Stability and H_∞ Performance for 2-D Discrete Systems with Time-Varying Delays

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Abstract. This paper investigates both the stability and H_∞ performance for a class of 2-D discrete systems with time-varying delays described by Fornasini-Marchesini (FM) second model. A new sufficient condition for asymptotic stability with H_∞ performance of these systems is developed based on differences of Lyapunov functionals proposed through introducing free weighting matrices. The findings are later tested via linear matrix inequality (LMI) feasibility. A numerical example is presented to demonstrate the effectiveness and benefits of the result obtained in this study.

Keywords: 2-D discrete time-delayed systems · H_∞ performance · FM second model · LMI

1 Introduction

The 2-D systems are found in various practical and physical processes where the information propagation occurs in two independent directions such as gas absorption and water stream heating. During the last decade, the research on 2-D systems both in practice and theory has enticed a large number of scholars due to their extensive applications [16], for example, image data processing, circuit analysis, transmission and other areas.

It has been well recognized that time-delay often occurs in practical systems, particularly in 2-D systems due to data transmission and finite speed of information processing among various parts of the system [20]. In addition, the reaction of realistic systems to external signals is seldom instantaneous and always affected by time delays. The time-delay frequently degrades the system performance and even causes the system instability [13]. Therefore, the exploration of time-delay systems stability plays a key role in applied models which has caused quite a stir in recent years.

The H_∞ technique introduced in [9] has been in the spotlight in recent years which attracted researchers, for example [5,7,8,12,17,22,23]. One of it many advantages is that it is insensitive to the exact knowledge of the statistics of the noise signals.

In this paper, 2-D systems are described by Fornasini-Marchesini (FM) second model [6,10,11] and based on the free-weighting matrix approach, proposed in [14,15,21] and by constructing a Lyapunov functional [18], a delay-dependent H_∞ performance analysis is reached by keeping some useful terms from the difference of Lyapunov functions.

This paper is adjusted to five sections: In Sect. 2, the problem under study is formulated. In Sect. 3, new criterion is obtained in terms of LMI, which ensure the asymptotic stability and the H_∞ performance of 2-D discrete systems described by the FM second model. Numerical examples are given to highlight the results in Sect. 4. Finally, some conclusions are provided in Sect. 5.

Notations: Throughout the paper, $\mathbb{R}^p$ denotes the p-dimensional real Euclidean space, $\mathbb{R}^{p \times q}$ denotes the set of all $p \times q$ matrices. $\mathbf{0}$ and I represent zero matrix identity matrix respectively. $diag\{...\}$ denotes a block-diagonal matrix in symmetric block matrices or long matrix expressions. X^T stand for the transpose and the matrix X. $Q > 0$ $(Q < 0)$ means that Q is real symmetric and positive (negative) definite matrix. The notation $||x||$ stands for the Euclidean norm of the vector x.

2   Problem Formulation

We consider the 2-D system with time-varying delays described by the following FM second model [10]:

$$x(s_1+1, s_2+1) = A_1 x(s_1+1, s_2) + A_2 x(s_1, s_2+1) + A_{1d} x(s_1+1, s_2-d_j)$$
$$+ A_{2d} x(s_1-d_i, s_2+1) + B_1 w(s_1+1, s_2) + B_2 w(s_1, s_2+1)$$
$$z(s_1, s_2) = C x(s_1, s_2) + D w(s_1, s_2) \tag{1}$$

where $x(s_1, s_2) \in \mathbb{R}^n$ is the state vector, $z(s_1, s_2) \in \mathbb{R}^m$ the signal to be estimated, $w(s_1, s_2) \in \mathbb{R}^s$ is the disturbance input. A_1, A_2, A_{1d}, A_{2d}, B_1, B_2, C and D are constant matrices with appropriate dimensions, d_i and d_j are time varying delays along horizontal and vertical directions, respectively, satisfying $\tau_1 \leq d_j \leq \tau_2$ and $\tau_3 \leq d_i \leq \tau_4$ where τ_1, τ_2, τ_3 and τ_4 are known positive integers $\tau_v = \tau_2 - \tau_1$ and $\tau_h = \tau_4 - \tau_3$.
The boundary conditions for the system are specified as:

$$\begin{cases} x(s_1, s_2) = \varphi_{s_1, s_2}, & \forall s_1 \geqslant 0, s_2 \in [-\tau_2, 0] \\ x(s_1, s_2) = \psi_{s_1, s_2}, & \forall s_2 \geqslant 0, s_1 \in [-\tau_4, 0] \\ \psi_{0,0} = \varphi_{0,0}. \end{cases} \tag{2}$$

