Normal Reduction Numbers of Normal Surface Singularities

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Dedicated to Professor András Némethi on the occasion of his sixtieth birthday

Abstract This article consists of two parts. The first part is a survey on the normal reduction numbers of normal surface singularities. It includes results on elliptic singularities, cone-like singularities and homogeneous hypersurface singularities. In the second part, we prove a new results on the normal reduction numbers and related invariants of Brieskorn complete intersections.

Keywords Normal reduction number · Normal surface singularity · Geometric genus · Elliptic singularity · Brieskorn complete intersection · Homogeneous hypersurface singularity

Subject Classifications Primary 14J17; Secondary 14B05, 32S25, 13B22

1 Introduction

In this paper, we survey results on the normal reduction numbers of normal complex surface singularities and some related topics [24, 26, 28, 29], and prove new results on the normal reduction numbers of Brieskorn complete intersections. The normal reduction number has appeared in the study of normal Hilbert polynomials from a ring-theoretic point of view (cf. [6, 14]). We study the normal reduction numbers of the local ring of normal surface singularities using resolution of singularities, and we wish to know what kind of geometric property of singularities relates to the normal reduction numbers.

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Let us briefly recall some basic facts about integral closure and reduction of ideals in a local ring. Let $(A, \mathfrak{m})$ be a Noetherian local ring and I an $\mathfrak{m}$-primary ideal (namely, $\sqrt{I} = \mathfrak{m}$). Let $\overline{I}$ denote the integral closure of I, that is, $\overline{I}$ is an ideal of A consists of all elements $z \in A$ such that $z^n + c_1 z^{n-1} + \cdots + c_n = 0$ for some $n \geq 1$ and $c_i \in I^i$ $(i = 1, \ldots, n)$. The ideal I is said to be integrally closed if $I = \overline{I}$. An ideal $Q \subset I$ is called a *reduction* of I if $I^{n+1} = QI^n$ for some $n \geq 0$. It is known that an ideal Q is a reduction of I if and only if $I \subset \overline{Q}$ (cf. [5, 1.2.5]). For a reduction Q of I, $\mathrm{r}_Q(I) := \min\left\{n \mid I^{n+1} = QI^n\right\}$ is called the reduction number of I with respect to Q.

Let (V, p) be a normal complex surface singularity[1] and $\mathcal{O}_{V,p}$ the local ring of the singularity with maximal ideal $\mathfrak{m}$. Let $I \subset \mathcal{O}_{V,p}$ be an $\mathfrak{m}$-primary integrally closed ideal. It is known that any minimal reduction of I is generated by two elements and that two general elements of I generate a minimal reduction of I (see [5, 8.3.7, 8.6.6]). Suppose that Q is a minimal reduction of I. We define two normal reduction numbers, which are analogues of the reduction number $r_Q(I)$, as follows:

$$\mathrm{nr}(I) = \min\{n \in \mathbb{Z}_{\geq 0} \mid \overline{I^{n+1}} = Q\overline{I^n}\},$$

$$\bar{\mathrm{r}}(I) = \min\{n \in \mathbb{Z}_{\geq 0} \mid \overline{I^{N+1}} = Q\overline{I^N} \text{ for every } N \geq n\}.$$

We note that $\mathrm{nr}(I)$ and $\bar{\mathrm{r}}(I)$ are independent of the choice of Q (see e.g. [4, Theorem 4.5], Proposition 3.2), though $r_Q(I)$ is not independent of the choice of a minimal reduction Q in general. It is obvious by the definition that $\mathrm{nr}(I) \leq \bar{\mathrm{r}}(I)$. We will show that $\bar{\mathrm{r}}(I) \leq p_g(V, p) + 1$ in general (see Proposition 3.2). We can also show that for any integer $g \geq 2$ there exists a singularity (V, p) with $\mathrm{nr}(I) = 1$ and $\bar{\mathrm{r}}(I) = p_g(V, p) + 1 = g + 1$ (Example 4.5). We define

$$\mathrm{nr}(V, p) = \max\{\mathrm{nr}(J) \mid J \text{ is an } \mathfrak{m}\text{-primary integrally closed ideal of } \mathcal{O}_{V,p}\},$$

$$\bar{\mathrm{r}}(V, p) = \max\{\bar{\mathrm{r}}(J) \mid J \text{ is an } \mathfrak{m}\text{-primary integrally closed ideal of } \mathcal{O}_{V,p}\}.$$

The invariant $\bar{\mathrm{r}}(V, p)$ naturally appears in several situation as follows. For any $\mathfrak{m}$-primary integrally closed ideal $I \subset \mathcal{O}_{V,p}$, there exist a resolution $\pi : X \to V$ and a divisor Z on X such that $\mathcal{O}_X(-Z)$ is π-generated and $I = \pi_*\mathcal{O}_X(-Z)_p$ (see Sect. 2). Let $r := \bar{\mathrm{r}}(I)$. By the definition of $\bar{\mathrm{r}}$ and Proposition 3.2, we have the following:

(1) Briançon-Skoda type inclusion (cf. [3, 13]): $\overline{I^{r+k}} \subset Q^k$ for $k \geq 1$.
(2) The natural homomorphism $\pi_*\mathcal{O}_X(-nZ) \otimes \pi_*\mathcal{O}_X(-Z) \to \pi_*\mathcal{O}_X(-(n+1)Z)$ is surjective for $n \geq r$.

[1] In our papers [24, 26, 28, 29], we treat a singularity $(\mathrm{Spec}\, A, \mathfrak{m})$, where $(A, \mathfrak{m})$ is an excellent normal two-dimensional local ring such that the residue field k is algebraically closed and $k \subset A$.

(3) The function $\phi(n) := \dim_{\mathbb{C}} H^0(\mathcal{O}_X)/H^0(\mathcal{O}_X(-nZ))$ is a polynomial function of n for $n \geq r$; note that $\phi(n) = \chi(\mathcal{O}_{nZ}) + h^1(\mathcal{O}_X) - h^1(\mathcal{O}_X(-nZ))$ by Kato's Riemann-Roch Theorem ([8]).

So we expect that the normal reduction numbers can characterize good singularities. For example, we see that (V, p) is a rational singularity if and only if $\bar{\mathrm{r}}(V, p) = 1$ (see Proposition 3.6). However, we can only show that $\bar{\mathrm{r}}(V, p) = 2$ if (V, p) is an elliptic singularity (see Theorem 3.9, Proposition 5.13). At present, we have computed the normal reduction numbers only for some special cases, and we do not know whether those invariants are topological or not.

This paper is organized as follows. Sections 2–4 are devoted to a survey of fundamental results on the normal reduction numbers and some related topics. We refer the reader to [20] and [32] for basic facts about normal surface singularities. In Sect. 2, we set up notation and briefly recall the basic results on the cohomology groups of ideal sheaves of cycles on a resolution space. Then we mention a question about the range of the dimension of those cohomology groups. In Sect. 3, we give a relation between the normal reduction numbers and the dimension of the cohomology groups associated with an $\mathfrak{m}$-primary integrally closed ideal in $\mathcal{O}_{V,p}$ and review fundamental results on the normal reduction numbers. Then we review the results on elliptic singularities. In Sect. 4, we consider the cone-like singularities, namely, those homeomorphic to the cone over a nonsingular curve. We give an upper bound of $\bar{\mathrm{r}}$ using the genus and gonality of the curve and the self-intersection number of the fundamental cycles. Then we show a formula for the normal reduction numbers of homogeneous hypersurface singularities. In Sect. 5, we prove an explicit formula for $\bar{\mathrm{r}}$ of the maximal ideal of a Brieskorn complete intersection and apply the formula to classify elliptic singularities, which are natural generalization of the results about Brieskorn hypersurface singularities in [28].

2 Cycles and Cohomology

Let (V, p) be a normal complex surface singularity, namely, the germ of a normal complex surface V at $p \in V$. We always assume that V is Stein and suitably small. Let $\pi : X \to V$ denote a resolution of the singularity (V, p) with exceptional set $E = \pi^{-1}(p)$ and let $\{E_i\}_{i \in \mathcal{I}}$ denote the set of irreducible components of E. We call a divisor on X supported in E a *cycle* and denote by $\sum \mathbb{Z}E_i$ the group of cycles.

