware of them and the (in)consistencies they embody through activities designed to bring them to the fore.

2.7 Pedagogical Content Knowledge: What Does 'Content' Mean Here?

Shulman (1986) described pedagogical content knowledge as 'the ways of representing and formulating the subject that make it comprehensible to others' (p. 9). It is clear that an MTE, as a teacher of prospective teachers, needs to represent the subject in such a way as to make it accessible to their students and hence must be in possession of PCK. In line with the reflections of Beswick and Chapman (2012), it is worth considering whether teachers' PCK is transferable to MTEs. For Abell et al. (2009), there is a duplicate of PCK for MTEs which comprises, as mentioned above, those aspects of knowledge indicated by Ponte (2012): knowledge, identity and professional practices. Consequently, mathematics teacher educators' PCK, in a parallel with teachers' PCK, could include, for example, different perspectives on mathematics learning (which we could identify as conceptions about mathematics teaching and learning or as knowledge about theories of learning mathematics), knowledge of how prospective teachers learn and their difficulties when faced with certain topics, knowledge of standards relating to teacher education, knowledge of teaching strategies for educating prospective teachers and knowledge of aspects relating to evaluation.

For the purposes of this chapter, we will follow the framework originally proposed by Shulman (1986) and more recently refined by various researchers (e.g. Ball, et al., 2008; Carrillo, et al., 2018), consisting of the three facets of pedagogical content knowledge, entailing knowledge about (1) teaching, (2) learning and (3) the syllabus, here reinterpreted in the light of the professional work of MTEs. The source for these areas is academic knowledge, that is, basic subject knowledge and meta-disciplinary knowledge, along with that gained by experience, directly intermingled with the disciplines which are usually considered sources of knowledge at an institutional level and are generally included in teacher education syllabuses (chiefly, psychology, pedagogy, sociology, mathematics, mathematics education and new technologies).

2.8 Knowledge of the Features of the Professional Development of Mathematics Teachers

As teachers, MTEs must be familiar with how their students' learning processes (concerning mathematics and mathematics education, in particular) evolve. It is thus natural for an MTE to ask himself or herself the following questions: How can the professional development of prospective teachers be characterised? What difficulties are they likely to encounter in terms of their specialisation as mathematics teachers? What kind of sequencing or focus might be the most appropriate with regard to constructing their knowledge and identity and developing their abilities? What (and how) do prospective teachers usually know before embarking on an initial teacher education programme? Such questions, couched in general terms here, can be given different degrees of specificity and can encompass different aspects of content making up the programme in question.

For example, the MTE must be aware of the characteristics of the knowledge with which the prospective teachers embark on the teacher education programme. This is often weak and fragmentary, as Olanoff (2011) illustrates with the topic of multiplying fractions. If it is intended that a primary teacher learns to multiply fractions at a significant level of depth, the MTE must take into account the fact that the prospective teachers will have preconceived ideas about the topic (in terms of the mathematics itself and how it might be taught), that their knowledge is likely to be procedural (which could be perpetuated through their teaching) and that in some cases it will be wrong. The MTE must be able to build on this knowledge, clarify mistaken concepts and present new ways of tackling the topic using appropriate models of representations. For example, having a wide knowledge of different ways of representing the multiplication of fractions (and being able to show the prospective teachers the advantages and disadvantages of each type of representation) enables the MTE to select examples that highlight these different representations which the prospective teachers need to know for their future work. This example illustrates the inter-connectedness of the various elements of the MTE's knowledge (features of the trainees' learning, types of teacher education content and aspects of teaching the content of teacher education programmes).

2.9 Knowledge of Teaching the Content of Initial Mathematics Teacher Education Programmes

Teaching people to teach mathematics requires the MTE to develop and set up tasks which fulfil the specific needs of the elements of the teaching profession he or she aims to develop. These tasks, and how they are used, have an impact on prospective teachers' learning (Grevholm, Millman, & Clarke, 2009), leading them to develop their knowledge, identity and professional skills. Hence, the MTE should have not only a profound knowledge of a repertoire of activities at his or her disposal for developing the three elements identified by Ponte (2012) but also an awareness of the limitations of each activity, the implications of each for learning and the resources which will make their deployment go more smoothly. Beyond specific tasks, the MTE should be able to use different types of activities, such as group discussions (Carrillo & Climent, 2011), video analysis (e.g. Coles, 2014) and narratives (Ivars & Fernández, 2018). As with the specific activities, this knowledge should include awareness of the potential and limitations of each with respect to the construction of knowledge and the development of skills and professional identities.

Alongside this, and following Zaslavsky and Leikin (2004), MTEs need to know how to set up scenarios in which the prospective teachers can acquire the desired professional knowledge. Such scenarios must provide trainees with a challenge relating to teaching and learning, their responses to which they can then reflect on (Cooney & Krainer, 1996). It is important to establish what knowledge enables this task. It is not simply a matter of distinguishing knowledge for teaching people to teach mathematics from knowledge for teaching mathematics pedagogy. It is important to see the possible associations with two of the components in Sánchez and García's work (2004): (1) knowledge of the different ways of characterising the process of learning to teach mathematics, referring to knowledge of the different theoretical perspectives on training for primary education, such as the knowledge of the dialectic between theory and practice as sources of professional knowledge (phronesis vs episteme; see Kessels and Korthagen, 1996) and (2) knowledge of the use of content in the context of mathematics teaching.

The MTE's knowledge of how to develop both professional identity and professional skills must also be borne in mind. For example, in the context of a discussion with prospective teachers about lesson planning, the MTE must identify the characteristic features of a particular methodology so that he or she can then identify potential inconsistencies in, for example, the use of materials favouring the meaning of a concept which is incompatible with the planned activity.

Finally, MTEs must be familiar with the design and use of evaluation methods for measuring the degree to which the programme objectives have been met. This kind of knowledge is deeply determined by both the MTE's previous teaching experience and the different contexts in which he or she has delivered teacher education, as well as the cycle in which teacher education takes place, whether initial or in-service or peer group discussions, each of which has its own particular demands in terms of the elements of knowledge to be mobilised. MTEs must be able to distil a topic into its most important features and to find the connections between these, as well as between mathematics and other disciplines, with the aim of helping prospective teachers deepen their understanding and achieve an holistic view of mathematics. Hence, to use again the example from Olanoff (2011) in the previous section, if the topic is the division of fractions, the MTE must be aware of the difficulties thrown up by the partitive division of fractions, which might lead him or her to encourage prospective teachers to engage with the invert and multiply method for dividing fractions. Such a capacity – deciding which aspects of a topic will help them to make the kind of connections which they themselves will have to help develop in their students – is especially important.

2.10 Knowledge of the Standards of Mathematics Teacher Education Programmes

As Beswick and Chapman (2012) point out, teacher education standards depend, in the first instance, on the educational level (primary, secondary or pre-university) of the teacher education programme. MTEs need to know the syllabus standards for the relevant educational level as well as the explicit principles guiding their design. However, there are also standards implicit within the system. For example, in each university, depending on which department is charged with teacher education programmes, the orientation will have a greater or lesser degree of purely mathematical knowledge. Being aware of this variation, which is a product of an implicit set of standards at the institutional level, enables the MTE to adapt to the context in which he or she conducts teacher education. There are also features in school mathematics that have an impact on teacher education. Hence, being aware that the work of school mathematics in Spain is centred on competencies could lead the MTE to take account of this in his or her planning and to focus the work on mathematics with their group of prospective teachers.

This knowledge is highly context dependent, as the focus in different countries could be quite different. Nevertheless, it might not be too much to expect that an MTE has studied the different perspectives of teacher education in other countries in order to enrich their work. Knowledge of standards for teacher education should also bring with it the ability to establish, explain and evaluate the learning objectives of the prospective teachers. Also to be included in this area of knowledge is an understanding of how to facilitate the professional development of mathematics teachers, such that consideration of this aspect of trainees' professional life is set in motion from the very start.

2.11 Three Profiles of MTE

In this section, we describe three MTE profiles with the aim of highlighting, according to each professional profile, the features of knowledge specific to the dimensions outlined above. Here we focus on the different types of knowledge that the different professionals can be expected to have and the sources of this knowledge. Our three profiles, consistent with the work of Beswick and Chapman (2012), are the following: professional mathematicians, researchers into mathematics education (didacticians) and practising teachers at primary and secondary levels.

Professional mathematicians command a deep knowledge of mathematics in terms of quality and specificity, regularly far exceeding that required for teaching prospective teachers. However, the very depth of their knowledge tends to lead them to view mathematics from a perspective which can often be quite remote from school mathematics. Likewise, their perception of the subject tends to be framed by a Platonic conception of mathematics (Ernest, 1998). Their awareness of teachers' PCK often derives from personal or informal sources of information such as their own teaching experience and the mathematics teacher education manuals they have at their disposal. They often adhere to the unspoken dictum that 'the more mathematics a teacher knows, the better' (Wu, 2011), without necessarily considering the specialised facets of this knowledge.

When we use the term 'didacticians', we usually refer to researchers into mathematics education, teacher MTEs who, from a theoretical or practical perspective, work with prospective teachers using ideas and research in mathematics education that enable us to understand modes of teaching (Jaworski & Huang, 2014). Unlike MTEs, whose profile is that of a professional mathematician and whose focus is educating prospective teachers in (advanced) mathematics, didacticians simultaneously keep in mind two processes by which mathematical knowledge is transformed, one focused on helping future teachers to understand mathematics as an object of teaching and learning (school mathematics) and one focused on helping them to transform mathematical content into forms which are comprehensible to students (at primary or secondary level).

The label 'researchers into mathematics education' covers a wide range of experts in the field of teaching and learning mathematics following specific lines of research, including work groups centred on the professional development of mathematics teachers. However, it is not only the specialist researchers in this area who are typically tasked with educating teachers. Rather, education programmes feature a range of experts in the field of mathematics education in general, on the conviction that they have usually acquired an in-depth knowledge of mathematics by virtue of their journey to a PhD in mathematics education and also that their own education is likely to have incorporated substantial reflection on the elements of both pedagogical and mathematical knowledge that a teacher might require. Such is the diversity of profiles and perspectives about the nature of mathematics which come together in this group that the kind of knowledge the teacher is assumed to be most urgently in need of depends in large measure on the professional trajectory and research interests of the MTE in question.

Finally, primary and secondary school teachers involved in teacher education typically take on two different roles: that of trainers, giving classes in the usual way at university, and that of tutors or mentors in the schools where they themselves work. The participation of these teachers in the training of their future colleagues brings with it one significant advantage: they can combine theory and practice, thus closing the gap that so often opens up between these two aspects (Goos, 2014). The single most important aspect of this profile is the opportunities it affords to reflect on teaching practice. When teachers are able to carry out a well-grounded analysis of their teaching practice, they become aware of their actions and make the tacit knowledge underpinning this practice explicit. In like fashion, they easily adapt to different groups of prospective teachers, acquire a global vision of teaching and the educational system and often achieve a maturity in their professional development that enables them to write research articles (Smith, 2005). The capacities they gain in this way put them in a privileged position to work in collaboration with didacticians in teacher education programmes (Goos, 2014).

It would seem that, given the diversity of perspectives and potentialities that each MTE profile can contribute, they have complementary roles to play in teacher education. In particular, experts in mathematics education and practising teachers contribute valuable knowledge of – respectively – the theory and the practice of teaching mathematics, while the professional mathematician can offer in-depth knowledge of the discipline. Nevertheless, we believe that a well-designed programme committed to providing comprehensive education for prospective teachers should combine different profiles of MTEs, guaranteeing that all of them hold a professionalising shared view of teacher education, tackling content assuming that their students will become teachers, meaning that the understanding of mathematics they have to develop should be linked to teaching and learning processes, to ensure its all-round quality. Absolutely necessary to achieving this would be the use of activities such as group discussions on both pedagogical and mathematical topics, the search for agreement regarding the ideal profile of the teachers being trained and the smooth coordination of this training. In this way, the strengths which each professional profile has to offer complement each other and result in effective teacher training.

2.12 Concluding Remarks

The knowledge that MTEs draw on in educating mathematics teachers is, as we have outlined above, multidimensional, complex, integrated, contextualised and dependent on mathematical content. In addition, as we have underlined, this knowledge can be understood in the light of the requirements necessary for prospective teachers to develop their professional identity and skills and to construct their professional knowledge. We have sketched out here the different aspects which an MTE should know, to a greater or lesser degree of depth, on the assumption that their knowledge will to a large extent be defined by the context in which they work (i.e. the country, particular school focus, specific mathematical focus of the respective school, training programme and so on). In this regard, the range of MTEs' dimensions of knowledge that we have mapped out is far from exhaustive. In addition, empirical research will help to determine more precisely these domains and their practical relevance to training, as well as their internal organisation.

Finally, the sphere of teacher education merits the development by its MTEs of their own specialised knowledge (Kilpatrick and Spangler (2016). This education could begin, for example, with obtaining a PhD in Mathematics Education, but in our view, specific MTE education with its corresponding range of activities is necessary. Such education should respect the autonomy that MTEs need to enjoy, but develop their knowledge in respect of the context of teacher development. In the case of teachers already holding a PhD in mathematics that are going to become MTEs, it might be worthwhile for them to undergo a process of discussion with expert MTEs (with different profiles, if possible), which allow them to understand the specificities of teacher education, as well as the very specialised kind of mathematical knowledge (Ball et al., 2008) that teachers need to build. Activities focused on the training of teacher MTEs can only have a positive impact on the quality of teacher education programmes, at the same time as they represent a privileged environment for researching the knowledge and professional development of MTEs.

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Chapter 3
Applying the Knowledge Quartet to Mathematics Teacher Educators: A Case Study Undertaken in a Co-teaching Context

Tracey Muir, Sharyn Livy, and Ann Downton

3.1 Introduction

As mathematics teacher educators (MTEs), our teaching and research is informed by frameworks and explanations of terms to guide our thinking about the knowledge an effective mathematics teacher might use. For example, Shulman's (1987) seminal study has guided many researchers as they consider important categories of a teacher's knowledge base such as *knowledge of content*, *pedagogical knowledge* and *knowledge of learners*. Others have elaborated by describing *specialised content knowledge* when referring to a unique kind of knowledge that mathematics teachers demonstrate (Ball, Thames, & Phelps, 2008). Rowland, Turner, Thwaites and Huckstep (2009) used the term Knowledge Quartet (KQ) to describe four categories of teacher knowledge: *foundation knowledge* (including knowledge of content and pedagogical knowledge); *transformation* (representing the mathematics); *connection* (e.g. coherence of planning, sequencing of instruction); and *contingency* (when the teacher responds to classroom events). The framework has been used elsewhere to investigate classroom practice (e.g. Livy, 2010), but its use has primarily been restricted to pre-service teachers (PSTs) and primary school teachers.

This chapter investigates whether or not the KQ framework can be applied to the work of MTEs. Using the context of a co-teaching situation, whereby an MTE taught a cohort of PSTs with a practicing primary school teacher, we examine the type of knowledge required by an MTE and whether or not it is different from that required by a primary school teacher. We then use the KQ to interpret the work of

T. Muir (✉)
Faculty of Education, University of Tasmania, Launceston, Australia
e-mail: Tracey.Muir@utas.edu.au

S. Livy · A. Downton
Monash University, Clayton, VIC, Australia

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_3

the MTE, using data collected from lesson observations, and interviews conducted with PSTs, the MTE and the co-teacher.

Our research questions for guiding our study are as follows:

- How does the knowledge needed by mathematics teacher educators differ from that required by primary school teachers?
- To what extent is the KQ applicable/transferable in describing the work of mathematics teacher educators?

In seeking to answer these questions, we are responding to Rowland's (2013) question of whether or not a framework for knowledge-in-teaching developed in one subject discipline can be legitimately adopted in another and, if so, what the conceptualisations of the dimensions would look like. As it is debatable whether or not mathematics teacher education could be considered a discipline, we have adapted the question to determine whether or not the KQ that was originally designed to examine the work of classroom mathematics teachers could be legitimately applied to the work of mathematics teacher educators.

3.2 Review of Literature

3.2.1 Mathematical Knowledge for Teaching

Research into the different types of knowledge required for teaching has been well documented (e.g. Chick, Pham, & Baker, 2006; Hill, Ball, & Schilling, 2008; Ma, 1999; Rowland et al., 2009; Shulman, 1986). Shulman's (1987) theoretical framework described seven categories of teacher knowledge, which became the foundation for describing the knowledge base for teaching. His conceptualisation of pedagogical content knowledge (PCK) is not subject-specific and is described as

> the blending of content and pedagogy into an understanding of how topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category [of teacher knowledge] most likely to distinguish the understanding of the content specialist from that of the pedagogue. (Shulman, 1987, p. 8)

In terms of mathematics teaching, PCK is needed for teaching different mathematical topics, in order to make these topics comprehensible to learners. This knowledge includes understanding student misconceptions; knowing how topics are organised and taught; having a repertoire of representations, explanations, and examples that illustrate concepts; and having the ability to adjust lessons to cater for all learners (Shulman, 1986). Ball et al. (2008) used the term 'mathematics knowledge for teaching' (MKT) to refer to the knowledge 'needed to perform the recurrent tasks of teaching mathematics to students' (p. 399). Their model distinguishes between subject matter knowledge; common content knowledge (CCK); specialised content knowledge (SCK); horizon knowledge and PCK; knowledge of content and teaching (KCT); knowledge of content and students (KCS); and knowledge of the curriculum (KCC). While it is beyond the scope of this chapter to discuss this model in detail, it is useful for considering the different types of knowledge required for teaching. In the model, CCK refers to common mathematical knowledge, which we would expect the 'average' person to possess, such as being able to correctly solve mathematical problems. SCK, on the other hand, refers to mathematical knowledge that is unique to teaching. It is knowledge that the 'average' person is not expected to have and refers to the knowledge required to unpack mathematics content in order to make it accessible to students. This knowledge is similar in nature to what Ma (1999) termed 'a profound understanding of fundamental mathematics' (PUFM) to describe understanding a topic with depth and breadth (p. 120). This involves connecting a topic with more conceptually powerful ideas of the subject, together with connecting it with those of similar or less conceptual power. For a primary school teacher, teaching the subject of subtraction, breadth would be demonstrated through connecting subtraction, including regrouping and renaming, with the topics of addition and associated regrouping and renaming, and subtraction without regrouping and renaming, and addition without regrouping and renaming. Connecting subtraction with concepts such as the rate of composing or decomposing a higher value unit, or the notion that addition and subtraction are inverse operations, is a matter of depth (Ma, 1999). Together with this knowledge, a primary school teacher would also be expected to know what representations and examples would be useful in teaching about this topic, which is all part of PCK (KCS, KCT, KCC). It is evident, therefore, in teaching primary mathematics to students that a teacher does require a specialised form of content knowledge, along with PCK to make the knowledge accessible to students. Does an MTE, who is required to teach PSTs how to teach subtraction, for example, therefore require additional knowledge to that required by a primary teacher?

Many teacher educators (TEs) transition into their role from a school teacher, but becoming a TE 'involves much more than applying the skills of school teaching in a new (and different) context' (Loughran, 2014, p. 272). Instead what is required is a form of 'meta-knowledge', which could be described as 'knowledge for teaching knowledge for teaching mathematics' (Beswick & Chapman, 2012, p. 2). Just like school teachers, MTEs need to study, for example, student misconceptions, analyse concepts and engage their students (Beswick & Chapman, 2012), but they also have to teach their adult students how to teach students in schools. This suggests that additional knowledge is required, beyond the knowledge previously required in their role as a mathematics school teacher. Interestingly, Beswick and Chapman (2012) also make the point that there may be elements of school teachers' knowledge that MTEs either do not need to know or need to know differently (e.g. detailed knowledge of how to assess according to the school curriculum).

Like Loughran (2014), Murray and Male (2005) claimed that TEs' work demanded new and different types of professional knowledge and understanding from that required of school teachers. Referring to the Standards for Dutch TEs, they list five inter-related competencies needed for the role of educating teachers: content competencies, pedagogical competencies, group dynamic and communicative competencies and developmental and personal growth competencies for working with adult learners. Although they do not explicitly answer the questions of ‘what professional knowledge TEs need, what pedagogical understanding and skills they require, and about how these things differ from the knowledge, skills and understanding of school teachers’ (Murray & Male, 2005, p. 136), they did identify that the development of pedagogy for teaching teachers and the generation of research and scholarship were key areas of development for beginning TEs.

Beyond an acknowledgement that TEs require an understanding of adult learning principles that school teachers do not (e.g. Murray & Male, 2005), we did not find compelling evidence in the literature that the (content) knowledge required by TEs is substantially different from that required by an experienced, competent school teacher. Chick and Beswick (2013, 2018) expanded an earlier PCK framework developed by Chick et al. (2006) to include mathematics teacher educator PCK (MTEPCK). Within the framework, examples are given as to how MTEPCK might be enacted for each element. As an example of profound understanding of mathematical content, the MTE might demonstrate PCK through ‘identif[ying] and explain[ing] the importance of identifying and addressing student misconceptions evident in a teaching episode’ (Chick & Beswick, 2018). This element for school teachers was exemplified by understanding why we invert and multiply when dividing fractions. The authors argued that in addition to the school mathematics knowledge required, MTEPCK is enacted in additional ways. Rather than demonstrating or providing examples of MTE knowledge in relation to mathematical content, the framework identifies aspects of MTE work (pedagogy), which proved to be useful in terms of analysing the moment-by-moment application of knowledge in the work of mathematics education (Chick & Beswick, 2018). We used a similar strategy to align elements of the KQ to the work of MTEs (see Table 3.2).

Returning to the subtraction example discussed earlier, does teaching PSTs how to teach a topic such as subtraction require knowledge beyond the depth and breadth of knowledge required by a school teacher? Chick and Beswick (2018) provided some examples of how school teaching can be transferred to teaching PSTs through replacing ‘students’ in their framework with ‘PSTs’, but most of the examples given in relation to MTEPCK could reasonably form part of a competent school teacher’s knowledge (e.g. ‘Contrasts different representations (e.g. MAB (multi-based arithmetic blocks) and LAB (linear-arithmetic blocks)) and what they offer for mathematics teaching’ (representation of concepts)) (Chick & Beswick, 2018). With this particular example, it could be argued that it would be reasonable to expect the school teacher to be aware of the affordances of both representations, how to use them and in what context. An MTE also needs to know this but arguably should also be cognizant of the theoretical underpinnings and current research behind the adoption of particular representations and use of manipulatives in order to convince their PSTs of the merits of using such representations and models. Similarly, with the subtraction example, like a school teacher, an MTE might be expected to know how to use appropriate materials such as MAB to demonstrate why the vertical subtraction algorithm ‘works’ but, in addition, be able to explain the theoretical underpinnings and research behind the use and appropriateness of the materials.

3.3 Theoretical Framework

3.3.1 The Knowledge Quartet

Developed from observations of 24 mathematics lessons by Rowland and colleagues, (Rowland et al., 2009), the Knowledge Quartet contains four 'units' or dimensions which describe teacher knowledge. Each dimension contains a number of elements that could be used to interpret classroom practice, including that as undertaken by PSTs. An overview of the framework is presented in Table 3.1.

Table 3.1 Overview of Knowledge Quartet and its elements

Category	Description	Elements	Examples of evidence
Foundation	Theoretical background, involving knowledge and understanding of mathematics, knowledge of mathematics pedagogy and beliefs about mathematics	Adheres to textbook Awareness of purpose Concentration on procedures Identifying errors Overt subject knowledge Theoretical underpinning Use of terminology	Concentrate on developing understanding rather than excessively using procedures Show evidence in planning of knowledge of common errors and misconceptions and take steps to avoid them Use mathematical language correctly
Transformation	Ways in which teachers transform or represent what they know for learners	Choice of examples Choice of representation Demonstration	Use equipment correctly to explain processes Select appropriate forms of representations Make use of interactive teaching techniques
Connection	The coherence of the planning or teaching across an episode, lesson or series of lessons; also includes the sequencing of topics of instruction within a lesson	Anticipation of complexity Decisions about sequencing Making connections between procedures Making connections between concepts Recognition of conceptual appropriateness	Make links to previous lessons Make appropriate conceptual connections within the subject matter
Contingency	Teacher's response to unplanned and/or unexpected classroom events	Deviation from agenda Responding to students' ideas Use of opportunities	Respond appropriately to students' comments, questions and answers Deviate from agenda when appropriate

Adapted from Rowland et al. (2009)

Table 3.2 Examples of primary teacher and MTE evidence for categories of KQ

Category	Elements	Examples of evidence – primary teacher	Examples of evidence – MTE
Foundation	Adheres to textbook Awareness of purpose Concentration on procedures Identifying errors Overt subject knowledge Theoretical underpinning Use of terminology	Critically selects, adapts and extends problems in textbook or teachers' guides Explains how the formal algorithm for addition is carried out; knows how and why a procedure 'works' Identifies that students often think 'longer is larger' when comparing decimals Identifies a 'rule' for solving an algebraic problem and then justifies why the 'rule' works Focus on explaining/ demonstrating why the formula for area 'works' Using correct terminology to describe operations and terms (e.g. sphere rather than ball)	Critically selects, adapts and extends problems in mathematics textbook and prescribed PSTs' texts (e.g. Reys, et al., 2012) Emphasises informal rather than formal algorithms; unpacks PSTs' use of procedures Identifies that PSTs often hold similar misconceptions to the students they will teach; addresses PSTs' beliefs about mathematics learning and teaching (e.g. algebra is not relevant for young children Identifies that arrays are integral for understanding multiplication, how and when they should be taught and the research that underpins their use Incorporate theories of learning (e.g. constructivism) into practice through explicitly referencing and modelling Use of correct mathematical terms and their precise meanings; use of correct mathematical pedagogical language (e.g. mathematical discourse)
Transformation	Choice of examples Choice of representation Demonstration	Use of MAB materials to model the formal algorithm for addition; use of balance beams to demonstrate equality Use of equipment correctly to explain processes Select appropriate forms of representations Make use of interactive teaching techniques	Use of balance beams to demonstrate equality; use of children's work samples to demonstrate common errors with adding fractions; awareness of range and purpose of current resources; emphasise importance of critical evaluation of the selection of appropriate resources

(continued)

Table 3.2 (continued)

Category	Elements	Examples of evidence – primary teacher	Examples of evidence – MTE
Connection	Anticipation of complexity Decisions about sequencing Making connections between procedures Making connections between concepts Recognition of conceptual appropriateness	Recognise that subtracting 11 or 21 requires less complex strategies than subtracting 9 or 19 When planning a sequence of lessons on geometry would refer to AC:M achievement standards Make links to previous lessons Make appropriate conceptual connections within the subject matter	Recognise that PSTs' prior knowledge of subtraction may be dominated by rules and procedures In addition to the sequences in AC:M, connections also made with geometric levels of thought (Van Hiele, 1986) and other frameworks and growth points Breadth and depth of understanding about how different mathematical topics are connected (e.g. volume of rectangular prisms and links with multiplication) Recognising that course is structured around discrete topics and that connections may be difficult to make Considering the appropriateness of course content as related to year course undertaken (e.g. designing task-based rubrics in final year vs second year of study)
Contingency	Deviation from agenda Responding to children's ideas Use of opportunities	Respond appropriately to students' comments, questions and answers Deviate from agenda when appropriate	Justify the learning and teaching approaches and content choices made Depth and breadth of knowledge about mathematics learning and teaching to respond to PSTs' questions Recognise when deviation is appropriate due to course constraints/expectations

The KQ was designed to be used as a framework for identifying and discussing the ways in which the use of mathematics content knowledge was observed in teaching. In the research reported in this chapter, the authors utilised this framework, not with a focus on school mathematics teaching per se but rather the knowledge required by an MTE when teaching PSTs how to teach primary mathematics. We were motivated to select this framework as Rowland et al. (2009) had developed a range of resources to assist with interpreting the various elements.[1] The framework has been adopted and reported on by other researchers (e.g. Livy, 2010; Muir, Wells, & Chick, 2017), and we wanted to determine whether or not it was appropriate for interpreting the work of experienced MTEs, rather than PSTs or in-service mathematics teachers. An earlier study by Muir et al. (2017) used aspects of the framework to interpret the work of two teacher educators but was limited in terms

[1] See http://www.knowledgequartet.org/

of providing examples of evidence across all the elements. The study discussed in this chapter adds to this research through providing corresponding examples of MTE knowledge, aligned with school teachers' knowledge (see Table 3.2) and through applying the KQ in practice in the context of a co-teaching arrangement.

Table 3.2 shows examples of evidence related to the work of an MTE. The examples have been drawn from our own experiences as MTEs and as a result of the observations and interviews conducted with the MTE discussed in this chapter.

3.4 Methodology

A case study was used to investigate an MTE's knowledge for teaching mathematics education to a cohort of PSTs. Case study was considered as a preferred methodology given that 'how' questions were being posed, with a focus on a contemporary phenomenon within a real-life context (Yin, 2009).

The participants in the study were an MTE, a primary school teacher and a cohort of third-year PSTs enrolled in a primary mathematics pedagogy unit. The MTE was an Early Career Researcher having worked at the university for the past 4 years since completing her doctoral studies. Prior to working in university settings, the MTE had worked as a mathematics consultant and also as a primary mathematics teacher. The primary school teacher had taught at a local school for the past 10 years and was now a leading teacher and numeracy coach. She shared a Year 4 class with another teacher and was also responsible for providing support to all teachers as a mathematics leader in her school. This included helping teachers to plan and implement their programs, as well as providing professional advice to guide their teaching.

Through a university initiative, Sarah (pseudonyms are used throughout this chapter for all participants), the MTE, had invited a practicing primary school teacher, Melissa, to co-teach her class of PSTs. Melissa was released for 1 day a week from her school to enable her to co-teach the weekly tutorials with Sarah throughout the semester. Each week Sarah and Melissa shared the teaching and after class reflected on their experience before planning the activities for the following week. As the lecturer, Sarah was responsible for assessment and marking of assignments.

As Sarah was interested in researching her own practice, she invited two colleagues (also authors) to observe her teaching. Both Julie and Mary were research colleagues of Sarah's, with a shared interest in PST mathematics education, and were happy to participate in the study. Mary taught in the same university as Sarah and had taught the same primary mathematics pedagogy unit in the past. Julie was from a different university where she taught a similar subject. Sarah invited each colleague/researcher to observe her teaching in weeks 7 (Julie) and 8 (Mary) of semester 1, 2018. Julie and Mary collected the data from the lesson observations and conducted the interviews. Table 3.3 provides a summary of the participants and the data collected.

Table 3.3 Participants and data

Date	Classes observed	Number of students	Data collected
20/4/18 (week 7 of semester 1)	A. 9.00–11.00 am B. 11.00–1.00 pm Topic: algebraic thinking	$n = 14$ $n = 11$	Pre-lesson planning notes; classroom observation notes; post-lesson interviews with MTE; post-lesson interviews with co-teacher; 25 PST post-lesson reflections; 1 post-lesson focus group interview with 5 PSTs; PST interviews (1 focus group); post-lesson reflection notes and recorded discussion between MTE and co-teacher
27/4/18 (week 8 of semester 1)	A. 9.00–11.00 am B. 11.00–1.00 pm Topic: measurement	$n = 18$ $n = 15$	Pre-lesson planning notes; classroom observation notes; post-lesson interviews with MTE; post-lesson interviews with co-teacher; 33 PST post-lesson reflections; 3 post-lesson focus group interviews with 6 PSTs; PST interviews (1 focus group); post-lesson reflection notes and recorded discussion between MTE and co-teacher

Semi-structured interview schedules were used to guide the interviews, and for the PSTs included questions as follows: What do you think an observer would take away from having observed your class? Can you tell me what experiences and teaching approaches in class today assisted with your learning? For the co-teacher and MTE, the questions were similar and included, for example: What experiences and/or activities do you think were most effective today? Which ones had most impact on their learning? How did you personally contribute to their learning today? The PSTs' post-lesson reflections included responses to questions related to how they learnt the content, describing an activity or mathematical understanding they needed help with and who helped them, and how their learning was facilitated in the lesson.

For the purpose of this chapter, two vignettes were composed from field notes taken from observations of two lessons. Post-lesson interviews conducted with all participants were fully transcribed, along with the post-lesson reflection conversation. The two researchers independently coded the data to find evidence of the categories and codes of the KQ, for example, use of balance beams to demonstrate equality, transformation (choice of representation); planning a micro lesson, making connections (decisions about sequencing); and providing definitions for area and perimeter, foundation knowledge (use of terminology). Open coding techniques were also used to identify instances relevant to the study that were not evident within the KQ. For example, when Sarah asked the PSTs to consider more than one strategy, this was coded as questioning; discussion related to assignment expectations was coded as assessment.

3.5 Results and Discussion

In this section, we illustrate the application of the KQ in the analysis of lesson excerpts from two lessons observed by the researchers, which provide typical examples of how Sarah taught her lessons and explored key content of the unit. They also illustrate how different aspects of the framework occurred in the lessons. Although the focus is on Sarah as the MTE, the lessons were co-taught, so sometimes incidental reference is made throughout the lesson episodes to both teachers' roles in order to accurately represent what occurred in the lesson.

3.5.1 Lesson Episode 1: Algebraic Thinking

3.5.1.1 Lesson Observations

The lesson began with Sarah welcoming the PSTs (there were 12 students sitting at 3 tables, 1 male and 11 female). She then asked them to reflect on the prescribed reading and to discuss their experiences with learning algebra at school. Sarah then directed the PSTs to discuss what algebraic thinking would mean for primary students, to post their thoughts on sticky notes and then to place them on the board at the front of the classroom. Sarah and Melissa both facilitated discussion around particular algebraic terms such as patterning and relationships.

The PSTs were then asked to explore ways the balance scales on their tables could be used to teach algebraic concepts (see Fig. 3.1). After some exploration, some PSTs volunteered to share their thinking, drawing diagrams on the board to show how the two sides balanced (see Fig. 3.2). During the sharing, Sarah referred to a poster in the room that explained 'talk moves' and cited how a PST was using one of the moves (revoicing) when explaining another PST's strategy. When asked what grade level the activity would be suitable for, Sarah deferred to Melissa and also stated that it could be appropriate for any grade level if suitably contextualised.

Following this, Sarah shared a work sample from one of Melissa's students who was demonstrating a common misconception of seeing the equals sign as a place to put the answer and referred them back to a similar example that was shared in last week's tutorial. Melissa had provided a number of other examples from her primary school classroom to also demonstrate this tendency, and these were also shared. Sarah encouraged the PSTs to discuss at their tables what the samples revealed about the students' thinking, with the general consensus being that they did not see the equations as being equal but were putting an answer in the box. They were also encouraged to think about what they would do to help students make connections, and one PST volunteered that using the dot patterns on a dice might be helpful. Sarah took this opportunity to remind them that this was called subitising.

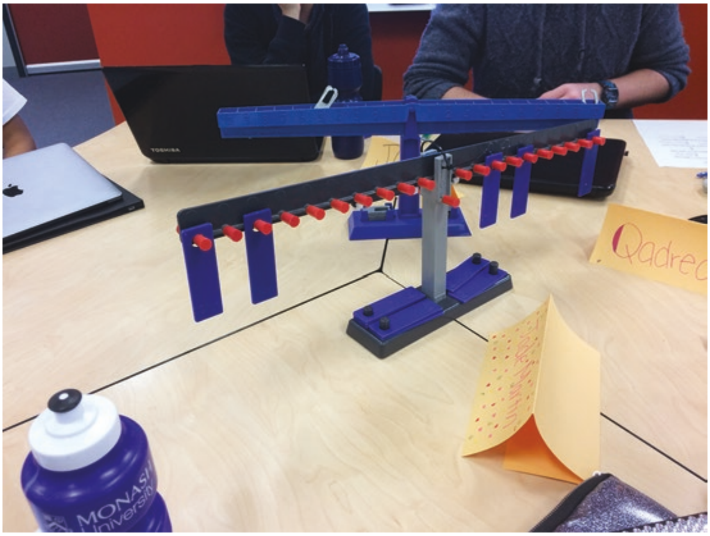

Fig. 3.1 Balance scales

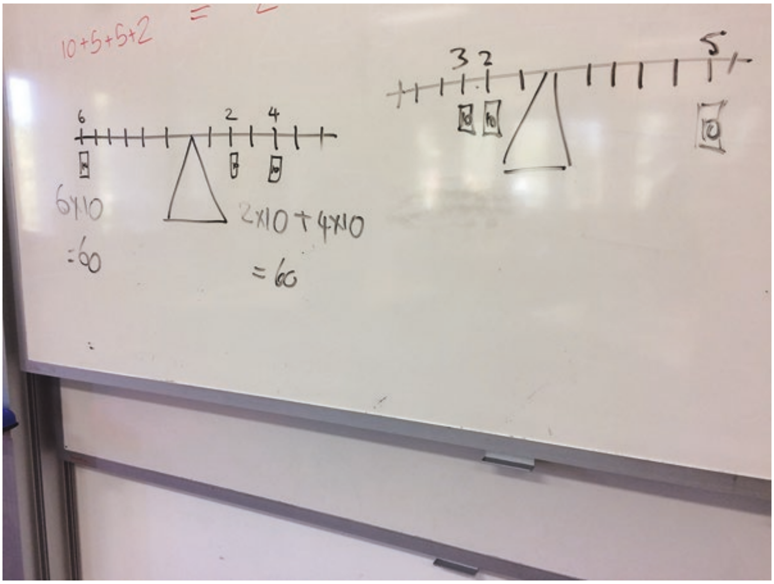

Fig. 3.2 Balance scale drawings

During the next part of the lesson, growing patterns were explored. There was no explanation given as to what a growing pattern was or how it differed from a repeating pattern. PSTs used counters at their table to create growing patterns and were then directed to pair up and ask their partner to continue their patterns (see Fig. 3.3).

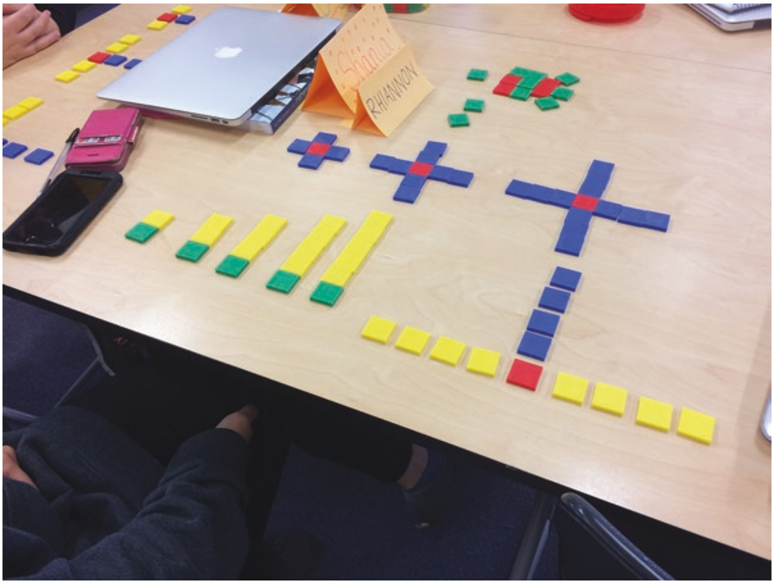

Fig. 3.3 Growing patterns

Sarah then took the PSTs on a 'numeracy walk' to look at others' patterns and to verbally describe them. At one point, one of the PSTs had difficulty explaining what their pattern was, and there was general confusion as to whether or not it was a growing pattern. Sarah asked questions about the pattern but did not attempt to explain what the pattern might be. Melissa then read the story *Two of Everything* (Hong, 1993) and described how it could be used as a stimulus for students to invent their own function machines. Due to time constraints, the PSTs did not get to participate in the planned activity of making their own function machines. The lesson concluded with the PSTs completing their post-lesson reflections.

3.5.1.2 Post-lesson Data

The PSTs' post-lesson reflection data showed that they identified a number of strategies that assisted their learning. Frequent mention was made of the use of manipulatives, particularly the balance beams (e.g. '[My learning was helped] through explaining, modelling, questioning, and facilitating learning through the use of manipulatives') and student work samples that provided examples of children's algebraic thinking and misconceptions (e.g. 'lots of student examples to assist us in understanding about misconceptions'). Mention was also made of the opportunities to contribute to discussions and to explain and justify their thinking (e.g. 'asking us to explain how we would explain to students').

The post-lesson interview conducted with the PSTs reinforced the practices mentioned in the post-lesson reflections. PSTs also had the opportunity to explicitly comment on the links they were able to make between theory and practice, for instance:

> I realised that algebraic thinking is at all stages of the curriculum. [Jill]

> I could see all the activities being used within a classroom and what terms we can use to explain the concepts. [Frida]

> I thought it was great how she showed us that the curriculum linked to the early ones because you don't ever think of it like that necessarily even though you may have seen it on placement [Sue]

> By actually physically doing the activity [balance beams], we've got more chance of remembering that…when we actually get in a classroom…because we actually participated, and we can say you could use balance scales to teach it but because we actually physically did it [Scott]

Although not asked to specifically comment on Sarah's knowledge, three of the PSTs interviewed said that they particularly valued Melissa's current experience as a teacher and would direct practically based questions to her. Fiona noted, for example:

> I probably would go to Melissa first because she's the in-service teacher. There's nothing wrong with Sarah but Melissa knows exactly what happens in schools. [Fiona]

3.5.1.3 Post-lesson Reflections: Co-teachers

Post-lesson reflective notes and discussion transcripts showed that the co-teachers were satisfied that the lesson had been effective. Again, mention was made of the effectiveness of the balance beams, for example:

> I liked exploring the number balances, even though it took some time, it was really good to connect a material to a concept, how you can use it, get them to explore different ways and explain that to other people is really good, and then making sense of it in the traditional way of learning about algebra, having things balanced on either side of the equation. So it's really nice to see that connect as they were working on it. [Melissa]

Post-lesson reflections also provided an opportunity for Sarah to reflect on her PCK and content knowledge:

> Normally I would do more on the growing patterns... So we did that briefly but I was still pretty happy that we showed them growing patterns and repeating patterns…probably we could've unpacked the mathematics in that a bit more but we've only got two hours.

> I've taught that [algebra] but when you start to think about the functions and the relationships and the groups and all that – well, see, I've never taught this before. So this was the first time I've done or taught all this. I've talked about different patterns and things but I've never taken it up to this level, but I'd done proportion and ratio which links to it - I try to then get on top of it … and I probably haven't got my roots in it … like all that proportional

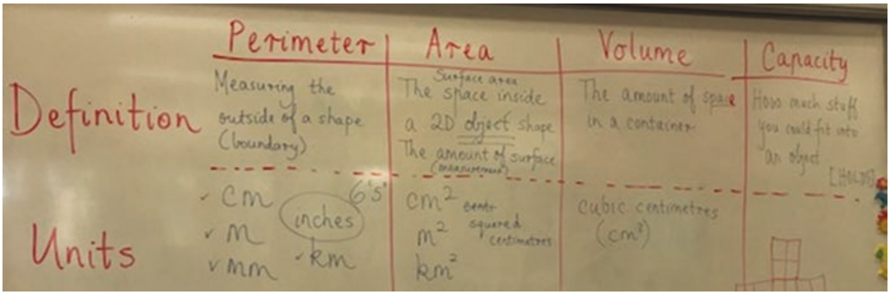

Fig. 3.4 Recording of PSTs' foundation knowledge

> reasoning that I've done days and days of and could just talk about. Depending on what you are researching sometimes becomes your strength.

Melissa was also able to comment on Sarah's PCK in her post-lesson reflections and discussions:

> You're [Sarah] very good at unpacking thinking, the thinking done by teachers and the learning that needs to happen. You're very good at pulling it apart and building their knowledge up again which is really, really good and you pose really great questions that challenges thinking…so I really like your technique when it comes to questioning and really extending them and thinking of it this way, what's another way of doing it, now you explain, that type of thing.

> I believe it was a very practical lesson. The PSTs can take away a lot of tips, techniques and ideas to use in their classroom. Made references to the Talk Moves and Knowledge Quartet; making connections to learning that has occurred in previous tutorials.

3.5.2 Lesson Episode 2: Measurement

Sarah welcomed the 17 PSTs (2 male, 15 female) who were seated at 5 tables, then explained the outline of the lesson, which included reference to enabling and extending prompts and use of talk moves.

The PSTs had been introduced to the KQ in week 1 and Sarah asked them what foundation knowledge would be required to teach perimeter, area and volume. After recording some ideas, Sarah invited the PSTs to turn and talk to the person beside them (modelling a talk move). During this time Sarah and Melissa roved, then Melissa led a discussion and recorded PSTs' responses on the board (see Fig. 3.4). Sarah then asked about appropriate measuring tools, and one PST used the term 'inches' in her response. Sarah asked Melissa if she used the term 'inches' with her class, and she indicated that it was not something her students could relate to. Sarah encouraged the discussion as it was of interest.

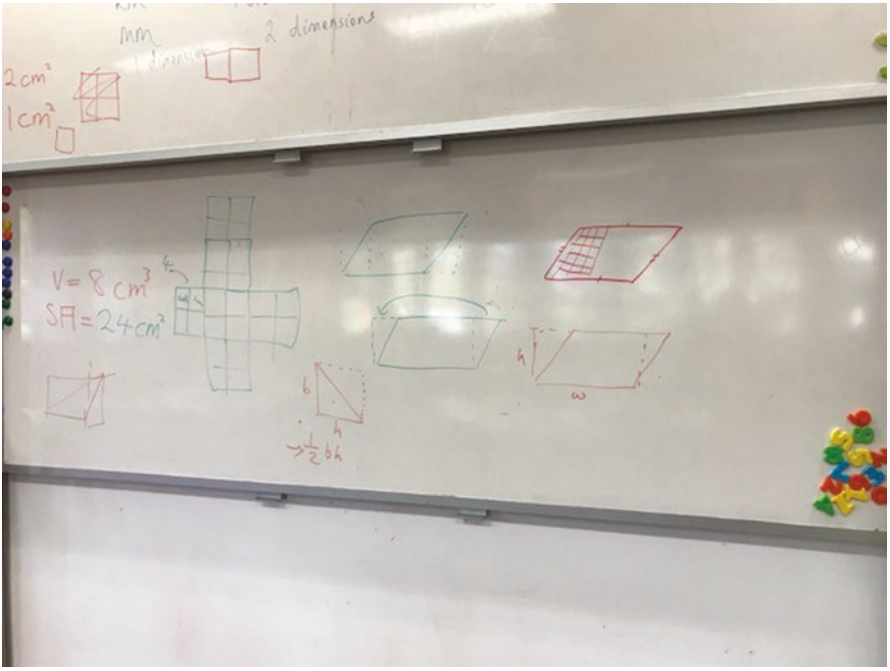

Fig. 3.5 Finding the surface area of a cube

In order to focus on measurement terminology, Sarah asked, 'is area a measurement?' This question led to a discussion about the correct terminology to use when interpreting the symbolic representation of square units. Sarah further challenged the PSTs' thinking of area by linking it to the surface area of a cube. She asked the PSTs to use the blocks on their tables to make a two by two by two cube and record the surface area of the construction. She left this for a moment and asked them how many cubic centimetres altogether, and they said eight. When asked how they worked it out, a PST replied length by width by height. Sarah stressed the importance of unpacking this rule with their own students to develop their conceptual knowledge, not just their procedural knowledge. One PST said she had no idea how to find the surface area. Sarah used the following prompts: 'How many faces do you see? What is the shape of each face?' Sarah asked them to work out the area of each face. She also drew a net, helping to unpack their thinking to find the answer of 24 square centimetres (Fig. 3.5). The PST who struggled initially said that she now got it.

Revisiting the discussion of formula, Sarah said that primary school students should understand why the rule works. She illustrated this by using a scenario of a Year 6 class who were going to explore the area of a triangle using an online learning object (ABC Splash: Maths). PSTs were encouraged to explore this resource and consider if this program would assist students to understand the rule 'half base

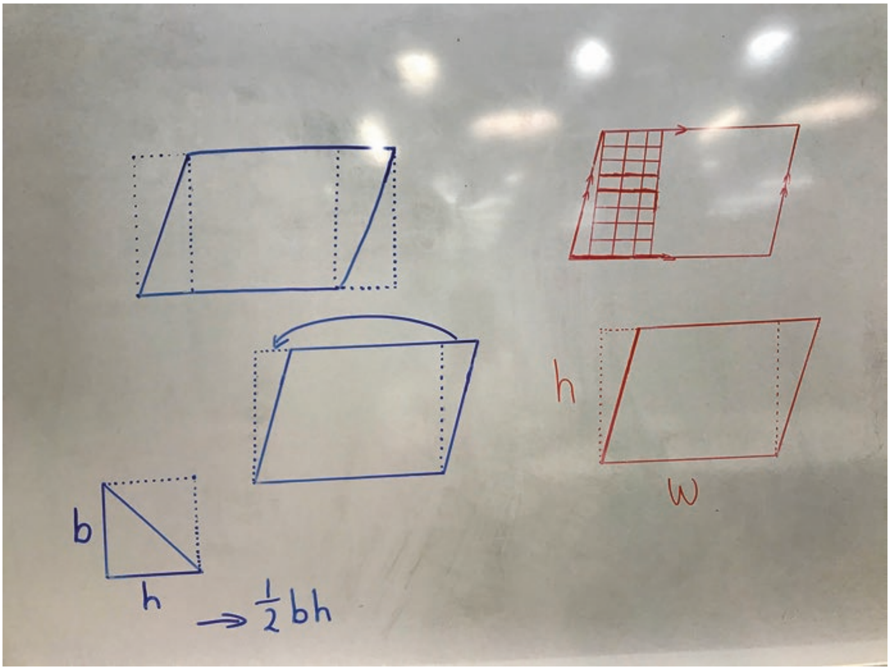

Fig. 3.6 Finding the area of a parallelogram

by height'. Some PSTs thought that getting the students to estimate what the area might be was good. Sarah mentioned Maths 300 lesson 44 (area of a triangle) provided a good hands-on approach with grid paper. The discussion included how to calculate the area of a parallelogram with understanding (see Fig. 3.6).

Finally, the PSTs used matchsticks to construct rectangles to compare their perimeters and areas. Sarah posed the following questions to prompt PSTs' thinking: Are the perimeters of the different rectangles the same? Why? Are the areas of the different rectangles the same? What is the smallest area? What is the largest area? There was also a discussion about whether a square was a rectangle, and some said no. Again, this was a situation in the lesson where the PSTs' foundation knowledge was being tested, with some saying it was a quadrilateral. The workshop concluded with PSTs completing their post-lesson reflections.

3.5.2.1 Post-lesson Data

The PSTs' post-lesson data showed that they valued the depth of discussion and choice of examples. Some PSTs mentioned the use of the open tasks and the 'talk moves' questioning, while others indicated the discussion helped with their foundation knowledge (e.g. 'being able to show the connections I made on the board about calculating the area of a parallelogram'; 'by providing explanation and working through step by step to get an understanding of the area of a triangle and time to explore with materials').

Several mentioned connections to primary school classroom practice through the use of photographs and work samples (e.g. 'using real examples of student learning from the classroom and Sarah backing this up with further practical tasks'). For others, it was clarification of the mathematics and terminology, such as $1cm^3$ is equal to 1 mL of water, and recognising misconceptions that they held. Some commented on the supportive environment in which they feel free to express their thinking while supporting each other (e.g. 'the way they scaffolded our learning by posing questions and challenging our thinking').

These comments were reinforced in the post-lesson PST focus interview when the PSTs were asked to comment on what helped their learning. They responded:

> There was a lot of discussion or a lot of time for discussion, and flexibility in the lesson. [Carol]

> I feel like there's a lot of hands-on activities always, especially in mathematics where there's always activities that we can do with hands-on materials. [Sam]

> And they are so practical and classroom ready, like Melissa comes in and shows us things she's already done which I think is really valuable because we know that it works. [Bob]

> There're a lot of opportunities where Sarah gets a few people up to show their examples on the board and so you see different people's thinking and how that was different from yours and how… I may have done it more complex and just how easy it could've been. [Katie]

In her post-lesson reflection Melissa observed that:

> Some pre-service teachers were challenged today as they were taught these concepts differently when they were in primary school. I hope they understood the importance of unpacking the concept with the students… building up their knowledge from the foundation.

> I thought the lesson was effective in that it highlighted the problems associated with learning the content based only on formulae.

> [What could we do differently?] Unpack the conceptual understandings with every activity would have been good. Showing them the sequence of learning involved.

3.5.2.2 Links to the Knowledge Quartet

There were applications of the KQ dimensions throughout the lessons. While most of these applications were evident in the descriptions of the lesson episodes, the post-lesson data were useful in terms of considering the effectiveness and/or impact of the elements.

3.5.2.3 Foundation

This category includes subject knowledge as well as beliefs about mathematics and mathematics pedagogy, which could be evident in both planning and teaching (Rowland et al., 2009). The data suggest that examples of evidence could be identified for each code of this category. For example, PSTs were allocated readings from the textbook *Helping Children Learn Mathematics* (Reys et al., 2012), and Sarah referred to the prescribed reading in both lessons observed (*adheres to textbook*). *Overt subject knowledge* was explicitly referred to when Sarah talked about algebraic thinking and how she felt more confident and prepared to teach proportional reasoning. This may have influenced her decision not to expand more on the growing patterns aspect of the lesson. Sarah's knowledge of subject knowledge and *use of terminology* was enacted in the first episode in relation to algebraic terms and in the second episode in the discussion about how to calculate and record the area of shapes. In much the same way as school teachers would identify *student misconceptions*, Sarah, as an MTE, also recognised that her PSTs may also hold algebraic, area/perimeter and volume/capacity misconceptions (*identifying errors*) and would also benefit from studying student misconceptions as demonstrated in the algebraic work samples shared. Previous experience with delivering the unit also meant that Sarah was able to anticipate that the PSTs were likely to believe that algebra was not relevant for young children and used her knowledge of curriculum to address this perception. This clearly had an impact on PSTs as evidenced by their post-lesson feedback. Sarah explicitly referred to particular practices and theoretical frameworks such as 'talk moves' and the KQ (*theoretical underpinnings*). Melissa was unfamiliar with these terms, indicating an example of MTE knowledge that was not part of a school teacher's knowledge.

It could also be expected that part of Sarah's foundation knowledge would include knowledge of the primary curriculum and actual primary teaching. Sarah was an experienced primary school teacher and maintained an active teaching role in primary school classrooms through modelling lessons and professional learning with teachers. In the lessons observed, however, and commented on by the PSTs, she often referred to Melissa when questions were asked about curriculum or implementation in the classroom. This seemed to create the impression that Melissa held greater expertise in this area and that Sarah's foundation knowledge in this area may be limited. We suspect, however, that this is more an indication of Sarah providing opportunity for Melissa to contribute, rather than an indication that Sarah lacked the foundation knowledge necessary to respond to PSTs' questions directly related to primary school classroom practice.

3.5.2.4 Transformation

When examining the work of trainee teachers, Rowland et al. (2009) looked for instances where teachers transformed what they knew in ways that made the knowledge accessible to students. This dimension included the codes *choice of examples*, *choice of representation* and *demonstration* and in the context of an MTE's work required Sarah to transform her knowledge of teaching mathematics in a way that developed her PSTs' understanding. As the post-lesson data showed, one of the representations and demonstrations that was emphasised as being particularly effective was the use of the balance beams to model equality. Sarah was able to transform her knowledge of common algebra misconceptions through the use of this model, the use of student work samples (e.g. incorrect interpretation of the equals sign) and interactive teaching techniques such as actively involving the PSTs in balancing equations and constructing growing patterns. The use of these techniques and others such as talk moves, sticky notes and table discussions were features of Sarah's practice. The use of the recording of diagrams on the board to work out the area of a parallelogram also illustrated a way to transform the PSTs' understanding. The post-lesson data showed that these techniques had a positive impact upon the PSTs as evidenced, for example, by Carol and Katie's comments.

Questioning and promoting classroom discussion was a critical aspect of Sarah's practice. Some of her questions served to challenge the PSTs' thinking, while others prompted them to think about their own future practice, facilitating connection. Questioning was not named as an element in the KQ but helped to demonstrate elements from the transformation and connection categories such as demonstrations and making connections between concepts and procedures.

3.5.2.5 Connection

Connection concerns the coherence of planning and teaching across an episode and includes the sequencing of topics of instruction within and between lessons (Rowland et al., 2009). Similar to the planning and sequencing of instruction that is undertaken by a school teacher, the MTE also has to make decisions about which tasks to use, in what order to present them and how to help her students (i.e. PSTs) make appropriate conceptual connections within the subject matter. When planning, for example, Sarah anticipated the complexity associated with understanding algebra and perimeter, area and volume. Decisions were made about sequencing that were particularly evident in the measurement episode where area and surface area were discussed before volume. During this same episode, Sarah also demonstrated the elements of making connections between procedures and between concepts in the discussion about area, calculation of area and its relationship with perimeter and volume.

A particularly important connection an MTE has to make that is not relevant to a school teacher is to link the 'theory' with the practice; the co-teaching arrangement facilitated the enactment of this as the PSTs had direct contact with an in-service practicing teacher. These connections were clearly made for the PSTs as evidenced in comments such as the following: 'I could see all the experiences being used in a classroom' (Pseudonym). Links were also made with previous lessons and experiences, including the consistent use of talk moves and reference to the KQ.

3.5.2.6 Contingency

This category concerns the teacher's response to unplanned or unexpected classroom events (Rowland et al., 2009). Although there were limited examples of deviating from the agenda, there was evidence of contingency moments, and at least one student gave feedback on Sarah's response to her ideas. Finding the surface area of a cube was one such instance. Sarah saw it as an opportunity to deepen the PSTs' foundation knowledge and to illustrate the importance of making connections to other mathematics concepts. Another example occurred when she referred to subitising in a discussion about patterning in the first lesson episode. It seemed that the co-teaching situation, along with the course structure of teaching a designated weekly topic, meant that contingency moments might not always be acted upon, even when they were recognised. Sarah later expressed in her interview that 'you can't take those teachable moments and go off on a tangent' as it may have resulted in not getting through the planned activities, which she was keen to do, particularly if Melissa had prepared them.

3.6 Conclusions and Implications

The work of the MTE is complex as it involves both the teaching of subject matter and appropriate pedagogical content knowledge. In addressing the first research question, it is evident that similar types of knowledge are required by MTEs and primary school teachers, such as knowing the appropriate representations and examples to use when teaching particular concepts or anticipating complexities and addressing student misconceptions. However, the MTE requires a deeper and broader understanding of the theoretical underpinnings behind the use of appropriate pedagogical practices (e.g. why we use MAB to teach place value) than arguably a school teacher needs to know. In addition, the MTE is required to justify the use of these practices to PSTs and to respond to their questions about them. Ongoing modelling of appropriate pedagogical practices, along with the accompanying commentary of why and how these practices can be used in a primary classroom, adds another layer to the knowledge MTEs require for teaching PSTs. In contrast, school teachers require a good understanding of each child's mathematics learning and how to respond to their needs on a daily basis and detailed knowledge of assessing and reporting student learning to parents, neither of which is an aspect of knowledge MTEs require. It could be argued that MTEs respond to PSTs' misconceptions; however, addressing specific needs within such a limited time frame of a semester is not possible. The knowledge of the co-teacher is invaluable in providing such specialised knowledge of planning, assessment and reporting practices. Both classroom teachers and MTEs require ongoing professional learning. In summary, while there are similarities in knowledge required by school teachers and MTEs, there are also differences as identified in this study.

As evident in the results and subsequent discussion, it was possible to identify elements from the KQ in the work of an MTE, indicating that it is transferable and applicable for describing MTE knowledge. As found by Muir et al. (2017), the framework proved useful in unpacking the complexity of the work of the MTE and in highlighting the somewhat subtle differences between being an MTE and a classroom teacher. Through providing examples of evidence from an MTE context, we have highlighted the similarities between classroom teaching and teaching teachers how to teach, acknowledging the 'meta-knowledge' (Beswick & Chapman, 2012) required by a teacher educator. Just as the KQ framework has been applied to interpret the work of classroom teachers, we can also see it being applied to interpret the work of teachers in a variety of contexts, including tertiary education. As Rowland (2013) intended, it provides a means of reflecting on teaching and teacher knowledge, with a view to developing both. Future studies could look at applying the KQ to other disciplines to further demonstrate its transferability.

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Chapter 4
The Research Mathematicians in the Classroom: How Their Practice Has Potential to Foster Student Horizon

Angeliki Mali, Georgia Petropoulou, Irene Biza, and Dave Hewitt

The requirements needed to become a mathematics teacher at secondary level vary in different countries and in different teacher education programmes. In some countries, mathematics teachers are required to hold a bachelor's degree in mathematics or other STEM fields (Cooper & Zaslavsky, 2017), while in other countries it is sufficient to pass a certain number of university-level mathematics courses. For example, in Greece, only mathematics graduates are eligible to teach mathematics at secondary level (i.e. grades 7 to 12); specifically, the majority of university mathematics students[1] aspire to become secondary mathematics teachers with no obligation for further training. In the UK, in contrast, the majority of mathematics graduates work in sectors other than education. Some of those who decide to become schoolteachers pursue a degree where teacher training is an integral part or register after graduation for a school- or university-based programme in teacher education.

Despite the differences, those undergraduates who are seeking to become secondary school mathematics teachers usually spend a significant part of their education in mathematics courses taught by mathematicians (Leikin, Zazkis & Meller,

[1] In this chapter, we use the term "students" to indicate undergraduate students who have the option to become mathematics teachers at school level and who are taught by research mathematicians.

A. Mali (✉)
University of Groningen, Faculty of Science and Engineering, Institute for Science Education and Communication, Groningen, The Netherlands
e-mail: a.mali@rug.nl

G. Petropoulou
National and Kapodistrian University of Athens, Athens, Greece

I. Biza
University of East Anglia, Norwich, UK

D. Hewitt
Loughborough University, Loughborough, UK

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_4

2017; Conference Board of the Mathematical Sciences, 2001). These research mathematicians educate school mathematics teachers, although they may not identify with the role of a teacher educator. Furthermore, the content of the mathematics courses taught at university is far beyond the content prospective mathematics teachers will teach at school, and the relevance of those courses to school teaching sometimes remains unclear to prospective teachers (Zazkis & Leikin, 2010). In some countries, it seems that there is a missing element between advanced mathematical knowledge, i.e. the "knowledge of the subject matter acquired during undergraduate studies at colleges or universities" (Zazkis & Leikin, 2010, p. 264), and the actual teaching practice when the prospective mathematics teachers are back in schools. Put another way, an advanced perspective on elementary and secondary mathematics that is applicable to teaching (Klein, 1908/1932) is obscure. We refer to this missing element between mathematics at university or college level and mathematics teaching at school as the "horizon", which we argue justifies this advanced perspective.

The metaphor of horizon is often used in literature in relation to strong content knowledge which is important for teaching school mathematics (Wasserman, 2016). In terms of the mathematical content, prospective mathematics teachers' horizon may include some elements of mathematical practices[2] and connections within mathematical content. Research mathematicians are experts in both, as they are experienced in using such practices and connections in their own research investigations. There is an increasing interest in research mathematicians' teaching practices at the tertiary level and the influence of their research practices on their teaching (Biza, Giraldo, Hochmuth, Khakbaz & Rasmussen, 2016). However, being experts does not necessarily imply they can share their expertise explicitly with their students (Alsina, 2001). We argue that in order for the mathematicians to initiate students into connections between mathematical areas and mathematical practices, they (the mathematicians) need to (a) be aware that their own mathematical research can influence the ways in which they teach mathematics at the undergraduate level and (b) be aware of the pedagogical potential of being explicit to students about the advanced mathematical practices[3] they use while working with the mathematical content. By being explicit, research mathematicians have the potential to foster students' own mathematical horizon because students are made aware of the mathematical connections and practices at the time they are used. So, students can witness from research mathematicians first-hand experience of how mathematics works in terms of practices, concepts, ideas and connections between them.

[2] We use mathematical practice to refer to the work of mathematicians or mathematics enthusiasts in terms of mathematical content (e.g. problem-solving, finding the least common denominator, using conventional notation), without taking into account considerations about teaching this content to students. As such, the mathematical practice is *not* considered here in the sense of publications by the National Governors Association Center for Best Practices (e.g. The Common Core State Standards for Mathematical Practice, 2010).

[3] Mathematicians use some mathematical practices in their own research investigations. We consider that those practices are advanced mathematical practices, which may influence their teaching.

In prior research, Mali and Petropoulou (2017) developed a new analytical framework of undergraduate mathematics teaching, which unpacks research mathematicians' teaching into four thematically connected categories—selecting, evaluating, explaining and extending—with detailed practices and tools within each. Of relevance to this chapter is the category extending, which has been operationalised as practices and tools used to support students' initiation into advanced mathematical practices such as proving. So, extending practices include mathematical content and teacher considerations for the students who learn the mathematics. The term extending practices originates from research in school mathematics education, referring to teaching practices such as generalising (e.g. Fraivilig, Murphy, & Fuson, 1999, and later, Cengiz, Kline, & Grant, 2011) that extend student mathematical thinking. In university contexts, Mali and Petropoulou (2017) found that extending practices include interpreting a task, asking a challenging question, transforming a mathematical representation, formulating a conjecture, generalising, justifying and providing mathematical heuristics, with associated teaching tools such as graphical representations, challenging questions and different types of proofs. In the same paper, Mali and Petropoulou distinguished aspects of advanced mathematical practices within extending practices, such as analysing, synthesising and providing a counterexample to refute an invalid claim. The very nature of such advanced mathematical practices is rooted in a mathematician's own mathematical research which deals with the discovery of connections between mathematical areas. Prospective mathematics teachers' deeper insights into and reflection on such advanced mathematical practices and the connections between mathematical areas have the potential to foster their horizon (Figueiras et al., 2011). But we think that such "fostering" could only take place if research mathematicians are aware of the potential of bringing advanced mathematical practices from their research into their teaching in the first place and of certain extending practices that they can teach prospective teachers in order to contribute to the improvement of these teachers' mathematics teaching.

Enacting extending practices that create mathematical connections and enhance deeper reflections on the mathematical practices requires more than knowledge of advanced mathematical practices. It requires being explicit about the reason for which a piece of mathematics is important (Frymier, 2002) or how a specific mathematical practice works. Jaworski, Treffert-Thomas and Bartsch (2009) and Thomas (2012) use the terminology of meta-comments about mathematics to refer to the statements a research mathematician makes to explain why a piece of mathematics is important or where and how it is useful. While commenting happens when a research mathematician uses her/his own words to explain a definition or a theorem, meta-commenting is at a level beyond commenting; meta-comments highlight explicit connections between mathematical content and relevant mathematical practices. A meta-comment could be, for example, a comment on the role of a definition or a theorem in mathematical theory and their connections to other areas of mathematics, thereby helping students recognise the underpinning principle behind the mathematical theory and practices. Those explicit connections also have the potential to foster students' horizon. Fostering the horizon of students is important as it increases the possibility of learning opportunities for their future pupils.

In this chapter, we offer insights into the nature of research mathematicians' extending practices that can foster students' mathematical horizon. In what follows, first, we offer a brief review of studies on the role that undergraduate mathematics education may play in the teaching profession and on research mathematicians' teaching practices when they educate teachers. Then, we offer a categorisation of extending practices related to research mathematicians' own mathematical research. Reflecting on the issue of explicitness in teaching, we offer research mathematicians' meta-comments on mathematics or associated mathematical practices that they use while they teach. We conclude with a discussion on potential implications for teacher education programmes.

4.1 Undergraduate Studies in Mathematics and the Teaching Profession: Teachers' Mathematical Horizon

The connection between the secondary and tertiary levels in mathematics education has been discussed in the literature in the context of the transition from one level to the other. Many studies investigate the transition from school to university, especially in relation to the differences between mathematical content, practices and institutional differences (e.g. Gueudet, 2008). Other studies discuss the transition of prospective teachers from their mathematics undergraduate studies to a school teaching profession (e.g. Winsløw, 2014). Although there is the assumption that a good mathematical background is necessary for high-quality teaching, research has reported disconnection between teachers' undergraduate studies in mathematics and what they use in their teaching profession (Even, 2011). Specifically, schoolteachers identify some aspects in which their undergraduate or college studies have influenced their teaching such as knowledge on specific topics, problem-solving and use of mathematics in other disciplines (e.g. Adler et al., 2014; Even, 2011; Zazkis & Leikin, 2010). Also, teachers declared the contribution these studies made to their confidence in, and their understanding of, pupils' difficulties through their own experience as students (e.g. Barton & Paterson, 2013; Even, 2011; Zazkis & Leikin, 2010). Although in some studies there are clear testimonials from teachers on how their engagement with the mathematical content contributed to the quality of their teaching (e.g. Barton & Paterson, 2013), in other studies this connection is not evident, and teachers express difficulty identifying how this content is used in their teaching (Zazkis & Leikin, 2010). It would appear that there are diverse views on how undergraduate and collegial courses influence teachers. We are interested in contributing to this area of research by investigating how research mathematicians' teaching practices have potential to assist prospective teachers towards their preparation for the classroom.

The mathematical background teachers need for their work in the classroom has been seen in parts of the literature as necessary for teaching knowledge, for example, subject-matter or pedagogical content knowledge (Shulman, 1986, 1987) or mathematical knowledge for teaching (Ball, Thames & Phelps, 2008). Other studies identify conditions and constraints that impinge on teaching and are beyond the control of the individual teacher, strongly related to the university institutions. Institutional differences may explain teachers' challenges in their transition from university to school (Winsløw, 2014). Other researchers focus on teachers' discursive practices as an inseparable part of their background (e.g. Cooper, 2014; Cooper & Karsenty, 2016). We consider that the background teachers bring especially from their undergraduate studies is important and valuable to their teaching profession. For this reason, we drew on the horizon knowledge (HK) theoretical construct of Ball et al. (2008); HK is a useful component of mathematical content knowledge for teaching that brings together mathematical and curricular content. In our perspective, we do not separate teachers' practice from their knowledge and the social and institutional environment in which they act. In fact, in our view, this knowledge is framed in this practice as it is shaped in teachers' teaching of mathematics either at the school or university level and is also shaped in teachers' work on mathematics.

Horizon content knowledge is a term introduced by Ball et al. (2008) to describe a dimension of mathematical knowledge for teaching related to the intended student horizon of mathematics: "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). Zazkis and Mamolo (2011) reconceptualised Ball and colleagues' idea of student horizon, positing teacher horizon as the place where advanced mathematical knowledge of a teacher appears to meet mathematical knowledge reflected in school mathematical content. Figueiras, Ribeiro, Carrillo, Fernandez and Deulofeu (2011) developed the idea of teacher horizon further; they emphasised that the practice of teaching mathematics to primary and secondary students, as well as to future elementary and secondary schoolteachers, requires not only horizon content knowledge but also a deeper reflection on the connections between mathematical concepts and ideas. We see teacher horizon as a unifying concept between university and school mathematics that includes elements of mathematical awareness beyond the content to be taught at school, elements of practice (mathematical and teaching) and, importantly, elements of reflection on both. This chapter explores how teaching could foster the horizon of university students by helping them to become more reflective teachers and create better learning opportunities for their future pupils. Next, we address how this horizon may be developed at university.

4.2 Research Mathematicians' Teaching Practices that Have Potential Implications on Teacher Education Programmes

In terms of research mathematicians' teaching practices in teacher education programmes, there are a few studies which investigate these practices, especially in juxtaposition to those of researchers in mathematics education. Research mathematicians and mathematics teacher educators differ in their approach to the education of pre-service mathematics teachers (Cooper & Karsenty, 2016; Cooper & Zaslavsky, 2017; Leikin, Zazkis & Meller, 2017). Leikin, Zazkis and Meller (2017) interviewed four research mathematicians on their views on mathematics necessary for schoolteachers. Those mathematicians taught courses in which prospective schoolteachers learn mathematics. The authors named them "teacher educators de facto" (Leikin, Zazkis & Meller, 2017, p. 452), suggesting that the mathematicians did not explicitly identify themselves as educators. The four mathematicians' focus was on "the development of logical thinking and rigor of language", and they saw the connection between university and school mathematics in terms of "enrichment and extracurricular support for strong students" (p. 471). In a different study, Cooper and Zaslavsky (2017) studied the co-teaching of a mathematician and a mathematics educator in an undergraduate course for prospective mathematics teachers. The mathematician brought the experience of how mathematics is taught and practised at university level, while the mathematics educator focused on how mathematics is taught at school level. The study highlights the affordances of such co-teaching, especially in bridging university to school teaching.

In terms of research mathematicians' research practices and their potential implications for teacher education programmes, Misfeldt and Johansen (2015) drew on Burton's (2004) work on professional mathematicians' thinking, learning and research practices. They interviewed mathematicians regarding their approaches to selecting mathematical problems for their research work. Their findings identified a variety of criteria that affect mathematicians' choices including "personal interest, continuity with previous work, the danger of getting stuck and how fellow mathematicians will respond to the findings" (Misfeldt & Johansen, 2015, p. 357). The authors see potentialities in using these criteria in facilitating (and getting more insight into) students' engagement with problem-solving. Furthermore, Lockwood, Ellis and Lynch (2016) conducted a study on how mathematicians choose and use examples to prove a conjecture through a survey of 220 mathematicians, 19 of whom were invited to participate in a proof-based interview. The analysis identified key traits in mathematicians' choice and use of examples. The study used these traits in the formation of pedagogical suggestions, such as the following: "[d]o not discourage or denigrate example use", although examples do not constitute proof; "[e]ncourage and foster deliberate awareness and discussion of example use"; "[e]xplicitly highlight example use in proving"; and learn from mathematicians' back-and-forth, non-linear, full-of-struggles proof activity and value the mathematical investigation (Lockwood et al., 2016, pp. 193–194). Finally, recent studies have raised the question of how and to what extent mathematicians' research backgrounds influence their extending practices (Mali, 2015, 2016; Petropoulou, Jaworski, Potari & Zachariades, 2015; Petropoulou, 2018). These studies posit that teacher education can learn from mathematicians' practices and is influenced in their teaching by the mathematicians' research. In our work, we take this point further by valuing the role of research mathematicians in teacher education programmes, for example, when they are explicit with students about their advanced mathematical practices and meta-comment on these practices.

4.3 Research Mathematicians' Teaching Practices with the Potential to Foster Students' Horizon

4.3.1 Methodology and Settings

In this chapter, we exemplify the nature of research mathematicians' practices for fostering students' horizon by drawing on the data of two studies conducted in the UK (Mali, 2016) and Greece (Petropoulou, 2018). In both studies, interviews with research mathematicians about their underlying considerations and thinking about their teaching practices were conducted, and their actual teaching of first-year calculus was observed. The interviews offered insights into research mathematicians' views on teaching undergraduate mathematics. Observations shed light on mathematicians' verbal communication about the use of mathematical practices in the form of meta-comments (Jaworski et al., 2009) addressed to students.

The Petropoulou (2018) study conducted observations of first-year lectures (45 lectures lasting an hour each) and interviews with 4 research mathematicians who lectured (24 interviews lasting an hour each) in the observed sessions. Fieldwork took place for over four academic semesters. The lectures were 2 large cohorts of 100 to 250 students. The interviews followed up each observed lecture. The Mali (2016) study conducted observations of first-year small-group tutorials (85 tutorials lasting an hour each) and follow-up interviews with 26 research mathematicians (55 interviews lasting between 10 and 90 minutes) who taught students in those tutorials. Tutorials included a small group of two to eight students. The interviews were one-to-one with the research mathematicians. Fieldwork lasted for over three academic semesters.

In both studies, the research mathematicians had a PhD in mathematics (mostly in real analysis but also in other domains such as geometry), and they were experienced in both research and teaching. The tutors had been teaching for more than 8 years and were lecturers for more than 20 years. A few tutors and lecturers (including a lecturer identified in this chapter as lecturer L2) conducted research in mathematics education. Tutors in the UK (especially those who were appointed to their faculty positions in recent years) had attended professional development courses in teaching usually at a Master's level. The specialisation of those courses was not necessarily on the teaching of mathematics. Attending such professional development courses was not mandatory for lecturers in Greece, who usually taught without further teaching qualification.

The data on which we draw for this chapter come from five research mathematicians—three in mathematics departments at Greek universities and two in a UK mathematics department—who volunteered for observations and interviews. We searched for common teaching practices across the two educational systems that have the potential to foster students' mathematical horizon. We found that these practices were extending in nature; for example, they intended to extend students' mathematical thinking to see connections between mathematical areas. We explored the nature of research mathematicians' extending practices first by understanding them and then by reflecting on how they were enacted in the university classroom.

The first two authors identified evidence of extending teaching practices that have the potential to foster student horizon—in interview data from Greece and the UK—and evidence of meta-commenting on those practices in observations. The excerpts of data we present are paradigmatic events selected as examples of a large corpus of extending teaching practices identified.

4.3.2 Teaching Work on Fostering Student Horizon

We start with an excerpt in which one of the research mathematicians offered her view on the relationship between mathematical research and teaching. With this excerpt, we seek to set up a discussion about the horizon of the span of mathematics and the teaching work on fostering this student horizon. In what follows, we label research mathematicians who teach in lectures at Greek institutions as Lecturers (L) and research mathematicians who teach in tutorials at the UK institution as Tutors (T). L1 explained that:

> For someone who teaches [mathematics] in the first year [of the programme], the research she has done in mathematics offers the direction, [in other words] the overall picture of the mathematics subject-matter: what is important and what is secondary, tertiary, and how you pass that to students. Each [mathematician] has a certain way to see mathematics. But that [way] can differ from one researcher to another.
>
> (Winter term, Year 2)

In the above excerpt, the lecturer described the horizon although not naming it as such; the horizon relates to the "overall picture" of the mathematics subject matter and the "ways" of seeing mathematics. An integral part of the overall picture is "what is important" and also, for this lecturer, "how you pass that to students". We think that research mathematicians' awareness that their own "overall picture of the mathematics subject matter" should be explicit to their students is key to their work on fostering students' horizon. In the following sections, we exemplify how such awareness is substantiated by meta-commenting in university classrooms, and we offer an account of extending practices which facilitate students' initiation into advanced mathematical practices. In particular, we found four categories of extending practices rooted in advanced mathematical practices that have the potential to foster students' horizon. These are drawing on examples, connecting mathematical areas, visualising and simplifying. We explore each of these four advanced mathematical practices, in turn, below.

4.4 Drawing on Examples

Drawing on specific examples to distil their essential characteristics forms the basis for the production of new mathematical knowledge for the research mathematicians. We found that drawing on examples is an important heuristic that research mathematicians had experienced in their research and brought to their mathematics teaching by commenting on its use.

In a tutorial, students did not know how to determine in a mathematical problem whether a composite function was injective. T1 asked the students to give her examples of functions which are injective in order to initiate a discussion with them about the comparison of characteristics that make those functions injective. At the end of the work of tutor and of students on the problem, T1 offered students the following meta-comment on using examples:

> [i]n mathematics, we do rigorously define things. And so, it's a situation where—from the set of examples that we have—we've come up with an ideal idea, and then we can actually rigorously then check that something is in that or not. And what happens in mathematics is that if we see a more general version of things that doesn't fit that, but that still has some things in common with it, then we create a new definition that's more general. We come up with new definitions any time we recognise that there are some sets of structures that have some relevance. But it really does emerge out of the examples. And if you look at the history of mathematics, it's not that people have had the idea of a function. It's that they've had lots of examples of functions and they've tried to distil what the critical characteristics of a function are. Does that make sense? So, I think it's a very natural way to think about the relationship between examples and theories – it's that we don't define definitions just off the tops of our heads. We define them because they capture a behaviour we see in examples that have interesting kinds of properties. (T1, Spring term, Year 1)

T1 used a set of examples of functions—polynomial, trigonometric and logarithmic—in order to elicit the definition of injectivity from her students. She then meta-commented with the above excerpt on her practice of drawing upon those examples. In particular, she referred to the history of mathematics to inform students about her view on the nature of mathematics. For her, mathematicians explore sets of relevant structures/properties distilled out of examples to extract a consensus of a mathematical object; that consensus forms a definition or a property of the mathematical object. At a later stage, that definition can be adapted to better describe the ideal object. Her meta-comment has the potential to foster students' horizon of how mathematicians develop mathematics, what the nature of mathematics is and how examples can be used to explore definitions of mathematical objects. After informing students about the benefit of drawing upon examples, this tutor stressed the confirmative role of a specific example as an instantiation of a general case which nevertheless is different from proving the general case. This is important for first-year students as many of them are unclear whether a confirming example constitutes a valid proof.

In another example from lecture observations, L2 used counterexamples to prove that the inverse of the theorem "if a sequence (x_n) converges to a real number x then the sequence $(|x_n|)$ converges to $|x|$" is not valid while the students struggled to prove by contradiction. We recorded him commenting to students: "you want to prove that something does not hold and not that it never holds". He asked them "how can we do this?" A student responded: "by giving a counterexample", and the lecturer meta-commented: "in general, if we want to prove that something does not hold, we find a counterexample". In interview with this lecturer, L2, he said:

> [i]n mathematics the generation of counterexamples is a common practice. You often make claims and get a sense about their validity through specific examples. When you suppose that a claim does not hold, you try to generate a counterexample and then you examine the critical characteristics of the counterexample that make the claim invalid. This could lead to the formulation of a new valid claim. However, in university teaching this process of thinking is not usually made explicit to the students. In the past, sometimes I followed this process in my teaching without being aware of students' difficulties to understand it. Now, I consider more what the students think and I am more conscious about what I need to emphasise. (L2, Winter term, Year 1)

L2 uses counterexamples, both in his mathematical research and teaching. He has experienced the mathematical practice of drawing upon specific counterexamples both for refuting invalid claims and for formulating "new valid" ones. So, he is aware of both possible uses. He is also aware that drawing on specific counterexamples is a practice that could be emphasised in teaching. Emphasising the use of a counterexample as a proof that a claim is not valid, in comparison with a proof by contradiction, has the potential to extend students' horizon with regard to ways of proving.

4.5 Connecting Mathematical Areas

Connections, mainly within mathematical areas, are at the core of mathematical discovery. Production of new mathematical knowledge is hardly advanced without connecting mathematical areas. Some research mathematicians speak about connections as an essential part of mathematics; such a central practice to mathematics can be explicitly highlighted to students in teaching. The following example shows that being explicit about the connections between mathematical areas has the potential to extend student horizon in a way that is relevant to their future teaching.

In a lecture, L1 challenged students to think of a reason why the number 0.3 recurring is a rational number (i.e. 1/3). The students knew from school how to prove that 0.333… = 1/3; they would set 0.333… = x and multiply by 10. However, in setting x = 0.333…, there is an implicit assumption that the number 0.333… is a rational number although decimals are infinite. Also, there is a problem in multiplying the infinite decimal 0.333… by 10 and proceeding as though it is finite (multiplying by 10 moves every digit one place to the left but this is implied in numbers with finite decimals). The lecturer made students aware of the inner reason why the number 0.333… is exactly equal to 1/3 with a meta-comment which explicitly connected the number 0.333… with the geometric series 3/10 + 3/100 + 3/1000 + … that converges to 1/3. So, he wrote the number 0.333… in the decimal system, 0.333 = 3/10 + 3/100 + 3/1000 + …, and showed that the infinite sum 3/10 + 3/100 + 3/1000 + … is a geometric series that converges to 1/3. By connecting 0.333… and the geometric series that converges, the lecturer argued why 0.333… is a rational number, thereby "extending" students' mathematical thinking from how the number 0.333… is written as the fraction 1/3 to why that number, despite having infinite decimal representation, is exactly equal to 1/3. In the interview after the lecture, the lecturer was articulate regarding his practice: "Students need to learn how to think mathematically" and "mathematics is nothing more than connections". The practice of connecting these different mathematical areas—rational numbers and series—made students aware of the mathematical reasoning for the equality 0.333… = 1/3, potentially extending their horizon with regard to different notations of the same numbers (e.g. 0.999… = 1).

4.6 Visualising

Visualising as a means to grasp abstract concepts is a common practice among research mathematicians. For those who produce mathematics, drawing graphs, diagrams and figures is a central heuristic. Visualising is also a common teaching practice especially at school level (and can be found in most textbooks with formal proofs) as it helps students understand the meaning of a theorem (e.g. the Bolzano theorem). In the next examples, research mathematicians are aware of the potential of using visualisation in their teaching.

L1 used a lot of graphs in his mathematics teaching. This research mathematician talked explicitly to his students about the potential of graphs as a way to work with the mathematics by offering them meta-comments on the graphs he drew. For example, in an introductory session on definite integrals, to answer the question of "how the series of the values of *f* is different from the integral of *f*" for a decreasing, non-negative function *f* which is defined in the interval [1, +∞), L1 introduced the relation:

$$\sum_{n}^{k=1} f(k) \geq \int_{n}^{1} f(t)\,dt$$

He subsequently drew the graph depicted in Fig. 4.1 on the board:

In the graph in Fig. 4.1, L1 represented the left and the right part of the relation for an appropriate function *f*. Then he asked students "how did we say that we comprehend a theorem?" and meta-commented "by drawing a graph!" In an interview with him about his extending practice of drawing graphs for students, he said:

> [w]hen I think of something in mathematics, I see mathematics as graphs which I later translate into something else. In teaching, I try to also draw graphs. So that the student remembers what I say… I draw graphs [for the student] to remember the graphs per se. The way I work [with the mathematics], even the order with which I talk [about the mathematics], is in order for me to draw graphs. (L1, Winter term, Year 2)

From this excerpt it seems to us that, for this lecturer, the horizon includes seeing mathematics as graphs. His view of teaching mathematics includes drawing graphs to help students remember the mathematics of the lecture. He revealed that he builds the whole design of his lectures around drawing graphs. In particular, with his graph for the relation

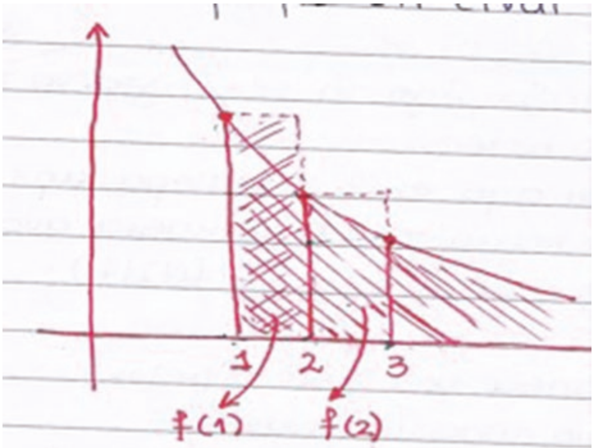

Fig. 4.1 Reproduction of L1's graph that demonstrates differences between series and definite integral (as recorded in the field notes from the lecture)

$$\sum_{n}^{k=1} f(k) \geq \int_{n}^{1} f(t)\,dt$$

he taught students a way to comprehend (and to remember) a (complex) relation. His meta-comment on drawing graphs indicates visualisation as the broad role of this mathematical practice which can be applied to any level of mathematics.

T2 sees the heuristic of drawing graphs as geometrically oriented and rooted in research practices. For example, a task in a tutorial was to re-write algebraically the expression $||x| - 1|$ without modulus signs. The tutor suggested to his students: "we can just sketch the graph of the function". In other words, his suggestion was to first think about the expression geometrically. In the follow-up interview, T2 responded to the question of why he chose a graphical solution when some mathematicians avoid choosing them with the following:

> [i]t depends on your research area. If you are a Geometer [he is a Geometer], you are happy with geometric solutions; it depends on your background I think. … You see to me it is easier to see the graph. …For instance, if you are a programmer writing computer programs, then it is more convenient to you to give an algorithm. (T2, Winter term, Year 2)

The tutor traced back to his research area and his geometric view of mathematics. Thus, for him the horizon also includes seeing mathematics geometrically as graphs. He said he does so because his "convenience" (meaning expertise) is in geometry. We interpret this as consistent with the following lecturer's view. L3 sees visualising as geometrically oriented and rooted in research practices which her teaching resembles. She stated in interviews:

> [w]hen I have a [mathematics] problem, I always look at it geometrically. The same is in research. I first look at it [i.e. a mathematics problem] geometrically, because my convenience is there… Next, I look at it from all other perspectives others looked at it, and then…you write those all others wrote in your own way. This is experience… You don't copy the other person's work. You write her central idea, but not in the way she [the other person] wrote it. In the way you made sense of it. In the way you interpreted it, OK? You do that also in teaching. (L3, Winter term, Year 2)

Although L3 does not offer a clear account of how students might use visualisation, there is a clear sense of the journey a teacher might make from approaching mathematical ideas to devising ways to present them to students. It seems to us that the horizon for this lecturer includes seeing mathematics geometrically as graphs. Her creative approach to proofs includes a clear personal structure: drawing graphs, using all other perspectives research mathematicians have used before and re-writing what has been written by others in her own geometric way. Her view on teaching mathematics, as observed in her classroom, also includes re-writing what is written in textbooks and other resources in her own way by drawing graphs.

4.7 Simplifying

Simplifying relates to reducing the complexity of a mathematical problem with the use of an analogous easier problem. It includes discerning a simple case which preserves the essential mathematical characteristics of the initial complex problem while removing irrelevant details. So, it requires a sense of what is essential and what can be replaced in a particular mathematical problem.

Simplifying is a heuristic used in both research in mathematics and teaching. For example, L1 referred to simplifying as a heuristic he brings from his research in mathematics and as a practice he consciously adopts in his teaching. He made students aware of the potential this practice has for solving complex problems: when students have difficulties in tackling a mathematical problem, they should know that simplifying is an appropriate way to address the problem. This lecturer tried to make simplifying relevant to his students by eliciting questions and appropriately selected examples. For example, in a lecture about series and convergence, L1 challenged students to "guess" when a series may converge (when the sequence inside the series tends to zero) by comparing two specific examples, the geometrical series $\sum_{+\infty}^{k=0} x^k = \frac{1}{1-x}, |x| < 1$ and the divergent series $\sum_{+\infty}^{k=1} (-1)^k$. Because the students focused on irrelevant details (the fact that "the first of these series had a variable in it"), L1 simplified the example of geometrical series to the series $\sum_{+\infty}^{k=1} \frac{(-1)^k}{10^k}$ so as "the two series resemble". In interviews, L1 described how he uses simplifying in his teaching:

> I do something, which in fact is research philosophy. What I have learned very well even when I do research is, I have a problem, I am continuously simplifying. [This is what] I [also] do in the classroom. When I teach any difficult concept [in mathematics], I discern a simple [case] and highlight it. Next, I build on it and make it complex. But I build on the very simple…so that what I do with the simplest problem is clear to everybody—by asking questions I sometimes try to make others figure it out, say it —next I slowly build towards the real problem I have. The [mathematical] examples are selected in such a way that drives me from what I consider as a level of understanding to the next one. In this way, I produce results for everything [i.e., research and teaching]. (L1, Summer term, Year 2)

This lecturer stressed his view that simplifying is a practice he uses both in teaching and research. In his teaching, he tries to involve students in simplifying by asking questions and by selecting appropriate problems.[4] Using such a practice in an explicit way through meta-commenting has the potential to foster prospective teachers' horizon in terms of first using analogous easier problems than those that should be solved in their own classrooms.

4.7.1 In a Nutshell

The participating research mathematicians had views of teaching undergraduate mathematics which included an awareness that their research informs their teaching. In particular, their extending practices are influenced by their own mathematical research practices, which include using examples (e.g. with a goal to identify commonalities and differences between examples), connecting mathematical areas (e.g. numbers and series), visualising (e.g. drawing graphs) and simplifying (e.g. using a less complex problem). The set of extending practices each research mathematician uses can produce a unique portrait of teaching in the classroom with different ways of seeing mathematics—geometrically, formalistically, etc.

We consider that the nature of research mathematicians' teaching practices that extend students' horizon is concerned with awareness of their use of certain mathematical practices coming from their own research along with explicitness to students about the use of those practices through meta-commenting (Jaworski et al., 2009). Students, who aspire to become teachers, have the opportunity to join, and work with, mathematics in undergraduate mathematics courses. Those opportunities form the basis for their future teaching in terms of their mathematical ways of working and the development of their own philosophy in furthering their future pupils' understandings (Jaworski, Mali & Petropoulou, 2017). We consider that research mathematicians' views on, and explicitness to students about, certain mathematical practices used constitute a type of awareness crucial for the development of prospective teachers' horizon. In the next section, we discuss how the students of the research mathematicians featured in this study, among whom are prospective mathematics teachers, could potentially carry on in their school classrooms the mathematical practices they have been taught.

[4] Polya (1971) introduced the "simpler analogous problem": a problem which includes certain relations of its respective parts with the original problem but is less complicated than the original problem. L1 selected a simple enough problem that had the potential to be clear to students; so, from a range of simpler analogous problems he selected a particular case of sequence with specific numbers he thought had potential to educate his students because of its less complicated mathematical nature.

4.8 Implications for Mathematics Teacher Education

There is a clear sense that the research mathematicians who participated in these two studies have key mathematical expertise which not only informs the way in which they go about conducting research in mathematics but also affects the way in which they carry out their practice of teaching undergraduates.

At a mathematics research level, the consideration of examples can bring an awareness of certain common characteristics. A new set of "objects" may come to light through a new definition which might classify otherwise disparate items into one concept. Lakatos (1976) offers a historical perspective of the changing definitions that emerged through efforts to prove that

$$V + F = E + 2$$

which came from people considering particular examples and counter-examples. At the teaching undergraduate level, we highlighted how two research mathematicians discussed the way in which the use of examples is central to the way in which they teach undergraduates. At the school level, examples are just as important. In the UK, for example, topics are commonly introduced not with a definition but with particular examples of the general case which teachers want school students to come to know. The addition of fractions, for example, is rarely introduced as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Instead it is introduced through examples such as

$$\frac{3}{8} + \frac{2}{8} \text{ and, later on, } \frac{1}{5} + \frac{3}{4} \text{ and } \frac{7}{12} + \frac{4}{9}$$

The generality is created through the collection of particular examples, with the choice of these examples being important (Rowland, 2008). In the Greek curricular context, proof and definitions are introduced in school mathematics teaching implicitly from grade 7 and explicitly from grade 10. Connections of examples (inductive reasoning) and proofs (deductive reasoning) are essential areas that challenge teachers and students.

At undergraduate level we have offered the example of connecting a fraction 1/3 with a geometric series, and this offered the opportunity to justify 0.333… as a rational number. At school level there is a danger that mathematics can be seen by school students as a collection of unrelated curriculum items. Askew et al. (1997) showed that effective teachers at primary school level were those who made connections between different areas of mathematics. For example, the rate of change within a linear graph can be seen as connected with work on ratio.

Mathematics can involve complex ideas. The ways in which these ideas can be grasped and understood very often involve the use of graphs. Feynman diagrams capture the essence of the mathematics involved with the behaviour of sub-atomic particles. We offered an example of a research mathematician who saw mathematics in terms of graphs which then got translated into “something else”. Graphs are a tool for conceptualising mathematics and can also be a way in which undergraduates come to initially meet mathematical ideas, such as the way in which $||x| - 1|$ might be viewed (Fig. 4.2).

At school level graphs can play a crucial role particularly with school students whose first language is not the language of instruction. Graphs can assist such students gaining a sense of what is being discussed even if they may have difficulty in understanding what is being said. This is an increasingly important consideration, given the significant number of refugees entering the school system and the increased flow of people from one country to another.

We showed a research mathematician who talked about always looking to simplifying when carrying out his research. At the university level, he talked about using a simple case of what might be a difficult mathematical concept. Simplifying as a way of working and simplifying as a teaching practice operate just as much at the school level as at the university level. As a way of working, simplifying is an important strategy when faced with problems which others feel too difficult. For example, a familiar problem in some UK classrooms is the question of how many squares there are on a chessboard. After realising that there are different sized squares present, the task can feel overwhelming for some students. So, simplifying the situation to, say, a 3 by 3 square rather than the 8 by 8 chessboard can allow

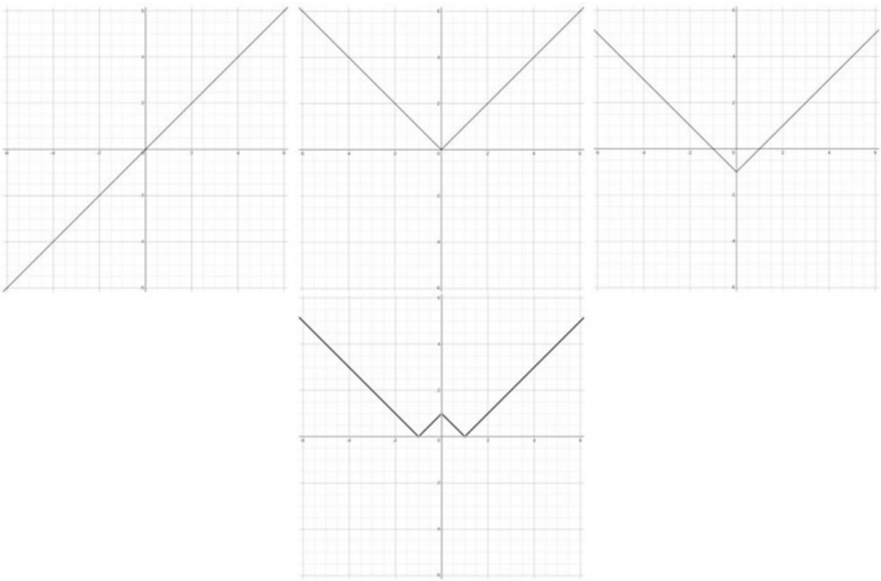

Fig. 4.2 Drawings to help give a sense what $||x| - 1|$ might look like as a graph and how it could be re-written

students to be successful at counting and beginning to see a more structured approach which they can later bring to the original 8 by 8 situation and even use to generalise about any number of squares (n by n) or rectangles (n by m).

As a teaching practice, the notion of initially simplifying situations in order to make an otherwise complex mathematical idea more accessible is a crucial aspect of a teacher's toolbox no matter what level they teach. This can affect the order in which certain topics are addressed. For example, the topic of area might consider rectangles first, then parallelograms and triangles before considering circles and other polygons.

Figure 4.3 summarises these key practices (use of examples, connecting mathematical areas, visualising and simplifying) operating at the different levels of mathematicians' research work: undergraduate teaching and school teaching.

The final arrow (dotted) on the right-hand side is crucial with regard to the influences which affect future teachers. In the UK there is a separate level where undergraduate students complete a teacher training course before starting as qualified teachers in schools. Thus, the picture is more like Fig. 4.4.

In countries where there is a teacher educator level, the courses focus primarily on teaching and learning issues. As such the students' experiences at university level still have a role to play in developing their mathematics horizon. In countries without the teacher educator level, what students take from their undergraduate course becomes more significant, and thus the role taken by the research mathematicians is crucial in shaping their students' future mathematics horizons.

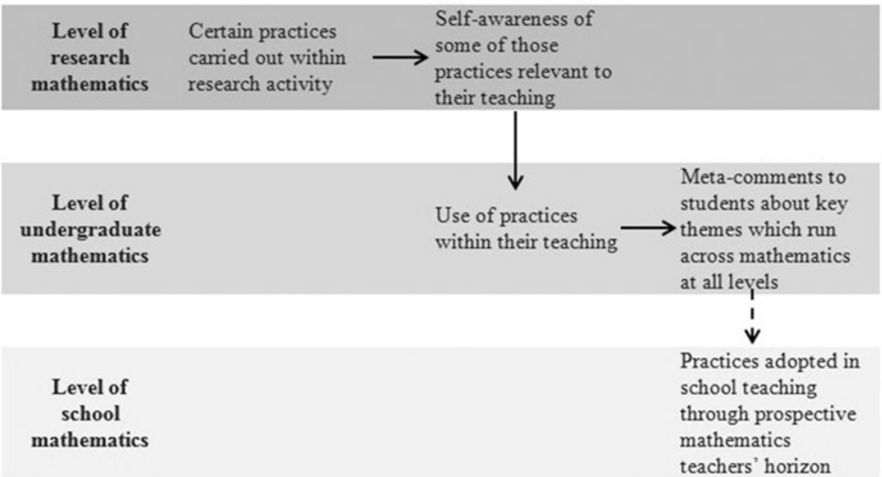

Fig. 4.3 Identified practices operating at different levels

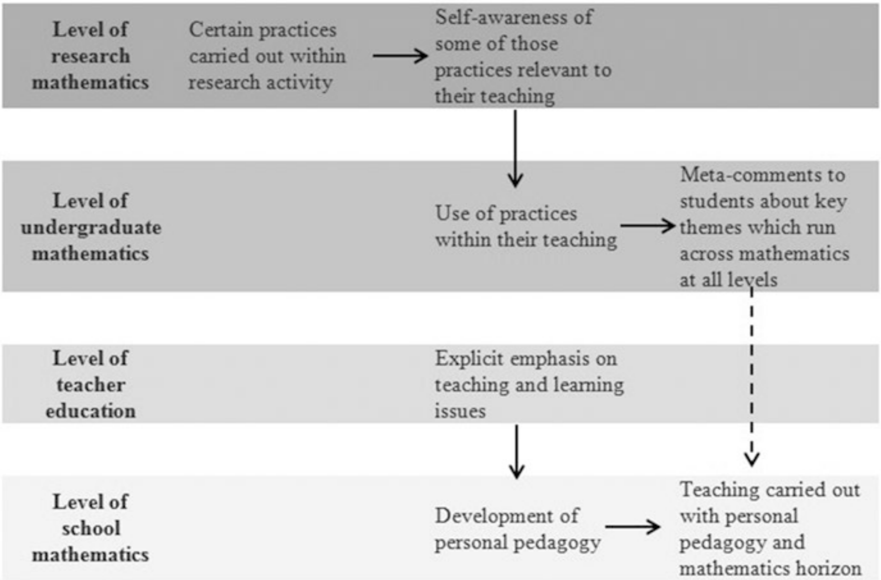

Fig. 4.4 The addition of a teacher educator level

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Chapter 5
Pedagogical Tasks Toward Extending Mathematical Knowledge: Notes on the Work of Teacher Educators

Rina Zazkis and Ofer Marmur

5.1 Introduction

The work of mathematics teacher educators is multifaceted and relies on extensive knowledge in various related areas. This includes knowledge in mathematics, pedagogy, didactics, school curriculum, assessment methods, education research literature, research methodologies, and different theories of learning and teaching (Jaworski, 2008).