In what follows, the boundary conditions assumed to satisfy:

$$\sum_{s_1=0}^{\infty} \sum_{s_2=-\tau_2}^{0} \varphi_{s_1,s_2}^T \varphi_{s_1,s_2} < \infty, \quad \sum_{s_2=0}^{\infty} \sum_{s_1=-\tau_4}^{0} \psi_{s_1,s_2}^T \psi_{s_1,s_2} < \infty$$

By considering the zero initial conditions, the $H\infty$ norm of the system in (1) is given by:

$$||G(z_1, z_2)||_\infty = sup_{w_1, w_2 \in [0, 2\pi]} \sigma_{max}[G(e^{jw_1}, e^{jw_2})]$$

where σ_{max} denotes the maximum singular value of the corresponding matrix, and

$$G(z_1, z_2) = C(z_1 z_2 I - A_1 z_1 - A_2 z_2 - A_{1d} z_1 z_2^{-d_j} - A_{2d} z_1^{-d_i} z_2)^{-1}$$
$$(B_2 z_2 + B_1 z_1) + D$$

is the transfer function from the disturbance input $w(s_1, s_2)$ to the output $z(s_1, s_2)$ for the system in (1).

To get the main results of this paper, the following definition and lemmas are needed.

lemma1 [4] For given symmetric matrices

$$S = S^T = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$$

where S_{11} and S_{22} are square matrices, the following conditions are equivalent

1. $S < 0$;
2. $S_{11} < 0, \quad S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$;
3. $S_{22} < 0, \quad S_{11} - S_{12}^T S_{22}^{-1} S_{12} < 0$;

Definition 1. *[24] The 2-D system given in* (1) *with zero boundary conditions in* (2) *is said to have H_∞ disturbance attenuation γ if it is asymptotically stable and:*

$$||z(s_1, s_2)||_2 < \gamma ||w(s_1, s_2)||_2 \tag{3}$$

For any nonzero $w(s_1, s_2) \in L_2\{[0, \infty), [0, \infty)\}$ where

$$z(s_1, s_2) = \begin{bmatrix} z^T(s_1 + 1, s_2) \ z^T(s_1, s_2 + 1) \end{bmatrix}^T$$
$$w(s_1, s_2) = \begin{bmatrix} w^T(s_1 + 1, s_2) \ w^T(s_1, s_2 + 1) \end{bmatrix}^T$$

3 Main Results

In this section, we consider the H_∞ performance analysis problem of system (1). For this case the following theorem holds.

Theorem 1. *Given integers $\tau_1 \leq d_j \leq \tau_2$ and $\tau_3 \leq d_i \leq \tau_4$ and scalar $\gamma > 0$ the system* (1) *with time varying d_i and d_j satisfying initial conditions given in* (2) *is asymptotically stable for all nonzero $w(s_1, s_2) \in L_2\{[0, \infty), [0, \infty)\}$ and* (3) *is*

satisfied if there exist appropriately dimensioned matrices $P = P_a + P_b = P^T > 0$,
for $i = 1, 2, j = 1, 2$, $Q_{ij} = Q_{ij}^T > 0$, $Z_{1j} = Z_{1j}^T > 0$, $Z_{2j} = Z_{2j}^T > 0$, for $j = h, v$

$$X_j = \begin{bmatrix} X_{11j} & X_{12j} \\ * & X_{22j} \end{bmatrix} \geq 0, \quad Y_j = \begin{bmatrix} Y_{11j} & Y_{12j} \\ * & Y_{22j} \end{bmatrix} \geq 0, \quad N_j = \begin{bmatrix} N_{1j} \\ N_{2j} \end{bmatrix}, \quad M_j =$$

$$\begin{bmatrix} M_{1j} \\ M_{2j} \end{bmatrix}, \quad S_j = \begin{bmatrix} S_{1j} \\ S_{2j} \end{bmatrix}, for \quad j = 1, 2 \text{ such that the following matrix inequalities}$$

hold:

$$\beta = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} & \beta_{15} \\ * & \beta_{22} & \beta_{23} & \beta_{24} & \beta_{25} \\ * & * & \beta_{33} & \beta_{34} & \beta_{35} \\ * & * & * & \beta_{44} & \beta_{45} \\ * & * & * & * & \beta_{55} \end{bmatrix} < 0 \qquad (4)$$

