

Cremona Transformations of Weighted Projective Planes, Zariski Pairs, and Rational Cuspidal Curves



Enrique Artal Bartolo, José I. Cogolludo-Agustín, and Jorge Martín-Morales

To András Némethi, source of inspiration in singularity theory

Abstract In this work, we study a family of Cremona transformations of weighted projective planes which generalize the standard Cremona transformation of the projective plane. Starting from special plane projective curves we construct families of curves in weighted projective planes with special properties. We explain how to compute the fundamental groups of their complements, using the blow-up-down decompositions of the Cremona transformations, we find examples of Zariski pairs in weighted projective planes (distinguished by the Alexander polynomial). As another application of this machinery we study a family of singularities called weighted Lê–Yomdin, which provide infinitely many examples of surface singularities with a rational homology sphere link. To end this paper we also study a family of surface singularities generalizing Brieskorn–Pham singularities in a different direction. This family contains infinitely many examples of integral homology sphere links, answering a question by Némethi.

Keywords Weighted projective planes · Homology spheres · Zariski pairs

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1 Introduction

This paper deals with curves in surfaces with normal singularities and the interplay between their topological and algebraic properties.

In this direction we provide a family of examples of curves in weighted projective planes using a generalization of the classical Cremona transformations. This allows us to construct infinitely many pairs of curves in weighted projective planes defining linearly equivalent divisors and the same local type of singularities, whose embeddings are not homeomorphic. Moreover, whose complements have non-isomorphic fundamental groups. This is known in the literature as Zariski pairs when referred to plane projective curves [2] since Zariski provided the first example of such a phenomenon in [36]. The curves are obtained from a smooth cubic and three tangent lines via a weighted Cremona transformation in Sect. 2.4. These groups are distinguished using two different techniques. In Sect. 3.1 a topological approach is given by obtaining presentations of the groups. These presentations, which in general are complicated to calculate, can be derived from those of the original curve after Cremona transformations in a very explicit geometric way. To complete this example, we also present a more algebraic approach via cyclic coverings as was originally used by Zariski and later developed by Steenbrink [32, Lemma 3.14], Libgober [21], Esnault-Viehweg [14], Vaquié [35], and the first author [2]. Our method uses a generalization of [14] given in [4], see Sect. 3.3. Section 4 is devoted to developing some methods to construct rational cuspidal curves in weighted projective planes which will be useful in the later sections.

The second part of the paper focuses on local properties of surface singularities. Our main goal is to provide examples of surface germs whose link is a rational (or even integral) homology sphere. A source of examples is given by superisolated singularities. In Sect. 5 we introduce the determinant of a surface singularity as the absolute value of the determinant of the intersection matrix of a resolution. This invariant of the surface singularity can also be calculated using a partial resolution, as shown in Sect. 5.1. Note that a surface singularity has a rational homology sphere link if and only if the dual graph of a (partial) resolution is a tree whose vertices are rational curves. Moreover, a rational homology sphere link is integral if the determinant of the singularity is one. We use this criterion to study weighted Lê–Yomdin singularities and to describe infinite families with rational and integral homology sphere links.

In particular, following the ideas in [1, 26], one can use the Zariski pairs obtained in Sect. 3 to construct weighted Lê–Yomdin singularities having the same Alexander polynomials, the same abstract topology, but different embedded topology. It would be hopeless to compute the Jordan form of the complex monodromy (the actual invariant that distinguishes the embedded topology) without the use of the techniques in this paper.

The last part is devoted to solving two problems on surface singularities with a rational sphere link. Namely, in Sect. 6.1 we study Brieskorn–Pham surface singularities $\{x^a + y^b + z^c = 0\} \subset \mathbb{C}^3$ as a special case of weighted Lê–Yomdin.

We illustrate how to recover classical results in a simple way, namely to characterize which ones have a rational sphere link and show that the only integral homology spheres occur in the classical case, that is, whenever (a, b, c) are pairwise coprime. Besides Brieskorn–Pham singularities, more examples are provided in Sect. 6.2 using weighted Cremona transformations and Kummer covers.

András Némethi asked us if it was possible to find singularities with integral homology sphere links in the realm of weighted Lê–Yomdin singularities. The only ones we found are the already known Brieskorn–Pham singularities. As an alternative, in Sect. 6.3, a family of surface singularities is presented following [27], see also [29] for the *splice diagram* approach. We give conditions for this family to have a rational homology sphere link. Moreover, this family provides infinitely many examples of integral homology sphere links which may answer the question by András Némethi in the affirmative.

2 Quotient Singularities and Weighted Cremona Transformations

The main objects of this work will be weighted projective planes (and lines) and quotient singularities. A quotient singularity is a normal space which is locally isomorphic to $(X, 0)$ where X is the quotient of \mathbb{C}^n by the action of a cyclic group $\mu_m \subset \mathbb{C}^*$ given by

$$\zeta \cdot (x_1, \dots, x_n) = (\zeta^{a_1} x_1, \dots, \zeta^{a_n} x_n), \quad \zeta^m = 1, (x_1, \dots, x_n) \in \mathbb{C}^n.$$

If $\gcd(m, a_1, \dots, a_n) = 1$, the action is faithful. We denote this singularity by $\frac{1}{m}(a_1, \dots, a_n)$. There are some trivial equivalences of quotient singularities such as $\frac{1}{m}(a_1, \dots, a_n) = \frac{1}{m}(da_1, \dots, da_n)$ if $\gcd(m, d) = 1$. A less obvious one is given by

$$\frac{1}{m}(a_1, \dots, a_n) \cong \frac{d}{m} \left(a_1, \frac{a_2}{d}, \dots, \frac{a_n}{d} \right) \text{ if } d = \gcd(m, a_2, \dots, a_n)$$

(see [12] as a general reference on the subject).