Our focus in this chapter is on teacher educators' work in professional development courses for secondary school mathematics teachers. On the one hand, our explicit goal in these courses is to extend the teachers' *mathematical* knowledge. On the other hand, as mathematics teacher educators, we are expected by teachers to address pedagogical issues associated with the *teaching* of mathematics. Consequently, to balance existing expectations of teachers with our goal of extending and strengthening their mathematical knowledge, we must engage the teachers in a delicate interplay between mathematics and pedagogy. We design tasks of pedagogical nature,

The original version of this chapter was revised due to some errors (listed below) in the text at page numbers 98 and 99.
- On pages 98 and 99, there are incorrect "+" symbols (with a circle around them), where this should have been an approximation symbol.
- Also on pages 98 and 99, there are two instances of wrong indentations.

The correction to this chapter is available at https://doi.org/10.1007/978-3-030-62408-8_23

R. Zazkis (✉)
Faculty of Education, Simon Fraser University, Burnaby, BC, Canada
e-mail: zazkis@sfu.ca

O. Marmur
Mathematics Education Unit, Department of Mathematics, Faculty of Science, The University of Auckland, Auckland, New Zealand

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_5

which are additionally aimed at extending the teachers' understanding of the underlying mathematics (e.g. Biza, Nardi, & Zachariades, 2007, 2018; Peng, 2007).

In order to advance teachers' mathematics, it is essential to first gain insight into the mathematical knowledge they possess, as teaching mathematics undoubtedly relies on prior knowledge of learners. School teachers usually have a relatively good grasp of their students' knowledge based on their prior work with the students and the curriculum sequence they follow. Teacher educators do not have such a privilege. This is particularly evident in postbaccalaureate teacher education programmes, to which students come from various undergraduate experiences, often educated in different decades and different countries. How is it possible to get a sense of where a group is in terms of their mathematical maturity and sophistication? How can a teacher educator be able to "scan" the group's knowledge of a mathematical topic in order to plan for, or adjust, subsequent instruction?

We address these questions by providing two illustrative examples: the first attends to the concept of a function, and the second deals with the concept of irrational exponents. In each case, we:

(a) Describe a task of a pedagogical nature that provides teacher educators with a window into strengths and weaknesses in teachers' knowledge of a particular mathematical topic.
(b) Provide a brief overview of the main themes that emerged from teachers' responses to the task, and exemplify several responses that informed our design of follow-up instructional activities.
(c) Illustrate the follow-up instructional engagements.

In the following section, we present the idea of script-writing, which served as the guiding principle for the design of our pedagogical tasks. We then discuss the *usage-goal framework* (Liljedahl, Chernoff, & Zazkis, 2007) that we employ to illustrate the symbiosis of mathematics and pedagogy in our approach. Subsequently we introduce the two examples as described above.

5.2 Script-Writing in Mathematics Education

Script-writing is a valuable pedagogical strategy and an innovative research tool that was adopted and developed in the context of mathematics teacher education. While script-writing is novel in mathematics education research, its roots trace to the Socratic dialogue and to the style of Lakatos' (1976) evocative *Proofs and Refutations* in which a fictional interaction between a teacher and students investigates mathematical claims.

Initially, script-writing was introduced in mathematics teacher education as a *lesson play*, a task in which participants script interaction between an imaginary teacher-character and student-character(s) (Zazkis, Liljedahl, & Sinclair, 2009; Zazkis, Sinclair, & Liljedahl, 2013). Juxtaposed to a classical lesson plan describing merely content and activities, the lesson play reveals how a teaching-learning interaction unfolds. In later research, the idea of a lesson play was extended to an activity of writing an imaginary dialogue that is not necessarily restricted to a lesson, referred to as script-writing. When used in teacher education, script-writing is a tool related to "approximations of practice" (Grossman, Hammerness, & McDonald, 2009), which "include opportunities to rehearse and enact discrete components of complex practice in settings of reduced complexity" (p. 283), and is advocated as an essential part of teacher preparation.

Script-writing is used as both an instructional tool for the advancement of pedagogy and mathematics and a research tool for data collection. It has been implemented in recent research (e.g. Brown, 2018; Koichu & Zazkis, 2013; Zazkis & Kontorovich, 2016; Zazkis & Zazkis, 2014) where participants had to identify problematic issues in the presented topics and subsequently clarify these by designing a scripted dialogue. The affordances of script-writing were detailed for script-writers (students, prospective teachers, and teachers), teacher educators, and researchers. In particular, for teachers, writing a script is an opportunity to examine a personal response to a situation, explore erroneous or incomplete approaches of a student, revisit and possibly enhance personal understanding of the mathematics involved, and enrich the repertoire of potential responses to be used in future real teaching. For teacher educators, the scripts provide a lens on planned pedagogical approaches that can be consequently highlighted and discussed in working with teachers. For researchers, the scripts form a rich data source that can be examined from various perspectives and provide a lens for examining images of teaching and insights into the script-writers' understanding of mathematics (e.g. Zazkis et al., 2013).

In this chapter, we highlight an additional use and benefit of script-writing tasks: they provide teacher educators with insight into the mathematical knowledge of their students, which can subsequently be used for planning follow-up instructional activities aimed at extending and strengthening this knowledge.

5.3 The Usage-Goal Framework

As mathematics educators, we draw on our knowledge of mathematics and pedagogy to design tasks aimed at advancing both the mathematical and pedagogical understanding of the teachers we teach. However, the interweaving of mathematics and pedagogy in professional development courses might create a complex educational setting in which the mathematical and pedagogical ideas become entangled and, consequently, harder to be discerned. The *usage-goal framework* suggested by Liljedahl et al. (2007) illustrates a way of examining the use of tasks in teacher education while attending to both mathematical and pedagogical perspectives. More specifically, the framework is organised in four cells of a 2 × 2 array (presented in Fig. 5.1), where the content of the cells should be read as "the use of **x** to promote understanding of **Y**", ***x*** and ***Y*** being either mathematics or pedagogy (e.g. mP is read as "the use of mathematics (m) to promote understanding of Pedagogy (P)"). The suggested array, according to Liljedahl et al. (2007), serves to disaggregate "our knowledge of mathematics and use of pedagogy [as teacher

USAGE		GOALS	
		Mathematics (M)	Pedagogy (P)
	mathematics (m)	**mM**	**mP**
	pedagogy (p)	**pM**	**pP**

Fig. 5.1 Goals and usage grid

educators] from the mathematical and pedagogical understandings we wish to instill within our students [teachers or prospective teachers]" (p. 240).

As we demonstrate in what follows, our scripting tasks are situated within the lower row of the grid (pM and pP). In short, these tasks present a prompt for a beginning of a dialogue between imaginary students and their teacher and ask the participating teachers to continue the dialogue in a way they find fit. These prompts typically attend to a problematic issue, a potential error or misconception, or an unexpected student question. This requires the participants to consider both the pedagogical issues of how to explain the topic and the mathematical knowledge involved. Accordingly, the scripting tasks utilise a pedagogical perspective to promote both pedagogy and mathematics.

The follow-up classroom activities in a teacher education course, however, are predominantly situated within the upper row of the grid (mM and mP). These activities utilise those mathematical themes that require further attention and deepening of understanding according to the tasks. Accordingly, the mathematics involved is used as a basis for the promotion of related mathematical knowledge, as well as pedagogical considerations specific to the topic.

5.4 Context for the Examples

The two examples described below took place in the context of a professional development course for practicing secondary school mathematics teachers. In both cases, the course participants were given a scripting task in the form of a prompt that was the beginning of a dialogue between a teacher and students and were asked to continue it according to their mathematical and pedagogical understanding (Part A). In addition to writing a script that extends the dialogue, the participants were also asked to explain their choice of the presented instructional approach (Part B). Furthermore, they were asked to explain their personal understanding of the mathematics involved in the task and note whether their potential explanation to a "mathematically sophisticated colleague" differed from what they chose to include in the scripted conversation with students (Part C).

5.5 Example 1: Functions, Not Just Linear

The concept of a function is fundamental in mathematics, and it has been repeatedly regarded in the education literature as a central concept in the school curriculum (e.g. Ayalon, Watson, & Lerman, 2017; Dreyfus & Eisenberg, 1983; Hitt, 1998; Paz & Leron, 2009). Rather recently, Dubinsky and Wilson (2013) have conducted a longitudinal literature review, covering over 50 years of research on student learning of this concept. However, the examination of the vast amount of mathematics education literature related to the understanding of functions reveals that there has been relatively little research performed specifically in relation to teachers' and prospective teachers' understanding of the concept. The following example provides a glimpse into this issue.

5.5.1 The Scripting Task: Functions

Figure 5.2 presents a prompt for a scripting task alongside a particular table of values. In this task, the participants were invited to explore an imaginary student question, whether there are functions other than $y = 3x$ that satisfy the same table of values. The task was designed to address several known misconceptions regarding the function concept (elaborated below), which are attributed in the literature to either secondary school students, undergraduate students, or prospective teachers. Accordingly, the pedagogical task could promote either mathematical growth, in case the teachers' existing understanding of the function concept was challenged (pM), or pedagogical growth, in case the teachers decided to address possible student mistakes that might arise (pP), or both.

The above task addresses two issues described in mathematics education research. Firstly, the task attends to the phenomenon of linear functions as "overpowering" prototypical examples, both for undergraduate students (e.g. Dreyfus &

Teacher:	Consider the following table of values. What function can this describe?
Alex:	y = 3x
Teacher:	And why do you say so?
Alex:	Because you see numbers on the right are 3 times numbers on the left
Jamie:	I agree with Alex, but is this the only way?
Teacher:	…

x	y
1	3
2	6
3	9
4	12
5	
6	

Fig. 5.2 A prompt for the table of values scripting task

Eisenberg, 1983) and secondary school students (e.g. Markovits, Eylon, & Bruckheimer, 1986; Schwarz & Hershkowitz, 1999). For example, Markovits et al. (1986) reported that half of the participating ninth-grade students in their study claimed that there is only one given function that passes through two given points and that this function is a straight line. In the current presented task, the table of values contains four points that satisfy the line $y = 3x$, which in comparison to the case presented by Markovits et al., where there were only two given points, further "strengthens" the idea of the line as the only available option. Secondly, the task addresses the issue of teachers' potential lack of understanding of the arbitrary nature of how a function may be defined (e.g. Even, 1990). More specifically, Thomas (2003) reported on teachers' need for an algebraic formula to describe a function that has a tabular representation, as is the case with the current "table of values" task.

Through Jamie's question in the task, "but is this the only way?", we expected the consideration of various suitable functions that constitute an example space (Watson & Mason, 2005) applicable to the task, which in turn could shed light on the teachers' understanding of the function concept.

5.5.2 Snapshots from the Scripts: Functions

Our analysis identified the main themes that emerged from the scripts and examined the structure of the exhibited example spaces of functions that were given by the participants. We distinguished between examples used in Part A, which could have been purposefully restricted in the scripts based on pedagogical and instructional considerations, and the examples mentioned in Part B or Part C, which pointed to the participants' own understanding of the task. In the context of the current chapter, however, we do not provide a detailed analysis of participants' responses. Rather, we highlight several themes that emerged, which served as a motivation for developing follow-up instructional activities.

5.5.2.1 On the Notion of Function

The following excerpt from Taylor's[1] script exemplifies several features evident in the participants' perception of the function concept:

Teacher: Excellent question Jamie, what's your instinct, are there other ways?
Jamie: Well I don't know, I guess there could be, but how could we tell?
Teacher: Why don't we start by plotting these points. And by we I mean you.

[Students plot the points]

Teacher: Good, so how would it look if we used Alex's function?

[1] All participant names are pseudonyms.

Jamie: It would have a straight line through all the points.
Teacher: Yes, but how else can we connect these points?
Jamie: I suppose we could do a zig zag line.
Teacher: Sure, that would work. But we want this to be a function, so what rule do we need to follow?
Jamie: The vertical line test.
Teacher: Which is the easy way of remembering what?
Jamie: Each output can only have 1 input.
Teacher: Correct, so how can we connect these points then?
Jamie: Any way we want as long as we don't break the vertical line test.

In this excerpt, the teacher's request to plot the points explicitly leads to the consideration of a graphical representation. The question "but how else can we connect these points?" leads students to explore alternative options to the straight line. However, all examples in the script explicitly or implicitly regarded the domain as the set of all real numbers.

In Taylor's script, we note the reference to the "vertical line test" as an implicit working definition of a function. We further note that the teacher-character agrees with a student's incorrect definition, "Each output can only have 1 input". This could be either a misconception or lack of attention on the part of the script-writer.

The features of this script – infinite and unbounded domain, continuous function ("connected points"), vertical line test, and lack of a correct definition – were typical in this group of participants. This led us to design an activity that focused on different definitions of a function and a historical evolution of the contemporary definition.

5.5.2.2 Polynomial Expressions

While some script-writers expressed difficulty in considering examples other than the linear function, a possibility to fit a polynomial function to the given table of values was featured in several scripts. This is illustrated in the following excerpt from Logan's script:

Teacher: Well in all of these cases we have assumed something subtle. If we filled the table of values what would we get for the remaining y entries?
Alex: 15 and 18
Teacher: Does it have to be those values? What if I put 16 and 23?
Jamie: … Can you do that?
Teacher: Why not? The points could be modeling anything! There is nothing there that says it has to be a line.
Jamie: Can we find an equation for that though?
Teacher: Certainly, but I need to talk about degrees of freedom. In our table of values we could make up 6 values of y and therefore we have 6 degrees of freedom. Simple enough?
Jamie: Mhmm.
Teacher: So we need to find a polynomial with at least 6 degrees of freedom to describe it, that is a polynomial with at least 6 terms.
Alex: So a 5^{th} order polynomial?

Teacher: Exactly Alex, we could find a polynomial of the form $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ that fits the table of values.

Jamie: But how can we ever assume that any patterns we see in a table of values continues?

Teacher: An excellent question, short answer is we don't. When we make these equations we are assuming that the trend we observe will continue. When making this assumption we need to look for reasons to explain the trend and then ask if we expect those factors to stay the same. Maybe the data was showing the population of a species but at x = 5 more food is introduced or a predator is removed and the species can grow at a faster rate.

While general solutions are usually considered in mathematics as more valuable than specific ones, Zazkis and Leikin (2007) noted that often general examples point to an individual's inability to generate a specific one. In this case, the presented example of a polynomial function points to Logan's awareness of a possibility of a polynomial function, while it may also indicate Logan's difficulty in producing an explicit formula for the polynomial.

While Logan noted the existence of a polynomial function, Corey provided such a function "out of the blue" and left it for the students to verify that it is consistent with the entries in the table of values.

Jamie: It's kind of obvious that it's y = 3x. What are we learning here?

Alex: I guess it's making us think outside the box a little, but yeah, our other answers are kind of lousy.

[…]

Teacher: Then let me give you an extension. Check out this function $y = x^4 - 10x^3 + 35x^2 - 47x + 24$

Alex: Where did you get that from?

Teacher: You tell me.

Jamie: Ug. This works doesn't it? That is so annoying! How did you get this?

[…]

Alex: But it's not a line!

Jamie: Who cares? It's a function. And I guess it takes going to the power of four to hit all four points.

Teacher: I'll leave you to it. Figure out how to derive that equation! I didn't just pull it out of thin air.

In his commentary in Part C, Corey added that the polynomial was generated by a computer program, using matrices to solve systems of linear equations. He felt, however, that this material was inappropriate for his students. Corey wrote: "[t]he level of math needed to determine the final function is beyond what I consider high school level math. After being given the function the answer can be easily revealed, but it still is not easy".

We agree with Corey that generating polynomials from a manipulation of matrices may be beyond high school student capabilities. However, the scripts of Corey and Logan informed a follow-up instructional activity aimed at extending the teachers' knowledge of mathematics. In the next section, we present and discuss an instructional exploration on how such polynomials can be found and how this approach could be utilised to extend teachers' connections between tertiary and school mathematics.

5.5.3 Follow-Up Activities: Functions

In the current section, we present two follow-up activities that were designed in response to the participants' scripts: (1) classifying different definitions of functions for the deepening of the conceptual understanding of the function concept and (2) seeking to "create" a polynomial function that fits the original table of values. The rationale behind the activities, their mathematical details, and the resulting classroom events are elaborated below.

5.5.3.1 Function Definition

The scripts revealed that the participants mostly considered functions as continuous, written as a single formula, defined on an infinite and unbounded domain, and continue the given pattern. More specifically, the examples that the teachers used lacked any referral to the *arbitrary* nature of functions, that is, that there is no need for regularity or a representative expression for the correspondence, nor are there rules for what the sets of the domain and codomain must be (Even, 1990; Steele, Hillen, & Smith, 2013). As explained by Even (1990), and in line with the scripts, the rejection of the arbitrary characteristic of functions may be the result of "a prototypical judgement whether an instance is a function, combined with a limited concept image" (p. 528). Accordingly, we decided to initiate a follow-up classroom activity that "goes back to the basics" and focuses on the function definition.

Initially, the participants were asked to provide a definition of a function in writing. While "function" is a familiar and frequently used term both in school mathematics and in undergraduate studies, only one half of the participating teachers succeeded in providing an accurate definition. The other half provided definitions that either did not allude to the necessity of a *unique* corresponding value to each value in the domain or, to the contrary, required *different* image values for different domain values (i.e. defined an injective function instead of a function).

After providing a personal definition, the participants were given a list of 17 definitions and were asked to classify them according to criteria of their choice. The classification itself was a subordinate goal (Hewitt, 1996) in order to invoke a thorough consideration of the details and features of each definition. Out of the given definitions, 11 definitions were taken from different textbooks and web-based sources, and the remaining 6 "definitions" were in fact incorrect, chosen to highlight frequent misconceptions (such as the aforementioned confusion between the definition of a "function" and an "injective function").

As a group, the participants succeeded in rejecting the incorrect or incomplete definitions. The subsequent conversation focused on two chosen clusters of definitions. The first cluster included set-based definitions, exemplified by Definition S below (based on Usiskin, Peressini, Marchisotto, & Stanley, 2003, p. 70), in which the sets consisting the domain and codomain were explicitly addressed:

> Definition S: The Cartesian product of two sets A and B, denoted A × B, is the set of all ordered pairs (a, b) such that a ∈ A and b ∈ B. For any sets A and B, a function f from A to B, f : A → B, is a subset f of the Cartesian product A × B such that every a ∈ A appears once and only once as a first element of an ordered pair (a, b) in f.

This continued into a discussion on the arbitrary choice of the two sets. More specifically, an important realisation that did not appear in any of the scripts was that the table of values in Fig. 5.2 already provides a function without any need for domain expansion: the set of ordered pairs S = {(1,3), (2,6), (3,9), (4,12)} given in the table determines a function with domain {1,2,3,4} and codomain {3,6,9,12}.

The second cluster consisted of the following two definitions, H1 and H2, where the reader may recognise their historical significance:

> Definition H1: A function of a variable quantity is an analytic expression composed in any manner from that variable quantity and numbers or constant quantities.

> Definition H2: y is a a function of a variable x, defined on an interval $a < x < b$, if for every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established.

Definition H1 belongs to Euler and dates to the year 1748 (see Euler, 1988, for a translated version of his original book *Introductio in Analysin Infinitorum*). As explained by Kleiner (1993), this early version of a definition of a function refers to a single analytic expression with an unrestricted domain (and corresponds to what we nowadays call "elementary functions"). Definition H2 belongs to Dirichlet from the 1820s and is considered as the basis of the current contemporary definition of a function (Kleiner, 1993). While this definition does not yet permit arbitrary sets as domain and codomain, it is nonetheless freed from the requirement of an analytic expression or a corresponding curve to define a function (e.g. the famous Dirichlet function, which is defined separately for rational and irrational numbers).

In addition to the meta-mathematical dimension of exposing the teachers to the developing nature of mathematical definitions (e.g. Kjeldsen & Blomhøj, 2012; Kjeldsen & Petersen, 2014), these historical definitions were discussed in order to highlight the arbitrary nature of the correspondence defining a particular function. In this regard, Euler's eighteenth-century definition was presented to illustrate the participants' own possible misconception of the function definition. As articulated by Even (1990), "[w]e cannot accept a situation where secondary teachers at the end of the 20th century have a limited concept of function, similar to the one from the 18th century" (p. 530). Juxtaposed with Euler's definition, the difference that emerged when alluding to Dirichlet's definition was that "it is *irrelevant* in what way this correspondence is established" (Definition H2). This led the class to the additional realisation that the function defined by the set of ordered pairs T = {(1, 3), (2, 6), (3, 9), (4, 12), (7.5, π), ($-\sqrt{2}$, 103.54)} is another possible solution to the task, in which not only are the domain and codomain sets arbitrarily chosen, but so is the correspondence itself. The latter example helped the participants realise that there are infinitely many arbitrary functions that fit the original table of values presented in Fig. 5.2.

While a set-based definition of a function is usually beyond school mathematics, we believe that exposure to this contemporary convention in disciplinary mathematics adds an important dimension to teachers' mathematical knowledge (mM, in the usage-goal terminology). Furthermore, we suggest that the engagement with a mathematical task that explores different function definitions may additionally affect teachers' future pedagogical approaches (mP). Even while attending to functions prescribed by the school curriculum, the raised awareness to the topic would (hopefully) result in teachers introducing students to rigorous definitions and highlighting the arbitrary nature of the function concept.

5.5.3.2 Fitting Polynomials

As became evident from the scripts, some teachers considered the option of polynomials when dealing with the mathematical aspects of the task yet struggled to come up with concrete formulas for these. Accordingly, an additional classroom activity focused on generating a nonlinear polynomial function consistent with the originally given table of values (see Fig. 5.2). Corey shared with the class his polynomial function $y = x^4 - 10x^3 + 35x^2 - 47x + 24$ for verification. While it is easily confirmed, both numerically and graphically (see Fig. 5.3), that this function indeed conforms to the table of values in Fig. 5.2, questions arose as to how this function could be generated.

To address this question, instead of producing a function that intersects the line $y = 3x$ at exactly four different points, the teachers were asked to create a function that intersects the line $y = 0$ (the x-axis) at exactly four distinct points. While the teachers easily generated a function that has zeros at 1, 2, 3, and 4 – $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$ – it was nonetheless a conceptual leap to combine it with the function $g(x) = 3x$ suggested by the table of values to generate a polynomial function $h(x) = f(x) + g(x)$ (see Fig. 5.4). However, this initial example naturally led to additional examples of the form $h_k(x) = kf(x) + g(x)$. Of note, for k = 1, $h(x)$ is simplified to $x^4 - 10x^3 + 35\,x^2 - 47x + 24$, which is the function that a computer program generated for Corey.

The creation of the above algebraic examples was accompanied with computer-generated graphs, providing yet additional visual evidence that the points from the table of values satisfied the generated functions (see Fig. 5.4). While, mathematically, such a confirmation was unnecessary, we suggest that this satisfied an aesthetic dimension of the teachers' mathematical experience, using various values for k.

Another polynomial function, $s(x)$, that appeared in Eric's script was presented for classroom consideration, with the (undeclared) goal of making connections between tertiary and secondary school mathematics:

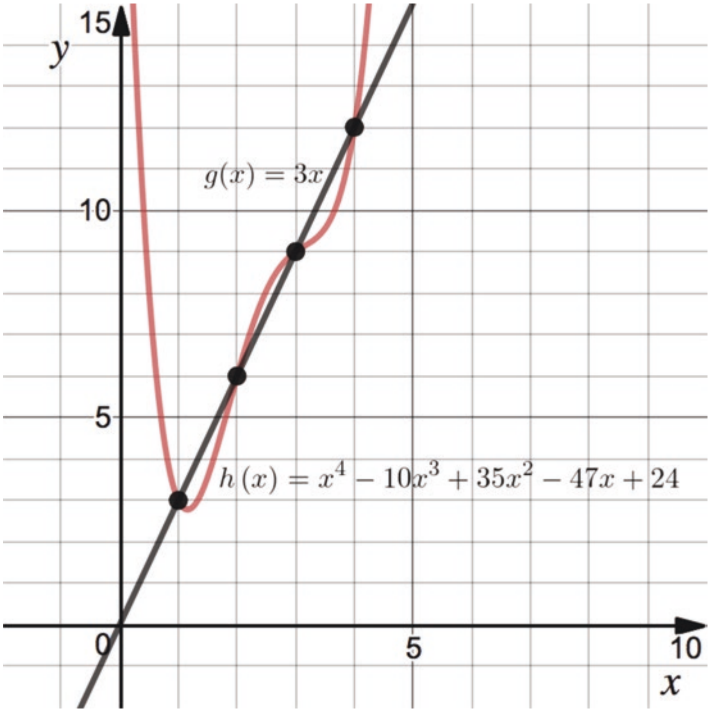


Fig. 5.3 A polynomial that passes through the points (1, 3), (2.6), (3, 9), and (4, 12)

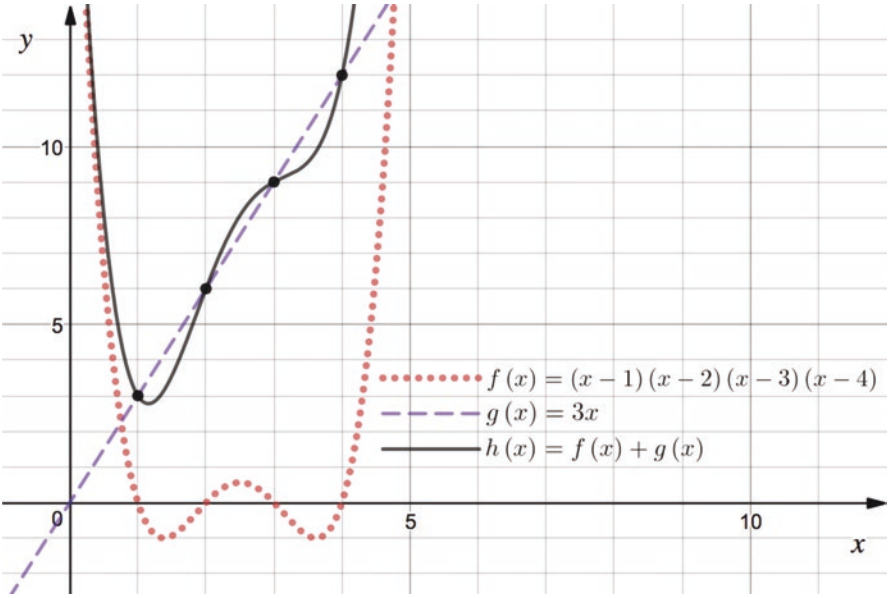


Fig. 5.4 Generating the polynomial $h(x)$ as the sum of $f(x)$ and $g(x)$

$$s(x) = -\frac{1}{3}(x-2)(x-3)(x-4) + 3(x-1)(x-3)(x-4)$$
$$-\frac{9}{2}(x-1)(x-2)(x-4)$$
$$+2(x-1)(x-2)(x-3)$$

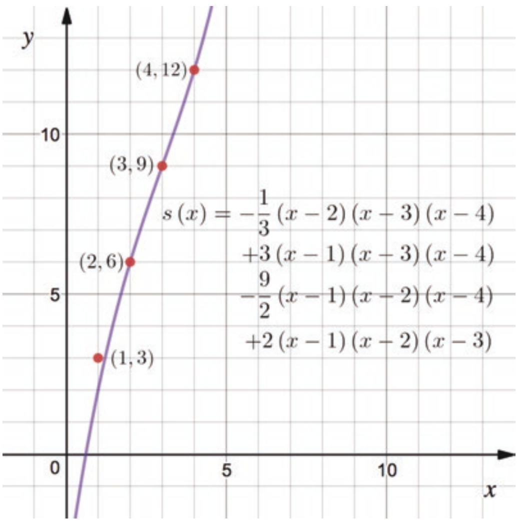
It was presented as a student's example, and the teachers were asked to consider the correctness of the example. Having examined the function and focusing on the factors in each component, the participants expressed a belief that "it should work". However, a close examination of the graph of $s(x)$ (see Fig. 5.5) revealed that one of the required points was in fact not on the graph.

In order to "fit" the point (1, 3) to the graph, the participants suggested replacing $\left(-\frac{1}{3}\right)$ with $\left(-\frac{1}{2}\right)$, suspecting that Eric's function was not copied properly. At that point, different participants expressed appreciation for Eric's idea of easily producing a suitable polynomial by "controlling" the values of the function at x=1, 2, 3, and 4. However, graphing the "corrected" function $q(x)$

$$q(x) = -\frac{1}{2}(x-2)(x-3)(x-4) + 3(x-1)(x-3)(x-4)$$
$$-\frac{9}{2}(x-1)(x-2)(x-4)$$
$$+2(x-1)(x-2)(x-3)$$

led to an unexpected result. The participants realised that $q(x)$ can be simplified to $3x$ and presents, as the participants referred to it, a "linear function in disguise".

Fig. 5.5 Eric's function



This realisation led to a follow-up question of whether (and, if so, how) it is possible to generate a polynomial function of the third degree that corresponds to the table of values in Fig. 5.2. After some consideration, Logan suggested that this would be impossible, as “third degree functions go up-down-up” (this was accompanied with a wave hand gesture), so they would not reach the fourth point on the line. While this was a reasonable explanation, a more rigorous one was sought in inviting the participants to consider possible zeros of a cubic function. This led to recalling the fundamental theorem of algebra and to acknowledging that the existence of three complex roots means that there are at most three real roots for any cubic function. Therefore, in analogy with the previous discussion, cubic functions can be generated to pass through any three points on the same line, but not four.

The above discussion of Eric’s example from his script clearly demonstrates how participants’ scripts can be utilised in the work of a teacher educator. In this case, a pedagogical task of considering the correctness of a student’s solution was used toward the goal of extending the teachers’ mathematics. The activity guided the teachers toward an explicit connection between undergraduate and secondary school mathematics, a goal considered by many researchers as highly valuable (e.g. Wasserman, 2016; Watson & Harel, 2013). Furthermore, it demonstrated the utility of advanced mathematical knowledge as a tool to instantly recognise student mistakes (mP): in this case, through the realisation that there is no need to check for calculation errors, as there could be no cubic function that intersects a line in four different points.

5.6 Example 2: Irrational Exponents, Not Just with a Calculator

The idea of irrational exponents lies at the juxtaposition of two mathematical concepts: irrational numbers and exponentiation. Both these concepts have been recognised in the literature as conceptually challenging for secondary school students and teachers alike (e.g. Confrey & Smith, 1995; Davis, 2009; Fischbein, Jehiam, & Cohen, 1995; Kidron, 2018; Sirotic & Zazkis, 2007a; Weber, 2002). However, we have not found any research that focuses on the combination of the two – that is, learners’ understanding of *irrational exponents*. The following case presents a step toward this goal.

5.6.1 The Scripting Task: Irrational Exponents

Figure 5.6 presents a prompt for a scripting task on the topic of irrational exponents. In the task, the teachers were encouraged to consider the mathematical meaning of irrational exponents, as well as how this concept could be explained to a group of secondary school students.

Your class learned how rational exponents are interpreted and practiced simplifying expressions that involved rational exponents.

Robin:	So when there is a fraction in an exponent you rewrite it as a root.
Teacher:	Indeed
Robin:	But what if there is a root in the exponent, do we make it a fraction?
Teacher:	What do you mean?
Robin:	So we learned that $x^{\frac{m}{k}} = \sqrt[k]{x^m}$, like $8^{\frac{2}{3}} = \sqrt[3]{8^2} = 4$. But what about $x^{\sqrt{k}}$, like $4^{\sqrt{2}}$, what would this mean?
Teacher:	This is a very interesting question, let us look at this together …

Fig. 5.6 A prompt for the irrational exponents scripting task

The task was designed to expand the teachers' mathematical knowledge of irrational exponents and strengthen their knowledge of rational exponents. We suspected that while the meaning of rational exponents would be familiar to them, this might not be the case for irrational exponents. This assumption was based on the following two considerations: (a) secondary mathematics textbooks typically introduce the idea of exponentiation as "repeated multiplication" and obtain the graphs of exponential functions without raising the issue of continuity or irrational exponents (Confrey, 1991; Davis, 2009; Davis, 2014); (b) previous research has pointed toward prospective secondary school teachers' reliance on calculators, rather than conceptual understanding, when dealing with irrational numbers (e.g. Zazkis & Sirotic, 2010) – a tendency we suspected to be even stronger in the case of two "irrationality layers" (i.e. the irrational number $4^{\sqrt{2}}$ that contains an irrational exponent $\sqrt{2}$). Accordingly, the formulation of the task encouraged the teachers to first make sense of irrational exponents, whether independently and/or by utilising external mathematical sources, and subsequently consider how to present the underlying ideas to students.

5.6.2 Snapshots from the Scripts: Irrational Exponents

Most of the script-writers presented the idea of irrational exponents to a group of imaginary students in an adequate manner. This included the creation of a sequence a_n that converges to $\sqrt{2}$ and subsequently considering the sequence 4^{a_n} to approximate $4^{\sqrt{2}}$ (we note that the terminology used in the scripts differed from the one presented here; some of the related nuances are elaborated below). Nonetheless, responses to Part C of the task revealed that the notion of irrational exponents was new to the teachers. Most of the teachers explicitly admitted they had never considered the meaning of irrational exponents before and were pleased with the opportunity to do so, as exemplified by the following excerpt by one of the teachers:

> I have taught Powers and Exponents to students through grade 8 to 10 a number of times so far, but strangely enough, the question such as 'what if the exponents are irrational numbers' has never crossed my mind, nor was ever asked by any of my students. […] As such, I sincerely appreciated this assignment for it guided me, or better yet, intrigued me to spend almost ridiculous amount of hours to think about the issues linked to this topic.

In the following sections, we focus on two prominent themes that emerged in the scripts, which led to follow-up instructional activities aimed at addressing the teachers' conceptions and difficulties related to the topic.

5.6.2.1 Irrationals Can Only Be Approximated

We acknowledged in the scripts a certain conceptual difficulty regarding the teachers' perception of irrational numbers. More specifically, most of the teachers claimed, both in the script and Part C of the task (referring to their personal mathematical understanding), that the value of $4^{\sqrt{2}}$ can be *approximated only* and that an accurate value either does not exist or cannot be described. Consider, for example, the following excerpt from Eden's script:

Teacher:	So things are much easier with an integer or a fraction in the exponent. How about a decimal in the exponent, such as $4^{1.2}$?
Robin:	That is easy. We can re-write 1.2 as a fraction, then we have a fractional exponent.
Teacher:	Excellent! So what makes $\sqrt{2}$ different? Why is it difficult to interpret it?
Robin:	I guess if we could write $\sqrt{2}$ as a decimal, then turn it into a fraction, then we would be able to work with it. But its decimal expansion is infinitely long…
Chris:	And there is no repeating pattern. I guess what makes $\sqrt{2}$ different is that it is an irrational number!
Teacher:	Exactly. We would not be able to find an exact answer. Now, if I draw a number line and ask you to put $4^{\sqrt{2}}$ on it, where would you put it?
Chris:	Since we can have a decimal in the exponent, and $\sqrt{2}$ is about 1.4, we can say that $4^{\sqrt{2}}$ is about $4^{1.4}$ which is about 6.96. So close to 7?
Robin:	If we use a more accurate approximation of $\sqrt{2}$, we can get a more accurate approximation of $4^{\sqrt{2}}$, right? If we use $\sqrt{2} \approx 1.414$, we have 7.101.
Teacher:	Excellent! As you use better approximation using $4^{1.4}$, then $4^{1.41}$, then $4^{1.414}$ which you can all make sense of as fractional exponents, or rational exponents, your estimated value is getting closer and closer to the real value of $4^{\sqrt{2}}$.

$$4^{1.4} \approx 6.964404$$

$$4^{1.41} \approx 7.061623$$

$$4^{1.414} \approx 7.100891$$

$$4^{1.4142} \approx 7.102860$$

And so on… So our best attempt to interpret an irrational exponent is to write it as adding a sequence of rational exponents.

While the above script correctly describes the process of approximating $4^{\sqrt{2}}$, which is at the basis of how to define irrational exponents, it nonetheless treats irrational numbers as numbers that can only be approximated. This is exemplified by the teacher-character's claim "we would not be able to find an exact answer" in regard to $\sqrt{2}$ and having solely a "best attempt to interpret an irrational exponent" in regard to $4^{\sqrt{2}}$. This approach was further supported by Eden's answer to Part C in which she claimed: "there are no exact answers, just approximations". Similar claims were found in the tasks of the other participants, such as the following assertion by Leslie in Part C: "this discussion leads the student that the value of the irrational power … cannot be stated 'definitively'. We can only give the bounds of the region where we would find the value". Accordingly, one of the goals of the follow-up activity was to demonstrate that irrational numbers could be perceived not only as approximations but also as exact values on the number line.

5.6.2.2 Attempting to Make Sense of Irrational Exponents with the Use of Graphs

One prominent method the teachers used to cope with the issue of "what $4^{\sqrt{2}}$ really means" was by providing additional explanations that used graphical representations, mostly of $y = 4^x$ or $y = x^{\sqrt{2}}$. As is illustrated by the following excerpt from Leslie's script:

Robin: I had lots of decimals in the answer when I used decimals instead $\sqrt{2}$. It is interesting that your calculator can answer $4^{\sqrt{2}}$; it doesn't give an error. That means $4^{\sqrt{2}}$ exists; now we need to figure out what it means.

Jo: When you were playing with decimal values, were you doing things like calculating $4^{1.4142}$?

Robin: Yes, I started with $4^{1.41}$, and kept increasing the number of decimal places that I used.

Jo: You realise that you were just using the ideas from today's class? $1.41 = 141/100$, which means you were finding the 100th root of 4 to the exponent 141, or $4^{\frac{141}{100}} = \sqrt[100]{4^{141}}$.

You know $\sqrt{2}$ is irrational, its decimal form will go on forever and you can't write it as a fraction. So, we can't understand $4^{\sqrt{2}}$ using roots. We are going to need another way of explaining irrational exponents.

[…]

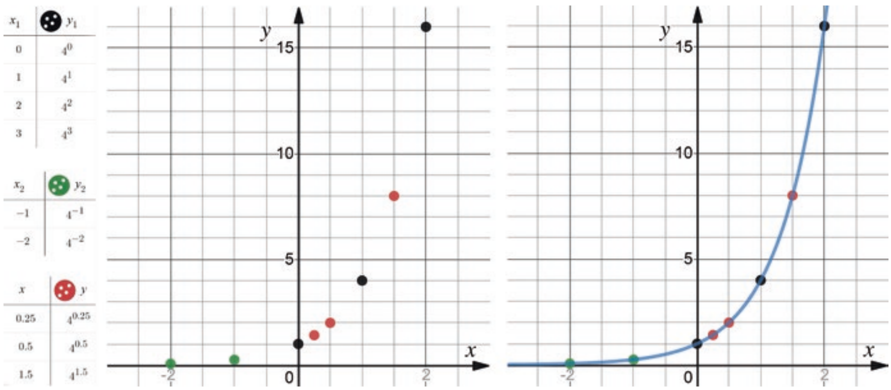


Fig. 5.7 The graphs shown in Leslie's script

Mel: (peer tutor) Do you folks need some help? (Jo recaps the discussion so far.)
What an interesting idea!
Let's pull up Desmos and look at a graph of the powers of four. Maybe that will help.

(Mel enters first table: whole number exponents.)

Robin: You should include negative exponents (Mel adds another table)

Jo: What about exponents that are decimals or fractions? (Mel adds another table)

[see Fig. 5.7]

Robin: The graph is sort of a line, with a weird bit at the bottom.

Mel: We can find the equation for this graph; what do you think it is?

Jo: Our "y" values are all 4 to the power of something, $y = 4^x$?

Mel: Let's see what that looks like.

[see Fig. 5.7]

Robin: All the points fit on the graph, you found the equation, Jo! We haven't seen graphs like that before, what is it?

Mel: It is the graph of an *exponential function*. We are just learning about this function in Pre-Calculus 12.

The need for additional ways to approach the topic is understandable when considering that this topic was new to the teachers (as they expressed themselves). However, the presented graphical approach and implied continuity avoid the issue of attending to an explicit definition of irrational exponents. Accordingly, in the follow-up activity, we wished to respond to the teachers' need for a graphical approach to address the idea of irrational exponents.

5.6.3 *Follow-Up Activities: Irrational Exponents*

In the current section, we focus on two follow-up activities that were designed to address the participants' tasks: (1) finding the exact location of irrational numbers on the number line and (2) graphing functions of the form $y = x^{\frac{m}{n}} \left(m, n \in \mathbb{N}\right)$ to deepen the learners' understanding of both rational and irrational exponents.

5.6.3.1 Finding Irrational Numbers on the Number Line

This activity was designed as a response to claims found in the scripts that irrational numbers can only be approximated. By demonstrating that $\sqrt{2}$ and other irrational numbers can in fact be placed in an accurate manner on the number line, we hoped to challenge the participants' aforementioned view. Sirotic and Zazkis (2007b) claimed that by placing irrational numbers on the number line, learners' understanding of irrational numbers can be improved in two ways. Firstly, it strengthens the distinction between an irrational number and its rational approximation; and secondly, it shifts learners' attention away from the conceptually challenging never-ending limit process associated with the decimal representation. As Sirotic and Zazkis (2007b) stated, "the geometric representation of irrational number may well turn out to be a very powerful and indispensable teaching tool for encapsulating a process into an object, especially in the case where the learner is on the verge of the reification stage in the development of the concept of irrationality" (p. 488).

At the onset of the activity, a number line was drawn on the board, and the participants were asked to place $\sqrt{2}$ on it. It took a few minutes of thought until one of the participants found and shared a solution: build an equilateral right-angle triangle of sides equal to one on the number line (where one of the sides rests on the number

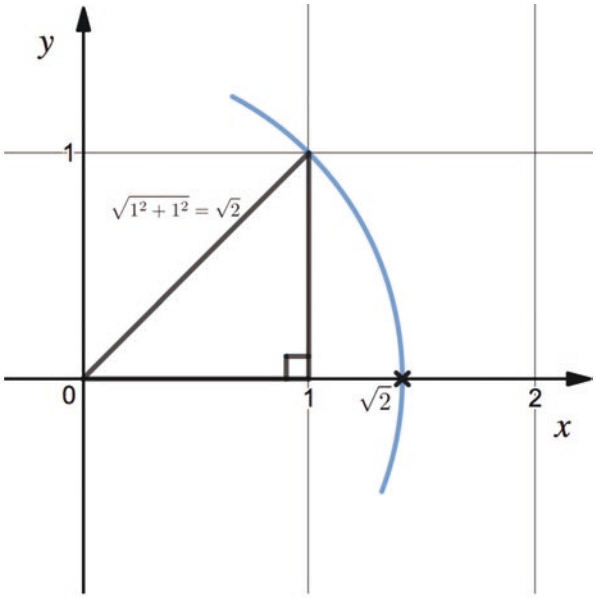


Fig. 5.8 Placing $\sqrt{2}$ on the number line

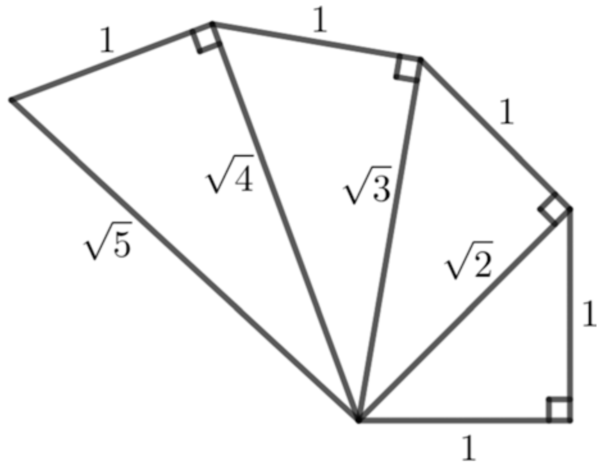


Fig. 5.9 Roots of natural numbers via successive triangles

line), and rotate the hypotenuse of length $\sqrt{2}$ until it is contained in the number line (see Fig. 5.8). This solution evoked enthusiastic and surprised responses in class and was received with clapping of hands. It is interesting to note that even though the solution based on the Pythagorean theorem was simple and clearly in the teachers' repertoire of knowledge, they did not immediately think of connecting this to the location of irrational numbers on the line.

Subsequent to finding $\sqrt{2}$, the participants were asked to find $\sqrt{3}, \sqrt{5}$, etc. on the number line. For $\sqrt{3}$, the immediate suggestions were to build a right-angle triangle in which one side equals $\sqrt{2}$ (as already found in the previous step) and the other equals 1. This leads to a hypotenuse of length $\sqrt{3}$, which again can be rotated until it sits on the number line. Similar logic was subsequently applied to find the location of other radicals, each step utilising the previous as a side in an appropriate right-angle triangle. One of the participants suggested that this could be visually done in a "manifold" of successive right-angle triangles (see Fig. 5.9). The general conclusion that in mathematics we do not always have to work with approximation of irrational numbers only was followed by exploratory participant questions, such as "how do we find π accurately?" and "what is the difference between transcendental and algebraic numbers and is there a relation between these properties and whether the number can be placed on the line?"

These questions demonstrate that while the original task regarded irrational exponents, responding to issues that arise may actively engage the participants and raise their interest in the topic. In this sense, the mathematical activity managed to promote the teachers' mathematical understanding and trigger their mathematical curiosity (mM). Furthermore, we suggest that the mathematical awareness of different representations for irrational numbers, as well as the distinction between irrational numbers and their approximations, may in turn raise the teachers' pedagogical attention to this issue when they discuss the topic with their students (mP).

5.6.3.2 Graphing Rational Exponents

As explained earlier, the scripts revealed a need for graphical ways to approach the issue of irrational exponents, manifested by using the graphs of $y = 4^x$ and $y = x^{\sqrt{2}}$ to explain what irrational exponents are. In the follow-up activity, we chose to focus on the latter, that is, on a possible method of how the graph of $y = x^{\sqrt{2}}$ could be created. More specifically, we planned to use a sequence of graphs of rational exponents (in the form of $y = x^{\frac{m}{n}}$, where $m, n \in N$) that (pointwise) converge to a graph of an irrational exponent (in this case, $y = x^{\sqrt{2}}$). The focus on $y = x^{\sqrt{2}}$ (over $y = 4^x$) was decided based on two learning affordances this option entailed. Firstly, this enabled us to emphasise the *same* approach to irrational exponents that the scriptwriters had already presented in their tasks (i.e. a sequence of rational exponents that converges to an irrational exponent), only this time via a graphical, rather than numerical, representation. Secondly, the focus on $y = x^{\sqrt{2}}$ was additionally planned to be used as a subordinate goal (Hewitt, 1996) in order to deepen the teachers' understanding of rational exponents and explore properties of their graphs.

In the classroom discussion, we invited the teachers to first consider graphs of rational exponents that are "close" to $\sqrt{2}$ and utilise computer software (Desmos) to do so. The teachers began with the estimation $1.4 < \sqrt{2} < 1.5$ and accordingly graphed the functions $f(x) = x^{1.4}$ and $g(x) = x^{1.5}$. Figure 5.10 shows the resulting graphs. While the teachers correctly predicted that the graph of $y = x^{\sqrt{2}}$ is located "in between" the graphs of $f(x) = x^{1.4}$ and $g(x) = x^{1.5}$, seeing these graphs seemed to have evoked reactions of puzzlement and surprise in class, and some responses of "What???" and "Wow!" were heard. The revelation that was surprising to the participants was that one of the functions was drawn for all $x \in R$, whereas the other function only for all $x > 0$.

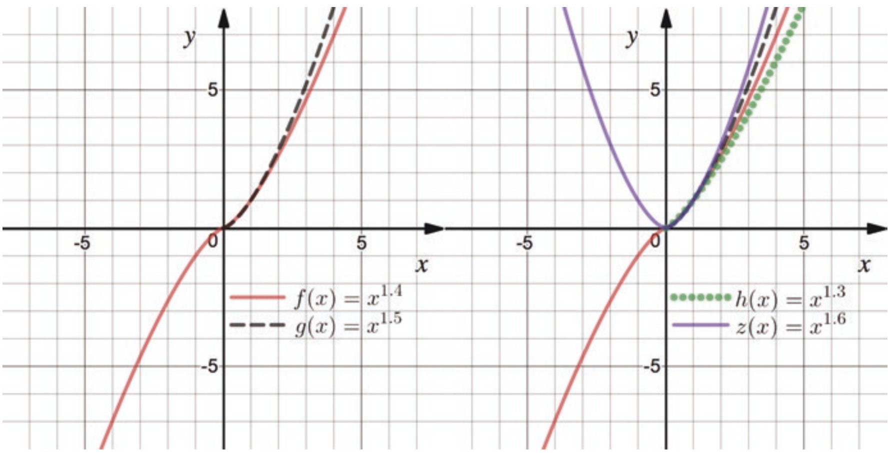


Fig. 5.10 Surprising behaviours in graphs of rational exponents

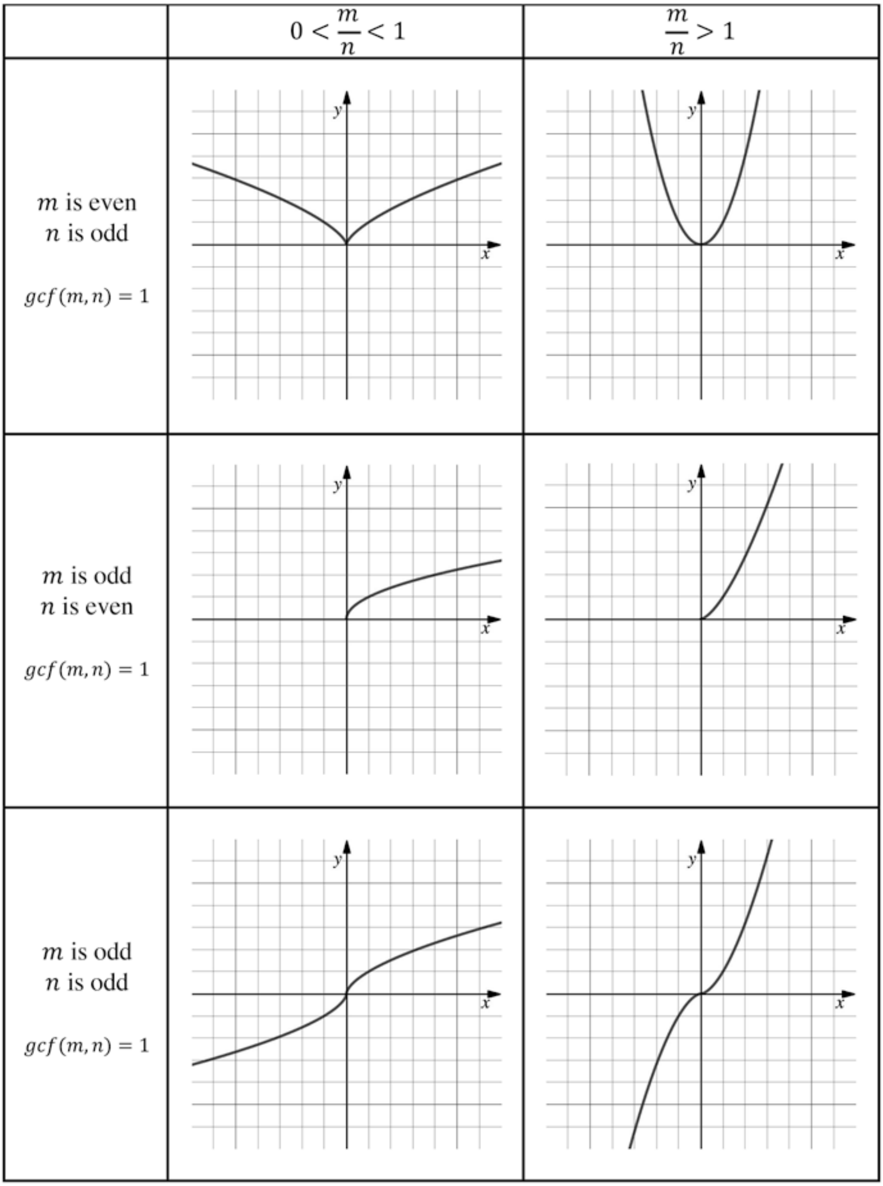


Fig. 5.11 The six different "shapes" of functions of the form $y = x^{\frac{m}{n}}$

Intrigued by the graphs, several participants asked to graph the functions of $h(x) = x^{1.3}$ and $z(x) = x^{1.6}$ as well (see Fig. 5.10). While $h(x)$ seemed similar to $g(x)$, the graph of $z(x)$, which according to the participants looked like a parabola, triggered additional surprise which invoked the need for a systematic investigation.

The participants subsequently worked in small groups on exploring the behaviour of the graphs, as well as other properties of the functions. Topics and conclusions that came up in the group activity and the following whole classroom discussion included the domain of the function as related to the evenness/oddness of the denominator, the convexity/concavity as related to the exponent being smaller or larger than 1, and functions that were each other's inverses (such as $y = x^{1.4} = x^{\frac{7}{5}}$ and $y = x^{\frac{5}{7}}$) and the graphic implication of reflection around $y = x$. In particular, the participants identified six possible shapes of a graph of the form $y = x^{\frac{m}{n}}$ and were able to predict the shape based on the given values of m and n (see Fig. 5.11). We note the participants' enthusiasm that portrayed itself in embodied gestures illustrating the different "shapes" of functions of the form $y = x^{\frac{m}{n}}$.

Subsequently, the discussion returned to irrational exponents and the underlying reason to define the domain of $y = x^{\sqrt{2}}$ as $\{x | x > 0\}$ as a common/joint domain to all approximating functions. Additionally, graphs of the form $y = x^{\frac{m}{n}}$ were plotted, where $\frac{m}{n}$ became "closer and closer" to $\sqrt{2}$, and were compared to the actual plotting of the graph of $y = x^{\sqrt{2}}$.

The mathematical activity regarding graphs of rational exponents served to advance the teachers' mathematical knowledge of both rational and irrational exponents (mM). This included observations made regarding properties of the functions, their geometric shapes, and reasoning behind their domain. Additionally, it raised the teachers' awareness to the possible limitations of using graphing computer software as a classroom pedagogical tool (mP). Whereas in the scripts the teachers used computer-generated graphs to explain irrational exponents, in the follow-up discussion, they became aware that these software tools cannot be designed without a pre-existing definition of irrational exponents. Such a pedagogical awareness of teachers is especially crucial when considering the strong technological dependence of their students.

5.7 Conclusion

The chapter provides a window into the work of mathematics teacher educators, which involves challenges in bringing together mathematics and pedagogy. In order to deal with this challenge, we utilised the usage-goal framework (Liljedahl et al., 2007), which highlights the interplay between mathematics and pedagogy involved in the tasks designed by teacher educators and accordingly can serve as a useful tool in the planning and refining of these tasks.

The usage-goal framework was originally illustrated by Liljedahl et al. (2007) through an account of an iterative process, where the *same* task went through a series of changes and adaptations, based on reflections on its implementation with *different* groups of teachers. The activities presented in this chapter, however, demonstrate a modified approach on how the usage-goal framework can be used in the work of teacher educators.

In our case, it served as a guide for a two-step instructional method, where the *same* group of teachers engaged with *different* tasks built upon each other, the design of which was informed by the usage-goal framework. In the first step, we made *usage* of scripting tasks – pedagogical tasks with the *goal* of promoting and revealing teachers' understanding of Mathematics and Pedagogy (pM and pP). The rationale was based on previous research, which illustrated that scripts generated by teachers provide a lens – for researchers and teacher educators alike – to examine teachers' mathematical knowledge and instructional choices. Extending this observation, we demonstrated that scripts can serve as a springboard for follow-up classroom activities aimed at strengthening teachers' mathematical knowledge. That is, we then made *usage* of mathematics in order to further promote the *goal* of Mathematics (mM), in particular by making connections between disciplinary and school mathematics. However, we believe these activities indirectly supported the goal of Pedagogy as well (mP), as the teachers' extended mathematical understanding of the related issues may well inform their future pedagogical approaches.

We suggest that this idea of role alternation of mathematics/pedagogy between usage and goal is an effective instructional method for mathematics teacher educators, since it both acknowledges the gradual and continual process of knowledge construction by learners (in this case, through a series of activities with a common theme) and responds to the needs of the *particular* group of teachers the teacher educator encounters (in this case, by planning the follow-up activities in accord with themes that emerge from the teachers themselves).

As a concluding note, we noticed that our pedagogical engagement in designing and implementing the instructional activities described above contributed to our personal and more nuanced understanding of the corresponding mathematical ideas. We relate this observation to Leikin and Zazkis' (2010) notion of teachers *learning through teaching*, which includes teachers "making connections between different components of previous knowledge, achieving deeper awareness of what concepts entail, and enriching their personal repertoire of problems and solutions" (p. 8). In this regard, our experience described in this chapter shows that the notion of learning through teaching applies not only to mathematics teachers but also to mathematics teacher educators.

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Chapter 6
Characterisation of Mathematics Teacher Educators' Knowledge in Terms of Teachers' Professional Potential and Challenging Content for Mathematics Teachers

Roza Leikin

6.1 Introduction

In this chapter, I suggest characterising mathematics teacher educators' (MTEs') knowledge and skills in terms of *students' mathematical potential* and *mathematical challenge*, which characterise students' learning and development, as well as in terms of the constructs of *teachers' professional potential* and *challenging content* for teachers associated with learning and professional development of mathematics teachers (MTs). There is a hierarchical relationship between teachers' potential, challenging content for teachers' potential and students' potential and mathematical challenge for students: MTs' professional potential integrates teachers' knowledge and skills associated with students' mathematical potential and mathematical challenge for students. Challenging content for mathematics teachers comprises mathematical, psychological and didactical factors and is a springboard for the realisation and advancement of the teachers' potential. Consequently, the teachers' and students' potential and the challenging content for teachers and students are the major elements of the suggested model of MTEs' knowledge and skills.

Like any human activity, the activities of MTEs have motives and goals and are composed from actions and tasks conducted under specific conditions (Leontiev, 1978). The major goal of mathematics education nowadays is to provide each student with learning opportunities directed at the advancement and realisation of his or her mathematical potential to the maximal extent. This is essential in order to create multiple opportunities for students' success and contentment in their future life, as well as to boost economic growth and the development of social justice

R. Leikin (✉)
Department of Mathematics Education, RANGE Center (Research and Advancement of Giftedness and Excellence), Faculty of Education, University of Haifa, Haifa, Israel
e-mail: rozal@edu.haifa.ac.il

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_6

(Pellegrino & Hilton, 2012). MTs are the agents of the educational system who are responsible for achieving this goal, which requires from teachers a high level of professionalism. In turn, MTEs' mission is the development of a high level of proficiency in MTs and the advancement and realisation of their professional potential at all stages of their studies and professional career. The focus of this chapter is on MTEs who work with in-service secondary school MTs.

In what follows, I start with an explanation of the concept of students' mathematical potential, discuss mathematical challenge as a springboard for the realisation of mathematical potential and connect teachers' proficiency with the ability to vary mathematical challenge in accordance with students' potential. Then, I extend the concepts of mathematical potential and mathematical challenge to the concepts of MTs' professional potential and challenging content for MTs. As mentioned earlier, I argue that mathematical potential and mathematical challenge are powerful constructs that have to be among MTs' knowledge and skills and thus have to frame MTs' professional development. Finally, I argue that these constructs, combined with the construct of challenging content for mathematics teachers, are integral components of MTEs' knowledge.

6.2 Background

6.2.1 Students' Mathematical Potential as Challenging Content for MTs

Developing MTs' comprehension of the complexity of students' mathematical potential and the role of mathematical challenge in the realisation of said potential is a very important goal of MTEs. The concept of students' mathematical potential is considered here to be one of the bases of the education and professional development of MTs; successful realisation of mathematical potential can serve as an indicator of teachers' proficiency. MTEs are required to understand the concept deeply enough to promote its understanding in MTs.

Mathematical potential is a complex function of a student's ability, his/her affective characteristics associated with the learning of mathematics, his/her personality and learning opportunities provided (Fig. 6.1) (Leikin, 2009). The construct of mathematical potential is related to the structure of mathematical promise (National Council of Teachers of Mathematics (NCTM), 1995; Sheffield, 1999), which was introduced as a substitute for the concept of mathematical giftedness in order to broaden a reference group and to express preference for a dynamic view of high mathematical ability. Similar to the construct of mathematical promise, mathematical potential integrates a person's cognitive, affective and personality factors and educational histories. However, in contrast to mathematical promise, mathematical potential is associated with a heterogeneous population with a range of low to high attainments in mathematics. In this context, mathematical potential is a dynamic characteristic that can and should be advanced with appropriate mathematical activities.

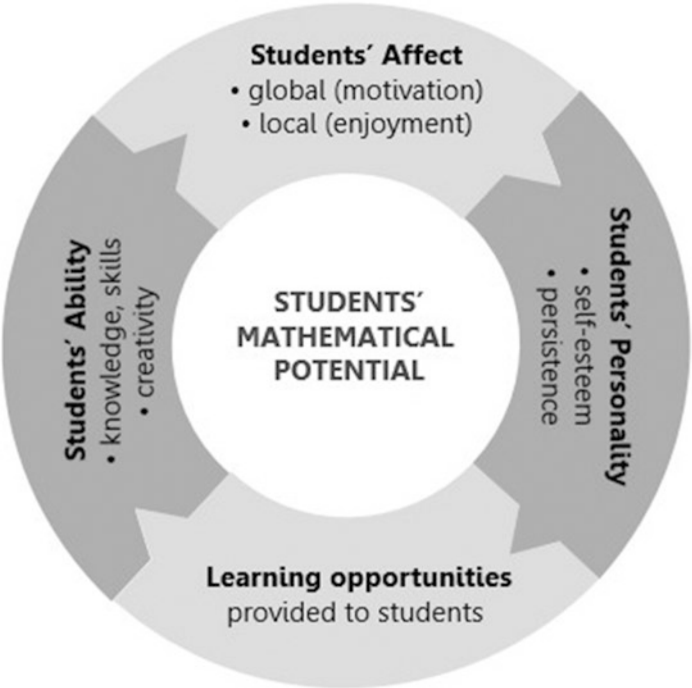


Fig. 6.1 A student's mathematical potential

The ability component of mathematical potential includes domain-specific and domain-general components. Domain-specific components comprise learning-based mathematical knowledge, problem-solving proficiency and argumentative and modelling skills. No less important is students' mathematical creativity, which can be considered one of the mechanisms of knowledge development (Vygotsky, 1930/1984). Domain-general characteristics include working memory, pattern recognition and visual processing, which have been demonstrated to influence mathematical processing.

Development of mathematical potential is associated with advancement of positive affect in a person. Affective characteristics include global structures like beliefs, motivation and attitudes towards mathematics and local characteristics such as joy or dissatisfaction associated with progress in solving a particular problem (Goldin, 2009). Motivation is one of the major conditions for the development of mathematical ability (Subotnik, Pillmeier, & Jarvin, 2009). Intrinsic motivation is associated with excitement, courage and joy in the process of mathematical activity and is an important construct that reflects the natural human propensity to learn and assimilate (e.g. Ryan & Deci, 2000). Extrinsic motivation refers to performance of an activity in order to attain some outcome, such as high test scores or pleasing parents or teachers. Personality characteristics are comprised of openness, conscientiousness, extraversion, agreeableness and neuroticism as well as self-concept and self-esteem. Subotnik et al. (2009) describe "teachability", self-evaluation, responsiveness, self-promotion and the ability "to play the game" as dynamic skills that can be developed in the course of teaching. Commitment and persistence are the most frequently addressed factors that determine success in mathematics learning.

Students' potential at a particular moment is the function of students' educational histories that formed their knowledge and skills, as well as their affective and personality characteristics. At the same time, the advancement and realisation of students' mathematical potential depends on the learning opportunities provided to them by the educational system and the extent to which these opportunities fit their potential. Learning opportunities are multifaceted due to both the complexity of the construct of mathematical potential and the variety of views on what mathematics should be taught and how it should be taught. However, there is a consensus in the mathematics education community that learning opportunities have to be challenging, that is, to include cognitively demanding tasks which are approachable for students and evoke in students internal motivation to overcome the difficulties. MTs should provide their students with mathematically challenging content.

6.2.2 MTs' and MTEs' Proficiency as a Function of Varying Mathematical Challenge

According to the *Cambridge Dictionary Online, challenge* is "something that needs great mental or physical effort in order to be done successfully" (Cambridge University, 2019, "challenge", para 1). In mathematics classrooms, mathematical challenge is an essential characteristic of an effective learning environment directed at developing the students' mathematical reasoning. *Mathematical challenge* is *mathematical difficulty that a person is able and willing to overcome*. This implies that the cognitive demand embedded in mathematical tasks determines whether there is mathematical challenge (Silver & Mesa, 2011). However, the concept of mathematical challenge goes beyond the cognitive demand of a task and acknowledges affective aspects (e.g. willingness, curiosity and motivation) associated with coping with the task.

The essentiality of challenge for the learning process is reflected in Vygotsky's (1978) notion of zone of proximal development (ZPD). For Vygotsky, the teacher's role in the instructional process is to provide a scaffold to students when they cope with a task which is "nearly approachable" for them. In terms of cognitive load theory (Sweller, Van Merrienboer, & Paas, 1998), mathematical challenge is linked to the cognitive resources a person must activate in order to satisfy task demands (*intrinsic cognitive load*) and to the cognitive resources needed for learning of new schema (*germane cognitive load*).

Jaworski (1992) introduced the construct of the teaching triad, composed of mathematical challenge, sensitivity to students and management of learning, which are core elements of any teaching situation. One of the central roles of MTs is the initiation of meaningful mathematical activities in their classrooms (Fennema & Romberg, 1999; Simon, 1997; Steinbring, 1998). When assigning challenging tasks to a particular classroom, teachers should "feel" their students, in order to ensure that the students are able to approach the task. Moreover, development of students' mathematical reasoning is linked to the knowledgeable choice (or even design) of challenging mathematical tasks and the integration of the tasks in appropriate settings (Choppin, 2011).

Silver and Mesa (2011) argue that expert MTs value and acknowledge students' individuality and address this individuality in their practice by systematically providing all students with equitable and complete access to mathematics. Proficient teachers create stimulating, caring and inclusive environments "in which students accept responsibility for learning, take intellectual risks, develop confidence and self-esteem, work independently and collaboratively, and value mathematics" (Silver & Mesa, 2011, p. 65). That is, proficient mathematics teachers' activities are directed at the realisation of students' mathematical potential. Effective integration of mathematical challenge in the instructional process is strongly connected to the equity principle of mathematics education (NCTM, 2000).

6.3 Framing Challenging Content for MTs Using Mathematical Challenge and Mathematical Potential

Mathematical instruction that integrates mathematical challenge requires from MTs continuous scaffolding acts which allow a learner to complete tasks "which would be beyond his unassisted efforts" (Wood, Bruner, & Ross, 1976, p. 90). However, while mathematical challenge is a necessary condition for an active, student-directed (as opposed to teacher-directed) and self-regulated learning process, there is a danger that scaffolding will withdraw the challenge. In contrast, varying mathematical challenge is defined here as a type of scaffolding that preserves mathematical challenge and thus opens multiple opportunities for student-directed mathematical activities. Instructional practices of varying mathematical challenge combine both mathematical and didactical mechanisms. Figure 6.2 depicts the three major factors of varying mathematical challenge. Teachers' choice or design of tasks inherently rely on these factors or their combinations. Figure 6.3 illustrates a task: "explore and solve in multiple ways" three open mathematical problems, each accompanied by multiple solutions. This task and the problems help in explaining factors that determine mathematical challenge.

The first factor in determining mathematical challenge is associated with conceptual and structural characteristics of mathematical tasks. These characteristics include the conceptual density of problems, defined by the number of concepts and properties needed to solve the problem, the logical relationships and the length of the solution and tasks (Csapó & Funke, 2017; Silver & Zawodjewsky, 1997). For example, Problems 1, 2 and 3 (Fig. 6.3) are conceptually dense and, if solved

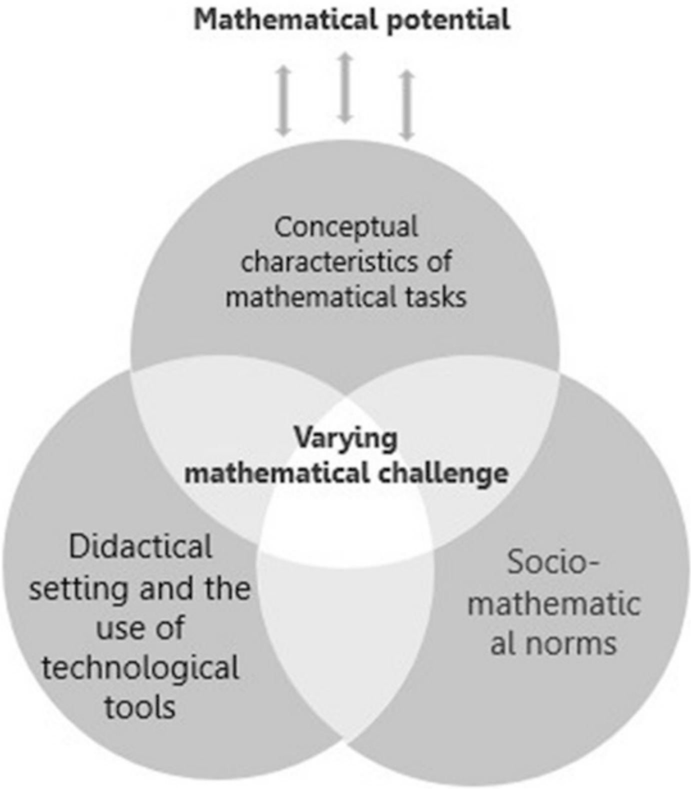


Fig. 6.2 Factors of varying mathematical challenge

algorithmically, require long solutions with algebraic manipulations. The task "explore and solve problems using as many different solution strategies as you can" that accompanies Problems 1, 2 and 3 is an explorative multiple-solution task (MST), a type of task which is creativity-directed, since it requires and develops mental flexibility and provides opportunities for the production of an original solution (Leikin, 2018). Such tasks are linked to mental flexibility, activation of a larger number of concepts and theorems and mathematical connections between different properties and representations of mathematical concepts as well as between different branches of mathematics. MSTs usually provide an opportunity for mathematical insight (Solutions 1.2, 2.3) and producing original solutions. Additionally, the task is of an explorative (open) nature since it asks solvers to discover properties that have to be proven in Problems 1, 2 and 3 and only then to prove the discovered property. It exemplifies creativity-directed tasks (Leikin, 2018), which eventually allow varying mathematical challenge such that each student can choose ways of exploring, solve problems in a number of ways according to their ability and choose the methods of solution that best fit their thinking style. No less important,

Task for Problems 1, 2 3: EXPLORE AND SOLVE PROBLEMS USING AS MANY DIFFERENT SOLUTION STRATEGIES AS YOU CAN

Problem 1: Given function $f(x) = \frac{cx}{2x+3}, x \neq -\frac{3}{2}$. Find for which ***c*** the function fulfills the condition $f(f(x)) = x$

Solution 1.1 *Algorithm-based solution*

$f(f(x)) = x;\ f(f(x)) = \frac{c\frac{cx}{2x+3}}{2\frac{cx}{2x+3}+3} \Rightarrow \frac{c\frac{cx}{2x+3}}{2\frac{cx}{2x+3}+3} = \frac{cx}{2x+3}$; … by solving equation with algebraic manipulations for $x \neq -\frac{3}{2}$

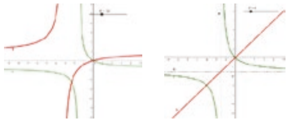
we get $c = -3$

Solution 1.2 *Exploring:* Drawing $f(x)$ in GeoGebra changing the value of parameter c and observing that for $c = -3$ the graph is exceptional

Meaning-based graphical solution

$f(f(x)) = x$ if and only if $f(x)$ is inverse to itself. This means that the graph of function $f(x)$ is symmetrical with respect to $y = x$.

$f(x) = \frac{cx}{2x+3} = 0.5c\,(1 - \frac{1.5}{x+1.5}) \Rightarrow$ the graph of $f(x)$ has two asymptotes $x = -1.5,\ y = 0.5c \Rightarrow c = -3$



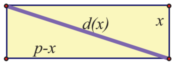
$c = 1.6$ $c = -3$

Problem 2: Of all rectangles with perimeter P which one has the shortest diagonal? To simplify solutions let's mark $p = 0.5P$

Exploring: For the solutions 2.1 and 2.2. the rectangle can be explored in dynamic environment connecting between graph of the length of diagonal
Solutions 2.3 and 2.4 are based on the comparisons of diagonals in different rectangles

Solution 2.1: *Calculus*: $d'(x) = 0 \Rightarrow x = \frac{1}{2}p$

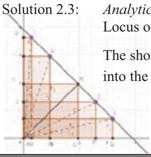
Solution 2.2: *Algebra* $(p-x)^2 + x^2$ is a parabola with vertex at $x = \frac{1}{2}p$



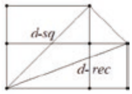
$d(x) = \sqrt{(p-x)^2 + x^2}$

Solution 2.3: *Analytical geometry*
Locus of vertexes of the rectangles $y = p - x$

The shortest diagonal corresponds to the altitude into the line $(y = x)$ thus $x = \frac{1}{2}p$



Solution 2.4: *Symmetry considerations*

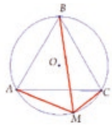


Let's compare the length of the diagonal's $d - rec$ in an arbitrary rectangle and $d - sq$ in a square, each with perimeter P:
$d - rec$ is the hypotenuse whereas $d - sq$ is a leg in the right triangle

Answer: Among all the rectangles with perimeter P the square has the shortest diagonal

Problem 3 The equilateral triangle ABC is circumscribed by the circumference with center O and radius R.
Point M is on the circumference. Find $MA^2 + MB^2 + MC^2$ (fig. 3a)

fig.3a



Solution 3.1: *Analytic geometry*:

Let ?'s circumference be $x^2 + y^2 = R$: *M(x, y), then* $B(0,R), A\,(-\frac{\sqrt{3}}{2}R; -\frac{1}{2}R)\,, C\,(\frac{\sqrt{3}}{2}R; -\frac{1}{2}R)$

$MA^2 + MB^2 + MC^2 = x^2 + (y-1)^2 + (x + \frac{\sqrt{3}}{2})^2 + (y + \frac{1}{2})^2 + (x - \frac{\sqrt{3}}{2})^2 + (y + \frac{1}{2})^2 = \cdots = 6R^2$

Solution 3.2 $MA^2 + MB^2 + MC^2 = (\overrightarrow{MO} + \overrightarrow{OA})^2 + (\overrightarrow{MO} + \overrightarrow{OB})^2 + (\overrightarrow{MO} + \overrightarrow{OC})^2$
Step a: Let's prove that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 0$

a.1 $\overrightarrow{OA} = -\overrightarrow{OD}$ (fig 3.a)

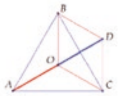


fig 3.b

a.2 Rotation of the triangle ABC by 120º about O does not change the figure
Thus vector $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$ does not change under rotation by 120o, that is $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 0$

Step b: $(\overrightarrow{MO} + \overrightarrow{OA})^2 + (\overrightarrow{MO} + \overrightarrow{OB})^2 + (\overrightarrow{MO} + \overrightarrow{OC})^2 = 3MO^2 + OA^2 + OB^2 + OC^2 + 3(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) = 6R^2$

Solution 3.2(a.2) leads to generalisation and analogy

Generalisation: For any equilateral n-polygon, if M is on the circumscribed circumference with radius R, the sum $MA_1^2 + MA_2^2 + \cdots + MA_n^2 = 2nR^2$

Analogy: The sum of the complex roots of the equation $x^n = R$ equals 0 for any natural number n.

Fig. 6.3 Creativity-directed tasks are ultimately challenging

creativity-directed tasks allow advancement of positive affect, since they usually evoke a feeling of surprise and often are exciting.

The other two factors that contribute to variations in mathematical challenge are socio-mathematical norms and didactical setting. For example, a socio-mathematical norm such as "no answer given without explanation" significantly raises the level of mathematical challenge in the classroom. From a didactical point of view, collaborative learning in which students are allowed to provide help to each other and share their ideas can both increase challenge, since explanations should be provided in the course of collaborative work, and decrease challenge, since the help of a peer is accepted (Leikin, 2004; Leikin & Zaslavsky, 1997). Technological tools (mathematics dynamic software) create opportunities for students to work with investigation tasks that lead to a true inquiry, which is usually challenging. Problems 1, 2 and 3 are formulated in a way that allows for mathematical investigations. Technological tools can reduce cognitive load by providing a visual scaffold to the thinking process, as well as raise mathematical challenge, when working with technological tools leads to the discovery of unfamiliar properties with different levels of complexity that the participants are then asked to prove.

The relationship between mathematical challenge and mathematical potential is complex. Since mathematical challenge is defined as approachable difficulty associated with positive affect associated with overcoming difficulties, all the components of mathematical potential (ability, affect, personality and previous experiences) determine both cognitive and affective processes activated when a person copes with the difficulty. The arrows from mathematical potential to mathematical challenge in Fig. 6.2 denote the relative nature of mathematical challenge and indicate that it should be linked to students' individual characteristics. The diagram in Fig. 6.4 indicates that "varying mathematical challenge" is one of the central components of learning opportunities that should be provided to students in order to advance and realise students' mathematical potential. These variations should be directed at both advancement of students' ability and evoking students' enjoyment.

To sum up, the concept of mathematical challenge and its variations frame numerous concepts, which are fundamental for teachers' knowledge, skills and practice. Varying mathematical challenge in the mathematics classroom is one of the most challenging tasks for MTs. Gaining personal experience associated with solving and designing mathematically challenging tasks for students, augmented by meta-analysis of the core elements of mathematical challenge, allows them to deepen their mathematical, psychological and didactical knowledge (Fig. 6.4). MTEs can use creativity-directed activities, which are ultimately mathematically challenging for teachers as well, accompanied by deep analysis of the ways in which the tasks can be monitored in classrooms with varying levels of mathematical challenge to fit students' mathematical potential. For example, the analysis of the "explore and solve problems using as many different solution strategies as you can" task outlined above can be used in a discussion that follows teachers' work on the task (Fig. 6.3). This implies that MTEs' knowledge and skills should integrate deep understanding of the concept of varying mathematical challenge and proficiency in designing and conducting challenging activities for MTs that combine mathematical challenge with psychological and didactical challenges that teachers meet in their everyday work.

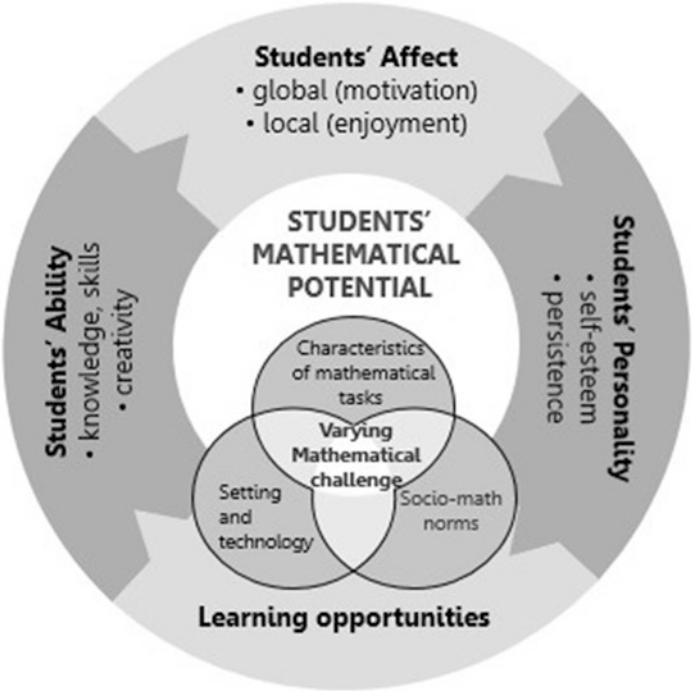


Fig. 6.4 Framing MTs' activity with students' mathematical potential and mathematical challenge

6.4 MTEs' Knowledge and Skills in Terms of MTs' Professional Potential and Challenging Content for MTs

A number of works on MTEs' competencies have indicated that there is a hierarchical structure to MTs' and MTEs' knowledge (e.g. Chick & Beswick, 2018; Jaworski, 2008; Zaslavsky & Leikin, 2004). The authors describe this hierarchy in different ways, which can be summarised as follows: to teach mathematics in school, MTs' competencies should include deep, broad and robust mathematical knowledge of school mathematics and far beyond (what to teach and why), accompanied by didactical and psychological knowledge and skills (who to teach and how to teach). In turn, MTEs are expected to be as competent as MTs in teaching mathematics when creating an environment suited for the learning and professional development of MTs. Here, I suggest considering the hierarchy between students' mathematical potential and MTs' professional potential as well as between mathematical challenge for students and challenging content for MTs.

It is broadly accepted that MTs' knowledge, skills and beliefs determine the quality of their mathematics teaching. While students' mathematical potential integrates their abilities, affective and personality characteristics and learning opportunities, I suggest *MTs' professional potential* be considered a construct that integrates teachers' knowledge and competencies, their beliefs and emotions, their personality and learning opportunities and teaching experiences.

Mathematics education literature suggests several models of MTs' knowledge and proficiency. Following Shulman (1986) who introduced the concept of teachers' content knowledge (including subject matter knowledge, pedagogical content knowledge (PCK) and curricular content knowledge), Ball, Hill, and Bass (2005) presented a concept of mathematics knowledge for teaching that provided a more precise categorisation of PCK and introduced a concept of horizon knowledge within the bounds of school mathematics. Zazkis, Leikin, and Jolfaee (2011) described advanced mathematical knowledge for teaching and stressed the importance of university mathematics knowledge, including knowledge of concepts and theorems that underlie school mathematics, as a critical component of secondary school teachers' knowledge, while Zazkis and Mamolo (2011) extended the concept of MTs' horizon knowledge to mathematics beyond school mathematics. The importance of the relationship between teachers' beliefs and their practice is also widely accepted (Calderhead, 1996; Pajares, 1992; Pepin & Roesken-Winter, 2015; Thompson, 1992). The importance of the affective domain can be seen in Radford's (2015) analysis, which demonstrates that emotions frame attitudes towards people and events and are implicated in mathematical thinking and, thus, are an important characteristic of teachers' proficiency. Less explored in mathematics education is the relationship between teachers' personalities and their instructional practices. However, Göncz (2017) argued that a teacher's personality has a significant impact on their interactions with their students and the extent to which students are involved in the learning process. Clearly, teachers' educational histories and their previous experiences are reflected in MTs' systematics and craft knowledge (Kennedy, 2002).

There is a clear hierarchy between MTs' professional potential and students' mathematical potential. Students' mathematical potential must be a fundamental part of teachers' knowledge. This is to say teachers need to be aware that students' learning not only is a function of their mathematical knowledge and skills but rather depends on domain-general cognitive traits, students' motivation, beliefs and their learning histories as well. Moreover, as presented earlier in this chapter, understanding of the concept of mathematical challenge and its variations and the ability to activate it are also components of MTs' professional potential. Figure 6.5 depicts this hierarchy.

Based on this theory, if MTs' goal is to realise students' mathematical potential by means of challenging mathematical content, MTEs' goal is to realise MTs' professional potential by means of diverse, challenging contents and practices. This perspective is consistent with Goos's (2009) view of "the teacher's ZPD as a set of possibilities for development that are influenced by their knowledge and beliefs, including their mathematical knowledge, pedagogical content knowledge, and beliefs about mathematics and how it is best taught and learned" (p. 212). The hierarchy is twofold: first, it assumes the inclusion of concepts of students' mathematical potential and mathematical challenge in MTs' knowledge and skills as well as in MTs' learning opportunities and teaching experiences. Second, it assumes that MTEs' knowledge and skills should integrate a concept of MTs' potential with

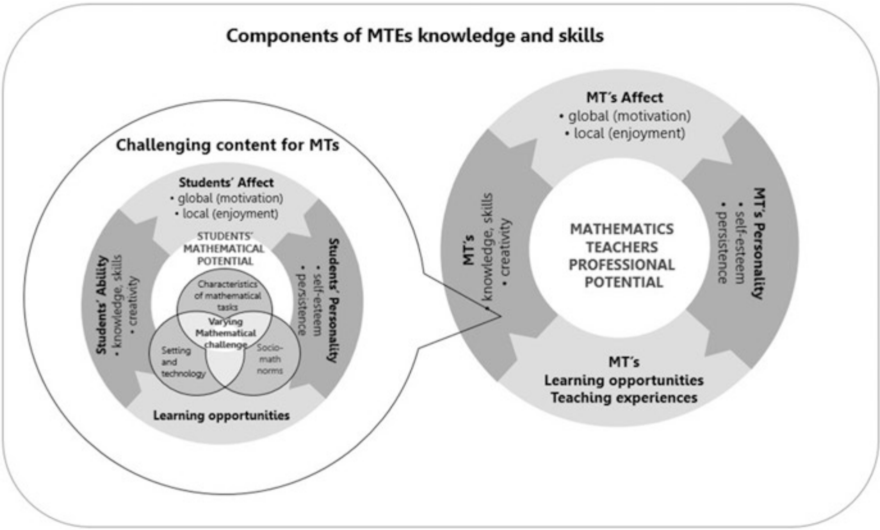


Fig. 6.5 Hierarchical structure of MTs' professional potential as a component of MTEs' knowledge and skills

precise understanding of all its components as well as an appreciation of different types of challenging content for MTs.

In sum, MTEs' professional potential integrates MTEs' knowledge and skills linked to students' mathematical potential and to MTs' professional potential along with MTEs' affect, personality and their learning opportunities and professional histories. Such a view of MTE as a complex profession with unique and extremely important goals leads to the understanding that MTEs should be provided with learning opportunities integrated in special programmes for the education of MTEs.

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Chapter 7
Learning to Teach Mathematics: How Secondary Prospective Teachers Describe the Different Beliefs and Practices of Their Mathematics Teacher Educators

Margaret Marshman

In secondary initial teacher education, mathematics teacher educators (MTEs) generally include mathematicians who teach the mathematical content courses and mathematics educators who teach the mathematics curriculum and pedagogy courses. In part because of this, mathematics teaching and learning in schools is usually different from mathematics in university. We also know that the way in which teachers teach is influenced by their beliefs. Prospective teachers' beliefs are influenced by their previous experiences of learning mathematics as a school student, the MTEs who teach them, the curriculum documents that they study, and their practicum experiences. This chapter begins by defining beliefs and then reviews the literature of beliefs about mathematics and its teaching and learning. The beliefs about mathematics, and mathematics teaching and learning of Australian MTEs and secondary mathematics prospective teachers are documented. The chapter explores how prospective teachers negotiate the different beliefs and practices of their MTEs and the impacts of this on the ways in which they plan to teach. This chapter reports on a study in which MTEs and prospective teachers were initially surveyed about their beliefs about mathematics and mathematics teaching and learning. Follow-up interviews further explored MTEs' beliefs, their decision-making about the pedagogy they used, the links between their practices of mathematics and their teaching, and the links with the practices of mathematics in schools. Interviews with the prospective teachers asked about how they were taught mathematics, how they were taught to teach mathematics, and how they negotiated any differences between the way they were taught mathematics and the way they were taught to teach mathematics.

There is a growing body of research on the influence of teachers' beliefs on their teaching practice and how these beliefs influence their students' beliefs about

M. Marshman (✉)
School of Education, University of the Sunshine Coast, Sippy Downs, QLD, Australia
e-mail: mmarshma@usc.edu.au

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_7

mathematics and their capability to learn mathematics (Grootenboer, 2008; McLeod, 1992; Mosvold & Fauskanger, 2014; Pajares, 1992). Despite this, there is not a clear definition of the concept of beliefs (e.g. Pajares, 1992). Generally, beliefs are seen as personally held assumptions which predispose the person to a particular type of action (Rokeach, 1968). Philipp (2007) defined beliefs as:

> psychologically held understandings, premises, or propositions about the world that are thought to be true. … Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions towards action. (p. 259)

Here beliefs will be used in the same way as Ajzen and Fishbein (1980) and Beswick (2005, p. 39), who maintain that a belief is "anything that an individual regards as true".

It is generally accepted that personally held beliefs are organised into some sort of structure (Green, 1971; Rokeach, 1968). Beliefs can exist in relatively independent clusters (Green, 1971), which can help to explain why individuals can hold seemingly contradictory beliefs about the discipline of mathematics, school mathematics, and how mathematics is best learned (Beswick, 2005, 2012; Jorgensen, Grootenboer, Niesche, & Lerman, 2010; Philipp, 2007). Beliefs cannot be directly observed and need to be inferred from people's words and actions (Grootenboer & Marshman, 2016; Pajares, 1992). Leatham (2006) described mathematics teachers' beliefs as a sensible system in which an individual's beliefs make sense to them, are internally consistent to them, and fit with their other beliefs. It does not, however, necessarily follow that an individual can express their beliefs or even be aware of them (Leatham, 2006).

We can infer someone's beliefs from their actions, but we cannot know with certainty which belief(s) they were acting on. When a teacher's actions appear to be inconsistent with the beliefs they have been inferred to have, it may be that we have "either misunderstood the implications of the belief, or that some other belief took precedence in that particular situation" (Leatham, 2006, p. 95). This can be complicated by tacit and powerful personal and social reasons (e.g. satisfying the schools' ethos or fitting in with a social group). Due to the contextual and clustered nature of beliefs, individuals may express different beliefs depending on the situation or context.

7.1 Beliefs About Mathematics and Mathematics Teaching

Teachers' beliefs about the teaching and learning of mathematics and the social context in which they teach, along with the degree to which teachers think about and reflect on their teaching, will determine what happens in their classroom (Ernest, 1989b). Ernest described three different views of mathematics: instrumentalist, Platonist, and problem-solving, each of which can be related to learning and teaching mathematics. From an instrumentalist perspective, mathematics is a collection of procedures, facts, and skills, and the teacher is an instructor whose role is to enable students to master the procedures and skills by carefully following the textbook or prescribed procedure. The Platonist view defines mathematics as a structured, unchanging body of knowledge that is discovered rather than created. Hersh (1997) described the Platonist view as follows: "mathematical entities exist outside space and time, outside thought and matter, in an abstract realm independent of any consciousness, individual or social" (p. 9). The teacher with a Platonist view is an explainer whose role is to support students to build conceptual understanding. The problem-solving view of mathematics is that it is human creativity and invention that drives a dynamic, growing field within a social and cultural context. In this view, the teacher is a facilitator helping students become confident problem-posers and problem-solvers (Ernest, 1989b).

Dreyfus and Eisenberg (1986) investigated the aesthetic value of mathematics and recommended that teaching include the "aha" of problem-solving and that "considerations of two or more solution paths could bring practical benefits by developing a familiarity with different solution methods, and deeper conceptual understanding" (p. 9). Schoenfeld and Herrmann (1982) explored differences in problem-solving by experts and novices, pre- and post-problem-solving course, showing that following the course, their students "perceived problem relatedness more like the experts" (p. 484).

Burton (1999) interviewed 70 mathematicians from the United Kingdom and Ireland who described mathematics as making sense of the world, seeing the connections between mathematics and the "real" world and between the different aspects of mathematics. Most mathematicians noted the collaborative or cooperative cultural climate of their research, describing mathematics as "personally- and culturally/socially-related" (Burton, 1999 p. 139). Many applied mathematicians and statisticians explained that "[y]ou know when you know, because it works, or, sometimes, because you can create a picture which convinces you" (Burton, 1999, p. 134). Mathematicians described mathematics as "a world of uncertainties and explorations, and the feelings of excitement, frustration and satisfaction, associated with these journeys, but, above all, a world of connections, relationships and linkages" (Burton, 1999, p. 138). This model of mathematics fits well with learners at any level but does not fit with the transmission model of teaching mathematics "where mathematics is presented to learners in disconnected fragments … [which] deprives them of the very pleasure of which these research mathematicians speak - the pleasure of making a connection" (Burton, 1999, p. 139). Although the interviews Burton conducted were not specifically about teaching, many mathematicians said they did not think much about their teaching, nor did they convey to their students "the struggle and the pleasure … of doing mathematics" (p. 140). According to the mathematicians, students needed to learn mathematics before they could begin mathematising, which Burton (1999) described as "objective mathematics they, as teachers, thrust towards reluctant learners" (p. 20).

Mura (1993, 1995) surveyed mathematicians and mathematics educators in Canada, asking open questions about their view of mathematics. Although many were reluctant to respond to philosophical and historical questions, mathematicians' definitions of mathematics were concerned with the design and analysis of models abstracted from reality; logic, rigour, accuracy, and reasoning; and the study of axiomatic systems (Mura, 1993). The most common themes to which mathematics educators alluded were patterns, logic, and models of reality (Mura, 1995). The views of mathematics educators and mathematicians differed in that the former were more concerned with patterns and mathematicians with logic. It may be that mathematics educators' views align with Schoenfeld's (1992) influential definition of mathematics as:

> an inherently social activity in which a community of trained practitioners (mathematical scientists) engage in the science of patterns – systematic attempts based on observation, study, and experimentation to determine the nature of principles of regularities in systems defined axiomatically or theoretically ("pure mathematics") or models of systems abstracted from real world objects ("applied mathematics"). (p. 34)

Carlson and Bloom (2005) studied how mathematicians solve problems and their emotional responses to doing so. For these mathematicians, it was important to make sense of problems and to manage their frustration and anxiety. Danish mathematicians chose mathematical problems strategically (Misfeldt & Johansen, 2015), ensuring problems contributed to their "identity as a mathematician" (p. 368), were interesting and potentially fruitful, fitted within their skills and competencies, and commanded an audience. Similarly, school teachers chose interesting problems for which their students had the skills and competencies and were potentially fruitful (Misfeldt & Johansen, 2015).

More recently Brandt, Lunt, and Meilstrup (2016) surveyed US and Canadian mathematicians and mathematics educators, asking them to rank, according to importance, processes used in doing mathematics. For lower-level mathematics courses at university (college algebra, trigonometry, or calculus), mathematicians identified problem-solving, acquiring content knowledge, and acquiring informal logical reasoning, whereas mathematics educators identified problem-solving, conjecture/generalisation/exploration, and making connections. For higher-level mathematics courses (abstract algebra, number theory, or topology), mathematicians valued proving, acquiring content knowledge, and conjecture/generalisation/exploration, whereas mathematics educators identified conjecture/generalisation/exploration, proving, and problem-solving (Brandt et al.). These mathematicians and mathematics educators described "doing mathematics" as investigating problems, looking for patterns, and understanding the mathematical ideas of others. "[s]imply mimicking procedures or reciting phrases with no understanding was not doing mathematics. Instead, doing mathematics required some understanding of the underlying mathematical principles" (Brandt et al., 2016 p. 765).

Lockwood, Ellis, and Lynch (2016) maintained that although students do not need to be aware of mathematicians' practices, those teaching them do. They showed that understanding how mathematicians think with examples is useful in teaching "to help [undergraduate] students learn to think critically about how they can draw upon examples as they engage in exploring and proving conjectures" (p. 194). Leikin, Zazkis, and Meller (2018) interviewed four mathematicians who taught prospective teachers as part of larger cohorts. While these mathematicians acknowledged that only some of the mathematical content, problem-solving strategies, and techniques of proof would be used by teachers in classrooms, they believed that the mathematical language, distinctions between problem-solving strategies and algorithms, the beauty of mathematics, mathematical history, understanding the meaning of theorems and definitions, and abstraction would be useful for school teachers (Leikin et al., 2018). However, these mathematicians were more interested in preparing professional mathematicians than teachers, which suggests that either they considered their roles as MTEs as less important than their role as educators of prospective mathematicians or they had not considered the possibly different needs of prospective teachers (Leikin et al., 2018).

Australian curriculum documents (e.g. *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority (ACARA), n.d.); *Mathematics K-10 Syllabus* (Board of Studies New South Wales, 2012)), as with international curriculum documents (e.g. Council of Chief State School Officers, 2010; Ministry of Education, Singapore, 2012), are informed by a constructivist view of learning where the teacher's role is "to facilitate, on the part of students, significant cognitive restructuring that goes beyond merely adding to and adjusting existing constructions" (Beswick, 2005, p. 4). This encourages a problem-solving pedagogy in a supportive classroom environment (Cobb, Wood, & Yackel, 1991).

7.2 This Study

This project used a mixed-methods methodology. A quantitative survey of Australian MTEs' and prospective secondary teachers' beliefs about mathematics and mathematics teaching and learning was conducted in 2017. MTEs and prospective teachers were then interviewed in order to explore in more depth the responses given in the survey.

The survey included demographic questions and 26 5-point Likert scale items, the aim of which was to elicit responses (strongly disagree to strongly agree) about the participants' beliefs. These items were replicated from Beswick's (2005) survey of teacher beliefs – *Beliefs about mathematics, its teaching and its learning.*

The online survey was sent to mathematicians, statisticians, and mathematics educators who were involved in initial teacher education programmes in Australia, inviting them to participate. Eighty-two academics (out of 120 who started the survey) completed all items. The respondents represented 35 Australian universities and 5 international universities, while 3 were seeking employment and 3 were retired. The overseas MTEs all reported having previously taught Australian secondary prospective mathematics teachers. Forty-nine (60%) were male, 33 (40%) were female, and the median age was 46 years. Sixty respondents (73%) taught mathematics content courses only, 8 (10%) taught mathematics pedagogy only, and 14 (17%) taught both pedagogy and mathematics (though it was likely that in most cases, this was not necessarily mathematics content courses but mathematics content as part of their pedagogy courses or to prospective primary teachers). The qualifications of the respondents included PhD in mathematics (44, 54%), PhD in education (12, 15%), PhD in mathematics and a Graduate Diploma in Education (GDE) (11, 13%), Master's or Honours in mathematics (7, 9%), Master of Education (3, 4%), and initial teacher education qualifications (5, 6%).

The online survey was also sent to prospective secondary mathematics teachers at three universities in south east Queensland. Twenty-five (of 39) prospective secondary mathematics teachers responded to all the statements. Nineteen were studying an undergraduate programme that included both mathematics and education courses, and six were undertaking a postgraduate programme in which only education courses were studied. The six in this last category had completed mathematics courses as part of a previous qualification.

The survey data were analysed using SPSS and included descriptive statistics and one-way between groups ANOVA with Bonferroni post hoc tests, eliminating those items that violated the Levene test for homogeneity of variance. Queensland survey respondents were invited to participate in a semi-structured interview to further explore their beliefs about mathematics and its teaching and learning.

Of the seven MTEs interviewed, five taught mathematics, one taught mathematics education, and one taught both mathematics education and mathematics content courses. The following questions were used as part of semi-structured interviews which were audio-recorded, transcribed, and analysed to identify concepts and themes related to MTEs' practices of doing and teaching mathematics and the ways in which mathematics is taught in schools:

1. Will you please describe how you teach mathematics in a lecture and a tutorial?
2. How would you describe any perceived differences (if any) between the way mathematics is practised and the way mathematics is taught?
3. How would you describe any differences between how mathematics is taught in schools and university?

Seven prospective teachers also participated in semi-structured interviews. Of these, six were studying an undergraduate qualification, and one was studying a postgraduate qualification after spending 10 years in the workforce in a non-teaching role. The questions to which the seven prospective teachers responded were:

1. How would you describe the difference, if any, in the way you are taught mathematics and the way you are taught to teach mathematics?
2. Do you feel any tension between the ways you are taught mathematics and the way you think you learn it best?
3. How would you best describe the different ways your lecturers and tutors view mathematics? Do you ever find it confusing? Please explain.