For a function $h \in H^0(\mathcal{O}_X(-E))$, we denote by $(h)_E \in \sum \mathbb{Z}E_i$ the exceptional part of the divisor $\mathrm{div}_X(h)$; so, $\mathrm{div}_X(h) - (h)_E$ is an effective divisor containing no components of E. We simply write $(h)_E$ instead of $(h \circ \pi)_E$ for $h \in \mathfrak{m}$.

An element of $\sum \mathbb{Q}E_i := (\sum \mathbb{Z}E_i) \otimes \mathbb{Q}$ is called a $\mathbb{Q}$-*cycle*. A $\mathbb{Q}$-cycle D is said to be *nef* (resp. *anti-nef*) if $DE_i \geq 0$ (resp. $DE_i \leq 0$) for all $i \in \mathcal{I}$. Note that if $D \neq 0$ is anti-nef, then $D \geq E$.

Definition 2.1 The *maximal ideal cycle* on X is the minimum of $\{(h)_E \mid h \in \mathfrak{m}\}$ and denoted by M_X. There exists a $\mathbb{Q}$-cycle Z_{K_X} such that $(K_X + Z_{K_X})E_i = 0$ for every $i \in \mathcal{I}$, where K_X is a canonical divisor on X. We call Z_{K_X} the *canonical cycle* on X.

In the following, we assume that $Z > 0$ is a cycle such that $\mathcal{O}_X(-Z)$ *has no fixed component*, namely, there exists a function $h \in H^0(\mathcal{O}_X(-Z))$ such that $(h)_E = Z$. We say that $\mathcal{O}_X(-Z)$ is *generated* if it is π-generated (i,e., $\pi^*\pi_*\mathcal{O}_X(-Z) \to \mathcal{O}_X(-Z)$ is surjective). For any coherent sheaf $\mathcal{F}$ on X, we write $H^i(\mathcal{F}) = H^i(X, \mathcal{F})$ and $h^i(\mathcal{F}) = \dim_{\mathbb{C}}(H^i(\mathcal{F}))$.

Definition 2.2 The *geometric genus* of the singularity (V, p) is defined by $p_g(V, p) = h^1(\mathcal{O}_X)$.

Definition 2.3 Let $A \geq 0$ be an effective cycle on X and let

$$h(A) = \max\left\{h^1(\mathcal{O}_B) \;\middle|\; B \in \sum \mathbb{Z}E_i,\ B \geq 0,\ \mathrm{Supp}(B) \subset \mathrm{Supp}(A)\right\}.$$

We put $h^1(\mathcal{O}_B) = 0$ if $B = 0$. There exists a unique minimal cycle C such that $h^1(\mathcal{O}_C) = h(A)$ (cf. [32, 4.8]). We call C the *cohomological cycle* of A. Note that $p_g(V, p) = h(E)$ and that if (V, p) is Gorenstein and π is the minimal resolution, then Z_{K_X} is the cohomological cycle of E ([32, 4.20]).

We define a reduced cycle $A^{\perp}$ to be the sum of the components $E_i \subset E$ such that $AE_i = 0$.

Remark 2.4 Let $F_1, \dots, F_k$ be the connected component of $Z^{\perp}$ and let (V_i, p_i) be the normal surface singularity obtained by contracting F_i. If C is the cohomological cycle of $Z^{\perp}$, we have

$$h^1(\mathcal{O}_C) = \sum_{i=1}^{k} p_g(V_i, p_i).$$

Definition 2.5 Let $q(Z) = h^1(\mathcal{O}_X(-Z))$ and $q_Z(n) = h^1(\mathcal{O}_X(-nZ))$ for $n \geq 0$. Let $s(Z) = \min\left\{n \in \mathbb{Z}_{\geq 0} \mid q_Z(n) = q_Z(n+1)\right\}$.

Proposition 2.6 (See [26, §3], [24, 3.6]) *We have the following.*

(1) $q_Z(n) \geq q_Z(n+1)$ *for every integer* $n \geq 0$.
(2) *If* $q_Z(1) = p_g(V, p)$, *namely,* $s(Z) = 0$, *then* $q(n) = p_g(V, p)$ *for* $n \geq 0$.
(3) *If* $\mathcal{O}_X(-Z)$ *is generated, then* $q_Z(n) = q_Z(s(Z)) = h^1(\mathcal{O}_C)$ *for* $n \geq s(Z)$, *where* C *is the cohomological cycle of* $Z^{\perp}$.
(4) $\mathcal{O}_X(-nZ)$ *is generated for* $n > s(Z)$.

We are interested in the range of the function q. Let $\mathcal{A}$ (resp. $\mathcal{A}'$) denotes the set of the pairs (Y, W) such that $W > 0$ is a cycle on a resolution $Y \to V$ such that

$\mathcal{O}_Y(-W)$ is generated (resp. has no fixed components). Clearly, $\mathcal{A} \subset \mathcal{A}'$. Let

$$q(\mathcal{A}) = \left\{h^1(\mathcal{O}_Y(-W)) \mid (Y, W) \in \mathcal{A}\right\}, \ q(\mathcal{A}') = \left\{h^1(\mathcal{O}_Y(-W)) \;\middle|\; (Y, W) \in \mathcal{A}'\right\}.$$

By Proposition 2.6, we have

$$q(\mathcal{A}) \subset q(\mathcal{A}') \subset \{0, 1, \dots, p_g(V, p)\}.$$

The proof of the following theorem is included in the proof of [24, 3.12].

Proposition 2.7 *We have the equality*

$$q(\mathcal{A}') = \{0, 1, \dots, p_g(V, p)\}.$$

Conjecture 2.8 *For every normal complex surface singularity, the equality* $q(\mathcal{A}) = q(\mathcal{A}')$ *holds.*

At present, we have the equality $q(\mathcal{A}) = q(\mathcal{A}')$ only for a few cases (cf. Proposition 3.11, Example 4.5). Some results related to Conjecture 2.8 are obtained in [16].

The next lemma is used in Sect. 5. For a $\mathbb{Q}$-cycle D, let $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$, where $\lfloor D \rfloor$ denotes the integral part of D.

Lemma 2.9 *Let* $C < E$ *be a reduced cycle and* $\{I_n\}_{n \in \mathbb{Z}_{\geq 0}}$ *a filtration of* $\mathcal{O}_{V,p}$ *such that* $(h)_E \geq nC$ *for all* $n \in \mathbb{Z}_{\geq 0}$ *and all* $h \in I_n \setminus \{0\}$ *and that* $\bigoplus_{n \geq 0} I_n/I_{n+1}$ *is reduced. Assume that there exists an anti-nef* $\mathbb{Q}$*-cycle* $\widetilde{C} = \sum a_i E_i$ *such that* $a_i = 1$ *for* $E_i \leq C$ *and* $\widetilde{C} E_i = 0$ *for every* $E_i \not\leq C$*. Moreover assume that there exists an integer* $d > 0$ *such that* $d\widetilde{C} \in \sum \mathbb{Z} E_i$ *and* $(h)_E = d\widetilde{C}$ *for some* $h \in I_d$*. Then* $I_n = \widetilde{I}_n := \pi_* \mathcal{O}_X(-n\widetilde{C})_p$*.*

Proof First we show that $I_n \subset \widetilde{I}_n$ for every $n \geq 0$. Let $h \in I_n$ and $\Delta = (h)_E - n\widetilde{C}$. We write $\Delta = \Delta_1 - \Delta_2$, where Δ_1 and Δ_2 are effective and have no common components. Since $(h)_E \geq nC$, by the assumption on $\widetilde{C}$, we have $\mathrm{Supp}(\Delta_2) \subset \mathrm{Supp}(\widetilde{C} - C) = \mathrm{Supp}(E - C)$, and hence $\widetilde{C}\Delta_2 = 0$. If $\Delta_2 \neq 0$, then $0 < -\Delta_2^2 \leq \Delta\Delta_2 = (h)_E \Delta_2$; it contradicts that $(h)_E$ is anti-nef. Hence $\Delta = \Delta_1 \geq 0$, namely, $h \in \widetilde{I}_n$.

From the arguments in §2.2–2.4 of [36], since $\bigoplus_{n \geq 0} I_n/I_{n+1}$ is reduced, we have a $\mathbb{Q}$-cycle $D > 0$ such that $I_n = \pi_* \mathcal{O}_X(-nD)_p$ for all $n \in \mathbb{Z}_{\geq 0}$, and we may assume that $dD \in \sum \mathbb{Z} E_i$ and $\mathcal{O}_X(-dD)$ is generated. The inclusion $I_d \subset \widetilde{I}_d$ implies that $dD \geq d\widetilde{C}$. Since there exists $h \in I_d$ such that $d\widetilde{C} = (h)_E \geq dD$, we obtain $\widetilde{C} = D$.