and

$$\Psi_{1j} = \begin{bmatrix} Xi & N_j \\ * & Z_{1j} \end{bmatrix} \geq 0, \quad \Psi_{2j} = \begin{bmatrix} Yi & M_j \\ * & Z_{2j} \end{bmatrix} \geq 0, \quad \Psi_{3j} = \begin{bmatrix} Xi + Yi & S_j \\ * & Z_{1j} + Z_{2j} \end{bmatrix} \geq 0, i = h, v, j = 1, 2$$

where

$$\beta_{11} = \phi, \quad \beta_{12} = \begin{bmatrix} \sqrt{\tau_4}\phi_1^T Z_{11} & \sqrt{\tau_h}\phi_1^T Z_{21} \end{bmatrix}, \quad \beta_{13} = \begin{bmatrix} \sqrt{\tau_2}\phi_2^T Z_{12} & \sqrt{\tau_v}\phi_2^T Z_{22} \end{bmatrix}$$

$$\beta_{14} = \begin{bmatrix} \phi_{e1}^T & \phi_{e2}^T \end{bmatrix}, \quad \beta_{15} = \begin{bmatrix} \phi_3^T P_a & \phi_3^T P_b \end{bmatrix}, \quad \beta_{22} = \begin{bmatrix} -Z_{11} & 0 \\ 0 & -Z_{21} \end{bmatrix}$$

$$\beta_{33} = \begin{bmatrix} -Z_{12} & 0 \\ 0 & -Z_{22} \end{bmatrix}, \quad \beta_{44} = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}, \quad \beta_{55} = \begin{bmatrix} -P_a & 0 \\ 0 & -P_b \end{bmatrix}$$

$$\beta_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} for \quad i = 2, 3, 4 \quad , j = 3, 4, 5 \quad and \quad i \neq j$$

the zero matrix is appropriately dimensioned.

$$\phi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \\ * & \pi_6 & \pi_7 & \pi_8 & \pi_9 \\ * & * & \pi_{10} & \pi_{11} & \pi_{12} \\ * & * & * & \pi_{13} & \pi_{14} \\ * & * & * & * & \pi_{15} \end{bmatrix}$$

$$\pi_1 = \begin{bmatrix} \phi_{v1} & 0 \\ * & \phi_{h1} \end{bmatrix}, \quad \pi_2 = \begin{bmatrix} \phi_{v2} & 0 \\ * & \phi_{h2} \end{bmatrix}, \quad \pi_3 = \begin{bmatrix} S_{12} & -M_{12} \\ 0 & 0 \end{bmatrix}, \quad \pi_4 = \begin{bmatrix} 0 & 0 \\ S_{11} & -M_{11} \end{bmatrix}$$

$$\pi_6 = \begin{bmatrix} \phi_{v3} & 0 \\ * & \phi_{h3} \end{bmatrix}, \quad \pi_7 = \begin{bmatrix} S_{22} & M_{22} \\ 0 & 0 \end{bmatrix}, \quad \pi_8 = \begin{bmatrix} 0 & 0 \\ S_{21} & M_{21} \end{bmatrix}, \quad \pi_{10} = \begin{bmatrix} -Q_{12} & 0 \\ * & -Q_{22} \end{bmatrix}$$

$$\pi_{13} = \begin{bmatrix} -Q_{11} & 0 \\ * & -Q_{21} \end{bmatrix}, \quad \pi_{15} = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}, \quad \pi_l = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, for \quad l = 5, 9, 11, 12, 14$$

with

$$\phi_{v1} = N_{12} + N_{12}^T + (\tau_v + 1)Q_{32} + Q_{12} + Q_{22} + \tau_2 X_{11b} + \tau_v Y_{11b} - P_b$$
$$\phi_{v2} = N_{22}^T - N_{12} + M_{12} - S_{12} + \tau_2 X_{12b} + \tau_v Y_{12b}$$
$$\phi_{v3} = -N_{22} - N_{22}^T - S_{22} - S_{22}^T + M_{22} + M_{22}^T - Q_{32} + \tau_2 X_{22b} + \tau_v Y_{22b}$$
$$\phi_{h1} = N_{11} + N_{11}^T + (\tau_h + 1)Q_{31} + Q_{11} + Q_{21} + \tau_4 X_{11a} + \tau_h Y_{11a} - P_a$$
$$\phi_{h2} = N_{21}^T - N_{11} + M_{11} - S_{11} + \tau_4 X_{12a} + \tau_h Y_{12a}$$
$$\phi_{h3} = -N_{21} - N_{21}^T - S_{21} - S_{21}^T + M_{21} + M_{21}^T - Q_{31} + \tau_4 X_{22a} + \tau_h Y_{22a}$$