Burton's (1995) theoretical framework for knowing mathematics, which she developed and tested in her study of research mathematicians (Burton, 1999), was used to further analyse the interviews and to test the applicability of the model to MTEs and prospective teachers. The model consists of five categories:

- Person- and cultural/social-relatedness
- Aesthetics
- Intuition and insight
- Different approaches (particularly to thinking)
- Connectivities (Burton (1995, 1999, p. 122))

Person- and cultural/social-relatedness recognises that knowing mathematics is "a function of who is claiming to know, related to which community, how that knowing is presented, what explanations are given for how that knowing was achieved, and the connections demonstrated between it and other knowings" (Burton, 1995, p. 287). Intuition and insight refer to being able to understand the idea instinctively and aesthetics to the beauty of the mathematics and different approaches to recognition of the different ways that ideas can be represented. Connectivities are the links between the mathematics at hand and other areas of mathematics, and/or with real-world data. In the following section, the survey results and discussion are organised around the major item categories in Beswick's survey, namely, participants' beliefs about mathematics, learning mathematics, and teaching mathematics.

7.3 Survey Results and Discussion

7.3.1 Beliefs About Mathematics

Most MTEs and prospective teachers (96%) agreed or strongly agreed that mathematics was a "beautiful, creative and useful human endeavour" and "both a way of knowing and a way of thinking", whereas only 10% of MTEs and 20% of prospective teachers agreed or strongly agreed that mathematics is "computation". These responses, shown in Table 7.1, indicate that participants were inclined to have problem-solving views (Ernest, 1989a, b) of mathematics as a discipline. The findings are consistent with those of Grigutsch and Törner (1998) that "mathematicians view mathematics as a discovery and understanding process" (p. 29).

Table 7.1 Survey responses on beliefs about mathematics collapsed into a three-point scale

		Educators			Prospective teachers		
No.	Item	D	U	A	D	U	A
9	Mathematics is a beautiful, creative, and useful human endeavour that is both a way of knowing and a way of thinking	0 0%	3 4%	79 96%	0 0%	1 4%	24 96%
20	Mathematics is computation	68 83%	6 7%	8 10%	14 56%	6 24%	5 20%

Note: *D* strongly disagree or disagree, *U* undecided, *A* strongly agree or agree

7.3.2 Beliefs About Teaching Mathematics

Table 7.2 summarises the combined MTE and prospective teacher responses to the survey items about teaching mathematics. Instances of 90% or more agreement have been highlighted for ease of viewing. Most MTEs (at least 83%) disagreed or strongly disagreed with traditional teaching methods (as reflected in Items 22, 23, 25, and 26 in Table 7.2) as did at least 80% of prospective teachers. Traditional teaching methods, including telling students how to solve mathematical problems,

Table 7.2 Survey responses about teaching mathematics (Beswick, 2005)

		MTEs			Prospective teachers		
	Item	D	U	A	D	U	A
13	Justifying the mathematical statements that a person makes is an extremely important part of mathematics	1 1%	2 2%	79 96%	1 4%	3 12%	21 84%
10	Allowing a student to struggle with a mathematical problem, even a little tension, can be necessary for learning to occur	1 1%	3 4%	78 95%	2 8%	4 16%	19 76%
15	Teachers can create, for all students, a non-threatening environment for learning mathematics	5 6%	13 16%	64 78%	1 4%	0 0%	24 96%
11	Students always benefit by discussing their solutions to mathematical problems with each other	7 9%	18 22%	57 70%	3 12%	3 12%	19 76%
12	Persistent questioning has a significant effect on students' mathematical learning	5 6%	22 27%	55 67%	2 8%	6 24%	17 68%
14	As a result of my experience in mathematics classes, I have developed an attitude of inquiry	8 10%	20 24%	54 66%	2 8%	9 36%	14 56%
19	Mathematical material is best presented in an expository style: demonstrating, explaining, and describing concepts and skills	25 30%	23 28%	34 41%	7 28%	4 16%	14 56%
26	If a students' explanation of a mathematical solution doesn't make sense to the teacher, it is best to ignore it	76 93%	5 6%	1 1%	25 100%	0 0%	0 0%
22	I would feel uncomfortable if a student suggested a solution to a mathematical problem that I hadn't thought of previously	75 91%	1 1%	6 7%	20 80%	3 12%	2 8%
23	It is not necessary for teachers to understand the source of students' errors; follow-up instruction will correct their difficulties	75 91%	3 4%	4 5%	24 96%	0 0%	1 4%
25	It is important to cover all the topics in the mathematics curriculum in the textbook sequence	68 83%	7 9%	7 9%	17 68%	6 24%	2 8%
24	Listening carefully to the teacher explain a mathematics lesson is the most effective way to learn mathematics	54 66%	21 26%	7 9%	17 68%	4 16%	4 16%

Note: *D* strongly disagree or disagree, *U* undecided, *A* strongly agree or agree are opposed to the social constructivist conceptions of learning mathematics for understanding by actively building on previous knowledge and experience espoused by the National Council of Teachers of Mathematics (NCTM) (2000). Items 10–15 describe teaching strategies that aim to support students to construct knowledge. MTEs and prospective teachers generally agreed or strongly agreed with these statements, particularly "allowing students to struggle" (Item 10), with which 95% of MTEs and 78% of prospective teachers agreed, and "the importance of justifying statements" (Item 13), with which 96% of educators and 84% of prospective teachers agreed.

Almost all prospective teachers (96%) and 78% of MTEs agreed or strongly agreed that the creation of a "nonthreatening environment" (Item 15) was desirable, and there was general agreement (67% of educators and 68% of prospective teachers) as to the importance of "persistent questioning in learning" (Item 12). Sixty-six per cent of MTEs and 56% of prospective teachers believed they had "developed an attitude of inquiry because of classroom experiences" (Item 14). However, 41% of MTEs and 56% of prospective teachers agreed that mathematics is learned best when taught using an "expository style" (Item 19), while 30% of MTEs and 28% of prospective teachers disagreed that was the case.

Overall, the survey results suggest that the respondents shared beliefs about the importance of supporting students to construct their own knowledge. However, both MTEs and prospective teachers were, on average, less comfortable with the use of questioning and less inclined to agree that they developed an attitude of inquiry in the classroom.

7.4 Beliefs About Learning Mathematics

Responses to items concerning beliefs about learning mathematics are summarised in Table 7.3. Items 1–3 and 5–8 describe approaches to learning mathematics that are consistent with Ernest's problem-solving view of mathematics. At least 90% of MTEs and prospective teachers agreed with statements that teachers needed to "motivate students to solve their own problems" (Item 1), to "give students opportunities to reflect on and evaluate their own mathematical understanding" (Item 3), and that "ignoring the mathematical ideas that students generate themselves can seriously limit their learning" (Item 2). Similarly, at least 90% of MTEs and at least 80% of prospective teachers agreed that "effective mathematics teachers enjoy learning and 'doing' mathematics themselves" (Item 5), and 88% agreed that "knowing how to solve a mathematics problem is as important as getting the correct solution" (Item 6). There was somewhat less agreement with the statement that teachers should be "fascinated with how students think" (Item 7) and "providing interesting problems to be investigated in small groups" (Item 8).

Only 2% of MTEs and 20% of prospective teachers agreed that "telling students the answer was an efficient way of facilitating mathematics learning" (Item

Table 7.3 Survey responses on beliefs about learning mathematics (Beswick, 2005)

			MTEs			Prospective teachers		
	Item	Number	D	U	A	D	U	A
6	Knowing how to solve a mathematics problem is as important as getting the correct solution		1 1%	1 1%	80 98%	2 8%	1 4%	22 88%
3	It is important for students to be given opportunities to reflect on and evaluate their own mathematical understanding		2 2%	1 1%	79 96%	0 0%	1 4%	24 96%
4	It is important for teachers to understand the structured way in which mathematics concepts and skills relate to each other		1 1%	2 2%	79 96%	0 0%	0 0%	25 100%
1	A vital task for the teacher is motivating students to solve their own mathematical problems		4 5%	1 1%	77 94%	1 4%	0 0%	24 96%
5	Effective mathematics teachers enjoy learning and "doing" mathematics themselves		0 0%	5 6%	77 94%	3 12%	2 8%	20 80%
2	Ignoring the mathematical ideas that students generate themselves can seriously limit their learning		4 5%	4 5%	74 90%	0 0%	2 8%	23 92%
7	Teachers of mathematics should be fascinated with how students think and intrigued by alternative ideas		5 6%	10 12%	67 82%	1 4%	3 12%	21 84%
8	Providing students with interesting problems to investigate in small groups is an effective way to teach mathematics		4 5%	23 28%	55 67%	2 8%	4 16%	19 76%
16	It is the teacher's responsibility to provide students with clear and concise solution methods for mathematical problems		19 23%	24 29%	39 48%	1 4%	7 28%	17 68%
17	There is an established amount of mathematical content that should be covered at each grade level		20 24%	22 27%	40 49%	1 4%	6 24%	18 72%
18	It is important that mathematics content be presented to students in the correct sequence		21 26%	23 28%	38 46%	1 4%	2 8%	22 88%
21	Telling the students the answer is an efficient way of facilitating their mathematics learning		61 74%	19 23%	2 2%	18 72%	2 8%	5 20%

Note: *D* strongly disagree or disagree, *U* undecided, *A* strongly agree or agree

21), and 74% of MTEs and 72% of prospective teachers disagreed with this statement. A quarter of MTEs and 8% of prospective teachers were undecided. Together these survey responses suggest the prevalence, among MTEs and prospective teachers, of a belief in the value of problem-solving for mathematics learning.

7.4.1 Differences Between the Beliefs of Subgroups of MTEs and Between MTEs and Prospective Teachers

The data were analysed for differences between MTEs who taught only mathematics content courses, MTEs who indicated that they taught both mathematics content and mathematics courses, MTEs who taught pedagogy only, and prospective teachers. Mean responses were also compared for MTEs categorised according to their highest mathematics and/or mathematics education qualification. The groups along with the numbers and percentage of MTEs in each group were as follows: no PhD (15, 18%), PhD in mathematics education (12, 15%), PhD in mathematics (44, 54%), and PhD in mathematics as well as a Graduate Diploma in Education (GDE) (11, 13%).

Survey responses differed among MTEs, depending on their teaching responsibilities and qualifications, as well as between some of these subgroups of MTEs and prospective teachers. Sixty (73%) MTEs taught only mathematics or statistics content courses, while eight (10%) taught only pedagogy courses. Fourteen (17%) taught both discipline content and pedagogy. It is unclear whether those who said they taught both mathematics and pedagogy taught separate content courses and pedagogy courses or whether they taught mathematics content as part of education courses primarily aimed at teaching prospective teachers how to teach mathematics.

Two of the items in Table 7.4 that concern beliefs about learning mathematics showed a statistically significant difference ($p < 0.05$) between some MTEs and prospective teachers (PSTs in Table 7.4). MTEs who were mathematicians had a higher mean agreement than did prospective teachers to Item 5: "teachers enjoy learning and 'doing' mathematics themselves". This suggests, unsurprisingly, that the prospective teachers tended to see themselves as teachers of mathematics rather

Table 7.4 Differences in beliefs of educators categorised according to their teaching responsibility and prospective teachers (PSTs)

	Statistic	MTECs	MTEPs	Both	PSTs	F (*p* value)
Effective mathematics teachers enjoy learning and "doing" mathematics themselves (5)	Mean SE	4.58^{b} 0.08	$4.75^{a, b}$ 0.16	$4.43^{a, b}$ 0.20	4.04^{a} 0.20	3.923 (0.011)
Mathematical material is best presented in an expository style: demonstrating, explaining, and describing concepts and skills (19)	Mean SE	$3.37^{a, b}$ 0.13	2.38^{a} 0.32	$2.79^{a, b}$ 0.24	3.48^{b} 0.22	3.708 (0.014)
Mathematics is computation (20)	Mean SE	$1.80^{a, b}$ 0.13	$2.00^{a, b}$ 0.27	1.57^{a} 0.25	2.52^{b} 0.19	3.898 (0.011)

PST prospective teacher, *MTEC* Content only MTEs, *MTEP* Pedagogy only MTEs

$^{a, b}$Mean values within a row with unlike superscript letters are significantly different ($p < 0.05$). For example, for Item 5, the (Bonferroni-adjusted) t-test results show a small p comparing "teaching content" with "prospective teachers", but neither was different from "teaching content and pedagogy" and "teaching pedagogy" than doers of mathematics. MTEs had a lower mean agreement than did prospective teachers as to the value of using an "expository style" of teaching (Item 19).

Each of the four groups, on average, disagreed that "mathematics is computation," (Item 20) but those who taught both mathematics content and pedagogy had a significantly lower mean agreement than did the prospective teachers. The MTEs tended to have more problem-solving belief about mathematics than did the prospective teachers.

7.5 Differences Related to MTEs' Qualifications

There were six items about learning mathematics for which there were statistically significant differences ($p < 0.05$), as shown in Table 7.5. Educators with a mathematics PhD had a lower mean agreement with the statement "ignoring students' mathematical ideas can limit their learning" (Item 2), compared to those who also had a GDE. MTEs with a PhD in mathematics education had a higher mean agreement than those with either no PhD or a mathematics PhD with Item 8: "providing students with interesting problems to investigate in small groups". Those with no PhD had a higher mean agreement for there being a "set amount of mathematical content to cover at each level" (Item 17). MTEs with a mathematics education PhD had a lower mean agreement than those with no PhD with "mathematics must be presented in the correct sequence" (Item 18). MTEs with a mathematics PhD and those with no PhD had a stronger mean agreement than mathematics educators that "mathematics should be presented in an expository style" (Item19).

These results suggest that gaining postgraduate education qualifications, either a PhD or a PhD and a GDE, provided MTEs with an opportunity to reflect on how

Table 7.5 Differences in beliefs of MTEs according to qualifications

Abbreviated item and number	Statistic	PhD M, S	PhD Ed	PhD M and GDE	No PhD	F (p-value)
Ignoring students' mathematical ideas can limit their learning (2)	Mean	4.02^{a}	$4.58^{a,b}$	4.82^{b}	$4.20^{a,b}$	3.228
	SE	0.144	0.149	0.122	0.262	(0.027)
Students with interesting problems to investigate in small groups (8)	Mean	3.68^{b}	4.50^{a}	$4.36^{a,b}$	3.53^{b}	4.729
	SE	0.121	0.195	0.203	0.322	(0.004)
An established amount of content to be covered at each level (17)	Mean	$3.41^{a,b}$	2.75^{a}	$3.00^{a,b}$	3.87^{b}	3.090
	SE	0.157	0.329	0.270	0.256	(0.032)
Content should be presented in the correct sequence (18)	Mean	3.57^{a}	$2.50^{a,b}$	$3.45^{a,b}$	3.40^{b}	3.451
	SE	0.154	0.314	0.312	0.254	(0.020)
Mathematics is best presented in an expository style (19)	Mean	3.39^{b}	2.50^{a}	$2.55^{a,b}$	3.53^{b}	5.130
	SE	0.135	0.359	0.247	0.236	(0.003)

M mathematics, *S* statistics, *Ed* education

^{a,b}Mean values within a row with unlike superscript letters are significantly different ($p < 0.05$). For example, for Item 19, the (Bonferroni-adjusted) t-test results show a small *p* comparing "a PhD in mathematics or statistics" with "a PhD in education", but not between the others mathematics is learned, such that their mean agreement for statements about mathematics learning consistent with problem-solving views of mathematics was higher than those of MTEs with mathematics PhDs or no PhD. Institutional pressures may mean that MTEs with mathematics PhDs spend less time than other groups reflecting on and developing their teaching since they are typically employed as research mathematicians.

7.6 Interviews with MTEs and Prospective Teachers

Interviews with three MTEs were conducted to deepen understanding of the survey data. Burton's (1995, 1999) categories for knowing mathematics (person- and cultural/social-relatedness, aesthetics, intuition and insight, different approaches (particularly to thinking), and connectivities) were used to identify ways in which these MTEs knew mathematics. Understanding how MTEs "know mathematics" may help with inferring their beliefs about mathematics and its teaching and learning. Following that, prospective teachers' responses to interview questions about their experiences of being taught mathematics at school, and learning to teach mathematics at university, are reported. In these sections, italics are used to show direct quotes from the interviews and to highlight Burton's (1995, 1999) categories.

7.6.1 The Case of Ryan

Ryan was a pure mathematician who believed the *aesthetics* of mathematics were important. He described mathematics as beautiful and bringing joy: "*There's a famous mathematician by the name of Hardy who said that all mathematics should be beautiful. There was no room for ugly mathematics. I would add … that the experience of doing mathematics should be one that brings joy.*"

This enjoyment extended to his teaching which he described as quite expository, although he wanted the students to understand:

> I enjoy … presenting problem solutions to students

Ryan thought that *intuition* was important and that one needed time for thinking. To him the practices of mathematics involved looking at problems and thinking about them before putting pen to paper:

> … this idea of being able to look at a problem and then sometimes deciding that the best way to advance the problem is to walk away from it for a while and just let the mind percolate on the challenges of the problem. …

This was reflected in his description of his teaching. He said that he "*advise[d] students to do [walk away from the problem for a while]*". Ryan's preference for using pen and paper *(*"*I much prefer to use pen and paper*"*)* was also reflected in his teaching with the use of the document camera to project his writing onto the screen for students to read. He believed this allowed students time to think as it slowed the pace of the lecture.

> I use the document camera to write out a fresh set of notes... writing at the speed of thought makes a much better connection … I encourage students to have a folio of worked solutions, so they have not a model solution but solution models.

Ryan talked about the traditional *culture* of mathematics – that is, "*the long tradition of mathematics to think logically with precision and without ambiguity*" – as something he shared with his postgraduate students rather than undergraduate students. He believed he was trying to cater to a range of students with diverse disciplinary backgrounds in engineering, science, and education, students with a wide range of abilities.

7.6.2 The Case of Paul

Paul was an MTE who taught mathematics pedagogy to prospective teachers. During the interview, he did not talk about his beliefs about mathematics. He began his courses by talking with his prospective teachers about:

> … what it might involve, teaching in a secondary school … get them to see the world from the eyes of a student.

Paul thought that it was important for prospective teachers to have a variety of activities ("*different approaches*") in their toolkits and that they considered how their students learned and that they adapted their teaching accordingly. He explained:

> [w]hat I would hope, is for any teacher, that they would think about how kids learn and then try and develop their pedagogy as best they can, based on what they think is going to work for their kids. … I'd encourage them to have as much variety [as possible].

Paul told his prospective teachers that sometimes they would use explicit teaching methods but that doing activities was valuable. For example, in his class for prospective senior secondary teachers, he had them use calculus to model the motion of Hot Wheel cars:

> I tend to show things which are more based around mathematical modelling and activities and different ways of approaching the mathematical ideas. … when we do an introduction to calculus for example, we will start by giving them Hot Wheel sets … do an experiment of sorts and then try and answer a question which will involve them using some sort of calculus. … [we] use the mathematical model to solve the problem. … I try not to give them answers. I just prompt them and ask questions.

Paul talked about the importance of building *insight*, which for him involved "*helping them to generalise some of the ideas*".

7.6.3 The Case of Sam

To Sam, an applied mathematician who taught mathematics courses to student cohorts that included prospective teachers, doing mathematics involves *different approaches* and *connectivities*. He talked about exploring different mathematical ideas and ways of working as he made decisions within a structured environment:

> … [When doing mathematics, one needs to be] prepared to play around a little bit with different ideas and ways of working that might shed different information on the same problems … you can have structure which still allows you choice and forces you to make decisions and then work out the consequences of those decisions. It gives you an understanding of what it is you're looking at.

Originally, Sam said that he had used a very transmissive way of teaching, using traditional lectures in which his students were given the information. He believed this was necessary because limited information was available from other sources. He described his teaching as:

> … running through the notes, running through examples with not much interaction from the students.

Not comfortable with how he had been teaching, Sam worked with another MTE who made him realise he could change the *culture and social relations* in his lectures and tutorials and how he taught. The MTE whom he consulted supported him by providing a variety of different examples of ways in which he could change his teaching. He said that, as a result, his classes became more interactive; students were encouraged to solve problems, to interact with the person sitting next to them, and to participate in whole-group discussions. When describing his working with the MTE, he said that she made:

> … me realise I had choices that I could make about how I was teaching. … she [the mathematics educator] was actually able to provide me with different ways that I could achieve that.
>
> [Now] I try to give students opportunities to talk during lectures. I'll give them a problem or ask them a question and ask them to discuss it with the person next to them or do a bit of work and then talk to their neighbour about what they did.

From their collaboration, Sam and the other MTE developed a mathematics course for prospective teachers designed to help students develop an understanding of the sociocultural and historical development of mathematical concepts and a deeper understanding of school mathematics and its connections to quantitative disciplines. One task was to regularly critique passages of the textbook they used in order to analyse the robustness of the mathematics as it was presented. Sam described his pedagogical approach in this context as follows:

> [w]e definitely try to model doing mathematics the way we would want them to think of mathematics as teachers and for them to think about the way we do our maths … A very typical example is to get the students to work with a textbook and for them to critique a passage in the textbook, a section or a bunch of questions for their robustness, if you like, in terms of presenting the mathematics.

7.6.4 Discussion of the MTE Cases

Ryan linked his beliefs about mathematics – that it was beautiful (Hardy, 2012) and involved working with problems, using pen and paper, and taking time to think – with his teaching. The constraints inherent in teaching large cohorts of undergraduate students from a range of programmes, and with varied abilities and mathematical backgrounds, meant that he used a more expository form of presenting problem solutions, reserving his knowledge of the traditions of mathematics for his postgraduate students.

In contrast to Ryan's expository style of mathematics teaching, Paul believed that it was important that prospective teachers understood how students felt. He encouraged prospective teachers to use different approaches to teaching, to think about how students learn, and to develop their pedagogy from there. Paul's prospective teachers performed experiments and developed mathematical models to solve problems and build insights that helped them to generalise.

Having collaborated with another MTE, Sam now linked his belief that mathematics involves exploring different ideas and ways of working – with his teaching. Students in his classes solved problems and interacted with others, building a mathematical *culture*. Teaching collaborations between MTEs working primarily as mathematicians and MTEs working exclusively with prospective teachers, such as that in which Sam was involved, are internationally rare (Fried, 2014). The sociocultural differences between these groups of MTEs can act as a boundary between them (Akkerman & Bakker, 2011). Akkerman and Bakker identified four processes that could lead to learning at the boundary between disciplines: (1) *identification*, whereby the specific ways of working of the two communities are challenged, (2) *coordination* of practices or perspectives through discussion to allow movement between the two worlds, (3) *reflection* on the differences of ways of working, and (4) *transformation* leading to significant changes. Sam used *identification* as he recognised the different ways of teaching in the two disciplines (university mathematics and mathematics education) and *coordination*, where dialogue with a mathematics educator allowed him to use some of the practices of educators, and this led to *transformation* of his practice.

These dialogues led to the creation of the course "Mathematics content for lower secondary school teaching" which was jointly taught by a Sam, a mathematician, and his MTE collaborator, as well as an awareness on Sam's part of how the pedagogies used in schools could be adapted for use in a university context.

7.6.5 Prospective Teachers' Views on Mathematics Teaching

This section presents the beliefs about mathematics teaching of five prospective teachers that they expressed in their interviews. One, Tim, was studying a 1-year Graduate Diploma of Education after having done some mathematics tutoring at university. Another, Dan, was studying a Bachelor of Education programme majoring in mathematics. The three other prospective teachers were studying for combined Bachelor of Education/Bachelor of Science degrees with mathematics as a teaching area.

The prospective teachers described differences between the way they were taught mathematics at university and the way they were being taught to teach mathematics. These related to methods of presenting material, assumptions about prior knowledge, ways of working with the students to help them to understand the mathematics, catering for the diversity of students in a class, the context in which content was presented, and the relevance of the content. Generally, the prospective teachers described being taught university mathematics via traditional lectures and tutorials in which information and worked examples were presented with the textbook as an important part of the process. Doug, for example, recounted that at university:

> the teacher [would be] up the front presenting the information and it's up to you to interpret it and try and make sense of it.

Tim described a similar approach that he experienced as:

> very text book heavy. Here's the content. Here's what this means. Here, go practice it.

When considering how they were being taught to teach mathematics, the prospective teachers described being encouraged to use a more problem-solving approach, encouraging and guiding students to work towards their own solutions so as to build their understanding. Doug explained his experience of learning how to teach mathematics as follows:

> We [were] taught to approach things teaching as a problem-solving approach, so to get students to try and discover things on their own and not just give them the data but guide them towards finding their own solutions.

The prospective teachers also discussed being taught to cater for the diversity of their students by considering their background knowledge rather than assuming that students had the prior knowledge that the curriculum might suggest. They described being urged to take time to explain concepts and to work with their students:

> … when you're taught to teach maths you can't assume what a student knows. We're taught to explain a lot more and to break it down a lot more … to make sure everyone is following each step if that makes sense … when you're a teacher you're taught to appreciate different learning styles and present things in ways that are relevant to the whole range of students.

(Dan)

As the prospective teachers were negotiating the different approaches they saw modelled in their mathematics courses and advocated in their mathematics education courses, they were thinking about how they wanted to teach. They believed it was important to engage their students by connecting the mathematics to the real world and giving their students a reason to be doing the mathematics by showing them its relevance. The aim, as Max put it, was:

> to try and develop a deeper understanding initially, connecting through the real world and connecting it to a reason for learning it.

For some prospective teachers, this different way of teaching was exciting. For example, Jett said:

> When I first started doing the maths education courses … [the mathematics education lecturer] would do sample lessons like inquiry lessons. I definitely thought that that was a way better way to learn

At university, the prospective teachers had experienced mathematicians presenting the mathematics to them. It was up to them how they engaged with and practised the material. In contrast, they were being taught to teach mathematics by engaging and motivating students to construct their own understanding of the mathematics. The prospective teachers were aware of the difference between how mathematics was taught at university and how they were being taught to teach it and described the tension between being taught to teach one way but being taught in a very different way. Anne described the two contexts as follows:

> [Some MTEs] were happy to allow discussions about different content and the way we should consider explaining or the activities that we choose or introduce to students to help synthesise or corroborate that knowledge.

Whereas being taught mathematics:

> One of my other course co-ordinators was very much of the, "You will write this down and you will understand it from having it written down and practicing it, and that's the only way you're going to learn it."

The prospective teachers appeared to accept whichever way they were taught, and because they were capable mathematics learners, they had the motivation and resources to find extra information and get help as needed. For example, Anne stated that:

> I can gain some understanding from that [the mathematics written down during the lecture] but, I will often take that further myself, in my own study time … I do have to engage in other learning practices to try and synthesise that knowledge.

They also believed that the way they were taught mathematics reflected the teacher's (MTEs with mathematics PhDs) beliefs. As Max put it:

> I think there are different interpretations and different views on it [how to teach mathematics]. I don't find it confusing because I see it as their makeup. There're university lecturers here that are more from analytical and more engineering backgrounds. Then you've got ones that are more from theoretical, pure mathematics backgrounds. To me it makes sense that they would have slightly different outlooks on it.

Some prospective teachers discussed their concern that in some of their classes, other university students in their cohort could not cope with the lecturing style. Doug explained this as follows:

> I imagine with a lot of students they'd have a lot of trouble just trying to put together that raw data they're given and actually understanding everything behind it.

Frustration was expressed that at one university, mathematics for engineering was taught without any consideration of prospective teachers and the mathematics and statistics they would be teaching in schools. Dan, for example, noted that:

> … I found the second year of uni extremely frustrating. Because we were learning maths well beyond what we needed to which in some way is useful but there's so much attention based on it. We weren't really covering all the maths in the curriculum too. We were just focusing on a small part of it. Things like statistics. We didn't touch statistics at all or very rarely in the two years of maths. … it's taught more like engineering rather than from the perspective of a maths teacher.

7.7 Conclusions

Students' beliefs about mathematics are directly influenced by their teachers' classroom practices (McLeod, 1992; Mosvold & Fauskanger, 2014). Therefore, one would expect that the practices of MTEs – teaching content or teaching pedagogy – would affect the beliefs of prospective teachers about mathematics teaching and learning. The survey of MTEs and prospective teachers reported in this chapter showed that generally they held problem-solving views of mathematics (Ernest, 1989b), consistent with Grigutsch and Törner's assertion (1998) that "mathematicians view mathematics as a discovery and understanding process" (p. 29).

In the main, respondents in this study held problem-solving views of teaching mathematics, which involved "allowing students to struggle" and highlighted "the importance of justifying statements". However, there were aspects of the problem-solving view with which not all respondents were comfortable, for example, the use of questioning.

Generally, prospective teachers believed that the aims of teaching mathematics at university were different from the aims of teaching mathematics in schools. At university, mathematicians presented the mathematics to the students, and the way in which they engaged with and practised the material was up to them. This contrasted with the way in which they were taught to teach mathematics to school students. The prospective teachers accepted the tension between how they were taught mathematics and how they were taught to teach mathematics. They acknowledged that they had the motivation and resources to find extra support to help them build their mathematical knowledge, but future research in this project will explore the nature of the beliefs that these prospective teachers take into schools and the extent to, and ways in, which the various groups of MTEs with whom they have worked may have influenced them.

The findings presented in this chapter suggest that preparing prospective teachers needs to include discussions on beliefs about mathematics and its teaching and learning, in which teachers are encouraged to reflect on the differing beliefs that may underpin the teaching of the various MTEs they have encountered and to consider how this may have shaped their own beliefs and could influence their own teaching. This would help prospective teachers develop the resilience and confidence to negotiate future school environments in which there are likely to be tensions between their beliefs about mathematics, and mathematics teaching and learning, and those of colleagues.

In Sam's case, engaging in discussion with an MTE with a different disciplinary background led to changes in his beliefs and teaching practices, resulting in a new approach that was more supportive of student learning. Boundary dialogues (Goos & Bennison, 2018) such as that in which Sam engaged can enable MTEs to identify and understand the practices of other MTEs and to reflect on their beliefs about mathematics and its teaching and learning. Such conversations could allow differing groups of MTEs to work together in order to develop greater coherence for the prospective teachers across their studies. Ultimately, improving the education of prospective teachers is likely to result in better outcomes for school students.

Acknowledgements The author thanks the educators and prospective teachers who completed the survey and interviews. This project is funded by the School of Education's Pilot Research Grant Scheme, University of the Sunshine Coast.

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Part II
Learning and Developing as a Mathematics Teacher Educator

Chapter 8
Supporting Mathematics Teacher Educators' Growth and Development Through Communities of Practice

Dana Olanoff, Joanna O. Masingila, and Patrick M. Kimani

8.1 Background

Mathematics teacher educators (MTEs) play a significant role in helping prospective teachers (PTs) to develop the mathematical knowledge that they need for teaching. However, research has shown that the majority of MTEs in the United States have little experience teaching students at the level of mathematics that they are preparing PTs to teach (e.g. elementary school) and that they receive little to no training or support either in their preparation programmes or in their jobs (Masingila, Olanoff, & Kwaka, 2012). In order to attempt to improve our teaching of mathematics content courses for prospective elementary school teachers, two novice MTEs (Dana and Patrick) worked with an experienced MTE (Joanna) as part of a mentored teaching experience in the Future Professoriate Program at Syracuse University (SU), where Dana and Patrick were graduate students and Joanna was a faculty member. Through this programme, graduate students can earn a Certificate in University Teaching by engaging in professional development experiences, including attending and presenting at conferences, preparing a portfolio, and participating in a mentored teaching experience. We designed Dana and Patrick's mentored teaching experience around our teaching of two mathematics content courses for PTs. We chose to study our teaching critically and form a community of practice (CoP) to support one another in improving our teaching and developing mathematical knowledge for teaching teachers (MKTT).

D. Olanoff (✉)
Widener University, Chester, PA, USA
e-mail: dolanoff@widener.edu

J. O. Masingila
Syracuse University, Syracuse, NY, USA

P. M. Kimani
Glendale Community College, Glendale, AZ, USA

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_8

We each entered into the teaching experience with different backgrounds and somewhat different motivations. Dana had been prepared as a secondary school mathematics teacher and had taught middle/high school mathematics for 1 year before enrolling in a PhD programme in mathematics education. At the point of the study, she had several years' experience teaching mathematics at the undergraduate level, including 1 year teaching mathematics content courses for PTs. However, she was not particularly happy with how she had taught the courses previously and wanted to do better at creating a supportive environment for PTs' learning. She had been studying how teachers were encouraged to reflect on their practice, and she wanted to incorporate this reflection into her own teaching.

Patrick had been prepared as a secondary school mathematics teacher in Kenya and had several years of experience teaching mathematics at the undergraduate level at SU and 4 years of experience teaching at a community college. His only experience with mathematics content courses for PTs came from an internship where he observed and assisted another instructor in teaching one of the courses. From the internship, he realised that if he wanted to be successful in creating opportunities for PTs to think deeply about elementary school mathematics, he needed to think carefully about creating and facilitating a learning environment that was different from what he had done in his teaching career to that point. In order to achieve this deep thinking, it would be helpful to engage with others who were also trying to do the same thing.

Joanna was an experienced MTE who had taught mathematics content courses for many years and had also taught mathematics in grades 7–12 for 6 years before pursuing a doctoral degree in mathematics education. She was involved in designing the two mathematics content courses at SU and was the course coordinator and a co-author of the textbook used to teach the two courses. Her motivation for participating in the research work was to mentor novice MTEs in learning to support PTs in developing mathematics knowledge for teaching (MKT), to think more deeply and gain understanding about what knowledge is needed to support PTs in learning via problem solving, and to add to her own MKTT.

8.2 Forming the Community of Practice

In order to facilitate our mentored teaching experience, we decided to form a CoP. Wenger, McDermott, and Snyder (2002) defined CoPs as "groups of people who share a concern, a set of problems, or a passion about a topic, and who deepen their knowledge and expertise in this area by interacting on an ongoing basis" (p. 7). Through research and participating in a CoP, members develop and articulate new knowledge in response to questions and problems they have about their practice.

A CoP offers a platform for its members to engage in negotiating shared understandings, learning, meaning-making, and identity. Wenger (1998) identified three dimensions of the community which has practice as the source of coherence: (a) CoP members interact with one another and determine norms and relationships through *mutual engagement*, (b) CoP members are held together by their understanding of a sense of *joint enterprise*, and (c) CoP members seek to produce, over time, a *shared repertoire* of communal resources (e.g. language, routines, artefacts, and stories). Hezemans and Ritzen (2005) studied a CoP in which they participated at their university, and they argued that CoPs can be "a place where the innovative energy of an organization is bundled: communities then perform an important role of adding value to the process of making the strategic policy operational and creating new and innovative solutions" (p. 46). We believed that forming a CoP would allow us to work together to gain knowledge about and improve our teaching of mathematics content courses for PTs.

For us, the fact that Dana and Patrick needed to have a mentored teaching experience with Joanna as the mentor provided the basis for our mutual engagement. We were each invested in the project, and our joint enterprise of improving our MKTT, as we all were interested in improving our teaching of mathematics content courses for prospective teachers. Through our work, we developed a shared repertoire of lesson plans, reflective memos, meetings, and ways of interacting with one another.

8.3 Theoretical Framings

Our CoP revolved around reflecting on the process of teaching mathematics content courses for PTs, part of our shared repertoire. We guided our reflections around research on reflection and inquiry, as well as research on MKT. Below, we briefly review the literature that guided our work.

8.3.1 Reflection and Inquiry

Chapman (2008) states that reflection on their own teaching is an inherent part of the work of teacher educators, with this reflection involving "examining, framing, and attempting to solve the dilemmas of classroom practice; and being aware of and questioning the assumptions and values [they bring] to teaching" (p. 121). We chose to situate our reflection using a position of *inquiry as stance* (Cochran-Smith & Lytle, 1999). Through this process, groups of teachers engage in joint construction of knowledge through conversation and other forms of collaborative analysis and interpretation. Through talk and writing, they make their tacit knowledge more visible, call into question assumptions about common practices, and generate data that make possible the consideration of alternatives (p. 294).

Researchers such as Cochran-Smith (2003) and Webb, Pachler, Mitchell, and Herrington (2007) propose that teacher educators take inquiry as a stance in examining "the enterprise of teaching, schooling, and teacher education" (Cochran-Smith, 2003, p. 5). Teacher educators in an inquiry community "generate local knowledge, envision and theorize their practice, and interpret and interrogate the theory and research of others" (Cochran-Smith & Lytle, 1999, p. 289); they produce the knowledge they need "to teach well … when they treat their work as a site for intentional investigation at the same time they treat the knowledge and theory produced by others as generative material for interrogation and interpretation" (Cochran-Smith, 2003, p. 16). Inquiry as stance is critical in examining both one's own work and the work of others. We used an inquiry as stance framing for examining our teaching practice and the practice of the members of our CoP; inquiry as stance provided a framing for our joint enterprise (Wenger, 1998) of developing MKTT.

In studying our practice through the CoP, we took on the role of "reflective practitioners" (Schön, 1983). Schön defines two ways to reflect on one's practice: *reflection on action* and *reflection in action*. The former refers to ways in which members of a community of practice (in our case, mathematics teacher educators) reflect on past experiences with the intention of refining their work to achieve their instructional goals, while the latter refers to "thinking on your feet" (Schön, 1983, p. 54), the reflection that occurs while one is in the process of teaching. Because part of our CoP focused on peer observation, we were able to add another type of reflection to our process: *reflection on the actions of others*. We observed each other's teaching both directly and through reading their memos, and we were able to think deeply about choices that they and we made and reflect on these decisions. We were bound together in mutual engagement (Wenger, 1998) through our reflective practitioner roles, and our reflections became part of our shared repertoire.

Students learn through engagement in tasks (Hiebert & Wearne, 1993); likewise, teacher educators learn through engaging with the tasks of teaching (Zaslavsky, 2005, 2007). Zaslavsky (2008) illustrates how both learners and facilitators learn through working with tasks. She proposes that teacher educators learn or construct knowledge by repeatedly participating in a cycle of designing or modifying tasks, supporting learners while they engage in tasks, and reflecting on learners' work. We modelled our CoP around this process, as we continually discussed and sometimes modified the tasks we used in our courses, facilitated the enactment of the tasks, and then reflected on both our enactments of the tasks and the enactments of our group members through memos, creating a shared repertoire (Wenger, 1998) of communal resources.

8.3.2 Mathematical Knowledge for Teaching

Research on teachers' knowledge has flourished following Shulman's (1986) presidential address at the 1985 American Educational Research Association's annual meeting, where he introduced the idea of pedagogical content knowledge (PCK). This knowledge, which Shulman called the "missing paradigm" (p. 6) in research on teaching, linked knowledge of teaching pedagogy with knowledge of the specific content that was taught. As a result of Shulman's speech and the research that followed, Ball and her colleagues introduced the term *mathematical knowledge for* *teaching* (MKT) (e.g. Ball & Bass, 2002; Hill & Ball, 2004), which describes the *mathematical* knowledge required by the work of teaching.

Ball and her colleagues (Ball, Thames, & Phelps, 2008) developed a framework for MKT in which they sought to expand Shulman's descriptions of content knowledge and pedagogical content knowledge to include subcategories of the mathematical knowledge that teachers need to know. They broke mathematical content knowledge into three subcategories: *common content knowledge (CCK)*, *specialised content knowledge (SCK)*, and *horizon content knowledge*. The first of these categories, CCK, refers to the mathematical knowledge that everyone needs to know, whereas SCK refers to the mathematical knowledge that is unique to the work of teachers. Examples of SCK include being able to evaluate student algorithms to determine their validity, explaining why we invert and multiply when we divide fractions, and understanding and being able to correctly use mathematical vocabulary. While these types of knowledge may be found in people other than teachers, the researchers argue that this knowledge is necessary for teachers, but is generally not needed by the typical learner of mathematics. Knowledge at the mathematical horizon involves knowing how mathematical topics are related throughout the curriculum and across year levels beyond that at which one is currently teaching.

The researchers (Ball et al., 2008) similarly broke pedagogical content knowledge into *knowledge of content and students (KCS)*, *knowledge of content and teaching (KCT)*, and *knowledge of the curriculum*. KCS combines knowing about students with knowing about mathematics and includes understanding students' reasoning and knowing common errors and misconceptions that students will have with specific material. KCT involves an understanding of sequencing of topics and the power and value of different mathematical representations. Knowledge of the curriculum involves knowing how a curriculum fits together and is related to knowledge at the mathematical horizon.

While much research has been done about the mathematical knowledge needed for teaching, significantly less research has looked at the mathematical knowledge needed by MTEs to help PTs develop MKT (Castro-Superfine & Li, 2014). A number of researchers have determined that there is a category of knowledge needed by teacher educators that goes beyond the knowledge needed by their students (PTs) (e.g. Jaworski, 2008; Perks & Prestage, 2008; Rider & Lynch-Davis, 2006; Zaslavsky & Leikin, 2004). Additionally, many frameworks for teacher educator knowledge provide a structure similar to a framework for teacher knowledge (Olanoff, Welder, Prasad, & Castro Superfine, 2018). In looking at the parallels between the ways in which MTEs learn and develop knowledge and the ways in which mathematics teachers learn and develop knowledge, we hypothesised that mathematical knowledge for teaching teachers (MKTT) would have similar knowledge categories to the MKT framework discussed above (Ball et al., 2008).

The initial joint enterprise of our CoP involved developing our MKTT. However, as we progressed through the semester, we added the additional enterprise of studying portions of our shared repertoire in order to better understand some of the aspects of MKTT we developed and how our CoP helped us in this enterprise. Thus, our CoP became dual purpose: to develop our MKTT (see, e.g., Hiebert, 2013; Van

Zoest, Moore, & Stockero, 2006 for other examples of this type of CoP) and also to research what we had learned (see Arslan, Van Zoest, & Ruk, 2017 for another example of a research CoP). In the remainder of this chapter, we will detail how we studied our collected data and what we learned about our knowledge development and the role that the CoP played in this development.

8.4 Our CoP Processes

We formed our CoP during the 2007–2008 academic year, when we each taught a section of the same two mathematics content courses for PTs during two consecutive semesters. During the first semester, the course content focused on whole number and operations, number theory, probability and statistics, and functions. The content for the second semester course focused on rational numbers, geometry, and measurement. These courses were taught using the same textbook and general lesson plans and with an emphasis on PTs learning mathematics via cooperative problem-solving. PTs worked together in small groups to solve problems with the goal of developing deeper understandings of the K-8 level (elementary and middle years) mathematics and their own MKT. The job of the instructors of the courses was to help facilitate the PTs' problem-solving and knowledge development. During the course of the two semesters, Dana and Patrick, the two novice MTEs, observed Joanna, the experienced MTE, teach her class before teaching their own classes later in the day. Joanna also observed Dana and Patrick's classes several times during each semester. In order for us to reflect on our teaching and observations, and to have a record of these reflections, each of us wrote a memo after each lesson session. We met briefly before each lesson and weekly after we had taught all of the lessons for the week to discuss our observations, what was going well, what was not going well, and where we thought we needed to go next. We audio-recorded the weekly meetings and transcribed them to add to our data set.

We used both ongoing and retrospective analyses of the data, which consisted of our daily memos and meeting transcriptions. The ongoing analysis, which occurred during the two semesters we were teaching, observing each other, and writing memos after each lesson, was the basis for continued reflection on our teaching and learning about our teaching, the testing of emerging hypotheses, and the strategies for promoting further development of the PTs' mathematical understandings. During the retrospective analysis, we examined the larger corpus of data through a carefully structured review of all the relevant data.

We began coding our data at the end of the first course. We used open coding (Corbin & Strauss, 2008) on the memos and meeting transcripts from the first 2 weeks of the semester to identify themes that emerged from that data. After looking at the data individually, we met and compiled a list of the themes that we had identified. Using this list, we then individually looked at 5 weeks' worth of data to see if the codes that we had identified matched our data. We met together again and as a group refined our list. During this meeting, we compiled a list of code definitions and examples, so that we were using the codes consistently. Since we were using open coding, we looked for any themes that emerged from the data, but as we developed our codes and definitions, we were able to categorise many of them around aspects of MKTT.

8.5 What Did We Learn?

Through coding our data, we discovered evidence that working in our CoP had helped us to develop our MKTT (the goal of our joint enterprise) in a number of ways, through our meetings and observations (our mutual engagement) and our reflections, memos, tasks, and revised lesson plans (our shared repertoire). As teacher educators, we developed (a) an enhanced understanding of the mathematics content of elementary school mathematics (MTE CCK and SCK), (b) a deeper understanding of the ways in which young adults (who may or may not already be familiar with the material) learn and how that affects our teaching and planning (MTE KCS), (c) different ways of questioning and facilitating students' problem-solving (MTE KCT), and (d) a better understanding of ourselves as MTEs through reflecting on both our own actions and those of others. Our mutual engagement and shared repertoire helped us achieve the goal of our joint enterprise.

8.5.1 Mathematics Content Knowledge

As former mathematics majors, we entered into teaching these courses feeling confident with the elementary school mathematics material. However, we soon realised that much of this material was more complex than we had originally thought. Ma (1999) described elementary mathematics as fundamental. She wrote:

> Fundamental mathematics is elementary, foundational, and primary. It is elementary because it is at the beginning of mathematics learning. It is primary because it contains the rudiments of more advanced mathematical concepts. It is foundational because it provides a foundation for students' further mathematics learning. (p. 124)

As teacher educators, we had not looked at some of the mathematics since we were in elementary school ourselves, and at that time, we were not studying it with the depth, breadth, and thoroughness that we would need in order to teach it and to help PTs develop the deep understandings that they would also need in their teaching. The following example on counting methods illustrates one instance where our mutual engagement in our CoP helped us develop mathematics content knowledge.

Thinking Differently About Counting Methods One activity in our curriculum involved using combinations and permutations to determine the number of possible outcomes in a given situation. Having taught this activity before, we discussed the observation that our students did not seem to develop a deep understanding of permutations and combinations. Instead, they learned the words and where the keys for permutations and combinations were on their calculators, and then they punched the appropriate keys and got an answer. Alternatively, if we did not allow them to use their calculators, they substituted numbers into a formula, but did not understand where the formula came from and why it was appropriate to use in a given situation. In discussing our plan for this lesson in our weekly meeting, we decided that rather than defining permutations and combinations during class, we would try to get our students to think about the situations in the problems and use the fundamental counting principle on every problem. If the order did not matter in a certain situation, they would need to divide their answer by the total number of orders. In the memo that follows, Dana describes how she introduced the problem: *At State University, a group of seven students wishes to select a committee of four to negotiate student activity fees with the Dean of Students. How many committees can be selected from the group of seven?*

> In the past, we have given notes for this activity, but ... this time, I didn't even really define permutation and combination in terms of their formulas. I talked about the first question [7C4] as we have 4 slots, so we have 7 × 6 × 5 × 4 ways of picking people to fill them, but then we have to divide by the number of ways of arranging 4 people. I'm not sure how it will work, but I liked it better than them just punching in 7C4 on their calculators. (DO, Memo 10/31/07)

At first, this method of talking about combinations seemed to be much better than what we had done in the past. The students really seemed to be thinking about whether or not the order mattered in a problem and when they needed to divide by the number of orders to figure out the answer. However, when Dana was asked a question about a homework problem, *An experiment consists of tossing a coin 8 times and observing the sequence of heads and tails. How many different outcomes have exactly 3 heads?*, she found it difficult to explain the problem without using the words "choose" or "combination". Here is an excerpt from her memo following this discussion:

> In my head, I knew the answer was just 8 choose 3. You have 8 coins and you want to choose 3 of them to be heads. This makes total sense to me and I usually teach it that way. However, since I am not doing "choose", I had a lot of trouble explaining it...I still have trouble working out why you would draw three slots instead of 8. It sort of makes sense, but I can't really explain it. I just don't know what the best way is. (DO, Memo 11/05/07).

Through talking with members of the CoP and reflecting through her memo (part of our shared repertoire), Dana was able to eventually make sense of how to explain the problem both to herself and to her students. Had she not had the benefit of the CoP, it would have been easy for her to go back to encouraging the PTs to use formulas. Patrick summarised this idea in a memo:

> The temptation here is very high for both the instructor and the students to fall into using the formulae for factorial, combination and permutation in which case the students may not get to critically think about what they are doing. The instructor needs to consciously work on avoiding getting trapped into that so that he/she can lead the students into actually thinking about why they are doing what they are doing. (PM, Memo, 10/31/07)

Joanna later mentioned the same thing in a CoP meeting where we were discussing how the CoP had been beneficial. "I think I hadn't thought about before, about how to teach, some of these ideas without referring to combinations and permutations" (JM, Meeting Notes, 1/31/08).

As this example illustrates, it can be difficult to teach in a way that supports students in developing deep, connected understandings. In general, we found that elementary school mathematics had a number of complexities of which we were unaware before we began teaching mathematics content courses for PTs and reflecting deeply on our teaching through our CoP. Like the majority of US teacher educators teaching these classes, we did not have experience teaching elementary school ourselves (Masingila et al., 2012). Through our mutual engagement and shared repertoire (e.g. reflections, memos, tasks, lesson plans), we came to realise the importance of looking deeply at the underlying mathematics behind the representations, algorithms, and definitions that we use, and we believe that understanding elementary school mathematics with strong levels of depth, breadth, and thoroughness is an essential component of MKTT. Being able to discuss and work through the challenging content with the CoP allowed us to expand our own MKTT (our joint enterprise) by learning new mathematics ourselves and discussing ways to encourage PTs to work with the content with high levels of depth, breadth, and thoroughness.

8.5.2 Working with Young Adult Learners

As teachers of young adults, we had to be cognisant of how their learning may be different from how children learn content. We also had to be prepared with the idea that these PTs had probably seen much of the material before, and we needed to think about how to build on their prior knowledge and prompt them to engage with the ideas meaningfully. For example, the majority of our students enter their mathematics content classes knowing how to add, subtract, multiply, and divide whole numbers. However, for most of them, this means being able to perform an algorithm, mostly without thinking, rather than knowing why the steps of the algorithm work. In order to make arithmetic operations more meaningful for our students, we require them to add, subtract, and multiply using bases other than ten, so that they cannot merely rely on the procedures that they have ingrained but instead must think about how the processes of addition, subtraction, and multiplication work. The following example shows the challenges that we faced when introducing multiplication in different bases.

Introducing Multiplication in Different bases Kazima, Pillay, and Adler (2008) describe designing meaningful "first encounters" with mathematical ideas as important tasks for teachers. They define these first encounters as "the first moment of the didactic process or process of study" (p. 285) and stress that these first encounters should be purposefully designed. In terms of this task for teacher educators, most of the encounters that prospective teachers have with the mathematical topics in their content courses are not “first encounters”. Thus, the job of the teacher educator is perhaps to provide a “deeper encounter” with the mathematics, or a first encounter into looking at the underlying mathematical features of a topic.

After successfully using base blocks to add and subtract with regrouping, our students were ready to move on to multiplication. The problems the students were asked to solve were $23_{four} \times 30_{four}$, $45_{six} \times 32_{six}$, and $78_{nine} \times 234_{nine}$. Joanna decided that she wanted her students to understand the role of place value in multiplication, so she would introduce multiplication using the partial products method: e.g. $45 \times 32 = (40 \times 30) + (40 \times 2) + (5 \times 30) + (5 \times 2)$. However, while this method had its advantages, it had some drawbacks as well, as evidenced by this excerpt from her memo:

> I hesitated with trying to decide whether to get into a deeper discussion of what larger than single-digit multiplication is when discussing multiplying two-digit numbers in bases other than ten. In the end, I did go into the meaning of multiplication using partial products. I think this did prompt the students to think more deeply about multiplication and their talk showed this, but it also slowed them down and was more difficult. Tradeoffs! (JM, Memo 9/12/07)

After watching Joanna teach her class (part of our mutual engagement), Dana also considered using partial products to introduce multiplication. However, in the end, she decided instead to use the standard algorithm, with which her students were already familiar in base ten, and translate it to the different bases:

$$\begin{array}{rrrr} & & 4 & 5_{six} \\ & \times & 3 & 2_{six} \\ \hline & 1 & 3 & 4_{six} \\ + \; 2 & 2 & 3 & 0_{six} \\ \hline 2 & 4 & 0 & 4_{six} \end{array}$$

She discussed her thought process in choosing an algorithm with which her students would be more familiar, since the different bases would already provide a challenge for her students, and the base block manipulatives had limitations when multiplying a two-digit number by another two-digit number.

> I think that the advantages to using partial products is that conceptually it makes a bit more sense. You can think about having 3 flats, 4 times and it makes sense. However, as we found, it doesn’t make sense to think about 3 flats times 4 flats—at least pictorially. Another disadvantage to using partial products is that it is further away from the algorithm that they are used to. (DO, Memo, 9/12/07)

In the meeting following these lessons, we had a long discussion (part of our mutual engagement through our CoP) of the advantages to introducing multiplication in different bases using each algorithm. Below are some excerpts from the discussion where Dana pointed out that Joanna had used partial products while she chose not to:

Joanna … the real problem that I see is that they don’t know which column it goes in.

Dana	A couple of reasons that I thought it made it harder is, one, it gives them a lot of things to add. … I liked it because it makes more sense conceptually. It makes multiplication more meaningful, at first, like when you're talking about, okay, four longs times two, that makes sense. But then when we're, when we …
Joanna	Longs times longs.
Dana	We got into the issue with longs times longs, and that doesn't really make sense, and ...
Patrick	You're forming rectangles or squares ...
Dana	I think that, it's a deeper level of mathematics, and that, to think about multiplication in that way, is a good idea. …
Patrick	For people doing it for the very first time, it's hard.
Dana	To give them the algorithm that I think they're more familiar with, my students, for the most part seemed to struggle less with it.
Joanna	Okay.
Dana	It was like, once they got it, they were like, yes, I remember how to multiply, and the mistakes that they were making, were, just transferring to the base a little bit.
Joanna	Because in fact, this, the point of this is to have them do it in another base ... In Chapter Three, we'll talk about what the algorithm means.
Dana	So, I'm going to wait until Chapter Three ... but, the way that [Joanna's students] were talking about it, I thought was more, deeper mathematically, or more mathematically mature than the way that mine were talking about it. … mine, learned the algorithm, and so now they can do it, and they understand how to do it, whereas, yours were saying, okay that's one unit and two longs.
Joanna	So, there's a tradeoff (Discussion Meeting, 9/13/07).

Having the CoP allowed us an opportunity to think about and discuss the pedagogical decisions that we made. Being able to watch and read reflections about multiple groups of students and approaches gave us a chance to observe and discuss how the students reacted to the content. We decided that both methods had their advantages and disadvantages and that while we were both happy with the method that we each chose, in the future, we would think about how to incorporate both methods, in order to gain the advantages of each.

8.5.3 Thinking About Our Questioning

We taught our mathematics content courses through problem-solving (Lesh & Zawojewski, 2007; Schroeder & Lester, 1989). This meant that rather than presenting mathematics directly to students, we presented them with problems that engaged them with mathematical ideas through participating in the problem-solving process. Therefore, it was important that, as MTEs, we thought about how we facilitated the problem-solving and the questions we were asking our students to help them in the problem-solving process.

As novice MTEs, it was helpful for Dana and Patrick to observe Joanna – an "old timer" (Lave & Wenger, 1991) – as she facilitated her own students' problem-solving activities and also to have her observe them while they taught. Using our memos and discussions (our shared repertoire) provided an opportunity for us to think about how we were interacting with our students and develop our KCT. The following excerpts from our memos illustrate some of our reflections on questioning.

> Jo, who was observing today, said that there were times when I asked questions and then did not wait very long before giving answers when there was silence. (DO, Memo, 2/4/08)

> One thing I noticed is that Jo tends to give an assignment and then hang by the front or leave the room for a few minutes before going to help her students. I generally start walking around as soon as I give the assignment. I think that Jo's method is probably good to get them talking to each other about the problems without asking questions right away. (DO, Memo, 10/22/07)

> It really makes me think about how much help we give the students versus how much they actually need. If that had been my class, I probably would have said something sooner, and Sarah wouldn't have been able to come to the same realisation—or at least not on her own. (DO, Memo, 11/26/07)

> Sometimes I think by helping them it encourages helplessness. I think that next year I will work more at hanging back a bit more like Jo does. (DO, Memo, 5/5/08)

> [My student's] question got me wondering if I have made a pattern of questioning them only on their incorrect responses. This is something that I need to pay more attention to and just be more aware about my questioning. While there is a general tendency to ask them questions about their incorrect solutions, I think it is also helpful to question them on their correct responses to ensure that they are making the connections intended by the activities. (PK, Memo, 2/13/08)

Through their observations of Joanna and our CoP discussions, both Dana and Patrick were able to reflect on their questioning. They noticed that Joanna often seemed to give her students more independence to struggle through problems and make mistakes, whereas they were more apt to try to help right away. Joanna's role as an old-timer (Lave & Wenger, 1991) in the CoP and the shared repertoire that we had established allowed Dana and Patrick to view their teaching practice in comparison to hers. Additionally, they were able to think about when they were probing our students' thinking and work on asking them to share their ideas, both when they were correct and when they were incorrect. Over the course of the academic year, Dana and Patrick both strove to better use their questions to encourage their students to become independent thinkers and doers of mathematics.

8.5.4 Learning from Our Community of Practice

Through our CoP, we developed a true community where each member felt comfortable asking questions of each other and of ourselves. Through our shared repertoire, we were able to experience multiple perspectives on teaching mathematics content to PTs, and we discussed these perspectives with others who were mutually engaged in the pursuit of developing MKTT (our joint enterprise). We have all found that participating in this study has changed us and helped improve our practice as MTEs. As a group, we think more about clearly communicating our goals and decisions to ourselves, to our fellow MTEs, and to our students. We all think about the mathematics content at a much deeper level, what makes certain topics challenging, and how do certain ideas/activities provide building blocks for others. We focus more on the process of helping PTs become mathematical problem-solvers, rather than worrying about covering all the material of elementary school mathematics (which is an impossible task.) Below, we highlight how each of us has personally benefited and learned from the CoP.

Dana Participating in the CoP helped me to develop my pedagogical content knowledge, specifically knowledge of content and students. Seeing how students in both Joanna's class and my class interacted with the material gave me twice as much data to look at. But the most important part was the reflecting part – writing down what happened with my students and thinking about how to help them construct knowledge and see problems with their work helped me make connections and figure out ways to help my students in the future. Being able to share the experience with the other members of the CoP also helped me develop my own knowledge in a way that reflecting on my own would not.

Additionally, I am much more comfortable with the material these days than I was when I was teaching during our CoP. I think this comes from experience and also from realising the importance of knowing what I am doing. It was clear from my reflections that on the days when I was not well prepared, the lesson did not go well. I think the original assumption was that elementary school mathematics is easy and that I should not really need to work hard to teach the content, but clearly we figured out that this was not true, and it has encouraged me to make sure that I am prepared for every lesson with a deep understanding of the material.

I think that a lot of the things I do with my students now come as a result of participating in the CoP. At this point, I have been teaching mathematics content courses for prospective teachers for more than 10 years, and many of the things I focus deeply on now were things that I struggled with and learned about from participating in the CoP. An example of this involves having PTs use models to represent operations:

> Other places where people had issues were that some people wrote that taking 1/3 of a number was subtraction, and a few still are having trouble modeling multiplication. They model the answer to the problem rather than the action taking place in the problem. I talked about this by drawing a rectangle divided into 12ths on the board. I shaded in one square

> and asked what was being modeled. A student replied that it was 1/3 times 1/4. I explained that just shading in 1/12 of a rectangle was a model of 1/12 or 1/12 of 1, but not of 1/3 times 1/4, or 1/4 times 1/3, or 1/2 times 1/6. If they wanted to model the process of 1/3 times 1/4 then they needed to do something to indicate the 1/3 and the 1/4, not just model the answer. A number of students in my class are struggling with this, and I have noticed this from students in Jo's class as well. (DO, Memo, 4/14/08)

In reading over this memo, I realise that this focus on modelling is something that I still do today. For example, I tell my students when we model something like 12 – 7 = 5, that we need to be able to see 12, 7, a model of subtraction, and 5 in the model. This idea is something that I developed from my reflections and work in the CoP.

Additionally, I still use CoPs in my teaching and research today. With the model of my original CoP as a guide, I have formed a new CoP with MTEs at institutions across the United States. Our joint enterprise is the creation, implementation, and modification of cognitively challenging tasks for mathematics content courses for PTs (e.g. Thanheiser et al., 2013). We meet regularly to develop a shared repertoire of work on our tasks and discussions of their implementation, as well as to support each other's teaching and research.

Patrick One of the main things I learned from this CoP was the importance of having PTs create their own understandings. I learned to support my students' development by carefully leveraging students' initial ideas to build a profound understanding of the concepts taught in these courses.

Something else I learned from the CoP was the importance of thinking carefully about the objective of the lesson. The CoP was designed to facilitate reflective practice about the efficacy of our efforts to engage PTs in thinking carefully about mathematics. That meant that we interrogated our teaching practices to determine how well they were aligned with this goal. For example, in this memo, I was documenting my reflection on the challenge of supporting PTs to think carefully about counting techniques:

> I think counting concepts are probably too complex for these students to be able to get a good understanding by doing one activity. While I am not advocating teaching counting by giving the students formulae and a significant number of practice problems, I think the students could benefit from a better selection of counting problems and a better sequencing of the problems. Also, as previously discussed in one of our meetings, the temptation here is very high for both the instructor and the students to fall into using the formulae for factorial, combination and permutation in which case the students may not get to critically think about what they are doing. The instructor needs to consciously work on avoiding getting trapped into that so that he/she can lead the students into actually thinking about why they are doing what they are doing. (PK, Memo, 10/31/07)

Having the opportunity to discuss these challenges through our mutual engagement in a CoP enhanced my development as a MTE.

As a result of my participation in the CoP, I am more reflective about my practice of preparing prospective teachers. While I no longer write memos like I did during the CoP, I find myself reflecting about how my students are experiencing my class and what I could do to afford them opportunities to deepen their understanding. This all started with my experiences in the CoP:

> I believe the co-teaching/observation experience and writing a memo is really helping me reflect on how I can make this course a better course for the students. By reflecting on the students' struggle, the goal of the activity and my actions as an instructor combined with my observation in Jo's class, I am getting an opportunity to think about my teaching more than I would normally have done. (PK, Memo, 10/29/07)

In the years since the completion of this research work, I have continued to engage in professional CoPs. For example, as the Developmental Education (Dev. Ed.) Lead for my department, I organised the Dev. Ed. instructors to form a CoP to discuss issues around Dev. Ed. The CoP met once a month and engaged in reading and discussing research literature about best practices in Dev. Ed. While the structure and goals of the Dev. Ed. CoP were different from the MKTT CoP, I was able to leverage my MKTT CoP experiences to facilitate the Dev. Ed. CoP.

Joanna I benefited from the mutual engagement of having other people to think carefully about how to support PTs in understanding the mathematical concepts underpinning the procedures they would be teaching in the future. For example, the CoP with Dana and Patrick caused me to rethink how I engaged PTs in thinking about tasks involving probability:

> I wonder why we do probability first before methods of counting. I find these activities very frustrating because of their disjointedness and lack of providing students with tools, such as tree diagrams, for solving problems. What I think we need to do is, after the semester is over, think carefully about what probability and statistics topics should be covered in the chapter and how they should be ordered. (JM, Memo, 10/31/07)

I also learned by observing Patrick and Dana teach and saw some things that they did (e.g. how Patrick engaged his students in thinking about necessary and sufficient conditions for definitions of plane figures) that provided me with new insight into my own teaching practices. The CoP discussions afforded the opportunity to rethink how the tasks were structured (e.g. Dana suggested that we change the order of the conditions given in the task where students attempt to construct triangles given the lengths of three sides, part of our shared repertoire) and to rethink teaching practices: "I've been challenged to think more deeply about my teaching practice, why I do what I do, when to change what I do, etc." (JM, Memo, 5/6/08).

In general, I found the opportunity to work closely with other colleagues teaching the same courses to be very helpful in my own thinking about the tasks and the courses: "Our preparatory meetings help me to clarify in my mind the big picture that I have for the day's lesson, and our weekly debriefing meetings help me as I form the bigger picture in my mind about the course" (JM, Memo, 2/4/08).

I have changed my practice as a result of participating in the CoP as I am more intentional in approaching my teaching through a stance of inquiry. I have implemented the changes in curriculum that we discussed during the time of the CoP (e.g. teaching probability through counting rather than as permutations and combinations) – implementing the shared repertoire we developed. Additionally, each year I work with instructors and interns in the courses to create a CoP where we meet in person twice weekly during the semester and use technological tools such as email, Blackboard™, and OneNote™ to share our thinking and planning related to the tasks and the PTs' learning and engagement.

Overall, the three of us all feel that we have benefited greatly from participating in the CoP, and we all currently participate in CoPs and use many of the ideas and the knowledge that we gained in our teaching. Although our study involved intense work, we believe that it was worth it to improve our teaching, to develop our MKTT, and to become more reflective practitioners overall.

8.6 Communities of Practice in the MTE Community

While the three of us believe that participating in the CoP was a worthwhile experience, we also understand that not everyone in the mathematics teacher educator community has access to a group at their own institutions. We suggest some questions and answers for people interested in forming their own CoPs based on both our own work and other successful examples of CoPs that focus on MTEs in the research literature.

Question How can MTEs work together to develop their mathematical knowledge for teaching teachers and improve their teaching?

Answer We believe that the most important part of developing a CoP is a group of willing and committed participants who have a desire to improve their teaching through mutual engagement around a joint enterprise. We suggest a group size of two to six people working towards a common goal of improving their teaching in some way. While it is possible for larger groups to work, we believe that a smaller group size is more beneficial to forming an actual community. Active participation in a CoP requires dedication from each of the members to play a role in the community and work to make change. Only if each member is dedicated to the group will a CoP be able to develop trust to question each other and offer constructive feedback. Along these lines, members of a CoP must feel safe participating. Although Joanna was Dana and Patrick's advisor and mentor, we were able to create an environment where we could talk to each other as equal members of the community who were all working to learn together.

Additionally, we believe that a successful CoP would require a way for the members to meet regularly (mutual engagement). The meetings would not need to be in person but would need to involve all of the members. The members of the CoP should be working towards a common goal (joint enterprise), so we suggest that all members of the community be teaching common course content and have common learning goals (Morris & Hiebert, 2013) at the time of the CoP. For our CoP, we were all teaching exactly the same course (other examples of this include Hiebert, 2013; Hiebert, Morris, & Glass, 2003; Van Zoest et al., 2006), but we envision that this could work for others who are teaching similar content over the course of a year or semester (e.g. Goos & Bennison, 2008).

Finally, in order to engage in a successful CoP, we believe that there must be a plan for documenting what you are doing (developing a shared repertoire); otherwise there will not be opportunities for meaningful change because the members will not remember what happened. We suggest that writing reflective memos is an important part of a CoP, in order to have a record of what happened, in order to put one's thoughts and ideas on paper (or screen) to help solidify thinking, and in order to share one's ideas with other members of the group. Alternatively or additionally, we audio-recorded and transcribed our CoP meetings, which helped achieve the purposes above. Similar types of data collection are found in Hiebert et al. (2003) as well as Van Zoest et al. (2006).

Question What are different models of a network of support for MTEs, including MTEs who are the only person at their institution teaching particular courses?

Answer We suggest that groups interested in participating in CoPs collaborate with others in small groups either inside or outside their institutions. Twenty-first-century advancements in technology have made it possible for self-formed teams to communicate within and across organisations distributed geographically. In particular, high-speed internet access, availability of sophisticated video conferencing capabilities and free or cheap file-sharing capabilities have greatly enhanced the ability of people to network and to collaborate professionally. This redefined context for learning, where participants can contribute, share, and learn in a virtual community of practice (VCoP), or "third space" (Hulme, Cracknell, & Owens, 2009, p. 539), has facilitated and extended community dialogue and reflection beyond the confines of physical meetings. Proponents of the "third space" theory argue that these spaces facilitate dialogue among VCoP participants that is safe, secure, and supportive (Hulme et al., 2009. A third space such as a virtual community of practice can be used "to create a community of practice and shared reflection on common experience around what the professionals do together" (Hulme et al., 2009, p. 541).

Additionally, Goos and Bennison (2008) present an example of an online CoP that began as part of a course but continued on after the students had graduated and began teaching in schools full time. The initial face-to-face interactions of the CoP helped its members develop trust and mutual engagement to allow them to continue their joint enterprise virtually. The mathematics education community through conferences, meetings, and workshops has the ability to establish groups such as this one that could continue virtually.

8.7 Conclusions

Overall, we found that participating in our CoP was extremely worthwhile for our development as MTEs. While the work that we did took a lot of time and effort, we believe that the benefits and opportunities for growth, both individually and as a group, outweigh the challenges. While it is possible for MTEs to reflect on their teaching alone by writing memos and examining student work, they would miss out on the benefits of the CoP. Without forming a CoP, there is no opportunity to reflect on the actions of others, to receive feedback on your teaching, or to see other ways of doing things. Additionally, participation in a group provides accountability for its members.

We would like to call on the Mathematics Education Community to encourage its members to form CoPs by providing a Special Interest Group or forum for interested people to meet with others who share this interest. Improving the teaching of mathematics for future teachers will benefit the community and the population at large.

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Chapter 9
Artifact-Enhanced Collegial Inquiry: Making Mathematics Teacher Educator Practice Visible

Laura R. Van Zoest and Mariana Levin

As a field, we do not yet have a road map for preparing mathematics teachers, let alone for preparing mathematics teacher educators (MTEs). *Principles to Actions: Ensuring Mathematical Success for All* (NCTM, 2014) provides six guiding principles for school mathematics and both a vision of effective mathematics teaching and concrete actions to achieve that vision. *Standards for Preparing Teachers of Mathematics* lays out what is needed to produce "well-prepared beginning teachers of mathematics" (AMTE, 2017, p. 3). These documents implicitly communicate a vision for the work of MTEs—to prepare mathematics teachers who can do these things, but the details and what it will take to do that work is beyond their scope. As described by Even (2014), historically there has been little focus on the professional development of those responsible for preparing teachers.

Fairly recently, the field of mathematics education has come to understand exemplary mathematics teaching as the result of effectively applying diverse forms of specialized knowledge, such as those delineated by Ball, Thames, and Phelps (Ball, Thames, & Phelps, 2008), and enacting high-leverage practices, such as those identified by TeachingWorks (2018). As has been discussed elsewhere (e.g., Zaslavsky & Leikin, 2004), in the same way that mathematics teachers must have command of the specialized knowledge and teaching practices required of their work in addition to the material that they are teaching their students, so too must MTEs. That is, MTEs must have command of the specialized knowledge and teaching practices required of the work of MTEs, as well as the specialized knowledge and teaching practices that they are teaching to mathematics teachers. The field has focused much attention on better understanding the nature of the knowledge and skills needed for teaching, and we can extrapolate some of what MTEs need from that work. However, the field has only begun to unpack the specialized knowledge and skills that MTEs

L. R. Van Zoest (✉) · M. Levin
Department of Mathematics, Western Michigan University, Kalamazoo, MI, USA
e-mail: laura.vanzoest@wmich.edu; mariana.levin@wmich.edu

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_9

need to prepare effective mathematics teachers, and even less is known about what it takes to become a highly effective MTE.

We do have some information about the scope of knowledge needed by MTEs. For example, Chauvot (2009) drew on Shulman's (1986) categories of teacher knowledge—subject matter content knowledge, pedagogical content knowledge, and curricular knowledge—as a starting place for mapping the knowledge that she sought out as a beginning mathematics teacher educator-researcher. Her study highlighted the variations of MTE knowledge needed in the many different facets of her work, including teaching preservice and in-service teachers and mentoring doctoral students. Chick and Beswick (2018) focused their investigations on the pedagogical content knowledge needed by MTEs for teaching preservice teachers (PSTs), drawing parallels to the knowledge needed for teaching mathematics. Others have focused on MTEs' work as professional development providers. For example, Selmer, Bernstein, and Bolyard (2016) focused on "elements, sub-elements, and components" of MTE knowledge that were evident in the planning of a professional development project. A common theme across this body of research is the complex, multilayered nature of MTE knowledge.

Although little is known about effective ways to support MTE learning, an interesting theme emerges from analyzing studies related to MTE development—the value of collegial reflection on artifacts. To illustrate, we briefly describe four quite different studies. Lovin et al. (2012) examined their MTE beliefs and practices in a collaborative self-study that "exemplif[ied] the power of examining and reflecting on other teacher educators' beliefs and practices as we analyze our own" (p. 65). They used both artifacts generated specifically for their study, such as personal narratives, and artifacts of practice that allowed them to analyze their beliefs in relation to their practices. Zaslavsky and Leikin (2004) studied MTEs' growth through participation in a community of practice. A thread throughout their work is iterative collaboration around artifacts of practice, particularly around designing the mathematical tasks that were an important focus of their project. In their elementary education courses, James Hiebert and colleagues have created "shared instructional products that guide classroom teaching" (Morris & Hiebert, 2011, p. 5). These lessons were collaboratively developed by the instructors of each course and are revised on an ongoing basis in response to conversations about data collected on the learning outcomes from enacting the lessons. Van Zoest, Moore, and Stockero's (2006) study of experienced school teachers and professional development providers learning to teach PSTs also highlighted the value of collaborative inquiry around artifacts of practice as a learning tool for MTEs. In addition, they emphasized the importance of focusing novice mathematics methods teacher educators' learning on the aspects of MTE practice that are different from their prior work as teachers and professional development providers, advocating the importance of "maintaining a focus on pre-service teacher thinking and having explicit conversations about the intent of pre-service teacher instruction" (p. 145). Thus, collegial conversations around artifacts seem an effective way to support MTE learning and development.

This chapter focuses on the particular approach we took to supporting MTE learning and development, what we call *Artifact-Enhanced Collegial Inquiry* (ACI).

Similar to much of the existing research on MTEs (e.g., Berry, 2007; Lovin et al., 2012; Tzur, 2001), we—Laura Van Zoest (LVZ) and Mariana Levin (ML)—turn our gaze on our own MTE practice. As is often the case, our collaboration was prompted by a practical concern. LVZ had been teaching and researching a course focused on learning to teach mathematics consistently since 2002 (e.g., Van Zoest, 2004; Van Zoest & Stockero, 2008; Van Zoest, Stockero & Taylor, 2012). Her impending sabbatical prompted the need for ML to teach the course. Although ML was experienced in teaching mathematics to teachers, she had not taught a course focused specifically on learning to teach mathematics—what we call a *methods* course.

In the following, we first describe the methods course that provided the context for our use of ACI and the theoretical perspective that undergirded both the course and our collaboration. We then describe ACI and illustrate how we used it to make visible important features of mathematics methods instruction. We conclude with a discussion of the implications of this work and how ACI might be used to support the learning and development of MTEs more generally.

9.1 The Methods Course

We first provide general information about the middle school mathematics methods course that provided the context of our collegial inquiry. We then describe two foundational aspects of the course that undergird the example that we use to illustrate our implementation of ACI: (1) the *Cycle of Enactment and Investigation* (Kazemi, Ghousseini, Cunard, & Turrou, 2016) used in the course and (2) the instructional activity *Contemplate then Calculate* (Routines for Reasoning, 2018) that was at the center of the course's first such cycle and is the context for the illustration provided in this chapter.

9.1.1 General Information

The middle school (ages 11–14) methods course is the first of a sequence of three courses devoted to the teaching of mathematics to students ages 11–18 and is a requirement both for mathematics education PSTs earning credentials for ages 5–14 and for those earning credentials for ages 11–18. The course runs for 15 weeks with two 100-minute meetings a week and includes three course-embedded field experiences in a local middle school.

The course focuses on teaching mathematics for understanding, as articulated by NCTM (2014), and instruction that incorporates student thinking is a recurring theme throughout the course. The course goals include acquiring mathematical knowledge for teaching; recognizing, valuing, and developing strategies for managing student mathematical learning; and developing skills and dispositions needed to access, interpret, and assess student thinking about mathematical ideas (for more information on the course, see Ochieng, 2018). Among other things, LVZ *models* (Loughran & Berry, 2005) the practice of attending to high-potential student thinking in a way that furthers the mathematical understanding of the class by engaging the class in making sense of that thinking in the moment in which it occurs—what Van Zoest, Peterson, Leatham, and Stockero (2016) call *building* on MOSTs. (For more information about MOSTs and the building practice, see BuildingonMOSTs.org).

The course sessions and embedded field experiences were routinely videotaped and audiotaped, and PST work, as well as middle school student work during field experiences, was routinely digitized. PSTs were expected to use these records of practice, particularly of their own teaching, as part of course assignments. Specifically, these records of practice were used as sites for analyzing teaching and as evidence to support claims made by the PSTs about their own learning and development as a teacher.

9.1.2 Cycle of Enactment and Investigation

The course provides opportunities for PSTs to engage in a *Cycle of Enactment and Investigation* (Kazemi et al., 2016; see Fig. 9.1) three times, each with a different *instructional activity* (IA; Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010). PSTs first engage in each IA as a learner to better understand the mathematics involved and to observe the instruction (Observation). They then unpack the IA as a class from the pedagogical perspective of a PST preparing to teach the IA (Collective Analysis). Next, they prepare for an in-class rehearsal in which each PST has the opportunity to rehearse the role of the instructor for a portion of the IA. Preparing

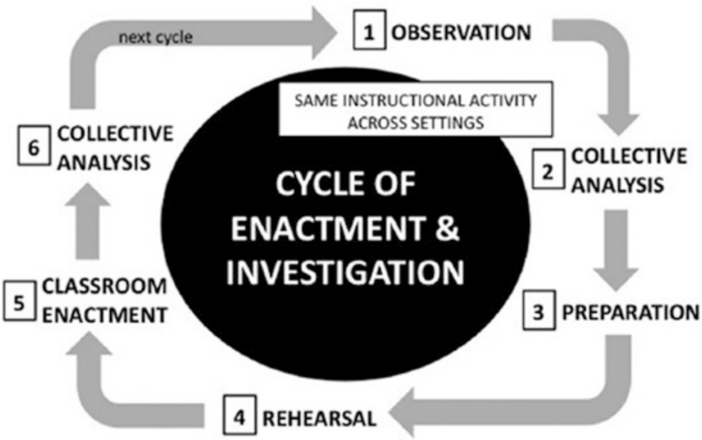


Fig. 9.1 *Cycle of enactment and investigation*. (Kazemi et al., 2016, p. 20. Reprinted by Permission of SAGE Publications, Inc.)

for the rehearsal involves anticipating student thinking, generating corresponding responses, and strategizing ways to maintain focus on the mathematics to be learned (Preparation). The PSTs then enact the rehearsal and debrief with their peers and the methods instructor (Rehearsal). Next they implement the IA in a local middle school classroom, having the opportunity to debrief their implementation with peers and the methods instructor immediately following their implementation (Classroom Enactment). Finally, the PSTs synthesize and reflect on what they have learned through their analysis of records of practice collected during the implementation (e.g., video of their teaching, written student work, audio of individual student conversations, and documentation of peer observations), both individually and as a class (Collective Analysis). The focus throughout is on developing PSTs' abilities to productively incorporate student thinking into instruction.

9.1.3 Contemplate then Calculate (CtC)

Contemplate then Calculate (CtC), created by Grace Kelemanik and Amy Lucenta (see Routines for Reasoning, 2018), is the IA we use in our first cycle. As Kelemanik and Lucenta explain, CtC "develops students' capacity to attend to mathematical structure when problem solving by using properties, rules of operations and relationships to uncover mathematical form and structure, and applying them to find calculation shortcuts or generalize results" (Routines for Reasoning, 2018). Thus, CtC shifts students' focus from mindless computing to meaningful consideration of mathematical objects and calculations as it develops *Common Core State Standards for Mathematics* Mathematical Practice 7, "Look for and make use of structure" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). CtC does this by providing a purposeful instructional sequence that can be repeated with different mathematical tasks at its center—tasks specifically chosen because they can be solved more quickly and easily by recognizing the structure underlying them. CtC's explicit instructional sequence (see Fig. 9.2 for an overview) embeds many aspects of effective instruction in a way that supports the PSTs in both eliciting and productively incorporating student thinking into their instruction. (See Kelemanik, Lucenta, & Creighton, 2016, for more information about reasoning routines and the practices embedded within them.)

For example, after part 1 of CtC (Launch Routine) sets the stage for the kind of activity that students will be engaging in—noticing patterns, finding calculation shortcuts, sharing and studying their shortcuts, and reflecting on their thinking—part 2 (Notice) begins by showing the students the task very briefly, just for the purpose of noticing patterns that may be mathematically important for coming up with a calculational shortcut. Students are then provided the sentence starters, "I noticed..." and "What did you notice…," to talk to their partner. These noticings of the pairs are then shared with the class and recorded by the instructor in a public place, forming a foundation for part 3 (Develop Shortcut). The built-in scaffolding for the students also serves as scaffolding for the PSTs as it creates concrete

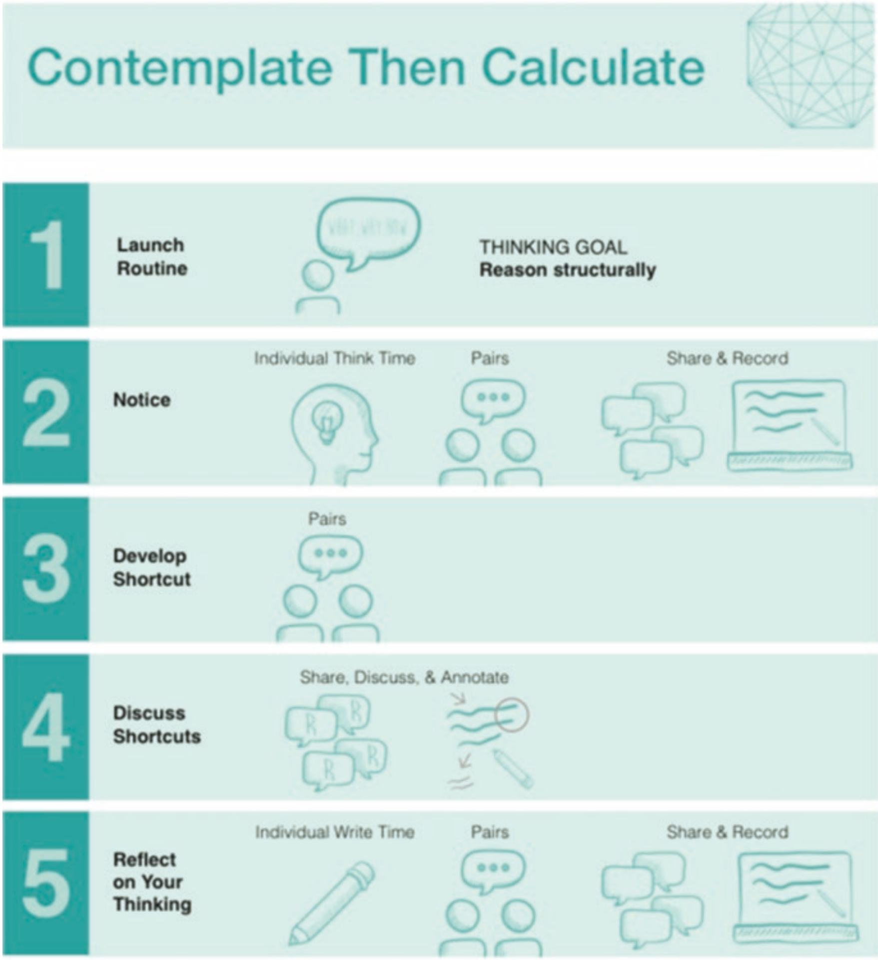


Fig. 9.2 Overview of the *Contemplate then Calculate* instructional activity (Routines for Reasoning, 2018)

opportunities for them to practice key teaching skills that they have encountered in the methods course, such as eliciting and making sense of students' thinking. This scaffolding is particularly important for our PSTs since the first cycle is often the first time that they have been responsible for the mathematical learning of a whole group of students.

9.2 Theoretical Perspective

We take an *inquiry as stance* (Cochran-Smith, 2003; Cochran-Smith & Lytle, 2001, 2009) perspective both for our own learning and development as MTEs and for the methods course that is at the center of that learning and development. This means that we are "both user and creator of knowledge, which is always regarded as generative and tentative, to be questioned, challenged, connected, tried out, revised, reshaped, and held problematic" (Cochran-Smith, 2003, p. 21). An *inquiry as stance* perspective is consistent with Jaworski's (2006) suggestions to use inquiry as "a tool to enable ourselves and others to engage critically with key questions and issues in practice" (p. 187). Taking this perspective has significant implications for both the methods course and our collaboration.

Approaching teaching a methods course with an inquiry as stance perspective means that LVZ does not see her role as imparting her knowledge to the PSTs. Instead, she uses what she knows to engage the PSTs in inquiry about what it means to be a mathematics teacher, specifically one who incorporates students' mathematical thinking into their instruction. Rather than present exemplars of teaching practice for the PSTs to emulate, she engages the PSTs in analyzing instances of teaching for their impact on student learning and to identify relationships between the actions of the teacher and the actions of the students. The PSTs and LVZ are positioned together as learners of teaching, with LVZ often noting the dilemmas she experiences while instructing the PSTs and explaining the reasoning behind her decisions, similar to the "talking aloud" and "debriefing teaching" described by Loughran and Barry (2005).

Similarly, approaching our collaboration with an inquiry as stance perspective meant that LVZ did not see her role as imparting her knowledge to ML. Although the desired outcome was that ML would be better prepared to teach the methods course, the expectation was that they would engage in joint inquiry and that LVZ would learn as well. Although LVZ had an accumulated "wisdom of practice" (Shulman, 1986, p. 9), she saw that wisdom as something that could be brought into the discussions to inform them, rather than a collection of ready-made solutions to be passed on to a novice instructor. LVZ welcomed the opportunity to bring ML into her ongoing inquiry and to adjust that inquiry to fit the needs of a novice instructor.

9.3 Artifact-Enhanced Collegial Inquiry (ACI)

We did not begin our work with a clearly articulated approach for our collaboration. Instead, the *Artifact-Enhanced Collegial Inquiry* (ACI) approach emerged out of our experience of putting *inquiry as stance* into practice during ML's experience of learning to become a methods instructor in a course that was predicated on that same stance. We describe ACI here so that the reader can benefit from the articulation of what, for us, was an intuitive process based on our perspectives on learning and our internalization of the literature base on teacher and teacher educator development. We specifically recognize the influence of Horn's (2005) work on collegial conversations as a site of informal learning opportunities for high school teachers and Lampert (1992; Lampert & Graziani, 2009) and the work of the *Learning Teaching in, from, and for Practice* group in general (e.g., Lampert et al., 2013) on building a pedagogy of teacher education through collaboration (Kazemi et al., 2016; Kazemi, Franke, & Lampert, 2009). We describe the setup and the three iterative phases of ACI here and then provide an illustration of our enactment of ACI.

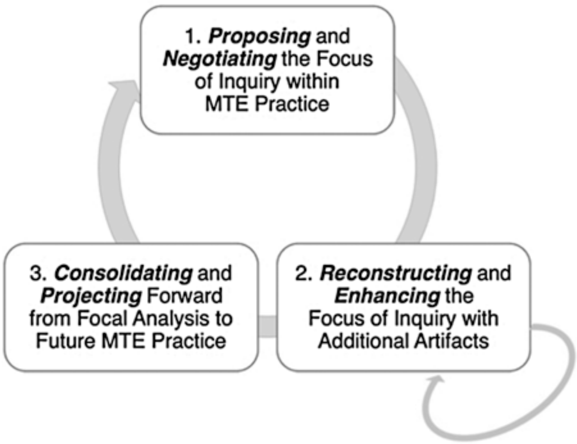
Before enacting ACI, it is important to identify a broad common goal for the collegial inquiry, such as whether the focus will be on instruction, curriculum, or students, and to determine who is going to participate. This initial step sets boundaries for iteratively engaging in the three phases of ACI, which are shown in Fig. 9.3.

The first phase of ACI, Proposing and Negotiating the Focus of Inquiry within MTE Practice, begins by *proposing* a specific problem of MTE practice to unpack. For full engagement, it is also important that the problem be rooted in a common experience of those participating. *Negotiating* the focus of inquiry involves both refining the problem and determining which instances of MTE practice to use to begin unpacking the problem.

The second phase of ACI, Reconstructing and Enhancing the Focus of Inquiry with Additional Artifacts, involves reviewing together the focal instances of MTE practice. This reviewing leads to jointly *reconstructing* what occurred and revealing the decision processes involved. This process of unpacking the problem often leads to questions that cannot be answered by the artifacts in the initial focus of inquiry. Thus, *enhancing* the focus by introducing additional records of practice to support the ongoing unpacking is often required in this phase. This enhancing leads to further reconstructing, which can, in turn, lead to further enhancing and reconstructing. Thus, phase 2 itself is iterative, as indicated by the smaller arrow in Fig. 9.3.

The third phase of ACI, Consolidating and Projecting Forward from Focal Analysis to Future MTE Practice, requires both *consolidating* the learning from

Fig. 9.3 Phases of Artifact-Enhanced Collegial Inquiry (ACI)



unpacking the problem in Phase 2 and *projecting* how this learning might inform future MTE practice. Although this projecting involves generalizing, it is important to note that we have no illusions that doing so will resolve the problem in general. Rather, it resolves the problem locally to the best of the participants' ability at that point in time.

9.4 Illustrating ACI

Our broad common goal for our collegial inquiry was to articulate what is involved in teaching methods in a way that would support a novice methods instructor in learning what they need to learn to be effective. Specifically, because ML had experience teaching mathematics courses for teachers, we were interested in highlighting what would be different about teaching a course that focused specifically on learning to teach mathematics. There were several graduate students observing the course as well as ML. Their goals for observing the course were different; however, thus it was only the two of us who participated in ACI. Once the goal and the participants were established, the iterative work of ACI began. It is important to note that there is variation in the length of an iteration of ACI, ranging from a single discussion to multiple discussions spread across time. Our selection of an illustrative iteration for this paper was based on the ability of the iteration to highlight a variety of critical features of ACI; the selected iteration also happens to be one of our lengthier ones. In the following we describe this particular iteration of our enactment of ACI.

9.4.1 Phase 1: Proposing and Negotiating the Focus of Inquiry Within MTE Practice

During the debrief immediately after the PSTs taught CtC to middle school students in the first embedded field experience of the course, ML learned that LVZ had intervened while a PST, Karry, was teaching. Karry had called on a student and got stuck documenting what she perceived to be his incorrect thinking. In looking toward the next iteration of the course and considering how to revise the CtC experience, ML wondered if perhaps her future PSTs would need more opportunities to practice rehearsing a broader range of scenarios, including incorrect student thinking. During our initial conversation when ML proposed this problem of practice as a focus of inquiry, LVZ prompted ML to recall that Karry did have the opportunity to engage with incorrect student thinking in her rehearsal (though in the role of a student as opposed to the role of a teacher). We clarified (for ourselves) the importance to the goals of CtC of selecting student thinking to share because it highlights the mathematical structure of the problem. That does not mean that it has to be correct or complete thinking, just that the thinking must illuminate the structure of the problem in some productive way. We made note to make the relationship between lesson goals and the type of student thinking that one selects for sharing a key point of discussion during the rehearsal preparation.

Our conversation about possible connections between Karry's preparation in the rehearsals and how her field experience teaching unfolded led naturally into a reframing of our inquiry to answer the question, "How does one know when and how to intervene in field experiences to support PST learning?" The case of Karry provided the opportunity to make sense of the decision to intervene when a PST is teaching and how to support PST learning after the intervention (both in the field and beyond). Thus we agreed to begin this iteration of our enactment of ACI with the instance of MTE practice where LVZ intervened during Karry's teaching.

9.4.2 Phase 2: Reconstructing and Enhancing the Focus of Inquiry with Artifacts

To investigate our focus of inquiry, we watched the video of Karry's teaching in which LVZ intervened. Working together from the video of the field experience, we reconstructed the following sequence of events leading up to the point where LVZ intervened. Karry and Karl (another PST in the course) were co-teaching CtC around the mathematical task 81-72+63-54+45-36+27-18+9 to a class of eighth graders (ages 13–14). The PSTs had successfully completed the first three steps of CtC—Launch Routine, Notice, and Develop Shortcuts (see Fig. 9.2)—with the class. As they had rehearsed in the methods class, they recorded on poster paper what the students noticed about the problem that might be useful in coming up with strategies for solving the problem and sent students back to work with their partners to use these noticings to generate shortcuts for solving the problem. Karry brought the class back together to have a discussion about the different shortcuts the class had generated (CtC Step 4). The first student Karry called on shared an expected, and correct, shortcut: Group the sum as (81-72)+(63-54)+(45-36)+(27-18)+9 in order to obtain a sum of five 9s and the solution of 45. Karry recorded the expected strategy step by step, using color to highlight the underlying structure, just as she had practiced in the rehearsal in the methods class.

Karry then prompted the class to share another shortcut. The class went quiet, and no one shared new thinking. Trying to stimulate some discussion, Karry promised the students that there were no wrong answers, and she tried to generate discussion by writing some additional things she thought of on the "noticings" poster paper that she had posted at the front of the room. Finally, she called on a specific student, AJ, who began explaining his shortcut, and, like before, Karry started recording it on the poster paper, utterance by utterance. At first, what Karry recorded appeared to correspond closely to AJ's utterances. AJ began by grouping 81-72 to find 9. What he said next was less clear but included focusing on -54, -36, and -18 and subtracting 9 from each term. Karry incorrectly recorded AJ's thinking at this stage as (-54-36-18)-9, and she immediately noticed that AJ seemed uncomfortable with the way that she had recorded his thinking. AJ reiterated that he subtracted 9 from each of -54, -36, and -18. Karry again asked if she could just write (-54-36-18)-9 to express what AJ meant. She attempted to press on even with the erroneous rendition of AJ's thinking recorded.

At this point, AJ said that this was as far as he got. Karry decided to push further, asking him for the answer to -54-9 in order to "finish his thought." AJ and the class both remained quiet. Karry was able to get some students to answer the specific questions she asked them: -36-9, -18-9, and -54-9. She reordered and rewrote these specific calculations as -63-45-27 on the board and prompted the class to regroup in some way. Frustrated that no one was contributing, she asked the class, "Do you want to scratch this and try something else or do you want to keep going?" She still received no response and seemed to have run out of ideas for what to do.

The decision to intervene As we reconstructed what had happened up until this point by watching the video for the purpose of making visible and unpacking LVZ's rationale for her decision to intervene, ML learned that LVZ had walked in right after Karry had successfully recorded the first shortcut on the poster paper. LVZ saw the exchange between Karry, the class, and AJ unfold. LVZ explained that she had observed that both Karry and the class were uncomfortable and that learning had halted; thus she felt some intervention was necessary. Not wanting to jump in prematurely, LVZ shared with ML that she first had asked Karry's co-teacher, Karl, if he would like to enter the discussion at this point. Karl told LVZ that he was not sure how best to intervene, but we did hear in the tape that Karl attempted to enter the discussion (Karl continued with AJ's thinking—subtracting 9 from each of the three terms—but did not engage directly with Karry's problematic rendition of AJ's thinking). Still, the students were very confused and not really sure what was being asked anymore. When the discussion stalled again, LVZ made the decision to intervene directly and said the following:

> I'm wondering—is this method going to get us where we need to get? What do you think? [There is a short silence with some indications of disagreement from both Karry and the students in the class]. Why not? It seemed like we were on to something and then all of a sudden it started to go awry—go wrong—like sometimes math problems do. [slight pause] Have you ever had that happen? You have a great idea and you're just cooking along and then all of a sudden it's just not working anymore? [LVZ laughs reassuringly.]
>
> I'm wondering if that's the kind of situation we're in. And then you've got to make a decision—do you try to undo what you did and try to fix it or do you try to start another way? [Slight pause.] Do you want to do another way? That is perfectly OK. It's OK to say, "All right—this doesn't seem to be getting us where we want to go and put a smiley face on it. [Karry draws a smiley face on the poster paper]. Because it was kind of fun while we were there but now we're going to move on to something else.

Karry seemed relieved to be offered another opportunity to restart the discussion. She went on to orchestrate the discussion of one additional student shortcut before turning the teaching back over to Karl for step 5 (as planned in their rehearsal of the field experience). LVZ's intervention was intended to be both a means out of what had turned into an unproductive learning situation—both for the middle school students and the teaching PSTs—and a new learning opportunity for the PSTs in the room (both those teaching and those observing) as she modelled a way to terminate unproductive sharing that upheld the principle of positioning the middle school students as legitimate mathematical thinkers.

Supporting continued PST learning beyond the field experience Immediately following the experience of teaching CtC, LVZ gathered all of the PSTs together in a conference room at the field site in order to debrief the experience. LVZ first asked the PSTs to debrief with their shoulder partners, giving them an opportunity to put into words the things that were most salient to them about the experience. LVZ shared with ML that her goal at this point was to listen to what the PSTs were sharing and to, in turn, reshape their contributions in a such way that the entire class could learn from them. Particularly in the case of Karry's contribution to the debriefing discussion, it was notable that LVZ did not emphasize at all any specific mathematical or pedagogical errors made in Karry's case. Instead, she chose to reframe Karry's contribution (and other contributions in general) in terms that all of the other PSTs might be able to access as learners (e.g., "How do you know when you should terminate a discussion and how might you do that?" "Does all student thinking offer opportunities for the entire class to learn from?").While it was tempting to dwell on Karry's "errors," particularly her misrepresentation of the student's solution that seemed to cause the sharing to become unproductive, LVZ felt that doing so would be akin to dwelling on a mathematics learner's arithmetic error made when struggling with a new higher level concept. Her reasoning was that if the work following the error cannot be quickly untangled by fixing the error, it is often best to ignore the error itself and focus instead on the learning opportunity that the error generated (e.g., If that was the answer, what would it tell us about the situation?). On the other hand, if the point is the details of a particular solution (or, as in this case, the recording of that solution), then it would be critical to pay attention to the error. Given the goals of helping the middle school students to understand mathematical structure and how it can work for them, and developing the PSTs' teaching skills, she decided that it would not be useful to either group to spend additional time on AJ's solution or Karry's rendition of it.

As we engaged in ACI around this instance, another important insight emerged that was not clear to LVZ at the time: the root cause of this unproductive situation could be traced back to Karry's focus on getting students to share. This led her to call on any student willing to share, rather than *selecting* (Smith & Stein, 2011, 2018) student work to be shared based on the ability of the work to illuminate the structure underlying the mathematical task. In hindsight, it seems that LVZ could have facilitated a discussion during the debriefing that would have supported the PSTs' developing understanding of the need to stay focused on one's instructional goal and to make decisions about which student work to share based on the extent to which it will support that goal. Furthermore, the PSTs may have been able to make connections both to the Smith and Stein (2011) work that they had read for the course and to a brainstorming discussion in an earlier course session about ways to support students in generating additional solution strategies. While these were missed opportunities for this particular group of PSTs, as a result of our ACI experience, we are better situated to recognize and respond to similar situations in our future teaching.

Following the field experience, the PSTs wrote immediate reflections (within 24 hours) on an electronic discussion board and uploaded supporting documentation of the field experience to the class google folder (e.g., their records of the thinking of the focal students they were shadowing, written artifacts of student work generated during CtC). On the discussion board, they were prompted as follows "What's foremost on your mind related to the first field experience? What things would you like to talk about in class?"

Curious to know more about how Karry processed the experience, we enhanced our focus of inquiry by looking at Karry's discussion board entry. Karry wrote that she "found the field experience very enlightening" even though she "had some hiccups." Karry also picked up on LVZ's reframing:

> [M]y inner self struggl[ed] to find the wording to fully redirect or to "terminate" this method all together without embarrassing the student. So one thing I would like to talk about in class is how exactly to word the "termination" if redirecting did not work.

As we jointly reconstructed this instance of MTE practice, LVZ remarked to ML that it was interesting that Karry did not seem to recognize LVZ's intervention as an example of exactly how to do what she was asking about in the online reflection—terminate an unproductive line of discussion without "embarrassing" the student. Even more, it was an example of doing so without undermining the development of the middle school students as legitimate mathematical thinkers. When LVZ had intervened in the moment, what she had said was something that Karry could have also learned from and added to her repertoire for similar situations in the future. We hypothesized that when writing her immediate reflection Karry may have still been in the grip of the stressful moment and wondered if she would recognize LVZ's actions when she watched the video to write her field experience reflection paper.