3 Cohomology and Normal Reduction Numbers

Let $\mathfrak{m} \subset \mathcal{O}_{V,p}$ denote the maximal ideal. In the following, we always assume that $I \subset \mathcal{O}_{V,p}$ is an $\mathfrak{m}$-primary integrally closed ideal, namely, I satisfies that $\sqrt{I} = \mathfrak{m}$ and $\bar{I} = I$. Let Q be a minimal reduction of I. Then there exist a resolution $\pi : X \to V$ and a cycle $Z > 0$ such that

$$I = I_Z := \pi_* \mathcal{O}_X(-Z)_p$$

and $I\mathcal{O}_X = \mathcal{O}_X(-Z)$ (cf. [12, §6]). In this case, we say that I is *represented* by a cycle Z on X. We use the symbol "I_Z" only when $\mathcal{O}_X(-Z)$ is generated. Conversely, such an ideal I_Z is $\mathfrak{m}$-primary and integrally closed. Note that $\overline{I_Z I_{Z'}} = I_{Z+Z'}$. Thus we can write

$$\operatorname{nr}(I_Z) = \min\left\{ n \in \mathbb{Z}_{>0} \;\middle|\; I_{(n+1)Z} = QI_{nZ} \right\},$$
$$\bar{\mathrm{r}}(I_Z) = \min\left\{ n \in \mathbb{Z}_{>0} \;\middle|\; I_{(m+1)Z} = QI_{mZ},\ m \geq n \right\}.$$

In the rest of this section, we always assume that I is represented by a cycle Z on X, namely, $I = I_Z$.

Definition 3.1 We put $q(I) = q(Z) = h^1(\mathcal{O}_X(-Z))$; this is independent of the representation of I (cf. [25, Lemma 3.4]).

Proposition 3.2 (Cf. [26, §2]) *Let $q_I(n) := q(\overline{I^n}) = q_Z(n)$ for $n \geq 0$. We have the following.*

(1) *For any integer $n \geq 1$, we have*

$$2q_I(n) + \dim_{\mathbb{C}}(\overline{I^{n+1}}/Q\overline{I^n}) = q_I(n+1) + q_I(n-1).$$

In particular,

$$\operatorname{nr}(I) = \min\left\{ n \in \mathbb{Z}_{\geq 0} \mid q_I(n-1) - q_I(n) = q_I(n) - q_I(n+1) \right\}.$$

(2) *We have*

$$\bar{\mathrm{r}}(I) = \min\left\{ n \in \mathbb{Z}_{\geq 0} \mid q_I(n-1) = q_I(n) \right\}.$$

In particular, $\bar{\mathrm{r}}(I) = s(Z) + 1 \leq p_g(V,p) + 1$ and $q_I(n) = q_I(s(Z))$ for every $n \geq s(Z)$.

Proof We write $H^i(Z) := H^i(\mathcal{O}_X(-Z))$. Let $h_1, h_2 \in H^0(Z)$ and $Q := (h_1, h_2) \subset \mathcal{O}_{V,p}$. Suppose that h_1, h_2 are sufficiently general so that Q is a minimal reduction of $I = I_Z$ and that the following sequence is exact:

$$0 \to \mathcal{O}_X(-(n-1)Z) \xrightarrow{(h_1\ h_2)} \mathcal{O}_X(-nZ)^{\oplus 2} \xrightarrow{\binom{-h_2}{h_1}} \mathcal{O}_X(-(n+1)Z) \to 0.$$

Taking cohomology, we obtain the long exact sequence:

$$0 \to \overline{I^n}Q \to \overline{I^{n+1}} \to H^1((n-1)Z) \to H^1(nZ)^{\oplus 2} \to H^1((n+1)Z) \to 0.$$

This yields (1). We write

$$\dim_{\mathbb{C}}(\overline{I^{n+1}}/Q\overline{I^n}) = \Delta_I(n-1) - \Delta_I(n) \geq 0,$$

where $\Delta_I(n) = q_I(n) - q_I(n+1)$. By Proposition 2.6 (1), $\Delta_I(n) \geq 0$. Therefore, if $\Delta_I(n-1) = 0$, then $\Delta_I(n+k) = 0$ for $k \geq 0$. Hence we have (2).

By the argument similar to the proof of Proposition 3.2, we have

Proposition 3.3 ([28, 2.9]) *Let* $r = \mathrm{nr}(I)$. *Then*

$$r(r-1)/2 + q(r) \leq p_g(V, p).$$

In [28, 3.13], the hypersurface $V = \{x^a + y^b + z^c = 0\} \subset \mathbb{C}^3$ with $p_g(V, o) = r(r-1)/2$ are classified.

Remark 3.4 Let $X \to Y$ be the contraction of $Z^{\perp}$ (cf. Remark 2.4). Then we obtain that $\bar{\mathrm{r}}(I) - 1 = \min\left\{n \in \mathbb{Z}_{\geq 0} \;\middle|\; H^1(I^n\mathcal{O}_Y) = 0\right\}$ (cf. [24, 3.8]).

Remark 3.5 The ideal I is called the p_g*-ideal* if $q(I) = p_g(V, p)$. It immediately follows from Proposition 2.6 that $\bar{\mathrm{r}}(I) = 1$ if and only if I is a p_g-ideal. Moreover, the following are equivalent (see [25, 3.10], [26, 4.1]):

- I is a p_g-ideal.
- $\mathcal{O}_C(-Z) \cong \mathcal{O}_C$, where C is the cohomological cycle of E.
- The Rees algebra $\bigoplus_{n \geq 0} I^n$ is a Cohen-Macaulay normal domain.

The p_g-ideals have nice properties and studied in [25–27]. For example, if I is a p_g-ideal and J an $\mathfrak{m}$-primary integrally closed ideal of $\mathcal{O}_{V,p}$, then $IJ = \overline{IJ}$ and $q(IJ) = q(J)$; in particular, p_g-ideals form a semigroup with respect to the product (cf. [25, 2.6, 3.5]).

The singularity (V, p) is said to be *rational* if $p_g(V, p) = 0$. Rational surface singularities can be characterized in many ways [1, 10, 12, 22, 27]. We have also a characterization in terms of the normal reduction numbers as follows.

Proposition 3.6 ([29, 1.1]) *The following are equivalent:*

(1) (V, p) *is a rational singularity.*
(2) *Every* $\mathfrak{m}$*-primary integrally closed ideal in* $\mathcal{O}_{V,p}$ *is a* p_g*-ideal.*

(3) $\bar{r}(V, p) = 1$.
(4) $\mathrm{nr}(V, p) = 1$.

Remark 3.7 The singularities with $\bar{r}(\mathfrak{m}) = 1$ ($\mathfrak{m}$ is a p_g-ideal in this case) have been characterized in [31, 5.2]. In case (V, p) is Gorenstein and $p_g(V, p) > 0$, the condition $\bar{r}(\mathfrak{m}) = 1$ implies that (V, p) is an elliptic double point (see [26, 4.3], [24, 4.10]).

The elliptic singularities were introduced by P. Wagreich, and the theory of those singularities were developed by Wagreich [37], H. Laufer [11], M. Reid [32, §4], S.S.-T. Yau [38–41], M. Tomari [34, 35], and A. Némethi [21], Nagy–Némethi [17, 18].

Let Z_f denote the *fundamental cycle* on X, namely, the minimal non-zero anti-nef cycle. The *fundamental genus* $p_f(V, p)$ is defined by $p_f(V, p) = p_a(Z_f) = 1 - \chi(\mathcal{O}_{Z_f})$. By the Riemann-Roch formula, $p_f(V, p) = Z_f(Z_f + K_X)/2 + 1$. This is independent of the choice of a resolution, and hence a topological invariant of the singularity (V, p).

Definition 3.8 The singularity (V, p) is said to be *elliptic* if $p_f(V, p) = 1$.

The following are well-known:

(1) For any positive integer m, there exists an elliptic singularity (V, p) with $p_g(V, p) = m$ (Yau [41, §2]).
(2) For any elliptic surface singularity (V', p'), there exists an elliptic singularity (V, p) with $p_g(V, p) = 1$ such that (V', p') and (V, p) have the same topological type (Laufer [11, Theorem 4.1]).