2.1 Curves in Quotient Surface Singularities

We introduce some notation for germs of curves in a quotient surface singularity $S := \frac{1}{d}(a, b)$ (with a, b, d pairwise coprime and $d > 1$). Let $\pi : \mathbb{C}^2 \rightarrow S$ be the quotient map. Any germ of curve $C \subset S$ is defined as the zero locus of a non-constant equivariant germ $f \in \mathbb{C}\{x, y\}$, that is, a germ satisfying $f(\zeta \cdot (x, y)) = \zeta^k f(x, y)$ for some $k = 0, \dots, d - 1$. For a fixed k , the collection of all such

equivariant germs inherits an \mathcal{O}_S -module structure as a subset of $\mathbb{C}\{x, y\}$ and will be denoted by $\mathcal{O}_S(k)$. Note that an equivariant germ is a function on S only when $k = 0$, that is, $\mathcal{O}_S = \mathcal{O}_S(0)$.

Definition 2.1 A germ of curve C is said to be *quasi-smooth* if C is smooth as an abstract curve. If, in addition, a defining germ for C can be found to have multiplicity one, then C is said to be *extremely quasi-smooth*.

Remark 2.2 There are simple characterizations of the above concepts in terms of a minimal resolution $\hat{S} \rightarrow S$; recall that its dual graph is a bamboo whose vertices represent smooth rational divisors. A curve is quasi-smooth if its strict transform in \hat{S} is a curvette of an exceptional divisor, that is, smooth and transversal to it at a smooth point of the exceptional locus. Moreover, it is extremely quasi-smooth if this divisor is either end of the bamboo. In the particular case $\frac{1}{d}(1, 1)$, any quasi-smooth curve is extremely quasi-smooth, and any linear form can be the multiplicity-one component of f . Otherwise, in $\frac{1}{d}(a, b)$ with $(a, b) \neq (1, 1)$ the equivariant part of multiplicity 1 of an extremely quasi-smooth f can only be given by the eigenspaces of the cyclic action, in our notation, either x or y .

2.2 Weighted Projective Planes

In this section we briefly describe weighted projective planes in order to fix some notation. A *weight* is a triple $\omega := (e_1, e_2, e_3) \in \mathbb{Z}_{>0}^3$ such that $\gcd \omega = 1$. The *weighted projective plane* \mathbb{P}_ω^2 is a normal surface obtained as the quotient of $\mathbb{C}^3 \setminus \{0\}$ by the action of \mathbb{C}^* given by

$$t \cdot (x, y, z) = (t^{e_1}x, t^{e_2}y, t^{e_3}z), \quad t \in \mathbb{C}^*, (x, y, z) \in \mathbb{C}^3 \setminus \{0\}.$$

Weighted projective lines are defined in a similar way. The symbol $[x : y : z]_\omega$ stands for points in \mathbb{P}_ω^2 , for orbits in $\mathbb{C}^3 \setminus \{0\}$ or their closure in \mathbb{C}^3 . This variety is covered by three *quotient charts*. One of them is

$$\begin{aligned} \frac{1}{e_3}(e_1, e_2) &\xrightarrow{\Psi_{\omega,3}} \mathbb{P}_\omega^2 \setminus \{z = 0\} \\ [(x, y)] &\longmapsto [x : y : 1]_\omega. \end{aligned}$$

The other two quotient charts are defined accordingly.

Define $d_k := \gcd(e_i, e_j)$ and $\alpha_k := \frac{e_k}{d_i d_j}$, $\{i, j, k\} = \{1, 2, 3\}$. Note that $\eta := (\alpha_1, \alpha_2, \alpha_3)$ are pairwise coprime. According to the properties described above, the map

$$\begin{aligned} \mathbb{P}_\omega^2 &\xrightarrow{\pi_{\eta,\omega}} \mathbb{P}_\eta^2 \\ [x : y : z]_\omega &\longmapsto [x^{d_1} : y^{d_2} : z^{d_3}]_\eta \end{aligned} \tag{2.1}$$

is well defined since

$$t \cdot [x : y : z]_\omega = [t^{e_1}x : t^{e_2}y : t^{e_3}z]_\omega \mapsto [t^{d_1e_1}x^{d_1} : t^{d_2e_2}y^{d_2} : t^{d_3e_3}z^{d_3}]_\eta = t^{d_1d_2d_3} \cdot [x^{d_1} : y^{d_2} : z^{d_3}]_\eta,$$

and $d_i e_i = \alpha_i d_1 d_2 d_3$. Moreover, one can easily check that it is an isomorphism.

One may consider \mathbb{P}_ω^2 and \mathbb{P}_η^2 in a slightly different way (see also [12]). The plane \mathbb{P}_η^2 has at most 3 singular points at $P_x := [1 : 0 : 0]_\eta$ (if $\alpha_1 > 1$), $P_y := [0 : 1 : 0]_\eta$ (if $\alpha_2 > 1$), and $P_z := [0 : 0 : 1]_\eta$ (if $\alpha_3 > 1$). The plane \mathbb{P}_ω^2 is an orbifold where the quotient charts $\Psi_{\omega,i}$ are not normalized; the associated analytic variety to \mathbb{P}_ω^2 is \mathbb{P}_η^2 since the normalization of the source of $\Psi_{\omega,i}$ is precisely the source of $\Psi_{\eta,i}$.

2.3 Weighted Blow-ups

Let us consider now $\omega := (e_1, e_2) \in \mathbb{Z}_{>0}^2$, $\gcd \omega = 1$. The ω -weighted blow-up of \mathbb{C}^2 at the origin is the map $\pi_\omega : \widehat{\mathbb{C}}_\omega^2 \rightarrow \mathbb{C}^2$ where

$$\widehat{\mathbb{C}}_\omega^2 := \{(\mathbf{x}, \mathbf{u}) \in \mathbb{C}^2 \times \mathbb{P}_\omega^1 \mid \mathbf{x} \in \mathbf{u}\}.$$

This normal variety is represented with two quotient charts. One of them is

$$\widehat{\Psi}_{\omega,2} : \frac{1}{e_2}(e_1, -1) \rightarrow \widehat{\mathbb{C}}_\omega^2, \quad (x, y) \mapsto ((xy^{e_1}, y^{e_2}), [x : 1]_\omega);$$

the other one is analogous and modeled on $\frac{1}{e_1}(-1, e_2)$. The exceptional divisor of π_ω is a weighted projective line which contains the singular points $(0, [1 : 0]_\omega)$ (if $e_1 > 1$) and $(0, [0 : 1]_\omega)$ (if $e_2 > 1$) of the surface $\widehat{\mathbb{C}}_\omega^2$. Note that the curvettes of this divisor are extremely quasi-smooth if either e_1 or e_2 equal 1.

