Approximate Efficient Solutions of Nonsmooth Vector Optimization Problems via Approximate Vector Variational Inequalities

Mohsine Jennane[1] and El Mostafa Kalmoun[2](✉)

[1] Department of Mathematics, Dhar El Mehrez, Sidi Mohamed Ben Abdellah University, Fes, Morocco
mohsine.jennane@usmba.ac.ma

[2] Department of Mathematics, Statistics and Physics, College of Arts and Sciences, Qatar University, Doha, Qatar
ekalmoun@qu.edu.qa

Abstract. In this work, we demonstrate the connection between the solutions of approximate vector variational inequalities and approximate efficient solutions of corresponding nonsmooth vector optimization problems via generalized approximate invex functions. The underlying variational inequalities are stated under the Clarke's generalized Jacobian.

1 Introduction

Various significant applications in engineering and economics can only be stated as a multiobjective optimization problem [1]. Nowadays, the connection of these problems to vector variational inequalities is well-established for differentiable convex functions [2]. In particular, results in this direction were developed under various assumptions of generalized convexity [3–7] and nonsmooth invexity [8–11]. On the other hand, relationships between a vector variational inequality and a nonsmooth vector optimization problem (NVOP) were established under the generalized approximate convexity assumption [12–14].

This paper is devoted to the case of NVOP involving generalized approximate invex multiobjective functions, which we have introduced in [15]. Our aim is to use approximate vector variational inequalities (AVVIs) of Stampacchia and Minty type in terms of Clarke's generalized Jacobian to characterize approximate efficient solutions. It is worth mentioning that, as generalized approximate invexity is an extension of generalized approximate convexity, the results obtained in our work are improvements and generalizations of the main results in [14].

The paper is organized as follows: in Sect. 2, we give some preliminary definitions, notation, and auxiliary results. In Sect. 3, we introduce the concept of approximate efficiency for NVOPs, and derive their relationships to AVVIs using the assumption of approximate invex functions. In Sect. 4, we give an example to illustrate our main results. Finally, we conclude our paper in Sect. 5.

Z. Hammouch et al. (Eds.): SM2A 2019, LNNS 168, pp. 91–101, 2021.
https://doi.org/10.1007/978-3-030-62299-2_7

2 Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean spaces, $S \subseteq \mathbb{R}^n$ be a given nonempty set and $C \subseteq \mathbb{R}^m$ be a solid pointed convex cone. We use the following partial ordering relations:

$$u \geq_C v \Leftrightarrow u - v \in C;$$

$$u >_C v \Leftrightarrow u - v \in intC.$$

Definition 1 ([16]). Let $F : S \to \mathbb{R}^m$ be a vector-valued function. F is locally Lipschitz if for each $w \in S$ there is $k > 0$ and $\rho > 0$ such that, for all $u, v \in B(w; \rho)$

$$\|F(u) - F(v)\| \leq k\|u - v\|.$$

Throughout this paper, we let $F := (F_1, ..., F_m) : S \to \mathbb{R}^m$ be a locally lipschitz function, $\theta : S \times S \to \mathbb{R}^n$ be a mapping and $\tau >_C 0$ be a vector.

Definition 2 ([16]). The Clarke's generalized Jacobian of F at $u \in S$ is given by

$$\partial F(u) = conv\{\lim_{i \to +\infty} JF(u^{(i)}) : u^{(i)} \to u, u^{(i)} \in D\},$$

where *conv* denotes the convex hull, $JF(u^{(i)})$ indicates the Jacobian of F at $u^{(i)}$, and D is the differentiability set of F.

We note that the Clarke's generalized Jacobian is not equal to the cartesian product of the components' Clarke subdifferentials. Nevertheless, one has

$$\partial F(u) \subseteq \partial F_1(u) \times ... \times \partial F_m(u).$$

Note also that $\partial(-F)(u) = -\partial F(u)$.

We recall some definitions given in [15] which are a generalization of the concepts of generalized approximate convexity provided in [12,14,17].

Definition 3. F is called approximate $(\theta, \tau)-$invex (A(θ, τ)I) at $w \in S$, if there is $\rho > 0$ satisfying

$$F(u) - F(v) \geq_C A_v\theta(u, v) - \tau\|\theta(u, v)\|, \quad \text{for each} \quad u, v \in B(w, \rho),\ A_v \in \partial F(v).$$

If F is A(θ, τ)I at each $w \in S$, we say that F is A(θ, τ)I on S.