also

$$\phi_1 = \begin{bmatrix} A_1 & (A_2 - I) & A_{1d} & A_{2d} & 0 & 0 & 0 & 0 & B_1 & B_2 \end{bmatrix}$$
$$\phi_2 = \begin{bmatrix} (A_1 - I) & A_2 & A_{1d} & A_{2d} & 0 & 0 & 0 & 0 & B_1 & B_2 \end{bmatrix}$$
$$\phi_3 = \begin{bmatrix} A_1 & A_2 & A_{1d} & A_{2d} & 0 & 0 & 0 & 0 & B_1 & B_2 \end{bmatrix}$$
$$\phi_{e1} = \begin{bmatrix} 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D \end{bmatrix}$$
$$\phi_{e2} = \begin{bmatrix} C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D & 0 \end{bmatrix}$$

Proof. Let

$$\eta_1(s_1, s_2) = x(s_1 + 1, s_2 + 1) - x(s_1, s_2 + 1) \tag{5}$$
$$\eta_1(s_1, s_2) = A_1 x(s_1 + 1, s_2) + (A_2 - I)x(s_1, s_2 + 1) + A_{1d}x(s_1 + 1, s_2 - d_j) + A_{2d}x(s_1 - d_i, s_2 + 1)$$
$$+ B_1 w(s_1 + 1, s_2) + B_2 w(s_1, s_2 + 1) \tag{6}$$

$$x_{sys}(s_1, s_2) = \begin{bmatrix} x^T(s_1 + 1, s_2) & x^T(s_1, s_2 + 1) & x^T(s_1 + 1, s_2 - d_j) & x^T(s_1 - d_i, s_2 + 1) \end{bmatrix}$$
$$..\, x^T(s_1 + 1, s_2 - \tau_1) \; x^T(s_1 + 1, s_2 - \tau_2) \; x^T(s_1 - \tau_3, s_2 + 1) \; x^T(s_1 - \tau_4, s_2 + 1) \; ..$$
$$.. \, w^T(s_1 + 1, s_2) \; w^T(s_1, s_2 + 1) \end{bmatrix}^T$$

$$\Rightarrow \qquad \eta_1(s_1, s_2) = \phi_1 x_{sys}(s_1, s_2)$$

The same for the vertical direction:

$$\Rightarrow \qquad \eta_2(s_1, s_2) = \phi_2 x_{sys}(s_1, s_2)$$

Choose a Lyapunov functional candidate to be

$$v(s_1, s_2) = v_1(s_1, s_2) + v_2(s_1, s_2) + v_3(s_1, s_2) + v_4(s_1, s_2)$$

$$v(s_1, s_2) = v_{11}(s_1, s_2) + v_{12}(s_1, s_2) + v_{21}(s_1, s_2) + v_{22}(s_1, s_2)$$
$$+ v_{31}(s_1, s_2) + v_{32}(s_1, s_2) + v_{41}(s_1, s_2) + v_{42}(s_1, s_2)$$

$$v_{11}(s_1, s_2) = x^T(s_1, s_2)P_a x(s_1, s_2), \quad v_{12}(s_1, s_2) = x^T(s_1, s_2)P_b x(s_1, s_2)$$

$$v_{21}(s_1, s_2) = \sum_{\theta=-\tau_4+1}^{0} \sum_{l=s_1-1+\theta}^{s_1-1} \eta_1^T(l, s_2)Z_{11}\eta_1(l, s_2) + \sum_{\theta=-\tau_4+1}^{-\tau_3} \sum_{l=s_1-1+\theta}^{s_1-1} \eta_1^T(l, s_2)Z_{21}\eta_1(l, s_2)$$

$$v_{22}(s_1, s_2) = \sum_{\theta=-\tau_2+1}^{0} \sum_{l=s_2-1+\theta}^{s_2-1} \eta_2^T(s_1, l)Z_{12}\eta_2(s_1, l) + \sum_{\theta=-\tau_2+1}^{-\tau_1} \sum_{l=s_2-1+\theta}^{s_2-1} \eta_2^T(s_1, l)Z_{22}\eta_2(s_1, l)$$