Let us study now three-dimensional weighted blow-ups. We recover the notation introduced in Sect. 2.2 for a weight ω and its normalization η , both in $\mathbb{Z}_{>0}^3$. We consider $\Pi_\omega : \widehat{\mathbb{C}}_\omega^3 \rightarrow \mathbb{C}^3$ where

$$\widehat{\mathbb{C}}_\omega^3 := \{(\mathbf{x}, \mathbf{u}) \in \mathbb{C}^3 \times \mathbb{P}_\omega^2 \mid \mathbf{x} \in \mathbf{u}\}.$$

The normal variety is now represented with three charts. One of them is

$$\widehat{\Psi}_{\omega,3} : \frac{1}{e_3}(e_1, e_2, -1) \rightarrow \widehat{\mathbb{C}}_\omega^3, \quad (x, y, z) \mapsto ((xz^{e_1}, yz^{e_2}, z^{e_3}), [x : y : 1]_\omega);$$

the other two charts can analogously be defined and have as domains the quotients $\frac{1}{e_1}(-1, e_2, e_3)$ and $\frac{1}{e_2}(e_1, -1, e_3)$.

Let us study the local structure of $\widehat{\mathbb{C}}_\omega^3$ at $E_\omega := \Pi_\omega^{-1}(0)$; since Π_ω is an isomorphism outside this exceptional divisor the points not in E_ω are smooth. Note

that E_ω is naturally isomorphic to \mathbb{P}_ω^2 ; in addition by (2.1), one has $\mathbb{P}_\omega^2 \cong \mathbb{P}_\eta^2$. For the sake of simplicity we will denote the elements of E_ω only by their ω -quasi-homogeneous coordinates. Let us denote:

$$P_x = [1 : 0 : 0]_\omega, \quad P_y = [0 : 1 : 0]_\omega, \quad P_z = [0 : 0 : 1]_\omega,$$

$$\check{X} = \{[0 : y : z]_\omega \mid yz \neq 0\}, \quad \check{Y} = \{[x : 0 : z]_\omega \mid xz \neq 0\}, \quad \check{Z} = \{[x : y : 0]_\omega \mid xy \neq 0\}.$$

In order to provide a stratification of E_ω in terms of the singular points of the ambient space we need a description of the singular locus.

Proposition 2.3 *Let $P = [x_0 : y_0 : 1]_\omega \in E_\omega \cap \widehat{\Psi}_{\omega,3}(\mathbb{C}^3) \subset \widehat{\mathbb{C}}_\omega^3$. The following properties hold:*

- (1) *If $x_0 y_0 \neq 0$ then $(\widehat{\mathbb{C}}_\omega^3, P)$ is smooth.*
- (2) *If $P \in \check{X}$, i.e. $y_0 \neq 0$ and $x_0 = 0$, then $(\widehat{\mathbb{C}}_\omega^3, P)$ is isomorphic to the germ at the origin of $\frac{1}{d_1}(e_1, 0, -1)$.*
- (3) *If $P \in \check{Y}$, i.e. $x_0 \neq 0$ and $y_0 = 0$, then $(\widehat{\mathbb{C}}_\omega^3, P)$ is isomorphic to the germ at the origin of $\frac{1}{d_2}(0, e_2, -1)$.*
- (4) *If $P = P_z$, i.e. $x_0 = y_0 = 0$, then $(\widehat{\mathbb{C}}_\omega^3, P_z)$ is isomorphic to the germ at the origin of $\frac{1}{e_3}(e_1, e_2, -1)$.*

Proof It is only necessary to prove (2). Note that P is obtained as the image by $\widehat{\Psi}_{\omega,3}$ of $(0, y_0, 0) \in \frac{1}{e_3}(e_1, e_2, -1)$. The isotropy subgroup of $(0, y_0, 0)$ by the action is the cyclic group of order $d_1 = \gcd(e_2, e_3)$. Hence at a neighborhood of $(0, y_0, 0)$ the space looks like $\frac{1}{d_1}(e_1, e_2, -1) = \frac{1}{d_1}(e_1, 0, -1)$.

Remark 2.4 A similar statement holds for the other charts. Note that a point satisfying property (2) above, say $P = [0 : 1 : 1]_\omega$ belongs in the image of $\widehat{\Psi}_{\omega,3}$, $P = \widehat{\Psi}_{\omega,3}(0, 1, 0)$ as stated in Proposition 2.3, but also in the image of $\widehat{\Psi}_{\omega,2}$, $P = \widehat{\Psi}_{\omega,2}(0, 0, 1)$. Note that the notation for the quotient types given above do no match, that is, $\frac{1}{d_1}(e_1, 0, -1)$ if considered in $\widehat{\Psi}_{\omega,3}(\mathbb{C}^3)$ and $\frac{1}{d_1}(e_1, -1, 0)$ if considered in $\widehat{\Psi}_{\omega,3}(\mathbb{C}^3)$. To avoid this ambiguity we will simply say that given $P \in \check{X}$, then $(\widehat{\mathbb{C}}_\omega^3, P)$ is isomorphic to the product of $(\mathbb{C}, 0)$ and the germ at the origin of $\frac{1}{d_1}(e_1, -1)$. A similar property holds for \check{Y} , \check{Z} .

Remark 2.5 If $d_1 = e_3$, i.e., if e_3 divides e_2 , in the proof of Proposition 2.3 the condition $y_0 \neq 0$ is not needed and P_z behaves as the points in \check{X} . A similar property holds for the other pairs of axes and vertices.