Taking $\theta(u, v) = u - v$, approximate invexity reduces to approximate convexity [18]. The counter-example given in [15, Example 2.2] shows that approximate invexity is still more general.

Definition 4. • F is approximate pseudo $(\theta, \tau)-$invex of type 1 (AP(θ, τ)I-1) at $w \in S$ if there is $\rho > 0$ such that, whenever $u, v \in B(w, \rho)$ and if

$$F(u) - F(v) <_C -\tau\|\theta(u, v)\|,$$

then

$$A_v\theta(u, v) <_C 0 \text{ for each } A_v \in \partial F(v).$$

- F is approximate pseudo $(\theta,\tau)-$invex of type 2 (AP(θ,τ)I-2) at $w \in S$ if there is $\rho > 0$ such that, whenever $u, v \in B(w,\rho)$ and if

$$F(u) - F(v) <_C 0,$$

then

$$A_v\theta(u,v) + \tau\|\theta(u,v)\| <_C 0 \text{ for all } A_v \in \partial F(v).$$

Proposition 1. *If F is AP(θ,τ)I-2 at $w \in S$, then F is AP(θ,τ)I-1 at w.*

Proof. Assume that there is $\overline{\rho} > 0$ satisfying for each $u, v \in B(w,\overline{\rho})$

$$F(u) - F(v) <_C -\tau\|\theta(u,v)\|,$$

then

$$F(u) - F(v) <_C 0.$$

Since F is AP(θ,τ)I-2 at w, then there is $\rho > 0$, $\rho < \overline{\rho}$, satisfying for each $u, v \in B(w,\rho)$

$$A_v\theta(u,v) + \tau\|\theta(u,v)\| <_C 0 \text{ for each } A_v \in \partial F(v),$$

which further implies that

$$A_v\theta(u,v) <_C 0 \text{ for each } A_v \in \partial F(v).$$

Hence F is AP(θ,τ)I-1 at $w \in S$.

Definition 5. • F is approximate quasi $(\theta,\tau)-$invex of type 1 (AQ(θ,τ)I-1) at $w \in S$ if there is $\rho > 0$ such that for each $u, v \in B(w,\rho)$

$$A_v\theta(u,v) - \tau\|\theta(u,v)\| >_C 0, \quad \text{for some } A_v \in \partial F(v),$$

implies

$$F(u) >_C F(v).$$

- F is approximate quasi $(\theta,\tau)-$invex of type 2 (AQ(θ,τ)I-2) at $w \in S$ if there is $\rho > 0$ such that, for each $u, v \in B(w,\rho)$

$$A_v\theta(u,v) >_C 0, \quad \text{for some } A_v \in \partial F(v),$$

implies

$$F(u) - F(v) >_C \tau\|\theta(u,v)\|.$$

The next proposition can be easily proven.

Proposition 2. *If F is AQ(θ,τ)I-2 at $v \in S$, then F is AQ(θ,τ)I-1 at v.*

Remark 1. • A(θ,τ)I $\Rightarrow \left[\text{ AP}(\theta,\tau)\text{I-1 and AQ}(\theta,\tau)\text{I-1 }\right].$

- There is no relation between AP(θ,τ)I-2 and AQ(θ,τ)I-2 and A(θ,τ)I (for examples, see [14]).

Now, we consider the following NVOP:

$$(NVOP) \quad \min F(u) := (F_1(u), \cdots, F_m(u)) \text{ subject to } u \in S,$$

where each $F_i : S \to \mathbb{R}$ are real-valued functions for any $i \in \{1, \cdots, m\}$.

Definition 6. Let $\zeta \in S$.

(i) ζ is an efficient solution of (NVOP) iff there is no vector $u \in S$ such that

$$F(u) \leq_C F(\zeta).$$

(ii) ζ is an τ-approximate efficient solution (τ-AES) of (NVOP) iff there is no $\rho > 0$ such that, for each $u \in B(\zeta; \rho) \setminus \{\zeta\}$

$$F(u) - F(\zeta) \leq_C -\tau\|\theta(u, \zeta)\|.$$

3 Relationships Between NVOP and AVVI

Consider the following AVVI of Stampacchia and Minty type in terms of Clarke subdifferentials as follows:

(ASVVI). To find $\zeta \in S$ such that, there is no $\rho > 0$ satisfying for each $u \in B(\zeta, \rho)$ and $A_\zeta \in \partial F(\zeta)$

$$A_\zeta \theta(u, \zeta) \leq_C -\tau\|\theta(u, \zeta)\|.$$

(AMVVI). To find $\zeta \in S$ such that, there is no $\rho > 0$ satisfying for each $u \in B(\zeta, \rho)$ and $A_u \in \partial F(u)$

$$A_u \theta(u, \zeta) \leq_C -\tau\|\theta(u, \zeta)\|.$$

The following theorems describe relations between AVVI and NVOP.