Theorem 3.9 (See [24, §3]) *If* (V, p) *is elliptic, then* $\mathrm{nr}(V, p) = \bar{r}(V, p) = 2$. *In fact,* $s(W) = 1$ *for any* $(Y, W) \in \mathcal{A}'$.

The point of the proof of Theorem 3.9 is as follows. Using Yau's elliptic sequences and Röhr's vanishing theorem [33], we have

Proposition 3.10 (Cf. [24, 3.11]) *If* (V, p) *is elliptic and* $W > 0$ *is a cycle on* X *such that* $\mathcal{O}_X(-W)$ *has no fixed component, then* $h^1(\mathcal{O}_X(-W)) = h^1(\mathcal{O}_{C_W})$, *where* C_W *is the cohomological cycle of* $W^\perp$.

This proposition implies that $h^1(\mathcal{O}_{C_Z}) = q_Z(n)$ for $n \geq 1$ (take $W = nZ$). If I is not a p_g-ideal, then $s(Z) = 1$, and $\bar{r}(I) = 2$ by Proposition 3.2 (2).

Proposition 3.11 (cf. [24, 3.12]) *If* (V, p) *is elliptic, then* $q(\mathcal{A}) = q(\mathcal{A}')$.

Proof By Proposition 2.7, there exist a resolution Y and cycles $W_0, \ldots, W_{p_g(V,p)}$ on Y such that $q(W_i) = i$. Since $s(W_i) = 1$, Propositions 2.6 and 3.10 imply that $\mathcal{O}_Y(-2W_i)$ is generated and $q(W_i) = q(2W_i)$.

Problem 3.1 Characterize the singularities (V, p) with $\bar{r}(V, p) = 2$. Is the converse of Theorem 3.9 true?

We define a topological invariant min-$p_g(V, p)$ to be the minimum of the geometric genus p_g of normal complex surface singularities homeomorphic to (V, p). For example, if (V, p) is elliptic, then $\bar{\mathrm{r}}(V, p) - 1 = 1 =$ min-$p_g(V, p)$ by Theorem 3.9 and Laufer's result mentioned above. Let us recall that $\bar{\mathrm{r}}(V, p) \leq p_g(V, p) + 1$ (Proposition 3.2).

Problem 3.2 For a normal complex surface singularity (V, p), does the inequality $\bar{\mathrm{r}}(V, p) \leq$ min-$p_g(V, p) + 1$ hold? Characterize singularities which satisfy $\bar{\mathrm{r}}(V, p) =$ min-$p_g(V, p) + 1$.

4 Cone-Like Singularities

If C is a nonsingular projective curve over $\mathbb{C}$ and D an ample divisor on C, then $V(C, D) := \operatorname{Spec} \bigoplus_{n \geq 0} H^0(\mathcal{O}_C(nD))$ is a normal surface with at most an isolated singularity at the "vertex"(cf. [30]). Such a singularity is called a *cone singularity*. The exceptional set of the minimal resolution of $V(C, D)$ is isomorphic to C with self-intersection number $-\deg D$. For example, if $R = \oplus_{n \geq 0} R_n$ is a two-dimensional normal graded ring generated by R_1 over $R_0 = \mathbb{C}$, then $\operatorname{Spec} R$ has a cone singularity.

Definition 4.1 Let $\pi_0 \colon X_0 \to V$ be the minimal resolution of the singularity (V, p) and F the exceptional set of π_0. We call (V, p) a *cone-like singularity* if F consists of a unique smooth curve. Note that in this case (V, p) is homeomorphic to the cone singularity $(V(F, -F|_F)$, vertex).

In the rest of this section, we always assume that (V, p) is a cone-like singularity. Let g denote the genus of the exceptional curve F of the minimal resolution $\pi_0 \colon X_0 \to V$ and let $d = -F^2$. Assume that $g \geq 1$. Let $\pi \colon X \to V$ be any resolution with exceptional set E as in the preceding section. Then we have a natural morphism $X \to X_0$. We denote by $E_0 \subset X$ the proper transform of F; this is the unique irreducible exceptional curve on X with positive genus. Note that $d = -Z_f^2$ because F is the fundamental cycle on X_0; the number d is sometimes called the degree of (V, p).

Definition 4.2 Let C be a nonsingular projective curve. The *gonality* of the curve C is the minimum of the degree of surjective morphisms from C to $\mathbb{P}^1$, and denoted by $\mathrm{gon}(C)$. It is known that $\mathrm{gon}(F) \leq \lfloor (g + 3)/2 \rfloor$.

Definition 4.3 For any $\alpha \in \mathbb{R}$, let $[[\alpha]] = \min\{m \in \mathbb{Z} \mid m > \alpha\}$. For example, $[[2]] = [[5/2]] = 3$.

We give an upper bound for $\bar{\mathrm{r}}(V, p)$ using the invariants $g, d, \mathrm{gon}(E_0)$. Note that g and d are topological invariant of (V, p), but $\mathrm{gon}(E_0)$ is not.

Theorem 4.4 ([29, 3.9]) *Let (V, p) be a cone-like singularity and let $I = I_Z$ be an $\mathfrak{m}$-primary integrally closed ideal represented by a cycle Z on the resolution X. Then we have the following.*

(1) *If $ZE_0 = 0$, then* $\bar{\mathrm{r}}(I) \leq [[(2g-2)/d]] + 1$.
(2) *If $ZE_0 < 0$, then* $\bar{\mathrm{r}}(I) \leq [[(2g-2)/\operatorname{gon}(E_0)]] + 1$.

In particular, $\bar{\mathrm{r}}(V, p) \leq [[(2g-2)/\min\{d, \operatorname{gon}(E_0)\}]] + 1$.

For the proof we apply Röhr's vanishing theorem (see [29, §3] for the details). The following example is a special case of [29, 3.10] (take $b = g$).

Example 4.5 Let C be a hyperelliptic curve with genus $g \geq 2$ and D_0 a divisor on C which is the pull-back of a point via the double cover $C \to \mathbb{P}^1$. Let $D = gD_0$ and $V = \operatorname{Spec} \bigoplus_{n\geq 0} H^0(X, \mathcal{O}_C(nD))$. Then $C \cong F \subset X_0$. We have $p_g(V, p) = g$ by Pinkham [30, Theorem 5.7].

If we take a general element $h \in H^0(\mathcal{O}_{X_0}(-F))$, then $\operatorname{div}_{X_0}(h) = F + H$, where H is the non-exceptional part and $F \cap H$ consists of distinct $2g$ points $P_1, \ldots, P_{2g}$. We may assume that $P_1 + P_2 \sim D_0$. Let $\phi\ X \to X_0$ be the blowing-up with center $\{P_3, \ldots, P_{2g}\}$ and let $Z = (h)_E$, the exceptional part of $\operatorname{div}_X(h)$. If we put $E_i = \phi^{-1}(P_i)$ for $3 \leq i \leq 2g$, then $Z = E_0 + 2(E_3 + \cdots + E_{2g})$. We can see that $\mathcal{O}_X(-Z)$ is generated since a general element of $H^0(\mathcal{O}_X(-2F))$ has no zero on H.

Then we have $h^1(\mathcal{O}_X(-(g-1)Z)) \geq h^1(\mathcal{O}_{E_0}(-(g-1)Z)) = h^1(K_C) = 1$ and $H^1(\mathcal{O}_X(-gZ)) = 0$. It follows from Proposition 2.6 (1) and Proposition 3.2 (2) that $q_Z(n) = g - n$ for $0 \leq n \leq g$. Hence we have $\bar{\mathrm{r}}(I_Z) = p_g(V, p) + 1 = [[(2g-2)/\operatorname{gon}(E_0)]] + 1$, $\operatorname{nr}(I_Z) = 1$, $q(\mathcal{A}) = q(\mathcal{A}')$.