Notation 2.6 *We fix the following notation for the strata of E_ω .*

- *two-dimensional stratum. The stratum \mathcal{T} is the intersection of E_ω with the smooth subvariety of $\widehat{\mathbb{C}}_\omega^3$; it contains $\{[x : y : z]_\omega \mid xyz \neq 0\}$. It contains also \check{X} (resp. \check{Y} , resp. \check{Z}) if $d_1 = 1$ (resp. $d_2 = 1$, resp. $d_3 = 1$) and P_x (resp. P_y , resp. P_z) if $e_1 = 1$ (resp. $e_2 = 1$, resp. $e_3 = 1$).*

- *one-dimensional strata.* Following Proposition 2.3 and the above remarks we set:

$$\mathcal{L}_x = \begin{cases} \emptyset & \text{if } d_1 = 1 \\ \check{X} \cup \{P_y\} & \text{if } 1 < d_1 = e_2 \neq e_3 \\ \check{X} \cup \{P_z\} & \text{if } 1 < d_1 = e_3 \neq e_2 \\ X & \text{if } 1 < d_1 = e_2 = e_3 \\ \check{X} & \text{otherwise.} \end{cases}$$

The remaining strata \mathcal{L}_y and \mathcal{L}_z are defined accordingly.

- *zero-dimensional strata.*

$$\mathcal{P}_x = \begin{cases} \emptyset & \text{if } e_1 \text{ divides either } e_2 \text{ or } e_3, \\ \{P_x\} & \text{otherwise.} \end{cases}$$

The remaining strata \mathcal{P}_y and \mathcal{P}_z are defined accordingly.

2.4 Weighted Cremona Transformations

The most well-known Cremona transformation of \mathbb{P}^2 corresponds to the birational map $[x : y : z] \mapsto [yz : xz : xy]$; geometrically, this map is the composition of the blow-ups at $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$ and the contractions of the strict transforms of the lines $x = 0$, $y = 0$, $z = 0$ which become pairwise disjoint (-1) -lines in the blown-up plane.

In this section we generalize this transformation to a birational map from a weighted projective plane to \mathbb{P}^2 . Let us fix \mathbb{P}^2_ω , $\omega := (e_1, e_2, e_3)$, where e_1, e_2, e_3 are pairwise coprime, i.e., $\omega = \eta$. In order to stress this property we will use the notation $e_i = \alpha_i$, $i = 1, 2, 3$. Consider two positive integers β_1, β_2 such that $\alpha_1\beta_1 + \alpha_2\beta_2 = \alpha_3 + \alpha_1\alpha_2$ (they exist from standard semigroup properties). These arithmetic data provide the following map

$$\begin{array}{ccc} \mathbb{P}^2_\omega & \xrightarrow{\Phi_{\omega, \beta_1, \beta_2}} & \mathbb{P}^2 \\ [x : y : z]_\omega & \longmapsto & [y^{\alpha_1}z : x^{\alpha_2}z : x^{\beta_1}y^{\beta_2}], \end{array}$$

which is a well-defined rational map (not a morphism) since the three coordinates have ω -degree equal to $\alpha_1\alpha_2 + \alpha_3$. It is in fact a birational map whose inverse is given by

$$\begin{array}{ccc} \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2_\omega \\ [x : y : z] & \longmapsto & \left[y^{\frac{1}{\alpha_2}} z^{\frac{\alpha_1}{\alpha_3}} : x^{\frac{1}{\alpha_1}} z^{\frac{\alpha_2}{\alpha_3}} : x^{\frac{\beta_2}{\alpha_1}} y^{\frac{\beta_1}{\alpha_2}} \right]_\omega. \end{array}$$

We will show that this map is well defined as long as the radicals $x^{\frac{1}{\alpha_1}}$, $y^{\frac{1}{\alpha_2}}$, $z^{\frac{1}{\alpha_3}}$ are chosen consistently throughout the formula. Assume x_0 (resp. y_0, z_0) is such that $x_0^{\alpha_1} = x$ (resp. $y_0^{\alpha_2} = y, z_0^{\alpha_3} = z$) and choose for instance $x_1 = \zeta_{\alpha_1} x_0$. Let $\hat{\alpha}_2 \in \mathbb{Z}$ be such that $\alpha_2 \hat{\alpha}_2 \equiv 1 \pmod{\alpha_1}$. As a consequence, the following congruences hold: $\alpha_3 \hat{\alpha}_2 \equiv (\alpha_3 + \alpha_1 \alpha_2) \hat{\alpha}_2 \equiv (\alpha_1 \beta_1 + \alpha_2 \beta_2) \hat{\alpha}_2 \equiv \beta_2 \pmod{\alpha_1}$. Then

$$\begin{aligned} [y_0 z_0^{\alpha_1} : x_1 z_0^{\alpha_2} : x_1^{\beta_2} y_0^{\beta_1}]_{\omega} &= [y_0 z_0^{\alpha_1} : \zeta_{\alpha_1} x_0 z_0^{\alpha_2} : \zeta_{\alpha_1}^{\beta_2} x_0^{\beta_2} y_0^{\beta_1}]_{\omega} = \\ [(\zeta_{\alpha_1}^{\hat{\alpha}_2})^{\alpha_1} y_0 z_0^{\alpha_1} : (\zeta_{\alpha_1}^{\hat{\alpha}_2})^{\alpha_2} x_0 z_0^{\alpha_2} : (\zeta_{\alpha_1}^{\hat{\alpha}_2})^{\alpha_3} x_0^{\beta_2} y_0^{\beta_1}]_{\omega} &= [y_0 z_0^{\alpha_1} : x_0 z_0^{\alpha_2} : x_0^{\beta_2} y_0^{\beta_1}]_{\omega}. \end{aligned}$$

A similar argument applies to other choices of roots of $y^{\frac{1}{\alpha_2}}$ and $z^{\frac{1}{\alpha_3}}$. These equations completely determine the birational map, but a more geometric description will be useful.