$$v_{31}(s_1, s_2) = \sum_{l=s_1-\tau_3}^{s_1-1} x^T(l, s_2+1)Q_{11}x(l, s_2+1) + \sum_{l=s_1-\tau_4}^{s_1-1} x^T(l, s_2+1)Q_{21}x(l, s_2+1)$$

$$v_{32}(s_1, s_2) = \sum_{l=s_2-\tau_2}^{s_2-1} x^T(s_1+1, l)Q_{12}x(s_1+1, l) + \sum_{l=s_2-\tau_1}^{s_2-1} x^T(s_1+1, l)Q_{22}x(s_1+1, l)$$

$$v_{41}(s_1, s_2) = \sum_{\theta=1-\tau_4}^{1-\tau_3} \sum_{l=s_1-1+\theta}^{s_1-1} x^T(l, s_2+1)Q_{31}x(l, s_2+1)$$

$$v_{42}(s_1, s_2) = \sum_{\theta=1-\tau_2}^{1-\tau_1} \sum_{l=s_2-1+\theta}^{s_2-1} x^T(s_1+1, l)Q_{32}x(s_1+1, l)$$

where $P = P_a + P_b = P^T > 0$, $Q_{ij} = Q_{ij}^T > 0$, i=1,2, j=1,2, $Z_{1j} = Z_{1j}^T > 0$ j=1,2, $Z_{2j} = Z_{2j}^T > 0$ j=1,2 are to be calculated. Defining $\Delta V(s_1+1, s_2) = V(s_1+1, s_2) - V(s_1, s_2)$ and $\Delta V(s_1, s_2+1) = V(s_1, s_2+1) - V(s_1, s_2)$ yields

$$\Delta_{v_{11}(s_1, s_2)} = x_{sys}^T(s_1+1, s_2)\phi_3^T P_a \phi_3 x_{sys}(s_1+1, s_2) - x_{sys}^T(s_1, s_2)P_a x_{sys}(s_1, s_2)$$

$$\Delta_{v_{12}(s_1, s_2)} = x_{sys}^T(s_1, s_2+1)\phi_3^T P_b \phi_3 x_{sys}(s_1, s_2+1) - x_{sys}^T(s_1, s_2)P_b x_{sys}(s_1, s_2)$$

$$\Delta_{v_{21}(s_1, s_2)} = x_{sys}^T(s_1, s_2)\phi_1^T(\tau_4 Z_{11} + \tau_h Z_{21})\phi_1 x_{sys}(s_1, s_2) - \sum_{l=s_1-d_i}^{s_1-1} \eta_1^T(l, s_2)Z_{11}\eta_1(l, s_2)$$

$$- \sum_{l=s_1-d_i}^{s_1-1-\tau_3} \eta_1^T(l, s_2)Z_{21}\eta_1(l, s_2) - \sum_{l=s_1-\tau_4}^{s_1-1-d_i} \eta_1^T(l, s_2)(Z_{21}+Z_{11})\eta_1(l, s_2)$$

$$\Delta_{v_{22}(s_1, s_2)} = x_{sys}^T(s_1, s_2)\phi_2^T(\tau_2 Z_{12} + \tau_v Z_{22})\phi_2 x_{sys}(s_1, s_2) - \sum_{l=s_2-d_j}^{s_2-1} \eta_2^T(s_1, l)Z_{12}\eta_2(s_1, l)$$

$$- \sum_{l=s_2-d_j}^{s_2-1-\tau_1} \eta_2^T(s_1, l)Z_{22}\eta_2(s_1, l) - \sum_{l=s_2-\tau_2}^{s_2-1-d_j} \eta_2^T(s_1, l)(Z_{22}+Z_{12})\eta_2(s_1, l)$$

$$\Delta_{v_{31}(s_1, s_2)} = x^T(s_1, s_2+1)(Q_{21}+Q_{11})x(s_1, s_2+1) - \xi^T(s_1-\tau_3, s_2+1)Q_{11}x(s_1-\tau_3, s_2+1)$$

$$- x^T(s_1-\tau_4, s_2+1)Q_{21}x(s_1-\tau_4, s_2+1)$$

$$\Delta_{v_{32}(s_1, s_2)} = x^T(s_1+1, s_2)(Q_{22}+Q_{12})x(s_1+1, s_2) - x^T(s_1+1, s_2-\tau_1)Q_{12}x(s_1+1, s_2-\tau_1)$$