4.1 Homogeneous Hypersurface Singularities

Assume that $V \subset \mathbb{C}^3$ is a hypersurface defined by a homogeneous polynomial $f \in \mathbb{C}[x, y, z]$ with degree $d \geq 3$ ($\deg x = \deg y = \deg z = 1$) having an isolated singularity at the origin $p \in \mathbb{C}^3$. Then $F \cong \{f = 0\} \subset \mathbb{P}^2$, $g = (d-1)(d-2)/2$. Let $D = -F|_F$. Then $V = \operatorname{Spec} \bigoplus_{n\geq 0} H^0(\mathcal{O}_C(nD))$. Since $\mathfrak{m} = I_F$, we have

$$q_F(n) = h^1(\mathcal{O}_Y(-nF)) = \sum_{m\geq n} h^1(\mathcal{O}_F(mD)) = \sum_{m=n}^{d-3} \binom{d-1-m}{2} = \binom{d-n}{3}.$$

Hence we have $\operatorname{nr}(\mathfrak{m}) = \bar{\mathrm{r}}(\mathfrak{m}) = d - 1$ by Proposition 3.2. By the definition, $\bar{\mathrm{r}}(V, p) \geq d - 1$. On the other hand, by Namba's theorem (Max Noether's theorem) [19, Theorem 2.3.1], we have $\operatorname{gon}(F) = d - 1$. By Theorem 4.4, we have

$$\bar{\mathrm{r}}(V, p) \leq [[(2g-2)/(d-1)]] + 1 = [[d - 2 - 2/(d-1)]] + 1 = d - 1.$$

Hence we obtain

Theorem 4.6 ([29, 4.1]) $\mathrm{nr}(\mathfrak{m}) = \bar{\mathrm{r}}(\mathfrak{m}) = \mathrm{nr}(V, p) = \bar{\mathrm{r}}(V, p) = d - 1$.

Remark 4.7 (See [29, §4]) Suppose that $R = \oplus_{n\geq 0} R_n$ is a normal graded ring generated by R_1 over $R_0 = \mathbb{C}$ and $V = \operatorname{Spec} R$. Then $\mathfrak{m}^n = \overline{\mathfrak{m}^n}$. Let $a(R)$ denote the a-invariant of R (see [2]). If Q is a minimal reduction of $\mathfrak{m}$ generated by elements of R_1, we can see

$$\mathfrak{m}^{a(R)+2} \neq Q\mathfrak{m}^{a(R)+1} \quad \text{and} \quad \mathrm{nr}(\mathfrak{m}) = a(R) + 2 = \bar{\mathrm{r}}(\mathfrak{m}).$$

If $R = \mathbb{C}[x, y, z]/(f)$ as above, then $a(R) = d - 3$ (cf. [2, (3.1.6)]).

5 Brieskorn Complete Intersections

In [28], we obtained an explicit expression of $\bar{\mathrm{r}}(\mathfrak{m})$ for Brieskorn hypersurfaces using ring-theoretic arguments and gave a classification of Brieskorn hypersurfaces having elliptic singularities. In this section, we extend these results to the case of Brieskorn complete intersections, using resolution of singularities.

In the following, we assume that $V \subset \mathbb{C}^m$ is a Brieskorn complete intersection define by the following $m - 2$ polynomials:

$$q_{i1}x_1^{a_1} + \cdots + q_{im}x_m^{a_m} \quad (q_{ij} \in \mathbb{C}, \quad i = 3, \ldots, m),$$

where a_i are integers such that $2 \leq a_1 \leq \cdots \leq a_m$. We also assume that V has an isolated singularity at the origin $p \in \mathbb{C}^m$. Then, since every maximal minor of the matrix (q_{ij}) does not vanish (see [7, §7]), we may assume that

$$(q_{ij}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & p_1 & q_1 \\ 0 & 1 & \cdots & 0 & p_2 & q_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & p_{m-2} & q_{m-2} \end{pmatrix}, \tag{5.1}$$

where $p_i, q_i \neq 0$ and $p_i q_j \neq p_j q_i$ for $i \neq j$.

5.1 The Maximal Ideal Cycle, the Fundamental Cycle, and the Canonical Cycle

We summarize the results in [15] which will be used in this section; those are a natural extension of the hypersurface case obtained by Konno and Nagashima [9].

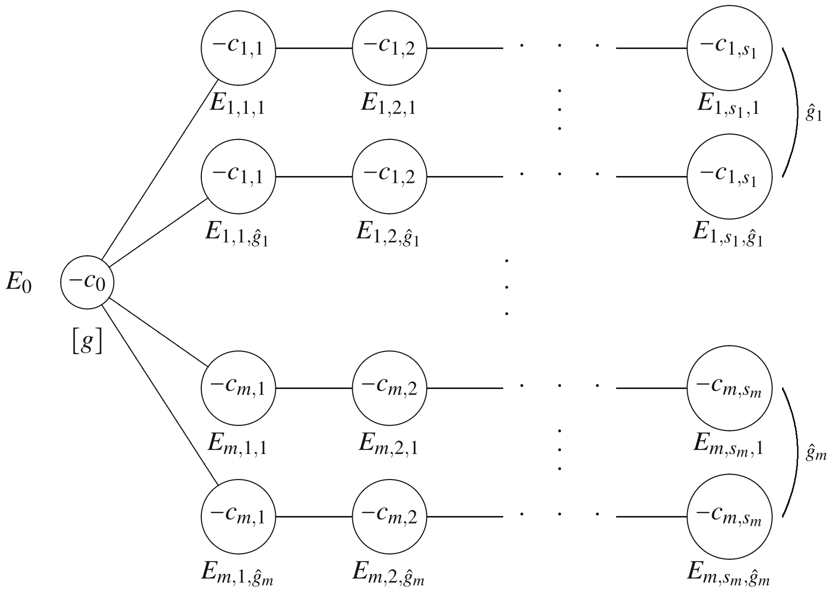

Fig. 1 The weighted dual graph of a Brieskorn complete intersection

In the following, we assume that $\pi : X \to V$ is the minimal good resolution. Since (V, p) is Gorenstein, the canonical cycle Z_{K_X} is an effective cycle.

We define positive integers ℓ, ℓ_i, α, α_i, $\hat{g}$, $\hat{g}_i$, and λ_i as follows:[2]

$$\ell := \operatorname{lcm}(a_1, \dots, a_m), \quad \ell_i := \operatorname{lcm}(a_1, \dots, \hat{a_i}, \dots, a_m), \text{ where } \hat{a_i} \text{ is omitted,}$$
$$\alpha_i := \ell/\ell_i, \quad \alpha := \alpha_1 \cdots \alpha_m, \quad \hat{g} := a_1 \cdots a_m/\ell, \quad \hat{g}_i := \hat{g}\alpha_i/a_i, \quad \lambda_i := \ell/a_i.$$

We easily see that the polynomials $x_i^{a_i} + p_i x_{m-1}^{a_{m-1}} + q_i x_m^{a_m}$ are weighted homogeneous polynomials of degree ℓ with respect to the weights $(\lambda_1, \dots, \lambda_m)$. Then the weighted dual graph of the exceptional set E is as in Fig. 1, where

$$E = E_0 + \sum_{w=1}^{m} \sum_{\nu=1}^{s_w} \sum_{\xi=1}^{\hat{g}_w} E_{w,\nu,\xi},$$

g denotes the genus of the central curve E_0, $c_0 = -E_0^2$, and $c_{w,v} = -E_{w,\nu,\xi}^2$ (see [15, 4.4]).

For any $\mathbb{Q}$-cycle B on X and any irreducible component $F \subset E$, let $\operatorname{cff}_F(B)$ denote the coefficient of F in B. Let $Z^{(i)} = (x_i)_E$.

[2]Using the notation of [15, §3], we have $l = d_m$, $\ell_i = d_{im}$, $\alpha_i = n_{im}$, $\lambda_i = e_{im}$, $\lambda_m = e_{mm} = e_m$.