Proposition 2.7 *The map $\Phi_{\omega, \beta_1, \beta_2}$ is the composition of the following blow-ups and downs:*

(1) *Three simultaneous blow-ups:*

- (a) *Type (α_1, α_2) at $[0 : 0 : 1]_{\omega} \cong \frac{1}{\alpha_3}(\alpha_1, \alpha_2)$.*
- (b) *Type $(1, \beta_1)$ at $[0 : 1 : 0]_{\omega}$ isomorphic to*

$$\frac{1}{\alpha_2}(\alpha_1, \alpha_3) = \frac{1}{\alpha_2}(\alpha_1, \alpha_1 \alpha_2 + \alpha_3) = \frac{1}{\alpha_2}(\alpha_1, \alpha_1 \beta_1 + \alpha_2 \beta_2) = \frac{1}{\alpha_2}(1, \beta_1).$$

- (c) *Type $(1, \beta_2)$ at $[1 : 0 : 0]_{\omega}$ isomorphic to*

$$\frac{1}{\alpha_1}(\alpha_2, \alpha_3) = \frac{1}{\alpha_1}(\alpha_2, \alpha_1 \alpha_2 + \alpha_3) = \frac{1}{\alpha_1}(\alpha_2, \alpha_1 \beta_1 + \alpha_2 \beta_2) = \frac{1}{\alpha_1}(1, \beta_2).$$

(2) *Three simultaneous blow-downs:*

- (a) *Type $(1, 1)$ at $[0 : 0 : 1]$.*
- (b) *Type (α_2, β_1) at $[1 : 0 : 0]$.*
- (c) *Type (α_1, β_2) at $[0 : 1 : 0]$.*

Proof Let us start with the three blow-ups in \mathbb{P}_{ω}^2 . We obtain a normal rational surface S . The preimage of the three axes appear in Fig. 1, containing the strict transforms L_x, L_y, L_z of the lines and the exceptional components E_x, E_y, E_z . The self-intersections and the type of the singular points are computed using [8, Theorem 4.3].

The strict transforms of the lines coincide with the exceptional components of a (α_1, β_2) -blowing-up (L_y), a (α_2, β_1) -blowing-up (L_x) and a standard blowing-up (L_z). The result of the triple blowing-down is \mathbb{P}^2 .

This geometric expression will be useful for the study of curves in \mathbb{P}_{ω}^2 via their transforms in \mathbb{P}^2 .

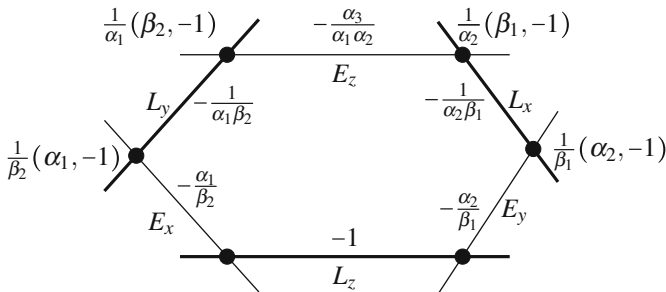


Fig. 1 Weighted blow-ups of \mathbb{P}^2 in S

3 Zariski Pairs on Weighted Projective Planes

In this section, we are going to use the Cremona transformations in Sect. 2.4 to produce Zariski pairs in weighted projective planes. By a *Zariski pair* we mean two curves embedded in the same surface whose *combinatorics* are the same, but whose embeddings are non-homeomorphic. As in the classical case of curves in the projective plane, the combinatorics of a curve in a weighted projective plane is encoded by the degrees of its irreducible components and the dual graph of a minimal resolution of the curve (where the strict transforms of the irreducible components of the curve are marked).

In this section we will produce families of Zariski pairs of irreducible curves. Let us start with the combinatorics defined by a smooth projective cubic and three tangent lines at inflection points. Note that a generic choice of a smooth cubic can be made so that such lines are non-concurrent and hence the remaining singular points are three nodes. This combinatorics admits a Zariski pair of sextics, see [2], and their embeddings are distinguished by the algebraic property of whether or not the inflection points of the cubic, that is, the three non-nodal singular points of the sextic, which have type \mathbb{A}_6 , are aligned. The image by a standard Cremona transformation of the smooth cubics (using the three tangent lines at the axes) produces a Zariski pair of irreducible sextics with three \mathbb{E}_6 -points. In this case, the embeddings can be proven to be different showing that the fundamental group of their complements are not isomorphic.

Our strategy is to replace this Cremona transformation by the inverse of those described in Sect. 2.4.

3.1 Fundamental Groups of Complements

Let us start by recalling the two possible fundamental groups of the complements of the sextic curves given as the union of a smooth cubic and three tangent lines at inflection points.

Proposition 3.1 ([3]) *Let C be a smooth cubic with three tangent lines X, Y, Z at inflections which are not aligned. Then, $\pi_1(\mathbb{P}^2 \setminus (C \cup X \cup Y \cup Z))$ is abelian.*

In [3], the fundamental group of the other member of the Zariski pair is also computed; since it is non-abelian, this invariant distinguishes the two members. For our purpose, we need a more geometrical presentation of the group involving meridians for all the irreducible components and such that the meridians close to the nodes are made explicit. Let us recall the concept of meridian in order to clarify what we mean by *meridians close to a singular point*.

Definition 3.2 Let Z be a connected quasi-projective manifold and let H be a hypersurface of Z . Consider $P \in Z \setminus H$ and K an irreducible component of H . A homotopy class $\gamma \in \pi_1(Z \setminus H; P)$ is called a *meridian about K with respect to H* if $\gamma = [\delta]$ for some loop δ satisfying the following:

- (1) there is a smooth complex analytic disk $\Delta \subset Z$ transverse to H such that $\Delta \cap H = \{P'\} \subset K$ (transversality implies that P' is a smooth point of H).
- (2) there is a path α in $Z \setminus H$ starting at P and ending at some point $P'' \in \partial\Delta$.
- (3) $\delta = \alpha * \beta * \bar{\alpha}$, where the operation $*$ here means concatenation of paths from left to right, β is the closed path obtained by traveling from P'' along $\partial\Delta$ in the positive direction and $\bar{\alpha}$ represents the path α traveled in the opposite direction, that is, $\bar{\alpha}(t) := \alpha(1 - t)$.

It is well known that meridians with respect to the same irreducible component define a conjugacy class of members of the fundamental group.

Example 3.3 Let $Z = \mathbb{C}^2$ and $H = \{xy = 0\}$ and let $P := (1, 1)$. The paths $\mu_x, \mu_y : [0, 1] \rightarrow Z \setminus H$ defined by

$$\mu_x(t) = (e^{2i\pi t}, 1), \quad \mu_y(t) = (1, e^{2i\pi t}),$$

define meridians with respect to the irreducible components of H (for which the path α is trivial). They commute as elements in the fundamental group $\pi_1(Z \setminus H; P)$. If Z is quasi-projective surface and H is a curve containing a node, *two meridians are close to the node* if there is a common path α from the base point of $\pi_1(Z \setminus H; P)$ to a point *close* to the node such that the β -paths look like in this example.