The field experience reflection paper, completed as part of the Collective Analysis stage of the Cycle of Enactment and Investigation, prompted the PSTs to connect their experiences in the field with the readings they had been doing in the methods course. The PSTs were not required to choose any particular readings, but they did need to tag moments of their practice to reflect upon and make connections to the literature in specific areas, including moments in which they made (or could have made) decisions toward a clear mathematical goal. In order to unpack the affordances of the different learning experiences for PSTs and also observe how LVZ worked with the PSTs as they engaged in these cycles of processing and reflection on their practice, we also went back to see how Karry framed the experience in her field experience reflection paper. In her paper, Karry wrote:

> Another place I could have been able to fully articulate better towards a clear mathematical goal is when one student's way went awry. What I have learned is that I need to know when to "funnel" and when to "focus" (Herbel-Eisenmann & Breyfogle, 2005, p. 485). Initially, I started using the "focus" line of questioning in which I listened to the student's responses and guide them based on what the student was thinking (Herbel-Eisenmann & Breyfogle, 2005, p. 486). So, initially the student had a thought on how to solve the 9 s, but had not

> fully finished it; that is when [I thought] let's finish it as we go, so [I asked], "What did you [d]o first? [timestamp of the video]. A few steps later there was a realization from me that there is an error in his logic, there was negation and distribution of nine. That is when I attempted to steer the student into seeing this flaw. However, after multiple attempts to redirect to what I noticed was wrong with the student's train of thought and kind of beating around the bush, this is when I should have switched from "focus" to some form of "funnelling."

LVZ chose not to comment on the specifics of Karry's rendering of the experience, some of which were questionable in comparison with the video of what happened, though she did tag the need to revisit the terms "focusing" and "funnelling" and their relationship to orienting toward a mathematical goal in the course. Instead, she made the following comment on Karry's final sentence in which Karry claims that what she should have done was "funnel" instead of "focus." LVZ:

> That is actually quite dangerous; switching to a funnelling line of questions when you have started with student thinking sends the message that the students' thinking isn't valued. Think about other options.

Although there are many possible issues that LVZ could have raised with Karry, ML noted that LVZ's comment was completely consistent with the inquiry as stance approach to the methods class. Instead of a long comment detailing "what Karry should have done," LVZ pressed on something problematic in Karry's conclusion about what she herself says she should have done. The issue is not resolved yet for Karry. Instead of allowing for a false sense of closure with a problematic resolution, LVZ pressed Karry to continue to think about this issue.

If this episode took place in a mathematics course for teachers, a top priority may have been to address Karry's inaccurate rendering of AJ's mathematics—specifically, that there is no distributive property of subtraction [(-54-36-18)-9≠(-54-9)+(-36-9)+(-18-9)]. There are important mathematical understandings that the PSTs might have developed as a result of engaging with this piece of mathematics. However, overgeneralizing the distributive property of multiplication in this way is not a common mistake that PSTs make, and it seemed likely that Karry could easily correct her mistake if it was pointed out to her. Thus this piece of mathematics did not rise to the level of a mathematical issue worthy of taking up limited time in the methods course. The question, "When does one foreground mathematics in methods course discussions?", was a problem of practice that we had unpacked in another ACI iteration. Our experience with that iteration provided principles for discussing LVZ's (lack of) interaction with the mathematics in this episode. Even though the "intervening" iteration of ACI had a different focus from that of the "foreground mathematics" iteration, it provided us the opportunity to revisit the earlier problem of practice and clarify our conclusions.

Although we stop our elaboration of our illustrative iteration at this point, our ongoing enactment of this iteration of ACI provided the opportunity for us to explore what, if anything, Karry might have learned. Recall that this was the first of three cycles of enactment and investigation in the methods course. Though Karry did not emerge from the class an accomplished teacher, she did continue to learn and was well on her way to becoming a "well-prepared beginning teacher" (AMTE, 2017).

9.4.3 Phase 3: Consolidating and Projecting Forward from Focal Analysis to Future MTE Practice

Following the process of reconstructing and enhancing our focal inquiry to encompass Karry's field experience and associated opportunities to interact with her around what happened, we attempted to consolidate our learning from unpacking the MTE practice of intervening in field experiences to support PST learning.

A main takeaway was the primacy of the learning and development of both the middle school students and the PSTs. The discussion in the middle school classroom had become an unproductive learning situation for both the teaching PSTs and the middle school students. For example, there was student thinking that had been publicly shared that was no longer being engaged with either by the student who had contributed that thinking or the class. A problematic rendition of student thinking had reified the student's contribution in a way that the student no longer recognized as his own and that no one else in the class had taken up either.

While it would have been preferable for the PSTs responsible for the class to resolve the teaching dilemma on their own, the teaching PST seemed visibly shaken and without an exit or redirection strategy, and her partner PST had also struggled to intervene. Thus, for the sake of the learning of both the middle school students and the teaching PSTs, it seemed like an intervention was required.

LVZ shared some further rationale: Without an intervention, there may have been serious consequences for the rest of the lesson (i.e., if no additional shortcuts were discussed, identifying connections in the structures of the shortcuts would have been impossible since there was only one strategy in which the role of structure was clear). Further, a departure of this magnitude from achieving the lesson goals could in turn have serious consequences for the PST's developing identity as a teacher. All of these features seem to support the decision to intervene.

Taking this information into account, we arrived at the following criteria regarding when to intervene during a field experience:

1. *The school students are likely to develop a misconception, either about mathematics or about themselves as mathematics learners.* In our case, Karry's actions were sending the message that the problem was with the students (for not responding) and particularly with the thinking of the student who had contributed the thinking (because it was not working). Although as MTEs we could see that the problem was in Karry's interpretation and representation of the student's thinking, this distinction was not likely to be clear to the students.
2. *The PST(s) are likely to develop a misconception about students, teaching, or about themselves as teachers.* Because learning teaching is so complex, when something does not go as planned, it is very easy for PSTs to draw any number of unproductive conclusions related to teaching. A common misconception that LVZ was trying to avoid in this situation is that instruction based on student thinking just does not work.

3. *Something has happened that will undermine the rest of the lesson*. In this case, it was the inability to compare solutions. Another situation related to teaching CtC that has required quite a different intervention occurred when a PST failed to get the students' attention prior to flashing the problem for them to notice things that might help them with finding shortcuts (Step 2). If the students do not see the problem, they have nothing to notice and thus nothing to share. In contrast to the intervention during Karry's teaching, the intervention in this situation was a quick comment to the PST to flash the problem again to give the students another chance to participate.

The opportunity for ML to unpack LVZ's decision to intervene with her was a critical point in ML's development as a novice instructor and LVZ's development as her mentor. Prior to their collegial discussions, ML had conceptualized the role of the MTE at the field site as primarily related to (1) behind-the-scenes logistical work to allow the PSTs to focus on their field teaching experience and (2) monitoring PST teaching so that she could engage with the PSTs in reflection following the experience. Although LVZ had developed wisdom of practice around the MTE work that occurs during field experiences, until this enactment of ACI, she hadn't realized the importance of making that in-the-moment work explicit to ML (and to other novice methods instructors). Engaging in the ACI process allowed both of us to see the work more clearly and to analyze the rationale for, and results of, various MTE actions.

9.4.4 Coda

It is worth noting that some of our ACI iterations were based on a shared experience and did not require as much reconstructing as the provided illustration. Although we found the video corpus and other artifacts of practice valuable in all ACI iterations, because ML had not witnessed Karry's teaching in the field experience directly (she was observing the teaching of other PSTs at the field site), these artifacts were essential for supporting ML's learning and development in this particular iteration. By the end of phase 2 in this ACI iteration, we had drawn upon (1) video of Karry's episode of teaching, (2) audio of the students who were contributing to the whole class discussion, (3) Karry's immediate posting on the course discussion board about her experience, and (4) Karry's description of the critical interaction in her reflective paper (and LVZ's written feedback). The opportunity to review the focal instance of MTE practice collaboratively with LVZ at a later time using the video really impressed upon ML the kinds of choices LVZ was making in the moment and the fact that there was a whole plethora of other things that LVZ might have foregrounded but had explicitly chosen not to. ACI supported our discussion about some of these alternative issues that had lower priority in that moment. For example, in addition to the mathematical error that Karry had unwittingly introduced into the discussion through the way she chose to record the student strategy, LVZ pointed out that another general pedagogical issue with Karry's teaching in her field experience was her repeated attempts to engage students one on one, rather than engaging the entire class in the discussion. LVZ chose not to make Karry's teaching an example of this issue because she knew that there would be many other opportunities in the class to address it. In terms of supporting ML's learning as a new methods instructor, discussions around the full corpus of data related to this episode provided a window into both MTE decision-making and the PST experience of their opportunity to teach. We note that such windows into both teacher decision-making and PST experience do not usually precede one's first experience teaching a course. Thus, this illustration indicates the power of creating and sharing records of practice and engaging in conversations between novice and experienced MTEs around these artifacts.

9.5 Discussion

In this chapter, we have exemplified a pedagogy for MTE learning and development designed to make MTE practice visible through collegial inquiry that is enhanced by artifacts of practice. This pedagogy emerged out of our experience putting *inquiry as stance* (e.g., Cochran-Smith & Lytle, 2009) into practice as an MTE learned about becoming a methods instructor. In the hopes of generalizing from our case, we articulated *Artifact-Enhanced Collegial Inquiry (ACI)* as an approach for doing this work. Here, we revisit ACI, considering both what seemed important for our case of MTE learning and development and what is generalizable.

As discussed earlier, identifying a common goal for collegial inquiry is foundational. In our case, the goal was to elaborate what is involved in teaching mathematics methods courses built around pedagogies of enactment and inquiry. That said, we see ACI being useful for promoting MTE learning and development in contexts other than supporting a novice methods course instructor. For example, ACI could be used to structure discussions between two experienced methods instructors who are collaborating around the design and revision of a course. More generally, this approach could be the basis for collaborations among teacher educator colleagues at the level of a department.

In the case of developing a novice methods course instructor, proposing as a focus of inquiry a problem of MTE practice that was rooted in a shared experience was crucial. A novice instructor may not yet have their own course experience from which to generate "re-plays" and "rehearsals" (Horn, 2005), so observations of the experienced MTE's practice provide an opportunity to ground the discussions. Negotiating the focus of inquiry is a process that involves both refining the problem of MTE practice and determining which instance(s) of MTE practice can be leveraged to begin to illuminate what is involved in that practice. We found that a useful practical heuristic for prioritizing which problems of practice to unpack was to start from noticings or questions of the novice methods instructor. We observed that many things had become "intuitive" to the experienced instructor, and since the common goal of this inquiry was supporting the learning of the novice instructor, it seemed productive to start from issues that were puzzling to her. However, collegial inquiry that has a different common goal may use different criteria or heuristics for establishing the problems of practice that the participants will collaboratively unpack.

The process of unpacking the focus of inquiry that occurs in phase 2 is where ACI really goes beyond previous explications of informal learning opportunities present in collegial conversations (e.g., Horn, 2005). We observed that the iterative process of reconstructing the focal instance of MTE practice and enhancing the focus of inquiry with additional artifacts generated rich opportunities for the learning and development of both the novice and the experienced methods instructor. We suspect that this would be true for MTEs more generally.

Finally, just as we supported PSTs in our methods class in reflecting on and generalizing from their field experiences, we found that consolidating our learning and projecting forward from our focal analysis into future MTE practice was extremely productive. In fact, projecting forward to hypothetical similar instances supported both MTEs. Articulating principles for aspects of MTE practice made visible in phase 2 provided a structure for the novice instructor to take into her future practice and honed the experienced instructor's wisdom of practice. Taken together, these observations suggest the benefits of enacting ACI to support a broad range of MTE learning and development.

ACI is a formalization and articulation of the process that we intuitively engaged in to improve our MTE practice. One of its strengths is that it resonates so strongly with our existing practices as MTEs (e.g., collecting and reflecting on artifacts of practice, collegial conversations). Because of the ability to co-construct a truly shared rendition of focal instances of MTE practice (using artifacts of practice to ground, elaborate, and extend), ACI appears especially valuable for making visible the rationale and wisdom of practice that supports in-the-moment decision-making. Finally, articulating the structure of ACI aided us in creating and optimizing learning opportunities within our conversations, for example, pushing ourselves to not only unpack a particular instance of practice of interest to us but to consolidate and project forward from our discussions and analyses. We hope that our articulation of ACI will support other MTEs in engaging in productive collegial conversations that tap into the wisdom of MTE practice and create rich opportunities for MTE learning and development.

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Chapter 10
Working with Awareness as Mathematics Teacher Educators: Experiences to Issues to Actions

Julian Brown, Laurinda Brown, Alf Coles, and Tracy Helliwell

10.1 Introduction

We, the authors of this chapter, teach, or have taught, on a 1-year, postgraduate, initial teacher education course in the UK. We have a range of experiences in this role from 27 years to 7 years to 2 years to having just started. Each of us taught mathematics for over 10 years, in secondary schools, before coming to the university mathematics teacher educator (MTE) role. In this chapter, we explore both how we work with our prospective teachers and how we work together in becoming more comfortable in the MTE position. We believe that some of our ways of working are both unusual and powerful, in terms of the learning of our prospective teachers. We offer them here, in the context of discussions related to our planning of MTE teaching sessions, in the spirit of "expanding the space of the possible" (Davis, 2004, p.184). These discussions are, of course, in part for ourselves, part of our praxis as MTEs working together to develop awarenesses that we use, enacting our planning. In putting together this writing, we illustrate that the processes we use as MTEs to develop our practices are the same as those our prospective teachers are offered to develop their practices. These processes have emerged from the way learning is seen within an enactivist perspective and underpin the design of the teacher education course. We believe experts and novices can learn in the same way through staying with the detail of their practices and attending to "what is the same and what is different" to expand their range of possibilities to act (Brown & Coles, 2011, p.866).

After brief discussions of the important ideas for us of (1) "awarenesses", (2) "metacommunication" and (3) "second-person perspectives", each of the authors of the chapter, in order of years of experience from most to fewest, offers a discussion

J. Brown · L. Brown · A. Coles (✉) · T. Helliwell
University of Bristol, Bristol, UK
e-mail: alf.coles@bris.ac.uk

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_10

of their planning for a session at the university with prospective mathematics teachers on the course. There is then a section of reflecting, where similarities and differences are discussed and analysed.

10.2 Background Ideas

10.2.1 Working with Awarenesses

An important word for us, which is in the title of this chapter, is "awareness", which was made into a countable noun by Gattegno (1987): "awarenesses" (p.25). A conviction that is expressed strongly at various stages of our MTE course is that there is no one model of good mathematics teaching. Planning does not focus, therefore, so much on a model, or even models, of mathematics teaching but rather on creating opportunities to develop awareness. In one form, awareness can be taken as experiential and self-referential. In this sense, "a person's awareness is the world as experienced by the person" (Marton & Booth, 1997, p.108), and there are similarities here to the use of awareness as a synonym for consciousness and as a framing of levels of articulation of mental states (Winkielman & Schooler, 2011). Here, though, we make use of the work of Gattegno (1970, 1987) and use "awareness" to indicate specifically the potential for and enabling of activity. In this sense, awareness is used to describe a core action or function that must be present in order to learn (Mason, 2008), so that, for instance, an awareness of counting squares covered by a shape might allow attention to be drawn to a definition of area and an awareness of tangents to a curve might allow attention to be drawn to stationary points of the curve (Wheeler, 1975). In particular, Gattegno (1987) describes a necessary condition for being a mathematician as the "awareness of relationships" (p.26) and, further, suggests it is when we become aware of such an awareness that we move forward, or, as we would say, we learn.

A teacher of mathematics can become engaged in a project of offering contexts in which learners' experience provokes them to make connections, giving the possibility of new actions; the assertion of Gattegno (1970) that "only awareness is educable" suggests this is the chief role of the mathematics teacher while keeping open the way in which it might happen. A movement into awarenesses as "that which enables action" (Mason, 2011, p.43) can be a powerful enabler for classroom practice (Coles, 2013). As Hewitt (2001) says:

> By educating awareness the mathematician inside a student is being educated, which would not be the case if everything were treated as if it were to be memorised. Awareness informs decisions and how to act using information which is known. (p.38)

The focus for teachers becomes the awarenesses that are present and might be brought to mind in their students. They are recognising when and how students experience the shifts in attention that indicate becoming "aware that what used to be attended to was only part of a larger whole" (Mason & Davis, 1988, p.488). This attention requires that teachers become aware of their own awarenesses, of what is present in the classroom and what is not, allowing action on their part (e.g. offering or not offering further prompts, new questions or different heuristics). By extension, the focus for MTEs is to become aware of the awareness of awarenesses, a guiding principle in planning for this course, in allowing both MTEs and prospective teachers to expand their possibilities for action. To illustrate this extension, imagine a school student in a classroom, who acts in a way that indicates they have not considered negative numbers as possible solutions to a particular problem. As a teacher, becoming aware of (in this instance, the absence of) an awareness might lead to a comment such as "you seem to be considering positive numbers only" to explore whether there is any awareness of the possibility of using negatives. Now imagine a prospective teacher, in a teacher education session, displaying exactly the same behaviour. In this case, the MTE might want to follow up any comment about the mathematics (which comes from a position of awareness of awareness of the mathematics) with a comment such as "so, as a teacher, how will you work with your students so they are able to question the assumptions they make in problem-solving?". Now the comment is coming from a position of an awareness that the teacher needs to be operating with an awareness of mathematical awarenesses (present or absent) of their students.

10.2.2 Metacommunication

We follow Bateson's (1979) use of "metacommunication" (p.107) to denote communication that is *about* communication. Bateson (1972) was among the first to bring to our attention the distinction between message and metamessage. He suggested that message and metamessage interact in meaning making and metacommunication and claimed that an essential function of metacommunication is to direct interpretation, as frames within which the speaker's comments are to be understood. In this chapter, we refer to our own use of verbal metacommunication when something is said *about* the communication that is taking place. As MTEs, we use verbal metacommunication explicitly in response to what our prospective teachers are saying in sessions we are running, to point to a range of ways of behaving as learners of mathematics, mathematics teachers and in schools. We believe their own explicit metacommunication will help them establish their mathematics classrooms, through pointing to the range of behaviours they value from their students and that they believe will support the learning of mathematics. In the imagined examples at the end of the previous section, the comments to both the school student and the prospective teacher would be examples of metacomments.

In a classroom, the teacher's metacomments are about their students' learning of mathematics (while the teacher is learning about the students' learning about mathematics). For MTEs, our metacomments are about our prospective teachers' learning of how to teach mathematics; our learning is about their learning (to teach mathematics, as well as about mathematics). A teacher's or MTE's metacommenting (rather than, say, directly answering, or offering leading hints) can act as a powerful mechanism to establish desired patterns of working as a group. A metacomment may require an awareness of what is absent. For instance, a common pattern for novice teachers is that when they talk about lessons they have just taught, their attention is only on themselves and what they did or did not do. As an MTE, awareness of the absence of discussion of the school students might provoke a metacomment. A metacomment about observing a desired behaviour can equally be powerful in establishing that behaviour as something others might do, or that might be done again (e.g. see Coles, 2013, for more illustrations of this phenomenon).

10.2.3 Second-Person Perspectives

Drawing on roots in introspection and phenomenology, Varela (who, along with Maturana, is one of the influential figures in the birth of enactivism, where knowing and doing are equivalent) offers the notion of "gestures of awareness" (Varela & Scharmer, 2000, p.1), in the process of elucidating first-person experience. His gestures are "suspending", "redirecting" and "letting go" (p.4), envisaged as a cycle that allows for learning from first-person experience. Suspending involves a break in our typical processes of sense-making in the world and may need an active determination not to be caught in habitual patterns of perception-action, for example, attending to the detail of a classroom event rather than evaluating. Redirecting is a process of directing attention towards something perhaps previously unnoticed, for example, provoked by the articulation of the awareness of another. Letting go refers to the gesture of non-attachment to previous modes of thinking-doing-being, to allow for a continuation of the cycle into suspension and redirection, for instance, in allowing a reinterpretation of an incident in a lesson that might have been experienced as "wrong" or "bad", to accepting alternative views.

For us, what is particularly significant, in Varela's characterisation of awareness, is the importance he places on the second-person perspective, the more experienced "other" who is able to recognise the awarenesses being elucidated during the cycle of suspending, redirecting and letting go (see Metz & Simmt, 2015, for a methodological use of the second-person perspective) and, as illustrated earlier, to recognise awarenesses that are, or are not, present. We cannot have access to each other's first-person awareness. However, an empathic "second person", who is an expert (in mathematics, or in teaching mathematics, or in the MTE role), is able to observe, not just externally. A second person who is an expert can recognise, empathise and become a "partner in the process" (Varela & Scharmer, 2000, p.7) of becoming aware. At the end of this chapter, we return to the theme of the second person, to elucidate the role of this "other", through the stories that now follow.

In the next section, Laurinda describes the origins of a cycle we refer to as "experiences to issues to actions", which informs all of our teaching on the teacher education course. Although not teaching on the course any more, she illustrates the ideas with an example of her own planning using this cycle. Following this section, the three MTEs currently teaching on the course, in each individual voice and in descending order of years of experience, offer accounts of their own teaching at the university, focusing on the planning. These accounts are offered from a first-person perspective. We then come together in a concluding section to look across these accounts to draw out similarities, differences and implications.

10.3 A Way of Working: Experiences to Issues to Actions (Laurinda)

No idea is original. In planning for writing this chapter, *experiences to issues to actions* emerged as important for the three other authors of this paper as they discussed how they teach prospective teachers on this course. The tutors work with prospective teachers both in the university and on visits to observe them teaching mathematics in school. My immediate reaction to the emergence of the phrase was that I had worked with it having read a book published by Barbara Jaworski through The Mathematics Association (1991) (one of the associations supporting teachers of mathematics in the UK), documenting the work of a group that she chaired. I offer here a historical perspective to the idea leading to a related action, a story of how I planned using the cycle. Although not currently teaching on the course having retired, I had, at some point in the early 1990s, designed the course in its current form.

I had originally started to use the phrase in working on Master's mathematics education courses and wrote up the sessions in a chapter in a book *Liberating the Learner: Lessons for Professional Development in Education* (Claxton, Atkinson, Osborn, & Wallace, 1996). Although I had thought that I had taken the ideas from Jaworski, in the chapter appeared:

> The way in which I planned to work in the session was by progressing from a consideration of *experiences*, via the formulation of *issues*, to the delineation of possible *actions*. The methodology is adapted from Jaworski (1991) […]. (Brown & Dobson, 1996, p.214)

I had adapted the ideas but needed to see the original to know how. I asked Barbara Jaworski if she still had a copy of her book from 1991 since I could not now find my copy. She kindly posted a copy to me, and I looked for what had been the original stimulus. The whole book was called *Develop Your Teaching* (The Mathematical Association, 1991) and was written to support the professional development of teachers. The process was based on what were called "anecdotes", which could be spoken or written and might provoke others to recall incidents. In the book (p.26) is a diagram for the ongoing work of a group of teachers. Anecdotes from a number of teachers lead to identifying issues, and classroom action is then implemented after which there is feedback into more anecdotes, and the process is then cycled. For anecdotes I had focused on spoken stories of experiences.

What follows is my planning using "experiences to issues to actions" for a session early in the teaching year of the course. Given that the session was repeated for many years, the discussion also illustrates how my own learning followed the same pattern of "experiences to issues to actions" as my awarenesses of using the activity and these particular tasks developed.

10.3.1 Story: Planning for the 4-Minute Workshop

The 4-minute workshop first appeared in my diaries on 26 September, 1991, and I then went on to lead the session during the first week of the new academic year (late September) for the next 25 years. As the person who had designed the course around "experiences to issues to actions", I did not have to write this down in my planning. I looked for an experience, in this case an activity that the group would experience together, that would have many purposes given how early in the course the session was given. The previous year, in the summer term, the prospective teachers, in groups, had created resources that would fill a need for the partner schools of the project. One of these resources was a workshop to support teacher assessment in mathematics for low-achieving year 8 students (aged 12–13 years old). The resources had been placed on tables that were in a circle around the walls of the classroom with a resource island in the middle. Two chairs were placed at each table facing the wall. The two students sitting at each table were labelled A and B. Every 4 minutes, hence the name of the workshop, the teacher would say, "Move", and the As went clockwise, and the Bs went counter-clockwise. In adapting this organisation as a session for prospective teachers, as the size of the group varied, I would add to or subtract from tasks in the original workshop so that there were enough tasks for pairs and perhaps, in some years with an odd number of participants, a singleton to be catered for.

This seemed a useful activity that would serve a whole range of purposes: introducing teacher assessment and supporting the prospective teachers in learning each other's names when meeting and working with their peers. There would also be issues, such as some of the activities would not take an adult 4 minutes, and they would like and not like particular activities for different reasons. It is important to work with all the members of the group, as they will have to do with all colleagues in school; but this activity would work, given how they reacted to working with different people, to uncover aspects of themselves that being aware of would prove useful when they went to school, for example, being used to working on mathematics by themselves and not being comfortable working in a pair. My focus is on supporting their individual developing awarenesses through learning about how they interact with the tasks.

The activity illustrates "experiences to issues to actions". The experience that we can all share is doing the workshop. Issues arise in reflecting together after the event, and we can then think through actions we could take as teachers to address those issues. From my perspective as leader of the activity, I get a lot of time to learn the individual behaviours of a new group of prospective teachers as they work.

Setting up the activity with the prospective teachers takes a lot of care and time because it includes giving them meta-tasks to work at, alongside the doing of the mathematics:

- Think about issues arising from working with another, and learn their name.
- Which tasks do you like and why, dislike and why?
- What if? Finding their own extension to the problem as written if they finish before the 4 minutes is up.

- What is each task assessing?

The activity is still done today, although the tasks are different. It is an example of an activity that takes quite a while to get ready on the day, prior to the arrival of the group, but once underway gives a lot of space for interaction and noticing of how individuals work, important for my learning of the individuals' strengths and areas for development in the group. There are particular points I want to get across for each task, illustrated by the following three examples from the set of 15.

10.3.1.1 Task 1: Limitations We Put on Ourselves

Two triangles, cut out of card, are provided for the task below. The triangles are congruent, obtuse-angled and scalene.

Triangles

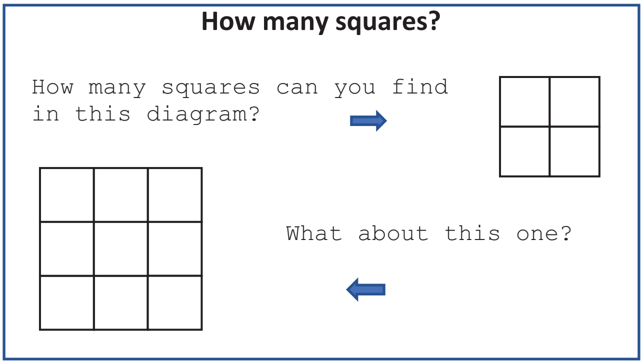
Using the two triangles can you make: a rectangle; a parallelogram; a kite; a pentagon; a hexagon?

What about a heptagon?

What is the biggest number of sides possible?

Observing prospective teachers doing this task, the first few can be made with sides of the same length touching corner to corner. Some pairs then get stuck. In the end, I would be looking for when this happens from the awarenesses built up through past experience and, when the issue is noticed, would then act, picking up one triangle and laying it across the other, overlapping. "Is that allowed?", is often asked. In response to such a comment, I would metacomment, saying something about, "Beware of limitations you put on yourselves and notice them in your students". Another limitation is to assume that "pentagon" means "regular pentagon".

10.3.1.2 Task 2: What to Do When Students Have Finished?



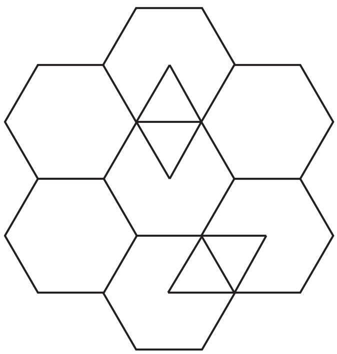
Answers of 5 and 14 come quickly from the prospective teachers for the task above. It does not take them 1 minute to agree on those. Some pairs discuss what to do next, often generalising for a square of side *n* smaller squares. Some pairs think that they have finished and start to make notes on whether they like the problem or not. The previous experiences with this workshop led me to notice the issue, for me, of "having finished" and the related actions of the prospective teachers. I move to act.

"What will you do when students you are teaching say that they've finished?", I ask. I ask them what the constraints are in the question. Small squares in a square? Only cases 2 × 2 and 3 × 3. What if it's not a square? What if it's $n > 3$? Identifying what's changing and then "what if-ing" and "what if not-ing" are strategies for generating new questions (Brown & Walters, 1969, p.38). This intervention gives the prospective teachers an action or something to offer to the students to whom they teach mathematics if they finish early. As an experienced MTE, my attention is fully on the prospective teachers' learning, and I am learning about them while also acting to provoke their learning.

10.3.1.3 Task 3: What's the Purpose of the Activity?

Make a daisy

Fit the seven hexagons together to make a shape like this:



Where the hexagons touch the numbers have to match!

In designing Task 3, the original prospective teachers had the idea of using hexagons with fractions, decimals or percentages on each edge being fitted together so that the numbers matched. To make the original "daisy", they drew hexagons in that pattern and then wrote numbers on the sides that they wanted the workshop to assess the equivalence of. Having cut up the hexagons and tried to fit them together again, they could not do it. Initially the group thought they would have to redo the making of the hexagons, but then one of them realised that for this task, the more practice that the students had on equivalent forms of fractions, decimals and percentages, the better, all with the spurious purpose of making the daisy. For some prospective teachers, this is their favourite activity and, for others, their least favourite because they could not finish it in the time. The same discussion has happened after the workshop has finished down the years: Is the purpose of the activity to make the daisy, or to get more practice than their future students would probably be prepared to do if given a set of questions from the textbook?

In keeping the activities of the 4-minute workshop the same over years, I became more and more skilled at making points and noticing where to intervene with the minimal of fuss, for example, moving one of the triangles to show that fitting corners together did not matter or saying, "What are the variables? What could be changed?", for the number of squares in the square problem. This is my learning. It seems to me that I am going through the same process as the prospective teachers in the sessions are. My *experiences* of teaching the activity repeatedly raise *issues* for me that become the focus of my observations leading to *actions* and *metacomments* that feed back into my teaching.

10.4 Current Stories and Discussions of Planning

10.4.1 Alf: Session on Using ICT

The use of ICT and technology in the classroom is an element of teaching practice that seems to change from year to year. We had timetabled a session of 90 minutes as an "Introduction to ICT" in the Autumn Term. I wanted to introduce prospective teachers to two software packages. This was a judgement made from wanting to give them some experience of comparison but also to allow enough time in the session (i.e. 45 minutes per package) where they could get deep enough into the package to hopefully mean they got a sense of its potential in the classroom and therefore had the motivation to explore one package in more depth in their own time (ultimately leading to them incorporating it into lessons).

Having one dynamic geometry package felt an easy choice. Just for ease of access to the software and also for the fact that it can act as a graphic tool, *GeoGebra* was the one I picked. For a second tool, I chose *Scratch* (which is a programming language developed out of the Logo microworld). This choice was made, perhaps partly because I know how I used to introduce children to work on *Logo* in a classroom and I could do the same introduction here. *Scratch* also links to programming, which is a relatively new focus in schools.

The two packages also felt important because having a contrast would allow me to make the focus of the session both learning the packages and saying this session was about them needing to choose one ICT package in which they were going to become expert over the year. The start would therefore be this "meta"-task, and hopefully they would like at least one package of the two. In the text below, for reasons of space, I focus just on the introduction to *Scratch*.

I planned to get the prospective teachers to clear a space at the front of the room and form a circle with chairs and sitting on desks. I put two desks in the space, making a square obstacle. I would ask for two volunteers, one to be a robot and one a controller. The controller has to direct the robot around the desks, but the robot has a very limited vocabulary (that I will help moderate). This start forces prospective teachers to think themselves "into" the robot's perspective. To turn requires a command of "left" or "right", and moving needs a "forward" or "backward" command. The task gives prospective teachers an entry into the programming language of *Scratch/Logo*, and I will show how they can use exactly those commands to control their own "robot" on the *Scratch* screen. The challenge for prospective teachers would be to try and generate different regular polygons, followed by trying to cover the screen with one of them. I imagined I might introduce the prospective teachers to how to generate variables and how to repeat and perhaps how to set up recursive instructions, as they got into the task.

In this introduction, there is a task beginning I have used in the classroom. I use it here, not to model good practice, but because I believe this is an efficient and potentially energising way to get into working with the piece of software. Unlike the classroom, I do not have any particular areas of mathematical content I want the prospective teachers to work on. The aim at the university is to consider the potential for ICT in their teaching and for them to commit to one programme on which they will do more work. I planned to end the session with a discussion of these issues.

I would be on the lookout for any mathematical awarenesses exhibited or perhaps seeming to be lacking, in the prospective teachers, and would comment on the issues I noticed, as they arose (e.g. I recognise how "natural" it can seem to be to think the exterior angle of an equilateral triangle is 60 degrees and might comment on this as the error arises). I am also aware of being on the lookout for how the prospective teachers handle their own emotional reactions. I am aware that certain individuals will respond to using ICT in a heightened manner (e.g. highly positive or negative), and, again, I might act on my awareness to comment on an issue which is them needing to work with students in their own classrooms who may have the opposite reaction. There is also an important learning, for me, about the way in which the prospective teachers approach their learning. Finally, it is an aim of this beginning not to set up an expectation that "good" practice in their placement schools would involve the use of these, or other, ICT tools (which could lead to a sense of what they are offered at university not being relevant to the reality of classroom life).

10.4.2 Tracy: Session on "Algebra"

The timetable for the course has included many of the same session titles for a number of years. Some session titles suggest a focus on specific areas of the mathematics curriculum (e.g. probability, algebra, proof), and others imply a focus on issues from teaching (e.g. assessment, English as an additional language (EAL) issues, topic planning). The "Algebra" session was one such session that had featured on the timetable for many years. The session is scheduled for around 90 minutes and takes place in week eight of the course when the prospective teachers are 3 weeks into their first extended placement in school and they return to university for a week.

I planned to begin with the question, “What is algebra?” or “What is algebraic activity?*”*. By beginning with a list of how the prospective teachers are seeing algebra, the idea was to return to the list at the end and add to it in light of the activities done in the session. I saw this to be a way of demonstrating the expansion of an initial set of views through offering the prospective teachers a common experience on which to reflect.

In planning any university session, one awareness I have is not wanting to offer any one particular model of mathematics teaching, and in this case a particular model of teaching algebra, that might be seen as a model for prospective teachers to try out in school. In order to talk at a meta-level *about* the algebra activities (detailed below) worked on by prospective teachers in the session, I wanted to provide a framework. I imagined the framework could support a way of thinking and talking about the activities from a more neutral position. I decided to introduce a set of distinctions of algebraic activity from Kieran (2004, p.22), which consists of three types of algebraic activity: *generational activities* involving generating expressions, equations and expressions of generality from geometric patterns or numerical sequences; *transformational rule-based activities,* for example, factorising and simplifying expressions and solving equations which are predominantly concerned with equivalence; and *global/meta-level activities,* for example, an awareness of the structure of mathematics and constraints of problem situations, prediction, justification and proof (which are therefore not exclusive to algebra). I also envisaged that using a theoretical framework in this way might support the prospective teachers with their Master’s level thinking and writing, and I planned to make this link explicit to them. The prospective teachers would need to make sense of this framework through firstly reading it and then being asked how they are seeing these distinctions. I planned to give them time reflecting briefly on where they would place their own responses to the original question, “What is algebra?”, within this set of distinctions.

Given “Algebra” is one of our long existing session titles, the common feature of this session over the years is that the prospective teachers are offered a variety of algebra tasks (often by a variety of tutors – in this case, it was going to be Julian and me). There is therefore a pre-existing list comprising of different tasks that have been offered before over the years, some of which we used. However, *Painted Cube* was not on the pre-existing list. I was keen to use visualisation at some point over the year and an activity where algebraic symbolism can be drawn out directly from a structure “from geometric patterns” (Kieran, 2004, pp.22–3). *Painted Cube* is an old coursework task used when I was in the classroom over 10 years ago, so I was very familiar with it having used it many times since then. I planned to be explicit about the old coursework task context, so it felt real and again not about me and *my* classroom but a well-known, much used, task. The meta-task while working on each of the activities would be to consider which, if any, of Kieran’s headings is most fitting for that particular activity.

Usually, when I am going to teach a session involving a mathematical activity, part of my preparation is working on the mathematics. Given my familiarity with the problem and with the algebra, I spent some time practising the visualisation on

Alf and Julian. I was aware that, in working with a visualisation with a group of individuals, it is likely that some prospective teachers would see something quite differently from what I had intended. At the end of the visualisation of an $n \times n \times n$ cube made of cubelets ($1 \times 1 \times 1$) painted red on the outside, I planned to ask the following questions:

- How many cubelets are there with 3 red faces?
- How many cubelets are there with 2 red faces?
- How many cubelets are there with 1 red face?
- How many cubelets are there with 0 red face?

I imagined that these questions would be likely to expose any differences in what the prospective teachers were seeing and would provide an opportunity to offer the group an experience of what can happen if you choose to use visualisation, that is, working with the group immediately after the visualisation so that we all see the same. Having worked on these questions with the group to the point where we can agree on some answers, I planned to allow them to extend the problem for themselves. This idea of allowing the prospective teachers to follow their own lines of enquiry when working on a problem like this is something I would do in a number of different sessions. For me, this is about offering them an experience of being motivated through working on their own mathematical questions.

Having spent some time working on a series of algebra activities, I planned to end the session returning to the meta-task by asking the prospective teachers to consider the activities in light of Kieran's framework, where they would place each activity and why. Having experienced a number of different activities together, it felt important to return to their original thoughts about algebra as a way of expanding what they are thinking are possibilities for their classroom, not staying with their original ideas.

10.4.3 Julian: Session on "Assessment"

Before planning individually, we met as a team of three and looked at resources from the equivalent session in the previous year of the course. The assessment session was scheduled to last for 90 minutes, and I planned four main sections:

(a) Beginning with school experiences of assessment
(b) Collecting experiences as a small group and then as a whole group
(c) Experiencing the use of a questioning and listening task as an opportunity for assessment
(d) Implementing ideas to design an assessment activity for a defined purpose

I made use of activities that had been refined over many iterations of the course, choosing not to change the substance of these. Quite quickly, my planning became populated with phrases that I intended to speak, and, at some point, the planning activity became writing a script for the session. The primary intention behind this scripting was to document what I would say and when but also to monitor what I would not say. A particular focus of these considerations was setting up the small group activity (step (b)), in which I would ask prospective teachers to work in groups of five (or six) to create a "poster" using a single sheet of flip chart paper, gathering their school-based observations of methods of assessment. It would have been possible to set up the group activity in any of a number of ways, and my thinking was concerned with how much to reflect with the group on the process of setting up the activity. I decided to draw attention to my instructions as a way of offering something to the group, but to leave the primary focus on thinking about assessment. Similar considerations applied to the mechanism used for sharing outcomes of each group, and a similar approach was used: drawing attention to the instructions while not inviting comments on the process.

In my script, I chose to adopt a feature of interaction I had noticed each of Alf and Tracy employ with the group, namely, use of a leading "So" at points of transition. My feeling about this detail was that it addressed an issue of stepping between the frames of the activity itself and of metacommenting. The verbal marker became a deliberate part of my delivery.

This was to be my first "solo" teaching session on the course. In addition to thinking about the group, I was also aware that in the room would be the two established tutors who would be able to offer reflections afterwards and that this would happen naturally as part of a debrief conversation between the three of us. These conversations take place routinely on days when we work with the mathematics group, over coffee and lunch, with an imminence that supports access to the experiences themselves.

A large part of the decision to use existing resources was my awareness of "experiences to issues to action" as an approach to the whole course that was well-understood and of great significance within the course. This awareness was informed by conversations with the other tutors in preparation for other sessions, in which attention was focused on the influence of experiences on the emergence of group and individual frames of reference.

During previous sessions with the group, I had adopted the practice modelled by the other university tutors of noting down what was said by the tutor leading the session. This activity had focused my attention on the language used and certain patterns of speaking. My feeling was that these patterns of speaking had a significance in forming spaces of attention in the room and guiding the attention of the students, as they do in school classrooms. In this way, the words and phrasing (the "So" that creates a space for commenting) took on a significance that matched, and perhaps exceeded, that of the "content". This feels in keeping with an enactivist positioning, since it is in doing that we change our knowing. Moving to writing a script created a short-circuit to my own recalled experience as a prospective teacher; I have a clear sense of writing scripts for my lessons when first on placement as a prospective teacher on the course myself. Many of the same motivations run through both situations, although with a different balance. In both cases, I was processing my own reluctance to let details arrive in the moment, lest I say something that was not what I intended. (In my school-based classroom practice, after some 13 years, scripting happens rarely now; generally, I would let ideas emerge from the members of the class or access descriptions I have used before.) There is a sense of freeing my attention to be on what is happening in the room, in the moment. This aspect was much more explicit and significant for me now than as a prospective teacher. I cannot ignore the personal significance of this being the first session I had led “solo” on the course, and, undoubtedly, some of my decisions were about taking control of my role in the session, of doing what I could in advance. Again, this is a counterpoint to my journey as a mathematics teacher, where I have worked on changing student perceptions of the locus of control within lessons.

I remember using these tasks as a prospective teacher myself, on this course. I have used some of the “listening” tasks with other teachers when in school, as a head of department. The mathematics in the activities has proved to be engaging, but the key aspect of using the activity is the quality of the listening (Ginsburg, 1981), so the mathematics needed to have sufficient complexity to provoke a need to reason (aloud) while providing opportunities to begin quickly. While I was struck by the similarities of the approaches I took as a beginning teacher and a beginning MTE, my awareness of my purposes in using the approaches was now in a different place, informed by considering “experiences to issues to actions” in discussion with the other university tutors. Through the processes and content of the session, opportunities were created for students to engage with practical issues related to assessment and to reflect on ways of being in the classroom. For me, the session gave clear opportunities to reflect on my own experiences as a beginning MTE.

10.5 Reflecting on Similarities and Differences in the Learning of Prospective Teachers and MTEs

This section will point to the way experiences, issues and actions work on the course, from the evidence of these stories, in the learning of both our prospective teachers and ourselves. We interweave discussion of metacommenting and second-person perspectives, before a final section returning to the theme of layers of awareness.

Experiences: There are a number of ways in which the word “experiences” is exemplified in the examples of planning above. Julian begins his session, related to assessment, with prospective teachers’ experiences of assessment in schools. The prospective teachers (in pairs or individually) have placements in different schools, and, although there is a national curriculum in place for mathematics in England, schools will have some similar assessment practices and some different ones. No one individual prospective teacher could have observed all the different assessment practices in their own school either, so there is an opportunity for a group of prospective teachers to share and, in this case, make a poster to illustrate the range they have discussed. Here the “experiences” are in different schools, but there are ways of working with these experiences in sessions at the university, one of which is described in Julian's story. Another way "experiences" can be used is by the prospective teachers having a common experience of an activity that they can then use to discuss issues raised. Laurinda, Tracy and Alf's planning is for mathematical tasks that are used to raise issues. The 4-minute workshop is a range of mathematical tasks experienced for a short amount of time; Alf introduces an ICT package actively, and Tracy works with prospective teachers on the task *Painted Cubes*. There are many ways of using such common mathematical experiences, for instance, being able to extend awarenesses of a concept through application of a framework (e.g. for algebraic activity) and becoming aware, as with the 4-minute workshop, that, within the group of prospective teachers, as with a group of students in a classroom, the actual experience of doing the mathematics and how you feel about it is different from person to person (one likes the challenge of making the daisy; another gets frustrated at not completing the task in the time; another likes the way the activity gave lots of practice with number skills). As MTEs, our experiences are within the sessions we offer prospective teachers, noticing similarities and differences in their responses. These are alluded to in all the stories.

Issues: A number of ways of organising sessions to support prospective teachers sharing their experiences to raise issues exist on the course. When asked about planning, Laurinda commented that often, in travelling to the university to lead a session, she was focusing most on how many in a group today and how to organise the seating in the room. For Julian's posters, there might be five or six in a group. Another common grouping, the first session back after a block of school practice, is a reflecting team of three. In a reflecting team, each prospective teacher is given a fixed time to explore the detail of an experience, while the other two prospective teachers ask probing questions, helping to get at the issues arising from the experience. Pairs are used in the 4-minute workshop to highlight issues of working with others. In a further parallel between prospective teacher learning and our learning as MTEs, Julian, Tracy and Alf also act as a reflecting team for each other, making time to explore the detail of our own work with prospective teachers, raising issues and asking probing questions.

Expertise as an MTE allows the move from our experience of prospective teacher behaviours to the explicit raising of an issue. This can be observed in stories from Laurinda's planning (such as noticing participants *not* overlapping triangles in Task 1 of the 4-minute workshop) and Alf's ICT task (e.g. raising issues linked to emotional reactions to packages). Raising these issues is dependent on a second-person awareness. Both Laurinda and Alf notice particular awarenesses (present or absent) in part because they recognise times when such awarenesses are present or absent in themselves, when working on mathematics or when teaching. Tracy indicates her awareness of typical behaviours (e.g. that some prospective teachers will interpret her visualisation differently) and is perhaps on the cusp of wanting to use such occurrences as an opportunity for metacommenting about issues. Julian described the first "solo" session he had taught as an MTE. He therefore had no patterns of expected behaviours, from the prospective teachers, on which to draw, and it is to be expected that his reflections focus on his own learning (e.g. comparing his learning as an MTE to his learning as a teacher). Work as a reflective team can support the placing of behaviours of prospective teachers in any particular session within a context or range of likely responses.

Actions: In the descriptions of planning there, are, of course, actions performed by prospective teachers and MTEs. The "actions" in the cycle "experiences to issues to actions" refer to actions that follow, and are linked to, the raising of an issue. So, for the prospective teachers in Laurinda's story of the two triangles in Task 1, the significant "actions" will be what they do in their own classrooms, for example, in response to Laurinda's prompt: "Beware of limitations you put on yourselves and notice them in your students". As MTEs on the Bristol course, we are fortunate (compared to some other colleagues internationally) in having the opportunity to observe our prospective teachers, teaching in placement schools. So, while there will be no immediate way of knowing what "actions" (if any) a particular issue might provoke, over time we do get a sense of this movement.

As MTEs, our "actions" are related to the learning of the prospective teachers. At its most immediate, as described above, our "experiences" are of the learning of those prospective teachers. "Issues" are linked to our awareness of the behaviours of the teachers; and our actions are the making explicit, via metacommenting, of these issues. Our learning is therefore focused directly on the learning process of the prospective teachers and is linked to our second-person awareness of that process, of learning to teach and of learning mathematics. However, when we work as a reflective team of MTEs, debriefing each other's experiences of teaching a session, there is a process we engage in which is much closer to what we offer our prospective teachers, for instance, when we invite them to work in groups of three, debriefing their experiences in schools. We might invite a story from our own (MTE) teaching (an "experience", as Julian also provoked in his session about assessment) and then gather other similar (or different) stories from each other. Having gathered a collection of stories, we would then move to identifying the "issue(s)" raised (as Julian invited prospective teachers to do in creating a poster about assessment). From here we would then consider implications, that is, "actions", for our own future practice – as we invite our prospective teachers to do, at the end of MTE sessions.

Our different experiences as MTEs also mean we can provide a second-person perspective for each other. We lay open, to each other, some of the "intelligent awareness" (Varela, 1999, p.32) behind our actions, and, in recognising and perhaps labelling some of the awarenesses of each other, we support further noticing. It is the second-person perspective that is often crucial to the "issues" phase of the cycle of learning, both for our prospective teachers and for ourselves. While it is possible to identify issues for ourselves, it can often take a more experienced and empathic "other" to recognise a similarity or pattern or connection.

10.6 Layers of Awareness

When we work on developing as MTEs, we do not have an image of ourselves as experts in teaching, transmitting our knowledge to our prospective teachers. The process of learning (for us, and our prospective teachers) is through awarenesses that can be metacommented upon. Our planning is, therefore, focused not only on the content of the session, such as *Painted Cubes,* but also on the meta-tasks, which for *Painted Cubes* are related to using a framework for algebraic activities and creating a space in which the prospective teachers are expanding their own awarenesses of how algebra might look in their classrooms. Working with our awarenesses is directly linked to our metacommenting, pointing to gaps and patterns in our prospective teachers' learning. The students in classrooms work on their mathematical awarenesses; the prospective teachers use their awarenesses of mathematics and mathematics teaching and learning to support the learning of those students by offering experiences, observing, listening and commenting. As MTEs, we are working with our awareness of the awarenesses of teaching mathematics. Awarenesses can rarely be communicated or pointed to directly. An empathic, second-person perspective allows the non-judgmental arising of potential issues, linked to the behaviours of the other (be they in a classroom, a prospective teacher or an MTE) and therefore the possibility of metacommunication about those behaviours and the occasioning of new possibilities for action.

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Chapter 11
Mapping the Territory: Using Second-Person Interviewing Techniques to Narratively Explore the Lived Experience of Becoming a Mathematics Teacher Educator

Alistair Bissell, Laurinda Brown, Tracy Helliwell, and Toby Rome

11.1 Introduction

Does being a strong mathematician make you a strong mathematics teacher? Does being a strong mathematics teacher make you a strong mathematics teacher educator (MTE)? There are first-person accounts "conceptualising the terrain" (Tzur, 2001) and using narrative inquiry (Chauvot, 2009) that use self-reflective analysis and self-study, respectively, what we would term first-person techniques, to contribute to the literature on developing as a university MTE. We will use the term "university MTEs" when talking about MTEs working with prospective teachers. In this chapter, the focus is on exploring the lived experience of Alistair, an experienced (over 10 years) mathematics teacher, who is also a strong mathematician, when moving from being a teacher of mathematics in a school for 11–18-year-old students to working in a national role as an MTE. Alistair is now running a year-long professional development course, *Teaching Advanced Mathematics* (TAM), for groups of teachers who want to develop their teaching of mathematics at advanced-level (A-level Mathematics is a course for students from 16 to 19 years old, often a preparation for university-level studies). The course is provided by *Mathematics in Education and Industry* (MEI), a charity committed to improving mathematics education, and involves eight course days and two lesson observations in each teacher's school. What changes in this journey from teacher to teacher educator? What is gained or lost in the transition? To explore these questions, Laurinda, his doctoral supervisor and herself an experienced university MTE, interviewed Alistair three

A. Bissell · T. Rome
Mathematics in Education and Industry (MEI), London, UK

L. Brown (✉) · T. Helliwell
University of Bristol, School of Education, Bristol, UK
e-mail: laurinda.brown@bristol.ac.uk

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_11

times, at the beginning, middle and end of his first 6 months in his new post using what is termed "empathic second-person interviewing" (Metz & Simmt, 2015), which will be discussed in more detail later in the section on second-person interviewing. We will illustrate, through this process, how enactivism, as a theory of learning, can be used to investigate how MTEs learn and develop.

In investigating how the transition from mathematics teacher to mathematics teacher educator (MTE) is made, it is important to collect case studies from a range of contexts. Culturally, the work of MTEs is different across country boundaries; for instance, in Bristol, England, it is usual for university MTEs working with prospective teachers to visit them whilst teaching on their school placements, and in Alicante, Spain, university MTEs do not visit their prospective teachers whilst on placement. Many differences become apparent when working in international groups, such as in the thematic working group (TWG18) on *Mathematics Teacher Education and Professional Development* at the Congress of European Research in Mathematics Education (CERME). For instance, in TWG18 at CERME 10, there were:

> Different points of views about errors in different teacher education programmes and how we use/understand errors in our teacher education programmes. Also differences in practices of "noticing". (Zehetmeier, Brown, Mellone, Santos, & Akar, 2017)

The differences in our language use when using common words, such as errors, problem-solving, discussion or even MTE, become apparent when we talk in detail about what we do, our practices, rather than when expressing theories more generally.

In this chapter, there will first be sections on some theoretical underpinnings: the background theoretical stance of enactivism that seems particularly appropriate for researching the learning of MTEs with its focus on knowing being equivalent to doing; a discussion on what learning from experience is to us; and how it is possible to explore first-person experience through second-person interviews. A section on methodological issues, including what we did to generate data, is followed by an extended case study, uncovering similarities and differences between being an expert classroom mathematics teacher and a novice MTE over the first 6 months of transition. This case study of Alistair will contribute to extending our awarenesses of how MTEs learn. There is then a focus on exploring these awarenesses, written as straplines, through inviting narratives, details of experiences, from two other MTEs, one, Toby, having recently made the transition to working with teachers in professional development and one, Tracy, a university MTE who works with prospective teachers.

11.2 Theoretical Underpinnings

11.2.1 Being an Enactivist

Being an enactivist is underpinned by an acceptance of a biological basis of being, where we have evolved and continue evolving to act in our environment. (If you are interested in exploring more on the theory and practice of enactivism, see the ZDM issue on *Enactivist Methodology in Mathematics Education Research*, edited by Reid, Brown, Coles and Lozano in 2015.) In essence, "All doing is knowing, and all knowing is doing" (Maturana & Varela, 1992, p. 26). This perspective is important in considering how we adapt to changes in our working practices, such as moving from teaching mathematics to students in school to working with groups of experienced mathematics teachers. In enactivist terms, our history of structural coupling with our environment leads to patterned actions. Varela's (1999b) first key point of enaction is "Embodiment: The mind is not in the head" (p. 73) given that our frontal cortex only becomes active when we do not know how to act (Varela, 1999a, p. 18). As Clark (1997) put it, "Minds make motions, and they must make them fast – before the predator catches you, or before your prey gets away from you" (p. 1).

In the moment, there is no time for reflecting. In moving to a new job, therefore, we act using what we have done previously. As Maturana and Varela (1992) phrase it, "Knowing is effective action, that is, operating effectively in the domain of existence of living beings" (p. 29). Using what we have done previously in a new environment will be followed by adapting when what happens is not effective or good-enough (Zack & Reid, 2003, 2004) for the situation. Identifying feelings of being uncomfortable and staying with the detail of what happened can support our learning by opening up new possibilities for acting, whether we are novice or expert (Brown & Coles, 2011). The difference between a novice and an expert is that the expert can reconstruct with "deliberate analysis" (Varela, 1999a, p. 32; Brown & Coles, 2012), after the event, the awarenesses that led to action. However, "even the beginner can use this sort of deliberate analysis to acquire sufficient intelligent awareness to bypass deliberateness altogether and become an expert" (Varela, 1999a, p. 32).

In this chapter, we are exploring how a new MTE adapts, bringing forth new behaviours and keeping some. How is this adapting done?

11.2.2 What Is Learning?

The link between language and action is through basic-level categories (Varela, Thompson, & Rosch, 1993, p. 177). How do we come to recognise a chair, when there are so many different varieties? A chair is most often a "sitting-on object". When there is a need to sit, we notice possibilities in our environment and act. For an experienced teacher of mathematics, walking into their classroom, much of what happens is already established through routines although the interactions in the moment are infinitely variable. When starting a new job related to teaching mathematics but working with teachers, the behaviours and routines fit for teaching children are what exist. How is it possible to learn in the new situation? Basic-level categories are positioned between the details of particular behaviours, say, sitting in our favourite comfortable chair, and superordinate categories, say, furniture, where there is no such clear link to behaviours in the praxis of living of most human beings. (Furniture would be a basic-level category for furniture removers, however, given that they know what to do with it.) Learning is done by changing or extending the basic-level categories by adapting to the new environment. The process, that can be carried out after the lived experiences by novices and experts, is to focus on a time of feeling comfortable or uncomfortable and, staying with the detail of what happened, without judgements, open up the possibility of acting differently. There is through this process the potential for new basic-level categories to emerge and, over time, avoid the move, in the case we are interested in, into automatic behaviours from a previous job. This process is learning to act in the new environment. The collection of data for this chapter, through interview conversations focused on staying with the detail of Alistair's lived experiences of a new job as an MTE, seeks to answer the questions of what changes and what stays the same in the transition from school teaching to being an MTE.

11.2.3 Second-Person Interviewing

So, in wanting to write a case study of one expert teacher's move from their classroom to the first year of working with teachers, what seemed important was to access in some way the changes that led to new behaviours or what of their previous actions could be effective in the new environment? We are interested in first-person accounts such as used in phenomenology, but there was not time to train Alistair to become a phenomenologist. Claire Petitmengin (2006), a doctoral student of Varela, was in the same position when working with epileptics. Scans had shown that there were changes in brain function before the epileptic seizure took place, and Claire Petitmengin's challenge was to find a way of developing first-person accounts of what was happening at that time. The process developed was that of second-person interviews, and there was a protocol for the interviewer in our study, Laurinda, to work with. In Alistair's case, he would begin by talking about some lived experience. We recognise three fundamental ways of acting as such an interviewer, adapted from Petitmengin's (2006) paper:

- Stabilising attention: A regular reformulation by the interviewer of what the interviewee has said, asking for a recheck of accuracy (often in response to a digression or judgement). Asking a question that brings the attention back to the experience.
- Turning the attention from "what" to "how" (never "why").
- Moving from a general representation to a singular experience. This is what we term "story" in the case study that follows, a re-enactment, reliving the past as if it were present. Talking out of experience, not from their beliefs or judgements of what happened, often involves teachers in a move to the present tense. Staying with the detail is important, a maximal exhaustivity of description that allows access to the implicit.

  With a colleague, Alf Coles (Brown & Coles, 2019), and being mindful of the enactivist take on learning through adapting basic-level categories, we have added a fourth fundamental way of acting:

- Getting to new basic-category labels: After dwelling in the detail, telling stories and exploring without judgement or digressions, the invitation is to elicit statements of what is being worked on. [...] In this way, new basic-level categories might be identified, such as the straplines (a word used in editing newspapers, memorable, usually less than five-word phrases) from this research of "listening for" or "setting up the culture". These awarenesses, triggering and being triggered by the environment, can allow adapted and new behaviours to emerge.

11.3 Methodology and Methods

To develop the case study, there were three interviews where the extended Petitmengin protocol was used by Laurinda to support Alistair in staying with the detail of times that had been comfortable or uncomfortable. After the interviews were transcribed, again by Laurinda, Alistair was invited to highlight what seemed to be important aspects, what we have called straplines. Alistair identified 6 straplines from the transcript of interview 1 and 11 from interview 2. To look for resonance, the six straplines, from interview 1, were shared by e-mail with two other MTEs: Toby, starting a new role as an MTE working with the professional development of teachers mainly online, and Tracy, who 3 years ago left school to become a university MTE of prospective teachers and is a doctoral student of Laurinda focusing on a first-person narrative study of becoming a university MTE. They received the following message, having agreed to take part:

> What I am interested in are any stories triggered by reading the straplines where either the issue seems similar to what you have experienced or where you feel uncomfortable because your experience is different. Try to tell any stories with a little context but then staying with the detail of experience without judgement or explication followed by talking about what you have just written to point the reader to the issue for you.

Involving the other MTEs serves to remind us of the many varied contexts in which MTEs work. A definitive answer to the questions raised is not possible, but reading case studies supports others in expanding their own range of possibilities to act, and the straplines begin to map the territory of potential development. Toby and Tracy both commented in detail on two of the six straplines, "setting up the culture" and "listening and listening for".

What follows next are some exemplars from the transcripts illustrating the way the interview protocol worked including, unusually perhaps, the interviewer telling stories, when triggered, to seek further resonance from the interviewee. There follows a piece of writing by Alistair related to the strapline, "setting up the culture", creating a narrative of a sequence of stories over the time of the interviews to illustrate learning and raise issues. This writing is largely taken from the interview transcripts, a support for him in producing this first-person account, telling stories that illustrate his change over time. Without the interviews, we suspect that this level of detail would have been lost.

11.3.1 Using the Protocol for Second-Person Interviewing

For each of the items in the protocol, a sequence from the transcripts has been chosen to give the detail of how the protocol is used. For the first three items, Laurinda is not saying very much. In the fourth item, however, a story from her own experience arises and is shared. This triggers another contribution from Alistair who identifies what is being talked about at the basic level.

11.3.2 Stabilising Attention

From Transcript 3:

Alistair: I feel like I'm getting to know them a bit better until you go into their classroom […] sometimes it's surprising and other times it's not.

Laurinda: Can you give me a story of something that's surprising?

Discussion: This extract is taken from near the beginning of the last transcript. Alistair begins by talking about his experiences rather than being in the detail, so Laurinda's contribution attempts to support a transition into the detailed layer of what happened.

11.3.3 Turning the Attention from What to How?

From Transcript 1:

Alistair: For [the teachers] the course entails 8 days. I work with another person to deliver these days for the teachers.

Laurinda: How do you get to the point of delivery? What happens before that?

Discussion: This extract is taken from the beginning of the first transcript. Alistair's first contribution is related to what he is involved in. Laurinda aims to turn the "what" into "how" but realises that Alistair might need to say what happens before delivery starts before saying anything about "how". From Petitmengin's (2006) perspective:

> Throughout any interview of this type, it is the question "how" which triggers the conversion of the attention of the interviewee towards […] pre-reflective internal processes, and permits the awareness of these processes. This may be contrasted with the question "why", which deflects […] attention to the description of objectives and abstract considerations, and must therefore be avoided. (p. 241)

It is hard to avoid "why" questions as a novice interviewer, but Laurinda is experienced enough to often recognise them arising and so asks something else. If a "why" question is asked without awareness, since Alistair is aware of the protocol, he comments "No, 'why' questions!"

11.3.4 Moving from a General Representation to a Singular Experience

From Transcript 1:

Alistair: Later in what I was asking the teachers to do after this, they became different actions for the teachers. But what this teacher offered that wasn't those things, was to take a point, if on the curve there's a point at which you want to find the gradient, they suggested taking two points an equal distance either side of that point and constructing a chord and finding the gradient of that.

Laurinda: What do you mean by distance either side?

Discussion: Here is an example of the move from a general statement ("Later in what I was asking the teacher to do after this, they became different actions for the teachers".) to staying with the detail of experience (from "… if on the curve there's a point …"). As an interviewer, Laurinda is concerned with supporting the interviewee to get to a maximal exhaustivity of description. To keep the focus on the detail, Laurinda has found it useful to be aware when she does not know what is meant. One way of becoming aware of this is when an image presents itself for which the detail has not been given. It is her own interpretation of what is being said. Her question, following Alistair beginning to focus on the detail, asks for more detail about what the image being described looks like. Many of our decisions, when teaching, are from theories that are implicit. We may not even be aware of them ourselves. Staying with the detail gives access to the implicit through the uncovering of basic-level categories.

11.3.5 Getting to New Basic-Category Labels

From Transcript 2:

Alistair: I was observing the first day; it was delivered by Simon, my line manager. But I remember being struck by how strong that message was. Whenever anybody offered something that was not from what was in there, they got challenged to justify it, every time, more strongly than I would have been able to do. I think I said I was blown away by that session and it was the recognition of how powerful they were setting up culture.

Laurinda: What I am personally interested in is where that comes from – the conviction of that person you observed. I was a head of mathematics […] invited to do a professional development session. We did a visual activity, talking about what we see, that sense of maths not being about me standing there telling them things. There was a man who did not say what he saw but gave a label, above the heads of most of the staff, something like Lissajous figures. I wanted him not to be able to use what he'd already known. It did happen. […] He realised that his initial statement didn't fit with where they had got to through talking. He couldn't use his memory anymore and he was at sea.

Alistair: That brings to mind a teacher who […] would make strong assertions and I thought long and hard about how I was going to react to that. I followed Simon's lead, "Why, can you justify that?" Over lunch time, the teacher came to speak to me and said, "I was thinking about you getting me to justify that and what I wanted to do was question and sometimes it might be better to just ask the question rather than making a strong assertion that something else is the case".

Discussion: A statement from Alistair following this interchange highlighted a basic-level category, convincing, that arose out of this interchange, "A useful word that I said was convincing. Your role here is to convince us. Can you convince us of this that you've said?" The sections of transcript have been shortened, but in each case the focus was on staying with the detail of the experiences. The discipline is to tell a story that arises from listening to another person's story, and at some point, although Laurinda did not specifically invite a shift to talking about the descriptions of experience, Alistair moves to seeing "working with others to see convincing" as being part of mathematics teaching and learning. This is a basic-level category that can accrue a range of behaviours at the implicit level that he can apply in his new role. There is evidence in the final contribution from Alistair that the teacher he was working with has moved to have a potential new basic-level category for herself, "just asking the question", with the old implicit one, "making a strong assertion" becoming questioned. In writing this chapter, Alistair's current purpose is to act so that the teachers he works with try asking the children in their mathematics classroom to be convinced and convincing.

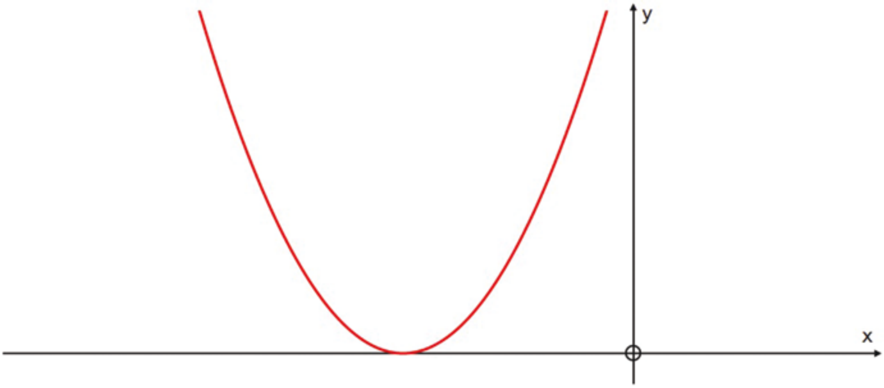
11.4 Case Study Written by Alistair: Becoming a Mathematics Teacher Educator

11.4.1 Narrative for Strapline: Setting Up the Culture

The narrative is told in three sections, each story, or detail from experience, being followed by reflections, also written by Alistair. After the three sections, there will then be a discussion pointing to other straplines and what might be considered to be findings.

1. Being fluent as a maths teacher and then doing those things when working with maths teachers but it not being the same

Story 1: I asked two people to stand outside the room whilst the rest of the group looked at a graph and were to design some clues.



I wanted to then take the graph off the board, invite the two teachers back in the room with the clues available, and see if they could get back to an equation and a graph. This was in the context of work introducing integration as the reverse of differentiation, and the plan was for the teachers that were still in the room to find an equation that could lead to that graph, differentiate to find a gradient function and say a point on the curve. Those two clues would be what would remain when the two teachers came back in the room. Now, the first question I asked the group was, "What clues could we give them [the two people outside] that might allow them to get back to the graph?" What came back were all sorts of things that I didn't know how to handle. Things like, "It's in the first quadrant."; "There's a line of symmetry."; "It's a parabola." I felt uncomfortable.

Reflections 1: I wanted to use the pre-existing plans for the course delivery because I felt that by using these I would be forced to consider new ideas and new ways of working, but I was also conscious of working with the plans in ways that develop what I care about. In planning for the days, I placed importance on my opportunities to listen, because this is where I get a chance to show that teacher contributions are valued, by listening to them and using their contributions as we work together.

Despite having cared so much about my opportunities to listen, I found that in this case I wasn't interested in the responses that were coming back from the teachers – I was only waiting for the responses that were in the plan for the day, which felt immediately uncomfortable.

A difference from my mathematics classroom is that there were time pressures within this session from there being two teachers waiting outside and there was a particular answer that was needed in order to invite them back (a gradient function and a point on the line). Comments like "It's in the first quadrant" would not provide the two teachers outside the room with much information to narrow down the possibilities and also aren't mathematically accurate. In my maths classroom, I would have wanted the students to take responsibility for deciding whether their suggestions were correct, but this new situation meant that we didn't have time to think about suggestions that were mathematically incorrect. There were mathematically correct statements that I still needed to reject because they wouldn't lead into the next activity on the plan.

This raises the question of what was different about my maths classroom that made this a natural question for me to ask. One difference is the nature of things that I intended to be learned. In my maths classroom, I take the role of deciding what it means to work mathematically and then set up tasks so that my students experience this. This kind of question might have helped to set up an openness to the questions that we might ask and explore, showing that mathematicians make choices and work on open tasks for extended periods. Somehow this feels less relevant to working with mathematics teachers, because they might have different views about the relevance of different ways of working mathematically for their students, which I don't want to influence. Instead, what I want to influence is how they work with their students and how they might communicate their own mathematical values (whatever they may be) to their students.

Another difference is the time pressure and regularity of sessions over the year. The setting up of a culture in my maths classroom was a larger priority at the start of the year because I had more time to work on the maths content once this culture was established. With only a few course days and bigger gaps between contacts, there is less time to establish a culture before having some specific aspects of A-level maths to work on and specific types of task to try out. There's less time to go off plan and explore.

2. What have I done to set up a culture?

Story 2: There was one teacher on the course who was strong about having a degree in mathematics, which is coming from a different place from most teachers on the course, and she would make strong assertions, offering methods that hadn't come out of the ideas and discussion within the room. I thought long and hard about how I was going to react to that, and I followed Simon's lead on challenging them to explain why and justify.

I can't remember what they offered now, but there was a point where I labelled it as a strong assertion, emphasised that the comments should be aimed at the audience of teachers in the room and asked, "Can you justify that?" She sank right down. There was a sense of people around the room recognising that this is quite nice actually, because she couldn't justify it. Over lunch time, the teacher came to speak to me and said, "I was thinking about you getting me to justify that and what I wanted to do was question and sometimes it might be better to just ask the question rather than making a strong assertion that something else is the case".

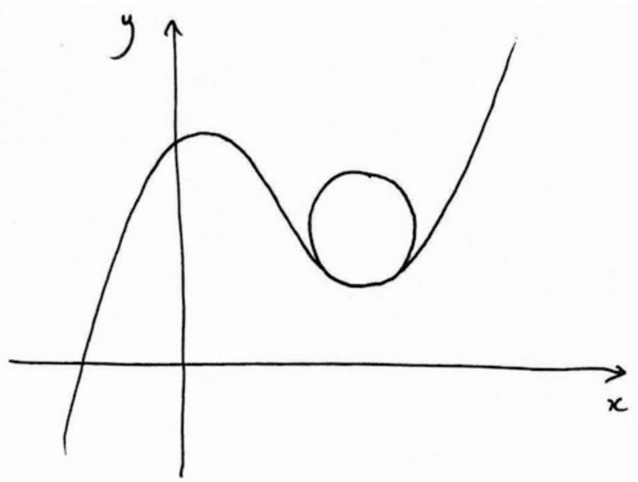
Reflections 2: A similarity between this situation and my maths classroom is the value placed on convincing others in the room of statements made, but a difference is that I was actively trying to find strategies to not allow someone to dominate. I was asking this teacher to justify her assertions with the hope that she wouldn't be able to do it, which feels unkind, and I'm not sure I would have done this to children in my maths classroom. I think I felt that there was more danger of a teacher with lots of conviction dominating and setting the tone for the course than there would be in my maths classroom, where I am the teacher and they are the students.

I was trying to establish a culture where people convince each other, so that someone who brings lots of rules and assertions doesn't have an advantage over others and doesn't gain an authority to tell others what maths is about. I was not establishing myself as a mathematical authority as I didn't involve myself in the maths content, but I did set an expectation about how people are to work mathematically in this space. I'm aware that I couldn't tell this person to ask questions rather than make assertions, but I could set up a culture of working mathematically where she can experience her assertions not being valued.

3. "I've got two more days of the course to do but I'm not worried about setting up culture – the teachers behave as I want them to now!"

Story 3: The teachers had been asked to consider a circle and a cubic function, and they were trying to work out how many points of intersection were possible between the two graphs. One teacher offered the idea that you could have a cubic function, looked at the curvature around one of the turning points and placed a circle to match the curvature in the turning points.

They had got a mini-whiteboard and were asking me what I thought of this. I didn't know how to respond to that but what I did do was try and draw it on the board. I thought there was something interesting about it.



I think the teacher had shared their image amongst their group. My sense was that there was a level of acceptance around that table of this idea and that then there were an infinite number of points of intersection, and they were suggesting this was the solution to the whole thing, I can get any number of points of intersection but that wasn't said explicitly. Having drawn it on the board and intervened with the group to say, "Can everybody look at this for a moment?" somebody shouted quite strongly, "No", in disagreement with the mathematics, which made everybody in the room laugh, and there was strong reaction to this image on the board. I just paused and let the image speak for itself and the "No" was hanging to some extent.

I think there was a sense of it causing disagreement or challenging each other. This table had been happy with their image, yet it caused real conflict with some other people quite immediately. I found something that got people's attention and has got people engaging and wanting to talk. It feels like I can back off a little bit. Something about the laughter was nice. There was a strange combination of relaxing and also feeling like eyes were on me to see what I'd do about the "No".

Reflections 3: This situation feels comfortable and similar to my maths classroom. The reaction of the room to the strong "No", with laughter and waiting to see what I would do about it, I believe is an indicative of this going against established norms of justifying any mathematical statements. My sense is that the teachers value the culture within which they're working, such that I'm not having to establish anything new anymore. I seem to have provoked difference of opinion such that teachers were reacting with energy, yet the group expect me to manage this energy such that the norms are maintained.

I find it interesting that there seems to be conviction about the different viewpoints, and yet people can't yet convince others of their convictions – there is work to be done. I like this combination of provoking difference of opinion, finding conviction and it being expected that people justify their views and convince others. This allows me not to involve myself with the maths but step back and allow the conviction and convincing to resolve itself.

11.5 Discussion of Case Study

Alistair is initially, story 1, grappling with the issue of setting up a culture in the new environment and feeling uncomfortable because of it not being the same as working with students in his classroom. However, through focusing on the detail of what is done, story 2, another awareness arises, in story 3, that the setting up of the culture has, in fact, now happened. The last paragraph of *Reflections 3* articulates, as basic-level categories, how the culture has emerged: provoking difference of opinion, finding conviction, justifying views and convincing others. These categories are not so different from in the mathematics classroom (comment at start of *Reflections 3*). Given the use of the interviewing protocol, the use of the present tense, "I found something that got people's attention and has got people engaging and wanting to talk", is striking. However, what is different is that what are being looked for are teacher behaviours, not student behaviours, and these felt different in story 1. Alistair felt uncomfortable. In story 2, Alistair felt uncomfortable but was able to act to challenge one teacher and yet support them to develop, and that felt different from the maths classroom because he did not want that teacher to influence others. In the maths classroom, there is less of a sense of one way being better than another. However, this teacher seemed to have conviction about how mathematics ought to be taught, and Alistair did not want them to influence less confident teachers. The teacher's intention was positive, trying to help the other teachers by giving them shortcuts. This was in conflict with setting up the culture of convincing and being convinced. The use of her strong assertions served to highlight what maths was about in this room. By story 3, a similar behaviour to the maths classroom emerges, with Alistair about to "not involve myself with the maths, but step back and allow the conviction and convincing to resolve itself".

Another strapline that feels important in this story is "listening and listening for". In *Reflections 1*, Alistair was planning for opportunities to listen. Listening was important in his own mathematics classroom. He was listening for the question or statement to keep coming back to over an extended period of time, testing out some clues to see if they work to come back to and adjust. The extended period of time was important. However, working on maths for an extended period of time in this group is not particularly relevant. By the end, Alistair was able to let go, asking open questions but not being interested in listening to the answers; he was listening for something else. These teachers are working on maths with Alistair, and that can feel the same as his maths classroom, but what is different is that he wants to open up possibilities of what their maths classroom might be like. So, sharing from his classroom is an offer. He does not want them to do what he does but wants them to see alternatives to what they currently do. He does not want them to leave what they did but to see alternatives by looking for what is different.

Findings in studies such as this are not general but are able to be used by others in their work seeking resonance. Alistair talks about his first experience observing Simon, an experienced teacher of teachers, and being struck by their conviction when setting up the culture for the course. The importance of such observations, of someone at the same level as you doing the same job, seems to set up possible behaviours when the job starts and you need to act. Alistair uses the challenge of convincing when he was unsure what to do, channelling his observation. This finding is reminiscent of Winter's (1996) finding that, with expert teachers on a course where there were a range of activities, the most powerful experience for the teachers was being able to go and observe one of their peers on the course teaching mathematics in a different school. We would offer the closeness to the actual doing as one explanation of why this might be so.

Another finding would be the way that, although initially Alistair did not know what to do and the classroom teaching seemed not to be useful, over time, his past experience and doings seem to be adapted to the new situation. The change or learning is not like putting on a new suit of clothes but is more expanding the range of possible actions.

11.6 Multiple Perspectives

After Alistair had identified straplines from the transcripts that were important to him, we invited Toby and Tracy, two other MTEs, to offer stories or writings about similarities with or differences from their own experiences for any of the straplines they were drawn to. The expectation is that such insights enrich the space of possibilities rather than that there is a search for definitive answers to what changes or what is gained or lost. Inevitably, the three MTEs work in different contexts, but we have chosen writing related to the two common straplines, “setting up the culture” and “listening for”, seen as important by all, to illustrate similarities and differences. To begin, setting up the culture was an important strand for all three MTEs. Toby’s and Tracy’s writing, on each strapline, is followed by thoughts about similarities and differences across all the authors’ experiences.

11.6.1 Strapline: Setting Up the Culture

Writing from Toby and Tracy on setting up the culture

Toby: One of the things that I loved about classroom teaching was having the opportunity to build relationships with students over time. Trust is so important in a teacher-student relationship, and it is far easier to take risks in a classroom when you know that students are prepared to take that risk with you. Even in the cases when the risks didn’t pay off and things don’t go to plan, it was only ever a short amount of time until I would see the class again and be able to rectify any issues. With professional development, however, the vast majority of my work involves only seeing teachers once. When I do work with teachers over a sustained period, most of the contact is through online sessions, so it can be hard to establish a rapport. Whilst I used to always cringe during ice-breaking activities at professional development I attended myself, I now appreciate the value of such measures. Teachers’ time is valuable, so to have a whole day, or even afternoon, of their time is a great responsibility. Although this makes me want to get on with the content of my session right away, I know from teaching that people learn best when they feel secure and comfortable, so I have developed an appreciation for building this aspect into my work.

Tracy: I feel compelled to write about an experience of working with a class of 10–11-year-old students as part of a “transition day” from their primary school (5–10-year-olds) to secondary school (11–18-year-olds). The day would always include a mathematics lesson, and the reason this particular experience came to mind was, in rereading the strapline “setting up the culture”, it felt like this began before first lessons at the start of the new school, in this initial experience of secondary school mathematics.

The lesson began with the following displayed on the board: $1 + 2 \times 3 + 4$. There would then be an invitation to comment on the calculation or offer an answer. This invitation would usually generate the following list of possible solutions to the calculation – 13, 21, 11 – and possibly a few other different answers. Students were invited to discuss how they came to the different answers, resulting in some comments about the use of brackets, the order of doing things, where to begin and where to end and so on.

My purpose (linked to setting up a culture) was to comment *about* the students’ comments. For example, following a set of comments from the group along the lines of “We need to multiply first” or “You start from the left and work across”, I would comment along the lines of, “Mathematicians need some conventions in order to be clear when they are communicating mathematically", which might be followed up with, "So, in order for us to communicate with one another mathematically we will need to agree on our own conventions". There is then time for discussion and agreement on the conventions we will be adopting for the next challenge, which is to find all of the numbers from 1 to 25 using values 1, 2, 3 and 4 along with the four basic operations: addition, subtraction, multiplication and division. All four values must be used and only once.

The class work on the challenge. They are given a board pen to write the calculation on a common board that is already set up with space next to the numbers 1–25. There is an opportunity to disagree publicly with any of the calculations on the board, and alternatives are written down. I am spending my time pointing out any differences that appear within a student's workings and prompting them to work on why it is different, encouraging conversations between those for whom the answers belong. If a student utters that one of the answers is impossible, I share this with the rest of the class, framed as a "conjecture" and written up for the class to see. A challenge to the students is to try and disprove the conjecture by counterexample or to try and convince if in agreement.

To offer a parallel, as a mathematics teacher educator working with prospective teachers of mathematics, there is the interview that happens before the chosen group of prospective teachers meets at the university. This is the setting where we first meet the prospective teachers so establishing a culture starts here. What follows is a piece from a diary entry I made on interviewing, written about 4 months into my new role as an MTE at the university. During the interview, the candidates work on a problem together.

> Interviewing is something I have done a reasonable amount of since starting here in February. We are still recruiting for September. I am conscious of the fact that there has been a strong philosophy and approach to the teaching on the course and this begins with the interview. I reflect constantly, alone and with my colleagues. What are the rules?
>
> During the interview, I take notes. I try to listen to what is said and capture that on my page. I find this difficult as I can't write quickly enough and my urge is to watch the body language in this performance. I think I will miss out if I don't watch, but notes are what we do. I become aware that I am not sure when it is OK to intervene in the group interview so I pay attention to my colleague who I am a little surprised by when he intervenes early on, not just once but a few times. I then feel like I can do the same. I say, "Try and focus on what the triangular number represents". I am a little frustrated with the progress on the problem. On reflection, I have felt like this before – that sense of not knowing when to intervene and when to just let things take their course. What is the purpose of the group activity? To watch how participants behave in a group? To make sure they can do some maths? To find out if they can communicate? To see how they reflect afterwards? If this is the purpose, why intervene at all? Because otherwise, I guess, we might be there for a long time.

In terms of establishing the culture, the interview is a time where we talk about models of good mathematics teaching, in that there isn't just one model. The course supports teachers in finding their own model. This not knowing how to act has been something I have become acutely aware of in the moment and has been the source of much deliberation within myself. It arouses a feeling of discomfort when it happens and prompts me to mark it as something to return to later on.

11.6.2 Thoughts on Similarities and Differences for Setting Up the Culture

Given that Toby and Tracy had been offered the straplines only, not the stories from Alistair's transcripts to write into, it was striking that having the time in classroom teaching to build relationships and culture was valued by all three MTEs. In moving from having professional development done to him to being the MTE, Toby has more conviction now in ice-breaking activities. Some changes in behaviour come out of personal histories, but the awareness of a range of experiences with ice-breaking activities seems important when offering one to teachers you are working with. Tracy's stories focus attention on how ways of working are set up before first lessons on a course or in school. Laurinda is reminded of the importance she attributed when she taught in a secondary school to the induction course, after the end of terminal examinations at 16 years, for students who wanted to enter the sixth form to take mathematics. Metacomments support the setting up of a culture in a classroom, such as, "Mathematicians need some conventions in order to be clear when they are communicating mathematically". In research carried out in Alf Coles's classroom, Laurinda observed that such comments were frequent at the start of the year but, over time, became fewer because the children knew what to do in their mathematics lessons, living "getting organised" or "generating conjectures" in what they did.

11.6.3 Strapline: Listening and Listening for

Another crucial basic-level category for Alistair is related to listening. This was also picked up by both Toby and Tracy in their responses. Tracy had already mentioned listening in her writing about interviews.

Writing from Toby and Tracy on listening and listening for

Toby: One issue I have faced is having to compromise between what I want to deliver in a professional development setting and what the teachers I am working with are looking for. Unlike most students, teachers have *chosen* to attend professional development sessions. Whilst self-selective participation has its benefits in terms of engagement, it also has the challenge of expectations being that much higher. I am naturally keen to explore pedagogy and encourage discussion about different ways teachers can approach new ideas with students. However, many of the courses I tutor on are designed to help teachers to understand the mathematical content. I therefore often find that they come on courses "wanting to get the knowledge" and see pedagogical discussion as a waste of time because they "already know *how* to teach". I can appreciate where they are coming from – they want to get the maximum learning from their professional development. However, I am conscious that they will be going back to work with students, and therefore whilst good subject knowledge is vital, so too is good subject *teaching* knowledge. The distinction seems to be most stark with different levels of teacher experience. I ran a professional development session on using games to enhance geometric skills such as transformations and finding bearings. The group of teachers were many in their first few years of teaching. They were open to reflecting on the approaches and how such activities could enrich student experience. In contrast, I recently worked with a group of experienced teachers on a day focusing on new content in the A level. Once again, we spent some of the time looking at using interactive materials to stimulate student discussion. In the feedback were comments that this part of the day was the least useful, as they were more interested in learning the content rather than exploring ways to approach it with students. I have not yet reconciled how best to compromise here; should I simply give them what they want or continue to try to sneak in pedagogical reflection by the back door?

Tracy: As a teacher of mathematics, I was *listening for* certain remarks made by students that might be identified as a mathematical behaviour. There are examples of this in my story from strapline 1, “Mathematicians need some conventions in order to be clear when they are communicating mathematically”. Other examples would include hearing a student say, “It’s going up half a square each time” and responding with, “That is a lovely example of thinking mathematically, mathematicians often look for patterns and generalise”.

As a teacher educator, I began not knowing what I was listening for and what the equivalent of “It’s going up half a square each time” would be. I remember running a session with prospective teachers called *algebra* and beginning with collecting responses from them to completing “Algebra is …”. Having created a list on the board of their contributions, I was not sure how to respond myself. Some of the responses were closer to something I might say myself than others, and I was conscious of not wanting this to become apparent. I think a response *about* the list might have been around the diversity of responses, same/different, or how rich a set of descriptions we have to work with. Some recognition of the complexity of the question, “What is algebra?”, that invites such diverse responses? I am aware of not knowing how to act sometimes in these sessions because I am searching for the *about*. For any comment or behaviour, how can I respond in a way that is a response about what has been said or done?

One thing I find myself doing, more automatically now, is not directly answering questions from my experience as a teacher (this is all I had to begin with) but using stories from my experience as a teacher educator of other schools, teachers, prospective teachers and so on, for example, in a session about jobs, being asked what I thought about being on interview and trying to negotiate more pay. Instead of responding to this question with my previously held head-of-maths hat on, I relayed two stories from prospective teachers in previous years. The two stories demonstrated the complexities of the issue, both stories conveying completely different outcomes. There was a sense that this was far more powerful than me just talking from my own experience.

11.6.4 Thoughts on Similarities and Differences for Listening and Listening for

Metacommenting is clear in Tracy's mathematics classroom, and she was listening for what to comment on. There is not a particular trigger that generates a particular metacomment, but there are behaviours that Tracy recognises as supporting the doing of mathematics in her classroom and she is listening for them. For all three MTEs, in the new situation, they do not know what to listen for as they start their new posts. Toby is hearing resistance to working on pedagogical issues and so does not know what to do or say in response. Alistair channels Simon in asking for justifications on his journey to feeling comfortable letting go and listening to his teacher group convincing and being convinced. Tracy collects responses from a group of prospective teachers to "What is algebra?" but then what? She reacts internally to what she would or would not have said, but this does not feel like a response. In a classroom, listening to a group of children working leads, for all three MTEs, to actions that are implicit. They have things to offer that come out of their past experiences and conviction in what they think mathematics teaching is. The awareness that listening to teachers is different brings the question, "What am I listening for?" Tracy's final story, as the most experienced of the three, reminds us that, with experience, it is possible to respond from that experience. So, although initially all she had to go on was her experience as a teacher, now she can respond with the lived experiences of other prospective teachers.

11.7 Final Discussion

Deliberate analysis allows us to work with each other in international groups and for novices to develop into their new roles becoming experts. Many years ago, Laurinda travelled to a seminar being given by a mathematics education researcher from the USA whose work she read and appropriated. For the first time, she was able to watch a video of the classrooms being described. She was in shock. When it was time for questions, she raised her hand and said, "Is that what you mean by discussion?" From her current enactivist perspective, this is a good example of how we bring forth our world. When she read the papers of this research group, she saw classrooms where discussions looked different to those on the video. We do not believe that it is wrong that this happens, just that we need to talk and write in the detail of our practice linked to the labels that we use so that we can explore such differences. We have tried to show ways how this works in detail in this paper.

In moving to be an MTE, a common theme was related to what role the doing of the mathematics has compared to being in the classroom. Both Tracy and Alistair are articulate about their practice as mathematics teachers; however, in moving to work with teachers both practising and prospective, there was a process of letting go of one image of mathematics teaching to support the people they are working with to extend their own images of teaching the subject. What they are listening for is different. In the case of moving to work with groups of teachers, the teaching of the mathematics in Alistair's case remains about convincing, but in Tracy's case, working with prospective teachers, she seems to have let go of the mathematics. More work needs to be done on case studies to begin to have some suggestions of differences between the role of working with practising teachers and working with prospective teachers. However, in both cases, there seem to be extra layers involved in being an MTE. As a teacher of mathematics, the children do the mathematics. You let go of that but support them in doing mathematics through metacommenting. What happens as a teacher of teachers of mathematics? The teachers and prospective teachers are now doing the teaching and you are doing something else. What are the equivalents to metacomments as an MTE? Tracy's story gives one suggestion, that she is now letting go of her own experience of teaching mathematics to be able to make comments about learning as a prospective teacher through the experiences of previous prospective teachers on the course. As an MTE you want the teachers and prospective teachers to extend their basic-level categories or teaching purposes. As a new MTE there are only your experiences in schools teaching mathematics to work with. Opportunities for the prospective teachers and teachers to be able to observe teaching of mathematics and talk about what they see in detail to identify issues seem an important part of the journey.

We have identified "setting up the culture" and "listening and listening for" as important aspects of mapping the territory and have a tool, second-person interviewing, that, from an enactivist perspective, supports first-person accounts to identify straplines or basic-level categories. The use of straplines by themselves, in this case by e-mail to other MTEs, seems useful when seeking resonance by triggering accounts or stories of experience without the need for more interview data.

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Chapter 12
From Researcher in Pure Mathematics to Primary School Mathematics Teacher Educator

Svein Arne Sikko and Yvonne Grimeland

12.1 Introduction

There is an emerging interest in learning more about who mathematics teacher educators (MTEs) are and how to become an MTE. This can, for instance, be evidenced by working sessions and discussion groups at recent conferences of the International Group for the Psychology of Mathematics Education (e.g. Beswick, Goos, & Chapman, 2014). In their paper about challenges concerning being a mathematics teacher educator in China, Wu, Hwang, and Cai (2017) also addressed this question and pointed out that very little is known about the development of MTEs and what kind of challenges MTEs face in their work but that it is important to investigate how MTEs develop into professionals. In the field of mathematics education research, student learning and understanding has been the focus of inquiry since the beginning, often building on constructivist or sociocultural learning theories. There is an abundance of literature on what knowledge and learning means in mathematics and on how to work with students to help them build knowledge. Examples include, but are by no means limited to, Skemp's notions of relational and instrumental understanding (Skemp, 1976), Hiebert and colleagues' notions of conceptual and procedural knowledge (Hiebert, 1986), Freudenthal's theory of realistic mathematics education (e.g. Freudenthal, 1991), and more recently also inquiry-based learning (e.g. Artigue & Blomhøj, 2013).

S. A. Sikko (✉)
Norwegian University of Science and Technology (NTNU, Norway), Trondheim, Norway

Department of Teacher Education, Faculty of Social and Educational Sciences, Norwegian University of Science and Technology (NTNU), Trondheim, Norway
e-mail: svein.a.sikko@ntnu.no

Y. Grimeland
Norwegian University of Science and Technology (NTNU, Norway), Trondheim, Norway

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_12

Teacher learning, teacher professional development, and what it means to be a mathematics teacher have been given growing attention in recent decades. Models describing mathematics teacher knowledge, such as Ball, Thames, and Phelps' (2008) theory of mathematical knowledge for teaching or Rowland, Huckstep, and Thwaites' (2005) notion of the knowledge quartet, have received notable attention and generated considerable amounts of research. How teachers work to develop professionally has likewise received a great deal of attention, including research on lesson design studies (e.g. Gravemeijer, 2004; Simon, 1995) and lesson studies in different parts of the world (e.g. Doig & Groves, 2011; Fernandez, 2005; Yang & Ricks, 2013).

However, much less is known about the learning and development of those who teach the teachers, the MTEs (Beswick & Chapman, 2013), and the theme is still underdeveloped.

Even (2008) found that "the education of mathematics teacher educators (of both prospective and practicing teachers) are rarely discussed in the scholarly literature" (p. 59). In addition, Even (2014) claimed that almost all published research in mathematics teacher education and professional development came from English-speaking countries (p. 330). Thus, as Even (2014) put it, there is a need to better understand what educators working with teachers (also referred to as didacticians) need to learn and when and how they should learn that (p. 332). As mathematics educators of prospective primary school teachers in a non-English-speaking country, we will contribute to the field by addressing the needs raised by Even.

Goos (2014) additionally made it clear how little is known about ways in which MTEs are prepared for their role and how they learn and develop throughout their careers (p. 454). We address this gap by examining how a newly appointed MTE, the second author, made the transition into becoming an MTE, what she had to learn, and how.

In this chapter, we investigate the transition from being a pure mathematics researcher to becoming a primary school mathematics teacher educator (primary MTE). This transition involves moving into teaching both practising and prospective teachers, and it also involves moving into doing research in mathematics education. Murray and Male (2005) identified two key challenges when moving into teacher education, namely, developing a pedagogy for teaching prospective teachers and becoming research active. While we investigate a transition from one field within higher education to another, the key areas where challenges are found are similar. This transition in both teaching and research is challenging on both a personal and an institutional level. Based on these observations, our research questions are "What does a pure mathematician need to learn in order to become an MTE? How and in which contexts does s\he learn it?"

We start by giving an overview of teacher education in Norway. This is necessary in order to understand how an increasing demand for MTEs has led to many research mathematicians moving into teacher education. We continue by reviewing relevant literature on the development of MTEs. The methodology used in this chapter is in the form of self-study and inner research, two terms that are next explained and justified. The transition from pure mathematician to primary MTE is then investigated as a boundary crossing using a four-dimensional framework.

12.2 Teacher Education in Norway

Ongoing reforms in teacher education in Norway have led to an increasing demand for mathematics teacher educators. The demand for pure mathematicians is, on the other hand, rather modest, resulting in people with a background in pure mathematics but none in teaching being drafted into mathematics teacher education posts. We will briefly outline the historical background to explain this situation.

Teacher education in Norway has traditionally been divided into two strands, with one strand catering for the education of primary school teachers and the other for secondary school teachers. Those who wanted to become teachers in primary school would attend teacher colleges, whereas prospective secondary school teachers would attend universities. Primary school teacher education focused on pedagogy and teaching methods. Unlike universities, teacher colleges did not focus on research. Staff at these colleges would typically be experienced teachers and not researchers, with people holding a PhD a rarity.

Throughout the last decades, there has been an increasing focus, both in Norway and other countries, on developing teacher knowledge in mathematics. Partially this has been driven by international test scores in PISA and TIMSS that have been considered “disappointing”. Politicians and policy-makers bluntly identified teachers as the “weak link” in the educational system, blaming the unsatisfactory results on teachers not having solid enough subject knowledge. This led to several reforms in teacher education.

In Norway, the primary school teacher education programme (grades 1–10) was increased from 3 to 4 years in 1992, including 15 mandatory credit points (ECTS) in mathematics. In 1998, this was increased to 30 ECTS. Since 2010, primary school teacher education has been divided into two strands, one for those wanting to become teachers for grades 1–7 and one for those wanting to become teachers for grades 5–10. For the 1–7 education, 30 ECTS of mathematics was kept as mandatory, while for the 5–10 education, the mathematics requirement was increased to 60 ECTS. Finally, since 2017, teacher education, including preparation for primary school, has been offered through a 5-year master’s programme. Parallel with the reforms in primary school teacher education, secondary school teacher education continued with the model where students first study content-based subjects, subsequently followed by a (now 1-year) course in pedagogy and subject didactics. In addition, a 5-year programme has been introduced, where subject content and didactics are more integrated throughout.

Each reform described above has led to an increasing recruitment of MTEs. Concerning the calls for reform in mathematics education, Zaslavsky and Leikin (2004) commented that it seems that these calls are based on the assumption that there exists a supply of well-prepared MTEs who are ready to work with teachers in professional development programmes. The main source able to meet the demand for teacher educators would traditionally have been either experienced school teachers with a Master's degree or persons with a PhD in mathematics education. However, in the past, teacher education has not led to a Master's degree. In addition, making the transition from being an experienced school teacher to become a university lecturer is financially not attractive in Norway. Furthermore, with PhD programmes in mathematics teacher education being relatively new in our country, the availability of well-prepared MTEs to handle the increasing number of pre-service teacher education students has become problematic. At the same time, the demand for research mathematicians has not increased to the same extent. There is limited availability of pure mathematics positions at universities, and the positions that are available are open to international competition, whereas in teacher education, it is seen as advantageous to be Norwegian/Scandinavian. As a consequence, many people holding a PhD or Master's degree in pure mathematics are now filling positions as MTEs in universities and colleges.

We next look at relevant literature on the transition into becoming an MTE which will help us to focus our study and identify gaps to be filled.

12.3 Literature on Becoming a Mathematics Teacher Educator

The literature offers personal stories of people who make the transition from being a school teacher to becoming an MTE. Examples include the book edited by Russell and Korthagen (1995) and articles or chapters by Tzur (2001) and Krainer (2008). Dinkelman, Margolis, and Sikkenga (2006) report on a study of how two classroom teachers made the transition to being teacher educators at the university. Likewise, the 28 teacher educators reported on by Murray and Male (2005) had a career background as teachers in primary or secondary school. There are also examples of mathematicians who have moved into teacher education, such as Hans Freudenthal, Alan Schoenfeld, John Mason, and Lingyuan Gu (to name a very limited number of well-known mathematicians who have contributed significantly to mathematics education research), and autobiographical descriptions of such transitions given by Gill Hatch and Tim Rowland (Hatch & Rowland, 2006; Rowland & Hatch, 2006). So even if there are exceptions, most of the stories analysing the transition into mathematics teacher education describe people moving from being a school teacher to becoming a teacher educator. In fact, Dinkelman et al. (2006) claimed that "most practicing teacher educators were practicing teachers at some point" (p. 5). Less is thus known or written about those people who make the transition from being an active researcher in pure mathematics to becoming a teacher educator and mathematics education researcher.

In their paper on the significance of mathematical knowledge in teaching prospective elementary mathematics teachers, Zazkis and Zazkis (2011) wrote that the mathematical knowledge of MTEs is often taken for granted (p. 249). However, what actually constitutes relevant mathematical knowledge is often not made explicit. Zazkis and Zazkis addressed this gap by illustrating cases in which mathematical knowledge is beneficial. In their interviews with five mathematics teachers (all deemed to have solid mathematical background, i.e. with Master's degrees in mathematics or a Bachelor's degree supplemented with graduate-level courses), it is apparent how the teachers' mathematics background gave them self-confidence to work with mathematical problems and problem-solving with their prospective teacher students. An important aspect was that these teachers saw their mathematical background as supporting their efforts to help students acquire a view of mathematics as an interconnected web of knowledge and not a more or less random set of formulas and procedures (p. 260). This account parallels the story of Gill Hatch (Hatch & Rowland, 2006), who found that her strong grasp of the mathematics meant that she could suggest more ways of approaching different mathematical themes, even without herself having tried them out in the classroom.

Artigue (1998) saw mathematics education research (mathematics didactics) as a field within applied mathematics and stressed the importance of tight bonds between mathematics and didactics. She emphasised that didacticians should have a strong mathematical background but also pointed out that didacticians coming from pure mathematics "have to try to preserve their present place within the world of mathematics production and mathematics education" (p. 483).

Our ongoing study aims to gain further insight into the particular communities of mathematicians and mathematics teacher educators on the one hand and the boundary relations and boundary crossings between these two communities of practice on the other hand. Where and how is the boundary located, when does one cross the border from being a mathematician to becoming an MTE, and can you be both? Is who you are dependent on your research, where, or what you publish?

Having conducted a thorough review of the literature on boundary crossing and boundary objects, Akkerman and Bakker (2011) concluded that the claims on boundary and learning found in the literature are of a general nature and that hardly any explication on how or what kind of learning takes place can be found (p. 133). The central questions to their research were (1) what is the nature of the boundaries between domains and (2) what dialogical learning mechanisms take place at boundaries. They identified four potential learning mechanisms that can take place at boundaries: identification (coming to know what the diverse practices of different communities are about in relation to each other), coordination (creating coordinated and routinised exchanges between practices), reflection (expanding your perspective on the practices of your own and others' communities), and transformation (collaboration and co-development of new practices) (p. 150).

Goos and Bennison (2018) explored the potential for learning at the boundaries between communities of disciplinary mathematicians and MTEs in pre-service teacher education. They point to workload formulas, financial models, and cultural differences between the disciplines as hindrances to broader collaboration. Of these, the cultural differences may be seen as the most difficult to overcome since they "are grounded in epistemological differences between the disciplines" (p. 272).

Goos and Bennison (2018) also found that the physical separation of discipline and education academics (in different buildings) caused a striking hindrance to more interdisciplinary collaboration (p. 266).

As we have seen, several authors have discussed problems and challenges concerning mathematics teacher education and the development of MTEs. Within the literature, we have found that our own paths from being active *research mathematicians* to becoming active *research MTEs* are rarely analysed. It is our intention that this chapter can contribute to the development of knowledge in this field. To investigate this transition, we apply the method of self-study and inner research.