Theorem 5.1 ([15, 4.4]) *We have the following:*

$$Z^{(i)} = \lambda_0^{(i)} E_0 + \sum_{w=1}^{m} \sum_{\nu=1}^{s_w} \sum_{\xi=1}^{\hat{g}_w} \lambda_{w,\nu,\xi}^{(i)} E_{w,\nu,\xi} \quad (1 \le i \le m),$$

where $\lambda_0^{(i)}$ *and the sequence* $\{\lambda_{w,\nu,\xi}^{(i)}\}$ *are determined as follows:*

$$\lambda_0^{(i)} := \lambda_{w,0,\xi}^{(i)} := \lambda_i,$$

$$\lambda_{w,s_w+1,\xi}^{(i)} := \begin{cases} 1 & \text{if } w = i \\ 0 & \text{if } w \neq i, \end{cases}$$

$$\lambda_{w,\nu-1,\xi}^{(i)} = \lambda_{w,\nu,\xi}^{(i)} c_{w,\nu} - \lambda_{w,\nu+1,\xi}^{(i)}.$$

The cycle $Z^{(i)}$ *is the smallest one among the cycles* $Z > 0$ *such that* Z *is anti-nef and* $\mathrm{cff}_{E_0}(Z) = \lambda_i$ *(cf. [15, 2.1]). In particular, we have* $M_X = Z^{(m)}$*, since* $\lambda_1 \ge \cdots \ge \lambda_m$*.*

Theorem 5.2 ([15, 5.3]) *We have*

$$Z_{K_X} = E + \frac{(m-2)l}{\alpha} Z_0 - \sum_{w=1}^{m} Z^{(w)},$$

where Z_0 *is the anti-nef cycle such that* $\mathrm{cff}_{E_0}(Z_0) = \alpha$ *and* $Z_0(E - E_0) = 0$*.*

Theorem 5.3 ([15, 5.1, 5.2, 5.4]) *If* $\lambda_m \ge \alpha$*, then* $Z_f = Z_0$ *and*

$$p_f(V, p) = \frac{1}{2}\alpha \left\{ (m-2)\hat{g} - \frac{(\alpha - 1)\hat{g}}{l} - \sum_{w=1}^{m} \frac{\hat{g}_w}{\alpha_w} \right\} + 1.$$

If $\lambda_m \le \alpha$*, then* $Z_f = M_X$ *and*

$$p_f(V, p) = \frac{1}{2}\lambda_m \left\{ (m-2)\hat{g} - \frac{(2\lceil \lambda_m/\alpha_m \rceil - 1)\hat{g}_m}{\lambda_m} - \sum_{w=1}^{m-1} \frac{\hat{g}_w}{\alpha_w} \right\} + 1.$$

5.2 The Normal Reduction Numbers

Since $M_X = (x_m)_E$ by Theorem 5.1, $\mathcal{O}_X(-M_X)$ has no fixed components; however, it is not generated in general.

Let $H = \mathrm{div}_X(x_m) - M_X$. Then $E + H$ is simple normal crossing and the set of the base points of the linear system $|\mathcal{O}_X(-M)|$ is an empty set or $\{t_1, \dots, t_{\hat{g}_m}\}$, where $\{t_\xi\} = E_{m,s_m,\xi} \cap H$ (see Theorem 5.1). Let us look in detail at a point. Let x, y be the local coordinates at $t_\xi \in X$ such that $E = \{x = 0\}$ and $H = \{y = 0\}$. We write $\eta_i = \lambda^{(i)}_{m,s_m,\xi}$ and $\delta = \eta_{m-1} - \eta_m$. Then $\delta \geq 0$ and $\mathfrak{m}\mathcal{O}_{X,t_\xi} = (x_{m-1}, x_m) = (x^{\eta_m} y, x^{\eta_{m-1}}) = x^{\eta_m}(y, x^\delta)$.

Proposition 5.4 ([15, 6.4]) *The following conditions are equivalent:*

(1) $\delta = 0$
(2) *The base points of the linear system $|\mathcal{O}_X(-M)|$ on E is empty.*

If $\delta > 0$, each base point can be resolved by a succession of δ blowing-ups at the intersection of the exceptional set and the proper transform of H.

Let $\phi \colon Y \to X$ be the minimal morphism such that $\mathfrak{m}\mathcal{O}_Y$ is invertible and let $F = \phi^{-1}(E)$. Let $W_i = (x_i)_F$ $(i = 1, \dots, m)$, and let M_Y denote the maximal ideal cycle on Y and H_Y the proper transform of H on Y. Then

$$W_i = \phi^* Z^{(i)} \quad \text{for } i \neq m, \quad W_m = M_Y = \phi^* Z^{(m)} + K_{Y/X}, \tag{5.2}$$

where $K_{Y/X} = K_Y - \phi^* K_X$. Now, $\mathfrak{m}$ is represented by M_Y and $\overline{\mathfrak{m}^n} = I_{nM_Y}$. Fix an irreducible component $F_\xi \subset F$ intersecting H_Y. For any cycle W on Y, we write $\gamma(W) = \mathrm{cff}_{F_\xi}(W)$. Note that $\gamma(M_Y)$ is independent of the choice of a component intersecting H_Y (see Theorem 5.1) and

$$\gamma(W_i) = \eta_i \quad \text{for } i \neq m, \quad \gamma(W_m) = \gamma(M_Y) = \eta_m + \delta = \eta_{m-1}. \tag{5.3}$$

Lemma 5.5 *Let $(u_1, \dots, u_m) \in (\mathbb{Z}_{\geq 0})^m$. For any positive integer n,*

$$\prod_{i=1}^{m} x_i^{u_i} \in \overline{\mathfrak{m}^n} \quad \textit{if and only if} \quad \sum_{i=1}^{m-2} \frac{u_i}{a_i} \geq \frac{n - (u_{m-1} + u_m)}{a_{m-1}}.$$

Proof We have $\left(\prod_{i=1}^m x_i^{u_i}\right)_F = W := \sum_{i=1}^m u_i W_i$. First we show that $\prod_{i=1}^m x_i^{u_i} \in \overline{\mathfrak{m}^n}$ if and only if $\gamma(W) \geq \gamma(nM_Y)$. Clearly, if $W \geq nM_Y$, then $\gamma(W) \geq \gamma(nM_Y)$. So we show the converse. Let $W - nM_Y = D_1 - D_2$, where D_1 and D_2 are effective cycles without common components. By the assumption, D_2 has no components of F intersecting H_Y. Thus $M_Y D_2 = 0$. Then $0 \leq D_1 D_2 - D_2^2 = W D_2 \leq 0$. Hence $D_2 = 0$. We have proved the claim.

We have the following (see [9, Lemma 1.2 (4)] for the first equality):

$$\eta_i = \lambda_i / \alpha_m = \ell / a_i \alpha_m \quad (1 \leq i \leq m-1). \tag{5.4}$$

Then we have

$$\begin{aligned}\gamma(W)-\gamma(nM_Y)&=\sum_{i=1}^{m-1}u_i\eta_i+u_m\eta_{m-1}-n\eta_{m-1}\\&=\frac{l}{\alpha_m}\left(\sum_{i=1}^{m-2}\frac{u_i}{a_i}+\frac{u_{m-1}+u_m-n}{a_{m-1}}\right).\end{aligned}$$

This implies the assertion.

Let $P\subset A:=\mathbb{C}[x_1,\dots,x_m]$ denote the ideal generated by the polynomials $\left\{x_i^{a_i}+p_ix_{m-1}^{a_{m-1}}+q_ix_m^{a_m}\mid i=1,\dots,m-2\right\}$ defining $V\subset\mathbb{C}^m$. For simplicity, let P also denote the ideal in $\mathbb{C}\{x_1,\dots,x_m\}$ generated by these polynomials; so $\mathcal{O}_{V,p}=\mathbb{C}\{x_1,\dots,x_m\}/P$. We easily see the following (cf. [23, Theorem 3.1]).

Lemma 5.6 *For any $1\le i\le m$, the quotient ring $A/(P+(x_i))$ is reduced.*

Proposition 5.7 *For $n\in\mathbb{Z}_{\ge 0}$, let $I_n\subset\mathcal{O}_{V,p}$ be an ideal generated by monomials $\prod_{i=1}^m x_i^{u_i}$ such that*

$$\sum_{i=1}^{m-2}\frac{u_i}{a_i}\ge\frac{(n/\eta_{m-1})-(u_{m-1}+u_m)}{a_{m-1}}.$$

Then $I_{n\eta_{m-1}}=\overline{\mathfrak{m}^n}$ for $n\in\mathbb{Z}_{\ge 0}$. In particular, $\overline{\mathfrak{m}^n}$ is generated by monomials.