Theorem 1. *Let F be $A(\theta, \tau)I$ at $\zeta \in S$. If ζ solves (ASVVI) w.r.t. τ, then ζ is a 2τ-AES of (NVOP).*

Proof. Assume ζ fails to be a 2τ-AES of (NVOP). It means that there is $\overline{\rho} > 0$ satisfying for each $u \in B(\zeta, \overline{\rho})$

$$F(u) - F(\zeta) \leq_C -2\tau\|\theta(u, \zeta)\|. \tag{1}$$

As F is A(θ, τ)I at ζ, it follows that there is $\widetilde{\rho} > 0$, satisfying

$$F(u) - F(\zeta) \geq_C A_\zeta \theta(u, \zeta) - \tau\|\theta(u, \zeta)\| \quad \forall\, u \in B(\zeta, \widetilde{\rho}),\ A_\zeta \in \partial F(\zeta).$$

By using (1) and the definition of approximate $(\theta, \tau)-$ invexity, and by taking $\rho := min(\overline{\rho}, \widetilde{\rho})$, we obtain

$$A_\zeta \theta(u, \zeta) - \tau\|\theta(u, \zeta)\| \leq_C -2\tau\|\theta(u, \zeta)\|.$$

Hence

$$A_\zeta \theta(u, \zeta) \leq_C -\tau\|\theta(u, \zeta)\|.$$

This means ζ does not solve (ASVVI) w.r.t τ.

Theorem 2. *Let $-F$ be $A(\theta, \tau)I$ at $\zeta \in S$. If $\zeta \in S$ is a τ-AES for (NVOP), then ζ solves (ASVVI) w.r.t 2τ.*

Proof. Assume ζ fails to be a solution of (ASVVI) w.r.t 2τ. It means that there is $\overline{\rho} > 0$ such that, for each $u \in B(\zeta, \overline{\rho}), A_\zeta \in \partial F(\zeta)$, we have

$$A_\zeta \theta(u, \zeta) \leq_C -2\tau\|\theta(u, \zeta)\|.$$

Then

$$-A_\zeta \theta(u, \zeta) \geq_C 2\tau\|\theta(u, \zeta)\|. \tag{2}$$

By $\partial(-F)(\zeta) = -\partial F(\zeta)$ we deduce that $-A_\zeta \in \partial(-F)(\zeta)$.
As $-F$ is A(θ, τ)I at ζ, it yields that there is $\widetilde{\rho} > 0$ satisfying

$$(-F)(u) - (-F)(\zeta) \geq_C -A_\zeta \theta(u, \zeta) - \tau\|\theta(u, \zeta)\| \qquad \forall u \in B(\zeta, \widetilde{\rho}).$$

By using (2) and by taking $\rho := min(\overline{\rho}, \widetilde{\rho})$, we obtain

$$-F(u)+F(\zeta)+\tau\|\theta(u, \zeta)\| \geq_C -A_\zeta \theta(u, \zeta) \geq_C 2\tau\|\theta(u, \zeta)\| \qquad \forall u \in B(\zeta, \rho)\setminus\{\zeta\},$$

which implies

$$F(u) - F(\zeta) \leq_C -\tau\|\theta(u, \zeta)\|.$$

Therefore ζ cannot be a τ-AES of (NVOP).