12.4 Methodology: Inner Research and Self-Study

Krainer (2008) outlined four possible options for research on mathematics teacher educators: (1) self-reflection by MTEs on their own learning, (2) a survey of MTEs conducted by a team of researchers, (3) an MTE writes about other MTEs' development, and (4) a commission or organisation collects data on MTEs on a mandate from a government or university authority. In this chapter, we report on work that is a combination of options (1) and (3).

Quoting Feyerabend's (1991, p.141) thesis that all you can do if you want to be truthful is to tell a story, Mason (1994) contrasted mathematics with mathematics education. Whereas in mathematics, knowledge is built by adding new theorems to old, education is a journey of self-discovery where each new traveller has to re-experience, re-learn, re-express, and re-integrate what previous generations have learned. Mason claimed that what researchers find out most about is themselves. By interrogating our own experiences, and addressing the questions on how to support teacher education students and teachers in developing their knowledge of mathematics and teaching, we report on transformations in ourselves. This "inner research" is about developing new types of "sensitivity" to the mathematical ideas, to the pedagogical and didactical possibilities, and to the students we are working with. The transformation arising from moving into a new field means noticing different things, since as members of a research community, we notice what we are attuned to notice (Mason, 1998, p. 368). The mathematician notices mathematical structures and concepts, whereas the mathematics educator may notice the struggle to come to terms with the concepts. The use of "the particular" in the form of ourselves thus resonates with Mason's notion of research from the inside and may still contribute to knowledge in the field at large.

As Mason (1998) wrote, research in education is different from research in other fields in that it is about being sensitive to others and transformation of other people than oneself. Therefore the "only certain place to stand is in the most unlikely place: ourselves" (p. 360). Our approach is thus to use ourselves, mainly the second author, as examples to shed light on the processes of transition from being a mathematician to becoming an MTE. With Mason's words in mind, we believe our approach may contribute to extending knowledge and raising new questions regarding the development of mathematics teacher education and MTEs. At the same time, we acknowledge Mason's thesis that "in mathematics education everything remains problematic" (p. 358).

We are thus situating ourselves within the paradigm of self-study, which has a strong history in teacher education research. Borko, Liston, and Whitcomb (2007) identified self-study as one of the sub-genres of practitioner research and one of four genres of empirical research in teacher education: "Practitioner research examines practice from the inside" (p. 5). LaBoskey (2004) identified five characteristics of self-study: (1) it is self-initiated and focused; (2) it is improvement-aimed; (3) it is interactive; (4) it uses multiple, mainly qualitative, methods; and (5) it defines validity as a process based on trustworthiness. We next describe how we fit ourselves within these five characteristics, and by so doing, we make our methodology transparent.

First, our research is self-initiated and focused. Both authors have PhD degrees in pure mathematics, more precisely in the subfield of abstract algebra called representation theory of Artin algebras. Both have made the transition into the field of primary mathematics teacher education. The first author made this transition two decades ago, the second author much more recently. In this paper, we focus on the second author's transition. The second author studied mathematics and informatics over a period of 10 years at the Norwegian University of Science and Technology (NTNU), one of the largest and most research active universities in Norway, gaining a Master's degree in mathematics leading to a PhD in September 2014. Immediately after graduating, she was offered a full-time position as a primary MTE at a local teacher college where she stayed for 1.5 years before returning to NTNU, this time to the department of teacher education as a primary MTE. During her Master's and PhD studies, she worked to a limited extent as a teaching assistant in undergraduate mathematics courses. This work involved tutoring groups of students during exercise sessions and assessing student assignments. During her PhD studies, she also undertook some substitute lecturing in pure mathematics courses. However, she had no experience of teaching at primary school level.

Regarding improvement, LaBoskey (2004) describes self-study methodology as "designed to understand and improve our professional practice settings" (p. 845). By engaging in reflective inquiry into our own experiences, we aim to improve our own practices and contribute to the learning of novice MTEs.

The third characteristic of self-study is its interactive nature. Interaction for us takes multiple forms. First, the two authors have collaborated directly for the writing of this chapter, of which more is detailed below. A second aspect of interaction is the discussions the authors have had with other colleagues in our institution. Third, we experience interaction with our own students, both directly in the classroom with all students and in meetings with selected students taking part in reference groups discussing the teaching and learning of the courses and also through anonymous student course evaluations in the form of questionnaires. In addition, the interaction with texts in various forms, such as educational research literature on mathematics, has been an important part of our work.

LaBoskey (2004) pointed out that self-study methodology uses multiple, mainly qualitative, methods. Such methods were evident in our use of narrative inquiry, taking the story of the second author's journey into teacher education as the starting point. Through dialogue and conversation between the two of us, we identified steps in the transition from pure mathematician to MTE that one or both of us found particularly prominent. During these meetings, notes were made, and our experiences compared and contrasted to what we could find in the literature, also leading to literature searches. To ensure that the stories emerging from the dialogues would be more reliable, we consulted "artefacts" from our past that could substantiate our data. These included course plans and lecture notes, including PowerPoint presentations, from courses one or both of us had been teaching; meeting notes from faculty meetings and seminars; notes from literature study sessions for new faculty (also called "reading groups"), including mentors' lecture notes/PowerPoints, handwritten notes made in margins of the articles/chapters provided as readings, and evaluation reports from the reading group; and reflective notes and reports from school visits following up prospective teachers and reports from school mentors. From biweekly meetings throughout one semester (autumn 2017), a narrative emerged that subsequently was made into an organised text (the first version of this chapter).

Finally, in self-study methodology validity is defined as a process based on trustworthiness. Hamilton, Smith, and Worthington (2008) claim that "triangulation of data establishes trustworthiness" (p. 21). Yin (2018) proposed that at least four types of triangulation are possible. While we did not make use of *methods* triangulation, our multiple data sources, as outlined above, constitute a form of *data* triangulation. In addition, two investigators looking at the same phenomenon, in our case the transition from pure mathematics to teacher education, constitute a form of *investigator* triangulation. An important part of our inquiring dialogues was looking at our experiences through different theoretical lenses. These included the (expanded) teacher educators' triad (Leikin, Zazkis, & Meller, 2018), cultural historical activity theory (e.g. Yamagata-Lynch, 2010), and Valsiner's zone theory (e.g. Goos, 2014). Having different theoretical perspectives on the same data set constitutes *theory* triangulation. Trying to view our data through different theoretical lenses all within a sociocultural frame helped us zoom in on what kind of analytical frame was best for analysing and presenting our data.

In the end, we made the decision to analyse the data using a four-dimensional framework proposed by Jaworski (2003). Within each of the four dimensions, Jaworski suggested questions that might be addressed. During our discussions, we kept coming back to these questions as they provided a kind of guide through our travels along each of the dimensions. In the next section, we go into detail about this and present our analysis.

12.5 Investigation of MTE Learning Within a Four-Dimensional Framework

The analytical framework we use to investigate the process of becoming an MTE is influenced by Wagner's (1997) discussion of cooperation between researchers and practising teachers. Wagner put forward co-learning agreements as one such form of cooperation where the roles of the participants are more ambiguous than in more traditional forms of cooperation. University researchers are outside, and practising teachers are inside the school, but at the same time, researchers are working inside and practitioners outside the university. While both the researchers and the practitioners are engaged in action and reflection and might learn something about the world of the other, it is equally important that "each may learn something more about his or her own world and its connections to institutions and schooling" (p. 16).

The transition from pure mathematics to mathematics teacher education involves several communities, researchers, and practitioners of different kinds. In each of the communities, people interact with both people inside the same community and people outside. A mathematician is part of the community of mathematicians but also interacts with different types of students as part of his or her teaching, faculty from other departments as part of cooperation or administrative work, and so on. These interactions may be seen in terms of being an insider in some situations and an outsider in others, and such interactions also involve learning from the different perspectives of those involved. Wagner's (1997) notions of co-learning and the insider-outsider perspective are therefore potentially useful in analysing the communities and their interactions. Jaworski (2003) extended Wagner's co-learning concept to include what she referred to as "insider researchers": practitioners who also engage in research into teaching and hence develop their own teaching. Jaworski stated that this situation often will mean teacher-researchers, but she pointed out that it can also include educator-researchers exploring processes and practices in teacher education. The latter is the case in this chapter where we investigate the transition from being a mathematics researcher to a primary MTE.

Jaworski (2003) proposed a four-dimensional framework (p. 263) that can be applied to research on development of mathematics teaching from insider or outsider perspectives. Each of the four dimensions consists of a reflexively related pair: knowledge and learning, inquiry and reflection, insider and outsider, and individual and community. Jaworski emphasised that the elements of the framework are deeply related and interlinked. Our own journeys from research mathematician to MTE involve not only development of mathematics teaching but also understanding of what mathematics teaching is. At different locations along the journey, we have experienced different insider and outsider roles. Our own knowledge has developed along with what it means to learn and what to learn. What a mathematician and an MTE inquire into and reflect upon differs. We thus find Jaworski's four-dimensional framework useful in analysing the transition from being a mathematics researcher to becoming a primary MTE. This can be seen as a way of doing inner research in the sense of Mason (1998) on the transitional phase.

We overlay our analysis using Jaworski's (2003) framework with consideration of learning mechanisms at the boundary between mathematics and mathematics education. The first learning mechanism identified by Akkerman and Bakker (2011) was identification, which concerns learning how the diverse practices of each community or domain relate to one another, "defining one practice in light of another, delineating how it differs from the other practice" (p. 142). To realise and explicate the differences between the two practices entails learning something new about both practices and can lead to reflection through perspective-making and perspective-taking as another of the four learning mechanisms involved in boundary crossing. In our analysis along each of Jaworski's four dimensions, we therefore start by clarifying the practices of the two communities between which we have moved.

12.5.1 Knowledge and Learning

For a researcher in pure mathematics, including PhD students, the focus is to develop new knowledge in mathematics itself, concentrating on solving open problems in a (usually) small subdomain of mathematics, unintelligible to those outside the particular subdomain. The second author's PhD work, for example, concerned classification problems for special biserial and gentle algebras (Grimeland, 2014).

In the community of MTEs, the knowledge in focus is how individuals, and in particular pupils, learn and gain understanding of mathematics. Pure mathematics lessons concern conveying the mathematical content itself and only to a lesser extent how this knowledge can be applied. By contrast, the mathematical concepts discussed in a session with prospective primary school mathematics teachers are concepts in, or directly related to, the primary school curriculum. The focus of the teaching and learning is to help prospective teachers develop their mathematical knowledge for teaching (Ball et al., 2008). This includes helping them to develop a "profound understanding" (Ma, 2010) of the mathematical concepts and also to develop knowledge of how children can work with and understand the concepts. The set of rational numbers is, for instance, a standard example used to shed light on certain algebraic structures such as fields of fractions and an example that both authors have studied in depth as mathematics students. However, this type of treatment does not give any immediate insight into how pupils can build a meaningful understanding of what a fraction is, what it can mean, how it can be used in different situations, and how operations on fractions can come to make sense for the pupils: What are useful interpretations, and appropriate representations, for thinking about division or addition of fractions?

Thus, in making the transition from pure mathematics researcher to primary MTE, both authors found it challenging to understand how to handle this changed lesson content. In her first years as a primary MTE, it gradually became clear to the second author that there were more issues than she had expected that needed attending to by a primary MTE. To support this realisation, two boundary practices were particularly helpful: reading mathematics education literature and discussing mathematics education with colleagues. An example concerns communication patterns in mathematics classrooms and how to lead productive mathematical discussions. Coming from research mathematics, the mathematical content did not pose any challenges, but how to work with prospective teachers on orchestrating productive mathematical discussions in the classroom is something for which she – as a mathematician – was not prepared. Since, to begin with, she lacked a background in the mathematics education literature, it was not clear to her either how this could be done in a classroom with pupils or how to work with prospective teachers on developing insight into this pedagogical strategy. As a result of lacking familiarity with the literature, her teacher education sessions were not founded on research in the mathematics education field but rather were informed by trial and error. This is one example of how the second author gradually became aware that, for the students to become mathematics teachers, it is not sufficient to focus on the mathematics content and try gaining deep understanding of the mathematics itself. She thus needed to develop her own awareness of the diversity of mathematical knowledge for teaching and sensitivity to both prospective teachers and pupils. This process included re-experiencing and re-learning the need for the diverse aspects of knowledge inherent in teaching.

12.5.2 Inquiry and Reflection

The focus of inquiry for the research mathematician is to try to describe concepts, connections, and relationships that are not already known. Reflection is on the mathematical content and how different concepts are related. For the primary MTE, the focus of inquiry centres on how the learning of mathematics takes place and how it can be facilitated. Reflection is on which actions can be taken in a mathematics classroom and how these actions support pupils' learning of mathematics. Furthermore, a primary MTE also needs to attend to how prospective teachers learn what they need to learn in order to become teachers. Therefore, a primary MTE also inquires into the nature of mathematics teacher knowledge and how it can be developed.

Through the discussions between the two authors about what it means to be a mathematician and what it means to be an MTE, the second author recognised that her areas of inquiry and reflection have changed, in the sense of having expanded. The reflections about mathematical concepts continue to be a part of the everyday activity of an MTE, but the reflections are now to a large extent centred around fundamental mathematical concepts that are related to the primary school curriculum. In this sense, the MTE's understanding of fundamental concepts grows, but in the process, an enhanced understanding is developed of more abstract concepts related to the particular fundamental concepts. An example is the concept of division, where neither of the authors was aware of the distinction between partitive and quotitive models of division prior to moving into teacher education. For the mathematician, this distinction is not important; the question would rather be whether we are working within a division ring or not. For the teacher, on the other hand, concepts like division ring are not interesting; the question is rather how to be able to help pupils extend their understanding of division from division of integers to division involving fractions and which representations and models are helpful in this extension. Reflection on different models of division may include thinking about which models are appropriate in which situations and within which number sets and thereby gaining deeper insight into the number systems themselves. Reflection may also lead us to think more closely on the connection between division and multiplication, the concept of inverse, both in the sense of inverse operation and inverse element, and thereby to develop other insights into group, ring, field, and function theory. This is helpful even if group theory is not the explicit topic, as MTE Rachel mentions in the study conducted by Zazkis and Zazkis (2011, p. 257). These renewed insights into the practices of the two communities result from the learning mechanism of identification (Akkerman & Bakker, 2011, p. 142) and, as such, constitute renewed sense-making within the separate communities rather than an overcoming of discontinuities between them (p. 143).

Inquiry is the norm in mathematics, even if this process has not always been the norm in school mathematics. It also has to be confessed that the inquiry of a mathematician rarely extends to inquiring into the teaching of mathematics. Having a background as a pure mathematician does not automatically provide an advantage going into mathematics teacher education. On the other hand, for an MTE, the capacity and disposition towards inquiry is fundamental. As an MTE you need to inquire into your own practice. This includes inquiring into the choice of models and representations, trying out new approaches, and not being “locked” into one particular way of doing things but instead continuing to reflect upon your own practice. It may also include letting students inquire “freely”, not being afraid they might get lost. As mathematicians we know that “getting lost” is also a sometimes necessary part of the learning process.

The background of a pure mathematician is not a disadvantage when stimulating prospective teachers to have an inquiring mind, hopefully resulting in school mathematics becoming more inquiry-based. A person with a PhD in mathematics has both a deep and broad knowledge of mathematics and also an understanding of the structure and coherence of mathematics. This understanding includes concepts or procedures that prospective teachers may not see as problematic or challenging because of their so far superficial knowledge. The knowledge a mathematician has may assist “students in acquiring a view of mathematics as an interconnected web of knowledge rather than a collection of unrelated facts and procedures” (Zazkis & Zazkis, 2011, p. 260).

In addition, the importance of knowing that mathematics is a living and developing field should not be underestimated. Pupils and students are often, surprisingly, unaware that mathematics is not a subject fully developed in ancient Greece, maybe due to the kinds of mathematics they have met in school and how they have worked with that mathematics. A researcher in pure mathematics, on the other hand, has first-hand experience in extending the field of knowledge within the subject and knows that the field is “continuously” developing. This realisation gives an insight into the kinds of questions that can be raised and explored by students. It also includes knowledge about conjectures that are still open in the field of mathematics itself, including those that can be formulated at a level intelligible to prospective teachers and their future pupils, such as Goldbach's conjecture or the twin prime conjecture.

MTEs in university are expected to do research as part of their profession. A particular difficulty experienced by the authors is the distance between the questions asked in mathematics and those asked in mathematics education. Questions in mathematics concern how to extend or develop a particular concept and how to build on already proven theorems in order to push the frontiers of research further. In mathematics education, questions may be about how to explain or help students and pupils explore a particular concept and how to help them build on their prior knowledge. And mathematics education is not an axiomatic-deductive discipline – unlike in mathematics, there are no theorems! However, in mathematics education research, questions similar to those in mathematics research also arise, for example, how to extend or develop concepts, and thereby push the frontiers of research. However, the nature of the research is different, leaning towards methods in the social sciences or the humanities. To be able to take part in research in mathematics education, it is important to know what the relevant literature is and where to find it. The research fields of mathematics and mathematics education seem to be separate at the level of independent searchable databases giving access to the up-to-date research literature, making the transition between disciplines less smooth. Thus, the second author struggled to find resources similar to MatSciNet (American Mathematical Society, 2018) or arXiv.org (Cornell University Library, n.d.) in mathematics education, resources that ideally could function as boundary objects connecting the two fields.

12.5.3 Insider and Outsider

In Norway, the community of research mathematicians and the community of primary MTEs are typically separated. Traditionally, education of primary school teachers was undertaken in "teacher colleges" separate from the universities. Even today, when primary school teacher education is also undertaken in universities, particularly after the recent university reforms in Norway, the preparation of teachers takes place in teacher education departments separate from the pure mathematics departments. This separation has been highlighted in the study described by Goos and Bennison (2018). Traditionally, then, an insider in the mathematics research domain remains an outsider to the teacher education world and vice versa.

Research in mathematics itself is either carried out alone (more rarely so today) or in collaboration with other research mathematicians (more common today). In this sense, doing mathematics research is an insider activity within the community of research mathematicians, being carried out outside of teacher education and not related to teaching. The purpose of this research is to expand mathematical knowledge within a well-defined area of pure mathematics. However, teaching mathematics at university often includes teaching mathematics courses for engineering students or others who need mathematics as a tool in their future studies or job. In this situation, a mathematician can be regarded as an outsider to the engineering programme while being an insider in the mathematics community. The mathematics being taught in such lessons is rarely related directly to the mathematician's research, and the mathematician does usually not conduct research on this teaching.

By contrast, in the Norwegian context, MTEs collaborate with practising teachers in several contexts. This collaboration can arise as part of prospective teachers' professional placements in schools, where groups of typically three or four students spend 2 or 3 weeks in the classroom of a practising teacher, supervised on a daily basis by this teacher but also by a university-based MTE. This experience creates opportunities for the learning mechanism of coordination (Akkerman & Bakker, 2011), whereby there is a communicative connection between the university and the school. Meetings at the boundary between school and university address one of the difficulties identified by Wu et al. (2017), concerning inconsistencies between the university teacher education courses and the prospective teachers' experiences during school practice (p. 1381).

Professional development for in-service teachers has been given high priority during the last decade in Norway, implying that MTEs and teachers meet at courses at the university and/or in schools. In addition, schools may host research and development projects supervised by university faculty. In these settings, the MTE has a dual role as an insider and outsider. The role of an MTE, both as an educator and as a researcher, is thus distinguished from the role of a mathematics researcher, which is more well-defined as pure researcher.

The MTE conducting research in schools and in cooperation with school teachers can make use of experiences and data from this research in his/her own teaching at the university. The purpose of the MTE's research in mathematics education is to expand knowledge about the teaching of mathematics and the professional development of mathematics teachers and push the frontiers of mathematics education. An MTE working closely with school teachers does not only conduct research into other people's practice but also creates a community of practice of which she/he herself/himself is a part. This collaboration and co-development of new practices exemplifies the learning mechanism at the boundary that Akkerman and Bakker (2011) refer to as transformation.

Making the transition from mathematician to MTE, one is at first an outsider in the MTE community, as experienced by the second author. Primary school prospective teachers are not necessarily interested in or intrigued by mathematics per se (which is the strength of the mathematician) but are often more concerned with learning methods that may work in school, with which the mathematician is not so familiar. Such an experience constitutes a confrontation that is the start of a transformation process (Akkerman & Bakker, 2011, p. 146). Familiarising herself with mathematics education research was a particularly important contribution to the transition from outsider to insider in this respect for the second author. In that way, she experienced a change, from the initial sense of being an outsider giving lessons to prospective teachers, and not having enough to offer, to being an insider, who actually has knowledge from which prospective teachers can benefit. Gaining knowledge about the literature on mathematics education research led the second author to reflect more on her own teaching, changing her teaching in ways that are informed by and more aligned with literature.

Moving from being an outsider to an insider in research in mathematics education is a necessary goal for a mathematician who is becoming a MTE in a university and naturally takes some time, not least since the relevant research methods in pure mathematics and mathematics education are so different. Both authors have experienced this as a gradual and difficult path that requires severe effort and does not happen overnight.

12.5.4 Individual and Community

The mathematics researcher is part of the community of mathematicians, sharing insights with other mathematicians. Moving into mathematics education involves a change in the community to which one (most strongly) belongs. Even if one tries to adhere to Artigue's statement (1998) that didacticians from a mathematical background should preserve their place within mathematics, doing research in either field is too complex to do "part-time", and therefore most of us have to make a choice of doing either one or the other. The second author experienced clear expectations from the community of mathematicians in the mathematics department that her interest should remain in doing research in pure mathematics, even after she had made the transition into the teacher education department. This expectation clashed with the expectations of the community of teacher educators who expect faculty holding a PhD, no matter in which field, to do research in education. The second author found it impossible to continue doing pure mathematics as her research activity at the expense of building knowledge as a primary MTE. The conflict between expectations on the two sides of the boundary echoes the findings of Dinkelman et al. (2006) that the two novice teacher educators in their study retained elements of their classroom teacher identities while struggling to construct their identity as teacher educators.

To overcome the challenges involved in becoming a primary MTE and mathematics education researcher, the second author found two "phenomena" (boundary objects) particularly helpful upon joining the department of education at NTNU. Attending an organised "reading group" on topics of mathematics education research, and research methods in the field, made a big contribution to her understanding of the nature of research in mathematics education and about relevant questions in mathematics education research. The reading group was organised to help newcomers in the mathematics section in the department of teacher education to gain insight into mathematics education research. The group was led by "more knowledgeable others" in the form of more experienced colleagues, including the first author. These “mentors” thus acted as brokers (Goos & Bennison, 2018, p. 260) facilitating the boundary crossing, and the readings in the form of journal papers and book chapters played a role as boundary objects. In this way, the second author became a participant in a community in which she was able to build a basis of knowledge that would have taken much longer to develop in a less organised setting, as experienced by the first author. Both authors found the reading group an opportunity to discuss research literature at the appropriate level in a community open to questions of any kind, providing learning for both the newcomers and the mentors.

The other helpful community “offering” was an archive, available to the mathematics section faculty, containing previously developed lesson plans. These plans had been developed by colleagues in the mathematics section of the teacher education department, to be used in a first-year mathematics education course for prospective primary school teachers. Based on the lesson plans in the archive, colleagues would work collaboratively on redesigning course plans. Working in this manner gave the second author the opportunity to focus on developing knowledge about pupils’ understanding of the particular topic of each lesson and which activities are relevant and possible to use with prospective teachers, using the lesson plans as a boundary object. So there was a dual type of learning: on the one hand, learning about how pupils learn mathematics and, on the other hand, learning about how to work with prospective teachers.

12.6 Conclusion

What does a pure mathematician need to learn in order to become a mathematics teacher educator? How and in which contexts does she learn it? As Murray and Male (2005) found, there are two paths of learning that need to be built. The first path concerns teaching, which involves teaching of prospective teachers, but it may also involve teaching of practising (in-service) teachers at further education courses (which has been given significant and continuing priority by the government in Norway the last decade). The second challenge is to do mathematics education research, which by its nature is very different from research in pure mathematics.

How do you learn to be a mathematics teacher educator? There are few systematic programmes aimed at educating the educators. This is not unique to mathematics teacher education or even to teacher education in general. In fact, in any university-level discipline, you are traditionally left to yourself to figure out how to do your teaching and your research. Traditionally, therefore, there are few boundary objects and brokers to help with the transition. Building learning communities that are open and inviting to newcomers, making co-learning partnerships, and working together on lesson planning and research make it easier to understand the meaning of knowledge and learning in teacher education. A central theme in mathematics teacher education, at all levels, is that of developing a profound understanding of the mathematics being taught. The MTE needs this understanding himself/herself and also needs to help prospective teachers develop it. Likewise, both the MTE and her students need to develop knowledge of the diverse aspects inherent in being a mathematics teacher.

Collaborating with colleagues was important for the second author to develop understanding of what was going on in teacher education. The reading group community was important in building knowledge of what research in mathematics education may involve. The archive of lesson plans and the collaboration with colleagues on teaching contributed in essential ways to knowledge about what teaching and learning constitute in teacher education.

Joining projects and learning communities, with partner schools and fellow researchers, is a way to understand the shift in what it is relevant to inquire into and what research in mathematics education contains. A systematic approach, like a "reading group" as described here, is one way to start addressing this. An awareness of what constitutes research in mathematics teacher education is one of the traits of the transition. Our reflections on this case point to ways of understanding and facilitating the transition.

Acknowledgement The authors wish to thank Professor Tim Rowland for helpful comments and suggestions during the preparation of the manuscript. We also thank our editor Merrilyn Goos and the two anonymous reviewers for their comments and suggestion on the first draft of this chapter.

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Chapter 13
Shaping our Collective Identity as Mathematics Teacher Educators

Judy-anne Osborn, Elena Prieto, and Edwina Butler

13.1 Introduction

Mathematics teacher educators (MTEs) play a crucial role in forming the mathematics teachers of the future and through them the quality of mathematics education in schools. The logic of this claim has two parts: the pivotal significance of mathematics schoolteachers in mathematics education and the importance of mathematics teacher educators in preparing school mathematics teachers.

The first part of our claim, referring to the importance of mathematics schoolteachers to student learning, has been extensively studied. A large body of research that relates teachers and their actions to student learning has emerged (Darling-Hammond, 1999). This includes general meta-analyses linking teacher effects to student learning (Hattie, 2008) and research relating teachers' mathematical knowledge to the quality of their instruction as directly observed and theoretically analysed (Hill et al., 2008).

The second part of our claim, concerning the specific importance of mathematics teacher educators to mathematics teacher learning, is an emerging area of research. Confirmation of this emergence can perhaps be evidenced by the creation of the *Journal of Mathematics Teacher Education* in 1998 and by the presence of policy-shaping works such as *The International Handbook of Mathematics Teacher Education*, Volume 3: Participants in Mathematics Teacher Education: Individuals, Teams, Communities and Networks (Krainer & Wood, 2008) and Volume 4:

J. Osborn (✉) · E. Butler
School of Mathematics and Physical Sciences, Faculty of Science, University of Newcastle, Callaghan, NSW, Australia
e-mail: Judy-anne.Osborn@newcastle.edu.au

E. Prieto
School of Education, Faculty of Education and Arts, University of Newcastle, Callaghan, NSW, Australia

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_13

The Mathematics Teacher Educator as a Developing Professional (Jaworski & Wood, 2008).

Research on mathematics teacher education has illuminated ways in which mathematics teacher educators (MTEs) are both similar to and different from the population of mathematics teachers whom they teach. Jaworski and Wood (2008) note similarities in needed knowledge, including mathematics and mathematical pedagogy, as well as differences, such as the MTE's need for "knowledge of the professional and research literature relating to the learning and teaching of mathematics" (p. 1) and the schoolteacher's need for knowledge of their particular students and schools. Furthermore, Llinares and Krainer (2006) indicate that "a domain which needs closer attention in the future [is] our own learning as teacher educators. It is the field where theory and practice of teacher education inevitably melt together and we thus face the challenge of self-applying our demands on teacher education" (p. 429).

This learning that Llinares and Krainer (2006) refer to can take place individually, or as a collective endeavour in a community of practice (Wenger, 1998). Communities of practice play a double role in mathematics teacher education praxis. The theory provides a lens through which to view and understand existing and emerging communities engaged in the practices of mathematics education and educator development, as well as providing a framework around which to explicitly foster such communities as productive and supportive of teacher learning (Goos & Bennison, 2002; King & Cattlin, 2017). Recent research has applied these dual potentialities of understanding and promoting communities of practice to MTEs specifically and shown it to be effective (Goos, 2014, 2015).

The complexity of communities of MTEs follows in part from diversity of membership. Jaworski and Wood (2008) write: "Mathematics teacher educators are professionals who work with practicing teachers and/or prospective teachers to develop and improve the teaching of mathematics. They are often based in university settings with academic responsibilities" (p. 1). Taking this description as a definition, MTEs include both individuals within Schools of Education specialising in mathematics education and individuals within mathematics discipline groups who teach pre-service teachers and who may or may not specialise in this endeavour.

Diversity of roles also contributes to the complexity of communities of practice of MTEs. Within a Western epistemology of division of labour, individual MTEs may or may not engage in the full range of roles and associated knowledge described by Jaworski and Wood (2008). Indeed, in the Australian context, it is common for pre-service mathematics teachers to study most of their mathematics content in Mathematics Departments and most of their pedagogical content in Schools of Education. Opportunities for pre-service teachers to productively link the two types of knowledge, in what Shulman (1986) seminally termed Pedagogical Content Knowledge (PCK), may be compromised by this dichotomy of presentation. This applies to varying degrees in both postgraduate and undergraduate training programs, depending on local context. The notion of a disciplinary division is part of the mathematics teacher education landscape, yet this is changing, with a gradual shift in the direction of more effective cooperation (Barton, Oates, Paterson, & Thomas, 2015; Bass, 2005).

In shaping the mechanisms by which a community of practice may operate so that its complexity can be used to enrich the learning experience of its members, the formation of a collective identity is sometimes seen as critical (Hökkä, Vähäsantanen, & Mahlakaarto, 2017). Collective identity is defined in sociology as "the shared definition of a group that derives from its members' common *interests*, *experiences*, and *solidarities*" (Whooley, 2007, p. 586, emphasis added).

We note that the concept of collective identity arose in sociology as part of understanding formation of politically active groups, for example, the Civil Rights Movement (Whooley, 2007, p. 587). According to Melucci (1995, p. 43), it addressed a gap in the literature that had previously taken such groups as starting hypotheses rather than phenomena to be understood.

For further explication of what "interests, experiences, and solidarities" are shared by people holding a collective identity, it is useful to turn to seminal work by Melucci (1995). In this work, framed within a political context, an action system is understood by its actors in terms of "ends, means and field" (p. 44). In our context, the field and end are both mathematics education, and the canonical means are teaching and actions that enable teaching – together these comprise the "common interests" of our definition. The "shared experience" aspect of our definition includes both the field, which includes "rituals, practices, cultural artefacts" (p. 44), and collective action, in which Melucci (p. 45) notes that "process" is key, including interaction, communication, mutual influence, negotiation, and decision-making. Thirdly, we recognise the "solidarity" aspect of collective identity as implied by Melucci's insistence that both collective action and emotional engagement are necessary parts of collective identity. In terms of the latter, Melucci writes: "Finally, a certain degree of emotional investment, which enables individuals to feel like part of a common unity, is required in the definition of a collective identity" (p. 45).

Identity matters because, as Palmer (2017) claims, "we teach who we are" (p. 1) and because "good teaching comes from the identity and integrity of the teacher" (p. 10). Day, Kington, Stobart, and Sammons (2006) extend the notions of the ways in which identity matters in teaching, in writing:

> If identity is a key influencing factor on teachers' sense of purpose, self-efficacy, motivation, commitment, job satisfaction and effectiveness, then investigation of those factors which influence positively and negatively, the contexts in which these occur and the consequences for practice, is essential. (p. 601)

These works are part of a large literature on the importance of identity in teaching; see, for instance, recent reviews by Beauchamp and Thomas (2009) and Carrillo and Flores (2018).

Collective identity matters in part because it relates to collective agency (Hökkä et al., 2017) and in part because of its impact on individuals (Day, Elliot, & Kington, 2005). This is particularly pertinent in mathematics teacher education because the field of mathematics education in Western countries has been undergoing a sequence of reforms or revisions since at least the 1950s (Davis, 2015, pp. 28–29). Reforms have been driven by a sense of crisis (Eacott & Holmes, 2010). An ongoing sense of crisis is a double-edged sword, capable of producing both passionate commitment and ennui. The sense of crisis has become a part of the identity of many mathematics teachers and teacher educators, as expressed by Dawson (1999) when she wrote: "this manifestation of in-service culture seems to have the following basic principle: there is something wrong with mathematics teaching world-wide, and that we, as mathematics educators, must fix it" (p. 148).

Relatively little is known about the development of identity amongst mathematics teacher educators working across disciplinary boundaries, though it is known that the nature of disciplines and disciplinary boundaries has an impact (Borwein & Osborn, 2020). Because of the relative paucity of research in our particular area of consideration, two related bodies of work also inform our conceptual framework.

The first related body of work investigates the development of teacher identity within a University context, independent of the discipline. A recent review by van Lankveld, Schoonenboom, Volman, Croiset, and Beishuizen (2017) found a rich and complex picture in which some aspects of University environments were typically conducive to the development of teacher identities and others typically constraining. Van Lankveld et al. contend that teacher identity development in a tertiary context has a specific contextual complexity and explain that this complexity arises from the tensions of combining the teaching and research roles typical in higher education institutions.

The second area for comparison concerns mathematics teacher identity development (as opposed to MTE identity development). This literature addresses the interdisciplinary divide between mathematics and education. For instance, Adler, Ball, Krainer, Lin, and Novotna (2005) wrote:

> An enduring problem in mathematics teacher education is its task to build both mathematics and teaching identities. […] We do not understand well enough how mathematics and teaching, as inter-related objects, come to produce and constitute each other in teacher education practice. (p. 378)

There are in the literature a number of case studies of identity development of mathematics teachers, such as that reported by Losano, Fiorentini, and Villarreal (2018). This chapter is, we hope, a contribution to the case study literature in the analogous context of identity formation for mathematics teacher educators.

13.2 Methodology

13.2.1 Methodological Framework

Our methodological approach in this work is a narrative inquiry process (Daiute, 2013) conducted at one node of a large multidisciplinary project. The eight team members at the institution where the study took place had disciplinary identities and backgrounds that included mathematics, statistics, computer science, science, secondary school teaching, and tertiary education; thus a methodology was required which was amenable to spanning multiple disciplines and which was consonant with the values and epistemologies of all participants.

The suitability of the narrative methodology to our team's diversity emerged from three factors: suitability for identity work, reputability, and intersectionality with values from all represented disciplines/participants. The first is beautifully expressed by Meretoja (2013) in writing: "We orient ourselves in the world by telling stories about who we are" (p. 99). We all felt this suitability, but still would not have chosen the methodology without the reassurance that it is a widely used and legitimate approach (Herman, Manfred, & Marie-Laure, 2010). In this aspect, the whole team relied upon the expertise of those members more closely aligned with the humanities in their daily work.

The team's appraisal of potential methodologies and subjects for investigation included explicit team discussion of the tension between the ideal of objectivity as often associated with science and quantitative research and the valuing of subjectivity, as often associated with the humanities and qualitative research, a tension explicated in Guba and Lincoln (1994). Our collective appraisal in favour of a narrative inquiry approach was concordant with the intraparadigmatic and extraparadigmatic critiques of positivism and its heir, post-positivism, in Guba and Lincoln (1994), in particular the "exclusion of meaning and purpose" and "theory-ladenness of facts", respectively (pp. 106–107).

It is noteworthy that the values and epistemologies that we needed to span did not necessarily fall along stereotypical lines of "qualitative methods with the humanities" and "quantitative methods with science and mathematics". Instead, the kind of difference that was often pertinent was between "experimental" and "theory building" work, with the statisticians and educationalists more commonly inhabiting the former space and the pure mathematicians more at home in the latter. We see narrative study as allowing a pleasing balance of both experiment and theory building.

In this study we focus upon the results of our narrative inquiry that relate to the creation of collective identity. A broader full thematic analysis (Riessman, 2008) of the entire narrative process of our team is given by Butler et al. (2019). In this chapter, we take a different lens inspired in part by similar work by Petersen (2014) and a deeper look at the stories of the people involved as reflected in the narrative process. We note that whilst Petersen draws on post-structural theorising as a broad conceptual framework, we are making a deliberate and more circumscribed use of that theory in the specific context of human subjective experiences and identity. This usage lies within existing traditions, as noted by Meretoja (2013), when she distinguishes between the use of narrative as a cognitive instrument and one with ontological significance.

Our conceptual framework allows us to use some of the quality criteria natural to a constructivist ontology, in the circumscribed context of human experiences and identity. Specifically we use "authenticity" as described by Guba and Lincoln (1994, pp. 106–107), which they conceive of as having four components: fairness (ontological authenticity), educative authenticity, catalytic authenticity, and tactical authenticity. The first two are related to understanding (respectively, of self and others). The second two are related to action (respectively, stimulating and empowering). Our methodology in this chapter makes explicit use of the understanding-related aspects of this sense of authenticity. Aspects of the action-related component of meaning may be implicitly present as well.

13.2.2 Actualising the Methodology

In this section, we briefly describe context, design, and implementation of our narrative study as relevant to the focus of this chapter. For a more detailed description with respect to the whole project, see Butler et al. (2019).

In relation to researching collective identity formation, the context of this narrative research has been as important as its design. The context, as explained above, is a large multi-institutional project focused around mathematics and science teacher education. Within this project, the teams at each institution spanned different disciplines and brought extensive and varied experience across tertiary education and research. The focus of the project at our institution was Mathematics and Statistics (where the broader project also included Science). The scheme that the grant was awarded within had a focus on teaching praxis that was unusual for the mathematics and statistics disciplines involved. Nevertheless, it was highly regarded, both for its funding scale and the nature of the collaborations possible within it.

If we see work spanning the mathematical sciences and the formal study of education as being interdisciplinary, then at the start of the project, there was already significant interdisciplinarity within individuals within the project team. Specifically, many of us had qualifications that meant we would be plausible candidates for jobs in either environment, with education faculty having held postdoctoral positions in mathematics and mathematics faculty with graduate qualifications in education (one Master's and one Graduate Certificate).

Throughout the life of the project, the project team met weekly to discuss ideas and plans, with occasional extra meetings to progress specific subprojects. The idea of a narrative research project arose at one of those weekly meetings and was further developed in the same context.

In the initial design phase, team members agreed that there would be interviews of all the original academic team members, conducted by the project officer. These interviews were to be recorded and transcribed and interviewees given the chance to make any corrections to the transcripts before they were shared amongst the team. Within project meetings, the team collectively drafted an initial list of interview questions, which were later refined by the project officer based on an extensive review of literature.

A first round of interviews was conducted in October 2014. Team members subsequently met and decided to write reflections on these interviews, which happened between November 2014 and January 2015. The second round of interviews was conducted in December 2015, and a second round of reflections was completed in February of 2016.

For the purpose of analysing data for this chapter, the narrative and reflective transcripts were concatenated into a single file and repetitively read and searched.

The length of the resulting concatenated file exceeded 55,000 words. This large size meant that we chose to implement some automated searching in addition to free reading. In particular, searches for keywords community, practice, collective, identity, interest, experience, and solidarity, and synonyms thereof, were employed.

In reporting quotes from interview transcripts and reflective text, the following conventions were employed. Project team members employed within the School of Education, together with the project officer, were assigned letters A, B, C, and D. Project team members employed as either mathematicians or statisticians were assigned letters W, X, Y, and Z. The labels 2014, 2014R, 2015, and 2015R were assigned to refer, respectively, to first-round interviews, first-round reflections, second-round interviews, and second-round reflections. Thus a quote labelled (W, 2014R) indicates its origin in a mathematician's or statistician's reflection on the first-round interviews. The purpose of this labelling is to illustrate features of interest, given the chapter's focus on commonality in the presence of interdisciplinarity, whilst appropriately preserving anonymity.

13.3 Analysis and Discussion

This chapter explores the narratives of the individual members of the team. In particular we examine the interviews and reflections looking for evidence of the formation of a collective identity (or identities) during the period within which this narrative study took place.

Our analysis delves into three related conjectures. The first one relates to the entwined nature of collective identity as a gestalt in our context, comprising more than the sum of its parts. The second involves a layering of collective identity on two levels: "as the project team" and "as mathematics teacher educators". A third concerns the relationship between disciplinary boundaries and collective identity. We conjecture that working across boundaries does not necessarily prevent collective identity, even when the different perspectives involved align with different and potentially conflicting values.

What follows, as well as being a story of our colleagues within our local project team, is also our own story that we now tell as authors of this chapter. Stories are all told from a certain viewpoint; we acknowledge that other tellings and meanings are possible and likely. In writing about collective identity, we further develop our own construction of collective identity. Thus, the act of reporting our findings influences our findings. As auto-ethnography, the telling of our story is part of its continuation.

Our results in a study of this nature are necessarily personal and subjective, but this does not make them arbitrary. Different lenses give different views of the same data. A measure of the effectiveness of a lens is the extent to which the view it provides affords an improved understanding of self and others. This improved understanding is part of *authenticity* as viewed within a constructivist paradigm. Specifically, according to Guba and Lincoln's (1994, p. 114) four categories of authenticity discussed previously, two are especially relevant here: *ontological authenticity* “enlarges personal constructions”, and *educative authenticity* “leads to improved understanding of constructions of others”.

It turns out that the lens of collective identity, with its three facets of interests, experiences, and solidarity (Whooley, 2007), is productive in terms of authenticity, in the sense of giving us an improved understanding of ourselves and each other. For instance, although “solidarity” is not a term that we initially used to describe our relationship (as indicated by its absence from transcripts and our own recollections), post-analysis, it is clear to us as authors that we did have considerable solidarity that helped to implement changes to programs (such as the inclusion of a new compulsory subject) that would not have happened otherwise. Thus one result of this analysis is the conclusion that collective identity is a highly effective lens in interpreting our narrative data and hence potentially other narrative explorations in similar contexts.

Collective identity does not mean collective identification. Individually and as disciplinary subgroups, we are not the same as each other. In seeing our narratives through the lens of collective identity, we gain insight into the ways and extents to which our self-understandings and our practices are similar and different, consonant and complementary, and aligned and potentially mis-aligned. The identity/identification distinction is illuminated in Whooley (2007) when in discussing “collective identity” he writes, “many movements face a conflicting set of identities among their members and must attempt to build solidarity *across* these multiple identities” (p. 587).

In the context under discussion in this chapter, we, the project team, are not collectively *identified* because, even though we share an identity as MTEs, we have diverse other identities which are also important to us and which, furthermore, differently colour our individual experiences of being MTEs. For instance, some but not all of us, in addition to being MTEs, include/included educational researcher as part of our identities, and some but not all of us include/included educator of future mathematicians and engineers as part of our identities.

In reporting our analysis below, we have chosen a number of quotes that exemplify and explain our findings. We have only included a relatively small number of such quotes and endeavoured to include representative quotes from members of the team across disciplinary boundaries.

13.3.1 Collective Identity: The Ingredients

To answer the question “Do we, the project team, indeed have a collective identity?”, we, the chapter authors, have used Whooley’s (2007) characterisation described in the Introduction section. We confirm that indeed the three elements that constitute a collective identity according to Whooley, interests, experiences, and solidarity, are prevalent within the transcripts. We found many quotes elaborating on each of these elements, confirming our hypothesis regarding the formation of a common identity during the project. The first category was identifiable through frequent use of phrases such as "common interests" and familiar synonyms thereof. The second category was recognised by frequent occurrence of common synonyms for "experience" and related keywords like "doing" combined with "together". The third category required deeper consideration, since the word "solidarity" did not occur in the transcripts. However, related words such as "allies" were present, as well as phrases that in context implied common values, such as "common beliefs". All three authors checked for the integrity of the categories. One author coded membership thereof, and the other two authors checked this and concurred in all cases. A selection of these quotes follows:

13.3.1.1 Common Interests

> I guess the thing that keeps us all working together, at least one of the things, all 3 disciplines share a common interest which is to improve the landscape of Maths education and the way that it's taught and to get wider and broader interest in Maths and Maths Education and Stats education. (Z, 2015)
>
> That whole vision about maths & science teaching as a creative activity; and communicating the wonder of maths & science. That aligns with my values and I can also see that would have the potential to make these teaching jobs/ careers as a more interesting exciting thing than just saying here is the syllabus here are your lesson plans: go! (W, 2014)
>
> I think we are all just interested in improving Maths education. That's the bottom line. (D, 2014)

13.3.1.2 Common Experiences

> By coming to the meetings – even a simple thing like last week looking at my colleague from Maths out of genuine interest was sitting there working something out on the back of something and that shows a genuine passion for maths and you don't always see that – so just seeing how mathematicians think and work and that has influenced me by highlighting the importance about being passionate about what you are doing – I am passionate about learning and teaching and so I am in the right space (A, 2015)
>
> I have been in schools and it is very, very similar … in terms of how much you are trying to do and chasing your tail and time limits and pressures and the diversity of the expectations and the high, high standards (B, 2014)
>
> The tasks we had been assigned had always been similar here in my discipline. And while I'm doing my courses they tell me about what is going on in their discipline area and I presume when they are in their discipline circles they tell them about what is going on here in this discipline. So we knew about each other's work and so forth. (Y, 2014)

13.3.1.3 Solidarity

> I think it has been a bit of a God-send … to have been able to find a group you are just happy to meet with … you can talk to them, you can email them, they seem to take things in the right way they all have a common belief or want to improve education and seemingly not about themselves individually for self gain … just a nice bunch (Z, 2014)

> I think they are a lot more open than I initially thought to pedagogies and to the idea that pedagogical content knowledge is as important as content knowledge. I think that has surprised me. I have been surprised how they include us, the education department, into all their daily workings. I did not expect it to be so good. (C, 2014)

13.3.2 Collective Identity as a Gestalt

Our first finding when exploring the narratives in search of identity formation cues is the inextricable interwoven-ness of aspects of common interests, experiences, and solidarity, in much of our talk about these matters. Although collective identity can logically be examined in terms of the three components separately as above, for our project team, those meanings were often entwined in twos and threes in ways that could not be separated without loss of meaning and thus formed a gestalt:

> One I have known for a long time; we have been allies from afar. Another is relatively new, but when they came along we started to see synergies if you like, when this came on. We had a natural affiliation because we are all interested in teaching. (Y, 2014)
>
> I think it is that we do have that common goal to improve mathematics teaching ... but I think it is also an attitudinal sort of thing. We all seem to like just discussing these ideas and I think it's also all of our sense of humour; nobody takes themselves too seriously and that creates a good bond and I think that we generally enjoy all the meetings we have – it is something I look forward to in the week. (D, 2014)

The nature of the intertwining of concepts in the above quotes gives clues to possible causes. Terms aligned with the idea of solidarity, like *allies*, *affiliation*, *bond*, *common philosophy of what we want*, and *common ideological approach*, are all presented as caused by common interests and/or experiences at the individual and personal level. Also, interests are presented with a connotation of values. For instance, in the phrase "we all have interest in improving Maths education", the phrase *have interest in* could grammatically be replaced by any of *engage in*, *enjoy*, or *value*, and we posit that this is so because shades of all these meanings are present in the speaker's use of the word *interest*.

Thus, we, the authors, conjecture that collective identity may be functioning as a gestalt in our context because mathematics education is simultaneously deeply personal (Palmer, 2017), value-laden (Bishop, 2001), and socially contested (Davis, 2015; Hersh, 1997; Tampio, 2017; Valero, 2017). For instance, the deeply personal aspect relates to two different senses of "interest": one relating to enjoyment and the other to valuing. The first pertains simply to "common interests", whereas the second has aspects of both "common interests" and "solidarity". The socially contested nature of mathematics education has echoes of the sociological origins of the notion of collective identity. Reminiscent of a gestalt, Melucci (1995) writes of people forming a "we" by continually adjusting actions and their personal meanings, means and a sense of associated possibilities and limits, and relationships with the field of action; and he refers to the need for individuals to create for themselves "a certain integration … between … contrasting requirements" (pp. 43–44).

13.3.3 Collective Identity as Partially Enabled by the Project

In this section, we claim that the participation in the project was instrumental in the formation of project team members' identities and collective identity.

In substantiating this claim, an associated question we ask is, "How might this claim be false?" One possibility, pointed to by frequent references in the transcripts, is that all the team members might have already had identities as MTEs before the project even began. In investigating this possibility, we begin to see indications of a layering of different kinds of collective identity. The following quotes are drawn from a combination of team members' original reflections within the narrative process and current reflections of the author team, post-project. Such a combination is needed to understand the ongoing effect of the project subsequent to its formal conclusion.

> It all started when I did my PhD – I realised I prefer the teaching side than the research side … I have an affinity with teaching teachers (Z, 2014)
>
> So my objective would be to produce all of these things in my teachers. I want them to be autonomous beings inside a community of practice, I want them to have a go at creativity, I want them to understand the nature and utility of maths, I want them to know how maths has contributed to society and so forth because they are all the things they need to know as custodians of the discipline. (D, 2014)
>
> The relationship between us was going to shape what teacher education for mathematics teachers was going to be about. (C, 2014)

In the above, we see that some members of the project were identified as MTEs long before the project began, yet there are also hints that the relationship to come within the team was to be personally significant in a way that relates to our roles as MTEs. Similar findings were observed in the study conducted by Barton et al. (2015). A window has opened: we now see a potential layering of collective identity for project participants, firstly as members of the project team and secondly as members of a more diffuse group, namely, mathematics teacher educators. A similar principle might apply to other collaborations. Thus the significance of this layering is both to the general theory of collective identity and to its particular implications in the work and challenges of mathematics education.

13.3.4 Collective Identity as Multi-layered

Collective identity as the project team is different from that as mathematics teacher educators. We see and analyse these as two different layers of collective identity. In the first layer, our personal identities are, potentially, drawn in the light of our relationship to a very specific set of individuals. In the second, the group involved is larger, more diverse and dispersed, and less well-defined. The second is more abstract, the first more concrete.

A very concrete sense of rapport and appreciation for the team as specifically constituted of particular individuals was evidenced both in quotes we have already seen above and many more, such as the following:

> Everyone is willing to listen, share, and to try to understand and accommodate the other team members. (D, 2014R)
>
> Common purpose I think. I mean we each have different views about what is ideal, but I think we are actually impressively open to each other's views. (W, 2014)
>
> … amazingness of each of my colleagues, and specifically the ways in which their talents and spirits contribute […] enabling what we can now say in retrospect is really Professor Chubb's vision for maths (and science) to be taught more like it is practiced.
>
> (X, 2014R)
>
> I think we are all on the same page as far as we want the project to work and be successful and to move that along, but I think it's the combination of our backgrounds that is going to actually make the project better than it would be if any of us tried to do it independently.
>
> (D, 2014)
>
> I really like them all for who they are and they are different, hey?
>
> (C, 2014)

The more abstract sense of identity as a mathematics teacher educator is something that we have already seen in quotes in the previous subsection. However, abstraction in this sense is a double-edged sword: more generically applicable but less indicative of collective action.

In analysing MTE collective identity, we expect to see all of Whooley's (2007) three components of interests, experiences, and solidarity, but in slightly different and more diffuse forms than in the context of project team identity. For instance, all MTEs would be expected to have a common interest in mathematics education, but not necessarily in the success of a particular grant or initiative. The experience of solidarity is also necessarily different. As an MTE, collective allegiance is likely to be around the value of mathematics education generally, whereas on the scale of our project, there was a sense that all team members were making a conscious and deliberate effort to make sure that every individual was supported in all of their endeavours within the project:

> … everyone wanted to play ball together and because there had been relationships established between multiple member groups, groups within this group, it made it a lot easier at the beginning, but there were still those initial stages of trying not to say the wrong thing accidentally (Z, 2015)
>
> I think it is a common approach to trying to improve things for the greater good. (D, 2015)
>
> I think we are all idealists, and I think that's nice, we are talking a common language, and then we have our pragmatism side of things which is different for each of us, whether it's the team-members in stats, in maths or in education, but we are helping each other see what their constraints are, the logistics are and that sort of stuff … the thing that is holding it together is the shared vision. (W, 2015)

13.3.5 Effects of Disciplinary Boundaries

Team members from both sides of potential disciplinary boundaries were interested in the theory of boundaries in the context of communities of practice (Wenger, 1998). One of our common interests was the boundary or boundaries between us.

> I don't think we have any shirkers in the group and I think there is a lot of mutual respect. But I also know there is a lot of disrespect in general terms between different faculties … you know, the only real science is physics everything else is stamp collecting – you know that famous quote – I mean, that's within the sciences, the Snow's two cultures and all the rest. (W, 2015)
>
> … this is getting at the idea of boundary encounters … I think it's great because it's adding to my knowledge about our teaching students and how they learn mathematics – by talking to the people that are teaching them mathematics. Otherwise it's very easy to stay in your silo … They are learning content and pedagogy and they have got to put it together and so I think if we can help put that together across the boundary. (D, 2014)

This is not to say that disciplinary boundaries had no effect. Even though amongst the initial team members for whom all of our PhDs were mathematical, disciplinary boundaries associated with our belongingness to education or mathematics or statistics did have practical impact.

There was evidence that the team members were trying to express respect for each other's areas of expertise, and not occupy what might be felt to be undeserved territory:

> I think the ability of the teacher to apply their knowledge flexibly … that knowledge can be the pedagogical stuff (which I don't have a formal handle on) and the mathematical knowledge. (W, 2015)
>
> I guess I don't want to speak on behalf of the other … because I don't see them as others, although I see them as experts in their space. (A, 2015)

Sometimes disciplinary boundaries were expressed in terms of different values. These kinds of different values have been problematic in other times and places (Tampio, 2017); however, the view is put forward without antagonism here:

> I think if there is a fundamental difference between us ... I mean I think we are all interested in improving mathematics teachers and the quality of teachers we produce and the quality of maths teaching in schools but I think fundamentally the reason underlying that is a bit different. For the maths academics they are really interested in the health of the discipline of mathematics […] whereas for the teachers I produce, I guess I am much more focused on the reality they face in schools, where they will be teaching not just those top students, but the large population …
>
> (D, 2014)
>
> Reflecting on colleague's claim … for me I don't think it is about the "best" students. Also when I'm thinking about the health of my discipline, it is about how the whole society sees it, and that includes the folks whose main passions are in entirely other areas of life. (X, 2014R)

We conjecture that although members of the project team were aware of disciplinary boundaries as being present and potentially problematic, this fact did not influence either the individual team members' personal identities as MTEs or their view of their colleagues as MTEs.

We make our conjecture on two bases. The first is that there is no evidence to the contrary that we have recognised in our very extensive narrative project transcripts. The second is the following expression of collective identity made precisely in the context of fond recognition of disciplinary boundaries.

> Actually it is really symbolic – crossing the campus – or crossing the discipline boundary – actually we should get a photo of ourselves on the bridge down there! (D, 2014)

We further wonder, is an identity as multidisciplinary protective against harmful disciplinary divisions? That is, might our own putative interdisciplinary identities have enabled us to form a collective identity as MTEs even within the context of potentially problematic differences in values?

13.3.6 Transitions Between Layers of Collective Identity

In our experience, identity and relationships formed within the project mediated the activities of team members. Something similar is described in the study of Barton et al. (2015). These activities influence the long-term impact of the project beyond what was institutionalised during the period when it was funded. Hence we are interested in the relationship between the collective identity as a project team that was formed within the project and collective identity as MTEs that may continue into the future.

The design of our narrative study does not facilitate definitive conclusions on the transition between layers, nor do we wish to imply that such a transition between layers will or should always happen; nonetheless insights can be gained by considering what team members expressed about project legacy.

A first observation is that there was a clear desire for project legacy. Two quotes illustrate that common desire:

> … a legacy or something you can put hand on your heart and say look at that, now the people that come through our teaching programs are now doing this whereas previously they weren't; and there is now this earlier collaboration between disciplines, they are now going out much more well equipped to handle what is going on in the classroom, plus they have also got a skill set which is not just defined by the classroom, but they are more worldly …
>
> (A, 2015)
>
> Perhaps we shouldn't be expecting anything more than what any other small group is achieving, but I would hope that from such a large and long collaboration that we would be able to be recognised for something that has made a significant impact to the landscape of maths education, to the point where there are greater numbers of people interested and participating in maths and maths related disciplines. (Z, 2015)

Secondly, within an extensive catalogue of desired legacies at the national and local level, one stood out as more commonly expressed than any other across the span of the narratives in time and people: it was the desire for a community of practice starting at undergraduate level. This desire was expressed within the first round of interviews.

> So, for teacher education, it is to have teachers that are confident in their maths skill, confident in their ability and that know how to collaborate with other teachers, know how to teach other teachers – not just their students and feel part of the community of practice with other teachers, with university people. Perhaps that is the strongest thing I have about this project and this vision: the community of practice; and a grass roots one for that matter, that is very important to me. (C, 2014)
>
> Changes that I would like to see the project and allied initiatives bring include: more stewardship and promotion of professional communities which (for individuals) start at University during their training, but extend far beyond … (X, 2014)

The same desire was expressed again in the second round of interviews.

> I would like a community of practice to be set up … I would like my students in 4th year to feel already part of the community. That's what I would like to achieve. (C, 2015)
>
> I would love to do something to leave a legacy. So that we can point to something and say that is because of the project that that happened. In particular to that end I would love to get the sense of community going … I mean the … undergraduate community (Y, 2015)
>
> I think the other thing which has come out, which is something which has been nicely informed by what has been happening in the other project nodes, is how community building works and how that supports teachers in their early years of teaching. I am hoping that we will make a difference there. (W, 2015)

The prominence and persistence of a desire to establish a community of practice has a pleasing twofold significance in our analysis. Firstly, it is simply a common value that we happen to know is still driving the activities of at least some team members beyond the project and is thus evidence of an enduring component of collective MTE identity. Secondly, that conclusion is further supported by the fact that the value is a valuing of community, with its entangled connotations of collective identity.

There was a clear desire expressed by some team members, in both rounds of interviews during the project, that the team's work together should continue beyond the end of the grant:

> I hope we keep on working together and I hope we take it to the full extent it can be taken and I hope that when this project finishes that we can continue the work that we have started into many other different projects – you know – sidekick projects and all that. (C, 2014)
>
> … we need to build a track record that will enable future funding to be obtained to support future hopefully common interests and collaboration of the group … My hope is this doesn't end when the grant ends. I think we should be continuing to have meetings beyond this otherwise things will just fall over and we go our own ways and we should be forward planning for that now to see how that is achievable. (Z, 2015)

At this stage, we can report that some of the "sidekick" projects and some of the legacy of impact that team members hoped for have come to pass. Whether that impact grows or diminishes with time, and what roles we may each play in the future, remains to be seen.

13.4 Conclusions

In this chapter, we have traced the learning journey of a team of mathematics teacher educators, of which we have been a part, and thus elucidated three main aspects of interest: a need within mathematics education for different kinds of practitioners to work together to address what may otherwise be a fraught pedagogy-content dichotomy, personal experiences of practitioners working at such an interface, and broader implications about the nature of collective identities. In particular, within our project, we formed productive collective identities in layers and in overlapping ways.

Further, in terms of theorising collective identity, in writing this chapter, we have formed some conjectures as to the ways in which collective identities can be shaped. Some of these conjectures may form the basis for further research. For instance, we hypothesised that the gestalt-like nature of MTE collective identity may arise from the personal, value-laden, and socially contested nature of mathematics education within society. We proposed that collective identity of a team within a particular project could promote long-term changes in broader collective identity beyond the project. We also conjectured that working across disciplinary boundaries, even those that are traditionally fraught, does not necessarily harm the formation of collective identity, in good circumstances. Further to this, we wonder if the prior or simultaneous formation of an identity as *interdisciplinary* or *multidisciplinary* might be an enabler in forming collective identity even in a contested space.

We conclude this chapter with reflections on the project and its influence in our own identity formation from the three authors of this chapter.

> Even though I do not see myself as a mathematics teacher educator, I do see myself as having been a mathematics teacher educator enabler in this work. The mathematics aspect of this identity was birthed and grew throughout the project. In addition to the administrative aspect of my role, I was in an educator role within the project, and that happened when I was given an open door to be creative, collaborative and contribute. I was able to select from my pedagogical smorgasbord to enable learning. (Project Officer, 2018)
>
> The project certainly shaped my identity as an MTE. Before, I saw myself as a mathematics educator (amongst other aspects of my mathematical identity), but not with that particular focus on educating mathematics teachers specifically. Now I see myself as having some expertise and some identity-stake in that area. This sense of myself is due both to the huge learning that I have done in that area, and is positively influenced by the recognition that the grant and members of our team's leadership in our School's practices within that area have had within the School. (Mathematics, 2018)
>
> For me participating in this project completely shaped my identity as a maths teacher educator. The project began the year after I started convening the mathematics teaching degree, so it has been a significant feature of most of my time in this role. The conversations that I have had with team members, one in particular from mathematics, have been such a huge and positive influence on my thinking. I think our maths teaching degree is much better because of this project. (School of Education, 2018)

Acknowledgements With deep gratitude, we acknowledge our fellow project team members, who have given us so much. We also acknowledge the Office for Teaching and Learning which funded our project. We are also grateful to other colleagues from the broader project who supported from afar and to supportive colleagues across disciplines from our own Institution and beyond.

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Chapter 14
The Influence of and Interactions Between Different Contexts in the Learning and Development of Mathematics Teacher Educators

Jenni Ingram, Katharine Burn, Jen Fiddaman, Colin Penfold, and Clare Tope

14.1 Introduction

Just as sociocultural theories of learning, first developed in the quest to make sense of young children's experience, have subsequently been applied to the professional learning of their teachers, so too can the chain be extended to consider the development of those responsible for supporting teachers' professional growth: the teacher educators. This chapter is co-written by two university-based teacher educators involved in teaching and supervision on the part-time Master's programmes to which it refers and by three participants in those programmes, each specifically seeking to develop their practice as a mathematics teacher educator (MTE). We examine the interplay between context and learner with reference to two theoretical models of professional learning. The first is the "interconnected model of professional growth", developed by Clarke and Hollingsworth (2002), to which the participants were deliberately introduced within their Master's programmes and on which they subsequently drew in reflecting on their experiences. The second is Goos's (2013) adaptation of Valsiner's (1997) "zone theory" of development, which

J. Ingram (✉) K. Burn
Department of Education, University of Oxford, Oxford, UK
e-mail: Jenni.Ingram@education.ox.ac.uk

J. Fiddaman
Bayards Hill Primary School, Oxford, UK

C. Penfold
Education Development Trust, Reading, UK

C. Tope
University of Winchester, Winchester, UK

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_14

we use as a lens through which to examine the way in which professional growth necessarily changes the context in which that growth occurs.

The chapter was constructed through an iterative process that began with the three participants in the Master's level programmes (Clare, Colin and Jen) drafting reflections on the nature and impact of their individual contexts on their learning as MTEs within their particular programme. They were asked to review what and how they had learned – with learning understood to encompass changes in knowledge and beliefs as well as in their professional practice and in their goals as teacher educators. Jenni and Katharine read these reflections, along with extracts from the participants' final research and development projects. They responded by posing two or three specific questions to each MTE about influences on their development to which they had alluded and by inviting them all to relate their reflections to the Clarke and Hollingsworth (2002) model to which Jen had, in her first reflection, already referred. All the participants' initial reflections and subsequent responses were then analysed by Jenni and Katharine, along with the extracts from the participants' research and development projects. The emerging drafts developed through this process were then worked on together and successively refined by all five authors.

14.2 Teacher Education Policy and Practice in England

Since two of the MTEs were working in England, one within higher education and the other in a school-based role, while the third had only recently moved from an advisory role in England to become an educational consultant in the Middle East, we begin with a brief description of teacher education policy and practice in England. This serves both to illuminate the nature of the Master's programmes and to contextualise the specific settings within which the MTEs were working.

From the mid-1980s until the accession of the coalition government in 2010, entry to the teaching profession in England was usually through one of two main training routes: a 1-year postgraduate programme leading to a Postgraduate Certificate in Education (PGCE) for primary and secondary teachers or a 4-year undergraduate Bachelor's degree in Education (BEd) offered only to primary teachers. In both cases candidates applied for places to higher education institutions that worked in partnership with schools. It is important to note that since the early 1990s, the *minimum* proportion of time that prospective teachers within the PGCE route spend in school has been set by statute (DFE, 1992) and now stands at 67% for both primary and secondary teacher education. The fact that prospective teachers spend 24 weeks of a 36-week teacher education programme in schools has had two important implications for the work of teacher educators. The first is the diminished status of *university-based* teacher educators, many of whom (in some institutions) may be employed part-time, or as "teaching-only" staff with little or no obligation to engage in research or to have undertaken previous postgraduate study. The second is the corresponding fact that there are many well-established roles for *school-based* teacher educators, acting either as programme coordinators within a particular school or as mentors at subject or class level. It was with these two groups of teacher educators in mind that the Master's in Teacher Education (MTEd) – undertaken by Clare and Colin – was first conceived. In its structure and delivery, it drew heavily on experience within the same university of running a part-time Master's in Learning and Teaching (MLT) – the course undertaken by Jen – which had itself been designed to allow qualified teachers to engage more fully with research than had been possible within their initial training.

The MTEd was given further impetus by two, more recent, government policies. The first was the launch of a new "School Direct" route (DFE, 2011) whereby a majority of teacher training places were allocated directly to designated teaching schools who selected their own partners. If such schools chose to award only Qualified Teacher Status, without a PGCE qualification, they did not need to work with universities at all. Although the ideological attack on universities with which this policy was originally associated (Brown, Rowley, & Smith, 2016) has since been tempered (Tatto, Burn, Menter, Mutton, & Thompson, 2018), the future of direct university involvement in initial teacher education appeared, for some years, to be in doubt. This made it even more important to ensure that teachers assuming significantly enhanced roles as school-based teacher educators had the opportunity to undertake a research-based postgraduate qualification in preparation for the role. The second involved the progressive dismantling of local government responsibility for education, as schools were variously encouraged or compelled to assume a new "Academy" status, independent of locally elected councils. Deprived of funding, as financial resources were transferred from central government directly to individual schools (or to groups of schools operating as multi-academy trusts), these authorities ceased to provide support services, such as specialist subject advisors. Schools seeking advice or professional development for their staff have therefore turned increasingly to one another or to private educational consultants, a role (like that of the former advisors) for which no particular postgraduate qualifications are formally required.

The professional contexts of the three participant authors reflect these developments in different ways. Clare had worked for many years as a university-based primary teacher educator before undertaking a Master's level qualification. Colin, who had previously been employed by a local education authority as an advisory teacher, had recently left England to assume a consultant role in the Middle East. Jen, who had qualified as a secondary mathematics teacher only a few years earlier, was already part-way through the MLT programme when her head teacher invited her to take on a new role at the primary school within their multi-academy trust, supporting the professional development of those teaching mathematics at that level.

14.3 Analytic Framework

Our initial decision to use Clarke and Hollingsworth's (2002) "interconnected model" as an analytical framework was influenced by Jen's explicit reference to it within her first reflective account. She regarded it as a valuable tool that had shaped her thinking

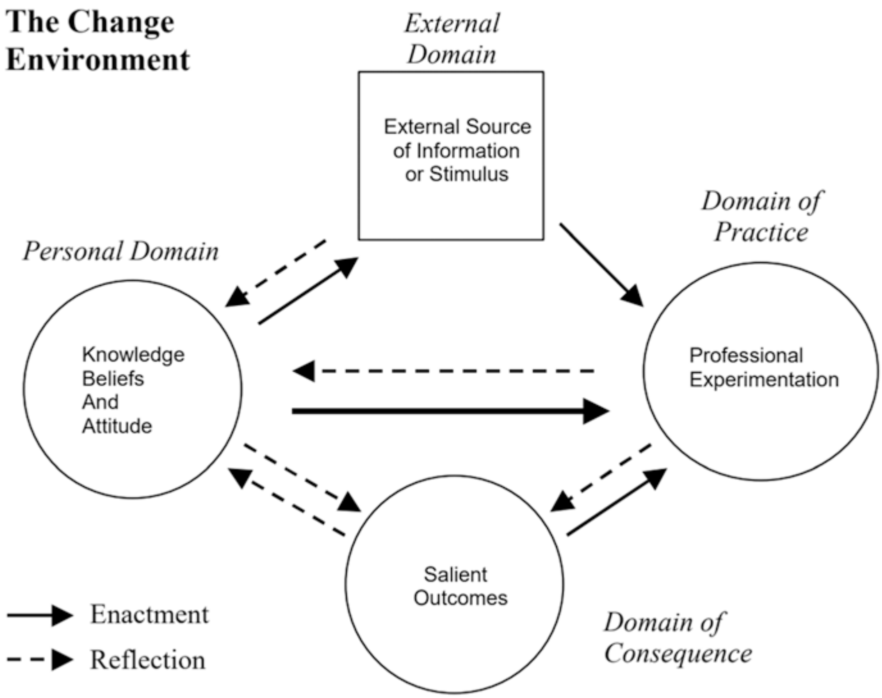

Fig. 14.1 Clarke and Hollingsworth's (2002) interconnected model of professional growth

both about the programme she developed for the primary mathematics teachers and about how to evaluate its impact. Since the model, shown in Fig. 14.1, had also been explicitly introduced to Clare and Colin and featured in both of their projects, there was evidence that all the participants had found it helpful in alerting them to the complexity of professional learning and to their scope for action in promoting it.

The interconnected model recognises the complexity of professional growth, acknowledging four distinct domains within which change may occur and identifying the mediating processes of "enactment" and "reflection" as the means by which change in one domain may be translated into change in another, along different pathways with different starting points. The "external domain", as the source of new information or stimuli, is clearly located outside the MTE's personal world. The other three domains together constitute the individual MTE's professional world of practice, encompassing his or her professional actions, the consequences that he or she infers as arising from those actions and the knowledge and beliefs that prompt and respond to those actions. Clarke and Hollingsworth (2002) chose the term "enactment" to represent the deliberate putting into action of a new idea or belief or newly encountered practice. "Reflection" carries the connotation that Dewey (1910) associated with it: that of "active, persistent and careful consideration". The model's clarity about these two mediating processes resonated powerfully with the way in which the participants reported on the developmental processes in which they had engaged as Master's students and as MTEs seeking to improve their professional practice and thus supported our assumption that it might provide an effective framework for analysis.

Two other important features made the interconnected model a particularly valuable analytical tool. The first is its acknowledgement of the possibility of several different patterns of interaction between the different domains, which also allows for clear distinctions to be drawn between simple "change sequences" and more sustained "growth networks". The term "change sequence" is used by Clarke and Hollingsworth (2002) to describe those instances where change in one domain can clearly be seen to lead to change in at least one other domain, but where there is no evidence of those changes becoming securely embedded. Where there is evidence of ongoing or enduring change, usually indicated by more complex patterns of enactment and reflection, then these mediating processes can be seen to establish a more powerful "growth network". The second important feature derives from the significance that the model ascribes to the "change environment": the wider contextual factors that shape and constrain each individual MTE's scope for action.

As teacher educators and part-time Master's students, Clare, Colin and Jen were each simultaneously working within two different contexts: the individual professional context of their work as MTEs and the academic context created by their Master's program, which required engagement with – and conduct of – research related to their MTE role. Within the interconnected model, the specific contexts within which they were working can most obviously be understood as constituting the change environment, giving each MTE varying scope to reflect on new ideas (and indeed on those previously held in the light of new information) and to enact and evaluate new practices. The Masters' programmes in which they were each engaged can most readily be equated with the model's external domain, introducing the MTEs to a variety of different kinds of stimuli (through directed readings, taught sessions, structured tasks and dialogue with fellow students). It is important to acknowledge, however, that the MTEs' professional contexts also provided other kinds of stimuli. These arose, for example, from the particular needs of teachers with whom they were working or from the demands of senior management or inspection regimes to which their provision was subject. Moreover, the very fact of enrolling on a Master's programme had its own impact on the MTEs' change environment, creating an expectation (acknowledged by their employers, colleagues and the mathematics teachers for whom they were responsible) of greater experimentation and reflection.

14.4 Specific Professional Contexts and Their Influence on the Change Environment

As noted above, Clare worked in a university setting in England. She had joined the primary teacher education team more than 10 years previously and was responsible for teaching mathematics modules to PGCE and BEd students who were preparing to teach at different levels across the primary age range (5–11). Clare's approach within university seminars was to explore students' practice as teachers through the subject content, effectively regarding subject knowledge and pedagogical knowledge as intertwined. She also supervised the extensive school-based placements of a number of prospective teachers across the full range of subjects. Beyond her role as a MTE, Clare acted as course leader for the whole primary PGCE program. This position of responsibility, combined with her 10 years' experience, gave her considerable scope to take action within her own seminars and to put forward suggestions for the whole PGCE team. However, the nature of this change environment also meant that any changes Clare made would carry quite high stakes; she therefore needed to feel confident about any new measures that she chose to implement.

Colin had been engaged in mathematics teacher education for over 25 years, in a variety of roles. He was acting as a mathematics consultant when he applied for the Master's, but 5 months into the programme became chief education officer of a group of 23 affordable private schools in Egypt. This role extended far beyond mathematics, making him responsible for providing academic vision, educational strategy and leadership across the whole group of schools, which catered for children from kindergarten to 12th grade (ages 4–19). These students were mostly Egyptian, but mathematics was taught through the medium of English. Just over 200 teachers had some responsibility for teaching mathematics, around 120 of these at primary level. Virtually none had taken part in any pre-service mathematics teacher education, and very few had participated in any subject-specific professional development or received any support since they started teaching. Indeed, there was little time and opportunity for teacher professional development: teachers were generally unavailable during the school day, and family commitments meant that after-school and weekend meetings were impossible. Teachers rarely met together or collaborated. Teacher turnover was extremely high, so many were in the first few years of teaching.

It is clear that despite his previous range of experience, Colin – in contrast to Clare – was new to the specific context in which he was working. The cultural setting of the mathematics teachers whose development he would seek to promote was essentially unfamiliar to him. This lack of familiarity, combined with the range of his responsibilities (many of which were unrelated to being a MTE), seems to have restricted his scope to take action, to experiment with his own practice as a MTE and to reflect on the outcomes.

Jen had worked for 2 years as a mathematics teacher in a comprehensive secondary school in England. Halfway through her 2-year Master's programme, she became primary mathematics achievement lead in her school's feeder primary school. This was initially a 1-year secondment, for 1 day per week. The role was an entirely new one, and Jen was given considerable freedom to make it her own. The school's intention in creating the post was to use the specialist subject knowledge of a secondary teacher in developing the teaching and learning of mathematics across the primary school. Jen was, however, acutely aware of her limited experience at this level. Since the school was also new to her, she deliberately spent her first few weeks getting to know the staff and their teaching styles through learning walks and informal conversations while familiarising herself with the primary stages of the newly revised National Curriculum. Again, the change environment in which Jen found herself was very different from each of those experienced by Clare and Colin. The freedom she had in developing the new role gave rise to a range of opportunities for action, experimentation and adaptation as she observed how the teachers responded. Yet she was also operating in an unfamiliar context, acutely aware of her own lack of experience in the primary school setting and of the fact that the teachers were much more comfortable and confident within it than she was.

14.5 The Master's Programmes: Operating as External Sources of Stimulus and as Influences on the MTEs' Change Environment

As explained above, Clare and Colin both completed the MTEd, a Master's qualification specifically designed for MTEs, whereas Jen undertook the MLT, essentially intended for practicing teachers. She chose, however, to focus her final-year research and development project specifically on mathematics teacher education. Both courses were part-time programmes, designed to be compatible with full-time professional commitments, and usually took 2 years to complete. The MTEd was primarily a distance learning course, taught through online tutorials and seminars, although each year included a week-long university-based summer school. The MLT used a form of blended learning, with face-to-face seminars on five weekends across the year, each preceded and followed by a school-based investigative task and a series of associated readings, shared and discussed online. There were fewer taught sessions in the final year, which was entirely focused on students' research and development projects. Academic support took the form of individual supervision, online or face-to-face, depending on the student's location and preference. Within both courses, the students' final research and development projects required them to design, implement and evaluate their own intervention within their roles as MTEs.

Like many experienced university lecturers working in initial teacher education in England, Clare had received limited formal academic preparation for her role beyond the completion of her own BEd. Her route into teacher education had been a professional one, moving from subject leadership (in a middle school, catering for students aged 8–12) into consultancy work to support the implementation of a new National Numeracy Strategy and then into a university role. She believed that the MTEd would give her the chance to look systematically at issues and challenges of which she had become aware in her practice as a MTE. At the start of the course, for example, she was particularly interested to explore ways of promoting greater debate among students who tended to embark on teacher education programmes with deeply rooted beliefs and assumptions about the nature of mathematics and about mathematics teaching and learning. The course would also allow her to examine beliefs and assumptions embedded in her own practice as a MTE. While she was particularly interested in the opportunity to focus on prospective teachers as teachers of mathematics, her supervisory work on school placements and her wider leadership role meant that she was alert to ways in which she could apply insights from mathematics teacher education more broadly to support prospective teachers' learning across the curriculum and to the development of the primary PGCE programme.

While Clare had a sense of obligation (and indeed pride) as a university-based teacher educator to root her practice in research-based knowledge, Colin's roles as mathematics consultant and then as chief education officer did not carry quite the same kind of expectation. Yet he was similarly inspired by the opportunities within the MTEd both to investigate issues of particular concern in his current practice and to reflect systematically on that practice. Working in the Arab states of the Persian Gulf and then in Egypt – contexts profoundly different from those in which he had taught mathematics – he specifically sought to understand how context might facilitate or inhibit teachers' professional growth. Aware that he had mostly gained his knowledge as a MTE through the practice of being a MTE, Colin now wanted to adopt an analytical approach to understand and improve that practice.

Jen had not been engaged specifically in teacher education when she embarked on her Master's programme. Unlike the MTEd, which focused on teacher education, almost exclusively within the domains of particular subjects (mathematics and science), the MLT had a generic programme at its core, accounting for two-thirds of the face-to-face teaching and half of the prescribed readings. Supervision for assignments and the final research and development project was, however, usually provided by subject specialists. Most of the school-based tasks (intended both to develop research skills and to explore different features of learning and teaching) were conducted within students' own classes – and thus also within their subject. While all Jen's work was concerned with mathematics, only for her final year research and development project did she choose to focus on teacher education, in order to support her new part-time role as primary mathematics achievement lead. She recognised at this point that she had much to learn, not only about primary education and the primary mathematics curriculum but also about the process of leading professional learning.

14.6 Analysis of Each of the Instances of Change Reported by the MTEs

In order to explore the professional growth of the MTEs in relation to their specific professional contexts and as stimulated by their Master's programmes, we focused on each of the instances of change that Clare, Colin and Jen reported in their reflections on their learning, using the Interconnected Model, shown in Fig. 14.1 (Clarke & Hollingsworth, 2002). In each case we sought to identify whether these instances constituted simple change sequences – in which change in one domain led to change in another, but with no evidence of those changes becoming securely embedded – or more sustained growth networks, for which there was evidence of ongoing or enduring change. While the set readings and small-scale investigative tasks that featured throughout the Master's programme were undoubtedly important stimuli for change, other stimuli also arose from the MTEs' professional work contexts. As the following accounts demonstrate, our analysis revealed that while the course readings and investigative tasks *sometimes* served to set in train patterns of reflection and enactment that formed enduring growth networks, this was not always the case. In contrast, the research and development projects undertaken by each of the MTEs *all* resulted in changes that the participants regarded as having had a sustained impact.

14.6.1 Clare

Clare's reflections included accounts of four specific instances of change stimulated by her work on the MTEd programme (although other external sources of stimulus occasionally also played an important role). The first instance, summarised in Fig. 14.2a, was related to a boost in confidence that derived from some of the set reading. Clare, who had long assumed that many prospective teachers had fixed ideas about mathematics as a subject and about the process of teaching and learning mathematics, was anxious to find ways of stimulating more active discussion – and thereby potential re-evaluation – of their ideas. Further thought about this issue was stimulated by two readings: one that demonstrated how deep-rooted these beliefs are (Liljedahl, 2008) and another which suggested that such beliefs might be held consciously or unconsciously (Thompson, 1984). The reading, as a stimulus in the external domain, prompted Clare to reflect on her existing beliefs (1), which were strengthened, giving her the confidence to suggest changes in practice to her team of tutors. Since Clare did not report specifically on whether or how these particular changes were enacted or any impact that they may have had on her own beliefs or practice or on those of the team, these developments only provide clear evidence of a change sequence.

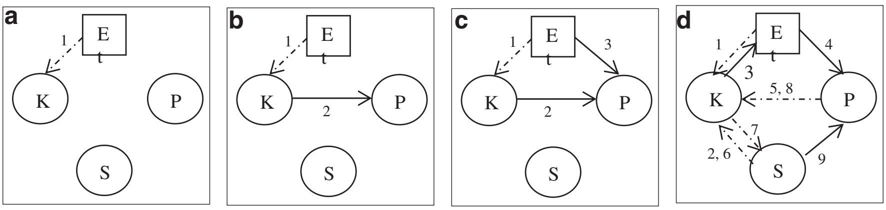

Fig. 14.2 Instances of change in Clare's experience analysed in relation to Clarke and Hollingsworth's (2002) interconnected model of professional growth (E external source of information or stimulus, K knowledge beliefs and attitude, P professional experimentation, S salient outcomes)

The second instance on which Clare reflected, shown in Fig. 14.2b, was also stimulated by reading within the MTEd. She had found her ideas challenged by a convincing line of argument (Watson, 2008), suggesting that school mathematics and the discipline of mathematics were so different that one could not even be considered a subset of the other. Clare had regularly emphasised the need for teachers to encourage their pupils to act like mathematicians, but what she understood by acting like a mathematician began to change when she reflected (1) on Watson's (2008) arguments and on those of Lockhart (2002). Stimulated by this challenge and inspired by the work of Brousseau (1997), Clare began to enact changes in her practice (2), working on transferring the warrant of authority from her as the tutor to the group of prospective teachers. Since she reported finding this change difficult, at times, to maintain, acknowledging that she sometimes reverted to her established ways of working, it did not yet appear to be sufficiently secure to be categorised as a growth network.

The third instance that Clare described, encapsulated in Fig. 14.2c, was, however, undoubtedly an example of sustained growth, prompted not only by shared discussion of particular readings within the MTEd programme but also by another stimulus within the external domain: an inspection visit from Ofsted (the national regulator for initial teacher education). Over the course of the MTEd, Clare had become aware of what she referred to as a "key message", derived from her reading and regularly advanced in discussions with other MTEs, that a teacher educator needed to label and make explicit to prospective teachers the decisions that teachers make (1). She began specifically working on making her own decision-making more explicit to the prospective teachers within her seminars (2), drawing on research by Rowland and Zazkis (2013) which identified and examined contingency subject knowledge and stated that there were three potential options for action "at the point of learning": ignore, acknowledge and set aside or acknowledge and incorporate (1). Clare was already considering how to act on this advice when an Ofsted inspection of the PGCE programme that she led gave rise to a recommendation that the PGCE team should "further develop student assessment/differentiation at the point of learning" (3). Clare took two kinds of action in response. The first was to formalise the process of making Rowland and Zazkis' three options clear to students at contingent moments in her own practice (3) – a sustained transformation that she could illustrate (when writing this chapter) with a fresh example. This had arisen in response to a prospective teacher's question about whether the number of lines of symmetry is always the same as the order of rotational symmetry. Clare reported that she took the opportunity to step aside from the mathematics of the seminar by labelling the prospective teacher's question as a contingent moment, listing the three options, and then exploring with the whole group the potential to enhance learning by amending her plans and taking up the new line of enquiry. That is what she then did, continuing to explore the conjecture using a range of examples identified by the students. Further evidence of the enduring nature of these changes, not only in Clare's practice but in that of her colleagues, could also be found in the proposals that the PGCE team presented to Ofsted. Their action plan set out a commitment to the approach outlined above and also highlighted Clare's second response, which was to allocate one of the course seminars to the work of Rowland and Zazkis (2013), requiring students to respond to it with examples from their own practice.

The final example on which Clare reflected was her research and development project. Here, so many changes could be traced moving from one domain to another, mediated in different ways by the processes of enactment and reflection, as shown in Fig. 14.2d, that the establishment of a growth network was in no doubt. The MTEd obviously required Clare to undertake a specific intervention in her practice, an intervention informed by research and systematically evaluated on the basis of a range of evidence. She chose to focus on a long-standing issue of interest, brought to her attention by two particular observations of prospective teachers: one achieving considerable success and the other much less effective. Clare identified that in both cases, the prospective teachers' choice of examples seemed to be critical to the success of the lesson, but she was also struck by the fact that even the successful prospective teacher could not articulate a rationale for her selection. Clare's reflection on these instances (1) led her to the conclusion that an important outcome of teacher education should be appreciation among new teachers of the critical role played by their choice of examples (2). Since this argument resonated with her reading of work by Watson and Mason (2006), Clare turned to resources explored within the MTEd programme for strategies by which she could help prospective teachers to evaluate different examples (3). Here she found two possible ideas: Ma (1999) suggested to her the potential of comparing textbooks as a way of helping prospective teachers to look systematically at the choice of examples and their effects, while Paine (2002) alerted her to the value of a collaborative approach. Clare therefore began to experiment (4) with a strategy of comparing examples, examining how the responses of a group of prospective teacher volunteers changed over the course of 6-hour-long sessions. In light of the data she collected, she judged that using textbooks in this way did indeed help students to look systematically at examples (5). This finding led her to conclude that ideas recently advanced by Cambridge Assessment (2014, 2016) and the National Centre for Excellence in the Teaching of Mathematics about the value of a good textbook were valid (6), while her observation of the ways in which the students supported one another convinced her about the value of working collaboratively. In reviewing the process, Clare also reflected that the approach (required by the MTEd) of looking systematically at how her students were responding gave her a much better understanding of those students (7). The value of the knowledge she had gained by attending so carefully to their responses prompted her to continue using the same approach even when the project was over (8). She also remained committed to using textbook comparisons as a way of helping prospective teachers to consider the quality of the examples that they were planning to use and of giving them good models of well-chosen examples.

14.6.2 Colin

In looking back over the course of the MTEd, Colin cited fewer specific instances of learning than Clare, with less evidence of sustained changes to his practice. With the exception of his research and development project (designed in response to a very particular practical concern), the stimuli for his learning were strongly derived from particular readings to which he was referred within the MTEd. The fact that the changes he described tended to follow the implicit model of teacher development – originally challenged by Guskey (1986) and subsequently by Clarke and Hollingsworth (2002) – that is, beginning with changes in knowledge and beliefs and sometimes not moving beyond them (as illustrated in Fig. 14.3a), is essentially unsurprising. Colin had quite consciously embarked on the Master's programme in order to engage with research literature and theoretical insights new to him that would support a more reflective approach to his practice. He drew specifically on Cochran-Smith and Lytle's (1999) distinctions between different kinds of teacher knowledge to explain that the tasks and assignments played an important role in developing his knowledge *of* practice: prompting (or rather "forcing") him to investigate, interrogate and interpret his beliefs and current practice, even if this process did not lead directly to change. Indeed, even when the stimulus provided by particular readings was reinforced by the distinctive features of his own professional environment as a MTE (as illustrated in Fig. 14.3b), Colin still did not necessarily know how to respond to the knowledge he had gained. Set readings about the impact of context on teachers' professional growth (1), fused with personal experience of how his own context was constraining his learning (2), led to a new awareness of the issue, but initially to little more than a sense of frustration. Since none of the literature available to him dealt with contexts that were similar to his own, they could not provide him with relevant models.

It was only when stimulated by a very specific problem in practice, in which he invested considerable time and effort in order to meet the requirements of the research and development project, that Colin's learning could confidently be characterised as a growth network (Fig. 14.3c). One stimulus came from a school visit in which Colin's attention had been directed to concerns about the practice of a recently appointed mathematics teacher. In response, Colin chose to observe a range of lessons across the department (rather than simply focusing on the newcomer), and senior colleagues within the school joined him in doing so. Their observations prompted a request for further support, giving Colin scope to act on previous reading within the course that had struck him as significant (1) about the factors that

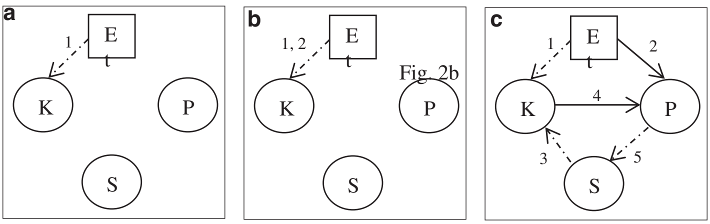

Fig. 14.3 Instances of change in Colin's experience analysed in relation to Clarke and Hollingsworth's (2002) interconnected model of professional growth (E external source of information or stimulus, K knowledge beliefs and attitude, P professional experimentation, S salient outcomes)

impact on teachers' capacity to act in the moment – including "noticing" (Mason, 2002). With encouragement and advice from his supervisor, he set up a video-based, professional development programme for this group of teachers (2), focused essentially on the quality of their interactions in the classroom. Having been prompted by the research and development project requirement within the MTEd to set up this kind of sustained intervention, Colin could now exploit what he had learned about the influence of context (3). Of the aspects identified by Askew (2012) as important in planning for communities of mathematicians – "task, tools and talk" – the last was the only one that he could seek to manipulate (4). The curriculum was tightly prescribed by the Ministry of Education, and the dominance of the textbook was such that (unlike Clare) he could not see any realistic prospect of inviting teachers to reflect critically on the tasks that they set.

The Ministry of Education also disrupted the schedule that Colin had planned for his intervention by moving forward the scheduled end-of-grade exams. As a result, five of the seven teachers who had participated in the first phase of his project, reflecting on their current beliefs and practices in relation to questioning, dropped out before the second phase – the intervention itself. This would have involved video analysis of an alternative, dialogic approach; participation as learners in such an approach; discussion of how they might apply it; and then the opportunity to teach their own class using such ideas and to reflect on it together, using video recording. Even for the two teachers who agreed to continue their participation, Colin had to reduce the planned programme, from 14 hours spread over 2 months to 8 hours spread over 3 weeks. While he was undoubtedly able to learn a great deal from his initial observations of seven teachers about the nature of beginning teachers' questioning (all of those invited to take part were untrained teachers in their first year of practice), he could only examine and reflect on the impact of his intervention as it was experienced by two of the teachers (5). This examination was enough, however, to demonstrate both the kind of changes that were possible and to identify possible causes of the difficulties that persisted, convincing Colin of the value of the way in which he had worked (6).

14.6.3 Jen

Since it was only Jen's research and development project that focused on her work as a MTE, we obviously cannot compare its impact on her professional growth in that role with that of other tasks and set readings. There is, however, no doubt that her engagement in this final project, as summarised in Fig. 14.4, constituted another powerful growth network. We must, of course, acknowledge that analysis of change is somewhat complicated by the fact that Jen had not previously been assigned any formal role as a MTE. She thus had no established practice on which to reflect or with which to experiment, and the vast majority of the action that she took within her new role was construed as part of her project.

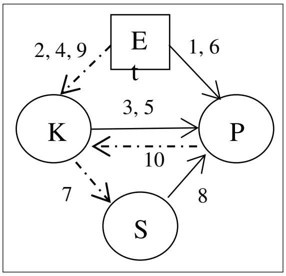

Fig. 14.4 Instances of change in Jen's experience analysed in relation to Clarke and Hollingsworth's (2002) interconnected model of professional growth (E external source of information or stimulus, K knowledge beliefs and attitude, P professional experimentation, S salient outcomes)

The specific focus for her intervention was identified not by Jen but by senior leaders in the primary school, and their choice clearly constitutes an external stimulus. They had highlighted fractions as a topic that seemed problematic across the age range: pupil outcomes as they were formally measured were lower than for other topics, and staff reportedly found fractions challenging to teach. While Jen's position meant she had to respond to the senior leaders' recommendations (1), she welcomed the focus on fractions, a topic in which she had a strong personal interest, based on her experience with low-attaining pupils in the first 2 years of secondary school.

Many of Jen's decisions about how to act were informed not by her reading but by her previous experience as a participant or subject (rather than the leader) of various professional development initiatives. Anxious not to impose on the teachers, she chose to run training sessions within "directed time" (i.e. time that had already been formally assigned to continuing professional development) so that it would not represent an additional burden. She also insisted on working with all teachers in the school, giving teachers across all year groups direct access to the same theoretical and practical input and scope for discussion. Here she was acting on (3) insights from research findings encountered in the course of writing her literature review (2), about the "watering down" that can occur when expert teachers who have received training are expected to share their new knowledge and practice with others (Bobis et al., 2005) While the intervention was therefore mandatory for all staff, participation in the second research phase – which involved video capture of a lesson taught soon after the training and discussed, subsequently, along with artefacts such as the teacher's slides and the pupils' work – was voluntary. In making this decision, Jen was again acting (5) on insights she had gained from Clarke and Hollingsworth's (2002) model itself (4) about the importance of teacher agency.

Jen's decision to use video as both a research and a professional development tool represented a pragmatic response to the fact that she was only in the school 1 day each week and all mathematics lessons were taught simultaneously. But it was also, perhaps, no coincidence that Jen's supervisor was engaged in her own research about the role of video in supporting teachers' professional learning (6). It was, however, Jen's own reflection (7), as a secondary practitioner, on the importance of the understandings about fractions that are established at primary level that prompted her to focus the training that she offered (8) not merely on classroom interactions but more emphatically on the nature and structure of the curriculum and the teachers' choice of tasks. She particularly emphasised the need to build deeper conceptual understanding through multiple representations and experience with different sub-constructs. In developing this vision of her desired outcomes, Jen was again influenced by the reading about fractions to which she was directed by her supervisor (9). A number of readings, most notably about the "knowledge quartet" elaborated by Rowland, Huckstep and Thwaites (2005), informed her observation and evaluation of the teachers' practice and (more indirectly) her reflection on the value of the strategies she had employed (10).

In her written reflections for this chapter, Jen tended (perhaps unsurprisingly) to focus on what the primary teachers and school leaders had learned from the intervention, noting the school's commitment to a new "mastery" approach to mathematics teaching for the following year. Nonetheless, her own professional growth as a MTE was evident from recommendations she went on to make about the nature of the school's future CPD provision. It was also reflected in her school's decision to second her full-time for another year to the primary school, both to continue the work in relation to mathematics teaching and to implement specific kinds of inquiry-based professional learning for all newly qualified teachers.

14.7 Reflections on the Interactions Between Different Contexts and Their Impact on Individual MTEs' Learning

Our use of Clarke and Hollingsworth's (2002) interconnected model to analyse Clare, Colin and Jen's experience has helped to focus attention on the varied kinds of external stimuli that arise in the context of MTEs' work as well as those that derive from professional Master's programmes. Its application has also allowed us to distinguish between potentially short-lived change sequences and more enduring growth networks. It has first demonstrated that sustained growth can result from many different kinds of stimuli (ranging from prescribed readings of research literature, through focused guidance from knowledgeable supervisors, to the assumption of a new role or the imperious demands of Ofsted) – depending on the individual and their purposes and on the nature of the change environment in which the stimuli are encountered. It has also shown, however, that the particular demands of a research and development project (i.e. a deliberate intervention undertaken with a commitment to inquiry and evaluation) make genuine growth much more likely. As a nervous newcomer, taking on an unfamiliar and under-theorised role, Jen found sufficient structure and support within its framework to enable her to develop and evaluate a new approach to professional learning, subsequently embraced by the whole school. The same framework gave Colin, a much more experienced and senior figure, the impetus necessary to find a new way of working around long-standing obstacles to providing more focused, sustained provision.

Where the interconnected model is perhaps slightly less illuminating as a theoretical framework for analysing the relationship between the different contexts in which MTEs' learning occurs is in the clear distinction that it seems to draw between the professional growth that occurs and the wider change environment within which that growth happens. Although the external domain is represented within the model both as a source of stimuli and as a domain that may be changed as the result of deliberate action (enactment) informed by new knowledge, beliefs or attitudes, the model does not explicitly acknowledge the fact that such changes may, in turn, actually alter the "change environment" itself. Within the model the latter is simply presented as the context within which the growth is happening, a context which shapes and constrains the nature and extent of the changes that can occur, but that does not seem (at least, as it is represented in Fig. 14.1) to be susceptible itself to change. It is for this reason that we turn in the final stages of our reflections to an alternative analytic tool that others have found valuable in relation to the professional growth of MTEs: "zone theory" as developed by Valsiner (1997), particularly as it has been applied by Goos (2013). As she has argued, the particular "contribution of zone theory is to permit analysis of [the] interactions between people and their environments while still emphasizing individual agency" (Goos, 2013, p.523). In focusing specifically, as we have done, on the interaction between the two different contexts within which each MTE was working, it is important not to overlook the agency that each exercised in reshaping those contexts. The interplay that occurred did not simply take place between the specific contexts of each MTE's professional work and their particular Master's programme, nor did it operate as a one-way process, with the contexts essentially determining the outcomes. Those outcomes were negotiated by the MTEs in the interaction that occurred between the two contexts.

This constant interplay and the process of negotiation can be illustrated with particular reference to the dynamic and interrelated complex created by what Valsiner has described as the "zone of free movement" (ZFM) and the "zone of promoted action" (ZPA). In the cases presented here, the ZFM can readily be identified with the MTEs' professional work contexts: the well-established practices and cultural norms that shaped others' expectations of them or, for example, the curriculum and assessment practices that constrained their scope to introduce new practices. The context of their Master's programmes can, within this theory, be similarly equated with the ZPA. The set readings, taught seminars, investigative tasks and final research and development projects all served to promote alternative ways of interpreting or responding to their teachers' (or prospective teachers') particular needs. As Goos has emphasised, however, it is not helpful to regard the ZFM and ZPA as two distinct entities. Valsiner (1997) actually regarded them as "fuzzy abstractions without sharp or continuous boundaries", subject at any moment "to further transformation" (Goos, 2013, pp. 523–4). That transformation might derive from an external source – such as the Ofsted inspection of Clare's PGCE programme – but it might equally arise from within an individual. The most obvious illustration of this fact is the decision made by each of the MTEs to enrol on a Master's programme: a decision deliberately taken (particularly by Clare and Colin) to provide them with the stimuli and the structured support necessary to reshape their contexts. They hoped that it would give them licence and, indeed, a warrant to undertake new practices and thereby to think of themselves differently. The status of the programme and its academic credibility also conferred a high degree of authority on them, making experimentation not just possible but well regarded. Clare was proud to be able to defend her action plan to Ofsted on the basis of its warrant within the literature.

That is not to suggest that the process of transformation was easy. Particularly for those already employed as teacher educators, choosing to assume the identity of a learner inevitably gave rise to complex and nuanced shifts in identity. Clare was deeply unsettled at first by the challenges to her long-held assumption that pupils could legitimately be urged to act as mathematicians. Colin faced huge – and in some cases insurmountable – difficulties in accommodating the programme's demand for a sustained intervention over time. But in both cases, these were the "productive tensions" described by Valsiner (1997) that give rise to self-initiated change and serve, in turn, to transform the "change environment" itself. While Colin failed to enact the video-based professional development programme that he had planned with all seven original participants, those who withdrew in the face of the new ministry-imposed examination schedule agreed to undertake the course with him the following semester. He had succeeded in creating new ways of working as a MTE.

Some contextual constraints, of course, cannot be overcome – as Colin conceded in choosing to focus on talk rather than tasks or tools – and thus will always shape what it is possible for MTEs to learn, no matter how well-structured or flexible and responsive their professional development programme might be. Yet the very process of embarking on a Master's degree can bring about change within both the ZFM and the ZPA, opening up new powerful new possibilities for individual MTEs. While the status of the degree may play an important role in this respect, the examples that we have presented here suggest that, within a Master's programme, the particular demands of a substantial research and development project appear to offer the best prospects for genuine professional growth.

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Chapter 15
Mathematics Teacher Educators' Learning in Supporting Teachers to Link Mathematics and Workplace Situations in Classroom Teaching

Dionysia Bakogianni, Despina Potari, Giorgos Psycharis, Charalambos Sakonidis, Vasiliki Spiliotopoulou, and Chrissavgi Triantafillou

15.1 Introduction

The chapter aims to provide insight into the development of mathematics teacher educators' (MTEs') professional learning by reflecting on their attempts to facilitate primary and secondary school teachers' professional development in the context of a European-funded project, Mascil (Mathematics and Science for Life). The Mascil project brought together 18 partners from 13 countries in order to promote inquiry-based learning (IBL) and connect school mathematical activity and authentic workplace situations (Mascil project, 2013, https://mascil-project.ph-freiburg.de/). To achieve these goals, in Greece, professional development (PD) activities were designed where science and mathematics teachers collaborated in groups to design, implement and analyse lessons in the spirit of lesson study approaches (Hart, Alston, & Murata, 2011). MTEs were predominantly academic researchers, teachers with Master's studies or school mentors in mathematics or science education.