Proof First we show that $G:=\bigoplus_{n\ge 0}I_n/I_{n+1}$ is reduced. It follows from (5.3) and (5.4) that the inequality is equivalent to the following (cf. the proof of Lemma 5.5):

$$\gamma\left(\left(\prod_{i=1}^m x_i^{u_i}\right)_F\right)=\sum_{i=1}^{m-1}u_i\eta_i+u_m\eta_{m-1}\ge n. \tag{5.5}$$

Therefore the filtration $\{I_n\}_{n\in\mathbb{Z}_{\ge 0}}$ is induced from the weight filtration of the power series ring $\mathbb{C}\{x_1,\dots,x_m\}$ with weight vector $(\eta_1,\dots,\eta_{m-1},\eta_{m-1})\in\mathbb{Z}^m$. Let $I\subset A=\mathbb{C}[x_1,\dots,x_m]$ denote the ideal generated by the leading form, with respect to these weights, of the polynomials $\left\{x_i^{a_i}+p_ix_{m-1}^{a_{m-1}}+q_ix_m^{a_m}\mid i=1,\dots,m-2\right\}$. Then A/I is complete intersection and isomorphic to G (cf. the proof of [23, Theorem 2.6]). If $a_{m-1}=a_m$, then $G=A/P$. If $a_{m-1}<a_m$, then $G\cong(A/P+(x_m))[x_m]$, and thus G is reduced by Lemma 5.6.

Let $C=\sum_{\xi=1}^{\hat{g}_m}F_\xi$, the sum of the irreducible components of F intersecting H_Y. From (5.5), every $h\in I_n$ satisfies $(h)_F\ge nC$. Now we can apply Lemma 2.9. Since $\eta_{m-1}\widetilde{C}=M_Y$, we obtain that $I_{n\eta_{m-1}}=\overline{\mathfrak{m}^n}$.

Let $Q=(x_{m-1},x_m)\subset\mathcal{O}_{V,p}$. Then $x_i^{a_i}\in Q$ for every i, and thus Q is a minimal reduction of $\mathfrak{m}$ (cf. [5, 8.3.6]).

Theorem 5.8 *We have the following.*

(1)

$$\mathrm{nr}(\mathfrak{m}) = \bar{\mathrm{r}}(\mathfrak{m}) = \left\lfloor a_{m-1} \sum_{i=1}^{m-2} \frac{a_i - 1}{a_i} \right\rfloor.$$

(2) *The image of the monomials* $\prod_{i=1}^{m-2} x_i^{u_i}$ *such that*

$$\sum_{i=1}^{m-2} \frac{u_i}{a_i} \geq \frac{n+1}{a_{m-1}} \quad \textit{and} \quad 0 \leq u_i \leq a_i - 1 \quad (i = 1, \dots, m-2)$$

in the vector space $\overline{\mathfrak{m}^{n+1}}/Q\overline{\mathfrak{m}^n}$ *form a basis. In particular,* $\dim_{\mathbb{C}}(\overline{\mathfrak{m}^{n+1}}/Q\overline{\mathfrak{m}^n})$ *is a non-increasing function of* n.

Proof Note that $Q\overline{\mathfrak{m}^n}$ and $\overline{\mathfrak{m}^{n+1}}$ are generated by monomials for every $n \geq 0$ by Proposition 5.7. Let $N = \left\lfloor a_{m-1} \sum_{i=1}^{m-2} (a_i - 1)/a_i \right\rfloor$. First we prove that $Q\overline{\mathfrak{m}^n} = \overline{\mathfrak{m}^{n+1}}$ for $n \geq N$. Let $v = \prod_{i=1}^{m} x_i^{u_i} \in \overline{\mathfrak{m}^{n+1}}$. By Lemma 5.5, we have

$$\sum_{i=1}^{m-2} \frac{u_i}{a_i} \geq \frac{n+1-(u_{m-1}+u_m)}{a_{m-1}} = \frac{n-(u_{m-1}+u_m-1)}{a_{m-1}}.$$

Therefore, if $u_{m-1} \geq 1$ or $u_m \geq 1$, we have $v/x_{m-1} \in \overline{\mathfrak{m}^n}$ or $v/x_m \in \overline{\mathfrak{m}^n}$, and hence $v \in Q\overline{\mathfrak{m}^n}$. We consider the case that $u_{m-1} = u_m = 0$ and $u_i \geq a_i$ for some $1 \leq i \leq m-2$; we may assume that $i = 1$. Then it follows that $x_1^{u_1} \in x_1^{u_1 - a_1}(x_{m-1}^{a_{m-1}}, x_m^{a_m})$, since $x_1^{a_1} + p_1 x_{m-1}^{a_{m-1}} + q_1 x_m^{a_m} = 0$. We show that $w_1 := (x_1^{u_1-a_1} x_{m-1}^{a_{m-1}}) \prod_{i=2}^{m-2} x_i^{u_i} \in Q\overline{\mathfrak{m}^n}$. Let $w' = w_1/x_{m-1} = (x_1^{u_1-a_1} x_{m-1}^{a_{m-1}-1}) \prod_{i=2}^{m-2} x_i^{u_i}$. Since

$$\frac{u_1 - a_1}{a_1} + \sum_{i=2}^{m-2} \frac{u_i}{a_i} \geq \frac{n+1}{a_{m-1}} - 1 = \frac{n-(a_{m-1}-1)}{a_{m-1}},$$

we have $w' \in \overline{\mathfrak{m}^n}$ by Lemma 5.5. Thus $w_1 = x_{m-1} w' \in Q\overline{\mathfrak{m}^n}$. In a similar way, we also have that $w_2 := (x_1^{u_1-a_1} x_m^{a_m}) \prod_{i=2}^{m-2} x_i^{u_i} \in Q\overline{\mathfrak{m}^n}$, since $\frac{n-(a_{m-1}-1)}{a_{m-1}} \geq \frac{n-(a_m-1)}{a_{m-1}}$. Hence we obtain that $v \in (w_1, w_2) \subset Q\overline{\mathfrak{m}^n}$. Next assume that $u_{m-1} = u_m = 0$ and $u_i < a_i$ for $1 \leq i \leq m-2$. Then we have

$$\sum_{i=1}^{m-2} \frac{a_i - 1}{a_i} \geq \sum_{i=1}^{m-2} \frac{u_i}{a_i} \geq \frac{n+1}{a_{m-1}}.$$

However this implies that $n \le N-1$. Hence we obtain that $Q\overline{\mathfrak{m}^n} = \overline{\mathfrak{m}^{n+1}}$ for $n \ge N$.

Next we prove that $Q\overline{\mathfrak{m}^{N-1}} \ne \overline{\mathfrak{m}^N}$. Let $v := \prod_{i=1}^{m-2} x_i^{a_i-1}$. Then $v \notin Q$, because $\mathcal{O}_{V,p}/Q = \mathbb{C}\{x_1,\dots,x_m\}/(x_1^{a_1},\dots,x_{m-2}^{a_{m-2}}, x_{m-1}, x_m)$. However, since

$$\sum_{i=1}^{m-2} \frac{a_i-1}{a_i} \ge \frac{N}{a_{m-1}},$$

we have $v \in \overline{\mathfrak{m}^N}$ by Lemma 5.5. Hence we obtain that $\bar{\mathrm{r}}(\mathfrak{m}) = N$.

From the arguments above, we see that (2) holds, because any non-trivial linear combinations of those monomials is not in the ideal $P + (x_{m-1}, x_m) = (x_1^{a_1},\dots,x_{m-2}^{a_{m-2}}, x_{m-1}, x_m)$. Since $\dim_{\mathbb{C}}(\overline{\mathfrak{m}^{n+1}}/Q\overline{\mathfrak{m}^n})$ is a non-increasing function of n, we have $\mathrm{nr}(\mathfrak{m}) = \bar{\mathrm{r}}(\mathfrak{m})$ (cf. Proposition 3.2).

Example 5.9 If $m = 3$, we have

$$\mathrm{nr}(\mathfrak{m}) = \left\lfloor \frac{a_2(a_1-1)}{a_1} \right\rfloor,$$

$$\begin{aligned}\dim_{\mathbb{C}}(\overline{\mathfrak{m}^{n+1}}/Q\overline{\mathfrak{m}^n}) &= \sharp\left\{ u \in \mathbb{Z} \;\middle|\; \frac{a_1(n+1)}{a_2} \le u \le a_1 - 1 \right\} \\ &= \max\left(a_1 - \left\lceil \frac{a_1(n+1)}{a_2} \right\rceil, 0 \right).\end{aligned}$$

The formula for $q(\mathfrak{m})$ in [28, 3.8] is generalized as follows.

Proposition 5.10 *Let* $p(n+1) = \dim_{\mathbb{C}}(\overline{\mathfrak{m}^{n+1}}/Q\overline{\mathfrak{m}^n})$ *and* $q(n) = h^1(\mathcal{O}_Y(-nM_Y))$ *for* $n \ge 0$. *Then we have the following:*

$$q(n) = p_g(V,p) + \frac{n}{2}(M_Y^2 - M_Y K_Y) + \sum_{i=1}^{n}(n+1-i)p(i).$$

(Note that the same formula holds for any normal surface singularity.)