Theorem 3. *Let F be $A(\theta, \tau)I$ at $\zeta \in S$ and $\theta(u, \zeta)+\theta(\zeta, u) = 0$ for any $u \in S$. If ζ solves (AMVVI) w.r.t τ, then ζ is a 2τ-AES of (NVOP).*

Proof. Assume ζ fails to be a 2τ-AES of (NVOP). It means that there is $\overline{\rho} > 0$ satisfying for each $u \in B(\zeta, \overline{\rho})$

$$F(u) - F(\zeta) \leq_C -2\tau\|\theta(u, \zeta)\|. \tag{3}$$

As $-F$ is A(θ, τ)I at ζ, it yields that there is $\widetilde{\rho} > 0$ satisfying

$$(-F)(\zeta) - (-F)(u) \geq_C A_v \theta(\zeta, u) - \tau\|\theta(\zeta, u)\| \qquad \forall u \in B(\zeta, \widetilde{\rho}), A_v \in \partial(-F)(u),$$

then

$$F(u) - F(\zeta) \geq_C A_v \theta(\zeta, u) - \tau\|\theta(\zeta, u)\|.$$

By using (3) and by taking $\rho := min(\overline{\rho}, \widetilde{\rho})$, we obtain

$$A_v\theta(\zeta, u) - \tau\|\theta(\zeta, u)\| \leq_C -2\tau\|\theta(u, \zeta)\| \qquad \forall u \in B(\zeta, \rho) \setminus \{\zeta\}.$$

From $\partial(-F)(u) = -\partial F(u)$, there is $A_u = -A_v \in \partial F(u)$. Consequently, using $\theta(u, \zeta) + \theta(\zeta, u) = 0$ together with the above inequality, we deduce

$$A_u\theta(u, \zeta) \leq_C -\tau\|\theta(u, \zeta)\|.$$

This means ζ does not solve (AMVVI) w.r.t τ.

Theorem 4. *Let $-F$ be $A(\theta, \tau)I$ at $\zeta \in S$ and $\theta(u, \zeta) + \theta(\zeta, u) = 0$ for all $u \in S$. If $\zeta \in S$ is a τ-AES for (NVOP), then ζ solves (AMVVI) w.r.t 2τ.*

Proof. Assume ζ fails to be a solution of (AMVVI) w.r.t 2τ. Thus, there is $\overline{\rho} > 0$ satisfying for any $u \in B(\zeta, \overline{\rho})$, $A_u \in \partial F(u)$

$$A_u\theta(u, \zeta) \leq_C -2\tau\|\theta(u, \zeta)\|. \tag{4}$$

As F is A(θ, τ)I at ζ, it yields that there is $\widetilde{\rho} > 0$, such that

$$F(\zeta) - F(u) \geq_C A_u\theta(\zeta, u) - \tau\|\theta(\zeta, u)\| \quad \forall u \in B(\zeta, \widetilde{\rho}), A_u \in \partial F(u).$$

Since $\theta(\zeta, u) = -\theta(u, \zeta)$, then

$$F(u) - F(\zeta) - \tau\|\theta(u, \zeta)\| \leq_C A_u\theta(\zeta, u).$$

By using (3) and by taking $\rho := min(\overline{\rho}, \widetilde{\rho})$, we obtain

$$F(u) - F(\zeta) \leq_C -\tau\|\theta(u, \zeta)\|.$$

We conclude that ζ cannot be a τ-AES of (NVOP).

Theorem 5. *Let F be AP(θ, τ)I-2 at $\zeta \in S$. If ζ solves (ASVVI) w.r.t. τ, then ζ is a τ-AES of (NVOP).*

Proof. Assume ζ fails to be a τ-AES of (NVOP). It means that there is $\overline{\rho} > 0$ satisfying for all $u \in B(\zeta, \overline{\rho})$

$$F(u) - F(\zeta) \leq_C -\tau\|\theta(u, \zeta)\| <_C 0. \tag{5}$$

As F is AP(θ, τ)I-2 at ζ, it yields that there is $\widetilde{\rho} > 0$, such that, whenever $u \in B(\zeta, \widetilde{\rho})$

$$F(u) - F(\zeta) <_C 0 \Rightarrow A_\zeta\theta(u, \zeta) <_C -\tau\|\theta(u, \zeta)\|, \quad \forall A_\zeta \in \partial F(\zeta).$$

By using (5) and the definition of approximate quasi $(\theta, \tau)-$invexity type 2, and by taking $\rho := \min(\overline{\rho}, \widetilde{\rho})$, we obtain

$$A_\zeta\theta(u, \zeta) \leq_C -\tau\|\theta(u, \zeta)\|.$$

This means ζ does not solve (ASVVI) w.r.t. τ.