The major challenge of the project for the MTEs' group was to link workplace situations with mathematics teaching in the context of PD activities. Although workplace settings can be seen as rich and meaningful contexts for students' mathematical understanding (e.g. Hoyles & Noss, 2001; Wake, 2014), connecting these contexts to classroom teaching appears to constitute a complex task for mathematics teachers (Nicol, 2002; Potari et al., 2016; Triantafillou, Psycharis, Potari, Zachariades, & Spiliotopoulou, 2017). Additionally, the linkage of workplace situations and

D. Bakogianni · D. Potari (✉) · G. Psycharis · C. Triantafillou
National and Kapodistrian University of Athens, Athens, Greece
e-mail: dpotari@math.uoa.gr

C. Sakonidis
Democritus University of Thrace, Komotini, Greece

V. Spiliotopoulou
The School of Pedagogical and Technological Education (ASPETE), Athens, Greece

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_15

mathematics teaching has been studied mostly in vocational school settings (e.g. Bakker, 2014). Thus, very little is known as to how innovations of that kind can be introduced into existing educational contexts and teaching realities. As regards the Mascil project and the Greek educational context, the formalistic view of the official curriculum, the lack of accessibility to workplace settings and resources and teachers' unfamiliarity with inquiry approaches constituted some added concerns for the national MTE group. Moreover, the collaboration between participants from different teaching subjects, although this might be seen as creating meaningful PD opportunities for all of the participants (Frykholm & Glasson, 2005), was a rather demanding task for the MTEs' group to deal with in the Greek educational context, where cooperation among teachers is not encouraged. The MTEs' own professional differences in terms of research, teaching and experiences in educating teachers provided some extra challenges that the MTEs' group had to deal with.

All the above complexities that stem either from the project itself or the Greek educational context provide a challenging site for MTEs' learning, the study of which can contribute to the growing research field focusing on forms of knowledge, competencies and challenges related to MTEs' practice and development (Ball & Even, 2009; Jaworski & Huang, 2014; Jaworski & Wood, 2008). The present study aims to trace the path of a group of MTEs endeavouring to support teachers through PD activities to employ inquiry-based teaching approaches targeting the connection between mathematics and workplace situations. Particularly, we want to investigate the following research questions:

What are the MTEs' concerns expressed in the design and enactment of the PD activities?

What emerging tensions were faced by the MTEs, and how did these tensions contribute to their professional learning?

15.2 Literature Review and Theoretical Background

Facing the challenge of supporting the Mascil project ideas in the Greek context, our work is framed by the term reflective practitioners (Shön, 1987) in two directions, namely, examining the role of teachers as well as the role of MTEs in the development of teaching practice. We view teachers as co-producers and consequently co-responsible in the research process as well as in the development of scientific knowledge (e.g. Ponte & Chapman, 2006). Teachers in this respect are key stakeholders (Kieran, Krainer, & Shaughnessy, 2013), advancing their role to be informed by research findings, to design and evaluate teaching material, to investigate their own practices and to use their own teaching experiences to produce new research findings. Such a demanding role can be cultivated and supported in collaborative contexts, where mathematics teachers, or mathematics teachers and researchers, co-learn in developing teaching practice, such as communities of inquiry (Jaworski, 2006; Potari, Sakonidis, Chatzigoula, & Manaridis, 2010), lesson study (Huang & Bao, 2006) or action research settings (McNiff, 2010).

Jaworski and Huang (2014) emphasised reflective practice as a principal goal for effective development of both mathematics teachers and mathematics educators. They also discussed the competences that mathematics educators need in order to be reflective in what they do. Such competences include being self-aware, reflective and articulate in action and able to explain tacit knowledge of teaching but also comprehensive, rich and deep knowledge, based on theory and theory testing in practice (Smith, 2005). Moreover, it is important for MTEs to develop adaptability; to cope with problems, dilemmas and problem situations; to select and use appropriate tools and resources for teaching; as well as to learn from the study of practices (Zaslavsky, 2008).

An important part of the limited research focusing on forms of knowledge, competences and challenges that are related to professional learning of MTEs is based on contexts where teachers and researchers are collaborators and co-learners in developing opportunities for students' learning and additionally on large-scale programmes of teachers' PD. To bridge the gap between research findings and the actual needs of teaching practice, Goos (2014) reported on ways in which this can be achieved by teachers' and researchers' collaboration. She considers mutuality and complementarity as central in developing expertise within the communities of both teachers and researchers and hence theoretical and practical knowledge in mathematics education. Potari et al. (2010) reported on conflicts and tensions in a 4-year collaboration between secondary school teachers and academic researchers that gradually led to an apprenticeship of both groups in inquiring into mathematics teaching and to a self-understanding and reconceptualisation of mathematics teaching and PD. Research findings thus indicate that collaboration between researchers and teachers, despite constituting a fruitful ground for the professional learning of both groups of participants, is also a terrain of continuous challenges and emerging demands that need to be addressed by the community of academic researchers. The increased demands concerning MTEs' practice have turned the lens of research towards their own professional development (e.g. Krainer, 2008). However, studies that provide relevant empirical evidence related to the development of learning in the case of mathematics teacher educators are still scarce.

This chapter aims to contribute to this open and unexplored field of discussion concerning the development of MTEs by investigating their professional learning as teacher educators in the context of an innovative project.

When MTEs and mathematics teachers collaborate to develop teaching, each brings to the emerging community new forms of mathematics learning and teaching discourse and practice. MTEs might bring the critical and reflective stance and modes of discourse that are valued within the academic community, whereas teachers can bring craft knowledge about pedagogical practices and the sociocultural contexts of their classrooms. Together, these two groups of participants can learn new ways of thinking about their practices and simultaneously create new forms of discourse and practice about mathematics learning and teaching, that is, new communities. These communities, while potentially powerful tools for developing pedagogical practice, may also introduce tensions into the PD experience. These tensions are often due to the mismatching and even conflicting goals of the practice itself but also of the activity within which the “old” and the emerging communities are situated. Focusing on the different goals of the practice, sources of tensions can be identified in the participants’ efforts to align their practice, while taking the view of the different goals of activity, tensions can be traced in participants crossing boundaries between different practices (Wenger, 1998).

Boundaries are dynamic constructions denoting co-location of practices and co-existence of competing discourses. Efforts by individuals or groups at boundaries to restore continuity in action or interaction across practices trigger dialogical engagement and collective reflection, compelling people to reconsider their assumptions and look beyond what is known and familiar (Akkerman & Bakker, 2011). Through collaboration/negotiation at boundaries between different practices, new and hybridised ideas and practices emerge where mutual understanding of shared tasks and problems develops (Edwards & Fowler, 2007). Described as boundary crossing (Engeström, Engeström, & Kärkkäinen, 1995), this process involves moving into unfamiliar territories and requires cognitive retooling. People who cross boundaries are called brokers or boundary crossers, and they are simultaneously members of multiple communities, while objects that cross boundaries are called boundary objects (Akkerman & Bakker, 2011). These objects can be, for example, curriculum materials, representations, school or workplace records that facilitate interactions and crossings at the boundaries.

Boundary crossing between different practices is seen as a way to address learning through four mechanisms: identification, coordination, reflection and transformation (Akkerman & Bakker, 2011). These mechanisms concern the different ways in which learning can occur when people interact with, move across and participate in different practices.

1. Identification: Boundary crossing can lead to a renewed insight into what the different practices concern.
2. Coordination: Boundary crossing can also lead to establishing minimal routine exchanges between two practices so as to facilitate transitions.
3. Reflection: Reflection involves going deeper into the specificities of two practices (perspective-making) and learning to consider one practice by taking on the perspective of the other practice (perspective-taking).
4. Transformation: Transformation leads to changes in practices or even the creation of a new practice that stands between the established ones.

15.3 Methodology

15.3.1 The Context of the Study

The context of this study is the European project Mascil aiming at supporting teachers in using IBL and workplace situations in mathematics and science teaching. In Greece, 11 MTEs (academic researchers, teachers and mentors) with different research and teacher education experiences worked for 1 academic year with 13 groups each comprising about 10 mathematics and science practising teachers (1–2 groups for each educator), who were meant to collaborate in developing shared teaching experiences. The PD activities aimed at promoting both the development of teaching in the direction of the innovative ideas of Mascil, as well as teachers' continuous reflection. Instructional materials in the form of exemplary tasks were provided by the project as a basis for teachers' designs (http://www.fisme.science.uu.nl/publicaties/subsets/mascil/). These tasks were available to the teachers through the project website. However, the teachers could modify them according to their teaching goals or even design new ones aligning with the same philosophy. MTEs could also use a teacher education toolkit provided by the project involving ideas and strategies for organising the PD activities. MTEs used this tool as a resource to design the PD activities, especially during the initial meetings with the teachers. In addition, a communication platform for teachers was available, although this was not widely used in Mascil implementation in Greece.

Overall, the Mascil project aimed to offer professional development to a large number of mathematics and science teachers in the participating countries. Most of the developed resources were translated into the language of each country. Although the project had specific goals, as mentioned in the introduction to this chapter, teacher educators and teachers in every country were flexible in using these resources and adapting them to their national educational context.

15.3.2 The Group of MTEs

In this chapter, our focus is on the group of MTEs of which the authors were members. The profile of each participant is briefly presented in Table 15.1. Sophia was the coordinator of the programme.

Although the group of MTEs consisted of science and mathematics teacher educators, we refer to them as MTEs due to our special focus on mathematics teaching practice.

MTEs collaborated for a period of 1 academic year (October 2014 to June 2015) to develop a mutual plan for the PD activities. We collected data consisting of audio and video recordings based on MTEs' discussions in their meetings (five in total lasting about 3 hours each). A brief description of the focus of the discussion in each meeting is presented in Table 15.2.

15.3.3 Data Analysis Process

The analysis of the data was based on grounded theory approaches (Charmaz, 2006), and it was carried out in two steps. Firstly, following an inductive content analysis approach, we investigated the main concerns of MTEs and issues triggering the group's attention and described them through a systemic network (Bliss, Monk,

Table 15.1 MTEs' profiles

Participants	MTEs' professional status	Research interests
Sofia	University teachers	Development of mathematics teaching and learning and teacher development with experiences on the use of mathematics in workplace situations and its transfer into mathematics teaching
Tim		Teaching and learning calculus in secondary and undergraduate education
Jason		Design of learning environments for mathematics with the use of digital tools and IBL approaches
Ben		Development of mathematics teaching and learning and teacher development in primary education
Elsa		Development of science teaching and learning and teacher development
Anna	Postdoctoral researcher in mathematics education	Use of mathematics in workplace situations and its transfer into mathematics teaching
Diana	PhD student in Mathematics Education	Development of teaching and learning of statistics in secondary education
Ken	Mentor (public schools' advisor offering practice-based professional support to teachers at school)	Mathematics learning in primary and secondary education
Marko and Chloe	Secondary mathematics teachers with Master's degrees in Mathematics Education	Curriculum development and action research
Rose	Secondary science teacher with Master's degree in Science	

Table 15.2 Brief description of the MTEs' meetings

Meeting	Main focus of the discussion
First	Familiarisation with the Mascil ideas and development of resources for the introductory meeting with the teachers
Second	Sharing insights from PD experience and adjusting the PD design
Third	Developing structures to facilitate teacher collaboration and co-design
Fourth	Sharing insights from the teachers' classroom implementations and developing materials to promote teacher reflection
Fifth	Connecting the Mascil ideas with the actual practice. Issues related to the classroom reality, the Greek context, the PD aims, the project's sustainability

& Ogborn, 1983). The network presents the different dimensions in the emerging concerns that co-exist throughout the MTEs' discussions; some of them appeared early, others later.

In the second step of the analysis, we identified tensions inherent in various categories of concerns in the systemic network. Tensions indicated either explicit divergent views among the MTEs or dilemmas implicit in these views. Considering that the identified tensions indicated a boundary, each tension was described, coded and traced in the data in different instances in which it appeared, and it was characterised in terms of the participants, the practices involved and boundary objects. In the present paper, we present two dominant tensions throughout MTEs' discussions: (a) authenticity of workplace situations versus classroom teaching and (b) high versus low degree of teacher autonomy. Next, we coded the process of MTEs' dealing with the boundaries inherent in these tensions by using the four types of learning at the boundaries.

15.4 Results

15.4.1 MTEs' Concerns

Figure 15.1 shows the categorisation of MTEs' concerns. Two categories appear: making sense of how workplace situations and IBL can be linked to mathematics teaching and the enactment of workplace situations and IBL in PD meetings. The categories and subcategories are discussed through illustrative examples below.

15.4.1.1 Making Sense of How Workplace Situations and IBL Can Be Linked to Mathematics Teaching

How to link workplace situations with mathematics teaching was a central issue in all MTEs' meetings, while IBL was discussed in a less extensive way. Workplace situations were seen by some MTEs through the use of tasks based on realistic or scientific contexts: "The ideal would be to have workplace situations related to physics or chemistry and to be able to solve problems in this area and somewhere there will be mathematics." (Jason). Some MTEs with experience from research on mathematics in workplace situations emphasised the need to sustain workplace authenticity in the classroom. Nevertheless, a number of issues that need to be considered seriously when exploiting workplace situations for classroom teaching emerged. One concern related to the complexity of the workplace context indicated by the unfamiliarity of context, representations, symbols and language. For example, Ben argued that "The student needs to learn extra things from the workplace context". Anna addressed complexity and limited accessibility of the workplace context: "We said that these authentic examples take you out from what you are

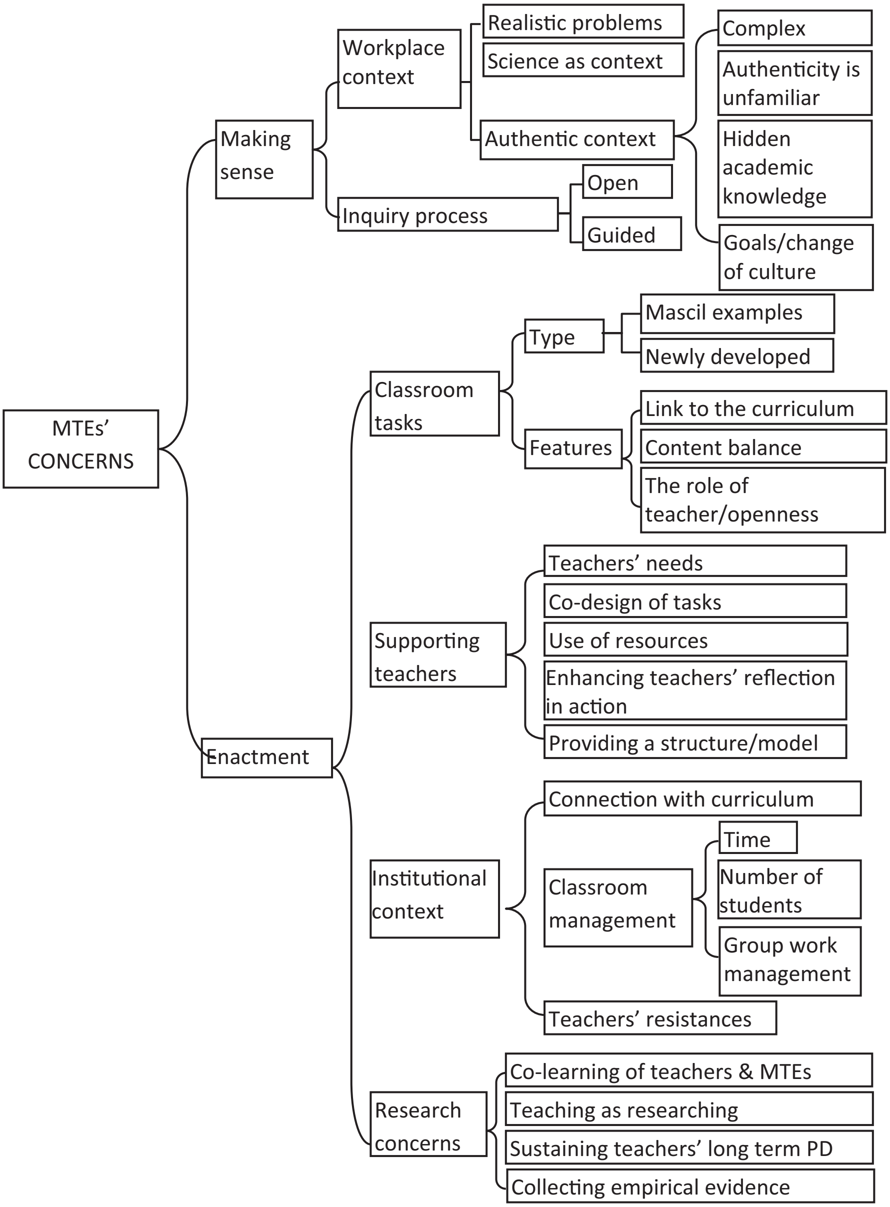

Fig. 15.1 Concerns that emerged during MTEs' meetings (the Bar ([) notation signifies that all the categories are mutually exclusive, while the Bra ({) notation signifies that the categories might co-exist)

familiar with. You see it and you say that I do not want it. Why do I need to understand what they say here?" She also pointed out that mathematics is hidden in the workplace context, but she considered it as a challenge to inquiring into mathematics: "The workplace setting, because it hides a lot of academic mathematics, it gives itself elements of inquiry. This hidden thing helps the inquiry. What is hidden? Why does this relation hold?" Yet another concern was the distance between the workplace culture and the established culture of school mathematics teaching, the former being seen as inferior to the latter: "What shall we do when the teacher says that the tasks that you give me are technical things? I have high goals for my students" (Marcos, 1st meeting).

IBL was considered a familiar construct both for MTEs and teachers: "IBL is more familiar to teachers and teacher educators than workplace settings. Thus, it gives us a basis for developing our PD activities" (Marcos, 1st meeting). This explains the limited focus on IBL in the initial meetings of MTEs. It was initially seen through the use of open tasks and then more related to workplace situations as the process of discovering the hidden mathematics.

15.4.1.2 Enacting Workplace Situations and IBL in PD Meetings

MTEs' concerns in the design and implementation of the PD programme were related to the use of classroom tasks, ways of supporting teachers, the role of the institutional and classroom context and the MTEs' research focus.

The nature of classroom tasks became the focus of the discussion from the beginning, referring to the type and the features of the task. MTEs wondered to what extent the tasks developed in the context of the project could be used in PD meetings and in the classroom. The example below illustrates the above concern: "Even in Mascil tasks, the workplace context is not integrated in a realistic way. It is role playing. In a few cases where the workplace context appears in a realistic way, it seems to exist as an idea" (Diana, 2nd meeting).

Another concern was whether the teachers themselves could develop their own tasks aligned to the project's perspective. The example that follows reveals MTEs' exchange of ideas to motivate teachers in developing their own tasks.

> Teachers in my group proposed a task referring to factors that are related to AIDS. One science teacher sent me some ideas, but he was not able yet to propose a specific task… How can we support teachers to complete their own designs based on contexts that are not familiar to us? (Anna, 4th meeting)

The link between the tasks and the curriculum; the content balance between science, mathematics and workplace situations; and the role of the teacher in designing authentic or open tasks were concerns addressed in the discussions. The following extract illustrates the above concerns:

> When I proposed the Photovoltaic task [an exemplary Mascil task] in the first PD meeting, the mathematics teachers were very negative in using this task… they could not see any mathematics there. In the second meeting though, a science teacher proposed some very nice ideas about this particular task (Tim, 4th meeting)

Supporting teachers in the PD meetings was another concern of the MTEs throughout the discussions. Identifying teachers' needs, and supporting them in the design of tasks and lessons, we considered what kind of literature readings and specific examples from the workplace setting could be helpful. Also, we cared about promoting teacher collaboration in PD meetings and especially the co-design of tasks between science and mathematics teachers. The following example is characteristic of how co-design could result in an interesting experience for the participating teachers:

> The Earthquake task designed together by mathematics and science teachers indicated how mathematics is used for the study of earthquakes. In this task the students had to play the role of a seismologist responsible for studying the main features of a specific earthquake, for example the epicentre. (Jason, 5th meeting)

After teachers' initial classroom implementation, MTEs' concerns were related to how to enhance teachers' reflection on teaching. A characteristic example is Sofia's concern for supporting teachers' reflection:

> I asked teachers to present the reports from their lessons in the meetings. What they wrote was somehow descriptive, I posed questions on what they noticed… But finally the discussion was between them and me…. so, I suggested them some research articles on teacher noticing (Sofia, 5th meeting).

Providing structures/models was a basis for mathematics teachers to make sense of the IBL dimension of teaching and of its connection to workplace settings. The example below is characteristic of the above concern:

> It is very difficult to identify the mathematics in an authentic context. But when teachers start from a workplace situation I insist to return to it at the end of the lesson. To follow the process of modelling, I return to the context, I reflect on it and I move forward … is the solution we found reasonable? (Anna, 5th meeting)

Institutional factors were addressed through MTEs' interaction with the teachers in PD meetings and in the school. Connections to mathematics curriculum and classroom management (short teaching time, large number of students in the classroom, complexity of the group work setting) were concerns that emerged and were debated. Many of these concerns were also expressed by teachers indicating their resistances to designing and using inquiry and authentic tasks in their classroom. In the following extract, two MTEs discuss the connection of a specific task with the curriculum on the basis of teachers' expressed doubts about the appropriateness of the task.

> Anna: They [teachers of group 10] are working now on a new task. I suggested to them, "the tournament of ping-pong". I found it in the Mascil toolkit. You see, teachers in vocational schools find Mascil activities as complex and they look for something simpler… I consider this is a good example…
>
> Chloe: This task has been also considered by the teachers in my group. However, during the discussion they claimed that it is not related to mathematics at all…, they considered it as a quiz and not connected to the school curriculum. They said that they could use it in the future when combinatorics will be taught. I liked it and I spent time on it, but when I discussed with teachers, all of them were very negative, asking what mathematics are involved in this? (4th meeting)

Issues central to research in mathematics teacher education beyond the specific project emerged in almost all the meetings and guided MTEs' actions. These issues were teachers' and researchers' collaboration, the role of teacher as researcher and the sustainability of teachers' professional development. The following extracts illustrate two of the above concerns:

> The involvement in supporting the teachers was a learning experience, teachers, educators and students, we are all learners. This is what we are doing. We are learning how to communicate (Chloe, 4th meeting).

> Teachers have to be reinforced to communicate through the platform between themselves… to inquire by themselves… to search for resources (Sofia, 5th meeting).

Finally, the data collection process (i.e. observation of PD meetings and classroom sessions, interviews with teachers and artefacts produced by students and teachers) was another of MTEs' concern throughout the meetings. This empirical evidence was important to our practice as researchers and teacher educators.

15.4.2 Tensions and Attempts to Deal with Them

In this section, we will describe and analyse two emerging tensions. The first one is related to the role of the authenticity of workplace situations in mathematics teaching, while the second one concerns the degree of guidance offered to teachers in different phases of the PD activities. Below, we exemplify these tensions through different instances of PD meetings indicating the boundaries that were encountered, the practices that were involved and the boundary crossing that occurred.

15.4.2.1 Tension: Authenticity of Workplace Situations Versus Classroom Teaching

In the first three meetings, the MTEs attempted to conceptualise the workplace-related innovation and think of ways of introducing it to the teachers. Divergent views were expressed as regards the potential of authenticity in workplace-based classroom tasks. Supporting views considered the importance of using authentic tasks in mathematics teaching as a means to promote inquiry, motivate students and develop students' mathematical meanings through rich representations. The rather sceptical views concerned the complexity of the workplace context, the different epistemological nature of school and workplace mathematics and the pedagogical difficulty of linking these two in the context of PD and mathematics classroom. For example, Chloe, a mathematics teacher, supports the use of workplace situations in teaching as a basis for inquiry: "Since the workplace context hides a lot of academic mathematics, it gives itself elements of inquiry". On the other side, Tim, a mathematician and mathematics education researcher, sees inquiry in mathematics and mathematics teaching as not necessarily related to workplace context: "Inquiry in mathematics does not necessarily involves a realistic context". Similar debates took place throughout the first two meetings.

In the following extract, we see a debate between Sofia and Ben. Sofia supports the view that workplace contexts can promote students' understanding, while Ben points out the complexity of the workplace context to an outsider (teacher or student). In particular, Sofia emphasises the importance of linking informal and formal learning and the flexibility of representations of workplace situations and practices that can be compared with the formal mathematical representations; so in this way, students' informal activity can gradually be mathematised and eventually lead to more formal mathematical activity. Ben challenges this development by arguing on the complexity of the workplace context:

> Sofia: First, the students can see a flexibility in the representations which can be found in the workplace context and through the connection between the formal and the informal mathematics that the workplace context offers, maybe develop more flexible problem solving strategies and decision making.

> Ben: What I do not understand is in what ways the school can exploit the informal knowledge for making connections and build coherent mathematical knowledge. Do we have some tasks?

> Sofia: I think that we have. When we say that the knowledge is hidden and the formulas and the symbols are different, it gives me the opportunity to discuss in what ways the typical formula differs, and this helps me to get the meaning of the formula. The formula is not something else than an expression of a relation. (1st meeting)

These views appear to be mainly originating from the research and teacher education practices of the participants. The supporting views about the connection between workplace situations and classroom teaching were mainly expressed by participants who were members of the research community involved in projects related to mathematics and science at work (Anna, Elsa, Sofia). Views doubting this connection were expressed by participants with research on primary mathematics teaching (Ben) and university mathematics teaching (Tim). MTEs supporting the use of authentic workplace situations in the classroom tended to indicate means (e.g. resources) and procedures (e.g. problem-solving strategies) allowing workplace practices and mathematics teaching practices to potentially cooperate efficiently in the classroom. This group of MTEs facilitated boundary crossing between research on workplace mathematics and mathematics teaching as coordination. MTEs questioning the connection of workplace situations to classroom teaching consider the distance between the two practices in terms of their epistemological and pedagogical differences. This engages them in an identification process where MTEs become uncertain of the possibility of crossing the two practices.

The aforementioned tension started to become less distinct in the last three meetings when MTEs interacted with the teachers in the PD meetings and visited mathematics classrooms to observe teachers' implementation of tasks in the spirit of the innovation. Discussing how to support teachers to further develop their teaching practice and to develop professionally led to reformulation of the meaning of workplace and IBL. For example, Ben, who initially doubted the connections of workplace contexts to primary mathematics classrooms, argues after working with the teachers:

> For me, the main issue in the primary school is how to make connections with the world of work. One way to overcome this problem is to look for important things of any human activity. For example, it is very important to search what social workers do to support unaccompanied refugee children … we do not know anything about these children's cultural or mathematical background. (Ben, 3rd meeting)

> In our team… the task was collaboratively devised, taught by one of the teachers and observed by the others. The teachers thought that the process of working together would imitate professionals' collaboration at work to reach an outcome …. Mathematics in human activities and actions that are of importance. Workplace in authentic terms! Back to something that was raised in the beginning (Ben, 4th meeting).

Although Ben belongs to the second group, here he highlights the potential of boundary crossing by broadening the meaning of "workplace" to involve a range of human activities. Building on his research perspective characterised by inclusive mathematics teaching and teacher collaboration, he appears to coordinate the mathematics teaching practice with the workplace context.

Teachers' difficulties in enacting the innovation in the classroom made MTEs aware of the complexity involved in relation to the existing educational context. Even Anna, who was in favour of using authentic workplace tasks in mathematics teaching, appears to reconsider her view in light of the inferiority attributed to practice as against theory in the Greek educational system and the wider society:

> What I understand is that the workplace context does not fit to the classroom's world! It is true that there exists this view in the Greek reality, that is, that the workplace context is a realm of practice far away from school…inferior to it. What a worker does is more practical/ practice oriented… That is, I think it has to do with the whole philosophy of the system, not alone the educational system. This explains why teachers have difficulty to integrate workplace situations into their teaching practice. (Anna, 4th meeting)

Along similar lines Jason, a researcher in mathematics education, was challenged by the teachers in his PD group as to whether authentic workplace contexts can promote challenging mathematical ideas (content) for students:

> …when the teachers raised questions related to whether this is trivial mathematics, I was not sure what to do or how to respond… There are organisational issues here … There is pressure on the teacher educator… I felt that I should provide answers compatible to the innovation but also operational! Hence, the issue of what workplace and IBL is acquiring less importance! (Jason, 4th meeting)

Anna and Jason, with rather little experience as teacher educators, were challenged by two dipoles, theory versus practice and context versus mathematics content, respectively. The PD practice mediated through these dipoles supported them to reconsider the relation between workplace situations and mathematics teaching in terms of systemic and epistemological features (perspective-making). The boundary crossing is also evidenced in the development of their awareness of the complexity surrounding mathematics teaching and teachers' work framed by these features (perspective-taking). It could be claimed that the learning mechanism is evident here.

Summarising, MTEs' attempts to understand the relation between workplace situations and classroom teaching brought to the fore tensions that progressively faded. The tension that we examined here stemmed from the multi-membership of MTEs in current and prior communities (research, teaching, educating teachers), and dealing with it facilitated boundary crossing. Different types of boundary crossing include identification, coordination and reflection, which supported MTEs to develop awareness about epistemological, pedagogical and systemic features shaping the meaning of the innovation.

15.4.2.2 Tension: High Versus Low Degree of Teachers' Autonomy

Although there were not strong divergent views among MTEs promoting opposing ways to work with teachers (high versus low degree of guidance), this tension seemed to underlie MTEs' decision-making. This was evident in the selection of appropriate resources for the teachers, the role attributed to the teachers in task design, the management of the diversity of the teachers' groups and the ways of supporting teachers' reflection. The MTEs were not sure about the level of teacher autonomy in designing tasks and lessons connecting workplace contexts and school mathematics. There was some debate in the group as to whether this responsibility can be given to the teachers from the beginning or if the MTEs should provide in the first PD meetings more direct ways of how this integration can be facilitated. For example, Ken, a school advisor, pointed out teachers' needs for some guidance before being involved in designing tasks for their lessons:

> What will we do if teachers want us to propose to them tasks related to the workplace context? We could discuss with the teachers some of the tasks coming from research and then to start to explore the emerging issues together. This might help them to start to develop some tasks. (Ken, 1st meeting)

Taking a similar view, Chloe suggested the provided Mascil tasks as a starting point in the PD meetings to smooth teachers' engagement in exploiting authentic workplace situations in their designs: "In the first meeting we can start with a Mascil task and in the second meeting we can support teachers to explore more authentic situations" (Chloe, 1st meeting).

In the second meeting, MTEs brought experiences from their first interaction with teachers and reported teachers' preferences to design their own tasks in the spirit of the Mascil project. MTEs started to develop more elaborated ideas about how to support teachers in their attempts to design their own tasks. Jason considers teacher collaboration as an important condition to engage teachers in developing and sharing ideas as a basis for their didactical designs:

> Collaboration is very important. Even if they have initial ideas I do not think that they will have a full idea of what they will finally implement. We [as teacher educators] concentrate on two of the proposed ideas so as the teachers to have time until the next meeting to communicate these ideas. It is not good to provide five strictly defined ideas. I think it is more

> important to cultivate a culture of discussion and communication around the final formulation of the tasks. (Jason, 2nd meeting)

In the above extracts, Ken and Chloe refer to mathematics education research and in particular to workplace mathematics and look for tools that could facilitate boundary crossing between research and mathematics teacher education. In this direction, Mascil tasks or other authentic workplace situations seem to play the role of boundary objects between the research and the teacher education practice. Jason builds on both his research practice and teacher education practice in early PD meetings. He targets the same boundary crossing by suggesting teacher collaboration. In terms of boundary crossing mechanisms, these actions indicate learning through coordination.

Task design or choice and its classroom implementation and/or management were shown to be central components of teaching practice in the first classroom implementations. These implementations were rather informative for MTEs as regards teachers' needs for support to enact their designs to facilitate mathematical inquiry and connections to workplace situations. A rather "instrumental" teaching approach adopted by the teachers became evident, characterised by a vague conceptualisation of IBL and connection to the workplace. It seems that a boundary was raised between the targeted project innovation and the existing teaching reality. This boundary challenged MTEs to reconsider their goals and actions to promote teachers' autonomy. They started to modify their working agenda for PD activities and to appreciate the need to extend the provisional resources, recognising their limited functionality in PD meetings. This became evident in the last two meetings where MTEs were able to describe clearly PD strategies and resources so as to make innovation accessible to teachers through reflection and PD.

For instance, in the fourth meeting, two main literature-driven ideas were discussed and used in developing schemes of action for PD activity: co-learning contexts for teachers and teacher educators and teaching as researching. A distinct feature of these ideas is that MTEs seem to degrade their role as "experts" and take a more global consideration of all the participants in PD meetings as "learners". This allows them to reflect more deeply on their approaches and use their PD experiences as the basis for combining teacher education and research-informed actions to facilitate teachers' PD.

In different parts of the data, MTEs refer to reflection as a PD practice and co-learning activity: "We need to help teachers develop ways of reflecting on their own teaching practice. However, what is a good practice like?" (Rose, 4th meeting). MTEs questioned their role as evaluators targeting participatory ways to engage teachers in reflecting on their own and/or other teachers' practices:

> We clearly cannot tell the teacher whether it was good or not. At this stage, I would say "what do you think? What was it that you didn't like? What was it that you didn't like?" Because we don't have the role of an evaluator … I would like to ask them to bring in the meeting a critical incident of their lesson and discuss it … I do not know whether we can determine (some) minimum elements expected to be there for the practice to be innovative! Because it depends too much on the group, its enthusiasm … Let them bring us something that was important for them... We can also present something that impressed us. (Sofia, 4th meeting)

The idea of teaching as researching was further promoted and concretised in the last two MTEs' meetings where central directions of action have been proposed: studying literature, sharing experiences and ideas and reflecting on teaching practice. These actions were discussed in relation to the identification of structures for helping teachers to reflect on their practice, indicating a much deeper concern of MTEs as regards teachers' PD in the long run. The above points are shown in the following extract where Ben tackles directly the theory-practice problem in mathematics teacher education: "We need to give them a framework to think, how to discuss what they did, which is not necessarily easy. A structure that they can modify as they please, which will contribute to the way they understand their PD" (Ben, 4th meeting).

In the last meeting, discussion about providing PD structures for supporting teachers targeting a balance between autonomy and guidance indicated a gradual distancing from the innovation itself and its usage. MTEs referred more explicitly to PD approaches reported in the literature, but now they connected them to their own practice as teacher educators in more specific/operational terms. The following extracts explicate two such approaches:

> Because I was worried I guided the teachers to organise a scheme of three phases. In the first phase, a familiarisation with the workplace context is taking place… whatever is this, the storekeeper, etc. in order to see the agents and understand how it works. In the second phase, to give the tasks, to see what this profession is about and in the third phase to enter in this profession and practice as a professional… to become an apprentice… This is like an agenda to follow. (Ben, 5th meeting)

> When they start from a workplace context, I insist to return to it at the end of the lesson. To follow the process of modelling. (Anna, 5th meeting)

The above extracts indicate relations between research, teacher education and mathematics teaching. In devising a practice-informed scheme of action for PD activity, several features are employed, some driven by the relevant research literature and the project's objectives and resources and others by MTEs' recent experiences in working with teachers. The research-informed ideas they expressed act as boundary objects among these three practices attempting to normalise the integration of workplace and IBL into mathematics teaching and make it part of the everyday teaching. MTEs take into account teachers' perspectives concerning mathematics teaching and professional needs and link them to their research and teacher education practice. This reflection process is characterised by an openness to take up teachers' perspectives to look at MTEs' own practice (perspective making/perspective taking).

Summarising, the tension concerning the level of teachers' autonomy in designing their lessons and the way that MTEs dealt with it throughout the PD meetings revealed several instances of boundary crossing. As in the previous tension, the multi-membership of MTEs in different communities (research, teacher education and mathematics teaching) influenced boundary crossing in terms of the practices involved and their professional learning. The dominant crossing was between research and mathematics teacher education where Mascil tasks and research-informed structures and constructs operated as boundary objects. The intersection of mathematics teaching, research and teacher education practices facilitated the emergence of reflection processes (perspective-making/perspective-taking) allowing a smooth integration of workplace contexts and IBL in PD and actual classroom teaching. These processes were characterised by the development of MTEs' awareness of the existing contradiction between teachers' autonomy and the targeted innovation and of teacher education strategies closer to mathematics teachers' needs.

15.5 Conclusion

In this chapter, we studied a group of MTEs as they designed and enacted PD activities to support teachers adopt IBL approaches and make connections between workplace situations and mathematics teaching. A number of concerns emerged regarding the meaning of IBL and workplace in mathematics teaching and the design and enactment of PD activities. Making sense of the connections between workplace situations and classroom teaching required MTEs to develop understanding about the complexity of the workplace context and the specificity of mathematics in it, as well as the differences between school and workplace culture (Nicol, 2002). Designing and enacting PD activities generated many concerns as the targeted innovation was rather new for both MTEs and teachers. Selecting/designing classroom and PD tasks, finding ways to support teachers, overcoming institutional constraints and linking research to PD were challenges faced by MTEs throughout their work with the teachers.

We focused on two dominant tensions, namely, the teaching potentiality of authentic workplace-based classroom tasks and the level of autonomy in teachers' work. These tensions brought to the fore three main practices enacted in the MTEs' meetings: research, mathematics teaching and teacher education. MTEs' participation in these practices and the negotiation of perspectives among them fuelled the raising of boundaries and facilitated boundary crossing including the learning mechanisms of identification, coordination and reflection. This process supported MTEs to develop a deeper awareness of meaning and of the materialisation of the targeted innovation and resulted in the development of their own professional learning. The objects that seemed to facilitate the process of boundary crossing were the tasks and objectives of Mascil, the authentic workplace situations and the relevant research literature regarding mathematics teaching and mathematics teacher education.

The work of MTEs, especially in large scale programmes, constitutes a very complex and challenging research issue which requires various skills and competences for which there are no professional programmes to support them (Jaworski & Huang, 2014; Smith, 2005; Zaslavsky, 2008). The complexity stems from the MTEs' activity, the specificities of the innovation and issues related to the large-scale character of the programme. The results revealed the multifaceted and systemic character of teacher education programmes targeting educational innovations. The MTEs' close collaboration and the high degree of teacher autonomy seemed to be crucial both for supporting the integration of innovation into actual practice as well as for MTEs' learning and professional development. Our study contributes to the existing literature in two ways. Firstly, the systemic network provides a tool that can help MTEs and researchers to gain deeper insights into the difficulty of bringing teaching innovations into mathematics classrooms and developing ways to bridge the gap between research and practice (Boaler, 2008). Secondly, the lens of boundary crossing to analyse MTEs' tensions and how they were dealt with offers a way to highlight the role of different practices in MTEs' professional learning becoming visible through the continuous transitions of MTEs across them. Becoming aware of how these crossings can be facilitated seems to bring the work of MTEs and researchers in mathematics education closer to the teachers' and students' reality.

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Chapter 16
Mathematics Teacher Educators Learn from Dilemmas and Tensions in Teaching About/Through Culturally Relevant Pedagogy

Kathleen Nolan and Lindsay Keazer

16.1 Introduction

Lindsay: First, I thought, "Oh, I'm defining culture too broadly or too narrowly" and then someone told me that they count recycling as culturally relevant! And I thought, "Oh, okay!" But then later, I think I just felt unsatisfied by that.
Kathy: … that reminds me, in my course when I was giving them projects to do, they could decide whether it focused in areas of ethnomathematics or social justice or Indigenous education or... in order to inform the bigger picture of what it meant to be culturally relevant or culturally responsive. And one of the students decided to do a project on deaf culture, and I really felt like I needed some kind of convincing at first, like, "What is deaf culture? How is that a culture?" I don't know how we define culture.
Lindsay: I don't know if everyone would define that as a culture, but [it makes] people feel more connected to the math. So, sometimes students' projects really trigger our learning.

The research for this chapter is situated in the nexus between prospective and practising mathematics teachers' (PTs') expectations for technical-rational classroom "tips and techniques" and mathematics teacher educators' (MTEs') desires to disrupt dominant discourses in mathematics education. The chapter is grounded in research by two MTEs (Nolan & Keazer) as they teach courses in culturally relevant pedagogy (CRP) in mathematics. In many ways, this work reflects a response to the challenge put forth by Averill et al. (2009): that educators "critically reflect on their own culturally responsive practices, ideally in discussion with other practitioners, teacher educators, and students" (p. 181).

K. Nolan (✉)
University of Regina, Faculty of Education, Regina, SK, Canada
e-mail: kathy.nolan@uregina.ca

L. Keazer
Sacred Heart University, Department of Teacher Education, Orange, CT, USA

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_16

In general, MTEs have tended to focus their research on what PTs learn from their instructional approaches, rather than on their own learning as they conduct research (Chapman, 2008; Goos, 2008, 2014). In this chapter, we attend to our own learning as a result of studying PTs' development of CRP. In other words, this chapter is the story of two MTEs who are learning (from their students and from each other) to understand and work through the questions and tensions that emerge when dominant discourses of mathematics are disrupted in teacher education courses. The focus herein is on our learning as MTEs from studying our own contexts, but also as we share experiences with each other through research conversations.

In this chapter, we each share a research/teaching narrative from our own university context. The narratives situate our learning and development as we struggle with self-questions of who and what is "relevant" and analyse how students respond to CRP in mathematics. Each of our narratives is followed by a response written by the other MTE, reflecting an additional layer of learning from each other. Also present throughout the text are brief transcript excerpts which are drawn from (recorded) conversations that we had with each other over the past 2–3 years, as we each designed, taught, researched and reflected upon courses in CRP in mathematics at our respective universities. Together, the narratives, responses and transcripts reflect how we are learning and developing as MTEs, with a layer of emphasis on the role that PTs play in MTE learning when dominant discourses are disrupted. Prior to introducing the narratives, we provide an overview of relevant literature in the areas of MTE learning, teacher education, and culturally relevant pedagogy.

16.2 Literature Review

16.2.1 Mathematics Teacher Educator Learning

MTE researchers, who are often engaged in research related to their teaching, are in a good position to investigate their own learning. When MTE researchers have investigated their own instructional practice, they typically focus on the learning of PTs, rather than describe *their* learning as they conduct research. Chapman (2008) reviewed MTE research on instructional approaches and found that learning outcomes focused on three factors: change in teachers' knowledge, prospective teachers' reflection, and guidelines for instruction. Chapman notes:

> Most of the studies were reported as if the teacher-educator researchers were 'outsiders'… teacher educators engaged in research of their practice more for broader purposes of the production, enhanced understanding, and advancement of knowledge about mathematics teacher education practices than for purposes of their own personal-professional development. (p. 130)

Similarly, Goos (2008) states, "It is rare to find in research reports a theorized discussion of our own teaching or how our research contributes to our learning as mathematics teacher educators" (p. 88). Goos (2008) describes the multiple layers of learning that the work of MTEs could investigate: teacher-as-teacher, teacher-educator-as-teacher, and teacher-educator-as-learner. Goos (2014) interrogates why the field has prioritised some opportunities to learn over others, leaving largely unexamined the learning opportunities for teacher-educator-as-learner. She considers the potential of sociocultural frameworks for understanding MTE learning.

Existing examples of sociocultural research on MTEs studying their own learning have examined contexts such as teaching mathematics, task design, and teaching in online learning environments. Other inquiries into MTE learning have attempted to outline and apply a framework of MTE knowledge, surveying MTEs for their knowledge (Anthony, Cooke, & Muir, 2016), but these have not yet delved into what MTE knowledge might be needed for developing PTs' equity-based and culturally relevant pedagogies. While our research into our own learning as MTEs is motivated by pedagogical concerns of a sociocultural nature, we also acknowledge the need to broaden our perspectives and ground our concerns in the political and the critical. In other words, we believe that to develop and reflect upon equity-based and culturally relevant pedagogies, we must challenge dominant paradigms in mathematics in our work with PTs and in teaching and learning mathematics in general.

16.2.2 Teacher Education

As MTEs and researchers, we are in agreement with others who claim that teacher education is coupled to a technical-rational model, reflected in the normalised language of "training" and "preparation" (Britzman, 2003; Brown, 2008; Loughran, 2013). This technical-rational model of teaching practice often functions to (re) produce an existing educational system rather than question and deconstruct modes of domination in the wider structures of schools and schooling (Nolan & Tupper, 2016). According to Selkrig and Keamy (2015), practices that reproduce an over-reliance on technical-rational approaches limit the possibility for deep and critical reflection amongst PTs. And yet, if PTs' previous learning experiences in schools have been situated in technical-rational discourses, then they may be compelled to expect the same of teacher education programs; that is, they expect teacher education programs to provide clear directives and techniques to address concerns of pedagogy, performance, and classroom management. While such expectations are prevalent throughout many areas of teacher education, there may be no area more replete with technical-rational understandings (of both content and pedagogy) than that of mathematics education.

Technical-rational mathematics education, or what Bishop (1991) refers to as technique-oriented curriculum, reflects a dominant approach to curriculum that is "essentially based on the assumption that a 'top-down' approach to mathematics education is optimal" (p. 12). Research in the field of mathematics teacher education calls for MTEs to introduce more critical and culturally responsive pedagogies. Mukhopadhyay and Roth (2012) note that "[m]eaningful integration of culturally based knowledge into school mathematics inevitably creates a strong tension" (p. 5), partly due to the unquestioned nature and culture of the mathematics taught in schools. That is, school mathematics (or, what has been referred to as Euro-Western (E-W) mathematics by Bishop, 1990, 1991, or "near-universal, conventional (NUC)" mathematics by Barton, 2008) has a history of being "culturally defined as objective, value-free, logical, consistent and [a] powerful knowledge-based discipline which students must accept, understand and manipulate" (Burton, 1994, p. 207). Bishop (1990) even goes so far as to accuse mathematics of being "one of the most powerful weapons in the imposition of western culture" (p. 51), calling it "a secret weapon of cultural imperialism" (p. 51).

Efforts to disrupt the dominance of NUC mathematics in classrooms have emerged through research on power, privilege and oppression (Gutiérrez, 2017; Nasir, 2016; Willey & Drake, 2013). As Gutiérrez (2017) notes, "[t]here is a robust domain of scholarship dedicated to chronicling the relationship between mathematics and power/domination in society stemming back more than 50 years" (p. 9). Willey and Drake (2013) indicate that subtle signs of power and privilege within dominant traditions are manifested in the form of "neglecting students' cultural and intuitive mathematics knowledge; granting mathematical authority to only the teacher, the textbook, or a few outstanding students; leaving unchallenged current constructions of what it means to do and learn mathematics" (p. 62). These same authors urge us, as MTEs, "to sharpen our sociopolitical lenses in order to notice and disrupt manifestations of privilege and oppression in mathematics education" (p. 68).

In this chapter, our intent is to communicate our learning as we worked to notice and disrupt technical-rational teacher education and the dominance of NUC mathematics. Specifically, we discuss our endeavours to foster disruptions through a critical study of our own learning as we teach courses in CRP in mathematics teacher education. While our critical dispositions risk setting up a dichotomy between technical-rational and CRP approaches, we claim that it is incumbent upon us to examine and critique dominant discourses that may implicitly structure and inform modes of operating across society while examining alternative discourses as a "a pedagogy of opposition" (Ladson-Billings, 1995) in order to increase awareness of our pedagogies and their impact on students.

16.2.3 Culture, Mathematics, and CRP

> When asked why I loved math in grade 12, I responded with "because no matter who you are, where you live, or the differences in your life, math is always the same, 1+1 will always equal 2". Looking back I cannot believe how wrong I was, but in looking forward I can also not believe how much I have learned and how much learning I have left in my future. (Practising teacher)

The above comment (made by a student enrolled in the CRP course taught by Nolan) reflects a commonly held belief about the perceived lack of complexity of mathematics. Embedded in the belief is the notion that mathematics is value-free and culturally neutral. Greer and Mukhopadhyay (2015) assert "that, far from being culturally neutral, mathematics education, and indeed mathematics itself, only make sense when considered as embedded in historical, cultural, social, and political—in short, human—contexts" (p. 261).

According to Tillman (2002), culture can be defined as "a group's individual and collective ways of thinking, believing, and knowing, which includes their shared experiences, consciousness, skills, values, forms of expression, social institutions, and behaviors" (p. 4). In the context of curriculum and classrooms, acknowledging and incorporating the cultures of students in the teaching and learning of school subjects is a critical step in moving beyond the unquestioned presence of only the dominant culture, as if it reflects "no culture". Teachers and researchers refer to this incorporation of culture using a variety of terms; for example: "cultural synchronization" (Irvine, 1990), "culturally congruent" (Mohatt & Erickson, 1981), "culturally appropriate" (Au & Jordan, 1981), "culturally revitalizing" (McCarty & Lee, 2014), "culturally sustaining" (Paris, 2012), as well as several others (Aronson & Laughter, 2016). The most widespread terms, however, are culturally responsive and culturally relevant.

The concept of *cultural responsiveness* emerged largely from cultural difference literature, challenging schools to meet the needs of marginalised students and to improve academic performance (Castagno & Brayboy, 2008). Gay (2010) defines culturally responsive teaching as "using the cultural knowledge, prior experiences, frames of reference, and performance styles of ethnically diverse students to make learning encounters more relevant and effective for them" (p. 29). Ladson-Billings (1995) defines *culturally relevant pedagogy* as "a pedagogy of opposition... not unlike critical pedagogy but specifically committed to collective, not merely individual, empowerment" (p. 160). She proposes that culturally relevant pedagogy must have three aims for students: (1) academic success, (2) cultural competence, and (3) critical consciousness (p. 160).

In the specific context of mathematics education, a similar collection of terms is used and defined. For example, Aguirre and Zavala (2013) discuss culturally responsive mathematics teaching (CRMT), which "involves a set of specific pedagogical knowledge, dispositions, and practices that privilege mathematical thinking, cultural and linguistic funds of knowledge, and issues of power and social justice in mathematics education" (p. 167). Here, in this chapter, we primarily use the term culturally relevant pedagogy (CRP) in mathematics.[1]

In addition to multiple names for CRP, there are also many questions about how teachers should enact CRP in mathematics classrooms (Ukpokodu, 2011). Leonard, Brooks, Barnes-Johnson, and Berry (2010) propose that teachers first work to

[1] The course taught by one author (Nolan) was named Culturally Responsive Pedagogy in the Mathematics Classroom, so *responsive* is used in the sections of the paper referring to her teaching and research.

"understand the cultural, social, political, and economic contexts that affect the lives of students" and, second, "mathematize these contexts" (p. 267). Not all research agrees, however, with an approach characterised by mathematising the cultural contexts. Some claim that mathematics itself must be deconstructed to suit culture, not fit culture into the current dominant paradigm. For example, with respect to Indigenous culture, Doolittle (2006) warns against mathematising cultural artefacts, claiming that one risks oversimplification of the artefact and its cultural meanings. Doolittle worries that "Indigenous students who are presented with such oversimplifications feel that their culture has been appropriated for the purpose of leading them away from the culture... and toward some strange and uncomfortable place" (p. 20). Doolittle argues that such "pale oversimplifications" could actually result in pushing the complex and powerful cultural traditions "underground" (p. 20). Other research calls for the work of culturally relevant mathematics education to take place in a "third space" (Gutiérrez, Rymes, & Larson, 1995). Lipka et al. (2005) propose that a "third space" is a space/place where "historically silenced knowledge… is privileged alongside traditional academic discourse" (p. 369).

Sleeter (2012) suggests that it is difficult, even "risky", to use culturally responsive practices in the classroom in these times of imposing "standardized and scripted curricula on teachers" (p. 579). In some ways, this chapter reflects our attempts, as MTEs, to embrace this "risky-ness" in our mathematics classrooms by wading into the waters of this highly fluid cultural dimension to teaching and learning mathematics and to our own MTE learning. The risky-ness reflects our commitment, as MTEs, to respond to the call for "a pedagogy of opposition" (Ladson-Billings, 1995, p. 160) and a mathematics education that privileges "issues of power and social justice" (Aguirre & Zavala (2013), p. 167). Thus, the theoretical premise of our research and teaching as discussed in this chapter is grounded in efforts to disrupt and decolonise NUC mathematics (Aikenhead, 2017; Lunney Borden & Wiseman, 2016; Meaney, Edmonds-Wathen, McMurchy-Pilkington, & Trinick, 2016; Stavrou & Miller, 2017); we are much less interested in settling for "an add and stir approach to systemic change" (Battiste, 2017, p. xi).

16.3 Lindsay's Narrative: Who and What Is "Relevant" in CRP?

> *Lindsay:* Why do we keep questioning? Is this a sign of our learning that we keep questioning "what is culturally relevant pedagogy?" the whole time we're teaching about it?
> *Kathy:* That's a good line. I think that's going to be the opening line of your narrative.

As part of an undergraduate course on mathematics and diverse cultures, my students (PTs) and I explore mathematics and its connections to themes of ethnomathematics and teaching mathematics for social justice, and we connect these to culturally relevant pedagogy. These are complex ideas, and each PT is on a journey of growth along a developmental continuum, making sense of what these ideas mean for us as teachers of mathematics. As an MTE, I often wonder, "How do my PTs conceptualise CRP as a result of my teaching?". By listening to PTs' discussions and reflecting on their course projects, I get a glimpse of their thinking. Through the process of reflecting on their ways of understanding, and comparing it to my own understanding, I found engaging opportunities that prompted my own learning.

This CRP course was taken by mathematics-focused elementary and secondary PTs. I borrowed a series of three connected projects from modules developed by the TEACH Math project (Drake et al., 2015): (a) "Getting to Know You" Student Interview, (b) community walks, and (c) mathematics lesson development. Each PT applied the knowledge learned about a student and the community from the projects into a culminating project, where they designed a mathematics lesson that built on student and community knowledge. The mathematics lesson and its rationale was illuminating evidence for me of how PTs made sense of what it means to connect to student and community knowledge in relevant ways.

As I reflected on PTs' culminating projects, some mathematics lessons felt somehow closer to my understanding of CRP than others, though they all offered opportunities to engage my internal critique. My preferences for some lessons over others pushed me to clarify my own understanding of CRP through self-questioning about which aspects could be arguably culturally relevant and what could be missing. As Rubel (2017) described, the mathematical contexts chosen for many PTs' projects represented general experiences that were familiar to the PT. The first time I taught the course, many PTs' lessons were typical mathematics problems resituated within locations related to a child's interest, such as a trampoline park or Chuck E. Cheese[2]. These projects posed layers of concern for me. While connected to a child's interest, the chosen contexts often related to experiences of the PT's culture, and the dominant culture, and offered no evidence of whether or not PTs could stretch themselves to learn about knowledge or experiences different from their own, nor did they tap into the powerful potential that mathematics holds to serve as a mirror or a window (Gutiérrez, 2007; Styles, 1988) to reflect a culture that is marginalised. While some could argue for the value of opportunities for students to look through "windows" at dominant ways of studying mathematics as a cultural artefact in its own right, the over-prevalence of NUC mathematics presents the injustice of an imbalanced "educational diet":

> [D]emocracy's school curriculum is unbalanced if a black student sits in school, year after year, forced to look through the window upon the (validated) experiences of white others while seldom, if ever, having the central mirror held up to the particularities of her or his own experience. Such racial imbalance is harmful as well to white students whose seeing of humanity's different realities is also profoundly obscured. (Styles, 1988, p.5)

In many cases, it was unclear whether or not the child (whose experiences were the focus in designing the mathematics lesson) would actually find the PT's lesson

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[2] Chuck E. Cheese is an American-based restaurant and entertainment centre for children.

engaging. Even when one PT developed distance, rate and time problems by inserting familiar locations and collecting distance data from mapping software, I noticed missed opportunities to use the power of mathematics to explore the experiences of students which differ from the PT's own experience. In other words, I felt that their ways of designing culturally relevant mathematics lessons did not move towards deconstructing dominant NUC-based views of what it means to know and do mathematics.

Some mathematics lessons, however, felt closer to my own understandings of CRP in some ways, which led me to further interrogate my thinking. For example, one PT described their student interview as a "failure" due to being unable to extract substantial information from an unengaged child, other than lyrics to a favourite song, *Watch Me* by rapper Silentó. Despite the PT's frustration, she stretched herself to explore this song further and developed a lesson exploring patterns and proportions situated within the song lyrics. Another PT sought opportunities for a conversation with local parents at a park during her community walk and developed a mathematics lesson that connected to parents' concerns: exploring the cost of replacing the wood chips covering the irregular polygon-shaped playground with rubber bits, to match the quality of parks these parents observed in nearby (more affluent) areas.

As PTs developed their understandings of CRP, it was easy to focus on their deficits. I could fill a narrative with critiques of their failures to deconstruct the dominance of NUC mathematics, or to build on students' funds of knowledge, or to develop lessons that encourage students to see through mirrors and windows (Styles, 1988) to understand others' cultures. But these "failures" point to the needs of PTs for further learning; they also point to my own opportunities for growth as an MTE who aims to create learning opportunities that perturb PTs' current understandings and trigger their learning about CRP.

Reflecting on PTs' projects prompted my own critical reflections on CRP. I began to notice features of PTs' mathematics lessons that served as evidence of a developing CRP. As I looked across two semesters of projects, I interrogated my own tensions and preferences. As Chapman (2008) suggests, when we explore the thinking of PTs, we experience conflict and tension that offers provocative opportunities for us to learn. I realised that understanding each PT's rationale behind the choices made helped me to understand how they developed the lesson from knowledge of child and community. For example, a general context such as basketball courts, which I may judge as acultural, could appear convincingly relevant when a PT explained how the lesson was not only situated in a community location, but that mathematics was being used as a tool to critique the unequal distribution of resources. While two PTs developed lessons related to local basketball courts, I recognised important distinctions between them as to whether (a) the task involved mathematics that was practically useful within the context and (b) the context extended beyond the PT's familiar knowledge or cultural experiences. Yet what became the most important to me was whether (c) the PT approached "teaching as mining" (Freire, 1974), that is, mining the child's and community's knowledge resources rather than approaching the context using their own knowledge resources (or the knowledge found from an Internet search). According to Ladson-Billings

(1995), a salient feature of culturally relevant teachers is that they view "teaching as mining", or pulling knowledge out of their students. This feature emerged, for me, as an important way to distinguish between PTs' lessons—noticing how much of the PT's rationale involved digging into knowing the child and community helped me recognise evidence of a developing CRP.

These reflections prompted me to wonder: How might I help PTs sense an urgency to include mathematical contexts that represent knowledge and experiences other than their own? How might I prompt PTs to consider the injustice of a curriculum that displays an imbalance of mirrors and windows? As Styles (1988) described, there is harm in having a curriculum that prioritises some people's knowledge (or views of mathematics) over others'. PTs' (and my own) missed opportunities to know students and communities better point to our privileged perspectives. Perhaps those of us with successful mathematics experiences who see ourselves in the mirror all the time struggle to imagine the experiences of a child who feels marginalised by the unbalanced mathematics curriculum.

To broaden PTs' experiences with regard to the potential for mathematics curriculum to connect to cultural knowledge, I created an assignment where PTs used mathematics to explore a context that was a mirror to their own family and community experiences. To encourage them to move beyond typical textbook problems, I requested that they connect mathematics to experiences not typically reflected. I called this assignment "Ethnomathematics and Me", and I also developed a parallel assignment, "Social Justice and Me", where they used mathematics to understand a context that connected their personal experiences with an issue of injustice.

I vividly recall the moment when a PT stood up to present her ethnomathematics assignment. I had asked PTs to start the presentation by sharing a narrative to convey the personal relevance. This PT proudly stated, "One thing that is really important to my family is… tattoos". This moment fostered an unexpected perturbation in me that broadened my own ideas of culturally relevant contexts. The idea of tattoos affiliated with an important family value surprised me and helped me to see how my students' experiences could be quite different from my own. She convincingly described the thoughtfulness put into self-expression through tattoos and explored geometry within designs. Prior to this moment, I had made assumptions that this PT probably had experiences similar to mine, but this experience taught me that there is always more to know about my students. This left me with the conviction that I must mine knowledge out of my own students and build on that knowledge as I teach them about mathematics and CRP.

16.3.1 Kathy's Response to Lindsay's Narrative

When Lindsay claimed that the PTs' "mathematics lessons did not move towards deconstructing dominant NUC-based views of what it means to know and do mathematics", I found myself wondering whether we, as MTEs, are at all successful in expanding views on what mathematics *is,* or we are only successful in tweaking the mathematics problems being solved through a superficial insert of names and locations into the dominant NUC mathematics. One of my own PTs expressed this same critique, stating: “simply adding words like ‘drums’ or ‘headdress’ into math problems (that may be significant to others) is not a sufficient way to be responsive because it’s just a replacement rather than a reinterpretation of the math” (N).

From Lindsay’s narrative, I also learned how powerful the metaphors of “mirror” and “window” are. They indicate to me the importance of both reflective and refractive perspectives when learning about/through cultural relevance. Lindsay used these two metaphors as lenses through which she assessed the value of the context and content of the mathematics lessons developed by her PTs, and they helped her recognise the need to push PTs to understand teaching through CRP as “mining the child’s and community’s knowledge resources”.

As Lindsay described, PTs’ ideas and course outcomes are valuable opportunities to prompt our learning as MTEs, particularly about “features of PTs’ mathematics lessons that served as evidence of a developing CRP”. While trying to avoid viewing her students’ work through a deficit lens, Lindsay works at “finding” CRP in her PTs’ lessons. This choice of focus offers two opportunities: first, it fosters her own learning about CRP through her students, and, second, the positive feedback to students encourages PTs to keep growing. At one point in her narrative, Lindsay claimed that “these ‘failures’ point to the needs of PTs for further learning; they also point to my own opportunities for growth as a MTE”. I can identify with her compassion coupled with her desire to push PTs—to “perturb PTs’ current understandings and trigger their learning about CRP”. I too wrote about my desire to push PTs to grow while being aware of the tensions between pushing for growth and being patient with their efforts. *My* patience, however, would likely have been pushed to its limit if one of my PTs had focused her culturally relevant project on Chuck E. Cheese.

16.4 Kathy’s Narrative: How Do Students Respond to CRP?

In thinking critically about and planning for my *Culturally Responsive Pedagogy (CRP) in the Mathematics Classroom* course, I broadly defined CRP to encompass five key areas, or themes: social justice, equity, Indigenous education, ethnomathematics, and linguistically diverse learners (using Greer, Mukhopadhyay, Powell, & Nelsen-Barber, 2009, as the course textbook). During my first offering of the course, I conducted a research study to understand how students’ ideas, experiences, and knowledge of CRP in mathematics were changing/growing/evolving throughout the semester. During the semester, I collected data from the seven student participants (most of whom were practising elementary school teachers) through a personal response journal (a blog) assignment. After each class, students were asked to respond in their blog to questions about their understandings of culture, CRP, mathematics knowledge, critical awareness, and other topics from class discussions.

In constructing this narrative, I draw on data from the blogs of the seven participants (referred to by the pseudo-initials: B, C, E, K, N, R, and S) while also viewing the data through the lens of my own MTE and researcher reflections written throughout the semester. I present here three significant themes that I constructed out of the data which speak to my own learning as an MTE and also offer points of reflection for other MTEs presented with similar concerns and contexts. The learning themes, which are also being offered as tensions within my own learning, are summarised here using the three words of *push*, *patience*, and *pressure*.

16.4.1 Push

In my CRP course, I wanted to push my PTs to grow, but not push so hard that they would become disinterested in exploring new ways of thinking about mathematics and culture. In one blog question, I asked students to reflect on how one might share with others (colleagues and students) that mathematics is actually *not* value- and culture-free. One PT suggested starting with "the ones that you can influence to becoming culturally responsive in their pedagogy" (K), which for her included those who have either similar teaching beliefs or those "who can be convinced with evidence, readings, and logic" (K). Another PT's response to this same question proposed that one needs to frame these new ideas or suggestions "in such a way that current practices were respected and acknowledged but [to suggest] that new ways can offer increased benefits to students as well as to the teacher" (C). She stressed the importance of gaining their trust in knowing that she wasn't suggesting "change for the sake of change" (C).

In general, the PTs in my course embraced (what I would call) a much *softer* approach to challenging and changing practice than I had in mind. While I understand the soft approach might gain more voluntary "buy-in", I also feel that once educators notice privilege and injustice associated with the dominant culture, they are obliged to act. Ladson-Billings (2014) discusses the two-pronged approach that must occur with CRP:

> In our attempt to ensure that those who have been previously disadvantaged by schooling receive quality education, we also want those in the mainstream to develop the kinds of skills that will allow them to critique the very basis of their privilege and advantage. (p. 83)

In other words, CRP is not just for those students/teachers who have been disadvantaged by a lack of (their or other) cultures in the mathematics classroom; CRP is also for those students/teachers who have much to lose if their privilege and advantage is challenged. My dilemma is in knowing how to push PTs to notice (and act on) the injustices in the outside world as well as the privilege inside their own lives, while at the same time not deter them completely from *any* engagement with the issues. In other words, I struggle with finding a balance between pushing their thinking to a more critically aware place without pushing them too far.

One PT expressed the dangers of becoming more critically aware: "... a certain angst can follow the beginning of questioning what has been accepted long term. It's definitely treading into unknown waters and one may not feel equipped to go down that path" (C). On the other hand, one PT seemed equipped to go down that path: "I thought of myself as culture free when I started in the education program at the University... I now realise that that attitude was a symptom of the privilege my whiteness affords me. I realised that my cultural values were not universal and that I indeed had cultural values" (N).

16.4.2 Patience

Through the semester, I learned the importance of having patience for others' growth in learning (including my own). Near the end of the semester, one PT shared in her blog: "Students need to be able to understand and make sense of what is going on around them. Perhaps the varying degree of understanding seems to be limited only by what information the teacher is ready to tackle with the students" (E). While the PT may be referring to other practising teachers, her words prompted reflection on my role as MTE. These words signified to me the importance of encouraging PTs to learn at their own pace, moving in their own directions, and acknowledging that PTs may not be "ready to tackle" everything at once. In other words, as an MTE, I need to relax into the fluidity of how PTs take up the various CRP issues. In another research study on the integration of Indigenous/Aboriginal education into mathematics curriculum, Nolan and Weston (2015) advise that one needs to explore "the uniqueness of each teacher, each classroom, and each interpretation of what it means to teach mathematics through a distinctly Aboriginal focus" (p. 20). I think this same advice applies for the five CRP issues explored in this course.

Leonard et al. (2010) advise that CRP-novice teachers should not be judged by their first few attempts at designing and implementing culturally responsive lessons, or even by their slowly evolving attitudes and dispositions towards CRP. They call attention to the amount of work required "in understanding the teaching and learning process in educational spaces where the lives of teachers and students from different cultural, ethnic and socioeconomic backgrounds intersect" (p. 267). One PT in the course touched on this idea of slowly evolving attitudes and dispositions when she proposed that "[a]uthentic integration of culture come in two parts: recognizing tokenism and planning for deeper, meaningful lessons" (K).

The matter of finding balance between pushing forward and patiently holding back is an ongoing tension for me. It leads me to reflect on when/how PTs' doubts about teaching through CRP become "legitimate" reasons for holding back and when they merely serve as rationalisations for not moving forward with CRP. Consider the following data where PTs expressed perceived challenges to teaching through CRP:

- “there are a lot of different cultures who’s [sic] mathematics you would need to include in the curriculum” (B).
- “I feel as though many teachers would have issues with parents who believe that “the old way worked for me so why change it?” (B).
- “I also feel that it would be extremely difficult to ensure that every single lesson we ever made was culturally responsive” (B).
- “one certainly sees that building the groundwork for CRP is labour intensive and can potentially take a teacher out of his/her comfort zone” (C).
- “I’m afraid that I could turn into a teacher that knows the materials and knows the [cultural] perspectives but doesn’t act because [s/he] is afraid of offending” (N).
- “[with] the added dimensions of responsiveness to equality, social justice, ethno-mathematics and linguistics... there is a LOT to consider when building a culturally responsive program. Truthfully, these are not really issues that I had ever considered could be discussed in mathematics” (C).

My concern is that articulating such challenges may actually be serving as “rationalising discourses” for PTs, that is, reasons which support a belief that CRP in mathematics will not work, or is too much trouble to even begin. I fully accept, however, that the comments (“rationalising discourses”) may be emerging out of fear, discomfort, and feeling overwhelmed with the accountability that comes with “noticing” and becoming critically aware of a new reality in which mathematics is no longer their point of reference for objectivity and cultural neutrality.

16.4.3 Pressure

> It’s hard to believe that just over one month ago, I had no idea what culturally responsive pedagogy was or why I needed to know about it. Math is math, isn’t it? .... (R)

Connected to the earlier theme of pushing students to grow, this theme relates to the pressure, or push back, I felt from students when teaching the CRP course. The act of disrupting or deconstructing a normalised discourse (e.g. like the one embedded in the phrase “math is math, isn’t it?”) can be met with much student discomfort, with students expecting a concrete suggestion for “what is better”. In other words, there can be student resistance to disrupting technical-rational practices that have clear steps or recipes to follow and replacing them with practices that involve ongoing discernment, ambiguity, and uncertainty. The discomfort associated with spaces of ambiguity and uncertainty tends to result in avoiding/resisting such spaces by constructing them as non-productive spaces (where nothing is accomplished). One PT queried in her blog:

> If I should take the time to really explore the issue rather than finding a quick solution, what do I teach until I figure it out? Will I get it right before I retire? Will I mess up a generation of children along the way? The questions that haunted my sleep in my first years of teaching are popping up again. It’s uncomfortable to not have the answers. (R)

Ambiguity and uncertainty come with the territory of introducing new theory to disrupt normalised practices. Several blog entries during the semester, however, articulated that the course had too much theory and not enough "this is how you teach mathematics through CRP". PTs commented on not being able to see how the course could directly impact their teaching and/or their classroom. After only the first two classes, I became frustrated by students' expectations that we move quickly from theorising disruptions of practice to proposing clear and practical suggestions for new classroom practices that reflect CRP. One PT wrote about the importance of "[c]oncrete examples of how CRP in math has been integrated in numerous ways/places and being able to present an illustration that personally connects..." (C). I felt pressure from PTs to explain exactly what to do (steps to follow) in order to *be culturally responsive*.

Ladson-Billings (2014) aptly proposes that "[i]f we are to help novice teachers become good and experienced teachers become better, we need theoretical propositions about pedagogy that help them understand, reflect on, and improve their philosophy and teaching practice" (p. 83). The significance of theory, however, is a hard sell for students who expect technical-rational techniques. As one PT noted:

> When it comes to resistance to pedagogical change, I think that most teachers would be resistant because of both inertia and a general attitude that pedagogical practices coming out of the universities are wonderful in theory but don't work in the "real world". (N)

By introducing CRP in mathematics teacher education courses, I hope to encourage PTs to shift from discursively producing the university and teacher education as out of touch with "real" classrooms and towards noticing and challenging the "rationalising discourses" that sustain dominant school mathematics practices (Nolan, 2009; Nolan & Tupper, 2016).

16.4.4 Lindsay's Response to Kathy's Narrative

Kathy's idea of "patience" seems to be taking a developmental perspective on PTs' learning about CRP and recognising that they, like us, are on a learning continuum. "Push" seems like it is about the dilemma of how do I push each PT to grow, or can I push them without getting too much push back of rationalising discourses? This reminds me of the work of teaching mathematics in a student-centred way, pushing students towards specific learning objectives. In developing CRP, however, student learning objectives are more ambiguous, and do not fit neatly into a technical-rational agenda. "Pressure" seems like it is about the combined pressure that Kathy feels to meet PTs' expectations for quick resolutions, answers, and recipes, as well as pressure from herself living in an environment dominated by rationalising discourses, thinking, "I am the teacher and isn't giving answers what a teacher is supposed to do?"

I wonder, though, about Kathy's efforts to push PTs to challenge dominant discourses, if perhaps she is using her power as an instructor to create her own hegemonic discourse by pushing PTs to drop their prior ways of thinking and adopt hers.

In fact, I found myself agreeing with PT "C", whom Kathy felt had too "soft" an approach to challenging and changing practice. C's suggestion that a teacher introduce new pedagogies "in such a way that current practices were respected and acknowledged" seems essential to respecting PTs as autonomous professionals. I wonder if this could be considered a humanising approach, which does not contradict the urgency we have as MTEs to "develop the kinds of skills that will allow them to critique the very basis of their privilege and advantage" (Ladson-Billings, 2014, p. 83).

Rather than providing answers, Kathy's narrative is powerful in how it uses PTs' own words to draw parallels between the work of MTEs and the work of PTs. Their words illuminate the ambiguity of fostering CRP and make me want to voice the same questions as PT (N): "What do I teach until I figure it out? Will I get it right before I retire? … It's uncomfortable not to have the answers". We are teaching about/through CRP *as* we figure it out. We are taking the risk of teaching within ambiguous spaces of learning from our teaching and from our students. From whom do we get our "push"? Perhaps it is from our students, who push us by creating tensions that foster our learning. And perhaps the push comes from our developing knowledge of this unjust world in which we live.

16.5 Concluding Thoughts

Kathy: *But the classroom itself might still be... where students are in rows and they have to put up their hand and they have to make eye contact and all these things which may not be culturally relevant for all of the students. And that's where I think we're really leaving out a lot of students, in the way in which we operationalise our classroom and all of its various traditions and rules.*

Lindsay: *Yeah. I think all teacher educators are sort of trying to teach that kind of pedagogy, but I don't usually see it called culturally relevant. It seems like we just call that good teaching. But when you look at Ladson-Billings' definition of culturally relevant pedagogy, she includes that... I don't know, I'm just babbling.*

Kathy: *Yes. Well, I think that's what our chapter's going to be, a lot of babble.*

[Laughter]

As MTEs, our learning from teaching, researching, and writing has certainly moved beyond "babble". This chapter has provided an opportunity to "critically reflect on [our] own culturally responsive practices, ideally in discussion with other practitioners, teacher educators, and students" (Averill et al., 2009, p. 181). Throughout this chapter, we have highlighted the individual and collaborative processes involved in coming to understand, question, problematise, and deconstruct our own practices as MTEs and CRP instructors. We close here by summarising our learning in three key areas: our learning about CRP and mathematics, negotiating this learning through working with PTs, and learning from each other in growing and developing as MTEs.

Firstly (our learning about CRP), questioning how to define and teach CRP has underscored our belief that "[c]ultural responsiveness is not a practice; it's what informs our practice so we can make better teaching choices for eliciting, engaging, motivating, supporting, and expanding the intellectual capacity of ALL our students" (Hammond, 2015, p. vii). We acknowledge the importance of taking this perspective beyond teacher education; that is, that we educate parents, teachers, and education leaders that culturally relevant pedagogy "is not cultural celebration; it does not trivialize differences; it does not essentialize identities; it does not shy away from political analysis" (Aronson & Laughter, 2016, p. 200).

In the second area (the context of working with PTs), we take seriously the warning to not turn CRP into "a buzzword or checklist of steps" (Aronson & Laughter, 2016, p. 198); that CRP must "be embraced more fully as a guiding ethos for every aspect of the classroom" (Aronson & Laughter, 2016, p. 198). As discussed in this chapter, we recognise the need to resist the technical-rational expectations of PTs for a checklist of steps in order to *be* culturally relevant in teaching mathematics; succumbing to these expectations risks superficial treatment of a complex situation. As MTEs, we have learned to be careful that we do not promote CRP which will merely make the dominant (NUC) mathematics more *accessible*; we want our courses to be disruptive enough to broaden awareness that mathematics can be *different* in an enduring manner.

And finally, in the third area, we stress the value of learning from each other, as MTE colleagues. While we focus in this chapter on our learning from tensions created in our work with PTs, we also recognise the contributions we each made to one another's professional practice as MTEs. Without this additional layer of collaboration between us—where we could discuss the tensions and dilemmas emerging in our practices—we would have remained isolated in our own individual contexts, struggling on our own with similar questions about who and what is "relevant" and how students are responding to CRP in mathematics. While we are still emerging "learners" in the context of teaching CRP courses, at least we do not wonder alone, as one PT did, "[w]ill I get it 'right' before I retire?".

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Chapter 17
Supporting Secondary Mathematics Teacher Educators in China: Challenges and Opportunities

Yingkang Wu and Jinfa Cai

17.1 Introduction

Education is the foundation upon which a nation's long-term development rests, and teachers play a critical role in the quality of education. The theory and practice of teacher education and professional development have become an important issue internationally, and China is not an exception. In early 2018, the central government of China released two national documents related to teacher education. The first document, *Opinion on Comprehensively Deepening the Reform of Teachers' Team Construction in the New Era* (Ministry of Education of China, 2018a), highlights teachers as an important resource for educational development. According to this document, teachers represent a cornerstone of the prosperity of the country, the rejuvenation of the nation, and the happiness of its people; thus, improving teachers' professional competency becomes a priority. The second document, *Teacher Education Revitalization Action Plan (2018–2022)* (Ministry of Education of China, 2018b), presents the objectives of and ten action plans for teacher education at both the pre-service and in-service stage. Both documents note the role of teacher educators and provide some detailed measures for advancing their development from a policy perspective, including policies on the evaluation, organization, and promotion of teacher educators.

Shanghai students' excellent performance in mathematics on PISA in 2009, 2012, and 2015 has attracted educators and researchers around the world to investigate Shanghai mathematics education. In addition, the Ministry of Education of China (2016) organized a group of experts to examine the underlying features that

Y. Wu
School of Mathematical Sciences, East China Normal University, Shanghai, China

J. Cai (✉)
Department of Mathematical Sciences, University of Delaware, Newark, DE, USA
e-mail: jcai@udel.edu

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_17

contribute to the success of Shanghai students' mathematics performance. Of the three underlying features identified, one was Shanghai's strong mathematics teaching research team and mathematics teachers (Gu & Gu, 2016). The mathematics teaching research team, composed mainly of mathematics teaching researchers (数学教研员; MTRs), provides practical guidance to classroom mathematics teachers by commenting on and improving their teaching plans before a lesson and facilitating gradual improvement via evaluations and reflections after the lesson. In fact, MTRs are a special type of mathematics teacher educators (MTEs) in China (Paine, Fang, & Jiang, 2015).

Although greater attention is being given to MTEs due to their influential role in preparing and developing mathematics teachers, research on MTEs is just emerging in the field of mathematics education (Novotná, Margolinas, & Sarrazy, 2013; Tzur, 2001) and represents a mostly untouched area of research in China (Wu & Cai, 2016). Moreover, there is little discussion in China on how to prepare and develop the expertise of MTEs (Kang, 2012) despite China's large population of mathematics teachers. This chapter addresses this issue by focusing on secondary MTEs in the context of China, where there is increasing emphasis on the role of teacher educators in policy and practice, a unique teaching research system to support in-service teachers' professional development, and a lack of discussion and research on teacher educators in general and MTEs in particular.

This chapter aims to answer the following two research questions: (1) What are the types, responsibilities, and developmental trajectories of secondary MTEs in China? (2) What challenges do university-based secondary MTEs in China face in their teaching work, and how do they respond to these challenges?

In answering the first research question, we will paint a portrait of secondary MTEs in China by describing who they are, what responsibilities they are expected to take, and how they are prepared and developed to fulfill their duties. To answer the second research question, we will provide information about how university-based secondary MTEs cope with the different types of teaching work they encounter in their teacher education practice. In answering these two questions, we hope to provide insight on and implications for better preparation and support for secondary MTEs both theoretically and practically, not just in China but also around the world.

17.2 Secondary MTEs in China: Composition, Responsibilities, and Developmental Trajectories

The composition of the team of MTEs in China is somewhat complex. It comprises university-based MTEs, school-based mathematics mentor teachers, and MTRs from teaching research offices at different levels. Although they are all teachers of mathematics teachers, these three types of MTEs have relatively different responsibilities in teacher education and are generally varied in their developmental trajectories, as shown in Fig. 17.1. In this section, we present and discuss these issues in

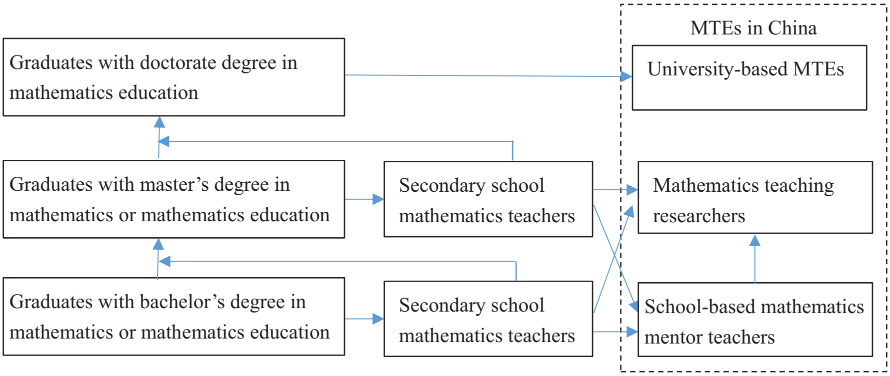

Fig. 17.1 The developmental trajectories of secondary MTEs in China

detail after a brief overview of the teaching research system and teacher promotion system in China.

17.2.1 A Brief Overview of the Teaching Research System and Teacher Promotion System in China

The Chinese teaching research system is unique to China and was established in the early 1950s. It consists of a four-level teaching research network composed of teaching research offices at the provincial, city, and county levels as well as teaching research groups in schools.

The *Provisional Regulation for Secondary Schools (Draft)*, issued by the Ministry of Education of China in 1952, stipulated that secondary schools must have teaching research groups for different subjects with the goal of improving teaching (cited in Lu & Shen, 2010). These groups are responsible for working out course plans as well as studying teaching content and methods (Lu & Shen, 2010; Yang, Li, Gao, & Xu, 2013). The *Report on National Secondary Education Conference*, produced by the Ministry of Education of China in 1954, stated that "in order to strengthen the quality of secondary school, teaching research offices can be set up to manage and be responsible for issues related to teaching research and teacher learning" (cited in Lu & Shen, 2010, p. 69). In 1956, the Ministry of Education further proposed that "teaching research offices should be established and strengthened at the province level, city level and county level" (cited in Lu & Shen, 2010, p. 69), leading China to accelerate the speed with which it established teaching research offices at different levels. Exemplary school teachers were selected to work in the offices as teaching researchers.

Teaching research offices at different levels organize and conduct various teaching research activities, such as demonstration lessons, lesson explanations, discussion and evaluation (Peng, 2007), and teaching contests (Li & Li, 2013), to support in-service teachers' professional development. As the basic units of teaching research activities, teaching research groups organize and encourage routine and school-based teaching research activities such as exchanging ideas, sharing notes, preparing lessons collectively, and visiting one another's classes.

The teacher promotion system in China was first implemented in 1986 (Gao, 2016). It is an incentive system designed to motivate teachers to improve their teaching quality. The recently released official document *Guidance on Deepening the Reform of the Primary and Secondary School Teachers' Promotion System* (Ministry of Human Resources and Social Security of China, 2015) outlines the promotion system across primary and secondary schools, according to which the professional titles ranking from low to high are third-class teacher, second-class teacher, first-class teacher, senior teacher, and professorship teacher. The document explicitly states the evaluation standards for each teacher professional title, and the review process involves expert peer evaluation. The number of professorship teachers is strictly controlled by the central government (Gao, 2016).

17.2.2 School-Based Mathematics Mentor Teachers

The professional title of school-based mathematics mentor teachers is generally senior or professorship teacher. In fact, one of the prerequisites for promotion to senior or professorship teacher is prior experience guiding and advising lower-ranked teachers. Aside from ordinary teaching responsibilities, school-based mathematics mentor teachers typically assume extra roles such as head of a teaching research group, mentor for novice teachers, and cooperative teacher of pre-service teachers.

As the head of a school-based mathematics teaching research group, the school-based mathematics mentor teacher takes the lead in constructing the group and performs a variety of activities such as demonstrating classroom teaching, guiding teaching research, developing curriculum resources, cultivating the culture of the group, and facilitating teacher training (Xu, 2016).

The mentoring system is an important resource for supporting the growth of novice teachers. A smooth and productive induction has a significant impact on new teachers' self-identification, confidence, and further development in their teaching career. Novice teachers are usually required to observe mentor teachers' classroom teaching as well as open their own classroom teaching to mentor teachers for guidance and advice. The mentoring takes a variety of forms, including one mentor and one novice, one mentor with several novices, or one novice with several mentors. Secondary mathematics teachers in China generally teach twelve to fourteen class periods per week with each period about 45 minutes (Wang & Zhang, 2016; Yu, 1997). This ensures time is allowed for them to grade assignments, prepare lessons, and conduct teaching research activities including the abovementioned mentoring activities.

Pre-service secondary school teacher preparation programs in China are specialized and discipline based (Ding et al., 2014). Field placement is an indispensable component in secondary school teacher preparation programs. *Curriculum Standards for Teacher Education Program* (Ministry of Education of China, 2011) recommends 18 weeks of teaching practice, which includes school fieldwork and student teaching. Similar to mentors of novice teachers, experienced school teachers function as cooperative teachers of pre-service teachers with university-based teacher educators by providing practical guidance on how to teach.

Becoming a school-based mentor teacher requires a history of rich teaching experiences. For example, graduates with a bachelor's degree in mathematics or mathematics education take at least 5 years to become first-class teachers, another 5 years to become senior teachers, and another 5 years to become professorship teachers (Ministry of Human Resources and Social Security of China, 2015). In short, it takes at least 10 to 15 years for a graduate with a bachelor's degree to become a school-based mentor teacher. School-based mathematics mentor teachers' development is mainly based on their accumulation of knowledge from the experiential learning process and their reflections from their own and their colleagues' mathematics teaching practices.

17.2.3 Mathematics Teaching Researchers (MTRs)

MTRs are housed at teaching research offices at different levels. They are closely connected to school mathematics teachers due to the nature of their duties. The most prominent duties of MTRs are providing guidance to in-service teachers on mathematics classroom teaching and promoting teachers' professional expertise (Huang, Xu, Su, Tang, & Strayer, 2012; The Professional Committee of Secondary Mathematics Teaching Under the Chinese Society of Education, 2014; Wang, 2011). In addition, MTRs are required to provide guidance on school-based teaching research activities, help develop school-based mathematics curriculum, conduct teaching research activities such as organizing public lessons and exemplary lessons, support implementation of new mathematics curriculum and textbooks, develop mathematics examinations at the district level, investigate and monitor mathematics teaching quality within the district, and so on. They are a proven asset in building a professional learning community for teachers from different schools to promote their development and in improving the quality of mathematics education of the entire district (Wang, 2011).

MTRs generally have extensive prior school mathematics teaching experience. The majority of MTRs have more than 15 years of prior school teaching experience, and almost all of them have professional titles of senior or professorship teacher. Moreover, they are required to have experience teaching the entire cycle from grade 7 to grade 9 for junior secondary or from grade 10 to grade 12 for senior secondary or both if possible (The Professional Committee of Secondary Mathematics Teaching under the Chinese Society of Education, 2014). In addition to a history of rich school teaching experiences, the requirements for becoming an MTR include having a solid knowledge base of mathematics and mathematics education, organization and coordination abilities, good command of secondary school mathematics teaching methods and strategies, knowledge of the current status of secondary school mathematics teaching and its reform, the ability to conduct a demonstration lesson and to explain and evaluate mathematics lessons, skills in developing and constructing examination papers, the ability to provide guidance on teaching research, and a high quality of teaching and ability to conduct teaching research (The Professional Committee of Secondary Mathematics Teaching Under the Chinese Society of Education, 2014, p. 64). These requirements are appropriately aligned with the duties that MTRs need to fulfil.

Although the role of MTRs in improving mathematics teaching has been highly valued (Gu & Gu, 2015; Huang, Peng, Wang, & Li, 2010; Paine & Fang, 2007), only a handful of studies have investigated the professional competence required to be an MTR and how MTRs work with school teachers to improve classroom teaching in China (Yu, 2015). For example, via analysis of questionnaire data completed by 549 secondary MTRs, Zhang et al. (2017) proposed a six-dimensional model to characterize MTRs' professional knowledge and competence. This six-dimensional model covers subject and cross-subject knowledge, student learning, teaching competence, assessment of mathematics teaching and learning, guidance provided on teaching and teaching research, and training of mathematics "backbone teachers."[1] Gu and Gu (2015) analyzed discourses in which MTRs provided guidance to teachers on classroom teaching and found that the primary content of the guidance was in the domain of mathematics pedagogical content knowledge rather than mathematics or general pedagogy knowledge. In addition, the guidance was mainly based on the MTRs' own teaching experiences, and the focus of the guidance was on lesson design prior to teaching and on recursive improvement after teaching. This is consistent with the findings reported by Zhang et al.'s (2017) study, in which the MTRs mentioned that they relied heavily on their own teaching experiences to help practicing mathematics teachers improve classroom teaching. Moreover, the MTRs were aware of the limit of their own teaching experiences and would like to acquire theoretical knowledge and research techniques related to observing, analyzing, and evaluating mathematics lessons, which could be applied to their teaching research activities to provide pertinent and evidence-based help to teachers.

However, little is known about how MTRs develop their own professional knowledge and competence to better cope with their duties (Yu, 2015). The general suggested approach to MTRs' professional development includes carrying out teaching research activities, mentoring teachers, observing and discussing classroom teaching, reading and reflecting, learning in an MTR community, and attending training specially designed for teaching researchers (Huang, Su, & Xu, 2014; Wang, 2011;

[1] Backbone teachers are recognized and developed by the government, considered to be better than average teachers in professional qualification, and supposed to act as a model and to play an important role in promoting collective improvements on teaching quality of the district they belong to.

Zhang et al., 2017). The development of MTRs in China as a special group of MTEs is likely to be a fruitful direction of research in the near future.

17.2.4 University-Based MTEs

University-based MTEs are professionals who work at universities to prepare prospective mathematics teachers and to improve practicing teachers' mathematics teaching. They generally specialize in mathematics education, and the majority of them have doctorate degrees in mathematics or mathematics education. Some MTEs may have taught mathematics at secondary schools before pursing their doctorate study.

As is commonly acknowledged, the responsibilities of university-based MTEs are multidimensional (Association of Teacher Educators, 2008; Koster, Brekelmans, Korthagen, & Wubbels, 2005; Smith, 2005) and could be summarized into four categories as follows (Wu & Cai, 2016). First, MTEs are educators and developers of mathematics teachers. They prepare pre-service teachers to become qualified mathematics school teachers, and they help practicing teachers to become competent teachers. They not only instruct teachers in how to teach mathematics but also demonstrate what good mathematics teaching is. This comprises their most crucial responsibility. Second, MTEs are designers, implementers, and evaluators of mathematics teacher education programs for pre-service mathematics teacher preparation and in-service mathematics teacher professional development. They develop and maintain quality programs that are comprehensive, relevant, and grounded in theory and practice and that serve as foundations for teacher education curriculum. Third, MTEs are researchers of mathematics teacher education. They aim to deepen their knowledge base for teaching mathematics by conducting educational research and to validate research findings with their own teaching in teacher education courses. Finally, MTEs facilitate their own professional development by reading theories and the most recent research on teacher education, and by reflecting on teacher education teaching practices and their own research practices on issues relevant to mathematics teacher education and mathematics classroom teaching. Reflection is considered to be a core component in promoting MTEs' self-development and ensures the ongoing growth and development of MTEs' knowledge and expertise.

It should be noted that the abovementioned responsibilities are connected and influenced by each other. Mathematics teacher education research findings provide support to MTEs to act as educators of mathematics teachers and developers of mathematics teacher education programs, the development of MTEs' knowledge and expertise is based on their own practices of teaching and research related to teacher education, and their research on teacher education is conducted around MTEs' teaching practices and professional development activities.

As shown in Fig. 17.1, there are generally two different approaches to becoming a university-based MTE. A doctorate degree is a prerequisite for both approaches.

In the first approach, university-based MTEs move from obtaining their bachelor's degree to obtaining their master's and doctorate degrees without any prior school teaching experience. In the second approach, university-based MTEs teach mathematics at secondary schools prior to obtaining their doctorate degree. The latter group of university-based MTEs generally has a deeper understanding of and direct experience with mathematics teaching at the school level, which appears to be an advantage to them as they cope with their MTE duties (Perks & Prestage, 2008).

17.2.5 Preparing Secondary MTEs in China

In China, programs exist for preparing primary and secondary mathematics teachers, but no specific program exists for preparing MTEs. This section describes the preparation of secondary MTEs in China in institutions in higher education.

Generally speaking, secondary MTEs in China, including school-based mentor teachers, MTRs, and university-based MTEs, usually have a strong knowledge base in mathematics. This, to a large extent, is related to their tertiary education. As indicated in Fig. 17.1, secondary MTEs in China are expected to have a bachelor's degree in mathematics or mathematics education (teacher education oriented). Secondary mathematics teacher preparation programs place a high value on mathematics content knowledge and problem-solving skills (Li, Huang, & Shin, 2008; Wu & Huang, 2018; Yang et al., 2013). At the graduate level, students enrolled in a mathematics education master's program are still required to take a certain number of mathematics content courses.

To give a more detailed picture of the course requirements for MTEs, coursework examples of the mathematics education master's and doctorate programs from a prestigious normal university in China are shown in Tables 17.1 and 17.2. This university was chosen because its mathematics teacher education program is regarded as superior in China and because it is one of the first universities in China to offer a doctorate program in mathematics education, which makes the program indicative of high quality. The course requirements for both the master's and doctorate program from this university are from 2017.

Regarding the university's master's program in mathematics education, a minimum of 36 course credits are required for graduation in addition to a thesis. Among the required credits, 7 credits (19.4%) are for three common courses, 12 credits (33.3%) are for three mathematics content courses, 15 credits (41.7%) are for five mathematics education courses, and the remaining two credits (5.6%) are for an interdisciplinary course not listed in Table 17.1. The mathematics content course credits comprise one-third of the total credits, demonstrating that mathematics content knowledge is highly emphasized in this program. The list of specialized courses in Table 17.1 serves as a platform for master's program students to build a solid and extensive knowledge base in mathematics education and acquire basic knowledge and skills to conduct educational research in mathematics education.

Table 17.1 Coursework requirements for the master's program in mathematics education

Category	Name of course	Credits
Common courses (compulsory)	Theory and Practice of Socialism with Chinese Characteristics	2
	The Outline of Dialectics of Nature	1
	Foreign Language	4
	Discipline and Ethics in Academic Research	/
Basic courses (compulsory)	Algebra	4
	Real Analysis and Complex Analysis	4
	Geometry and Topology	4
Specialized courses	An Introduction of Mathematics Education Research	3
	Psychology of Mathematics Education	3
	Research Methods in Mathematics Education	3
	Mathematics History and Mathematics Education	3
	Measurement and Assessment in Mathematics Education	3
	Elementary Mathematics from an Advanced Standpoint	3
	International Comparison in Mathematics Education	3
	Principles and Methods in Solving Problems	3
	Mathematics Teaching Design and Teaching Case Study	3
	Study of Mathematics Education Literature	3
	Mathematics Textbook Analysis and Development	3

Table 17.2 Coursework requirements for the doctorate program in mathematics education

Category	Name of course	Credits
Common courses (compulsory)	Chinese Marxism and Contemporary World	2
	Foreign Language	4
Specialized courses	Frontiers of Research in Mathematics Education	3
	Mathematics Learning Theory	3
	Quantitative and Qualitative Methods in Conducting Mathematics Education Research	3
	Mathematics History and Mathematics Philosophy	3
	Mathematics Gifted Education	3
	Elementary Mathematics from an Advanced Standpoint	3
	International Comparison in Mathematics Education	3

As for the university's doctorate program in mathematics education, a minimum of 15 course credits are required for graduation in addition to a thesis. Among the required credits, six credits (40%) are composed of common courses, eight credits (53.3%) are composed of mathematics education courses, and the remaining two credits (13.3%) are for an interdisciplinary course not listed in Table 17.2. The coursework for the doctorate program is less demanding and more focused than that of the master's program, which is understandable given that doctorate students need sufficient time to conduct educational research and write dissertations, all of which are aimed at developing their academic interest, innovative ability, and critical and independent thinking skills.

In short, these master's and doctorate mathematics education programs demonstrate that the tertiary education experienced by MTEs emphasizes theoretical knowledge related to mathematics education as well as abilities in conducting academic research specialized in the area of mathematics education but lacks sufficient consideration toward the duties that MTEs must fulfil. This implies the existence of a gap for MTEs between obtaining their master's or doctorate degree in mathematics education and growing into a qualified secondary MTE.

17.3 University-Based Secondary MTEs: Challenges and Responses

In this section, we present a portrait of Chinese university-based secondary MTEs' challenges and strategic responses in their teaching work based on secondary analysis of the data from Wu, Hwang, and Cai's (2017) study.

Teaching work is considered to be among the most important tasks that teacher educators perform in their work (Association of Teacher Educators, 2008; Koster et al., 2005). This suggests a need to examine the particular challenges that arise in MTEs' teaching work. There is a clear distinction between the teaching work of mathematics teachers and that of MTEs because they operate at different levels (Zaslavsky & Leikin, 2004). This study investigated the challenges that university-based secondary MTEs reported facing in their teaching work as well as their strategies for dealing with these challenges.

This study addressed four activities that constitute the main teaching work of university-based secondary MTEs: teaching pedagogical courses, teaching courses on mathematical problem-solving, teaching undergraduate mathematics courses, and supervising student teaching. Pedagogical courses, such as mathematics pedagogy, deal with issues related to how to teach school mathematics. These courses are similar to what would be called methods courses about the teaching of mathematics. Courses on mathematical problem-solving involve mathematical ideas, mathematics methodology, and mathematical thinking at the school level, but are not oriented toward teaching specific mathematics content. For example, problem-solving strategies like pattern recognition, working backward, and using diagrams are typically included in these courses. These courses often include a focus on developing teachers' mathematical problem-solving skills and their understanding of school mathematics from an advanced perspective (Cai & Nie, 2007). Undergraduate mathematics courses refer to content-focused courses like calculus and linear algebra that are typically also open to undergraduates from across the university. Student teaching is a required field experience for pre-service teachers in China, and MTEs are typically responsible for supervising pre-service teachers. Other responsibilities of university-based MTEs in China, such as supervising undergraduate students' theses and training in-service mathematics teachers, are not considered in this study.

17.3.1 Methodology

Participants The study's participants were chosen from among the participants of a research conference on mathematics education in China. Ninety-five of the conference participants were identified as university-based secondary MTEs and were invited via email to participate in the study and complete a questionnaire. Responses from a total of 68 MTEs were retained for data analysis finally, representing views from MTEs who were actively involved in mathematics education activities and research in China. However, because this is a sample of convenience, any generalization of the results reported here should be made with caution.

Most of the 68 MTEs (85%) worked in a mathematics department rather than a school of education or teacher education department, more than half of them (53%) had taught mathematics in secondary schools, and most of them (90%) held their bachelor's degree in mathematics or mathematics education (teacher education oriented).

Instrument We developed a questionnaire consisting of two parts to investigate the challenges that MTEs face in the four activities that comprise their teaching work as well as the strategies they have developed to cope with these challenges. The first part of the questionnaire collected background information including demographic data, level of education, years of teaching experience both in schools and as MTEs, and types of institutions worked in. The second part of the questionnaire consisted of four sets of open-ended questions that probed the MTEs' experiences in the four activities of teaching work (teaching pedagogical courses, teaching courses on problem-solving, teaching undergraduate mathematics courses, and supervising student teaching). Figure 17.2 shows a sample set of questions related to teaching pedagogical courses. We used open-ended questions rather than multiple-choice or Likert-type items to prompt rich and in-depth responses, as suggested in other studies (Cai, Ding, & Wang, 2014). This section focuses on their perceptions of the challenges encountered by these secondary MTEs as well as their approaches to handling those challenges. However, their responses to the kind of support they obtained and their suggestions for preparing MTEs are not included in the analysis in this section.

1. Have you ever taught pedagogical courses like mathematics pedagogy or mathematics instructional design? If not, please skip this question. If yes, please answer the following items:

(1) What challenges have you encountered in teaching pedagogical courses?
(2) How did you respond to these challenges?
(3) What kind of support and help have you obtained?
(4) How do you think future mathematics teacher educators should be trained so as to avoid the kind of challenges you have encountered based on your own experiences?

Fig. 17.2 Sample questions

Data Analysis The coding scheme was developed using a grounded approach working from the responses of the participants. We created initial codes to record the participants' responses to each set of questions. If a participant's response did not fit into any of the available codes, a new code was added. In this way, we obtained a fine-grained coding scheme. This scheme was carefully examined to capture the meaning of the codes in a compact and explanatory way.