Proposition 3.4 ([6]) *Let C be a smooth cubic with three tangent lines X, Y, Z at inflections which are aligned. Then, $\pi_1(\mathbb{P}^2 \setminus (C \cup X \cup Y \cup Z))$ is*

$$\langle c, \ell_x, \ell_y, \ell_z \mid [\ell_x, \ell_y] = [\ell_y, \ell_z] = [\ell_z, \ell_x] = [c, \ell_x^{-1}\ell_z] = [c, \ell_y^{-1}\ell_z] = c\ell_x c\ell_y c\ell_z = 1 \rangle \tag{3.1}$$

where c is a meridian of \mathcal{C} , and ℓ_x, ℓ_y, ℓ_z are meridians of X, Y, Z , respectively; moreover the meridians of the lines correspond to meridians close to the double points.

Let us fix $\Phi := \Phi_{\omega, \beta_1, \beta_2}$ as in Sect. 2.4, and let us denote by $\tilde{\mathcal{C}} \subset \mathbb{P}_\omega^2$ the strict transform of the smooth cubic \mathcal{C} by Φ , where the lines X, Y, Z have equations $x = 0, y = 0, z = 0$, respectively. Consider the following three homogeneous polynomials of degree 3

$$H_\lambda(x, y, z) := x^3 + y^3 + z^3 + 3xy(\lambda^{-1}x + \lambda y) + 3xz(x + z) + 3yz(\lambda^{-1}y + \lambda z),$$

where $\lambda^3 = 1$. The curve $\mathcal{C}_\lambda = \{H_\lambda = 0\}$ is a smooth cubic which is tangent to the line L_x at the inflection point $[0 : 1 : -\lambda]$ and analogously for Y at $[-1 : 0 : 1]$, and Z at $[1 : -\lambda : 0]$. Note that for the cubic \mathcal{C}_1 the three inflection points are contained in the line $x + y + z = 0$. However, for the smooth cubic $\mathcal{C}_{\exp \frac{2i\pi}{3}}$ the three inflection points are not aligned.

Corollary 3.5 *In the non-aligned case, $\pi_1(\mathbb{P}_\omega^2 \setminus \tilde{\mathcal{C}})$ is isomorphic to $\mathbb{Z}/3(\alpha_1\alpha_2 + \alpha_3)$.*

Proof The space $\mathbb{P}_\omega^2 \setminus \tilde{\mathcal{C}}$ is homeomorphic to $S \setminus (\widehat{\mathcal{C}} \cup L_x \cup L_y \cup L_z)$ (see Fig. 1) and the space $\mathbb{P}^2 \setminus (\mathcal{C} \cup X \cup Y \cup Z)$ is homeomorphic to $S \setminus (\widehat{\mathcal{C}} \cup L_x \cup L_y \cup L_z \cup E_x \cup E_y \cup E_z)$, where $\widehat{\mathcal{C}}$ denotes the strict transform of \mathcal{C} in S . As a consequence of [17, Lemma 4.18] the kernel of the epimorphism

$$\pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup X \cup Y \cup Z)) \twoheadrightarrow \pi_1(\mathbb{P}_\omega^2 \setminus \tilde{\mathcal{C}}) \tag{3.2}$$

is the normal subgroup generated by the meridians of E_x, E_y, E_z in S . Since the source is an abelian group by Proposition 3.1, the group $\pi_1(\mathbb{P}_\omega^2 \setminus \tilde{\mathcal{C}})$ is abelian as well. Hence it coincides with $H_1(\mathbb{P}_\omega^2 \setminus \tilde{\mathcal{C}}; \mathbb{Z}) \cong \mathbb{Z}/\deg(\tilde{\mathcal{C}})$, since $\tilde{\mathcal{C}}$ contains the vertices of \mathbb{P}_ω^2 .

In order to compute the other fundamental group we need a technical result.

Lemma 3.6 *Let $\pi : \widehat{\mathbb{C}}_{(\alpha_1, \alpha_2)}^2 \rightarrow \mathbb{C}^2$ be the (α_1, α_2) -blow-up of the origin in \mathbb{C}^2 and let E denote its exceptional component. Let $X, Y \subset \mathbb{C}^2$ be the axes (curves of equations $x = 0, y = 0$, respectively), and let us keep this notation for their strict transforms. Let $U := \mathbb{C}^2 \setminus (X \cup Y) \cong \widehat{\mathbb{C}}_{(\alpha_1, \alpha_2)}^2 \setminus (E \cup X \cup Y)$.*

If μ_X, μ_Y, μ_E denote meridians of the respective curves in $\pi_1(U) \cong \mathbb{Z}\mu_X \oplus \mathbb{Z}\mu_Y$, then (multiplicative notation) $\mu_E = \mu_X^{\alpha_1} \mu_Y^{\alpha_2}$.

Proof Consider $(1, 1)$ as the base point, then μ_X is the loop $t \mapsto (e^{2i\pi t}, 1)$, while μ_Y is the loop $t \mapsto (1, e^{2i\pi t})$. Let us pick a chart of $\widehat{\mathbb{C}}_{(\alpha_1, \alpha_2)}^2$, say

$$\begin{aligned} \frac{1}{\alpha_1}(-1, \alpha_2) &\longrightarrow \mathbb{C}^2 \\ [(x, y)] &\longmapsto (x^{\alpha_1}, x^{\alpha_2}y). \end{aligned}$$

The base point in the chart is the class of $(1, 1)$; the equation of E is $x = 0$ and hence μ_E is represented by $t \mapsto [(e^{2i\pi t}, 1)]$. Hence, in \mathbb{C}^2 is represented by $t \mapsto (e^{2i\alpha_1\pi t}, e^{2i\alpha_2\pi t})$ and the result follows.