$$- x^T(s_1+1, s_2-\tau_2)Q_{22}x(s_1+1, s_2-\tau_2)$$

$$\Delta_{v_{41}(s_1,s_2)} = (\tau_h + 1)x^T(s_1, s_2 + 1)Q_{31}x(s_1, s_2 + 1) - \sum_{l=s_1-\tau_4}^{s_1-\tau_3} x^T(l, s_2 + 1)Q_{31}x(l, s_2 + 1)$$

$$< (\tau_h + 1)x^T(s_1, s_2 + 1)Q_{31}x(s_1, s_2 + 1) - x^T(s_1 - d_i, s_2 + 1)Q_{31}x(s_1 - d_i, s_2 + 1)$$

$$\Delta_{v_{42}(s_1,s_2)} = (\tau_v + 1)x^T(s_1 + 1, s_2)Q_{32}x(s_1 + 1, s_2) - \sum_{l=s_2-\tau_2}^{s_2-\tau_1} x^T(s_1 + 1, l)Q_{32}x(s_1 + 1, l)$$

$$< (\tau_v + 1)x^T(s_1 + 1, s_2)Q_{32}x(s_1 + 1, s_2) - x^T(s_1 + 1, s_2 - d_j)Q_{32}x(s_1 + 1, s_2 - d_j)$$

From (5), we have
$$\eta_1(s_1, s_2) = x(s_1 + 1, s_2 + 1) - x(s_1, s_2 + 1)$$
$$\Rightarrow$$

$$\sum_{l=s_1-d_i}^{s_1-1} \eta_1(l, s_2) = \sum_{l=s_1-d_i}^{s_1-1} x(l + 1, s_2 + 1) - \sum_{l=s_1-d_i}^{s_1-1} x(l, s_2 + 1)$$

$$\Rightarrow$$

$$0 = x(s_1, s_2 + 1) - x(s_1 - d_i, s_2 + 1) - \sum_{l=s_1-d_i}^{s_1-1} \eta_1(l, s_2)$$

$$0 = x(s_1 - d_i, s_2 + 1) - x(s_1 - \tau_4, s_2 + 1) - \sum_{l=s_1-\tau_4}^{s_1-d_i-1} \eta_1(l, s_2)$$

$$0 = x(s_1 - \tau_3, s_2 + 1) - x(s_1 - d_i, s_2 + 1) - \sum_{l=s_1-d_i}^{s_1-\tau_3-1} \eta_1(l, s_2)$$

the same for vertical direction.
then the following equations are true for any matrices N_1, N_2, M_1, M_2, S_1 and S_2 with appropriate dimensions for $\Delta V(s_1, s_2 + 1)$:

$$0 = 2 \times [x^T(s_1, s_2 + 1)N_{11} + x^T(s_1 - d_i, s_2 + 1)N_{21}$$

$$\times [x(s_1, s_2 + 1) - x(s_1 - d_i, s_2 + 1) - \sum_{l=s_1-d_i}^{s_1-1} \eta_1(l, s_2)] \tag{7}$$

$$0 = 2 \times [x^T(s_1, s_2 + 1)M_{11} + x^T(s_1 - d_i, s_2 + 1)M_{21}$$

$$\times [x(s_1 - d_i, s_2 + 1) - x(s_1 - \tau_4, s_2 + 1) - \sum_{l=s_1-\tau_4}^{s_1-d_i-1} \eta_1(l, s_2)] \tag{8}$$

$$0 = 2 \times [x^T(s_1, s_2 + 1)S_{11} + x^T(s_1 - d_i, s_2 + 1)S_{21}$$

$$\times [x(s_1 - \tau_3, s_2 + 1) - x(s_1 - d_i, s_2 + 1) - \sum_{l=s_1-d_i}^{s_1-\tau_3-1} \eta_1(l, s_2)] \tag{9}$$

The same thing for $\Delta V(s_1 + 1, s_2)$.