Because the instructional process generally involves interactions among curriculum, students, and teachers (Cohen, Raudenbush, & Ball, 2003), the codes for challenges and strategies were initially grouped based on which of these three categories they applied to. Note that in this categorization, Teacher refers to the participants themselves. Some codes, like "how to improve students' teaching ability," did not fit neatly into any of the three categories. A fourth category, Pedagogy, was added to resolve the issue. Therefore, *Curriculum*, *Students*, *Teacher*, and *Pedagogy* were used as categories to organize the codes for the challenges and strategies described by the MTEs for the first three activities (those dealing with teaching courses). The codes for challenges and strategies regarding student teaching were slightly different, reflecting the more practice-oriented nature of this activity. Supervising student teaching does not have specific teaching content in the same sense as the other courses do, but it does require MTEs to deal with logistical issues during the period of student teaching. Therefore, *Logistics* replaced *Curriculum* in the four categories of response codes, with the other three categories kept the same.

We established inter-coder reliability to determine the extent to which different coders using the same coding scheme would obtain consistent results. Twenty-seven responses (40%) were randomly selected and coded independently by two coders for 98% agreement, indicating a quite high level of inter-coder reliability.

17.3.2 Results

Challenges Encountered in Teaching Work The number of participants who responded to each set of questions, in descending order, is 60 (Q4, supervising student teaching), 58 (Q1, teaching pedagogical courses), 44 (Q3, teaching undergraduate mathematics courses), and 39 (Q2, teaching mathematics problem-solving courses), indicating that the MTEs had more experience in teaching pedagogical courses and supervising student teaching. For each question, a small percentage of participants responded that they had not encountered any challenges in their work. In descending order, these percentages were 18% (8/44, Q3), 13% (5/39, Q2), 7% (4/60, Q4), and 5% (3/58, Q1). This suggests that teaching pedagogical courses and supervising student teaching were generally more challenging for the MTEs than teaching undergraduate mathematics courses or mathematical problem-solving courses.

Most of the participants reported that they encountered challenges in their work. Table 17.3 shows the number of participants reporting each challenge category across the four types of teaching work. Overall, these values follow the trend of percentages for participants who claimed that they did not encounter any challenges, namely, that there were more participants who reported challenges in teaching pedagogical courses and supervising student teaching than participants who reported challenges in teaching undergraduate mathematics or mathematics problem-solving courses.

Table 17.3 shows that challenges related to *Curriculum* (49.1%) and *Students* (43.6%) were mentioned more frequently in teaching pedagogical courses. Challenges related to *Curriculum* included concerns that there were not enough appropriate teaching cases that could effectively illustrate the content and pedagogical issues in the courses as well as concerns about the gap between course content and actual teaching in school classrooms. Challenges related to *Students* involved concerns about students' negative attitudes. Over 90% of the respondents (22 out of 24) referred to students' lack of interest in the course, lack of participation in the classroom activity, or refusal to pay attention to the lecture. Some of the responses included participants' views on the reasons for these negative student attitudes, with one participant attributing it to the value of the courses: "Courses like mathematics pedagogy and mathematics instructional design contain many theories. When we talk about these theories, students did not see the value of the theories, and thus they did not have a positive attitude and were not interested in the courses." *Teacher* and *Pedagogy* were relatively less frequently mentioned categories of challenges related to teaching pedagogical courses. The most frequently reported challenge related to *Teacher* (i.e., the MTEs themselves) was their lack of school teaching experience.

In supervising student teaching, challenges related to *Pedagogy* were mentioned most frequently (60.7%) by the MTEs. They described challenges regarding how to help pre-service teachers prepare for a lesson (including textbook analysis and analysis of school student thinking as well as writing a lesson plan) and deliver a lesson, how to improve pre-service teachers' teaching skills, and how to help them manage the classroom and deal with the inconsistencies between what had been taught in the pedagogical courses and what happened in real classrooms. In addition, some of the challenges in supervising student teaching were related to *Logistics*. The

Table 17.3 Distribution of participants reporting different challenge categories across the four types of teaching work

Challenge category	Q1, teaching pedagogical courses ($N = 55$)	Q2, teaching math problem-solving courses ($N = 34$)	Q3, teaching undergraduate math courses ($N = 36$)	Q4, supervising student teaching ($N = 56$)
Curriculum/ Logistics[a]	**27 (49.1%)**	**16 (47.1%)**	3 (8.3%)	**19 (33.9%)**
Students	**24 (43.6%)**	4 (11.8%)	**14 (38.9%)**	4 (7.1%)
Teacher	13 (23.6%)	**13 (38.2%)**	8 (22.2%)	9 (16.1%)
Pedagogy	16 (29.1%))	5 (14.7%)	**16 (44.4%)**	**34 (60.7%)**

Note: Values in bold show the more frequently mentioned challenge categories
[a]Challenges related to Logistics are specific to Q4

participants described a number of such challenges, including not having a stable set of schools for student teaching, pre-service teachers not having enough chances to deliver mathematics lessons, the lack of assessment and management policies to monitor student teaching, and the need to mediate the relationship between pre-service teachers and their cooperative teachers.

As shown in Table 17.3, there were comparatively more challenges related to the categories of *Curriculum* (47.1%) and *Teacher* (38.2%) in teaching courses on problem-solving. Many participants expressed that these courses usually focus too much on problem-solving techniques rather than functioning as methodological courses aimed at understanding the nature of mathematics. Ideally, courses like these should foster connections between advanced and school mathematics. Several participants mentioned the lack of good textbooks for courses on problem-solving. The challenges under the *Teacher* category fell into two subtypes. One concerned the MTEs' own insufficient school mathematics problem-solving skills. The other was related to teachers not having a solid and in-depth understanding of mathematical thinking and ideas.

In teaching undergraduate mathematics courses, challenges related to *Students* (38.9%) and *Pedagogy* (44.4%) were mentioned more frequently. The participants expressed concerns about students' negative attitudes toward courses like calculus and linear algebra, including their lack of interest in the topic and their fear of difficulties encountered in the learning process. The most frequently mentioned challenge in the *Pedagogy* category was how to help students understand and appreciate undergraduate mathematics.

MTEs' Strategies for Responding to Challenges The four categories used to classify the challenges reported by the MTEs were also used to organize their reported strategies for responding to those challenges. Table 17.4 shows the numbers of participants who reported using a strategy related to each category to address challenges in each of the four types of teaching work. Across these four activities, it would appear that participants tended to use strategies related to *Teacher* (i.e., the MTE) and to *Pedagogy* more than strategies related to *Curriculum/Logistic*s or *Students*.

In responding to the challenges related to teaching pedagogical courses, strategies related to *Teacher* (41.8%) refer to different approaches mentioned by the MTEs aimed at increasing their own practical experience and theoretical knowledge base, including observing school classroom teaching, communicating with school teachers, reading and self-reflection, discussions with colleagues, attending seminars and conferences, and practices such as editing textbooks and carrying out educational research. Strategies related to *Pedagogy* (47.3%) involve making changes in the design of teaching content such as using more examples or using real classroom videotapes to make connections between theory and practice, making changes in teaching methods by incorporating group work, incorporating classroom teaching observations and discussions, and introducing alternative assessment methods such as performance assessments and project work. For example, one participant wrote, "I will make connections between lecturing on educational theories and

Table 17.4 Distribution of participants reporting different categories of strategies for responding to challenges

Strategy category	Q1, teaching pedagogical courses (*N* = 55)	Q2, teaching math problem-solving courses (*N* = 34)	Q3, teaching undergraduate math courses (*N* = 36)	Q4, supervising student teaching (*N* = 56)
Curriculum/ Logistics[a]	14 (25.5%)	9 (26.5%)	0	11 (19.6%)
Students	7 (12.7%)	0	7 (19.4%)	2 (3.6%)
Teacher	**23 (41.8%)**	**23 (67.6%)**	**11 (30.5%)**	12 (21.4%)
Pedagogy	**26 (47.3%)**	9 (26.5%)	**21 (58.3%)**	**36 (64.3%)**

Note: Values in bold show the more frequently mentioned strategy categories
[a]Strategies related to Logistics are specific to Q4

discussing teaching cases, and arrange activities through observing lessons, studying lessons, explaining lessons and discussing lessons alternatively."

About two-thirds (64.3%) of the participants shared strategies related to *Pedagogy* to address challenges in supervising student teaching. Their approaches could be classified into two categories based on when they would be applied: before or during student teaching. Before student teaching, the participants provided course-based training like "mathematics textbook analysis" and "teaching cases in mathematics classroom teaching" and conducted practical training such as simulation teaching, microteaching, and observations of real classroom teaching. During student teaching, the participants responded that they would advise pre-service teachers on how to write lesson plans, attend their rehearsals and actual teaching, provide feedback on how to adjust instructional design based on rehearsals and actual teaching, give suggestions on how to observe school teachers' teaching, and guide pre-service teachers to polish a lesson through lesson studies.

To address their challenges related to teaching mathematical problem-solving courses, around two-thirds (67.6%) of the participants mentioned strategies related to the *Teacher* category. They expressed that they needed to improve their own mathematical problem-solving abilities and mathematical knowledge base and to do more reading and reflection. In dealing with the challenges related to teaching undergraduate mathematics courses, nearly one-third of the participants (30.5%) responded that they study the textbooks and expand their own mathematical knowledge base. Nearly 60% of the participants mentioned that they would address their challenges in teaching these courses by changing their teaching method.

The Role of Prior Secondary School Teaching Experience Teaching pedagogical courses appears to be the most challenging among the four types of university-based MTEs' teaching work. We conducted further analysis to probe the relationship between prior secondary school teaching experiences and the challenges encountered in teaching pedagogical courses as well as strategies that the participants reported having used to cope with these challenges. Tables 17.5 and 17.6 summarize the results of this analysis. The participants were categorized in three groups according to their amount of prior school teaching experience: none, less than 5 years, and equal to or more than 5 years.

Table 17.5 shows that the majority of the participants with equal to or more than 5 years of school teaching experience (63.1%) reported challenges related to *Students*, but no one reported challenges related to the *Teacher* category, whereas the more frequently mentioned challenge categories reported by the participants who had no prior school teaching experience were *Curriculum* (68.2%) and *Teacher* (36.4%). This observed pattern suggests that prior school teaching experience seems to have influenced the MTEs' perception of the challenges that they encountered in their teaching of pedagogical courses.

Table 17.6 shows the strategies that the participants with different amounts of prior teaching experience reported having used to cope with their challenges. Most of the strategies shared by the participants with no school teaching experience fell into the *Teacher* category, whereas most of the strategies mentioned by the participants with prior school teaching experience fell into the *Pedagogy* category. This implies that the learning needs of MTEs may vary according to how much prior school teaching experience they have.

One finding apparent from these two tables is that the participants with no school teaching experience mentioned more challenges related to *Curriculum* and *Teacher* and responded with more strategies related to *Teacher* to deal with their challenges, whereas the participants with equal to or more than 5 years of school teaching experience reported more challenges related to *Students* and shared more strategies related to *Pedagogy* to deal with these challenges.

17.3.3 Summary

Based on the results of the questionnaire analysis, we can formulate several conclusions. First, it is evident that the MTEs reported more challenges in teaching

Table 17.5 Distribution of participants with different prior school teaching experience in reporting different categories of challenges in teaching pedagogical courses

Prior school teaching experience	Curriculum	Students	Teacher	Pedagogy
None (*N* = 22)	**15 (68.2%)**	6 (27.3%)	**8 (36.4%)**	6 (27.3%)
<5 (*N* = 14)	4 (28.6%)	6 (42.8%)	5 (35.7%)	6 (42.8%)
> = 5 (*N* = 19)	8 (42.1%)	**12 (63.1%)**	**0**	4 (21.1%)

Table 17.6 Distribution of participants with different prior school teaching experience in reporting different strategies related to challenges in teaching pedagogical courses

Prior school teaching experience	Curriculum	Students	Teacher	Pedagogy
None (*N* = 22)	3 (13.6%)	2 (9.1%)	**13 (59.1%)**	8 (36.4%)
<5 (*N* = 14)	6 (42.8%)	2 (14.3%)	4 (28.6%)	**8 (57.1%)**
> = 5 (*N* = 19)	5 (26.3%)	3 (15.8%)	6 (31.6%)	**10 (52.6%)**

pedagogical courses and supervising student teaching than in teaching undergraduate mathematics courses and mathematical problem-solving courses. This finding reflects a key difference in teaching demands between teaching mathematics and teaching how to teach mathematics. Second, the strategies mentioned by the MTEs to address their challenges were often related to pedagogy and the teacher (i.e., the MTEs themselves), although the challenges they described also involved the curriculum and students. It could be the case that the *Pedagogy* and *Teacher* categories are more easily manipulated by the MTEs as compared to the *Curriculum* and *Students* categories. Third, there were some indications that prior school teaching experience may have an effect on the ways that MTEs perceived and dealt with challenges in their teaching of pedagogical courses. This finding informs the content and goals of professional development for MTEs with different amounts of prior school teaching experience.

17.4 Discussion

In this chapter, we have presented a portrait of Chinese secondary MTEs by describing who they are, what responsibilities they have, how they develop, what challenges they face in their work, and how they respond to these challenges. This is a particularly timely portrait given that research on MTEs is just emerging, especially in China, where the roles of teacher educators in preparing and developing school teachers have been emphasized in recent documents such as the *Teacher Education Revitalization Action Plan (2018–2022)* released by the central government of China. Moreover, the description of the current situation of types, responsibilities, and developmental trajectories of secondary MTEs in China and the challenges encountered by university-based secondary MTEs as well as the strategies that help them cope with these challenges may serve as a useful reference to inform, reflect on, and improve the preparation of MTEs both in China and in other countries.

China has a large population of school teachers. According to its educational statistics (Ministry of Education of China, 2017), there were approximately 5.2 million secondary school teachers in 2016. Pre-service teacher education and in-service teacher professional development in China is based on a unique, long-standing tradition. Secondary mathematics teacher preparation programs in China place a strong emphasis on mathematics content knowledge, with less attention given to mathematics pedagogical knowledge, whereas in-service mathematics teacher professional development programs highlight the acquisition of practice-based mathematics teaching knowledge within a four-level teaching research network in the context of a teacher promotion incentive system. School-based mentor teachers and MTRs give strong support and guidance to help practicing mathematics teachers move from novice, to qualified, to excellent via the organization and implementation of various teaching research activities such as planning lessons in a group and exemplary lesson observation and evaluation. A consensus exists that it is impossible to develop pedagogical content knowledge without solid and appropriate content knowledge, and China's tradition of emphasizing content knowledge at the pre-service stage and practice-based pedagogical content knowledge at the in-service stage within the four-level teaching research network appears reasonable and sound. The strong content knowledge base acquired at the pre-service stage is treated as the foundation for teachers' continued growth in their pedagogical content knowledge in and from real teaching practices at the in-service stage.

MTEs are professionals who guide mathematics teachers through this developmental process. Although three groups of secondary MTEs are described in this chapter, school-based mentor teachers and MTRs could be clustered together because they both have extensive practical knowledge for mathematics teaching embedded in real classroom teaching contexts. In contrast, university-based MTEs generally have a deep understanding of theoretical knowledge related to teaching and learning in general and to mathematics teaching and learning in particular, and they have methodological knowledge of and rich experiences in mathematics education research. MTEs' contribution to the development and growth of mathematics teachers has been noted and acknowledged (Even & Krainer, 2014; Gu & Gu, 2016; Huang et al., 2010).

As seen in this chapter, however, both MTRs and university-based MTEs face challenges in providing guidance to prospective and practicing mathematics teachers. On the one hand, the MTRs mentioned that their help relied heavily on their own teaching experiences, without attending much to relevant theoretical views and evidence-based guidance in light of findings from related research (Gu & Gu, 2015; Zhang et al., 2017). On the other hand, among the four types of teaching work, the university-based MTEs reported greater challenges in teaching pedagogical courses. Specifically, the MTEs without prior school teaching experience mentioned more challenges related to the *Curriculum* and *Teacher* (i.e., themselves) categories. This finding implies that the MTEs without prior school teaching experience were not ready to teach pedagogical courses, possibly due to their lack of practical knowledge in mathematics teaching.

Two issues stand out from the above observation. First, this finding suggests that MTEs, whether university-based MTEs or MTRs, are working at the intersection of the theory and practice domains. Any deficit in either domain may result in challenges in their teaching work. Second, this observation implies that there is inadequate communication taking place between university-based MTEs and MTRs. Indeed, what is needed of one group is actually possessed by the other group. Building up a learning community composed of both university-based MTEs and MTRs could be a feasible way to resolve the challenges mentioned by each group. In fact, there are communities in which mathematics teachers, mathematicians, textbook writers, and mathematics education researchers work together to produce innovative curriculum materials and to organize teacher professional development activities by taking advantages of each other's strengths (Llinares, Krainer, & Brown, 2014). These communities provide examples of how different types of MTEs can work collaboratively and productively.

A fundamental question underlying this discussion, one which deserves further exploration, is what knowledge and expertise secondary MTEs need. Is the ideal model of secondary MTEs a combination of an MTR and a university-based MTE? In other words, does the ideal model of secondary MTEs need to have rich practical knowledge embedded in the context of school mathematics classroom teaching and learning as MTRs and theoretical knowledge including knowledge of theory and research involving teaching and learning in general and teaching and learning of mathematics in particular and knowledge of educational research methodology as university-based MTEs? If so, how do we prepare secondary MTEs to meet the requirements of this model? Does theoretical knowledge come first, as in the development of university-based MTEs who could begin with no prior school teaching experience and pursue a bachelor's degree followed by a master's and doctorate degree, or does practical knowledge come first, as in the development of MTRs who are consistently immersed in real classroom teaching practices? Is there a compromise that could effectively combine the development of practical and theoretical knowledge? Lin, Yang, Hsu, and Chen (2018) explored the perspectives of mathematics teacher educator researchers on the use of theory in facilitating teacher growth, and they proposed that the key to the development of MTEs' professional expertise is the process of decontextualizing and recontextualizing theory between research and practice. Their view opens a new window into the relationships among theory, research, and practice, and it may prove interesting, relevant, and informative in identifying how to develop both the practical and theoretical knowledge of MTEs.

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Part III
Methodological Challenges in Researching Mathematics Teacher Educator Expertise, Learning and Development

Chapter 18
What Influences Mathematics Teacher Educators' Decisions in Course Design: Activity Theory and Professional Capital as an Investigative Approach

Greg Oates, Tracey Muir, Carol Murphy, Robyn Reaburn, and Nicole Maher

18.1 Introduction and Impetus for the Study

The Bachelor of Education (Primary) course at the University of Tasmania is a four-year programme (two semesters per year, March to June and July to November) for pre-service teachers preparing to teach children in the early years to year six of primary school, with an enrolment of between 800 and 900 students. Most pre-service teachers come directly from school, and our experience is that many are not confident with their knowledge of mathematics and have negative dispositions towards mathematics. The degree structure has 32 units in total (each unit is one semester long with 13 weeks of teaching; a unit may also be commonly called a paper or a course in other institutions), which are a mix of curriculum content units (e.g. English, mathematics, creative arts, health and physical education, science and technology), general pedagogy units (e.g. inclusive practices, assessment, child development) and practical classroom experience (some 80 days over the 4 years, often referred to as practicum).

Mathematics education teaching staff in the course held a preliminary meeting in September 2017 to consider the three core mathematics units in the programme, taught sequentially in semester one of the first year, semester two of the second year and semester one of the final fourth year. Thus, the interpretation of mathematics teacher educator (MTE) we use here refers to university-based academics who work in a School of Education, rather than mathematicians who teach mathematical content to prospective teachers or supervisors of the school-based practicum. Each unit has a principal coordinator, but we also have input into each other's units through tutoring and sometimes assuming the coordinator's role during leave. The meeting was driven by internal and external factors, both pragmatic and aspirational, that

G. Oates (✉) · T. Muir · C. Murphy · R. Reaburn · N. Maher
School of Education, University of Tasmania, Newnham Campus, Launceston, TAS, Australia
e-mail: greg.oates@utas.edu.au

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_18

fuelled a desire to critically examine the content, delivery and coherence between the three units. Internal factors included the need to set teaching allocations for 2018, our belief in the value of periodic review as reflective practice, changes in staff over the previous 2 years that made us suspect the alignment of our units may have drifted somewhat and the peer review of assessment process undertaken by the wider School of Education. Externally, there has been a move in Australia, largely driven by the Australian Institute for Teaching and School Leadership (AITSL), towards a national, highly regulated approach to initial teacher education. AITSL has published professional standards for teachers (APST) that set out what graduates of initial teacher education programmes need to know and be able to do, and these standards are used to accredit programmes and regulate their content (Australian Institute for Teaching and School Leadership [AITSL], 2018).

Discussions at this early meeting raised many questions about the knowledge and beliefs we bring to the decisions we make, individually and collectively, in relation to content, pedagogy and assessment practices. We realised that there were a range of factors underpinning these decisions that influenced the design of our overall courses and teaching units. As researchers, we were naturally interested in exploring these factors and decided to record our future conversations and investigate possible theoretical frameworks that might guide our decisions. A literature search suggested that research into the knowledge and practices of mathematics teacher educators is somewhat limited: for example, studies examining pedagogical content knowledge (PCK) (Ball, Thames, & Phelps, 2008; Chick & Beswick, 2018; Shulman, 1987; 2015) are predominantly focused at the primary school, secondary school or pre-service teacher level (Goos, 2013).

It was clear to us all that our PCK as MTEs plays a significant role in underpinning the decisions we make about course and unit design and, at this time, we lacked an appropriate theoretical approach to analyse these effects. We decided to deepen the extent of the review, document and interrogate the process we undergo in the collective redevelopment of our units and explore theoretical bases for the decisions we make. The process we followed is described more fully in Sect. 4. Next, we consider the nature of MTEs' practice and the theoretical perspectives we adopted to examine our deliberations.

18.2 Researching the Work of Mathematics Teacher Educators

Whilst many teacher educators in Australia transition into higher education from school teaching, their work in this new domain requires more and different types of knowledge and understanding compared with that required of school teachers (Murray & Male, 2005). Teaching others to teach is about "thoughtfully engaging with practice beyond the technical; it is about using the cauldron of practice to expose pedagogy (especially one's own) to scrutiny" (Loughran, 2014, p. 274). The work of teacher educators, and particularly mathematics teacher educators, remains largely unexplored, with most of the research in this field being dominated by accounts involving self-study (e.g. Alderton, 2008; Schuck, 2002) that may not necessarily lead to changes in institutional practice. Chapman (2008), in a review of such studies, found that the authors were not explicit about what they learned and how they learned from their research. She recommended formal systematic study that focused on MTEs' teaching approaches such as a course, a programme or specific activities/tasks given to their students to determine their effectiveness.

Brown (2009) suggested that teaching is essentially a design activity whereby the teacher makes decisions about how to sequence content and which instructional strategies and materials to use. These strategies, materials and content are artefacts of teaching that mediate their decisions, and MTEs' beliefs, content and pedagogical content knowledge mediate their decisions. Li and Superfine (2018) further suggested that it is crucial to understand the design of such courses. They found that the MTEs in their study shared the goal of deepening pre-service teachers' mathematics knowledge and believed that the creation of a positive and collaborative learning environment was important for pre-service teachers. Other common themes included the need to expand the mathematics content needed for teaching, ensure pre-service teachers have a positive experience with mathematics, create collaborative and safe learning environments, implement worthwhile mathematical tasks, connect to teaching practice, encourage collaborative group work and assess pre-service teachers' learning using comprehensive assessments. Challenges identified by the MTEs included managing pre-service teachers' various backgrounds, reshaping pre-service teachers' perceptions about the value of the course and assigning grades to tasks that assess deep learning. Li and Superfine concluded that the six MTEs designed mathematics content courses from a learner-centred perspective and integrated various instructional approaches to support pre-service teachers' mathematics learning in pedagogically appropriate ways.

The principal focus in this chapter is on the gap in systematic study of programme design identified earlier by Chapman (2008). Building on studies such as that of Li and Superfine (2018) and in relation to Brown's (2009) view of teaching as a design activity, we aimed to examine the themes that emerged from our conversations and to explore how different theoretical perspectives might aid our examination. It is important to note that, in Australia, university-based educators are traditionally seen to have more autonomy and control over their work than school teachers (Loughran, 2014). Chick (2011) referred to this autonomy, emphasising how teacher educators make many of their choices in isolation, with plenty of scope for decisions, essentially acting as "god-like arbiters" of what should be included in mathematics education courses (Chick, 2011, p. 5). Thus, autonomy was intrinsic to our examination.

18.3 Theoretical Perspectives

We initially explored a number of theoretical perspectives in arriving finally at the two adopted here. These included communities of practice (Wenger, 2010), critical inquiry (Jaworski, 2006), discourse analysis (Forman, 2000) and self-study (Pinnegar & Hamilton, 2009). Each of these has been used successfully to examine MTE practice in some capacity but seemed less appropriate for our specific purposes. For example, whilst our group reflects elements of, and may well constitute, a community of practice within Wenger's definition, our focus on course design narrowed the scope of the study from the wider perspective that such studies usually adopt. Similarly, whilst our pre-service students' needs were clearly a key interest in our deliberations, the nature of how our decisions may explicitly affect student learning in the school classroom was less of a focus at this time than considered by Jaworski. A formalised self-study community (Lassonde, Galman, & Kosnik, 2009) may ultimately be a very effective outcome from our current course review and the subsequent decision to examine it from a research perspective. However, the organic way in which this study developed seemed to preclude such an approach, given that many of our preliminary discussions and decisions were conducted outside of an established self-study research framework.

Reflections on the nature of discussions at our September 2017 meeting and our examination of theoretical possibilities in the literature led us to settle on two theoretical perspectives to inform our analysis: activity theory and professional capital. Activity theory (Cole, Engeström, & Vasquez, 1997) arises from Vygotsky's cultural-historical theories and recognises how design of activity happens within a system (Hardman & Amory, 2015). Professional capital arises from the notion of human capital (Strober, 1990) and is concerned with the knowledge, skills, experiences and personal attributes of individuals. Undoubtedly, these skills and experiences will have arisen through systems, but there is no attempt, in this chapter, to relate the two theories. Our intention is to determine the appropriateness of each theoretical approach rather than to look for relationships between them. How does each of the theories highlight aspects of our decisions? What insights does each theory provide?

18.3.1 Activity Theory

Activity theory has developed from a collective group of methodological concepts as a way of analysing relations between personal, institutional and societal conditions (Fleer, 2016). Established by Engeström (1987) and rooted in the work of Vygotsky and Leont'ev, activity theory is concerned with learning and development in workplaces. Two aspects of activity theory suggested its relevance as a theoretical framework for this chapter. First, activity theory explores a collectively shared process within cultural and historical dimensions. Second, analysis focuses on activities within the context where they take place (Sannino, Daniels, & Guitérrez, 2009). In this respect, activity theory has emerged as an effective means of examining and transforming higher education pedagogy (Bozalek et al., 2014). Activity theory in this context provides a way to analyse our decisions within the practical social activities that organise our work whilst also recognising that through our activities, we transform conditions for our teaching and for our students as we generate new artefacts and practices.

The six essential elements of activity theory are presented in Engeström's (1987) second-generation model of activity theory (Fig. 18.1). A key element is the object of activity (Blackler, 2009). The object is defined as the problem space that the activity is addressing (Hardman & Amory, 2015) or the thing being done. Whilst fundamental to the activity, the object is not so easy to identify as it is socially constructed and often contested (Blackler, 2009). What might be seen as the object for one person is seen differently by another. A second element, the subject(s), refers to those responsible for the activity or the "doers" who engage in the activity. The subjects are distinct from the social group in which the activity is taking place, which is described by the third element, community. The fourth element, instruments or tools, is the artefacts that act as mediators. Such tools can be physical objects or concepts or beliefs that mediate the activity. The fifth element, the division of labour, refers to the roles of the subjects within a community, and the sixth element, rules, refers to the norms and conventions that mediate between the subject, the community and the object. The six elements interact, shape each other and result in the outcome(s), which can be intended or unintended. Within activity theory, there is a distinction between the object or motive of the activity and the intended or unintended outcome(s).

Activity theory provides a framework to identify the dynamics and tensions between these different elements. In this chapter, we explore how the framework enables us to analyse the reasons for our decisions and explore our agency within a system (Sannino et al., 2009, p. 18).

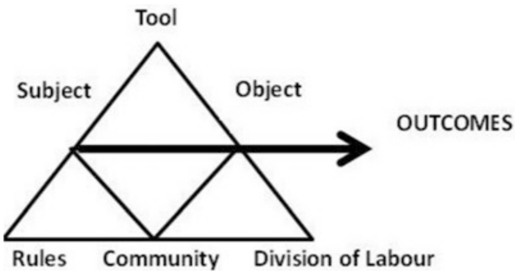

Fig. 18.1 Essential elements of activity theory. (Engeström, 1987, Fig. 18.1; with permission of Cambridge University Press)

18.3.2 Professional Capital

Based on the notion of capital as “one’s own or a group’s worth” (Hargreaves & Fullan, 2012, p. 1), human capital has become a dominant economic theory in policy decision-making in Western education (Gillies, 2015). In more recent times, the definition of human capital has widened beyond skills to include personal attributes such as “competencies,” “attributes” and “attitudes” (Becker, 1992, p. 6), and Hargreaves and Fullan (2012) have drawn on these more recent definitions in developing the conceptualisation of professional capital as the importance and effectiveness of professional work. The notion of professional capital in an educational context relates to the qualities and worth of professional individuals within education settings (Gillies, 2015). Teachers are seen as decision-makers whose decisions are informed by these personal attributes. Professional capital not only incorporates the talent of individuals but also recognises how talents are shared and networked over a group and provides a model for analysing decisions that are made both by individuals and by groups.

Hargreaves and Fullan (2012) proposed three aspects of professional capital: human capital, social capital and decisional capital. Human capital refers to the “worth” of each individual. Social capital refers to the quality, frequency and focus of interactions amongst a group of educators. A key element of professional capital is how social capital adds value to human capital. The third aspect, decisional capital, refers to the ability of individuals to make judgements and decisions, which link back to human capital and to social capital. In these regards, professional capital is the product of human, social and decisional capital and is essential for professional capital to be realised.

Hargreaves and Fullan (2012) stated that “making decisions in complex situations is what professionalism is all about” (p. 5). Hence, the concept of professional capital presents an opportunity to examine our decisions within a theoretical framework that focuses on the worth of individuals and the group, alongside our capacity to interact with each other within the institution. Professional capital has been used to explore teachers’ numeracy as a general capability and to assign value to aspects of their practice (Callingham, Beswick, & Ferme, 2015). We use this theoretical model to consider how we, as MTEs, make professional decisions and to what extent they might be explained in respect of their capital value.

18.4 The Course-Review Process

The first step leading to the review of our Bachelor of Education (Primary) course was an informal, anonymous, mostly Likert-scale survey of students midway through their final semester of study in August 2017. The survey sought their perspectives on the value and alignment of content and assessment in our three core mathematics units and how effective these units are in terms of preparing them to teach mathematics. Questions in the survey asked students about their perceived preparedness for teaching a range of mathematical content and strategies they might employ. Responses to three questions in particular supported the need to review our units:

Q27: How well do you think the units aligned with or flowed on from each other?
Q29: Do you think the assessment items provided a good balance throughout the units?
Q31: To what extent do you feel the teaching and design of these units reflects the teaching and learning principles we are asking you to develop in your classrooms?

Although the overall response rate was admittedly low (35 responses from 195 enrolments), we felt that the fact that no question received strong endorsement was concerning (a mean of 2.5, 2.5 and 2.6, respectively, on a five-point Likert scale, with "1" high/strong, "3" neutral and "5" low/weak). Question 31 received no responses in the "strongly agree" option. In September 2017, we held our first group meeting to consider the alignment of our units, in advance of preparing our units for 2018 and in light of student responses to the survey and the experiences of new staff teaching in our units. This first meeting was not recorded, as the decision to research our process had not been made at that point; indeed, the decision to conduct the research grew out of our awareness of the complexity of issues we had discussed and our limited progress at this meeting.

Thus, we decided to document and examine our review process. A second two-hour meeting was held in March 2018, which was audio-recorded and transcribed. We also captured images from our whiteboard recordings and collected documents from the meeting for later analysis. Subsequently, we held a one-hour focus group interview with all five participants, facilitated by a former teaching colleague in our mathematics education group. Questions for this focus group were formulated from a combination of issues identified in the literature and impressions from our initial meeting which had inspired the whole research initiative.

Apart from the preliminary student survey data which prompted our review, our data for the present investigation are qualitative in nature. Overall, our analysis incorporated a discourse analysis approach, although the thematic analysis we used to code our data and arrive at our categories was less rigorous than might be commonly employed in such an approach (Braun & Clarke, 2006; Riessman, 2008). Instead, the purpose of the discourse analysis was to identify emerging themes that might be examined using the two theoretical perspectives. Thus, the codes we arrived at here may be largely seen as descriptive, rather than explanatory. A more rigorous inductive thematic analysis is warranted and remains an area for further investigation.

Our first approach was to read through the transcripts from the meetings and interviews independently and to look for consistent and emerging themes. These themes were coded independently. We then met on two occasions (meetings three and four) to consider, compare and refine the themes we had identified. After meeting four, we used NVivo to further explore and strengthen the initial emerging themes agreed on at meeting three. These initial themes were gathered together and presented under four major themes: beliefs and values, agency and autonomy, PCK for pre-service teachers and alignment. Meeting four raised some questions about these themes, for example, whether they might be legitimately emerging from the data compared to what we were predisposed to look for or to what extent they might be complete in their representation. Whilst we acknowledge that these aspects may need more attention, we feel that the identified themes were appropriate for trialling our chosen theoretical perspectives.

The participants in the study are identifiable as the authors, but names have been changed to gender-neutral pseudonyms to avoid attributing quotes to specific persons.

18.5 Results and Analysis of Meetings and Interviews

As described in the previous section, there were four major themes that emerged from the discourse analysis of our meeting discussions and the focus group interview. These are used here to collect together supporting evidence from the transcripts, as a basis for reflection using our two theoretical perspectives in Sect. 6.

18.5.1 Beliefs and Values

As part of the focus group interview, we each presented three messages or big ideas that we wanted pre-service teachers to take away from our units. These provided insight regarding our beliefs and values about the teaching of our respective units. We identified four key subcategories in our analysis related to this theme: perceptions of mathematics, pre-service teachers' productive disposition, how to deepen pre-service teachers' content knowledge and what is important for pre-service teachers to learn regarding pedagogy in mathematics.

18.5.1.1 Perceptions of Mathematics

There was some reference to mathematics across the curriculum (Chris) and to mathematics in everyday life (Jordan):

Chris 1[1]: I want them to come out with a consistent message about mathematics, mathematics across the curriculum, mathematics and numeracy. I want them to see the connections and not isolated content descriptors from the curriculum.

Jordan 1: Maths appears everywhere. If they are looking at their bank account or looking at a bus timetable, all those things you do every day are maths,

[1] The transcripts for each participant are numbered sequentially to allow for ease of reference in later discussion.

and that means they are all doing maths even if they are not sitting there doing algebra.

These ideas led to a discussion regarding numeracy and mathematics. Our attempts to make distinctions suggested that, whilst we agreed mathematics was not just about *doing sums*, being able to carry out calculations correctly was an important aspect of functional numeracy:

Jordan 2: Numeracy to me is the everyday stuff, and that is part of mathematics….

18.5.1.2 Pre-service Teachers' Productive Dispositions

The need for pre-service teachers to develop productive dispositions was a key belief held by all of us:

Chris 2: Having productive dispositions, being keen and interested in it [mathematics], and having an awareness of what the mathematics is at every level they are doing, then I think they are going to be good maths teachers.

Jordan 3: According to the Melbourne declaration [on Educational Goals for Young Australians, Barr et al., 2008], one of the aims of education is to produce active and informed citizens and, if they are going to produce active and informed citizens, they need to be one themselves.

Taylor indicated how some pre-service teachers often appreciated the need for their students to develop a productive disposition but were not so aware of the need to develop this disposition themselves. Alexis also referred to pre-service teachers' lack of confidence in mathematics and their need to realise that everyone can learn mathematics:

Alexis 1: I like to immerse them in experiences, because I like the growth mindset stuff. I'll give them challenging tasks and I'll get them to try and put themselves in the role of the student … and so you have to give them a task that's actually challenging for them … participate in the activities … try and get them to actually do the activity, see the importance of it, then engage in discussion about that.

18.5.1.3 Deepening Pre-service Teachers' Content Knowledge

The need to deepen pre-service teachers' content knowledge was held as a key idea by all of us, and our comments suggested our perceptions of how content knowledge and PCK are connected:

Taylor 1: I'm concerned with the connection between their [the pre-service teachers'] own content knowledge and pedagogical content knowledge. One of the things I've learnt over the years is not to come in and just teach the content, because they won't automatically make the connections… Whenever we are addressing a piece of mathematics it is put into the context of a task that they might use with students… so there is always a pedagogical aspect… it is not just knowing the mathematics.

Alexis 2: I also recognise that their own content knowledge is not as robust as perhaps it should be so like Taylor what I'd like to do is to increase their own content knowledge which increases their confidence and I would do that through modelling good practice and modelling good pedagogical practice that will help them to learn the content but also to teach it as well.

18.5.1.4 What Is Important for Pre-service Teachers to Learn Regarding Pedagogy in Mathematics?

Chris, Taylor and Alexis each emphasised different aspects in relation to our key ideas. Chris' emphasis was on the mathematics, Taylor referred to pedagogy and Alexis related to planning:

Chris 3: If we are having a pedagogical discussion it should be connected to the mathematics, it shouldn't just be educational questions, the maths should be there… So, if they're doing a garden and it's about area, what do they need to know about area? The mathematics has to be a focus.

Taylor 2: Pre-service teachers look at appropriate effective teaching of mathematics that goes beyond delivery as the "I do, we do, you do" model… so thinking about how pre-service teachers can open up tasks… How to develop tasks and activities that engage students and allow them to approach them at various different levels.

Alexis 3: I also like to provide them with some practical ways of how they should plan important mathematical lessons and what are some fundamental basics that they need to know when planning a maths lesson, and how it can be different to planning generically.

18.5.2 Agency and Autonomy

The transcripts showed that we all acknowledged a level of agency when it came to deciding upon the content of our units. As Chris stated in the following two excerpts:

Chris 4: Autonomy is one of the reasons why we are doing this review because we do have that degree of autonomy and we need to keep it but it does mean that periodically you need to return to some consistency to try to bring things together and make sure you're not repeating or have any gaps.

… There's always naturally going to be a drift because we're going to pick up on things and introduce them into our units because of the autonomy that we have so therefore we need to keep regularly meeting.

We felt we had opportunities for designing teaching and assessment practices, which were done independently and with little collaboration with each other. However, we also recognised some constraints, such as ensuring that our units took account of what pre-service teachers were expected to teach in the Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2019):

Taylor 3: We are constrained by the curriculum, so we're not totally god-like. We have to keep referring back to content descriptors, but we can be god-like in the way that we interpret deliver the curriculum. How much is it to do with our PCK and how much is from our beliefs and experiences, feedback from students, what we've observed [in tutorials].

We also recognised how the historical design of the units could act as a constraint, although this historicity did not affect our autonomy in delivering the content:

Jordan 4: When it comes to autonomy I teach the topics that we all agreed on in a maths meeting years ago but what I do in those topics is up to me.

Taylor 4: When I picked it up [the unit] I immediately went to the unit outline from the previous tutor and saw what was there. So, there is that history there but, as Jordan says, there is also that opportunity to expand on some, contract some, and change the emphasis and the way it is delivered.

Alexis 4: So, the topics we wouldn't change without discussing but the autonomy comes from how we deliver those topics.

Other exchanges suggested that we often tailored the content and delivery according to our expertise and beliefs of what the pre-service teachers "needed to know," as the following shows:

Jordan 5: … I have made the decision to decide well, what should you know (in relation to being an active and informed citizen) so you should be able to read the media, you should be able to know issues about surveys.

Alexis 5: …ultimately, we are making the assumption and decisions that we know or think we know what it is that is important for you [the pre-service teachers]. You [the pre-service teachers] need to know this, and so we are telling you that.

Chris 5: We know our own personal interests and expertise and things that we like to concentrate on.

A further aspect of agency was accountability for the unit and for the pre-service teachers' evaluation of the teaching and delivery of these. At the University of Tasmania, every student has the opportunity to answer an online survey evaluating each unit they have experienced. As MTEs, we are primarily responsible for setting and marking assessment tasks for the teaching approach taken and ultimately are expected to reflect on and respond to student satisfaction with the unit. Jordan observed the following:

Jordan 6: If I'm going to get negative evaluations, then it should be related to something I've done not someone else. If I've done it, then I'll take responsibility for it … What I don't want to do is be in a position where I was presenting stuff that was really poor quality and I didn't agree with it.

18.5.3 PCK for Pre-service Teachers

Examination of the transcripts shows that the lack of mathematical content knowledge of some pre-service teachers was one of our major concerns. Our experiences suggested that poor content knowledge revealed itself in surprising ways:

Taylor 5: We were talking about sharing out one by one as being a very rudimentary way of doing division and shouldn't they use their multiplication facts. One student came up to me at the end and she said, "I don't know any other way of doing division other than sharing out one by one". That's all she can do. And when I've talked about the sort of inefficient counting on strategies, again some students have come up and said, "that's how I do my maths".

This frustration was also noted by Chris:

Chris 6: The reason I'm saying that is how can we expect anyone to even look at that [unusual solutions/pupil solutions] if they actually don't even know how to do that [the algorithm] themselves in the first place?

It seemed we all had an aim to deepen the pre-service teachers' content knowledge. Jordan described how she selects tasks for the first-year pre-service teachers:

Jordan 7: What I try and do is to get them to do mathematical tasks where each task has an important [mathematical] message but there is not much calculating to do. There's so little confidence, and this is a generalisation, so I'm trying to get all these mathematical principles into their minds [with the tasks].

Alexis and Taylor also encourage the pre-service teachers to get involved in mathematical problems and to be critical of these tasks. By the fourth year of their course, Taylor encourages the pre-service teachers to see themselves as teachers:

Taylor 6: In the fourth year there is a shift from a person who does maths to a person who teaches maths and I think that's probably the trickiest unit in this regard.

The discussion between the MTEs also illustrated that as a group, we found it difficult to strike a balance between teaching the content of individual mathematical topics whilst remembering the overall picture. For example, Chris asked the following:

Chris 7: How do we look at students' content knowledge within those areas where we're focusing on big ideas… in the sense of we're not worrying about learning little atomised bits of mathematics? How do they [the pre-service teachers] see them as big ideas? How do they see the connections between algebra and mental calculations, algebra and geometry? What are the really important key, mathematical concepts that are going to help in a primary classroom, and how do we help them see that and see the horizon knowledge aspect of it and see the connections between bits of maths?

Chris's comments connect how our beliefs, desire for agency and our concerns for PCK impact on the alignment of the units – a key aim of our discussion.

18.5.4 Alignment

Whilst we recognised it was not possible to include all the content we believed should be covered, we were sometimes unsure about what to include and what to exclude and also unsure in which unit some content should be placed. There was some discussion regarding each other's expertise and if we used our strengths effectively:

> Alexis 6: So, do we recognise and capitalise on people's strengths or do we just allocate them units because it's more convenient.

We also wondered how much we could share each other's expertise:

> Alexis 7: …if we have a particular passion for a certain topic or area, like probability or problem solving, but that is covered in Chris or Taylor's unit, could we be flexible, and say, for example, Alexis is going to do this week's lecture on whatever because she has that particular expertise.

There was consensus that pre-service teachers' beliefs, attitudes and dispositions about mathematics should be emphasised throughout all the mathematics education units. However, we were not always confident about how this could be managed without repeating content. For example, Taylor said the following:

> Taylor 7: So, if we're thinking now about [pre-service teachers'] beliefs about mathematics… we want quite an emphasis on that right from the start …but do we touch on that every time and, if we're going to do that in each unit… how do we do it in a slightly different way?

Alexis suggested that this repetition should be made explicit, which was echoed by Taylor:

> Alexis 8: It's also readiness… so [the pre-service teachers] might watch or read something and it doesn't resonate with [them] perhaps because of [their] experiences, and when [they] see it in the fourth year, it might mean more. And of course, you're also making the assumption that [the pre-service teachers] actually have read it or watched it the first time which they don't always do. So I think the key thing is that if we repeat, we make it look as if we are deliberately doing this and being explicit about it.
>
> Taylor 8: If repetition is there, then we [should be] aware of it and use it for reinforcement.

We also considered how we could make more consistent links regarding our key messages, but that we do not always communicate in our resources and materials. For example, in relation to deepening PCK, Taylor commented on a model she introduced, and Alexis pointed out how she could also have used this model:

> Taylor 9: I've been supplementing Shulman's PCK ideas by referring to the Rowland work on the Knowledge Quartet. It's got some really accessible video clips of student teachers falling into the traps that they themselves as pre-service teachers fall into.
>
> Alexis 9: But I don't think we have been very good at communicating [to each other] about the suite of units because I could be using the Rowland videos as well and Taylor wouldn't even know.

Another topic of agreement was that we do not always know enough of the content of each other's units if we have not taught in them. Whilst we all have access to each other's unit outlines, we might not always acknowledge new ideas in the outlines. For example, we may select an extract that we have seen on TV or in a video clip, not knowing that the same clip might already be used in another unit.

We all agreed that the students needed continuity across the units and, at present, it is not always there. There was also a strong consensus that these identified problems arise because we do not meet together regularly or often enough. Alexis indicated that due to a level of agency within our units:

Alexis 10: There's naturally going to be a drift … it just means we have to regularly meet.
Chris 8: It is part of being a professional learning community and it's very easy to sit in your silo.

Whilst we recognised the lack of collaboration, we realised that there were advantages for the pre-service teachers and for us in relation to making connections and sharing our expertise:

Chris 9: Well … it's about connectivity, the connection between units. I mean if we're not talking about connections amongst ourselves they're not going to be explicit to the students.
Taylor 10: It's for our own professional development and continued development of our PCK.

These review meetings have enabled us to collaborate, and we intend to continue meeting regularly to facilitate systematic review of the units. Morgan also commented how the opportunity to collaborate and review across the units had enabled her to change focus:

Morgan 1: Whenever I'm teaching in these units, any reflection I do is on my own teaching, what went on in that tutorial, what went wrong, what might have gone well, those sorts of thing. So, I have not given a thought to connections between units. … it hasn't crossed my mind to make those connections.

18.5.5 Summary of Analysis

The analysis of our meetings and interviews indicated that we felt we could act independently of each other in altering some aspects of our units. We would choose to make adaptations in relation to our beliefs and values about mathematics, how to encourage productive disposition and confidence and how to deepen PCK and provide pre-service teachers with practical strategies for planning and implementing authentic tasks that focused on mathematics. Adaptations are also related to our own interests and expertise and to the discovery of new materials, research and policies that we felt would be useful to pre-service teachers.

As such, we all felt we had a degree of agency; however, we were not acting entirely autonomously. Whilst we were able to adapt the units in relation to our own values and beliefs, we were also aware of the need to maintain some alignment with the other units. This alignment was mostly done through historicity, in that we aimed to keep the main content of the units consistent with previous years and previous coordinators. We were also aware of the need to maintain links to the Australian Curriculum. Furthermore, in making these adaptations, we were aware that the units could move out of alignment, creating a tension between our agency and the need for maintaining alignment within a programme and with curriculum needs. These tensions are considered next within our chosen theoretical perspectives of activity theory and professional capital.

18.6 Reflection on the Themes

The previous section presented analysis of our meetings and interviews in relation to the emerging themes. In this discussion section, we reflect on the value of activity theory and professional capital in illuminating how these may influence our decisions.

18.6.1 Reflection on Themes in Relation to Activity Theory

We consider how Engeström's (1987) activity theory enables us to reflect on the themes within an overarching system. In doing so, we reinterpret the model (Fig. 18.1) in relation to the activity within our system and propose an adaptation of this model for MTEs (Fig. 18.2). We have used these reflections to propose an activity model to depict our system as MTEs within an initial teacher education programme (Fig. 18.2), in a similar fashion to Engeström's (2000) example of a children's hospital.

The subject, that is, the doer in this model, relates to us, the MTEs who coordinate and teach on the units. Our early discussions and comments reported on in the previous section show how, in reflecting on our practice, we often focus on our actions as university academics and not on our activity as a group of MTEs within a system. Whilst action is observable within our tutorials, our discussions revealed how our activity is often covert but that reflecting on these actions made our actions more visible, particularly in relation to alignment. Our analysis made visible the tension between agency and continuity (Alexis 10; Chris 8), as well as the advantages in making connections as we shared expertise (Chris 9; Taylor 10). This visibility raised our awareness of our teaching outside of individual units (Morgan 1). Reviewing the theme of alignment within activity theory further emphasises the connectivity between our individual work and how the balance between agency and alignment can cause tensions, but that connecting with each other within the system can have advantages both for the students and for us.

The object, the problem space or what is being done, is the review of our BEd (Primary) programme. The activity model is helpful in highlighting some aspects we see as requirements or that are effective in the programme. These aspects relate to beliefs and values, what we see as important in mathematics (Chris 1, Jordan 1) and in supporting pre-service teachers' dispositions (Chris 2; Jordan 2) and pre-service teachers' PCK (Chris 7; Taylor 1). Reflecting on these different themes in relation to the object in activity theory suggests connections between our beliefs and values and pre-service teachers' dispositions and PCK. For example, we felt that the

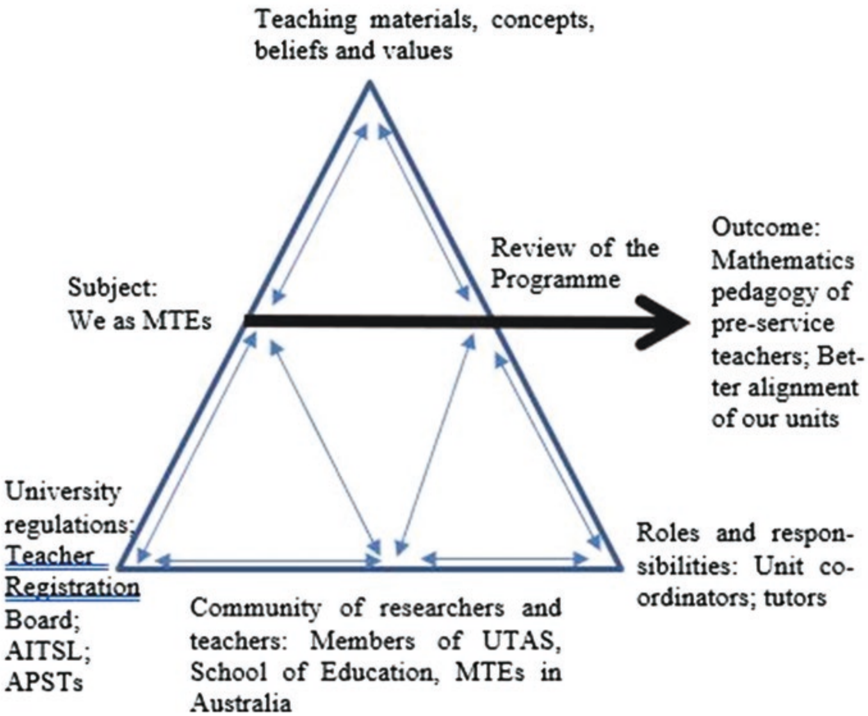

Fig. 18.2 Reinterpretation of Engeström's (1987) activity theory model for MTEs

ability of our students to see mathematical connections and to emerge as informed and active citizens (Chris 1, 7; Jordan 1, 2; Taylor 1) was important and also that the problem space within the primary school mathematics teaching domain is often related to poor content knowledge and negative disposition towards mathematics (Alexis 2; Chris 2, 7; Taylor 5; Jordan 6). The model helps us identify how we see this problem space in relation to deepening pre-service teachers' mathematics knowledge and providing pre-service teachers with a special kind of mathematics knowledge in order to teach (Chris 6; 7). What we see as important in mathematics can influence how we interpret what is a requirement or what is effective within the programme.

Instruments or tools are the mediating artefacts within the system. We view these artefacts not only as resources and materials that we use in our teaching but also as the concepts we uphold in relation to perceptions of mathematics and to effective teaching in mathematics. Hence, it is not just the tangible resources that mediate within the system but also our beliefs, values and motives. In relation to activity theory, we can examine our beliefs and start to determine how they mediate our decisions to use materials and in what ways they relate to practices that we perceive will afford learning. We all hold key ideas about our units and how they should fit within the programme. In this respect, our own beliefs and values are factors that influence our PCK as MTEs and hence our choices of tasks and content focus. For example, the key ideas that we hold influence how we support productive disposition (Alexis 1) and a focus on mathematics (Chris 3), the agency we have in tailoring content (Alexis 5), the deepening of pre-service teachers' PCK (Taylor 1,9) and how we might align units further by using our expertise (Alexis 7; Chris 5). These ideas were underpinned by our motives, values and beliefs in relation to our views of mathematics and numeracy, primary pre-service teachers' content knowledge and their disposition towards mathematics. Whilst we all have a common aim to deepen pre-service teachers' knowledge for teaching mathematics, we approach this aim in different ways and with different emphases.

The element of community recognises us as members of a team of MTEs and colleagues within the School of Education. We also see ourselves as academics within a community of MTEs and researchers in Australia and internationally. As such, we are participants in multiple communities and have multiple points of view within each community. As MTEs working within a university, we have roles and responsibilities between ourselves as a team of MTEs and within the institution as academic staff. Consequently, we engage in social practices and are involved in organising, managing and designing. However, we are also subject to regulations and professional standards that place constraints on our practices. Comments in our discussions related to the theme of agency and autonomy reflect how we manage some of these constraints by taking responsibility for how we deliver topics (Jordan 3, 5) and in managing the curriculum and historicity (Taylor 3, 4). Whilst we might share a degree of freedom with respect to decisions about our course structure and unit content and assessment (Chris 4, Taylor 4), we all collectively recognised we fell short of the "god-like" autonomy described by Chick (2011). A focus on community within a system then leads us to an awareness of the need for more regular and effective communication to support alignment and a better knowledge of others' units, for example, with respect to particular content knowledge, repetition and order of topics (Alexis 6, 7, 8, 9, 10; Chris 8; Morgan 1; Taylor 7, 8, 9).

In summary, we contend that activity theory, as an examination of the dialectic relationship between ourselves, the object and the other elements, provides a framework for reflecting on the emerging themes from the discourse analysis within a system adapted from Engeström (2000).

Whilst the object, that is, the review of the programme, is fundamental to the activity, it is socially constructed by us as MTEs. Reflection on the themes in relation to the elements of the activity model suggests that we all see the review of the programme differently and so the subject-object relationship is both objective and subjective. The subject-object relationship is mediated by our beliefs and motives, and these are in turn mediated by our different academic communities both in mathematics education and in the institution. Reflection on the themes using activity theory enabled us to interpret aspects of the activity and to explore the complexities that underlie our actions. The review meetings and interviews uncovered our activity to the extent that we can begin to see how we are influenced by the different aspects in the system. Hence, we now have a deeper view of the factors involved in determining the outcome of the activity, that is, the possible redesign to better develop effective pedagogical knowledge for teaching mathematics with primary pre-service teachers. We also realise an unintended outcome as the elucidation of the complexities involved in aligning the units.

18.6.2 Reflection on Themes in Relation to Professional Capital

The three aspects of professional development proposed by Hargreaves and Fullan (2012) were human capital, social capital and decisional capital. We further reflect on the themes from the discourse analysis in relation to these three aspects.

Our backgrounds regarding qualifications and academic standing suggest a high level of human capital. In our meetings, we recognised and respected each other's expertise and interests from both a research perspective and our expertise as teachers. In this regard, we respected the elements of autonomy that we have in the delivery of our units (Chris 4; Taylor 4; Alexis 4). In addition, in the context of this study, we perceive human capital as our worth: not just our qualifications, experiences and ability to teach mathematics, but our qualities as MTEs. Such worth relates individually to how we understand mathematics pedagogy and how we use this to conceive teaching materials and carry out tutorials with pre-service teachers. This view of human capital is evident in relation to the theme of agency in making decisions according to our individual expertise (Jordan 4; Alexis 5; Chris 5), to alignment and our differing expertise and interests (Alexis 6, 7) and to our beliefs and values in relation to deepening pre-service teachers' PCK (Taylor, 1).

We interpret social capital as our coherence and focus, as a group, within an institution: how we act together and how we act within the wider remit of systems and expectations within the institution. For example, within the theme of alignment, tensions were raised regarding the allocation of teaching units through convenience or through our expertise and strengths (our human capital) (Alexis 6, 7) and regarding our autonomy in managing continuity and repetition across the programme (Taylor 7). We reflected on our agency in tailoring the content and delivery to our beliefs and expertise, but we also accounted for agreed content and historicity (Jordan 3, 5; Taylor 4).

Decisional capital is interpreted as how we individually and collectively identify opportunities. Such decisions explain how and why we approach teaching in the way that we do and are evident across a range of themes. Examples include the decisions we make in relation to beliefs and values about mathematics (Jordan 1), deepening pre-service teachers' PCK (Alexis 2; Taylor 1) and determining what is important for pre-service teachers to learn (Alexis 3; Chris 3). Decisions are also evident in relation to developing PCK for pre-service teachers (Chris 6, 7) and how we have agency in determining our emphasis (Jordan 4) and in aligning the units (Alexis 8).

When taking these elements of professional capital together, human capital is seen as a key element of agency in making decisions and adapting our units, but human capital is evident in relation to beliefs and values about what mathematics is (Jordan 1) and the deepening of pre-service teachers' content knowledge (Taylor 1). However, decisional capital is also influenced by social capital, and Hargreaves and Fullan (2012) proposed that social capital is a key aspect that underpins professional capital. In the meetings, we referred to the lack of opportunities to collaborate in our teaching or in designing the content of our units (Alexis 9, 10; Chris 4, 8; Taylor 9, 10; Morgan 1). As a consequence, we perceive low social capital in our system, and this low level is a key hindrance in maintaining alignment between the units. Whilst we might acknowledge human capital and decisional capital as important elements, our professional capital as a group of MTEs is hindered by this low-level social capital. There was a sense that the tension between agency and the need for alignment can be explained as a tension between human capital and social capital and that this tension impacts on our professional capital.

18.7 Implications

The analysis and reflections in this study have illustrated the usefulness of two theoretical frameworks to examine our practice as MTEs within the system of our BEd (Primary) course. In using these two theoretical models, we begin to understand both the complexities of, and the tensions between, our agency and the need for alignment between the units we teach. From the analysis in relation to professional capital, we acknowledge our expertise and how we use this to adapt aspects of our units. However, the lack of social capital means that capacity to develop decisional capital is reduced with respect to aligning elements of the units. Hence, reflection on the themes using the professional capital framework has helped us to recognise that the main issue to be addressed is our access to a lower level of social capital and how our human capital almost works against improving our social capital.

Activity theory illustrates how our agency mediates our practice within the system and results in the unintended outcome of complexity and tensions. Whilst we have an overall common object, our beliefs and motives mediate the way we interpret this object and hence the duality of the intended and unintended outcomes. Our beliefs within our community as MTEs and researchers have created and shaped our practice. In reflecting on our practice, we often focus on our actions as individual academics and not on our activity as a group of MTEs within a system.

Whilst not attempting to draw the two theories together, one important implication is that we need to recognise the constraints within the system that limit our social capital and be more explicit in establishing mechanisms that facilitate better communication and awareness amongst our members. We can, for example, look more broadly at how our goals for our course design (see the six goals described by Li & Superfine, 2018) may be influenced by these tensions and hence realise the factors involved in both the intended and unintended outcomes.

Some limitations are evident in the study. First, we acknowledged earlier that the thematic analysis could be more rigorously conducted. Our conversations were rich, and the discourse analysis could benefit from deeper analysis than the largely descriptive approach used here. We also have yet to consider more closely the pre-service teachers' perspective in the review process, for example, with respect to a closer examination of the tasks and assessment techniques we use in our teaching.

Several questions remain for further exploration. From the analysis and reflection, questions arise in how we can maintain a level of autonomy but also collaborate further to manage the complexity of delivering an initial teacher education programme. How might we use common elements of our beliefs as mediators in order to achieve alignment and a coherent experience for the pre-service teachers? A more prolonged longitudinal study that compared the development of our units over time may shed more light on these questions. However, we feel confident that our findings here demonstrate that activity theory and professional capital may provide an effective means by which such developments might be analysed in future studies.

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Chapter 19
Researching Modelling by Mathematics Teacher Educators: Shifting the Focus onto Teaching Practices

Francisco Rojas, Helena Montenegro, Manuel Goizueta, and Salomé Martínez

19.1 Introduction

In the last 10 years, there have been considerable efforts to improve teacher education in Chile. For this reason, multiple public policies have been implemented to regulate teacher education, such as the introduction of new pedagogical and disciplinary standards for pre-service teacher education programmes and the creation of nationwide diagnostic tests for measuring the knowledge of pre-service teachers during their final year of study. These policies have led to the implementation of curricular changes in teacher education programmes, strengthening disciplinary and methodological aspects over general pedagogy (Mineduc, 2011). Currently, according to a study focused on characterising pre-service primary teacher education programmes in Chile, most of the students take at least four mathematics courses (Mineduc, 2016), while as reported by Varas et al. (2008), in 2008, over 80% of prospective teachers were required to take no more than two. Despite the implementation of these measures, major challenges remain, especially in mathematics. For instance, concerning learning opportunities for pre-service primary mathematics teachers, Rojas (2017) notes that these students receive more

F. Rojas (✉)
School of Education – Pontificia Universidad Católica de Chile,
Macul, Santiago de Chile, Chile
e-mail: frojass@uc.cl

H. Montenegro · S. Martínez
Center for Mathematical Modeling, UMI 2807 CNRS – Universidad de Chile,
Santiago de Chile, Chile

M. Goizueta
Institute of Mathematics – Pontificia Universidad Católica de Valparaíso,
Cerro Barón – Valparaíso, Chile

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_19

theoretical information (isolated mathematical concepts) than practical knowledge (strategies for teaching mathematics), regardless of the disciplinary contents studied.

Teacher educators are a key agent to consider when attempting to improve learning processes, because they are involved in curriculum design, its implementation and research into pre-service teacher education (Furlong, Barton, Miles, Whiting, & Whitty, 2000), although this key actor has been seldom discussed and researched in Chile (Cisternas, 2011; Montenegro, 2016). There is evidence that teacher educators' working conditions in Chile are not consistent with the importance of their role in preparing future teachers. Indeed, most educators have hourly contracts and lack professional development opportunities (Mineduc, 2016; Radovic, Peñafiel, San Martin, Bustos, & Martinez, 2018).

In this chapter, we conceptualise mathematics teacher educators as agents tasked with helping pre-service teachers improve their skills and facilitating the teaching of mathematics (Jaworski, 2008; Zaslavsky, 2009). Rojas and Deulofeu (2015) suggest two essential tasks for mathematics teacher educators: first, offering pre-service teachers the chance to learn the discipline in the same way as their students are expected to learn it (Chapman, 2008); second, promoting activities in university classrooms which allow pre-service teachers to learn how to teach mathematics (Watson & Mason, 2007), establishing a strong theory-practice link (Gellert, 2005).

The role as a model that mathematics teacher educators adopt when teaching how to teach becomes hugely relevant since it is a mechanism that can contribute to strengthening pre-service teacher education (Lunenberg, Korthagen, & Swennen, 2007). To understand the relevance of this role, the following sections introduce the concept of modelling and two methodological challenges to studying it. Firstly, we show the complexity and pertinence of including students in the modelling process and thus the need to devise ways to research modelling that take into account both mathematics teacher educators and prospective teachers. Second, we argue the need to understand modelling as a situated practice and thus the need to account for the complexity of the context in which it takes place. The discussion of both methodological challenges is supported by findings of specific research experiences with the purpose of exemplifying modelling as a relational and situated teaching practice.

19.2 Concept of Modelling

Every time a mathematics teacher educator teaches, he or she is enacting a way of thinking mathematically as well as a way of thinking about the teaching of mathematics, either intentionally or unintentionally. Therefore, mathematics teacher educators do not only organise and support the learning of their students; also, through their teaching, they model the practices that students learn (Korthagen, Loughran, & Lunenberg, 2005). Thus, the teaching process in which mathematics teacher educators engage appears to be as influential as the knowledge imparted. Russell (1997) famously summarised this view with the expression "how I teach IS the message," suggesting that teacher educators' teaching practices may be more relevant than the content of the said message when it comes to teaching how to teach.

Even though the implicit modelling of pedagogical reasoning conducted by mathematics teacher educators constitutes the first chance of showing best practices to prospective teachers, it does not necessarily generate substantial learning because it fails to identify such practices as an object of learning (Lunenberg et al., 2007). Thus, it is necessary to expand the concept of modelling teaching practices. Lunenberg et al. (2007) define modelling as a practice that involves intentionally deploying certain behaviours in one's teaching to promote the professional learning of prospective teachers. These authors have identified four types of modelling: implicit, explicit, transferred (facilitating the translation to the prospective teachers' practices) and connected (linking exemplary behaviour with theory). These types vary depending on their degree of explicitness, the connections made between theory and practice and the prospective teacher's role in the process.

For example, explicit modelling is conducted through the teacher educator's critical reflection on his or her practice (Boyd, 2014), which involves the use of teaching strategies that make explicit the decision-making process involved in the planning and implementation of his or her teaching, such as thinking aloud, co-teaching and meta-commentary (Swennen, Lunenberg, & Korthagen, 2008). On the other hand, transferred modelling is aimed at helping the prospective teacher see how the practices modelled can be applied to various teaching situations. Boyd (2014) points out that this type of modelling should make it possible for prospective teachers to reconstruct a teaching situation through learning activities that enable them to compare and analyse the teacher educator's teaching practices with their own. Finally, in connected modelling, the teacher educator links theory and practice whenever he or she treats his or her teaching as an object of reflection with his or her students. By connecting his or her teaching decisions with theories of learning, research evidence or even public policies, the teacher educator is expanding his or her modelling and sending prospective teachers a message: to perform well in a professional capacity, linking theory and practice is crucial.

Much of the research on modelling indicates that it can help prospective teachers learn based on their teacher educators' perspectives and teachings (Loughran & Berry, 2005). Prospective teachers would learn to teach more effectively if teacher educators shared and made explicit the pedagogical reasoning that supports their teaching, explaining the kinds of pedagogical decision that underpins their instructional practices (Bullock, 2009; Loughran, 2006). To do this, teaching must be intentional and congruent to connect prospective teachers' learning with teacher educators' teaching (Swennen et al., 2008), making clear the pedagogical rationality of the latter (Rojas & Deulofeu, 2015). However, the research on this topic has been focused on the teacher educator, leaving in the background critical aspects for the understanding of modelling. For instance, research has not taken into account that the modelling enacted by the teacher educator has an interactive nature and therefore is directly related to the prospective teacher. Furthermore, elements of academic communities and school classroom contexts take part in and mediate the instructional practice of the teacher educator (Goizueta, Montenegro, Rojas, & González, 2017).

Addressing these concerns, research on modelling presupposes new methodological challenges. As a way to advance in this discussion, in the following section, we link this new perspective with findings of studies conducted by the authors of this chapter to make sense of the ideas mentioned above.

19.3 Methodological Challenges in the Study of Modelling

This section will discuss two methodological challenges in the research of modelling practices enacted by mathematics teacher educators: the necessity of including the prospective teachers' perspective and the complexities of the educational context where future teachers will work.

19.3.1 Modelling as a Two-Sided Practice

Previous research on modelling has mainly focused on how teacher educators model (i.e. what teacher educators do) and on teacher educators' explicit claims about teaching (i.e. what teacher educators say). By contrast, there are few studies of what teacher educators model (i.e. techniques, values, dispositions, educational principles) and how prospective teachers interact with the contents of such modelling. The focus on teacher educators' performance neglects or even invisibilises prospective teachers as the necessary counterpart of the teacher educator's educational aims and actions (Boyd, 2014; Goizueta et al., 2017). We argue that the role of prospective teachers must be recognised and taken into account when modelling is used as a means of teaching how to teach.

To help prospective teachers identify the modelling practices enacted, it is essential that mathematics teacher educators ponder some crucial questions: what do prospective teachers look at in the teaching practices enacted by mathematics teacher educators? What instructional practices and teaching knowledge do prospective teachers incorporate into their pedagogical practices? Why do they make those particular choices? What impact do mathematics teacher educators have in these processes? These questions have in common that only prospective teachers can answer them. In other words, for achieving better understanding of these issues, we need to research the modelling practices enacted by mathematics teacher educators taking into account the prospective teachers' perspective. Hence, it seems reasonable to suggest that a first methodological approach to extend the research on modelling is to inquire what prospective teachers experience and think when the mathematics teacher educator is teaching.

To illustrate this methodological approach, we share two studies that consider the perspective of prospective teachers. The first study, conducted by Martínez (2017), focuses on the perceptions of prospective primary school teachers regarding the implementation of learning units for teaching mathematics. In the second study, Rojas and Montenegro (2018) explore how prospective secondary school mathematics teachers perceive a set of instructional practices enacted by their mathematics teacher educators according to the degree to which the latter make their pedagogical reasoning explicit.

Concerning the first study, Martínez (2017) leads a research and development project aimed at developing a system for supporting mathematics teaching in pre-service primary teacher education. In this project, learning units for teacher education are sequences of lessons around a mathematical topic and include mathematical tasks for teaching and supporting resources for mathematics teacher educators. A multidisciplinary team developed the learning units following an elaboration-testing-adjusting design cycle.

During 2017, four learning units were developed focused on topics selected for their high impact in initial teacher education. Two units deal with numbers. The first of these concerns addition and subtraction problems, covering the classification of these problems according to the actions involved and the place of the unknown (Lewin, López, Martínez, Rojas, & Zannoco, 2010). The second unit on numbers addresses representing addition and subtraction problems, which seeks to identify concrete and pictorial representations of these problems and discuss their pertinence (Veloo & Parmijt, 2017). In addition, two geometry units were developed. The first of these deals with definition of perimeter, which addresses the process of constructing a definition of the contour of a shape and problem-solving involving perimeters (Lu, Weng, & Tuo, 2013). The second geometry unit addresses variations of area and perimeter, which deals with the relationship between area and perimeter when changing geometric shapes (D'Amore & Fandiño Pinilla, 2006; Ma, 2010). In January 2018, the units reviewed were tested by mathematics teacher educators from the development team in two different short courses included in a summer programme for pre-service primary teachers.

Two focus groups were conducted (Flick, 2002) to assess the implementation of these four learning units, with pre-service teachers who took part in each of the two courses. These focus groups sought to examine in more detail the implementation and experiences associated with the numbers and geometry units. The discussion was guided by a set of questions aimed at evaluating the activities designed, as well as making explicit the teaching practices adopted by the mathematics teacher educators when implementing the learning units. The focus groups were also used to explore how the students perceived their teacher educators' modelling role. Both were recorded and transcribed in full for subsequent analyses. The transcripts were examined using thematic content analysis (Bardin, 2002).

Concerning the participants' perceptions regarding mathematics teacher educators' role as models, the prospective teachers in both groups pointed out that they learned not only the content imparted but also from the practices of the mathematics teacher educators. That is, they learned from the modelling in which they implicitly engaged, as the following extract shows:

> Also, not only... at least in my case, I learned from what we were taught about mathematics and also from the teacher herself (Numbers FG).

Specifically, the prospective teachers who took the geometry course pointed out that the mathematics teacher educator was able to anticipate students' questions, which he used to make it easier for them to learn the content:

> Yes, I feel that he [the mathematics teacher educator] anticipated our heuristics, so to speak, like our ways of thinking or tackling an exercise. He took our answers into account and knew how to use our ways of reasoning to construct the content of the course. I think that was quite admirable (Geometry FG).

For their part, the prospective teachers who took the numbers course stressed the mathematics teacher educator's ability to organise learning according to their mistakes. In other words, prospective teachers had a positive opinion of how the mathematics teacher educator managed the classroom climate to encourage them to share their answers without fearing criticism. This perception is observed in the following extract:

> In contrast, with her [the mathematics teacher educator], if I made a mistake it was the opposite, it was a good thing. Because I know she is going to clarify it for me, she is going to make it clear. I know that when she explained something… I think sometimes you can also learn from your mistakes, and you should… but I was not afraid of making mistakes (Numbers FG).

Lastly, prospective teachers in both groups mentioned that these pedagogical practices constituted another type of learning that they think will be essential in their own work as teachers in the future. For instance, a prospective teacher said the following:

> I wish I could do the same later, with the children, so they would not be afraid to make mistakes. Because sometimes children make a mistake one day and they do not want to work anymore (Numbers FG).

As these excerpts show, prospective teachers were able to see mathematics teacher educators as teaching models from whom they learned some teaching practices that they would like to implement in school classrooms. In addition, they considered that this type of learning was positive for their professional education. The above observations are especially relevant considering that all the teaching practices identified by the participants were enacted through implicit modelling. As a consequence, the professional role model that the mathematics teacher educator enacts while teaching has an impact on learning outcomes beyond the explicit pedagogical and disciplinary content at stake, and such effects relate to what prospective teachers notice about such role models. Precisely as Russell (1997) suggested, there is "a message" about teaching in teaching itself and thus the criticality of the teaching model's role enacted by the teacher educator.

Nevertheless, according to Loughran and Berry (2005), when prospective teachers learn about teaching, what is evident for teacher educators might not be so for their students. Thus, for modelling to be an effective teaching and learning tool, explicit attention must be intentionally directed to particular features of teaching practices, to make implicit content about teaching available and to address possible differing interpretations. However, despite the efforts made by the teacher educator to make explicit his or her pedagogical reasoning and thus to justify his or her teaching practices, prospective teachers still might not perceive what has been

Table 19.1 Descriptions of instructional practices

Mathematical task (MT)	Interaction (INT)	Consideration (CON)
Generation of mathematical reasoning opportunities through the design and application of mathematical tasks	Generation of teacher-student interactions to promote mathematical reasoning	Observation and consideration of students' actions, behaviours, responses and mathematical output

modelled by the teacher educator, or these practices might be perceived differently from what the latter intends.

In this regard, the second study developed by Rojas and Montenegro (2018) explores how prospective teachers perceive the modelling enacted by their mathematics teacher educators, specifically related to their instructional practices. These practices were defined using several standardised protocols for the observation of mathematics lessons (e.g. Boston, Bostic, Lesseig, & Sherman, 2015; Hill et al., 2008) and grouped into three categories (Rojas & Chandía, 2015) (Table 19.1).

These categories were transformed into a Likert-type questionnaire (Rojas & Chandía, 2015), in which prospective secondary school mathematics teachers were asked which modelling type – implicit, explicit, transferred or connected modelling (Lunenberg et al. (2007) – they identified in the mathematics teaching practices of their mathematics teacher educators.

Rojas and Montenegro (2018) analysed the results of the previous questionnaire applied to a subsample of 61 prospective teachers taking mathematics teaching methods courses at eight Chilean universities. Two-stage cluster analysis was conducted to characterise their perceptions of the instructional practices modelled by mathematics teacher educators. This approach made it possible to group together continuous and categorical variables and to form groups with a high degree of internal homogeneity and high heterogeneity with respect to each other (Hair, Anderson, Tatham, & Black, 1998).

This process yielded four groups[1], two of which show consistency in the type of perceived modelling (clusters 3 and 4) and two with greater divergence (clusters 1 and 2), according to different categories of teaching practices. In cluster 1, prospective teachers tended to recognise that mathematics teacher educators based their practices on a theoretical knowledge that informs their pedagogical decisions, specifically in categories of practice regarding mathematical tasks and consideration of students' productions. In contrast, in cluster 2, prospective teachers perceived that the mathematics teacher educator implicitly guides them through various mathematics teaching practices. In this group, prospective teachers were unable to identify a specific modelling practice for actions related to mathematical tasks (they tended to choose "Does not apply" in these cases). Cluster 3 comprises prospective teachers who perceived that their mathematics teacher educators explicitly support their actions as models of teaching practice, in all three categories. However, this does not necessarily mean that these explicit explanations have a theoretical basis or

[1] The analysis of the χ^2 test revealed statistically significant differences in the distribution within the groups ($\chi^2 = 28.685$, df = 12, $p < 0.004$).

Table 19.2 Type of modelling practices by cluster and category

	Dimensions of instructional practices		
	MT	INT	CON
Cluster 1	Connected	Transferred	Connected
Cluster 2	Does not apply	Implicit	Implicit
Cluster 3	Explicit	Explicit	Explicit
Cluster 4	Transferred	Transferred	Transferred

are linked to experiences in mathematics classrooms. Finally, prospective teachers in cluster 4 perceived that their mathematics teacher educators are permanently transferring to the school classroom that which is studied in the university classroom. As in cluster 3, the students in this group also considered that their teacher educators connect their practices in the three categories with the school context, but do not necessarily make decisions based on their knowledge of public theory. The following table summarises the type of modelling perceived in each dimension for the four clustered student groups (Table 19.2).

Since each cluster is composed of students from different universities, this analysis shows that the perception of prospective teachers is heterogeneous within the same university classroom. Regarding the type of instructional practice that the mathematics teacher educator promotes more explicitly, the analysis shows that those related to the consideration of student productions (CON) are those that the prospective teacher most easily discriminates. This result suggests that mathematics teacher educators can make explicit to various degrees the pedagogical reasoning that supports their teaching decisions, which prospective teachers are unable to see clearly. These results concerning prospective teachers' perceptions of the type of modelling employed by mathematics teacher educators highlight the importance of harmonising prospective teachers' learning and mathematics teacher educators' teaching (Swennen et al., 2008).

Although the prospective teachers' perspective can inform the modelling enacted by the mathematics teacher educators, there is still a question about why one type or another is perceived, besides knowing what kind of instructional practice they see most clearly. The answers to these questions should not tend to seek a homogenisation of the perception of modelling by prospective teachers. Heterogeneity tells us about the level of involvement and evolution of pedagogical thinking that prospective teachers have. Even so, and at a theoretical level, it is desirable that the interrelation between the mathematics teacher educator and his or her students tends to project and perceive, respectively, a modelling closer to what we define as connected practice.

19.3.2 Modelling as a Situated Practice

Various authors highlight the relevance of researching teaching and learning processes from a situated perspective, taking into account the context and how it shapes both individuals and teaching practices (Borko, 2004). From a sociocultural perspective, researching teaching and learning implies taking into account not only teacher educators' and prospective teachers' views on the setting in which they interact but also the broader social and school contexts in which the latter (will) teach. In this regard, the classroom setting and sociocultural context are fundamental to facilitate a more comprehensive and connected understanding of teaching and learning experiences in the pre-service teacher education classroom (Marton & Tsui, 2004).

Research on modelling teaching practices from a situated perspective presupposes methodological challenges for approaching the complexities associated with learning to teach from a relational perspective. On the one hand, we need to consider teaching and learning from the positioning of those engaged in teaching and learning processes. That is, the focus should be on studying both mathematics teacher educators' and prospective teachers' conceptions of teaching and learning and the perceptions of their educational settings with the purposes of enhancing the learning experience of school students. On the other hand, the situated character of teaching requires an understanding that prospective teachers learn to teach in a particular educational context (teacher education classroom), but in the future, their teaching will take place in a different educational context (school classroom). As Boyd (2014) suggests, learning to teach implies "becoming within a transitional process of boundary-crossing" (p. 53). This idea illustrates the challenging task that prospective teachers face in developing their teaching practice and professional identity inside teacher education programmes, a different workplace setting compared to the school system. Loughran and Berry (2005) state that for many teacher educators, this dual setting is an ever-present feature of their teaching context.

Hence, it is possible to argue that the value of research on modelling from a situated perspective depends on integrating descriptions of those elements in an interrelated way, giving a whole and complex picture of the educational phenomenon. In this regard, phenomenographic research inquires into how teachers and students in naturalistic teaching contexts approach their teaching and learning processes (Marton & Tsui, 2004). Phenomenography studies how an experience can vary by identifying the qualitatively different ways in which a phenomenon is experienced, perceived or conceptualised. The results of this variation are systematised using categories of description that are hierarchically organised to create an outcome space (Bowden & Walsh, 2000).

Montenegro (2018) is currently conducting a phenomenographic research project aimed at understanding how the notions of modelling held by mathematics teacher educators influence their teaching practices. As part of this study, phenomenographic interviews (Trigwell, 2000) have been held with a sample of 12 mathematics teacher educators teaching disciplinary and pedagogical courses in three programmes for pre-service teachers. The results of the preliminary analyses reveal findings that are interesting to examine. Four categories of descriptions, structured by complexity, emerged from the analysis. Table 19.3 reports the name of the categories of descriptions and representative quotations from the interviews.

Table 19.3 Categories of description of modelling

	Mathematics teacher educators model
A	Pedagogical activities that can be replicated in school classrooms "The use of the body is also relevant, especially in geometry. I don't know, angles, parallel lines, you can show all that using your arms. I said to them "everyone, show me an obtuse angle with your arms, an acute angle", that sort of thing... and I also said to them explicitly that it is good for them to do that with their students" (MTE1).
B	Pedagogical interactions to be conducted with students "There is also the emotional aspect... in my opinion, if there is no emotion, there is no learning. So I become emotionally involved with students, I mean, I tell them that they can do it, that they can generate changes. That they can change mathematics teaching" (MTE9).
C	Teaching connected with school classrooms "I try to model, with a theoretical basis, a way of thinking about designs and their objectives that is not unique… it is like thinking aloud about what I want to achieve in the classroom regarding a mathematical objective" (MTE6).
D	Teaching practices consistent with the context where they are carried out "Because otherwise there is no consistency, how can I… so if I am not a model, I can just babble about how I think students should learn mathematics. But if I am not [a model], students will not have a point of reference to observe how you can do those things that the teacher says you can do. So, I think discourse and practice must coincide" (MTE2).

Regarding the first category, mathematics teacher educators point out that they model pedagogical activities that prospective teachers will be able to replicate when they become teachers. In the second category of description, mathematics teacher educators model pedagogical interactions that can facilitate learning in the classroom, a process in which it is fundamental to establish an appropriate bond with students. Regarding the third category of description, mathematics teacher educators conceive modelling as a teaching practice linked to the school classroom. Here, mathematics teacher educators model a type of teaching that is aimed towards the mathematics taught in schools predicting the most frequent errors and difficulties observed in school students. Finally, the fourth category of description views modelling as the use of a consistent set of teaching practices that allow prospective teachers to experience mathematical learning and replicate it with students in the school system. In this category of description, mathematics teacher educators are interested in modelling teaching practices consistent with the theoretical model that they ascribe to since they regard this as essential for learning how to teach mathematics.

These results support the view that mathematics teacher educators have different notions of their role as models, which vary regarding the position that they adopt and their awareness of the effect that they can have on their prospective teachers' learning. When mathematics teacher educators see modelling as a practice with a focus on performing pedagogical activities and interactions with prospective teachers, they attempt to recreate the complexity of the school classroom inside teacher education programmes. In contrast, mathematics teacher educators who regard modelling as a teaching practice linked to the school classroom and supported by a corresponding modelling approach not only connect their teaching to the university classroom but also invite prospective teachers to reflect on the school classroom where they will work in the future. In other words, learning to teach is viewed as a complex phenomenon that can be only understood if it is discussed and pondered considering the context where it will take place (Boyd, 2014; Loughran, 2006).

19.4 The Next Step in Researching Modelling with a Focus on Teaching Practices Inside the Classroom

In this chapter, we have discussed some methodological challenges associated with research on modelling practices of teacher educators when they teach about teaching mathematics. These challenges are related to how to incorporate the prospective teachers and the critical role they play in the practice of modelling, as well as to understand that modelling is a practice situated in a university context but, at the same time, directed to the school context. Based on our experience in studying mathematics teacher educators' conceptions and teaching practices, we have shared some interesting findings as a way to contribute to this discussion. However, those studies focused on the pedagogical discourse and on perceptions, ideas and teaching practices of both prospective teachers and mathematics teacher educators. That is, they do not explore what the experience of modelling inside teacher education classrooms is like.

To move forward in research on modelling with a focus on displayed teaching practices, including prospective teachers and the educational context where it is materialised, it is fundamental to take into account new issues and methodological challenges in this field of research. For instance, it is necessary to think about the content of what is modelled, particularly in how the mathematics teacher educator makes visible the disciplinary reasoning that underpins his or her mathematical knowledge for teaching. Despite the acknowledgement of disciplinary differences in teaching and learning and how to learn to teach, previous research on modelling has mainly focused on general aspects and has not taken into account specificities and nuances associated with the disciplinary content at stake (see, e.g. Boyd, 2014; Lunenberg et al., 2007; Loughran & Berry, 2005). We claim that it is necessary to consider that, besides general pedagogical principles, values and knowledge, mathematics teacher educators deploy specific mathematics knowledge for teaching (Ball, Thames, & Phelps, 2008) associated with discipline-specific teaching practices. Thus, modelling should also be understood and researched with its discipline-specific features. In the same vein, we consider it essential to research how this disciplinary reasoning is perceived by prospective teachers regarding the possible improvement of their learning to teach mathematics as well as how prospective teachers might transfer this disciplinary reasoning to students when they become teachers.

The significance of modelling disciplinary reasoning is in line with our results. For example, Montenegro's findings highlight that mathematics teacher educators position themselves differently in terms of what they model. This positioning oscillates from teaching pedagogical activities that can be replicated in school classrooms (focus on mathematical procedures) to teaching styles connected with classrooms in schools (focus on school mathematical knowledge). Martinez's research shows that prospective teachers tend to pay more attention to pedagogical approaches to teaching. Particularly, when asked about the mathematics teacher educator's teaching practices, prospective teachers highlighted general pedagogical resources, such as learning from mistakes, anticipating answers and approaches and cultivating confidence among students. Nevertheless, they also highlighted the criticality of unpacking and making certain mathematics-related elements of teaching explicit for prospective teachers to notice and reflect about them. In the same way, Rojas and Montenegro's findings make evident that prospective teachers have difficulty in identifying the mathematical reasoning related to the specific mathematical task. In other words, prospective teachers do not recognise teaching practices specific for learning to teach mathematics as part of what mathematics teacher educators enact as a role model when they are teaching.

Furthermore, another methodological challenge that we consider crucial in research on modelling from this new approach is to explore it in a holistic and contextualised way. To strengthen approaches to research on modelling with a focus on teaching practices materialised within the classroom, we should move towards two new developments and levels of complexity. Firstly, it is required to inquire how prospective teachers engage with the modelling enacted by the mathematics teacher educator. To analyse the interactions between teachers and students, we need new methods and instruments for identifying patterns of interactions, widely developed for the school classroom but scarce in the teacher education programme contexts. Second, it is fundamental to think of teacher education programmes as a community of practice (Wenger, 1998) in which researchers, mathematics teacher educators and prospective teachers reflect on the complex task of learning to teach mathematics and how the modelling enacted by mathematics teacher educators might contribute to improving the learning experience when they work as future teachers.

Concerning the first new development, Lunenberg et al. (2007) argue that teacher educators have difficulties becoming aware of their role as models and the influence of their teaching practices and their pedagogical choices on prospective teachers' learning about teaching. Therefore, a challenge for mathematics teacher educators is to pay attention to what is being taught and to how it is taught, taking into account the need for congruency between the pedagogical theories they introduce and the teaching practices they enact (Swennen et al., 2008). Similarly, the shift in focus to enacted teaching practices turns such practices, pedagogical reasoning and rationale behind them into objects of collective conscious reflection. Such space for reflection constitutes an opportunity for mathematics teacher educators and prospective teachers to develop professional scrutiny and critique and to explicitly connect professional practice to the knowledge basis behind it (Loughran & Berry, 2005).

Unpacking teaching and learning activities in the classroom might also constitute an opportunity to discern different elements and aspects of the mathematics teacher's professional knowledge and thus an opportunity to introduce, make accessible and connect with actual practice the knowledge basis and theory behind it. Nonetheless, some mathematics teacher educators tend to overlook prospective teachers as the other side of the coin in the process of learning to teach because they do not realise that prospective teachers are learning both content and teaching strategies. Thus, modelling might be an effective way to connect and bridge different elements of the prospective teachers' and the mathematics teacher educator's professional knowledge. This approach might raise awareness about how the prospective teachers' and the mathematics teacher educators' professional practices are related, how they develop together and how modelling might contribute to such development. Because modelling promotes a reflective stance towards teaching and learning practices in the classroom, it might help prospective teachers to understand teaching mathematics as a layered activity related to both content knowledge and knowledge for teaching such content.

Regarding the second new development, the perspective of teaching education programmes as communities of practice reflects that teaching is a relational and interactive activity. However, at the same time, it generates a new methodological challenge: that teaching, being a relational practice, should include research by mathematics teacher educators and their prospective teachers, together. To advance in this matter, we propose that modelling is better understood as a collaborative practice in which both mathematics teacher educators and prospective teachers participate and for which explicitness about what is worth paying attention to, reflecting on and learning is needed and negotiated between the participants.

In this regard, collectively reflecting on teaching and learning mathematics through modelling is thus an opportunity for prospective teachers and their mathematics teacher educators to learn about how to teach mathematics. From this perspective, mathematics teacher educators can be seen both as facilitators of learning and as learners themselves (Zaslavsky, 2009), so that they and prospective teachers can be regarded as learning in two interrelated communities of practice improving each other's professional learning (Jaworski, 2008; Wenger, 1998). These communities of practice might allow the emergence of opportunities to learn that are not likely to materialise otherwise (Loughran & Berry, 2005).

Additionally, this collaborative learning between mathematics teacher educators and prospective teachers broadly contributes to the professional development of both. It allows for the explanation of the different roles and tasks carried out in the profession of school mathematics teaching (Jaworski, 2008) and to develop a sense of belonging in teacher education programmes (Loughran, 2006). A collaborative approach also allows to make explicit the tacit knowledge of teaching when it is verbalised and discussed with others (Loughran & Berry, 2005) and to consolidate a language which can be analysed with other teachers, mathematics teacher educators and researchers in the field (Ball et al., 2008; Lunenberg et al., 2007). As a result, we must start to see teacher education programmes as learning communities, not only for prospective teachers but also for the mathematics teacher educators who are part of them. It is the responsibility of both the programme and mathematics teacher educators to ensure that challenges associated with entry into this new educational context are discussed and scrutinised (Jaworski, 2008; Loughran, 2006).

To conclude, we believe that reconceptualising research on modelling from a more integrated, holistic perspective should take into account the complementary roles of mathematics teacher educators and prospective teachers and how they complement each other in the challenge of learning to teach mathematics. Furthermore, this new approach must consider modelling as having disciplinary specificities for teaching mathematics from a situated perspective. We hope that future studies will not only contribute to the improvement of this proposal but also generate and develop new strategies for enhancing the disciplinary and pedagogical development of mathematics teacher education.

Acknowledgements This work was carried out within the following research projects: VRI INICIO 23/2013 "Estudio de las Prácticas Instruccionales del Formador de Profesores de Matemática de Educación Media: Construcción de Indicadores de Coherencia" (*Study of the Instructional Practices of the Secondary Mathematics Teacher Educators: Construction of Coherence Indicators*) led by Francisco Rojas, FONDECYT POSTDOCTORADO 3170423 "Caracterización de las experiencias de enseñanza y aprendizaje en la formación docente inicial: un estudio exploratorio" (*Characterization of teaching and learning experiences in pre-service teacher education: an exploratory study*) led by Helena Montenegro, FONDEF ID16I10119 "Metodologías innovadoras para la formación inicial de profesores de educación básica en matemática: modelo basado en prácticas efectivas de aula y tecnologías de la información" (*Innovative methodologies for the pre-service training of elementary school teachers in mathematics: a model based on classroom practices and information technologies*) led by Salomé Martínez and CONICYT + PIA/Concurso de Apoyo a Centros Científicos y Tecnológicos de Excelencia con Financiamiento Basal AFB170001.

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Chapter 20
Mathematics Teacher Educators Within the New Technological Environments: Changing the Perspective

Ferdinando Arzarello and Eugenia Taranto

20.1 Introduction

The processes by which mathematics teacher educators (MTEs) learn and the forms of knowledge they require for effective practice have not been systematically investigated (Llinares & Krainer, 2006). In the literature, it is found that MTEs often engage in self-analysis. This brings with it the risk of producing too personal and contextually specific results, which are difficult to generalise. Zaslavsky and Leikin (2004) introduce the role of the MTE educator, a "super-partes" person who should analyse how the MTEs themselves work. However, even in this case, this figure is in turn an MTE and probably personally involved in research issues. A possibility that Lovin et al. (2012) suggest is research teams are made up of people who received the same training (similar experiences of a doctorate) but who then work in different institutions. Therefore, being in contact with realities that present different backgrounds, contexts, communications and knowledge should reduce the limits of self-referential research.

We are researchers in mathematics education and in particular MTEs, at the Department of Mathematics "G. Peano" at University of Turin, Italy, and with our experience, we share a practice that takes a step forward in the direction of analysing MTEs' learning. Our Department's primary mission is to promote excellence in research and teaching in all areas of mathematics. In mathematics education, the Department has produced sustained, deep and excellent work in the last 40 years. In

F. Arzarello
Department of Mathematics "G. Peano", University of Turin, Turin, Italy
e-mail: ferdinando.arzarello@unito.it

E. Taranto (✉)
Department of Mathematics and Computer Science, University of Catania, Catania, Italy
e-mail: eugenia.taranto@unict.it

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_20

the following, we show some examples that concern us and the other members of the research group in mathematics education.

Some members of the research group were members of national projects for the curriculum promoted by the Ministry of Education and by the Italian Mathematical Union[1] (UMI), as *Mathematics for the citizen* and m@t.abel (https://goo.gl/Q30Dn0), a plurennial National Programme that promoted innovation in mathematics teaching, based on concrete activities proposed to teachers and discussed with them in suitable professional learning programmes. Others are involved in the National Evaluation Institute for the School System, INVALSI[2] (Arzarello, Garuti, & Ricci, 2015). At the Department, the research group also deals with the education of prospective mathematics teachers in lower and upper secondary schools. Thanks to the GeoGebra Institute, the research group takes care of the education of in-service and pre-service teachers on the integration of technologies in mathematics lessons. An important Department resource for teachers is the DI.FI.MA. platform (http://difima.i-learn.unito.it), a Moodle platform containing all the projects' materials and the interactions of participants – more than 2500 Italian teachers of all school levels. Recently, from October 2015, the Department is engaged in an innovative initiative: the *Math MOOC UniTo* project. It consists of delivering massive open online courses (MOOCs) for Italian in-service mathematics teacher education, with the use of the platform DI.FI.MA. The aims of these online courses are to cover the main topics in the official Italian programmes for secondary school (arithmetic and algebra, geometry, change and relations, uncertainty and data) from a mathematical, didactical and methodological point of view and to give teachers an opportunity for professional development at the national level.

In the following, we will focus on the *Math MOOC UniTo* project, but first, we make some general comments. A peculiarity that characterises our community of MTEs is precisely the integration in it of two main components, namely, university researchers and *researcher-teachers*. These latter figures have been described by Arzarello and Bussi (1998) as a core aspect of Italian research in mathematical education (RME):

> The core of Italian RME is trying to overcome the distinction between theoretical and pragmatic relevance […] by means of developing their mutual relationships from the very beginning. This attempt is made initially in a very pragmatic way, by the joint work of researchers and teachers […]. Yet, it becomes clearer and clearer over time that a strong epistemological choice is involved in it (p. 250).

The main idea of the researcher-teacher "corresponds to the participant observer, who develops a split between observing and observed subjects in a dialogical relation" (p. 250–251). As a consequence in the researcher-teacher, "we find the presence of two types of contrasting issues: the first are more empirical, pragmatic, concrete and specific, whilst the second are more speculative, theoretical, abstract and general" (p. 251).

[1] http://umi.dm.unibo.it/

[2] http://www.invalsi.it/invalsi/index.php

In our case, the researcher-teachers are in-service teachers who have been collaborating with our research group in mathematics education for several years. In particular, in our MOOC project, the community of MTEs continues to assume a particular composition, due to the specificity of the Italian tradition: it is composed of both university researchers and researcher-teachers. This constitutes a successful conjunction between university and school and turns out to overcome the self-referential difficulties about the analysis of MTEs pointed out in the literature.

The research questions we intend to address in this chapter are the following:

- What are the dilemmas and opportunities associated with researching ourselves as MTEs?
- What methodologies might be effective in building such an evidence base?

The main claim of the chapter is that the figure of *researcher-teacher* can be a solution to the above self-referential problem of MTEs. We will develop our arguments analysing the role of the researcher-teacher and showing his/her function as a *broker* in planning and managing our MOOCs. Our analysis is based on a new theoretical framework, the *MOOC's Zone Theory*, elaborated by Taranto (2018) working in a massive online education context. She adapts to the MOOC phenomenon some theories already employed in the analysis of teacher education. A key point of this theoretical framework is the consideration of the MTEs both in the virtual context (i.e. within the MOOC) and in their habitual real context (the research environment).

In the next sections, we will briefly show the Meta-Didactical Transposition model, elaborated by Arzarello et al. (2014), and Valsiner's Zone Theory used by Goos (2013). These are the theories that Taranto has considered and adapted for the development of the MOOC's Zone Theory. The analysis will show how this MTE community worked in the design and management of our MOOCs, taking advantage of the presence of the researcher-teachers. Finally, we will discuss why this figure can be a solution to the dilemma of the MTE self-referential problem.

20.2 Meta-Didactical Transposition

Arzarello et al. (2014) have elaborated a model that dynamically features the intertwining of the practices of mathematics educators (researchers) and those of teachers, when engaged in teacher education activities. A source for the model is the Anthropological Theory of Didactics (ATD) of Y. Chevallard (1985), especially his notion of didactical transposition. The Meta-Didactical Transposition (MDT) is constituted by five intertwined features: (i) the institutional aspects, (ii) the meta-didactical praxeologies, (iii) the dynamics between internal and external components, (iv) the role of the broker and (v) the double dialectics. We will sketch here only components (i), (ii) and (iv). Components (iii) and (v) will be omitted because they concern aspects closer to the teachers whilst the focus of this chapter is on the MTEs.

Institutional Aspects MDT particularly focusses on two social institutions relevant to teacher education activities: the *community of researchers* (the MTEs) and the *community of teachers* that participate in an educational course. Both these communities involved in the MDT process are subjects within a certain institution. Teachers belong to the actual schools where they teach, and researchers refer to the School as a higher institution that decides curricula, has particular teaching traditions, produces textbooks and so on. Indeed, when the researchers in mathematics education come into contact with the teachers' community, they hold simultaneously two different positions. They belong to the university or the department where they work, but in that particular occasion, they act as teachers' educators.

Meta-didactical Praxeologies At the core of ATD are the notions of didactical transposition and praxeology:

> [the didactical transposition] formulates the need to consider that what is being taught at school (contents or knowledge) is, in a certain way, an exogenous production, something generated outside school that is moved – "transposed" - to school out of a social need of education and diffusion. For this purpose, it needs to go through a series of adapting transformations to be able to "live" in the new environment that school offers [...] (Bosch & Gascón, 2006, p.53).

A praxeology (or mathematical organisation) is structured in two main levels (García, Gascón, Ruiz Higueras, & Bosch, 2006): (a) the "know how" (*praxis*), which includes a family of similar problems (or *tasks*) to be studied, as well as the *techniques* available to solve them (e.g. 2nd-degree equations and their solution formulas); (b) the "knowledge" (*logos*) that is the "discourses" that describe, explain and justify the techniques used (e.g. the justification of the formula for 2nd-degree equations through the completion of squares or even the theory of algebraic equations).

In the MDT, the researchers have the objective of transposing a certain piece of knowledge, related to the teaching and learning of mathematics, to support the professional development of teachers, according to the reference institutions (national curricula, textbooks, etc.). This researcher knowledge to be transposed is the result of their research work, enriched by the comparison with the international research community in mathematics education or by working in collaboration with researcher-teachers. In this case, Arzarello et al. (2014) introduce the notion of *meta-didactical praxeologies*: they consist exactly of the tasks, techniques and justifying discourses that develop in teacher education processes. Because of this process, teachers' didactical praxeologies may change and develop into new ones. On the other hand, the community of researchers reflects on the nature of and reasons for changes produced by the teacher education project (and possibly shares such reflection with the community of teachers). The result possibly leads to new researchers' meta-didactical praxeologies. Of course, this process can go on and on, further refining itself: it is this complex transfer that constitutes exactly the *Meta-Didactical Transposition.* In a nutshell, a Meta-Didactical Transposition produces a dynamical change in the praxeologies of both communities that together tend to form a unique community.

Brokering The MDT framework uses the notion of broker as a professional who belongs to more than one community and makes possible exchanges between them: "Brokers […] are able to make new connections across communities of practice, enable coordination, and – if they are good brokers – open new possibilities for meaning" (Rasmussen, Zandieh, & Wawro, 2009, p.109).