Theorem 6. *Let $-F$ be $AQ(\theta,\tau)I$-2 at $\zeta \in S$. If ζ is a τ-AES of (NVOP), then ζ solves (ASVVI) w.r.t. τ.*

Proof. Assume ζ fails to be a solution of (ASVVI) w.r.t. τ, then, there is $\rho > 0$ satisfying for each $A_\zeta \in \partial F(\zeta)$ and $u \in B(\zeta, \overline{\rho})$

$$A_\zeta \theta(u,\zeta) \leq_C -\tau\|\theta(u,\zeta)\|.$$

Then

$$-A_\zeta \theta(u,\zeta) \geq_C \tau\|\theta(u,\zeta)\| >_C 0. \tag{6}$$

As $\partial(-F)(\zeta) = -\partial F(\zeta)$ it yields that $-A_\zeta \in \partial(-F)(\zeta)$.
Since $-F$ is AQ(θ,τ)I-2 at ζ, it follows that there is $\widetilde{\rho} > 0$ such that, whenever $u \in B(\zeta, \widetilde{\rho})$

$$-A_\zeta \theta(u,\zeta) >_C 0 \Rightarrow -F(u) - (-F(\zeta)) >_C \tau\|\theta(u,\zeta)\|.$$

By using (6) and the definition of approximate pseudo $(\theta,\tau)-$invexity type 2, and by taking $\rho := \min(\overline{\rho}, \widetilde{\rho})$, we get

$$F(u) - F(\zeta) \leq_C -\tau\|\theta(u,\zeta)\|.$$

Consequently ζ cannot be a τ-AES of (NVOP).

The following corollary can be deduced from Theorems 5 and 6.

Corollary 1. *Let F be $AP(\theta,\tau)I$-2 at $\zeta \in S$ and $-F$ be $AQ(\theta,\tau)I$-2 at ζ. ζ is a τ-AES of (NVOP) if and only if ζ solves (ASVVI) w.r.t. τ.*

Theorem 7. *Let F be $AQ(\theta,\tau)I$-2 at ζ and $\theta(u,\zeta) + \theta(\zeta,u) = 0$, $\forall u \in S$. If ζ is a τ-AES of (NVOP), then ζ solves (AMVVI) w.r.t. τ.*

Proof. Assume ζ fails to be a solution of(AMVVI) w.r.t. τ. Then, there is $\overline{\rho} > 0$ satisfying for each $A_u \in \partial F(u)$ and $u \in B(\zeta, \overline{\rho})$

$$A_u \theta(u,\zeta) \leq_C -\tau\|\theta(u,\zeta)\|.$$

From $\theta(u,\zeta) + \theta(\zeta,u) = 0$, we obtain

$$A_u \theta(\zeta,u) \geq_C \tau\|\theta(\zeta,u)\| >_C 0. \tag{7}$$

As F is AQ(θ,τ)I-2 at ζ, it yields that, there is $\widetilde{\rho} > 0$ such that, whenever $u \in B(\zeta, \widetilde{\rho})$

$$A_u \theta(\zeta,u) >_C 0 \Rightarrow F(\zeta) - F(u) >_C \tau\|\theta(\zeta,u)\|.$$

By using (7) and the definition of approximate quasi $(\theta,\tau)-$invexity type 2, and by taking $\rho := \min(\overline{\rho}, \widetilde{\rho})$, we deduce

$$F(u) - F(\zeta) \leq_C -\tau\|\theta(u,\zeta)\|.$$

This means that ζ is not a τ-AES of (NVOP).

Theorem 8. *Let $-F$ be AP(θ,τ)I-2 at ζ and $\theta(u,\zeta)+\theta(\zeta,u)=0$, $\forall u \in S$. If $\zeta \in S$ solves (AMVVI) w.r.t. τ, then ζ is a τ-AES of (NVOP).*

Proof. Assume ζ fails to be a τ-AES of (NVOP). It means that there is $\overline{\rho}>0$ satisfying for any $u \in B(\zeta,\overline{\rho})$

$$F(u)-F(\zeta) \leq_C -\tau\|\theta(u,\zeta)\|.$$

Thus

$$-F(\zeta)-(-F)(u) \leq_C -\tau\|\theta(u,\zeta)\| <_C 0. \tag{8}$$

As $-F$ is AP(θ,τ)I-2 at ζ, it yields that there is $\widetilde{\rho}>0$, such that, whenever $u \in B(\zeta,\widetilde{\rho})$

$$-F(\zeta)-(-F)(u) <_C 0 \Rightarrow A_v\theta(u,\zeta) <_C -\tau\|\theta(u,\zeta)\|, \quad \forall A_v \in \partial(-F)(u).$$