Proposition 3.7 *In the aligned case, $\pi_1(\mathbb{P}_\omega^2 \setminus \tilde{C})$ is isomorphic to $\mathbb{Z}/3(\alpha_1\alpha_2 + \alpha_3)$ if 2 divides $\alpha_1\alpha_2\alpha_3\beta_1\beta_2$ and to*

$$\langle \ell, u \mid \ell^{\alpha_1\alpha_2+\alpha_3} = 1, u^3 = \ell^2 \rangle \quad (3.3)$$

otherwise. This group is a central extension of $\mathbb{Z}/2 * \mathbb{Z}/3$ by a cyclic group of order $\frac{\alpha_1\alpha_2+\alpha_3}{2}$.

Proof Following the proof of Corollary 3.5, the epimorphism described in (3.2) also holds in this case. Hence a presentation of $\pi_1(\mathbb{P}_\omega^2 \setminus \tilde{C})$ can be given once meridians of E_x, E_y, E_z are written in terms of the generators provided in (3.1). Since the meridians ℓ_x, ℓ_y, ℓ_z of the lines in the presentation (3.1) are homotopic to meridians close to the double points, by Lemma 3.6 we have that $\ell_x\ell_y$ is a meridian of E_z , $\ell_x^\alpha \ell_z^{\beta_2}$ is a meridian of E_x , and $\ell_y^{\alpha_2} \ell_z^{\beta_1}$ is a meridian of E_y . Hence a presentation of $\pi_1(\mathbb{P}_\omega^2 \setminus \tilde{C})$ can be obtained by adding the relations

$$\ell_x\ell_y = \ell_x^\alpha \ell_z^{\beta_2} = \ell_y^{\alpha_2} \ell_z^{\beta_1} = 1 \quad (3.4)$$

to the presentation given in (3.1).

Finally, let us simplify this presentation. As a first step one can eliminate ℓ_x , since $\ell_x = \ell_y^{-1}$. Also, choose $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathbb{Z}$ such that $\alpha_2\hat{\alpha}_1 - \alpha_1\hat{\alpha}_2 = 1$. Note that ℓ_y, ℓ_z commute; then the remaining two relations in (3.4) become

$$\ell_y^{-\alpha_1} \ell_z^{\beta_2} = \ell_y^{\alpha_2} \ell_z^{\beta_1} = 1 \implies \begin{cases} 1 = \ell_z^{\alpha_1\beta_1 + \alpha_2\beta_2} = \ell_z^{\alpha_1\alpha_2 + \alpha_3}, \\ \ell_y = \ell_z^{-(\hat{\alpha}_1\beta_2 + \hat{\alpha}_2\beta_1)}. \end{cases}$$

In fact, this is an equivalence. Let us denote $\ell := \ell_z$ and $u := c\ell$. Since $[c, \ell_y\ell] = [c, \ell_y^{-1}\ell] = 1$, one has

$$\begin{aligned} 1 &= c\ell_y^{-1}c\ell_y c\ell = c\ell_y^{-1}c\ell_y \ell \ell^{-1}c\ell = c\ell_y^{-1}(\ell_y\ell)c\ell^{-1}c\ell \iff \\ 1 &= (c\ell)^2 c\ell^{-1} = (c\ell)^3 \ell^{-2} \iff \ell^2 = u^3. \end{aligned}$$

Hence $\pi_1(\mathbb{P}_\omega^2 \setminus \tilde{C})$ admits a presentation

$$\langle \ell, u \mid \ell^{\alpha_1\alpha_2+\alpha_3} = 1, [u, \ell^{\hat{\alpha}_1\beta_2 + \hat{\alpha}_2\beta_1 - 1}] = 1, u^3 = \ell^2 \rangle. \quad (3.5)$$

Note that, using $\ell^2 = u^3$, the relation $[u, \ell^{\hat{\alpha}_1\beta_2 + \hat{\alpha}_2\beta_1 - 1}] = 1$ can be either eliminated or replaced by $[u, \ell] = 1$ depending on the parity of $\hat{\alpha}_1\beta_2 + \hat{\alpha}_2\beta_1$. In addition, ℓ can also be eliminated using $\ell^{\alpha_1\alpha_2+\alpha_3} = 1$ and $u^3 = \ell^2$ in case $\alpha_1\alpha_2 + \alpha_3$ is odd. In

particular, if $\hat{\alpha}_1\beta_2 + \hat{\alpha}_2\beta_1$ is even or $\alpha_1\alpha_2 + \alpha_3$ is odd, then (3.5) becomes an abelian group. Otherwise, one obtains the presentation (3.3).

It is immediate to verify that $\hat{\alpha}_1\beta_2 + \hat{\alpha}_2\beta_1$ is odd and $\alpha_1\alpha_2 + \alpha_3$ even if and only if $\alpha_1\alpha_2\alpha_3\beta_1\beta_2$ is odd, which ends the proof.

Corollary 3.8 *The derived subgroup F of $\pi_1(\mathbb{P}_\omega^2 \setminus \tilde{\mathcal{C}})$ (in the non-abelian case) is the direct product of $\mathbb{Z}/(\frac{\alpha_1\alpha_2+\alpha_3}{2})$ and a free group of rank 2. The characteristic polynomial of the action of the monodromy on $F/F' \otimes_{\mathbb{Z}} \mathbb{C}$ is $t^2 - t + 1$.*

3.2 A Family of Zariski Pairs of Irreducible Weighted Projective Curves

Summarizing the previous section, let $\omega = (\alpha_1, \alpha_2, \alpha_3)$ be pairwise coprime positive integers, and β_1, β_2 such that $\alpha_1\beta_1 + \alpha_2\beta_2 = \alpha_1\alpha_2 + \alpha_3$. Consider \mathcal{C} a smooth projective cubic and Φ_1 (resp. Φ_2) the weighted Cremona transformation from \mathbb{P}_ω^2 to \mathbb{P}^2 with respect to three tangent lines to \mathcal{C} at aligned (resp. non-aligned) inflection points. Let us denote by $\tilde{\Phi}_i^*(\mathcal{C})$ the strict transform of \mathcal{C} by the Cremona transformation Φ_i .