In the Italian community of academics in mathematics education, the role of broker is often played by the researcher-teacher, who is part of the communities of researchers and of teachers as aforeomentioned. The role of a broker is fundamental in the exchange of information, techniques, justifications and theories, namely, all about praxeologies and their components. In fact, the role of the researchers is to manage a research project in which the educational programme is inserted and then to design the programme with its activities and actions. The role of the researcher-teachers is to collaborate in these phases and to participate also in the professional development programme as trainers, where the role of the teachers involved is to be learners in communities with colleagues. Participating simultaneously in the researchers' community and the teachers' community, the researcher-teacher acts as a broker between the two communities.

The MDT has recently undergone an evolution with the conceptualisation developed by Taranto (2018). By working in a massive online education context, we realised that the MDT theoretical framework was no longer suitable for effectively analysing educational phenomena involving both teachers and MTEs within such new environments. Therefore, the MOOC-MDT was born and networked with another theoretical adaptation, from the Zone Theory used by Goos (2013), to give life to MOOC's Zone Theory (Taranto, 2018). In the following, we briefly show how the MDT was adapted to the MOOC environment, obtaining the so-called MOOC-MDT. Next, we briefly show how the adaptation of Zone Theory was undertaken, with particular relevance to the role of MTEs. Subsequently, without entering into the detail of the further theoretical networking, we illustrate the parts of the MOOC's Zone Theory theoretical framework that are useful for analysing MTEs' expertise, learning and development.

20.3 MOOC-MDT: The MDT Framework Adapted to the MOOC

Previously, we have seen three intertwined features of the MDT model. Now, we recapitulate on them and explain the changes made to them in relation to the MOOC.

Institutional Aspects MOOC-MDT focusses on two social institutions active in a MOOC for teacher education: the *community of MTEs* (university researchers, who generally are the designers and facilitators of the course with the help of some computer technicians, together with the researcher-teachers) and the *community of teachers enrolled in the MOOC* (MOOC-teachers). As in the MDT, so too here, both these communities are constrained by institutional aspects. However, the MOOC-teachers are not subjected to the same institutional weight that the school or the principal impose on them. In fact, generally, the MOOCs are open (and free) to anyone who wants to enrol. The number of members is massive. They never meet each other in person; they do not have space or time constraints: they can access the MOOC whenever they want and wherever they are. The MOOC-teachers also are not required to finish the course. So, inside the MOOC, they experience the institutions as something not too invasive, which intervene only a little in the MOOC dynamics.

Meta-didactical Praxeologies Having clarified in the previous sections what is meant by didactical and meta-didactical transposition, in a MOOC, the purpose of developing such a meta-didactical transposition remains unchanged. Therefore, both communities could experience some evolution in their praxeologies.

The MOOC-teachers enrol themselves in the MOOC on a voluntary basis. All of them have different geographical origins; they teach in different school typologies and grades; therefore, in general, they have differing professional backgrounds. In a face-to-face professional development course, it is less likely that there will be a need to deal with this *heterogeneity* of the teachers' community. The MTEs must take charge of this a priori heterogeneity in transposing some ideal praxeologies to the MOOC-teachers. In fact, the MTEs have to implement meta-didactical praxeologies and invite the MOOC-teachers to reflect on them and therefore to consider whether these can be appropriate praxeologies for them. The design phase is indeed very demanding: first, because *the materials and contents* must all be prepared and defined *before the MOOC begins*. Second, the MTEs will never be able to personally compare themselves with the MOOC-teachers. *Everything happens online*, in a more asynchronous than synchronous way. In fact, the MTEs *do not know when or what a MOOC-teacher really viewed in the MOOC*. In addition, *they do not know if the MOOC-teachers have deeply understood the proposed materials*. There is not a moment of simultaneous interaction that clarifies doubts for everyone or that gives suggestions to reinforce ideas. There are the materials, where everything is written, such as suggestions and clarifications. However, it cannot be assumed that these are read by everyone or shared by everyone. There are spaces for online communication where the MOOC-teachers can interact mostly with each other, but not all at the same time. Moreover, it is not the case that every MOOC-teacher really wants to compare himself/herself to the others on an online platform. The particular steps of the MDT model do not happen: a community consisting of MTEs and MOOC-teachers does not tend to be formed because they cannot interact with each other on the same materials at the same time.

Broker In the MOOC-MDT, the previous description of the MDT elements remains valid. In fact, the researcher-teacher helps the university researcher in the design phases and participates in monitoring the MOOC actions whilst it is running. In addition, by the definition we have previously seen, it is precisely the MOOC itself that is able to create new connections between the MTEs' and MOOC-teachers' communities. In this sense, *the MOOC can be considered as another broker or at least a tool that has the same functions as a broker*. It facilitates the transposition of mathematical concepts from the MTEs' community to the MOOC-teachers' community and vice versa.

The MOOC-MDT also takes into consideration other aspects related to the MOOC-teachers that we will not examine in this chapter. For further information, see Taranto et al. (2017) and Taranto (2018).

20.4 MOOC's Zone Theory: Networking Between MOOC-MDT and Zone Theory

A finer analysis of the role of the MTEs involved in a MOOC for teacher education can be achieved by realising a networking (Bikner-Ahsbahs & Prediger, 2014) between the MOOC-MDT and the adaptation of the Zone Theory used by Goos (2013). Goos uses Valsiner's Zone Theory (1997) to study the learning of teachers (Table 20.1).

Goos (2013) interpreted the zone of proximal development (ZPD) as the set of possible ways in which a teacher might develop, the zone of free movement (ZFM) as the constraints and affordances provided by the teacher's professional context and the zone of promoted action (ZPA) as activities in which the teacher can be involved and which promote certain ways of teaching. The ZFM and ZPA are dynamic and interrelated, forming a ZFM/ZPA complex. Goos (2013) claimed that such an approach enables the complexity of teacher learning and development to be analysed whilst still allowing for the influence of the teacher to direct their own learning by seeking out professional development or modifying their environment.

Taranto has adapted the Zone Theory used by Goos applying it distinctly to the MOOC, the MOOC-teachers and the MTEs. For each of them, a ZPD and ZFM/ZPA complex has been defined. In this chapter, we focus only on the MTEs and in particular on the adaptation made to their ZFM/ZPA complex that Taranto has called *research environment's ZFM/ZPA* (Table 20.2) (for more details, see Taranto, 2018).

The research environment's ZFM/ZPA (Table 20.2) is an external part of the MOOC, so it is observable by everyone who is interested in analysing MTEs' behaviours. In this chapter, we are clearly more oriented towards the analysis of MTEs who are involved in the design and delivery of MOOCs for teacher education. In any case, the model that will be described below can also be understood and used for other educational experiences where MTEs are involved.

Specifically, for the *research environment's ZFM*, the definition given by Goos (Table 20.1), in its general sense, is valid, but it must be modified to make it more specific to this environment. Instead of *teachers*, one must consider the MTEs as learners. They could be researchers and possibly also researcher-teachers interested

Table 20.1 Zone Theory used by Goos

Zones Theory used by Goos	
ZPD: Zone of Proximal Development (possibilities for developing new teacher knowledge, beliefs, goals, practices)	- Mathematical knowledge - Pedagogical content knowledge - Skill/experience in working with technology - Beliefs about mathematics, teaching and learning
ZFM: Zone of Free Movement (structures teacher's access to different areas of the environment, availability of different objects within an accessible area, ways the teacher is permitted or enabled to act with accessible objects in accessible areas)	- Perceptions of students - Access to resources - Technical support - Curriculum and assessment requirements - Organisational structures and cultures
ZPA: Zone of Promoted Action (people, objects or areas in the environment in respect of which the teacher's actions are promoted)	- Pre-service teacher education - Professional development - Informal interaction with teaching colleagues

in teacher education and professional development in mathematics education. The *environment*, in this case, is represented by the space occupied by the MTEs: the University, Department of mathematics, national/international conferences, etc. In addition, the school becomes part of the research environment if amongst the MTEs there are researcher-teachers. Regarding the second columns, for the research environment's ZFM (Table 20.2), instead of *teachers' perception of students*, we consider MTEs' perception of MOOC-teachers, that is, the teachers who are imagined to follow the course (the MOOC in particular in our case) and then actually follow it. To understand *access to resources*, we consider as resources the existing literature which an MTE can access (thanks to the departmental library, internet searches and so on); the possibilities offered by participation in a conference, such as the proceedings and other related materials; and data that come from a teaching experiment (such as video, students' papers, teachers' logbooks, etc.), questionnaires and interviews. By *technical support*, we mean the possibility for an MTE to use the print centre of her department, to book classrooms (in the department, in a school, etc.), to organise meetings with her colleagues or for teacher education purpose, to ask for help from the computer technician of the department for access to the resources and so on. *Curriculum and assessment requirements*, in a sense, are exactly as Goos intends them. The teachers should take into account their national

Table 20.2 Research environment's ZFM/ZPA

Taranto's re-elaboration/interpretation *Research environment's ZFM/ZPA*	
ZFM: Zone of Free Movement (structures MTEs' access to different areas of the environment (university, department of mathematics, national/international conferences, etc.), availability of different objects within an accessible area, ways the MTEs are permitted or enabled to act with accessible objects in accessible areas)	(a) Perceptions of teachers (b) Access to resources (literature, proceedings of conferences, teaching experiment, data obtained via questionnaires or interview, etc.) (c) Technical support (print centre, classroom booking, etc.) (d) Italian curriculum and assessment requirements (e) Organisational structures and cultures
ZPA: Zone of Promoted Action (people, objects or areas in the environment in respect of which the MTEs' actions are promoted)	(a) Participation in national/international conferences (b) Involvement in actions to foster teachers' professional development (face-to-face meetings, MOOCs, etc.) (c) Professional development (d) Informal interaction with their peer and researching colleagues

curriculum and assessment requirements. For an MTE involved in teacher education, it is surely mandatory to take into consideration the same points. Then, as *organisational structures and cultures*, we consider the environment created to allow sharing and comparing amongst MTEs interested in the same topic and with the same culture.

For the *research environment's ZPA*, also in this case, the definition given by Goos (Table 20.1), in its general sense, is valid, but instead of *teachers*, we are considering the MTEs whose actions are being promoted. Regarding the second columns, for the research environment's ZPA (Table 20.2), instead of *pre-service education*, which certainly does not concern MTEs, we consider the events from which an MTE benefits in terms of personal learning. This might include participation in national/international conferences as a speaker or member of the audience or as an alternative when he/she is involved in delivering teacher's professional development, like a MOOC. In light of this, we can retain the term *professional development* also for MTEs, and *informal interaction with teaching colleagues* is replaced by informal interaction with research colleagues.

In the following, we propose some examples in which we show how operatively the networking process is used in the analysis. We will focus in particular on how the MTEs organised themselves in the design and management phases of some MOOCs for teacher education, also highlighting how the figure of the researcher-teacher is an important component of the MTEs.

20.5 The Math MOOC UniTo Project

In our department at the University of Turin, we are involved in a project called *Math MOOC UniTo*. It started in the spring of 2015 and focused on designing and delivering MOOCs for mathematics teachers, mainly from secondary schools, with the aim of increasing their professional competencies and improving their classroom practices. The slogan for our course is "MOOCs made by teachers for teachers." In fact, the idea of creating MOOCs for mathematics teacher education was born as a result of a second-level master's course that took place at our Department from September 2013 to June 2015. It addressed in-service secondary school mathematics teachers and was also attended by some researcher-teachers who had collaborated in previous experiences with our research group in mathematics education. At the end of the master's course, two researcher-teachers felt the desire to share the educational innovations they had analysed and studied. Therefore, in agreement with the master's MTEs, it was decided to try to take advantage of the emerging phenomenon of MOOCs during that period in Italy. Hence, it was decided to offer the opportunity of an authentic professional development experience designed for a larger group of teachers: this idea generated the *Math MOOC UniTo* project, namely, MOOCs for mathematics teacher education.

Four MOOCs were designed, one for each of the main topics in the official Italian programmes for secondary school: geometry, arithmetic and algebra, change and relations and uncertainty and data. So far, the first three have been delivered[3], and the fourth is a work in progress. These MOOCs are open, free and available online for teachers on a Moodle platform (http://difima.i-learn.unito.it/). Each MOOC is subdivided into modules of 1 or 2 weeks and lasts 8 weeks in total. Every week, the MOOC-teachers worked individually to become familiar with different approaches. In our MOOCs, these activities included watching videos where an expert introduced the mathematical topic of the week, reading the mathematical activities based on a laboratory methodology and, optionally,[4] experimenting with these in their classroom. The MOOC-teachers were invited to share thoughts and comments about the activities and their contextualisation within their personal experience, using specific communication message boards. The methodology of our MOOCs aims to create collaborative contexts for teachers' work, where they can learn from these kinds of practices.

The MTE team is composed of two university professors, a group of secondary school researcher-teachers who had attended the master's course (nine in MOOC Geometria and 20 in the other two MOOCs) and a PhD student. All of them are

[3] They are called, respectively, as MOOC Geometria (the one based on geometry contents, from October 2015 to January 2016), MOOC Numeri (the one based on arithmetic and algebra contents, from November 2016 to February 2017) and MOOC Relazioni e Funzioni (the one based on changes and relations concepts, from January 2018 to April 2018).

[4] If the MOOC-teachers liked them; if they in their classes were explaining at that time topics close to those proposed

involved in course design and delivery and in monitoring their evolution in terms of interaction amongst participants (both MOOC-teachers and MTEs themselves). In particular, the researcher-teachers proposed the topics to be discussed in each MOOC and then developed the materials that were subsequently revised by university researchers and then included in the MOOCs. In addition, the MTEs help the MOOC-teachers to solve technical problems (sending personal e-mail or showing some tutorials uploaded in the platform), and they remember the tasks to be done week by week with weekly e-mails.

The MTEs met regularly, both during the design of the MOOCs and at the end of each activated module, sharing what they had observed during that specific module. In fact, the most significant MOOC-teachers' interventions or sharing actions were discussed, in order to generate a fruitful exchange of ideas on the progress of the course and its development.

20.6 Analysing the MTEs Involved in Our MOOCs

As we mentioned, our MOOCs aspire to create collaborative contexts for teachers' work, where they can learn through sharing their practices. The MTEs put in place some techniques that are justified by the MOOC-MDT theoretical framework and also take into account the influence of the research environment's ZFM/ZPA (Table 20.2). Therefore, some MTEs' techniques and their evolution are presented and analysed in order to highlight and discuss their methodological and theoretical justifications.

We show here two essential meta-didactical types of tasks that, according to our experiences, any MTE in a MOOC for mathematics teacher education should address. Precisely, we consider two topics related to the design principles: (i) target and (ii) theme.[5]

The MOOCs we will consider here are the first two delivered: *MOOC Geometria* and *MOOC Numeri*.[6]

20.6.1 Target

Our MOOCs aim at increasing the MOOC-teachers' professional competencies and improving their classroom practices in a collaborative context. Given this aim, it is important to identify hypothetical target teachers. Our choice fell on in-service mathematics teachers of lower and upper secondary school (Table 20.3). However,

[5] There are seven meta-didactical types of tasks in total that any MTE of a MOOC for mathematics teacher education should address. For more details, see Aldon et al. (in press).

[6] The data of the third MOOC (*Relazioni e Funzioni*) are still under analysis.

MTEs cannot know in advance the teachers who will decide to enrol in the MOOC, and they will never meet them in person. For these reasons, as MTEs, we were forced to hypothesise a *mean* Zone of Proximal Development (ZPD) of our future teachers. The ZPD (Vygotsky, 1978) concerns an internal level and comes into play when the MTEs think about the praxeologies that they want to transpose to the MOOC-teachers. Therefore, the MTEs assume a certain level of prior knowledge (ZPD) of the MOOC-teachers' community (not of the individual teacher since they are forced to consider mean values). Here, it is evident how the MTEs are conditioned by the research environment's ZFM (Table 20.2). They have a certain *perception of the teachers* that they imagine will follow the MOOC, formulating some hypotheses concerning their mean ZPD. In making these hypotheses, the comparison with the researcher-teachers was important because in their role as brokers they possessed information on updating teachers who were not necessarily known in the university world. It is precisely thanks to the comparison with the researcher-teachers that we chose to deliver the MOOCs in the autumn. The researcher-teachers have pointed out that autumn is the most profitable period for an educational course because teachers at the school are less overburdened with deadlines than at the end of the school year and they are also more receptive and eager to learn new things to experiment with during the year. Therefore, the *informal interaction with research colleagues* – as detailed in the research environment's ZPA – whose protagonists were university researchers and researcher-teachers was fruitful for the identification of the target.

Regarding characteristics of the target teachers (Table 20.3), there has not been any evolution from one MOOC season to another. The imagined target was clearly stated, and the enrolled participants proved to be in line with MTEs' expectations. In fact, the MOOC-teachers were exactly mathematics in-service teachers of lower and upper secondary school from all over Italy. The contemporary presence of this heterogeneous group of teachers, with different cultural and professional backgrounds, has not been perceived as a limitation, but rather as bringing richness to the MOOC. As we will see in the next sections, at the design level, the proposed activities could be considered useful across lower and upper secondary school. This has also guaranteed rich exchanges of teaching practices amongst the MOOC-teachers from the different levels of schooling and who could hardly have confronted in the presence of such topics.

Table 20.3 The meta-didactical praxeology related to "Target"

	Target
Task	To identify a hypothetical target (lower and upper secondary school teachers)
Technique	To choose activities of a specific school level (according to the target), related to specific mathematics topics
Logos	To hypothesise a mean ZPD of the target
Evolution	None

20.6.2 Theme

Another essential aspect of a MOOC design for mathematics teacher education is the "Theme" (Table 20.4). The choice of the theme is naturally related to the identified target and to institutional purposes of the professional development programme. Both MOOCs aim to respond to teachers' specific needs identified in the institutional and social contexts (point *e* of the research environment's ZFM), referring to the national curriculum for teacher's professional development (point *d* of the research environment's ZFM). We see below how these points were considered.

In the design phase, the MTEs have to evaluate essentially two possibilities, according to their long-term educational aim: to keep the same theme and deliver the same content, considered as crucial in the professional development, in every season, or to change the MOOC theme from season to season trying to cover one by one different crucial aspects and educational objectives. Such a decision influences the potential MOOC audience. Indeed, with the former choice (see, e.g. Panero, Aldon, Trgalová, & Trouche, 2017), the opportunity for professional development is offered to an increasing group of teachers (including those who have not completed the previous season). With the latter choice, that is the one taken by our MTE team, the same group of teachers can enrol in every season of the MOOC to pursue their professional development. In fact, Taranto (2018) observed how being familiar with the MOOC environment reduces the cognitive effort that is involved in the phases of self-organisation: the mathematical contents of the MOOC change but its structure remains unchanged in terms of general timing (weekly modules) and tasks to

Table 20.4 The meta-didactical praxeology related to "Theme"

	Theme
Task 1	To identify the main theme to address in the MOOC
Techniques	To focus every season on a different core part of the curriculum (geometry, number) and to choose activities around specific topics according to the theme
Logos	To innovate methodology and strategies of teaching mathematics as highlighted in the Italian curriculum
Evolution	The first season was devoted to geometry whilst the second one to numbers. Once a topic is covered, the professional development programme moves onto another one, with the long-term aim of deepening the professional development of the same group of teachers. Fifty percent of enrolled in the second season came from the previous one
	Time
Task 2	To decide how much time is devoted to each module of the MOOC
Techniques	To estimate the time necessary to acquire the treated topic, taking into account an estimated engagement of 4 h per week: If necessary, to divide theoretical and practical parts If the material is too dense, to devote 2 weeks to the same topic
Logos	Average of estimated learning times of the target
Evolution	To reduce the quantity of the material provided; greater attention to differentiating the material for different school levels

be performed. These awarenesses were possible thanks to point *a* of the ZPA and point *b* of the ZFM in the research environment (Table 20.2). In fact, because MOOCs are an emerging phenomenon, there is still little literature on this. Therefore, the comparison with other researchers involved in MOOC design was important in order to be able to analyse our methodological choices more consciously.

As previously mentioned, the MOOC contents are carefully designed by the researcher-teachers, reviewed by the university researchers and then implemented in the MOOC. The involvement of the researcher-teachers in carrying out the activities was important. The university researchers, during the aforementioned master's course (note that this is point *b* in the research environment's ZPA), had carried out a meta-didactical transposition: they had presented some theoretical aspects and offered some practical examples to make the researcher-teachers reflect. The latter have adopted these praxeologies, and in preparing the materials of the MOOC, they have made a didactical transposition of similar contents to the MOOC-teachers. In other words, they have translated a knowledge of the research world into a knowledge at hand for the teacher, acting as brokers. This is why the slogan of our project is "MOOCs designed by teachers for teachers." The mathematics curricula to which the activities refer are in line with the Italian curriculum. For example, in MOOC Geometria, space was devoted to some activities designed to deal with students' common misconceptions (perpendicular vs vertical; angle vs arc). The activities do not exhaust all the topics of the curriculum, but have the ambition to provide detailed methodological information on how to deal with some mathematical topics of particular importance for the mathematical education of the students.

Once the theme and its possible evolution from season to season are decided, MTEs have to consider the "Time" variable (Table 20.4). There could be two possible approaches: decide how much time has to be devoted to each module of the MOOC or how much material it is possible to read and to work on in a module that has a fixed duration (e.g. 1 week). The MTEs chose the first approach, and according to the theme, they decided to devote 1 week or two to the same content or methodology because of its complexity or of the large amount of material. In fact, the required techniques are those listed in Table 20.4. It is important to note that in the design, it is necessary to make an average of the estimated learning times of the target (Carroll, 1963). After the first season of the MOOC Geometria, the MTEs decided to reduce the quantity of the provided material and to pay greater attention to differentiating the material for different school levels. In fact, although it was intended that the proposed activities could be used in both the lower and the upper secondary school, in the first MOOC, there were more activities for the upper secondary school. This is because the researcher-teachers involved in the first MOOC were mostly upper secondary school teachers. In MOOC Numeri, the second MOOC, as mentioned above, the number of researcher-teachers has more than doubled (from 9 to 20), and there has been a greater balance between those who taught at the lower and those at the upper secondary school.

We note how in the choice of the theme to be treated (Table 20.4), as well as identifying the target teachers to whom the MOOC should be addressed (Table 20.3), the MTEs are conditioned by the research environment's ZFM (Table 20.2).

Identifying the target teachers is then closely connected not so much to the choice of the theme itself, but to how to deal with such a theme taking into consideration the educational levels being addressed. Specially, the *Italian curriculum and assessment requirements* are points that cannot be ignored. Furthermore, the choice of the theme is closely linked to the task of time, to which it corresponds, in the research environment's ZFM, and to consideration of relevant *organisational structures and cultures*. Another aspect that has not been overlooked by designers has been to compare their work with that of other researchers involved in the design and delivering of MOOCs for mathematics teacher education. *Accessing resources* such as literature and proceedings of conferences but also – as detailed in the *research environment's ZPA – participating in national/international conference* and entertaining an *informal interaction with research colleagues*, the MTEs were able to confirm the methodological choices made, for example, to decide to change the theme from time to time in each MOOC season or decide how much time is devoted to each module of the MOOC (Table 20.4). Through all these activities, a close harmony between university researchers and researcher-teachers has developed and flourished.

20.7 Discussion and Conclusion

The main goal of the chapter has been to draw on the authors' experience in designing and managing a series of MOOCs within a distance education programme for in-service Italian mathematics teachers, in order to discuss a fresh approach to addressing the self-reference dilemma experienced by MTEs who wish to analyse their own learning and development. Namely, our MTE team not only is made of people who work in different institutions but also has two essentially different types of institutional backgrounds: that of the university researchers and that of the researcher-teachers. This assures a higher level of differentiation in backgrounds, contexts, communications and knowledge, which Lovin et al. (2012) recommend in order to reduce the limits of self-referential research in MTE teams.

We have discussed how the researcher-teacher is a typical figure in the Italian mathematics education tradition, which differentiates it from other cultural backgrounds. As such, our contribution is deeply marked by our national story in mathematical education, as discussed by Arzarello and Bussi (1998). However, the brokering features that distinguish our researcher-teachers' role have a more general value that could also be applicable in different environments. In fact, the crucial aspect of the researcher-teacher in our MTE team consists in his/her role of brokering between the academic component of the MTE and that of the teachers involved as trainees in the MOOCs. He/she knows directly the concrete background where these teachers are working and can so guarantee experiences according to what Darling-Hammond and Richardson (2009) call the new paradigm of professional learning opportunities. They claim that current research supports professional development that

(i) deepens teachers' knowledge of content and how to teach it to students;
(ii) helps teachers understand how students learn specific content;
(iii) provides opportunities for active, hands-on learning;
(iv) enables teachers to acquire new knowledge, apply it to practice and reflect on the results with colleagues;
(v) is part of a school reform effort that links curriculum, assessment and standards to professional learning;
(vi) is collaborative and collegial;
(vii) is intensive and sustained over time.

Within the objective limits of a MOOC, most of these items are present in our project because of the effective contribution of the researcher-teachers as brokers. They know these needs at the concrete didactical level of everyday professional engagement but have the background theoretical knowledge that allows them to elaborate these aspects at a meta-didactical level into suitable tasks for the MOOC-teachers. We have exemplified this claim, discussing selected meta-didactical types of tasks amongst the many involved in our project (target and theme).

The role of researcher-teachers as brokers is crucial in order to overcome the self-referential dilemma of MTEs involved in designing and managing teacher education programmes. Our project contributes to understanding and resolving this dilemma. On the one hand, since the field of mathematics teacher education is composed largely of academic researchers, it judges itself in academic reports about its actions; on the other hand, teachers can at most make an evaluation about the practical results of these programmes, but the theoretical elaboration is again inexorably developed by academics. Hence, we arrive at the so-called auto-referential dilemma. It seems not possible to follow Kant in his metaphors of reason's experiment upon itself or compelling itself to give testimony: contrary to Kantian reason, MTEs cannot have a safe *self-knowledge.* But our typology of MTE makes possible a sort of Kantian solution, because of the crucial role of brokering played by researcher-teachers in our MTE team. Researcher-teachers have a dual character: their status as teachers makes it possible to introduce into the MTE field an external, practice-based point of view; on the other hand, their role as researchers allows them to simultaneously adopt an academic point of view and to generate a MOOC design and management that can avoid the negative aspects of professional learning development (Darling-Hammond & Richardson, 2009):

(i) Relies on the one-shot workshop model.
(ii) Focuses only on training teachers in new techniques and behaviours.
(iii) Is not related to teachers' specific contexts and curriculums.
(iv) Is episodic and fragmented.
(v) Expects teachers to make changes in isolation and without support.
(vi) Does not provide sustained teacher learning opportunities over multiple days and weeks.

The role of the researcher-teachers as brokers additionally helps avoid several of these aspects within the MOOC environment:

- What are the dilemmas and opportunities associated with researching ourselves as MTEs?
- What methodologies might be effective in building such an evidence base?

Our first research question asked about the dilemmas and possibilities associated with doing research on ourselves as MTEs. From the literature and our discussion, the concrete needs have emerged that led to the emergence of theories (MDT and MOOC-MDT) relevant to the contexts in which we have worked, namely, the specific Italian institutional and cultural environment. Regarding the second research question, concerning methodologies that might be effective in building an evidence base, we claim that the function of brokering embodied in the researcher-teachers in the MTE team represents the aspect that effectively overcomes the auto-referential dilemma. Hence, it is exactly this function that should be further studied within different cultural environments to check if/how it frees the MTEs from the weight of self-referential statements, which are a serious obstacle for developing a scientific analysis of teacher education programmes.

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Part IV
Commentaries

Chapter 21
Mathematics Teacher Educator Knowledge for Teaching Teachers

Olive Chapman

While we are still in the early stage of research on mathematics teacher educators (MTEs), mathematics teachers have received significant attention in a variety of studies in the last two decades. In particular, the focus on mathematics teachers' knowledge for teaching became prominent as the field promoted the need to reform the teaching and learning of school mathematics, with the teachers being viewed as change agents. This expectation of teachers obviously has implications for MTEs who work with practicing and/or prospective teachers to develop and improve the teaching of mathematics and highlights the importance for us to understand what MTEs should know and do to support teachers' learning and change. Thus, it makes sense that researchers have started to investigate MTE knowledge based on what it is or ought to be, directly or analogously, in relation to the content and pedagogical content knowledge conceptualized for mathematics teachers in recent years. The six chapters in this section of the book provide insights on this perspective of MTE knowledge, which I discuss in this chapter. I, therefore, first consider some ways in which mathematics teacher knowledge has been conceptualized, then discuss MTE knowledge as indicated or implied by the studies in the six chapters, and finally consider some issues in conceptualizing MTE knowledge with implications for ongoing research.

O. Chapman (✉)
Werklund School of Education, University of Calgary, Calgary, AB, Canada
e-mail: chapman@ucalgary.ca

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_21

21.1 Mathematics Teacher Knowledge: A Basis for MTE Knowledge

Mathematics teacher knowledge (*specific to teaching mathematics*) (MTK) is an obvious basis for determining MTE knowledge specific to teaching mathematics teachers to support their development of mathematics knowledge for teaching. Thus, to understand MTE knowledge from this perspective, it is important to consider how MTK has been conceptualized.

What MTK is or ought to be has received a lot of attention that has resulted in a variety of ways of conceptualizing it. Shulman's (1986) classification of knowledge for teaching, in particular, as subject matter knowledge, pedagogical content knowledge, and curriculum knowledge provided a model that mathematics education researchers built on, refined, or extended to develop categories of knowledge that are distinctive to teaching mathematics. The result is a perspective of MTK that gained prominence in the field of mathematics education. This *category-based perspective* (Chapman, 2014), or *subject knowledge differentiated perspective* (Ruthven, 2011), identifies types of knowledge and "provide [s] an overarching heuristic framework that can guide the analysis, assessment and development of professional knowledge" (Ruthven, p. 83).

Table 21.1 (an expanded version of that in Chapman, 2014) contains examples of the category-based perspective of MTK. I focus on this perspective because of the connections to the six chapters in this section of the book. Collectively, the examples in Table 21.1 offer a landscape of types of knowledge that are relevant to mathematics teachers, which also have direct or indirect relevance to MTEs as reflected

Table 21.1 Models of category-based perspective of MTK

Ball, Thames, and Phelps (2008)	Krauss, Baumert, and Blum (2008)	Rowland, Turner, Thwaites, and Huckstep (2009)	Tatto et al. (2012)	Carrillo et al. (2018)
Common content knowledge Specialized content knowledge Horizon content knowledge Knowledge of content and students Knowledge of content and teaching Knowledge of content and curriculum	Knowledge of mathematical tasks as instructional tools Knowledge and interpretation of students' thinking Knowledge of multiple representations and explanations of mathematical problems	Foundation knowledge Transformation Connection Contingency	Mathematics content knowledge Mathematics curricular knowledge Knowledge of planning Knowledge for enacting mathematics	Knowledge of topics Knowledge of the structure of mathematics Knowledge of the practice of mathematics Knowledge of mathematics teaching Knowledge of the features of the learning of mathematics Knowledge of mathematics learning standards

in these six chapters, discussed next. In addition, although not explicitly a component or separate category of these examples of the category-based perspective, *beliefs* are also considered to be an important category of MTK as established by the relationships between beliefs and practice. These beliefs include beliefs about the nature of mathematics, beliefs about mathematics teaching and learning, and beliefs associated with problem-solving. Thus, beliefs are also represented as MTE knowledge in some of the chapters.

21.2 MTE Knowledge for Teaching Mathematics Teachers

MTEs are not a homogeneous group in terms of their work with mathematics teachers. This is reflected in the six chapters which collectively cover university-based mathematicians and mathematics-education educators/researchers and their work with prospective and/or practicing teachers. However, when appropriate (as in Table 21.2), I consider them as a group in addressing the types of knowledge (including beliefs) they hold or should hold collectively. Table 21.2 consists of a summary of categories or types of MTE knowledge I identified in each of the six chapters, which I discuss in terms of three themes: MTE knowledge as MTK, MTE knowledge as *knowledge of mathematics teacher education* (KMTEd), and MTE knowledge as beliefs.

21.2.1 MTE Knowledge as MTK

MTE knowledge as MTK refers to the knowledge school teachers need to know and use with their students that MTE should also know. Thus, categories or elements of MTK are being considered as representing a central domain of MTE knowledge. Three of the chapters offer insights into this relationship between MTE knowledge and MTK by making a case for it or identifying examples of categories or elements of the knowledge that overlap.

Dinazar Escudero-Ávila, Miguel Montes, and Luis Carlos Contreras, in their chapter, argue that decades of research about and with mathematics teachers should have an influence on what an MTE ought to know. Through a review of relevant literature on mathematics teachers, they suggest that MTEs should hold knowledge of the mathematical knowledge and pedagogical content knowledge (PCK) to be built by their students (prospective teachers). In particular, they should be aware of the characteristics of PCK in relation to theories of teaching, teaching strategies and methodological resources for teaching mathematics, key features of learning mathematics, and learning standards. In their chapter, Tracey Muir, Sharyn Livy, and Ann Downton also consider MTE knowledge in relation to MTK. However, unlike Escudero-Ávila et al., they established this connection based on a study they conducted involving a co-teaching situation with an MTE and a practicing primary

Table 21.2 MTE knowledge for working with mathematics teachers

Escudero-Ávila, Montes, and Contreras	Muir, Livy, and Downton	Mali, Petropoulou, Biza and Hewitt	Marshman	Leikin	Zazkis and Marmur
Mathematics knowledge Teachers' PCK Mathematics teaching practices and skills Teacher professional identity Features of professional development of mathematics teachers Teaching content of mathematics teacher education programs Standards of mathematics teacher education programs	Classroom teacher knowledge Theory and rationale Foundation Transformation Connection Contingency	Mathematical practices based on own mathematical research practices	Beliefs about mathematics, mathematics teaching and learning Beliefs and decision-making about pedagogy	Students' mathematical potential and mathematical challenge Teachers' professional potential and challenging content for teachers	Instructional tool for advancement of pedagogy and mathematics

school teacher teaching an education course for prospective teachers. They investigated how the knowledge needed by the MTE differed from that required by the primary school teacher based on their teaching of this course. They conclude that similar types of knowledge are required by MTEs and primary school teachers, such as knowing the appropriate representations and examples to use when teaching particular concepts or anticipating complexities and addressing student misconceptions. Finally, Roza Leikin, in her chapter, suggests a model of MTE knowledge and skills in which a major element is (school) students' mathematical potential and the challenging content for students, which is an integral aspect of MTK. Her view requires MTEs to know what school teachers need to know about their students' mathematical potential and to understand it deeply enough to promote its understanding in mathematics teachers. While these three chapters address MTE knowledge in different ways, they draw attention to a category-based perspective of it that includes categories or elements of MTE knowledge (summarized in Table 21.2), directly related to that required by an experienced, competent school teacher.

21.2.2 MTE Knowledge as KMTEd

While it makes sense for MTEs to hold knowledge that includes categories or elements of MTK, it is also obvious that they should hold knowledge different from that of teachers in their role of educators of teachers. Thus, for this theme, MTE knowledge as *knowledge of mathematics teacher education* (KMTEd) refers to the knowledge MTEs need to know and use to engage teachers in their learning, for example, PCK for supporting teacher learning. In this context, content is teacher education instead of mathematics. Most of the six chapters provide insights regarding the nature of MTE knowledge in terms of categories or elements of KMTEd.

Escudero-Ávila et al. discussed several categories of MTE knowledge that represent their KMTEd. These categories consist of knowledge of mathematics teaching practices and skills, teacher professional identity, features of professional development of mathematics teachers, teaching content of mathematics teacher education programs, and standards of mathematics teacher education programs. For example, the authors explain that MTE knowledge about mathematics teaching practices should include what constitutes effective mathematics teaching practices (e.g., National Council of Teachers of Mathematics, 2000, 2014) and about mathematics teaching skills that include professional noticing and classroom preparation. This knowledge should include how teachers use knowledge and how they focus their teaching practice. Also, MTE knowledge about the professional identity of prospective teachers, as part of the content of teacher education, should include different factors (e.g., beliefs, interaction with environment, attitudes, emotions) that influence the development of identity. The authors conclude that the most significant difference in terms of knowledge between teachers and MTEs is that MTEs need to be aware of and to understand a wide network of connections between both purely mathematical items and between mathematics and other professional elements.

While Escudero-Ávila et al. discussed broad categories of KMTEd (see Table 21.2), Muir et al., in their chapter, attended to elements that can be related to some of these categories. Muir et al. identified examples of specific elements of an MTE's KMTEd that illustrate differences between the MTE's and primary school teacher's knowledge based on their teaching of a course for prospective teachers. These elements of knowledge include MTE holding a deeper and broader understanding of the theoretical underpinnings behind the use of appropriate pedagogical practices than a school teacher needs to know, knowledge required to justify the use of these practices to prospective teachers and respond to their questions about them, and knowledge required to model and explain the why and how of appropriate pedagogical practices that can be used in a primary classroom.

Angeliki Mali, Georgia Petropoulou, Irene Biza, and Dave Hewitt's chapter also includes specific elements of KMTEd but from the perspective of research mathematicians who taught calculus to students who may become mathematics school teachers. They investigated how the mathematicians' own mathematical research can influence the ways in which they teach and the pedagogical potential of being explicit to their students about the mathematical practices they used while working with the mathematical content. Specifically, Mali et al. considered the extending practices rooted in advanced mathematical practices that have the potential to foster the mathematical horizon of the university students, including prospective teachers. They identified four categories of these extending practices related to their participants' own mathematical research: drawing on examples, connecting mathematical areas, visualizing, and simplifying. While these practices also represent practices that are proposed as part of MTK, Mali et al.'s consideration of them does not imply that mathematicians should hold knowledge of them as part of MTK but as part of KMTEd in terms of them originating from their research to support their teaching while at the same time expanding the mathematics horizon of their students who may or may not become teachers. As the authors concluded, research mathematicians' views on, and explicitness to students about, certain mathematical practices used constitute a type of awareness crucial for the development of prospective teachers' horizon, and the role taken by the mathematicians is crucial in shaping their students' future mathematics horizons.

MTE knowledge regarding designing appropriate tasks to support teachers' learning is another aspect of their KMTEd that is addressed in the chapter by Roza Leikin and the chapter by Rina Zazkis and Ofer Marmur. These authors draw attention to special types of tasks that are important to the MTEs' role in working with teachers. Leikin promotes the creation and use of challenging content consisting of creativity-directed activities suitable for mathematics teachers that integrate mathematical, psychological, and didactical components, while Zazkis and Marmur promote the design and use of tasks of a pedagogical nature that also aim at extending the teachers' understanding of the underlying mathematics. For Leikin, the major goal of MTEs is promoting mathematics teachers' professional potential and challenging content, which is based on knowledge of their students' mathematical potential and challenging content. She explains that both potentials and challenging content are integral components of MTE knowledge to frame teachers' professional development. Focusing on challenging content, Leikin argues that creativity-directed mathematical activities are ultimately challenging for students and teachers alike and thus are beneficial in the work of MTEs. She suggests that MTE knowledge and skills should integrate deep understanding of the concept of varying mathematical challenge and proficiency in designing and conducting challenging activities for mathematics teachers.

Zazkis and Marmur point out the importance for MTEs to obtain information on the knowledge of a group of teachers (their students) and their understanding of a mathematical topic in order to plan for, or adjust, subsequent instruction for them. They promote the use of a special type of task to determine the nature and extent of the teachers' prior mathematical knowledge. These tasks are of a pedagogical nature but are additionally aimed at extending the teachers' understanding of the underlying mathematics. Specifically, these tasks are based on the idea of *script writing* that involves the teachers writing an imaginary dialogue in which, for example, a fictional interaction between a teacher and students investigates mathematical claims. MTEs can use the scripts to examine the teachers' mathematical knowledge and instructional choices and to strengthen their mathematical knowledge and make connections to school mathematics and pedagogical approaches.

While these five chapters address MTE knowledge in different ways, they also draw attention to a category-based perspective of it that includes specific categories or elements of MTE knowledge (summarized in Table 21.2). As discussed later, while they differ in nature, there are parallels between some of these types of MTE knowledge as KMTEd and those associated with MTK.

21.2.3 MTE Knowledge as Beliefs

MTE knowledge as beliefs refers to the professional beliefs MTEs hold or ought to hold in working with mathematics teachers. I treat beliefs as a separate theme since they provide a unique perspective of knowledge and could include elements related to both MTK and KMTEd in similar ways. Beliefs are important because of the relationship to teaching. Some MTEs who have conducted self-studies on their practice have found how their beliefs limited or conflicted with their teaching. For example, Alderton (2008) became aware of a contradiction between her practice and beliefs that prompted her to question why she did not always live the values she professed, Marin (2014) uncovered her assumptions in her teaching that challenged her beliefs, and Bailey (2008) became aware of beliefs about mathematics and its learning that she did not know she held, which were contrary to what she espoused in the classroom. None of the six chapters attended to the relationship between beliefs and teaching, but Margaret Marshman's chapter provides insight regarding beliefs held by MTEs through a study that explicitly investigated their beliefs about mathematics and mathematics teaching and learning and their beliefs and decision-making about pedagogy used with prospective teachers.

Marshman's study included 60 (73%) MTEs who taught only mathematics or statistics content courses (i.e., mathematicians), eight (10%) who taught only pedagogy courses, and 14 (17%) who taught both discipline content and pedagogy. Marshman found that these MTEs were inclined to have a problem-solving view of mathematics as a discipline, a way of knowing, and a way of thinking. Most of them agreed that teachers should motivate students to solve their own problems and that knowing how to solve a mathematics problem is as important as getting the correct solution. Most also agreed that effective mathematics teachers enjoy learning and doing mathematics themselves, teachers should give students opportunities to reflect on and evaluate their own mathematical understanding, and, if teachers ignore the mathematical ideas that students generate themselves, it can seriously limit their learning. Most of the MTEs disagreed with traditional teaching methods but agreed with allowing students to struggle and with the importance of students justifying statements. In general, according to Marshman, these MTEs shared beliefs about the importance of supporting students to construct their own knowledge but were less comfortable with the use of questioning and less inclined to agree that they developed an attitude of inquiry in the classroom.

These examples of MTEs' beliefs (Marshman's chapter offers much more) are related to their knowledge about *mathematics*, mathematics teachers, and teaching mathematics, with connections to their KMTEd. Mali et al.'s chapter also offers a glimpse of MTEs' beliefs based on mathematicians' views on their teaching and how it is related to their own mathematical research. Their views embody their beliefs about what was important in their teaching that was meaningful to support their students' (including prospective teachers') learning. The authors concluded that the mathematicians' views about certain mathematical practices used constitute a type of awareness crucial for the development of prospective teachers' horizon.

21.3 Reflection on Conceptualization of MTE Knowledge

The preceding section (21.2) focused on categories or types of MTE knowledge suggested by the six chapters in this section of the book. While the chapters do not explicitly or intentionally offer a particular perspective to conceptualize MTE knowledge and while other ways may be possible, the category-based perspective seemed appropriate to address how they present MTE knowledge. Similar to the common use of this perspective to understand MTK (Table 21.1), it is likely to be commonly used for MTE knowledge particularly if researchers who studied MTK are also now studying MTE knowledge. The category-based perspective does offer a basis to understand MTE knowledge in relation to school teacher knowledge and to examine MTE knowledge from the perspective of being a teacher of school teachers.

There are two obvious ways in which elements, categories, or models of MTK, such as in Table 21.1, can form a basis for MTE knowledge. First, since they represent the knowledge MTEs should help teachers to develop, MTEs should hold knowledge of them to the extent that they are needed to frame the MTEs' teaching. Escudero-Ávila et al.'s chapter, for example, illustrates this. Second, the categories or models of MTK can be used to frame corresponding categories or models of MTE knowledge specific to teaching teachers by shifting the focus of the categories from teaching mathematics to teaching teachers with analogous meaning. For example, in their chapter, Muir et al. demonstrate through their study how the knowledge quartet (Rowland et al., 2009, and Table 21.1) is applicable or transferable in describing the work of MTEs. Similarly, Chick and Beswick's (2018) framework for mathematics teacher educator PCK (MTEPCK) is based on adapting school teacher PCK for teaching mathematics, building on existing research into PCK. Their adaptation involved substituting prospective teachers for school students and substituting teacher PCK for mathematics as the teaching domain. For example, in their framework, for knowledge of *student thinking or misconceptions*, in teacher PCK, the teacher addresses it for students regarding a mathematics concept, while in MTEPCK, the MTE addresses it for prospective teachers regarding a PCK concept. Similarly, for knowledge of *cognitive demand of task*, mathematics tasks for students become PCK tasks for prospective teachers. Other specific categories or elements of knowledge in Table 21.1 can be adapted in a similar way. Table 21.2 illustrates some analogous items, for example, knowledge of instructional tools such as script writing (Zazkis and Marmur's chapter) and creativity-directed activities (Leikin's chapter), while the research literature offers a variety of other tools used specifically in teacher education (e.g., Llinares & Chapman, 2019; Tirosh & Wood, 2008). Table 21.2 also includes beliefs in an analogous way as promoted for teachers. Marshman's chapter provides examples of these beliefs which deal with similar topics to those of teachers that include mathematics and teaching and learning mathematics. Similar to the view that teachers should be aware of their beliefs and hold productive beliefs to effectively frame their practice, MTEs should also be aware of and hold productive beliefs. As Schuck (2002) found in her self-study of her practice as an MTE, it was essential for her to be familiar with her students' beliefs as well as her own.

In general, the category-based perspective of knowledge for teaching mathematics and mathematics teachers is useful to establish connections and disconnections between MTE knowledge and mathematics teachers' knowledge, to investigate and understand models of MTE knowledge, to use such models to investigate or analyze MTEs' teaching, and to plan professional development activities for new MTEs. For example, Chick and Beswick (2018) used their framework of MTEPCK to identify aspects of an MTE's work (i.e., pedagogy), which proved to be useful in terms of analyzing the moment-by-moment application of knowledge in the work of mathematics education. However, the category-based approach of conceptualizing MTE knowledge does have issues, some of which are linked to concerns raised about the category-based perspective of MTK and the challenge in determining MTK. I address five possible issues that require consideration in future research.

First, Table 21.1 suggests a lack of a universal model or overarching framework of what MTK is or ought to be for effective teaching of mathematics. While there are similarities or overlaps among the categories of knowledge for the different authors, there are differences and perceived inadequacies that make an integrated model highly problematic to accomplish. Thus, with no universal agreement on a widely accepted category-based framework of MTK, it becomes problematic to conceptualize MTE knowledge based on this perspective of MTK. Table 21.2 suggests that separate models could possibly emerge for MTE knowledge that are similar to or different from those for MTK. For the category-based perspective, as Ruthven (2011) points out, "The goal is to provide an overarching heuristic framework that can guide the analysis, assessment and development of professional knowledge" (p. 83). More research is needed on MTE knowledge to determine whether such a goal is achievable, meaningful, or useful in moving forward the field of teacher education in mathematics.

Second, the category-based perspective, by its nature, is usually presented as consisting of separate or discrete categories within a model even though distinctions between categories are not always clear-cut. Such representations, as illustrated in Table 21.1 and Table 21.2, may not be appropriate to describe MTE knowledge and can provide a simplistic view of what it actually is. For example, Escudero-Ávila et al. concluded that the knowledge that MTEs draw on in educating mathematics teachers is multidimensional, complex, integrated, contextualized, and dependent on mathematical content. This complexity is also represented in Leikin's chapter that suggests MTEs should hold knowledge that enables them to design tasks for teachers that combine mathematical challenge with psychological and didactical challenges that teachers meet in their everyday work. Thus, while categories may be convenient to describe MTE knowledge similar to teacher knowledge, research needs to give attention to other ways of representing it as a complex system or way of thinking.

Third, MTEs are not a homogeneous group in terms of what they teach. For example, the six chapters covered MTEs who are research mathematicians and researchers and/or instructors of mathematics education (didacticians) and taught only mathematics or statistics content courses or only pedagogy courses or both discipline content and pedagogy courses. There are also other groups of MTEs who are school-based but not considered here. This diversity among MTEs creates a more complex situation in conceptualizing MTE knowledge than for teachers who all teach only mathematics. For example, in their chapters, Escudero-Ávila et al. make a case for how the knowledge of mathematicians who teach only mathematics to prospective teachers is different from the knowledge of MTEs who are didacticians, Marshman shows overlap in different groups of MTEs' professional beliefs, and Mali et al. identified teaching practices in mathematicians' teaching based on mathematical practices used in their research that are the same as practices promoted for teachers, that is, drawing on examples, connecting mathematical areas, visualizing, and simplifying. These examples suggest a complex relationship among possible categories of knowledge of different groups of MTEs and the challenge to conceptualize the knowledge needed to support teachers' learning and professional development.

Fourth, focusing predominantly on the category-based perspective of MTK as the basis of conceptualizing MTE is also limiting given that it is not the only perspective of MTK promoted in the field. While the category-based perspective is widely used in research on the mathematics teacher, other ways of thinking about MTK have different implications for what MTEs need to know to support teachers' learning. For example, in addition to subject knowledge differentiated (i.e., the category-based perspective), Ruthven (2011) identified three other perspectives of MTK: (i) *subject knowledge situated*, which is concerned with the "use and development of subject-related knowledge in teaching [that] is strongly influenced by material and social context" (p. 87); (ii) *subject knowledge interactivated*, which is concerned with the "epistemic and interactional processes through which mathematical knowledge is (re)contextualised and (re)constructed in the classroom" (p. 89); and (iii) *subject knowledge mathematized*, which is concerned with characterizing "those *mathematical modes of enquiry* which underpin any authentic form of mathematical activity, and to show how teachers employ them to foster such activity in their classrooms" (p. 91). Other researchers have suggested related perspectives of MTK that include a way of being and acting that develops and grows through doing mathematics and being mathematical (Watson, 2008); *how* teachers hold their knowledge, in particular their orientation toward mathematics, for example, embodying modes of mathematical enquiry (Barton, 2009); a participatory attitude toward mathematics (Davis & Renert, 2009); and pedagogical content *knowing* to stress pedagogical content knowledge as a dynamic concept meaning knowing to act that is inherently linked to and situated in the act of teaching within a particular context (Cochran, DeRuiter, & King, 1993). Thus, there is not only lack of consensus within the category-based perspective of MTK but also regarding how MTK is viewed in the field that further complicates determining an overarching framework to conceptualize MTE knowledge. Research needs to explore how or whether these alternative ways of thinking about MTK translate to MTE knowledge for teaching teachers or can lead to new ways to extend our understanding and conceptualizing of MTE knowledge.

Fifth, focusing only on MTEs of education (MTEEs), that is, MTEs who teach mathematics education courses for prospective teachers, regardless of which of the above perspectives of MTK is involved, there is another consideration that creates a problematic situation in conceptualizing MTEE knowledge: what really is their content? For mathematics teachers, the content (discipline) is mathematics. For MTEEs, the content is *mathematics education*, which tends to be treated analogously to mathematics as being a discipline when replacing mathematics with mathematics education in the category-based models for MTK. However, if a teacher says, "I teach mathematics," it is likely that most people will have some idea of what the teacher is talking about. But if an MTEE says, "I teach mathematics education," the reaction is likely to be "what is that?" So, assuming it is, what is mathematics education as a discipline? Understanding MTEE knowledge means understanding mathematics education as a discipline in its own right. For example, what does mathematical knowledge look like within mathematics education as a discipline? Ball et al. (2008) argue that mathematics teachers need to hold knowledge, such as *specialized content knowledge*, which is a different kind of knowledge from what mathematicians need, that is, for a category-based perspective "expert teaching requires more than what would ordinarily constitute expert knowledge of a subject" (Ruthven, 2011, p. 83). Do MTEEs need to hold a different kind of mathematical knowledge from mathematicians and/or mathematics teachers for their discipline of mathematics education? The answer seems to be obviously yes. For example, as Escudero-Ávila et al. point out in their chapter, MTEE knowledge:

> should encompass, not being limited to, the mathematical knowledge to be built by their students; it takes a panoramic and interrelated view, is fluid and intentional in nature, and so emphasises connections (the scope and organisation of knowledge) and depth.
>
> [M]athematical knowledge … does not depend solely on mathematics, but on the knowledge, abilities and identity necessary to teach it.

However, research-based evidence is needed to clarify what this actually means. In general, research should consider the nature of MTEE mathematical knowledge in the context of mathematics education as a discipline (or not) to establish how or whether it is unique to this context.

In addition to mathematical knowledge, what does teaching and learning look like for mathematics education as a discipline in its own right? Is it about pedagogy or andragogy (adult learning)? Is it about PCK (which teachers of children need) or andragogical content knowledge (ACK) (which teachers of adults should need)? One could argue that both are important knowledge for MTEEs, that is, they should know PCK as part of the content of mathematics education and ACK as a way of engaging prospective teachers in their learning of mathematics education. However, in the context of mathematics education, these two may not be separate entities but integrated to give a form of specialized P/ACK. For example, teaching teachers the way they should teach their students is one approach MTEEs may use to engage their prospective teachers as adult learners to experience a teaching/learning process to use with their future students. But whether this or other approaches used by MTEEs constitute some specialized combination of PCK and ACK distinctive to teaching teacher education (as a discipline or not) need to be explored in future research.

These five issues or situations are offered to encourage further discussion, consideration, and research of the nature of MTE knowledge and the perspectives and processes that are suitable to achieve realistic conceptions of MTE knowledge. The implication is that there is still a lot to be explored for us to understand the knowledge MTEs need to work with practicing and/or prospective teachers to develop and improve the teaching of mathematics. This is similar to what the Association of Mathematics Teacher Educators (2017) pointed out in their *Standards for Preparing Teachers of Mathematics* regarding the set of proficiencies they proposed for "well-prepared beginners and programs preparing mathematics teachers," which have implications for MTE knowledge:

> Although these proficiencies are grounded in available research, in many areas that research is not yet sufficient to determine the specific knowledge, skills, and dispositions that will enable beginning teachers to be highly effective in their first years of teaching. (para. 1)

21.4 Conclusion

Understanding MTE knowledge in terms of what MTEs do, can do, or ought to do is important to design or improve teacher education programs for prospective teachers and prospective MTEs and to support growth or change in the professional knowledge and practice of practicing teachers and practicing MTEs. This chapter highlighted how a category-based perspective of MTK can be useful as a basis to make sense of MTE knowledge. However, based on perceived inadequacies of this perspective in conceptualizing MTK, it is not sufficient to account for MTE knowledge or adequate to deal with it as a complex, integrated system and way of being/thinking and as knowing how to act in the teaching process. We are at the early stage of researching MTEs, and the category-based perspective makes sense at this point as a way to begin to understand MTE knowledge. However, it is important for research to explore other ways of understanding and representing MTE knowledge. This chapter suggests some issues that embody possible considerations in conceptualizing MTE knowledge, for example, how MTEs hold knowledge – the manner in which the knowledge is organized to make it usable in an effective and meaningful way and their ways of being and acting. The six chapters in this section of the book also have implications for future research, for example, attending to the nature of beliefs different groups of MTEs hold and the relationship to their teaching and students' learning; the quantity, quality, robustness, and depth of mathematical knowledge required by an MTE; and other factors involved with particular attention to how they interact in the teaching process. Thus, in the ongoing work to conceptualize MTE knowledge, researchers should aim to address areas of research that will move the field beyond a focus on identifying relationships between MTE knowledge and category-based perspectives of MTK.

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Chapter 22
Who Are We as MTEs: And How Do We Learn and Develop?

Barbara Jaworski

22.1 Background and Introduction

Throughout Themes 2 and 3 of this book, for me, issues of *identity* are very strong. Who *are* "we" as MTEs? Particularly, are we learners? Are we teachers? Are we researchers? Are we all three of these? And how do our identities sit alongside those of the teachers with whom we work in a complexity of contexts and cultures? There will be many more questions as I seek to unravel the richness of relationships and issues that have been revealed in these chapters.

In 1998, the first issue of JMTE, the *Journal of Mathematics Teacher Education*, was published. This was a strong acknowledgement that teacher education in mathematics had become an important field for research. Building on a long(ish) history of research into the learning of mathematics in schools and higher education, and a rather shorter history of mathematics teaching at these levels, it had become clear that these two fields of research interest were incomplete without consideration of how teachers come to know how to work with students to support "effective" learning of mathematics (Simon, 2008). In many countries, by this time, teacher education programmes were in place to educate or in some cases "train" new teachers of mathematics or to contribute to the professional development of practising teachers. In JMTE, much of the research into mathematics teacher education was conducted by the people who were responsible for teaching the prospective or practising teachers. In many cases, these people, the MTEs, were also the researchers seeking to illuminate this field of education. Research addressed how the new teachers learn to teach mathematics in a variety of programmes and contexts led by MTEs. However, few of the papers submitted to JMTE or other mathematics education journals raised

B. Jaworski (✉)
Mathematics Education Centre, Loughborough University,
Epinal Way, Loughborough, Leicestershire, UK
e-mail: B.Jaworski@lboro.ac.uk

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_22

questions about the learning of the MTEs in their programmes for educating teachers to teach mathematics effectively to students at a range of levels (Chapman, 2008; Jaworski, 2001).

When the editors of JMTE at that time agreed to edit the (first) *International Handbook of Mathematics Education* (Wood, 2008), it seemed imperative to recognise this general omission in the research and to build on what a smallish number of scholars were doing to remedy it. Thus, the fourth of four volumes of the handbook was entitled *The Mathematics Teacher Educator as a Developing Professional* (Jaworski & Wood, 2008). Its chapters offered a richness of experience of MTEs' learning and development from working to educate prospective and practising teachers and researching the associated programmes and courses, their models and theories. I believe that what we find in this book is indicative that MTE learning and development has become an important area of research since 2008. I therefore thank Merrilyn Goos and Kim Beswick for the opportunity to read and respond to these accounts in Themes 2 and 3 of this book.

What I write here is related to my own knowledge and experience and my reflections on what I read. Readers can perhaps recognise my projecting from what I read into my own experience and my own interests, both practical and theoretical. I expect that this is what you are doing as you read the chapters.

22.2 Theme 2: Learning and Development as an MTE

In their prospectus for the special issue, the editors for this book set out their focuses for Theme 2 including the following paragraph:

> Mathematics teacher educators are also well positioned to learn from their research with teachers, even though this learning is often left unacknowledged and unarticulated (Jaworski, 2001). Chapman (2008) suggested that an explicit goal of mathematics teacher educators' research of their practice should be self-understanding and professional development. Reports of such studies, therefore, need to include how the teacher educator-researchers reflected, what practical knowledge they acquired, and how this knowledge impacted or is likely to impact their future behaviour in working with their students. This will allow such research to contribute to greater theoretical understanding about mathematics teacher educator learning and to the improvement of practice.

This paragraph was insightful in its projection into the chapters of Theme 2 of this book, which together provide a most interesting and illuminative response.

I start my commenting with reference to the first chapter in the 2008 book, written by Martin Simon, who refers to "Two categories of Mathematics Teacher Education":

> Teacher professional development efforts can be sorted into two categories, those with process goals only and those that have content and process goals (Simon, 2008, p. 18).

According to Simon, the second category "consists of courses and workshops for teachers in which teacher educators aim to promote particular mathematical and pedagogical concepts, skills, and dispositions" (p. 18). The first category seems less well defined. Simon illustrates with reference to "the Japanese Lesson Study model … and programmes based on teacher inquiry or teacher research" (p. 18). He writes further:

> There are no a priori learning goals for teachers involved in these programmes (other than learning the processes of inquiry, reflection, etc.) (p. 18).

The premise behind the programmes in the first category is that teachers' engagement in reflective, inquiry-based, professional activity, including research, supports their professional development. I was curious to see whether the focuses of MTE involvement in teacher professional development discussed here would fit these two categories, and what might go beyond this, to new areas of study. A subtle difference here is that our focus is mainly on MTE learning as differentiated from teacher learning, although teacher learning remains a factor in several cases. First, the two categories.

Category 2: *teacher educators working with teachers in a course or workshop aimed to promote teachers' mathematical and pedagogical concepts, skills and dispositions.*

I start with Category 2 because I find it the one that is perhaps the most clear to recognise. In this book, teachers in several of the chapters learn to teach in some initial teacher education, or prospective teachers' programme or course, engaged in sessions or workshops led by MTEs. In Theme 2, we find such a setting in the chapters of Bissell, Brown, Helliwell and Rome; Brown, Brown, Coles and Helliwell and Ingram, Burn, Fiddaman, Penfold and Tope, in all of which the MTEs work in educating prospective teachers in a nationally guided initial teacher education programme in the United Kingdom. Learning and development for these MTEs is related to their work with teachers in such a programme and to their own reflections on their activity with teachers. Chapters of Nolan and Keazer; Olanoff, Masingila and Kimani; and Osborn, Prieto and Butler all relate the learning of MTEs to their teaching of teachers in a range of courses, in, respectively, Canada, the United States and Australia. The courses vary among "Culturally relevant pedagogies" (Nolan and Keazer), "Mathematical knowledge for teaching" (Olanoff et al.) and the "Disciplinary teaching of science, or mathematics, or statistics" (Osborn et al.). While all of these chapters vary in the ways in which MTEs' learning is related to their teaching of teachers in the prospective teacher course or programme, their commonality is that their learning is essentially related to their responsibility to the programme and their desire to develop their own activity with the teachers. I see a difference here from many articles in this category in the past that the MTEs address their own learning as much as, if not more than, they evaluate the teachers' learning.

Category 1: *there are no* a priori *learning goals for teachers involved in these programmes (other than learning the processes of inquiry, reflection,* etc.*).* (Simon*,* 2008*, p. 18*).

This category is much harder to define or determine. The four other chapters, in Theme 2, are harder to group – all have some elements that relate them to Simon's (2008) Category 1, although in different ways. Chapter 15, by Bakogianni, Potari, Psycharis, Charalambos, Spiliotopoulou and Triantafillou, is located within a large EU project (Mascil) which promoted research and development into inquiry-based learning and the connection of school mathematical activity with authentic workplace situations. Situated in Greece, ten teacher educators (mathematics and science) worked with groups of teachers to explore inquiry-based learning and authentic workplace learning with their students, in the spirit of Lesson Study approaches, and to share experiences and issues among their own MTE inquiry community. A characteristic of this programme concerns the large number of teachers and MTEs involved and the corresponding complexity of issues arising. Chapters 9 and 12 both involve two MTEs, one experienced and one novice, learning together in a specially focused project. Chapter 12 (Sikko and Grimeland) offers a self-study of two mathematicians developing their own practice in working with teachers, prospective and practising, exploring what a mathematician needs to learn to become an MTE. Chapter 9 (Van Zoest and Levin) offers multiple layers of MTE learning, in the United States, through "Artefact-enhanced Collegial Inquiry (ACI)" between two MTEs teaching teachers in a "methods" course. Chapter 17, rather different from the other chapters, focuses on educational structure in China and explores the roles of two groups of educators, the MTEs and MTRs, mathematics teacher researchers, who work both separately and collegially to support prospective or practising teachers.

An issue here is that while these chapters address MTEs' learning and development in relation to teachers' learning in some course or programme, the course or programme is very much in the background, subsumed in the more prominent focuses. These focuses have the "process goals" of which Simon writes and can be seen to involve research and development into new approaches and ways of thinking such as inquiry-based and workplace learning or how to teach mathematics in ways such that pedagogy supports mathematical concept development.

A question we might ask here is what are the similarities and differences between our learning as MTEs and the learning of teachers, both in content and in process, in the programmes in which we teach? How do our responsibilities in these programmes limit our learning opportunities?

As I read these chapters, in both categories, I became strongly aware of MTEs *reflecting* on their activity and learning in relation to their work with teachers, often in narrative style. I was reminded of a special issue of *ZDM Mathematics Education* which I edited with Rongjin Huang, addressing collaborations between mathematics teachers and MTEs, in which we observed the following:

> Reflective practice emerges as a principal goal for effective development and is linked to teachers' and didacticians' engagement with inquiry and research. (Jaworski & Huang, 2014, p. 173).

We drew on the work of Orit Zaslavsky (2008) who had emphasised that among the enormous and multifaceted demands on MTEs, in terms of knowledge and quality, the overarching demand is for MTEs (like teachers) to be reflective practitioners. The seminal work of Donald Schön (1987) comes to mind, in which he writes of reflection *on*, *in* and *for* action, where the *action* is that of the professionals engaging with their particular practice. Stephen Kemmis (1985), building on John Dewey (1933), has written about reflection as not only something quiet and personal but also being *action oriented* and *critical*. In our case, here, we are thinking of the practice of taking action as an MTE. I believe that, our over recent years, we have seen a transition through MTEs supporting *teacher* reflection to acknowledgement of our own reflection leading to us developing insight, awareness and knowledge. The literature has many examples (e.g. Chapman, 2008; Goos, 2014; Jaworski, 2001; Zaslavsky, 2008).

As MTEs, how does our reflection on, in or for practice affect overtly our learning and development and ongoing action in practice?

22.2.1 Reflection and Voice

In this book, many of the chapters offer narratives, dialogues or personal statements of MTEs in which their reflections provide readers with insight into the nature of their practice and the issues whose critical consideration *on* or *of* this practice leads to a learning that is compelling and motivating *for* or *in* future practice. Space limits what I can focus on particularly, but a few examples seem in order.

1. In Chap. 11 (Bissell et al.), Alistair writes the following:

   > Despite having cared so much about my opportunities to listen, I found that in this case I wasn't interested in the responses that were coming back from the teachers – I was only waiting for the responses that were in the plan for the day, which felt immediately uncomfortable (p.14).

2. In Chap. 16 (Nolan and Keazer), Kathy writes the following:

   > Prior to this moment, I had made assumptions that this PT probably had experiences similar to mine, but this experience taught me that there is always more to know about my students. This left me with the conviction that I must mine knowledge out of my own students and build on that knowledge as I teach them about mathematics and CRP.

In these two examples, we find the writer reflecting on his or her intentions, actions and, possibly, implications for the future that arose from the reflection. In both cases, I find myself saying, "YES, I recognise this, I've been there," which leads to my own reflections on my own practice.

In the third example, I was struck by the layers of learning that emerged from a classroom event.

3. In Chap. 9 (Van Zoest and Levin), we gain insight from a three-way interaction, over time, between MTE (LVZ), MTE (ML) and teacher Karry: During Karry's teaching, a mathematical explanation seems to cause confusion, and when discussion stalled, LVZ intervened:

   > LVZ (to the class): It seemed like we were on to something and then all of a sudden it started to go awry—go wrong—like sometimes math problems do. [slight pause].
   >
   > Have you ever had that happen? You have a great idea and you're just cooking along and then all of a sudden it's just not working anymore? [LVZ laughs reassuringly.]

In a written reflection, Karry makes reference to her experience of the intervention:

> Karry (written reflection): Another place I could have been able to fully articulate better towards a clear mathematical goal is when one student's way went awry. What I have learned is that I need to know when to "funnel" and when to "focus" ….
>
> However, after multiple attempts to redirect to what I noticed was wrong with the student's train of thought and kind of beating around the bush, this is when I should have switched from "focus" to some form of "funnelling".

Subsequent analysis reveals a range of issues; LVZ's choice of action, to intervene in the class; the focus of the intervention; analysis of various data from the event; and the wisdom of switching from focus to funnelling. Discussion between LVZ (expert MTE) and ML (novice MTE) leads to MLs becoming aware of the complexity of issues for both teacher Karry and MTE LVZ. I urge you to read the full account.

In all three examples, I was made aware of my own entering into the experience described and feeling for myself how it might have been for me in that situation. As John Mason has so vividly expressed (e.g. 2002), it is by entering into moments in our own experience, recognising and *noticing* issues we have faced ourselves, that we gain the possibility to externalise and to re-enter such experiences, both in and outside the event. Having access to the voice of the teacher or MTE (spoken or written) both enables us to *hear* that person and also *to enter the experience* in our own practice.

These examples of reflection draw attention to the matter of "voice." We can talk *about* what we do, or what others do, but *giving voice to someone* is about having that person speak (or write) their own reflections on what they have experienced and learned and how this does or can affect their work in the future. In the ICME 13 survey on "Teachers working and learning through collaboration," in our reading of many relevant papers, we reported that the teachers' voice was largely absent. What we read was written by researchers talking *about* the teachers with whom they worked and their learning and development (Robutti et al., 2016). As Chapman (2008) pointed out, in research reports from researchers who were also MTEs, the MTE voice was rarely heard reflecting *on their own learning and development*. Rather, we read what they observed of the teachers they studied and their analysis of the resulting development of teaching practice. In recalling an experience, talking *about* the experience is very different from talking *in* or *from* the experience, giving voice to the thoughts and feelings the experience created.

Do we, or how do we, engage with a methodology of promoting or giving voice to our teachers and to ourselves as a device for learning and development?

22.2.2 Collaboration and Inquiry

Although not always labelled as such, many of the chapters discuss "communities of inquiry" within a research and development setting. For example, in Chap. 15 (Bakogianni et al.), a group of 11 MTEs formed a community of inquiry to share experiences and learn from each other's practice with regard to supporting teachers to link mathematics and workplace situations in inquiry-based learning. In the examples from Chaps. 9 and 16, we see two pairs of MTE colleagues each acting as a small community of inquiry. The idea of community of inquiry (CoI) is that a group of practitioners (in these cases, the practitioners are MTEs) collaborate to explore activities and issues from their practice for mutual support, learning and development. They literally *inquire into* their activities and address the issues raised; this gives rise to identifying and exposing issues and tensions relating to inquiry in practice. Two quotations illustrate such tensions related to issues of teachers' and researchers' collaboration and the role of teacher as researcher:

> The involvement in supporting the teachers was a learning experience, teachers, educators and students, we are all learners. This is what we are doing. We are learning how to communicate (Chloe, 4th meeting).
>
> Teachers have to be reinforced to communicate through the platform between themselves… to inquire by themselves… to search for resources (Sofia, 5th meeting).

The words used to express relationships and issues are revealing of the speaker's focus – on their own learning or on the teachers' learning, or on both.

Chapter 12 (Sikko and Grimeland) presents a "self-study" in which two mathematicians form a community of inquiry to address what a pure mathematician needs to learn in order to become an MTE. Reflection and collaboration are evident in their inquiry. The nature of the self-study involved the two inquirers in lengthy discussions drawing on literature, a range of artefacts including lecture notes and presentations, conversations with other colleagues and personal reflections. They write the following:

> As an MTE you need to inquire into your own practice. This includes inquiring into the choice of models and representations, trying out new approaches, not being "locked" into one particular way of doing things but instead continuing to reflect upon your own practice.

One example illustrates the ways in which mathematics and pedagogy became linked:

> An example is the concept of division, where neither of the authors was aware of the distinction between partitive and quotitive models of division prior to moving into teacher education. For the mathematician, this distinction is not important … For the teacher, on the other hand, … the question is rather how to be able to help pupils extend their understanding of division from division of integers to division involving fractions, and which representations and models are helpful in this extension.

The idea of community of inquiry (CoI) can be linked closely to theory of community of practice (CoP; Jaworski, 2006; Wenger, 1998). In Chap. 8 (Olanoff et al.), we are told that three MTEs (two novices and one experienced) form a CoP to develop their mathematical knowledge for teaching teachers and improve their teaching of mathematics content courses for prospective elementary school teachers. With relation to CoP (Wenger, 1998), they write, "Through our mutual engagement and shared repertoire (e.g., reflections, memos, tasks, lesson plans), we came to realise the importance of looking deeply at the underlying mathematics behind the representations, algorithms, and definitions that we use." However, they also talk of inquiry, referring particularly to Cochran-Smith and Lytle's (2009) "inquiry as stance." As I read the chapter, appreciating reflective writing from the two novices on how the CoP theory had influenced their personal learning and development, it was not until reflections from the experienced MTE that inquiry became explicit in the reflection. The following quotations illustrate:

> Dana: Participating in the CoP helped me to develop my pedagogical content knowledge, specifically knowledge of content and students. … writing down what happened with my students and thinking about how to help them construct knowledge and see problems with their work helped me make connections and figure out ways to help my students in the future. Being able to share the experience with the other members of the CoP also helped me develop my own knowledge in a way that reflecting on my own would not.
>
> Patrick: I believe the co-teaching/observation experience and writing a memo is really helping me reflect on how I can make this course a better course for the students. By reflecting on the students' struggle, the goal of the activity and my actions as an instructor combined with my observation in Jo's class, I am getting an opportunity to think about my teaching more than I would normally have done.
>
> Joanna: I benefited from the mutual engagement of having other people to think carefully about how to support PTs in understanding the mathematical concepts underpinning the procedures they would be teaching in the future. For example, the CoP with Dana and Patrick caused me to rethink how I engaged PTs in thinking about tasks involving probability. … I also learned by observing Patrick and Dana teach and saw some things that they did (e.g., how Patrick engaged his students in thinking about necessary and sufficient conditions for definitions of plane figures) that provided me with new insight into my own teaching practices. … I have changed my practice as a result of participating in the CoP as I am more intentional in approaching my teaching through a stance of inquiry.

In these statements, although "inquiry" is not articulated until the very end, I believe these MTEs are all inquiring into their own teaching using a number of tools to aid them and finding out more about themselves and the teaching approaches they engage with. The position of "inquiry as stance" lies behind their statements, even if not uttered specifically. We see again here how collaboration and inquiry are important for these MTEs in researching their own developments in practice. Although not part of the theoretical basis of this chapter, I would encourage these researchers and readers of this chapter to consider the theoretical underpinnings of CoP and *inquiry as stance* as constituting a *community of inquiry* linking reflection and development through collaborative inquiry.

In what ways do we use theoretical terms like community, practice, collaboration or inquiry in our reflections and communications? What can they offer us for learning and development?

22.2.3 Theoretical Underpinnings

In the above examples, we see MTEs using *community of practice* as a theoretical basis for their inquiry. As Lawrence Stenhouse (1984) reminded us, *research is systematic inquiry made public*. Transitioning from CoP to CoI is a recognition that the inquiry basis of activity is fundamental to this research. The significance of acknowledging inquiry as a theoretical element of the community activity is that it brings with it the construct of “critical alignment.” Wenger (1998) speaks of three elements of belonging to a CoP*: engagement, imagination and alignment*. We engage with the practice alongside our co-participants, we use imagination to guide our own trajectories in the practice and we align with the norms and expectations within the practice. While CoI draws on many of the postulates within CoP, including “engagement” and “imagination,” the concept of “alignment” is tricky. Thinking of *teaching* as the practice under consideration, with the goals of supporting student learning (of mathematics), and then *aligning with* the practice as it exists might support elements of practice that many professionals would like to change (e.g. rote learning). However, some elements of practice are deeply ingrained in what schools and teachers do and have been doing for years; they cannot easily be changed overnight; the concept of *alignment* supports the lack of change. Thus, in a CoI, theoretically, the postulate of alignment is modified to become “critical alignment” (Jaworski, 2006). Critical alignment is a theoretical basis for the inquiry of change. In practice, it means that we do not align uncritically. By inquiring into our practice as we engage with it, we consider what ideally we should like to do and see: we *look critically* at the status quo, discuss with our colleagues and seek ways to bring in the changes that can lead to the outcomes we would like to see. Of course, this might be a lengthy process involving cycles of innovation and reflection through which we learn about what is possible as well as desirable. For example, in the quotations above, Karry might carry out her intention to “funnel rather than focus” and then discover other issues or tensions in the outcome, and further attention to the elements of funnelling could be necessary, prompting further innovation. In practice, the inquiry of critical alignment can be lengthy and challenging, requiring much reflection, sharing with supportive colleagues and willingness to sustain uncertainty, a significant process of learning and development and, ultimately, sustainable outcomes.

Other theoretical perspectives are used by researchers in these chapters. For example, in Chaps. 10 and 11, by Brown et al. and Bissell et al., researchers use *enactivism* as their theoretical foundation. I am reminded that Sandy Dawson (1999) referred to enactivism in practice as “a path laid while walking” (p. 148); literally, we achieve the path we want, as MTEs, teachers and students, as we engage with the practice of doing what we do and, I add this, looking critically at what we achieve. This suggests that the path laid while walking might be seen as a form of critical alignment in practice. Dawson quotes Bakhtin in writing, “we are completely responsible for our actions and it is in knowledge garnered through embodied action that ethical responsibility lies” (p. 149). He raises issues with those who judge teachers and teaching as “wrong” with associated claims of what *should* be done in classrooms – the “right” thing. He writes, “Part of the motivation behind the development of the enactivist view is a questioning of current views of the nature of knowledge development and acquisition in the mathematics classroom” (p. 149). As an alternative, he proposes the following:

> Consider for a moment a different approach … one based on becoming aware of what you are doing without judging it. … mathematics teacher educators and mathematics teachers could move from a culture based on judgment to one based on possibility (p. 148).

For me, this is entirely consistent with critical alignment.

Bissell et al., in Chap. 11, write of enactivism as “Using what we have done previously in a new environment will be followed by adapting when what happens is not effective or good-enough for the situation” and “Identifying feelings of being uncomfortable and staying with the detail of what happened can support our learning by opening up new possibilities for acting.” As we read on and encounter Alistair and his transition from being a mathematics teacher to becoming an MTE, we see how these words relate to the actions, experiences and feelings he reports.

In Chap. 10, Brown et al. acknowledge that enactivism guides the processes that they use as MTEs and, indeed, underpins the design of their teacher education course (for prospective teachers). They write, “the processes we use as MTEs to develop our practices are the same as those our prospective teachers are offered to develop their practices.” However, enactivism as a guiding force is less upfront in this chapter than in Chap. 11 (for some of the same researchers). Here, in Chap. 10, several further constructs are offered to describe/explain teaching/learning developmental processes. The first is *awareness*. With reference to a number of well-known scholars, awareness is used as a “synonym for consciousness,” as “the world experienced by the person” and as “a core action or function that must be present in order to learn.” They claim “Only awareness is educable,” suggesting that “this is the chief role of the mathematics teacher, while keeping open the ways in which it might happen.” They quote Dave Hewitt (2001) as follows:

> By educating awareness the mathematician inside a student is being educated, which would not be the case if everything were treated as if it were to be memorised. Awareness informs decisions and how to act using information which is known. (p.38).

Brown et al. exemplify awareness as follows: “an awareness of counting squares covered by a shape might allow attention to be drawn to a definition of area; and an awareness of tangents to a curve might allow attention to be drawn to stationary points of the curve.” Course design takes account of layers of awareness as MTEs and prospective teachers follow the same processes in teaching and learning, but in educating teachers’ awareness, MTEs need to become aware of the awarenesses of the teachers in relation to the teachers’ awarenesses of mathematics, not forgetting of course their own awarenesses – a complex set of relationships. I see these theories or constructs, enactivism and awareness, as integral to the nature of the project, providing a philosophy and methodology underpinning the research. As well as the construct of awareness, these researchers refer to other theoretical constructs including *metacommunication, second-person perspectives* and *experiences to issues to action*. I leave it to readers to follow these up in the chapter and to link them to the overarching perspectives of enactivism.

A range of other theoretical perspectives are evident in other chapters. For example, in Chap. 14, Ingram et al., within a broad sociocultural perspective, refer to “two theoretical models of professional learning”: Clarke and Hollingsworth’s “interconnected model of professional growth” and Merrilyn Goos’s adaptation of Valsiner’s *zone theory*. Here, the sociocultural perspective seems to be a philosophy underpinning the activity and research, while the models are used as a lens to examine or analyse the ways in which professional growth changes the context in which growth occurs. This seems a different use of theory from that in enactivism, awareness and other perspectives above. The models here are used in the analytical process to make sense of the data, rather than to provide constructs in the developmental process itself.

Two chapters use the concept of *boundary crossing* between communities of practice. The concept of boundary crossing seems to me to be both integral to the developmental process and a tool or provision of tools for analysis. Bakogianni et al., in Chap. 15, see *boundary crossing* between different practices as a way to address learning, using Akkerman and Bakker's (2011) four mechanisms: identification, coordination, reflection and transformation. These they apply to *boundary objects* such as curriculum materials, representations, school or workplace records that facilitate interactions and crossings at the boundaries. They recognise many "tensions" arising for MTEs from perspectives and activity across the various communities within the project; they use the lens of boundary crossing to analyse MTEs' tensions and to bring the work of MTEs and researchers closer to the teachers' and students' reality. Sikko and Grimeland, in Chap. 12, use the concept of boundary crossing to explore relationships between communities of mathematicians and mathematics teacher educators, using the learning mechanisms already mentioned above. They used Jaworski's (2003) framework for analysing teaching-learning development in co-learning partnerships and overlaid it with learning mechanisms in the boundary between mathematics and mathematics education. Thus, we see the framework used as an analytical tool, whereas the theory of boundary crossings seems to be both a developmental and an analytical tool.

Finally, three chapters report theoretical principles closely related to the philosophy of mathematics learning and teaching espoused by the researchers. For Nolan and Keazer, in Chap. 16, theory provides a basis for their course on *Culturally Relevant Pedagogies* in which they study their own teaching practice. They write, "the theoretical premise of our research and teaching as discussed in this chapter is grounded in efforts to disrupt and decolonise NUC [Near-Universal, Conventional] mathematics." Their desire is to challenge dominant discourses of "training" and "preparation" in mathematics education and the notion that mathematics is value-free and culturally neutral. In their teaching practice, they seek what can be seen theoretically as a pedagogy of opposition and a mathematics education that privileges issues of power and social justice. Their study draws on reflections and narratives from their own teaching and the dialogue that emerges between them as they look critically at tensions and dilemmas in a practice that embodies the theoretical principles on which it is based.

Situated within a broadly constructivist paradigm, Osborn et al. (Chap. 13) focus on collective identity relating to collective agency among MTEs working across disciplinary boundaries, here specifically mathematics and statistics. Researchers see collective identity both as a gestalt in their focus on identity and as having multiple layers of significance for the study, addressing the question "Do we, the project team, indeed have a collective identity?" In addressing this question, they take a narrative, storytelling approach, analysing narratives of individuals to discern commonalities of rapport and appreciation. They noticed differences between seeing themselves as members of the project team and separately and historically as MTEs, the latter perhaps challenging the construct of collectivity more than the former. However, their attentions to project legacy indicated a collective desire to form a continuing community of practice.

Van Zoest and Levin, Chap. 9, started from (consistent) perspectives of "Inquiry as stance" (from Cochran-Smith & Lytle, 2009) and "Inquiry as a tool" (from Jaworski, 2006) to address their own development as MTEs. Their approach to collaborative practice – *Artefact-enhanced Collegial Inquiry* (ACI) – emerged from their early experience of putting inquiry into practice as well as from their guiding literature. Thus, ACI was embedded in their practice and also provided a framework for analysing data with three phases of inquiry-based activity. Their roles as experienced and novice MTEs (similar to those in the chapter of Sikko and Grimeland) and their associated learning were both differently significant and commonly rewarding, enhanced through the ACI framework.

This panorama of theoretical perspectives seems broadly to be distinguishable in three ways: theories or theoretical constructs that guide developmental processes internally and allow a critical questioning of developmental outcomes, theories or theoretical constructs that provide an external analytical process to make sense of the outcomes of developmental processes and lastly theories or theoretical constructs that do both.

We have seen here a range of theoretical perspectives, their uses in research and for development by MTEs for themselves and their teachers. In what ways, if at all, do we see the theoretical perspectives and the outcomes of research activity to be related?

22.2.4 Methodology

While it seemed important to address some of the nuances as well as the detail of theoretical perspectives in these chapters, I am somewhat daunted when I consider doing the same for methodology. Thus, I have decided to focus on a few things that I have noticed that seem to permeate several of the chapters and some things that I think extend our ways of presenting ourselves to our MTE community as a whole.

One of the first things I noticed was the use of first names in reporting from the data. This was very obvious when the names used were names of chapter authors. For example, in Chap. 10 (Brown et al.), we meet, in the order of years of experience, Laurinda, Alf, Tracy and Julian who are both the MTEs reflecting on their practice and the authors of the chapter. Here, we gain insight into the personal narratives of these practitioners, reflected in the phrase "revealing the lived experience," which is achieved by both personal storytelling and what the authors call "second-person" interviewing. In Chap. 9 (Van Zoest and Levin), initials are used, so we meet authors LVZ and ML reflecting on their own learning and that of their teacher students. I have mused on what difference of effect it makes, revealing ourselves in first-person reflections addressing our learning and development versus a more distanced, third-person passive voice. In the former, we treat our own experiencing both as individuals with personal identity and as representatives of our (international) community of MTEs. In the latter, we try to offer a distanced, perhaps more rational, perspective, perhaps seeking for greater objectivity but missing the emotional and psychological impact that we experience.

How much are we prepared to reveal of our own perceptions and perspectives and our learning from them, capturing vividly our issues, tensions or contradictions, and to what extent are we more comfortable with presenting a general or common rationality, objectively argued? Where are we most likely to find one or the other?

Sikko and Grimeland, Chap. 12, refer to John Mason's work in stating the following:

> Whereas in mathematics, knowledge is built by adding new theorems to old, education is a journey of self-discovery where each new traveller has to re-experience, re-learn, re-express and re-integrate what previous generations have learned.

My own view, and one I have pursued myself in a number of publications, is that our willingness to reveal our personal learning and its associated challenges (the "lived experience") can be powerful in discerning insights and issues deeply germane to our community and especially instructive for its novices.

This brings me to the methods and modes by which we share our experiences and analyse their significance for the learning and development of our students and colleagues and as elements of wider theoretical understandings and practical guides. Again, I notice pervasiveness, this time of the use of narrative accounts, stories, either as data for further analysis or as a narrative analytical style. In the chapters here, we see both, and in some cases, it is hard to separate them. For example, in Chaps. 8 and 14 (Olanoff et al. and Ingram et al.), I think we see a form of narrative analysis, while in Chaps. 11 and 16 (Bissell et al. and Nolan and Keazer), we see raw narrative. When I say "raw," I don't mean it has not been worked on, but I see a (lengthy) story told in the "I" form, rather than selected extracts juxtaposed to illustrate some key analytical construct. Both are, of course, important analytical forms. I hurry to emphasise that these are my own views and that the authors might disagree. Let's say these are conjectures for consideration. I recognised the chapter by Bakogianni et al. as being different methodologically from some of the others. Here, the participants have pseudonyms which label extracts from their contributions in the project meetings. I can see that the large number of participants in the project and associated issues of confidentiality possibly influenced this choice.

How do we choose the modes through which to share our personal experiences and learning? How do we want the chosen mode to influence the response of others to what we try to convey?

22.2.5 Learning from the Literature

I expect that we all encourage our students to read, read and read again. It goes without saying that becoming familiar with the literature in our research areas and beyond is a principal plank in our research methodology. All chapters, unsurprisingly, include substantial referencing of the literature. Indeed, when we review papers, the literature review is both an indication that the author has attended to theory and research relevant to their focus and personally informative for the reviewer. In some of the chapters here, we see direct reference to encouraging our students to read. For example, in Chap. 14, Ingram et al. outline the professional development which a master's programme provides for practising MTEs, offering different kinds of stimulus which include "directed readings." We see some of the value of "directed" readings in the following quotation:

> Clare, who had long assumed that many prospective teachers had fixed ideas about mathematics as a subject and about the process of teaching and learning mathematics, was anxious to find ways of stimulating more active discussion – and thereby potential re-evaluation – of their ideas. Further thought about this issue was stimulated by two readings: one that demonstrated how deep-rooted these beliefs are … and another which suggested that such beliefs might be held consciously or unconsciously …. The reading, as a stimulus in the external domain, prompted Clare to reflect on her existing beliefs … which were strengthened, giving her the confidence to suggest changes in practice to her team of tutors.

In Chap. 12, Sikko and Grimeland, as mathematicians and MTEs, one experienced and one novice, found a reading group and the literature it addressed extremely valuable as seen in the quotation below:

> Attending an organised "reading group" on topics of mathematics education research, and research methods in the field, made a big contribution to her understanding of the nature of research in mathematics education and about relevant questions in mathematics education research. The group was led by "more knowledgeable others" in the form of more experienced colleagues, including the first author. … the readings in the form of journal papers and book chapters played a role as boundary objects. In this way, the second author became a participant in a community in which she was able to build a basis of knowledge that would have taken much longer to develop in a less organised setting, as experienced by the first author. Both authors found the reading group an opportunity to discuss research literature at the appropriate level in a community open to questions of any kind, providing learning for both the newcomers and the mentors.

As well as the significance of learning through reading the literature, these two examples emphasise the importance of some more formal approach to this reading: in the first case as an integral part of an accredited course with required reading (in this case a master's course) and in the second case as part of a reading group which provides both support and structure as well as recommended reading. Support through others of varying degrees of experience and structure through the course or reading group provide building blocks for all participants.

22.2.6 Another Chapter

In the sections above, I believe I made reference to all chapters in Theme 2, except for one, Chap. 17 (Wu and Cai). This chapter provides a fascinating introduction to teacher education in China. It provides an account of educational stages and their content in China, with a particular focus on the education of teachers, leading to a detailed discussion of MTE activity and development. In fact, the chapter emphasises important distinctions between three kinds of MTE with different roles and developmental routes: university-based MTEs, school-based mathematics mentor teachers and MTRs, mathematics teacher researchers. Together, these three groups, despite their different names, fulfil the roles of MTE that correspond to MTE roles in the chapters above.

Graduates with a bachelor degree progress to become teachers, later possibly mentor teachers or MTRs; some graduates progress through master's studies to become teachers and possibly mentor teachers or MTRs; master's graduates can also progress to become university-based MTEs. The system is complex providing a range of education and support for teachers. The system has a long history of research in schools, where MTRs lead research activity in which teachers engage to explore and learn from teaching-learning experiences. University MTEs design and implement mathematics teacher education programmes in the university for both prospective and practising teachers.

Master's programmes include courses in mathematics, so both MTRs and MTEs are well qualified mathematically; MTEs, with doctoral qualifications, are knowledgeable in educational theory. However, there is little support for either group in the roles they are expected to fulfil in educating teachers. A research study, described in the chapter, surveyed university-based MTEs, working mainly in mathematics departments, focusing on the challenges they faced in their teaching of pedagogical courses. Perhaps unsurprisingly, the results showed more challenges in teaching pedagogically related courses than teaching undergraduate mathematics, which links directly to what we learn from Sikko and Grimeland in Chap. 12. It would be great to read some personal reflections from the Chinese MTEs.

More telling for those of us in westernised educational contexts is the reported difference between the teaching and expertise of MTEs and MTRs. While the MTEs excel in theoretical knowledge but have little practical pedagogically focused teaching experience, MTRs have the experience of being teachers themselves and have developed research expertise through their experience as researchers in schools and classrooms. The two groups are complementary in their education, experiences and qualifications and, seemingly, could learn much from working together. It would have been interesting to read more chapters from such backgrounds, perhaps with MTEs and MTRs inquiring into and reflecting on their developing activity and its challenges.

I am impressed by the roles and facets of MTEs' activity and the challenges they face as revealed in these chapters. I wonder if the experiences revealed by others illuminate or challenge the situations and contexts we experience ourselves?

22.3 Theme 3: Methodological Challenges in Researching MTE Expertise, Learning and Development

As in my beginnings in addressing Theme 2, I extract what seems to be a guiding paragraph, from the editors' prospectus, focusing on Theme 3:

> Zaslavsky and Leikin (2004) introduced the role of *mathematics teacher educator educator* to describe a person responsible for the development of mathematics teacher educators. This introduces a new "layer" that could be seen as analogous to mathematics teachers researching their students, and mathematics teacher educators researching mathematics teachers. MTE educators could thus be the appropriate people to research mathematics teacher educators. In reality, however, those who take the role of MTE educators often are also mathematics teacher educators and hence, as was the case in the study reported by Zaslavsky and Leikin (2004), likely to be involved in the milieus that they are researching as well as personally engaged with the same issues with which their research subjects (mathematics teacher educators) are grappling.

While recognising that the authors of chapters included under Theme 2, and discussed above, are in many ways "likely to be involved in the milieus *that they are researching* as well as personally engaged with the same issues with which *their research subjects* (mathematics teacher educators) are grappling," this section of the book includes just three chapters explicitly. My first challenge was to think about how these three were "different" from the ten chapters in Theme 2. I decided to start by addressing overtly the roles of the researchers and those of their research subjects.

In Chap. 18, by Oates, Muir, Murphy, Reaburn and Maher, in Tasmania, Australia, the researchers are the authors of the chapter, who are MTEs in a teacher education programme educating prospective primary school teachers. These researchers' research subjects are themselves, as a group, working within their teacher education programme and exploring the factors that underpin decisions they make as MTEs. In this respect, their activity fits within Martin Simon's second category as discussed above. In contrast, while the researchers in Chap. 19, Rojas, Montenegro, Goizueta and Martínez, in Chile, are teacher educators; their research subjects are both the teachers who participated in the MTEs' courses on modelling and themselves in action with these teachers. The teachers were addressed through a questionnaire and interviews focusing on their experiences of participation in the MTEs' courses; thus, we might, again, locate the activity in Simon's second category. In Chap. 20, by Arzarello and Taranto, in Italy, the complexity of participation and relationships between the authors, researchers, MTEs and researcher-teachers makes this indeed a complex milieu in which to distinguish researchers and research subjects. It is also hard to locate in relation to Simon's categories, but I tend to see it also in Category 2 since the learning of the MTEs is related to the MOOC courses for teachers as well as the other practitioners in their construction.

One of the guiding questions raised by Arzarello and Taranto is the following: "What are the dilemmas and opportunities associated with researching ourselves as MTEs?" They regard this question as a self-referential problem creating dilemmas for MTEs. In the Italian tradition, they claim that the figure of *researcher-teacher* can be a solution for this problem. They tell us that the researcher-teacher is a common role: "In our case, the researcher-teachers are in-service [practising] teachers who have been collaborating with our research group in mathematics education for several years." They see researcher-teachers as "brokers" in the divide between teachers and university researchers. I found it interesting to compare the role of researcher-teacher in the Italian tradition with that of MTRs, mathematics teacher researchers, in the Chinese tradition, articulated in Chap. 17, although I have not the space to follow this up here.

This question, posed by Arzarello and Taranto, and the associated self-referential "dilemmas" led me to look more closely at the chapter of Oates et al. who, like researchers in several of the chapters in Theme 2, conduct research into their own practices as MTEs. Oates et al. make the following statement:

> It was clear to us all that our PCK as MTEs plays a significant role in underpinning the decisions we make about course and unit design, and, at this time, we lacked an appropriate theoretical approach to analyse these effects. We decided to deepen the extent of the review, to document and interrogate the process we undergo in the collective redevelopment of our units and explore theoretical bases for the decisions we make.

Thus, I see the "theoretical bases" being, potentially, a way in which these researchers overcome the self-referential problem raised by Arzarello and Taranto. For Oates et al., the two theoretical perspectives are *activity theory*, rooted in the work of Vygotsky and A. N. Leont'ev and developed by Yrio Engeström among many others, and *professional capital*, arising from Strober's notion of *human capital*. The research to which these perspectives are applied involved a study of these researchers' review of their *Bachelor of Education* course for educating primary school teachers. This review followed a survey of the previous cohort of students (prospective teachers) concerning the value and alignment of content and assessment in three core mathematics units, addressing how effective these units were in preparing students to teach mathematics. Their review began in a meeting to discuss the outcomes of the survey, which led to a recognition of a complexity of factors which deserved further attention. Thus, the data for their study consisted of recordings of further meetings and focus group interviews which were analysed through a discourse analysis to identify emerging themes. The chapter reveals these themes with reference to anonymised quotations from members of the MTE team.

The authors write, "Meeting four raised some questions about these themes, for example, whether they might be legitimately emerging from the data compared to what we were predisposed to look for." I see this question as addressing directly the dilemmas raised by Arzarello and Taranto. Readers will follow this up specifically in the chapter itself; however, I quote briefly on the effect of one of their theoretical perspectives, that of activity theory. The authors write the following:

> Reflection on the themes using activity theory enabled us to interpret aspects of the activity and to explore the complexities that underlie our actions. The review meetings and interviews uncovered our activity to the extent that we can begin to see how we are influenced by the different aspects in the system. Hence, we now have a deeper view of the factors involved in determining the outcome of the activity, that is, the possible redesign to better develop effective pedagogical knowledge for teaching mathematics with primary pre-service teachers.

Thus, we see that the researchers' use of activity theory helped them to see beyond their own extraction of themes, to a revealing of factors and relationships that took them more deeply into their own thematic analyses.

This use of external theory to allow an alternative way to inspect self-referential outcomes of research is clearly one of the focuses of Theme 3. Arzarello and

Taranto, focusing on the development of massive open online courses (MOOCS) for educating prospective teachers, also used two theoretical perspectives to analyse the contribution of MTEs and of the researcher-teachers to the preparation of the MOOCs. MOOCs are designed to engage students at a distance, and it is up to the students to design their own course through elements provided in the MOOC. The theoretical perspectives used here are meta-didactical transposition (MDT), as developed originally by Ferdinando Arzarello and colleagues, and Valsiner's zone theory as developed by Merrilyn Goos. The author, Eugenia Taranto, had modified these theoretical perspectives, adapting first MDT to the MOOC environment to produce MOOC-MDT and then networking with zone theory to produce MOOC's zone theory. Details are in the chapter. A key element of MDT "uses the notion of *broker* as a professional who belongs to more than one community and makes possible exchanges between them," a role similar to that of brokers in Wenger's community of practice theory and Engeström's activity theory. *MOOC's zone theory* adapts the brokering role to the nature of the MOOC, with the roles of MTEs in relation to the MOOC's content and structure being analysed through zones of free movement and promoted action mediated by the researcher-teachers as brokers. My brief account cannot do justice to the complexities here, but the point I want to make is that theory is being used here both to explain and to examine the ways in which MTEs both contribute to the MOOC but are also distanced from it by the brokering. The authors see this theoretical mediation as a "fresh approach to addressing the self-reference dilemma experienced by MTEs who wish to analyse their own learning and development." Thus, in both Chaps. 18 and 20, we see theory playing a methodological role in the self-referential dilemma acknowledged in both chapters.

In contrast, in Chap. 19 (Rojas et al.), the methodological challenges relate to theories of *modelling* which guide how MTEs "model" processes and actions for teachers in the teachers' learning and subsequently teaching of mathematics. Aims of the modelling process are for teachers not only to gain access themselves to the mathematical concepts being taught but also to be aware of what the MTE is doing that contributes to their learning and can be used subsequently in their teaching of students in school classrooms. These processes involve a complexity of factors – whether the MTE as teacher is aware of modelling for her students, the extent to which the modelling is explicit for the MTE, the extent to which students are overtly aware of the modelling processes and how modelling in a university context can be transferrable into school classrooms. Research has focused on all these elements in addressing both prospective teachers' perspectives and MTEs' perspectives. The chapter proposes "a new methodological challenge" that a community of practice involving prospective teachers and their MTEs might together explore modelling practices and their contribution to the learning of both the MTEs and the teachers. The authors conclude with the following words:

> we believe that reconceptualising research on modelling from a more integrated, holistic perspective should take into account the complementary roles of mathematics teacher educators and prospective teachers and how they complement each other in the challenge of learning to teach mathematics.

It strikes me that we might see here a community of inquiry involving MTEs and their prospective teachers inquiring together into the ways in which MTEs' modelling is effective in enabling teachers to develop their own learning and their teaching practices.

Related to all three papers in this section, we might ask the following:

What are the challenges that we MTEs face as teachers of teachers in the learning of mathematics and how we teach mathematics? How do these compare with our teachers' learning? What theoretical perspectives guide our inquiry, enable us to look critically at the self-referential nature of our own learning and development and allow our knowledge and practice to develop objectively?

22.4 In Conclusion

I have been inspired by my reading of chapters in these themes, and although I have exceeded my word limit, I have left unsaid much of what I have learned from my reading. It has been my pleasure and privilege to engage with some of the key ideas in these chapters and offer them here. Perhaps drawing attention to what has stimulated me to consider and question might lead you to delve more deeply into what is written and follow your own threads in these texts. Please be aware that what I say here is my own version of what is written, my responsibility, not that of the authors. I heartily recommend that you read the chapters themselves for a full enjoyment of the richness that is offered.

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Correction to: Pedagogical Tasks Toward Extending Mathematical Knowledge: Notes on the Work of Teacher Educators

Rina Zazkis and Ofer Marmur

Correction to:
Chapter 5 in: M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_5

The original version of this chapter was revised due to some errors (listed below) in the text at page numbers 98 and 99.

- On pages 98 and 99, there are incorrect "+" symbols (with a circle around them), where this should have been an approximation symbol.
- Also on pages 98 and 99, there are two instances of wrong indentations.

The author bio of Dr. Ofer Marmur has been updated in the front matter of this book under the section "Editor and Author Biographies."

The updated online version of this chapter can be found at
https://doi.org/10.1007/978-3-030-62408-8_5

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8_23

Author Index

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8

Subject Index

M. Goos, K. Beswick (eds.), *The Learning and Development of Mathematics Teacher Educators*, Research in Mathematics Education,
https://doi.org/10.1007/978-3-030-62408-8