In the other hand, for any appropriately dimensioned matrices $X_i = X_i^T \geq 0$, $Y_i = Y_i^T \geq 0$, for i = h, v, the following equations are true:

$$0 = \tau_4 \xi_1^T(s_1, s_2) X_h \xi_1(s_1, s_2) - \sum_{l=s_1-d_i}^{s_1-1} \xi_1^T(s_1, s_2) X_h \xi_1(s_1, s_2)$$

$$- \sum_{l=s_1-\tau_4}^{s_1-d_i-1} \xi_1^T(s_1, s_2) X_h \xi_1(s_1, s_2) \tag{10}$$

$$0 = \tau_h \xi_1^T(s_1, s_2) Y_h \xi_1(s_1, s_2) - \sum_{l=s_1-d_i}^{s_1-\tau_3-1} \xi_1^T(s_1, s_2) Y_h \xi_1(s_1, s_2)$$

$$- \sum_{l=s_1-\tau_4}^{s_1-d_i-1} \xi_1^T(s_1, s_2) Y_h \xi_1(s_1, s_2) \tag{11}$$

$$0 = \tau_2 \xi_2^T(s_1, s_2) X_v \xi_2(s_1, s_2) - \sum_{l=s_2-d_j}^{s_2-1} \xi_2^T(s_1, s_2) X_v \xi_2(s_1, s_2)$$

$$- \sum_{l=s_2-\tau_2}^{s_2-d_j-1} \xi_2^T(s_1, s_2) X_v \xi_2(s_1, s_2) \tag{12}$$

$$0 = \tau_v \xi_2^T(s_1, s_2) Y_v \xi_2(s_1, s_2) - \sum_{l=s_2-d_j}^{s_2-\tau_1-1} \xi_2^T(s_1, s_2) Y_v \xi_2(s_1, s_2)$$

$$- \sum_{l=s_2-\tau_2}^{s_2-d_j-1} \xi_2^T(s_1, s_2) Y_v \xi_2(s_1, s_2) \tag{13}$$

$$\xi_1(s_1, s_2) = \left[x^T(s_1, s_2 + 1) \; x^T(s_1 - d_i, s_2 + 1) \right]^T$$

$$\xi_2(s_1, s_2) = \left[x^T(s_1 + 1, s_2) \; x^T(s_1 + 1, s_2 - d_j) \right]^T$$

Then, if the terms of the right side of the Eqs. (7)–(13) are added to $\Delta V(s_1, s_2) = \Delta V(s_1 + 1, s_2) + V(s_1, s_2 + 1)$, we have:

$$\Delta V(s_1, s_2) + z^T(s_1, s_2)z(s_1, s_2) - \gamma^2 w^T(s_1, s_2)w(s_1, s_2) \leq x_{sys}^T(s_1, s_2)(\phi + \phi_1^T(\tau_4 Z_{11} + \tau_h Z_{21})\phi_1$$

$$+ \phi_2^T(\tau_2 Z_{12} + \tau_v Z_{22})\phi_2 + \phi_{e1}^T\phi_{e1} + \phi_{e2}^T\phi_{e2} + \phi_3^T(P_a + P_b)\phi_3)x_{sys}(s_1, s_2)$$

$$- \sum_{l=s_1-d_i}^{s_1-1} \xi_3^T(l, s_2)\Psi_{11}\xi_3(l, s_2) - \sum_{l=s_1-d_i}^{s_1-\tau_3-1} \xi_3^T(l, s_2)\Psi_{21}\xi_3(l, s_2) - \sum_{l=s_1-\tau_4}^{s_1-d_i-1} \xi_3^T(l, s_2)\Psi_{31}\xi_3(l, s_2)$$

$$- \sum_{l=s_2-d_j}^{s_2-1} \xi_3^T(s_1, l)\Psi_{12}\xi_3(s_1, l) - \sum_{l=s_2-d_j}^{s_2-\tau_1-1} \xi_3^T(s_1, l)\Psi_{21}\xi_3(s_1, l) - \sum_{l=s_2-\tau_2}^{s_2-d_j-1} \xi_3^T(s_1, l)\Psi_{32}\xi_3(s_1, l)$$

$$\xi_3(l, s_2) = \left[\xi_1^T(s_1, s_2) \; \eta_1^T(l, s_2) \right]^T$$
$$\xi_3(s_1, l) = \left[\xi_2^T(s_1, s_2) \; \eta_2^T(s_1, l) \right]^T$$

thus if $\Psi_{1j}, \Psi_{2j} \geq 0, \Psi_{3j} \geq 0$ for $j = 1, 2$ and

$$(\phi + \phi_1^T(\tau_4 Z_{11} + \tau_h Z_{21})\phi_1 + \phi_2^T(\tau_2 Z_{12} + \tau_v Z_{22})\phi_2 + \phi_{e1}^T\phi_{e1} + \phi_{e2}^T\phi_{e2} + \phi_3^T(P_a + P_b)\phi_3) < 0$$

which is equivalent to (4) by schur compliments, then

$$\Delta V(s_1, s_2) + z^T(s_1, s_2)z(s_1, s_2) - \gamma^2 w^T(s_1, s_2)w(s_1, s_2) < 0$$