By using (8) and the definition of approximate pseudo $(\theta,\tau)-$invexity type 2, and by taking $\rho := \min(\overline{\rho},\widetilde{\rho})$, we obtain

$$A_v\theta(u,\zeta) \leq_C -\tau\|\theta(u,\zeta)\|, \quad \forall A_v \in \partial(-F)(u),\ u \in B(\zeta,\rho).$$

Using $\partial(-F)(u)=-\partial F(u)$, there is $A_u=-A_v \in \partial F(u)$, then we have

$$-A_u\theta(\zeta,u) \leq_C -\tau\|\theta(u,\zeta)\|.$$

Since $\theta(u,\zeta)+\theta(\zeta,u)=0$, therefore,

$$A_u\theta(u,\zeta) \leq_C -\tau\|\theta(u,\zeta)\|.$$

This means ζ does not solve (AMVVI) w.r.t. τ.

The following corollary can be deduced from Theorems 7 and 8.

Corollary 2. *Let F be AQ(θ,τ)I-2 at $\zeta \in S$ and $-F$ be AP(θ,τ)I-2 at ζ and $\theta(u,\zeta)+\theta(\zeta,u)=0$, $\forall u \in S$. ζ is a τ-AES of (NVOP) if and only if ζ solves (AMVVI) w.r.t. τ.*

4 Example

Consider the following NVOP as an example to illustrate the obtained results.

$$\min_{u \in S} F(u) = \begin{cases} u^2+3u, & u \geq 0 \\ -u^2+4u, & u<0, \end{cases}$$

where $S=\mathbb{R}$, $C=\mathbb{R}^+$ and $\theta(u,v)=(u-v)^3$ for each $u,v \in S$.

The Clarke subdifferential of F at $u \in S$ is defined by

$$\partial F(u) = \begin{cases} 2u+3, & u > 0; \\ [3,4]\,, & u = 0; \\ -2u+4, & u < 0. \end{cases}$$

For $1 < \tau < 2$, there is $\rho = \frac{1}{2} > 0$ such that, for each $u, v \in B(\zeta, \rho)$, $\zeta = 0$, $A_v \in \partial F(v)$, we have

$$F(u) - F(v) = \begin{cases} (u-v)(u+v+3) > 0, & \text{if } v > 0, u > 0, u - v > 0; \\ (u-v)(u+v+3) < 0, & \text{if } v > 0, u > 0, u - v < 0; \\ -u^2 + 4u - v^2 - 3v < 0, & \text{if } v > 0, u \leq 0; \\ u^2 + 3u + v(v-4) > 0, & \text{if } v < 0, u \geq 0; \\ (u-v)(4-u-v) > 0, & \text{if } v < 0, u < 0, u - v > 0; \\ (u-v)(4-u-v) < 0, & \text{if } v < 0, u < 0, u - v < 0; \\ u^2 + 3u > 0, & \text{if } v = 0, u > 0; \\ -u^2 + 4u < 0, & \text{if } v = 0, u < 0. \end{cases}$$

Also,

$$A_v\theta(u,v) + \tau\|\theta(u,v)\| = \begin{cases} (2v+3-\tau)(u-v)^3 < 0, & \text{if } v > 0, u > 0, u - v < 0; \\ (2v+3-\tau)(u-v)^3 < 0, & \text{if } v > 0, u \leq 0; \\ (-2v+4-\tau)(u-v)^3 < 0, & \text{if } v < 0, u < 0, u - v < 0; \\ ku^3 < 0, & \text{if } v = 0, u < 0, \end{cases}$$

where $k \in [3,4]$. Hence, F is AP(θ, τ)I-2 at $\zeta = 0$.

Since for any $u > 0$, one has

$$A_\zeta\theta(u,\zeta) + \tau\|\theta(u,\zeta)\| = ku^3 + \tau u^3 > 0,\ k \in [2,3].$$

Hence, there is no $\rho > 0$ satisfying for each $u \in B(\zeta, \rho)$ and $A_\zeta \in \partial F(\zeta)$

$$A_\zeta\theta(u,\zeta) \leq_C -\tau\|\theta(u,\zeta)\|.$$

Thus, $\zeta = 0$ solves (ASVVI) w.r.t. τ.