Proof It is well-known that the multiplicity of $\mathcal{O}_{V,p}$ coincides with $\dim_{\mathbb{C}} \mathcal{O}_{V,p}/Q$ (e.g., [5, 11.2.2]). Thus we have $p(1) = \dim_{\mathbb{C}}(\mathfrak{m}/Q) = -M_Y^2 - 1$. From the exact sequence

$$0 \to \mathcal{O}_Y(-M_Y) \to \mathcal{O}_Y \to \mathcal{O}_{M_Y} \to 0,$$

we have

$$q(1) - q(0) = \chi(\mathcal{O}_{M_Y}) - 1 = \chi(\mathcal{O}_{M_Y}) + M_Y^2 + p(1) = \frac{1}{2}(M_Y^2 - M_Y K_Y) + p(1).$$

For $n \geq 1$, it follows from Proposition 3.2 (1) that

$$q(n) - q(n-1) = q(1) - q(0) + \sum_{i=2}^{n} p(i) = \frac{1}{2}(M_Y^2 - M_Y K_Y) + \sum_{i=1}^{n} p(i).$$

Hence we obtain

$$q(n) - q(0) = \frac{n}{2}(M_Y^2 - M_Y K_Y) + \sum_{i=1}^{n}(n+1-i)p(i). \qquad \square$$

Remark 5.11 The invariant $M_Y^2 - M_Y K_Y$ can be computed from $a_1, \ldots, a_m$ as follows. First we have $M_Y^2 = -\mathrm{mult}(V, p) = -\prod_{i=1}^{m-2} a_i$ (see [15, 6.3]). On the other hand, from (5.2), we have

$$M_Y^2 + M_Y K_Y = M_X^2 + M_X K_X + 2(K_{Y/X})^2 = 2p_a(M_X) - 2 - 2\delta \hat{g}_m.$$

We have seen a formula for $p_a(M_X)$ in Theorem 5.3.

5.3 Elliptic Singularities of Brieskorn Type

We classify the exponents $(a_1, \ldots, a_m)$ such that (V, p) is elliptic, applying the formula for $\bar{\mathrm{r}}(\mathfrak{m})$.

Theorem 5.12 *(V, p) is elliptic if and only if $(a_1, \ldots, a_m)$ is one of the following.*

(1) $(a_1, a_2, a_3) = (2, 3, a)$, $a \geq 6$.
(2) $(a_1, a_2, a_3) = (2, 4, a)$, $a \geq 4$.
(3) $(a_1, a_2, a_3) = (2, 5, a)$, $5 \leq a \leq 9$.
(4) $(a_1, a_2, a_3) = (3, 3, a)$, $a \geq 3$.
(5) $(a_1, a_2, a_3) = (3, 4, a)$, $4 \leq a \leq 5$.
(6) $(a_1, a_2, a_3, a_4) = (2, 2, 2, a)$, $a \geq 2$.

Proof For the case (1)–(6) in the theorem, we can check that $\alpha \geq \lambda_m$ and obtain $p_f(V, p) = 1$ using Theorem 5.3.

Assume that (V, p) is elliptic. By Theorems 3.9 and 5.8, we have

$$3 > a_{m-1} \sum_{i=1}^{m-2} \frac{a_i - 1}{a_i} \geq a_{m-1}(m-2)/2 \geq m - 2.$$

Hence $m \leq 4$. We first consider the case $m = 4$. We have $a_3 < 3$, and thus $a_1 = a_2 = a_3 = 2$. Then $\alpha/\lambda_4 = a_4/2 \geq 1$ and $p_f(V, p) = 1$ by Theorem 5.3,

Next assume that $m = 3$. Then we have $\bar{r}(\mathfrak{m}) = \left\lfloor \dfrac{(a_1 - 1)a_2}{a_1} \right\rfloor \leq 2$, and thus $(a_1 - 1)(a_2 - 3) \leq 2$. If $a_2 = 2$, then $a_1 = 2$ and (V, p) is a rational. Hence $a_2 \geq 3$ and the list of (a_1, a_2) is as follows:

$$(2, 3), (2, 4), (2, 5), (3, 3), (3, 4).$$

We can see that $\alpha \geq \lambda_3$ for those cases. So it follows from Theorem 5.3 that

$$p_f(V, p) = \frac{1}{2}\{a_1a_2 - a_1 - a_2 - (2\lceil \mathrm{lcm}(a_1, a_2)/a_3 \rceil - 1)\gcd(a_1, a_2)\} + 1.$$

Let us look at each case.

(1) The case where $(a_1, a_2) = (2, 3)$. We know that (V, p) is rational if $a_3 \leq 5$. Hence $a_3 \geq 6$. We have $\alpha/\lambda_3 = a_3/\gcd(6, a_3)$ and $p_f(V, p) = 1$.
(2) The case where $(a_1, a_2) = (2, 4)$, $a_3 \geq 4$. We have $\alpha/\lambda_3 = a_3/4$ if $4 \mid a_3$, $\alpha/\lambda_3 = a_3/2$ otherwise, and $p_f(V, p) = 1$.
(3) The case where $(a_1, a_2) = (2, 5)$, $a_3 \geq 5$. We have $\alpha/\lambda_3 = a_3/\gcd(10, a_3)$ and $p_f(V, p) = 3 - \lceil 10/a_3 \rceil$. Sine $p_f(V, p) = 1$, we have $a_3 \leq 9$.
(4) The case where $(a_1, a_2) = (3, 3)$, $a_3 \geq 3$. We have $\alpha/\lambda_3 = a_3/3$ and $p_f(V, p) = 1$ for all $a_3 \geq 3$.
(5) The case where $(a_1, a_2) = (3, 4)$, $a_3 \geq 4$. We have $\alpha/\lambda_3 = a_3/\gcd(12, a_3)$ and $p_f(V, p) = 4 - \lceil 12/a_3 \rceil$. Sine $p_f(V, p) = 1$, we have $a_3 \leq 5$.

Hence we have proved the theorem.

From the proof of Theorem 5.12, we obtain that $\bar{r}(\mathfrak{m}) = 2$ and $p_f(V, p) \geq 2$ if $(a_1, a_2, a_3) = (2, 5, a)$ with $a \geq 10$ or $(3, 4, a)$ with $a \geq 6$. For the cases $(2, 5, a)$ with $a \geq 10$ and $(3, 4, a)$ with $a \geq 8$, letting $Q = (y, z^2)$ and $I = \overline{Q}$, we have $\bar{r}(I) \geq 3$. Hence we obtain the following.

Proposition 5.13 ([28, 4.5]) $\bar{r}(V, p) = 2$ *if and only if* $p_f(V, p) = 1$*, except for the cases* $(a_1, \ldots, a_m) = (3, 4, 6)$*,* $(3, 4, 7)$.

For the reader's convenience, we put some information about the two exceptional cases above. Both singularities have $p_g = 3$ and $p_f = 2$. The weighted dual graph Γ_1 (resp. Γ_2) of $\{x^3 + y^4 + z^6 = 0\}$ (resp. $\{x^3 + y^4 + z^7 = 0\}$) is as in Fig. 2.

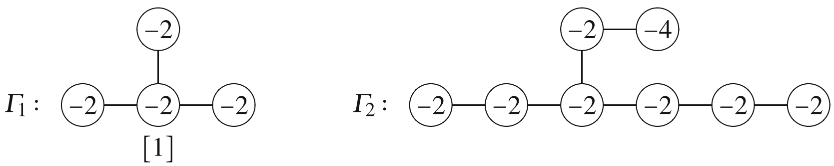

Fig. 2 The weighted dual graphs Γ_1 and Γ_2

As we have seen above, the equality $\bar{\mathfrak{r}}(V, p) = \bar{\mathfrak{r}}(\mathfrak{m})$ does not hold in general (see also Remark 3.7).

Problem 5.1 For a given normal surface singularity (V, p), characterize $\mathfrak{m}$-primary integrally closed ideals $I \subset \mathcal{O}_{V,p}$ (or, cycles which represent I) such that $\bar{\mathfrak{r}}(V, p) = \bar{\mathfrak{r}}(I)$. Characterize normal surface singularities (V, p) such that $\bar{\mathfrak{r}}(V, p) = \bar{\mathfrak{r}}(\mathfrak{m})$.

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