This ensures that (3) holds under zero-initial conditions for all nonzero $w(s_1, s_2) \in L_2\{[0, \infty), [0, \infty)\}$ and a specified $\gamma > 0$ following the similar line in [25]. On the other hand, (4) entails that the following matrix inequality (14) holds, which guarantees $\Delta V(s_1, s_2) < 0$, such that the system (1) with $w(s_1, s_2) = 0$ is asymptotically stable.

$$\beta = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{15} \\ * & \beta_{22} & \beta_{23} & \beta_{25} \\ * & * & \beta_{33} & \beta_{35} \\ * & * & * & \beta_{55} \end{bmatrix} < 0 \tag{14}$$

where

$$\phi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ * & \pi_6 & \pi_7 & \pi_8 \\ * & * & \pi_{10} & \pi_{11} \\ * & * & * & \pi_{13} \end{bmatrix}$$

$$\phi_1 = \begin{bmatrix} A_1 & (A_2 - I) & A_{1d} & A_{2d} & 0 & 0 & 0 \end{bmatrix}$$
$$\phi_2 = \begin{bmatrix} (A_1 - I) & A_2 & A_{1d} & A_{2d} & 0 & 0 & 0 \end{bmatrix}$$
$$\phi_3 = \begin{bmatrix} A_1 & A_2 & A_{1d} & A_{2d} & 0 & 0 & 0 \end{bmatrix}$$

This completes the proof.   □

Remarque: The terms $\sum_{l=s_1-\bar{h}_1}^{s_1-d_i-1} \eta_1^T(l, s_2)Z_{s_1}\eta_1(l, s_2)$ and $\sum_{l=s_2-\bar{h}_2}^{s_2-d_j-1} \eta_2^T(s_1,$ $l)Z_{s_2}\eta_2(s_1, l)$ are kept in Theorem 1 to overcome the conservativeness. On the other hand, τ_2 and τ_4 are split into two parts like d_j and $\tau_2 - d_j$, d_i and $\tau_4 - d_i$, for vertical and horizontal direction, respectively, in order to prove Theorem 1 thus showing the benefits of the suggested method.

4   Numerical Example

In this section, we will give a numerical example to illustrate the applicability of the proposed result.

Example 1. [19] Denote the design of 2-D delay-dependent H_∞ performance and filter for a stationary random field in image processing where the disturbances are a random process (noise), using LMI approach proposed in (4), the 2-D system can be converted to the 2-D FM model system (1) with the ensuing parameters:

$$A_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0.2 \end{bmatrix}, \quad A_{1d} = \begin{bmatrix} 0 & -0.03 \\ -0.08 & 0 \end{bmatrix}, \quad A_{2d} = \begin{bmatrix} -0.03 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Given $\tau_3 = 3$, $\tau_4 = 3$, $\tau_1 = 2$, $\tau_2 = 2$ by solving the LMIs, the minimum H_∞ norm bound for this example is $\gamma_{opt} = 6.9671$.

Fig. 1 shows the maximum singular values plot of the transfer function matrix of the system (1). In the figure, the grids denote the obtained H_∞ disturbance attenuations and its maximum value is 6.3725, which is below 6.9671.
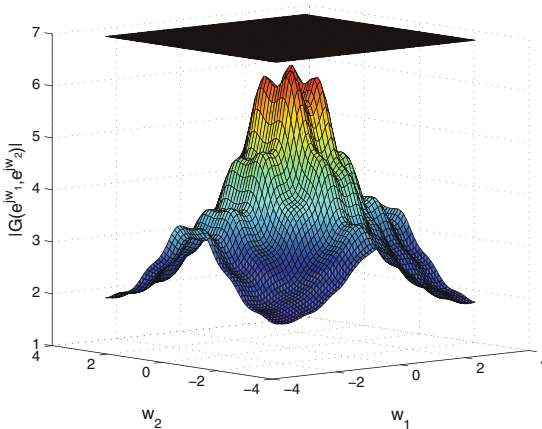

Fig. 1. Transfer function (Example 1).

5   Conclusion

This paper has explored the problems of stability and delay-dependent H_∞ performance analysis for 2-D discrete systems with time varying delay described by FM second model without ignoring any terms in the derivative of lyapunov functional by considering the relationship between the delay and it upper bound. The new criteria may be extended to systems with uncertainties.

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