Finally, as F is AP(θ, τ)I-2 at $\zeta = 0$, then, from Theorem 5, $\zeta = 0$ should be a τ-AES of (NVOP). Indeed, for all $u > 0$ we have

$$F(u) - F(\zeta) + \tau\|\theta(u,\zeta)\| = u^2 + 3u + \tau u^3 > 0.$$

Hence, there is no $\rho > 0$ such that, for each $u \in B(\zeta;\rho) \setminus \{\zeta\}$

$$F(u) - F(\zeta) \leq_C -\tau\|\theta(u,\zeta)\|.$$

Therefore, $\zeta = 0$ is a τ-AES of (NVOP).

Remark 2. In the above example, the function $-F$ is AQ(θ, τ)I-2 at $\zeta = 0$ and $\theta(u,\zeta) + \theta(\zeta,u) = 0$, $\forall u \in S$. We can easily show that it verifies the conditions of Theorem 6.

5 Conclusions

We have shown the relationships between AVVI in terms of Clarke's generalized Jacobian and NVOP using the concepts of approximate efficiency and generalized approximate invexity. Our work improves that of Gupta and Mishra [14] with respect to two aspects:

- If the generalized approximate invexity assumption is replaced by generalized approximate convexity assumption, then the proof arguments remain the same. Consequently, our theorems are more general since the concept of invexity includes that of convexity as a special case.
- In addition to necessary conditions of approximate efficient solutions of NVOP, we have also provided sufficient conditions using the generalized approximate invexity of $-F$.

Acknowledgments. The authors are most grateful to Dr. Lhoussain Elfadil for continued help throughout the preparation of this paper.

References

1. Eichfelder, G., Jahn, J.: Vector optimization problems and their solution concepts. Recent Developments in Vector Optimization, pp. 1–27. Springer, Berlin, Heidelberg (2012)
2. Giannessi, F.: On Minty variational principle. New Trends in Mathematical Programming, pp. 93–99. Kluwer Academic Publishers, Dordrecht (1998)
3. Yang, X.M., Yang, X.Q., Teo, K.L.: Some remarks on the Minty vector variational inequality. J. Optim. Theory Appl. **121**(1), 193–201 (2004)
4. Gang, X., Liu, S.: On Minty vector variational-like inequality. Comput. Maths. Appl. **56**, 311–323 (2008)
5. Fang, Y.P., Hu, R.: A nonsmooth version of Minty variational principle. Optimization **58**(4), 401–412 (2009)
6. Al-Homidan, S., Ansari, Q.H.: Generalized Minty vector variational-like inequalities and vector optimization problems. J. Optim. Theory Appl. **144**, 1–11 (2010)
7. Oveisiha, M., Zafarani, J.: Vector optimization problem and generalized convexity. J. Glob. Optim. **52**, 29–43 (2012)
8. Long, X.J., Peng, J.W., Wu, S.Y.: Generalized vector variational-like inequalities and nonsmooth vector optimization problems. Optimization **61**(9), 1075–1086 (2012)
9. Mishra, S.K., Wang, S.Y.: Vector variational-like inequalities and non-smooth vector optimization problems. Nonlinear Anal. Theory, Methods Appl. **64**(9), 1939–1945 (2006)
10. Yang, X.M., Yang, X.Q.: Vector variational-like inequalities with pseudoinvexity. Optimization **55**(1–2), 157–170 (2006)
11. Ansari, Q.H., Rezaei, M.: Generalized vector variational-like inequalities and vector optimization in Asplund spaces. Optimization **62**, 721–734 (2013)
12. Bhatia, D., Gupta, A., Arora, P.: Optimality via generalized approximate convexity and quasiefficiency. Optim. Lett. **7**, 127–135 (2013)
13. Mishra, S.K., Laha, V.: On minty variational principle for nonsmooth vector optimization problems with approximate convexity. Optim. Lett. **10**(3), 577–589 (2015)

14. Gupta, P., Mishra, S.K.: On Minty variational principle for nonsmooth vector optimization problems with generalized approximate convexity. Optimization **67**, 1157–1167 (2018)
15. Jennane, M., El Fadil, L., Kalmoun, E.M.: On local quasi efficient solutions for nonsmooth vector optimization. Croatian Oper. Res. Rev. **11**(1), 1–10 (2020)
16. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley-Interscience, New York (1983)
17. Aslam Noor, M., Inayat Noor, K.: Some characterizations of strongly preinvex functions. J. Math. Anal. Appl. **316**, 697–706 (2006)
18. Ngai, H.V., Luc, D., Thera, M.: Approximate convex functions. J. Nonlinear Convex Anal. **1**, 155–176 (2000)