Theorem 3.9 *Under the conditions above, if $\alpha_1\alpha_2\alpha_3\beta_1\beta_2$ is odd then $(\tilde{\Phi}_1^*(\mathcal{C}), \tilde{\Phi}_2^*(\mathcal{C}))$ is a Zariski pair of irreducible weighted projective curves of degree $3(\alpha_1\alpha_2 + \alpha_3)$ in \mathbb{P}_ω^2 .*

Proof Since both $\Phi_i, i = 1, 2$ are birational and \mathcal{C} is irreducible, then $\tilde{\Phi}_i^*(\mathcal{C}), i = 1, 2$ are both irreducible as well. Also, the singularities of $\tilde{\Phi}_i^*(\mathcal{C})$ are determined locally by the singularities of the union of \mathcal{C} and the lines used for the Cremona transformation Φ_i . Hence, $\tilde{\Phi}_1^*(\mathcal{C})$ and $\tilde{\Phi}_2^*(\mathcal{C})$ have the same combinatorics. Finally, if $\alpha_1\alpha_2\alpha_3\beta_1\beta_2$ is odd, then by Proposition 3.7 and Corollary 3.5 the fundamental groups of their complements are not isomorphic. This ends the proof.

3.3 Cyclic Covers and Their Irregularity à la Esnault–Viehweg

The purpose of this section is to prove Theorem 3.9 via a generalization of the Alexander polynomial method, that is, the calculation of invariants associated with cyclic covers of the weighted projective plane ramified along the curves. In particular, we will calculate the dimension of the eigenspaces of the homology in degree 1 of the cover with respect to the action of the deck transformation. This approach was originally used by Zariski [36] for sextics with six cusps in the projective plane. Later on, Libgober [21] and Esnault [13] made significant progress in this direction for cyclic covers and projective plane. Also Esnault–Viehweg [14] gave the tools that allowed the first author in [2], Sabbah [31], and

Loeser-Vaquie [23] to find descriptions of the irregularity of cyclic covers. This approach was extended by Libgober [22] for abelian covers. The approach presented here is a generalization of Esnault-Viehweg’s and was developed by the authors for cyclic covers of surfaces with abelian quotient singularities and \mathbb{Q} -resolutions (or partial resolutions) in [4].

Let $\rho : X \rightarrow \mathbb{P}_\omega^2$ be the cyclic cover of \mathbb{P}_ω^2 ramified along a reduced curve \mathcal{C} of degree d . Consider $\check{X} = \rho^{-1}(\mathbb{P}_\omega^2 \setminus (\mathcal{C} \cup \text{Sing } \mathbb{P}_\omega^2))$ the unramified part of the cover and let $\sigma : \check{X} \rightarrow \check{X}$ be a generator of the monodromy of the unramified cover.

Let $\pi : Y \rightarrow \mathbb{P}_\omega^2$ be a \mathbb{Q} -embedded resolution of \mathcal{C} . For $P \in \text{Sing } \mathcal{C}$, let Γ_P be the dual graph of the exceptional divisor of π over P . For any v vertex of Γ_P we will denote by E_v the associated exceptional divisor over P and by m_v (resp. $v_v - 1$) the coefficient of E_v in the divisor $\pi^* \mathcal{C}$ (resp. in K_π , the relative canonical divisor).

The following result describes a method to recover the dimension of the different eigenspaces of $H^1(X, \mathbb{C})$ with respect to the monodromy action (or deck transformation of the cover). A more general result can be found in [4, Theorem 4.4] for non-reduced divisors, but we state it here for covers associated with reduced divisors.

Theorem 3.10 ([4, Theorem 4.4]) *The dimension of the eigenspace of σ^* acting on $H^1(X; \mathbb{C})$ for the eigenvalue $e^{\frac{2i\pi k}{d}}$, $0 < k < d$, equals $\dim \text{coker } \pi^{(k)} + \dim \text{coker } \pi^{(d-k)}$ where*

$$\pi^{(k)} : H^0 \left(\mathbb{P}_\omega^2, \mathcal{O}_{\mathbb{P}_\omega^2} \left(kH + K_{\mathbb{P}_\omega^2} \right) \right) \longrightarrow \bigoplus_{P \in \text{Sing } \mathcal{C}} \frac{\mathcal{O}_{\mathbb{P}_\omega^2, P} \left(kH + K_{\mathbb{P}_\omega^2} \right)}{\mathcal{M}_{\mathcal{C}, P}^{(k)}},$$

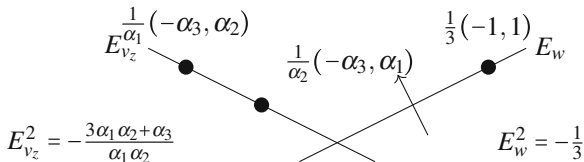
is naturally defined given H a divisor of degree 1, $K_{\mathbb{P}_\omega^2}$ denotes the canonical divisor, and $\mathcal{M}_{\mathcal{C}, P}^{(k)}$ is the following $\mathcal{O}_{\mathbb{P}_\omega^2, P}$ -module of quasi-adjunction

$$\mathcal{M}_{\mathcal{C}, P}^{(k)} := \left\{ g \in \mathcal{O}_{\mathbb{P}_\omega^2, P} \left(kH + K_{\mathbb{P}_\omega^2} \right) \mid \text{mult}_{E_v} \pi^* g > \frac{km_v}{d} - v_v, \forall v \in \Gamma_P \right\}.$$

Note that the module of quasi-adjunction $\mathcal{M}_{\mathcal{C}, P}^{(k)}$ is a submodule of the module of equivariant germs $\mathcal{O}_{\mathbb{P}_\omega^2, P}(\ell)$ for some $\ell = 0, \dots, d - 1$ as defined in Sect. 2.1, namely, ℓ is the local class of the divisor $kH + K_{\mathbb{P}_\omega^2}$ at P . Our purpose will be to calculate $\dim \text{coker } \pi^{(k)} + \dim \text{coker } \pi^{(d-k)}$ for certain k and $d = 3(\alpha_1\alpha_2 + \alpha_3)$ for the d -cyclic cover of the curves in the family presented in Sect. 3.1.

Under the conditions of Theorem 3.9, that is, $\alpha_1\alpha_2\alpha_3\beta_1\beta_2$ odd, let us consider the curve $\tilde{\mathcal{C}}_\lambda := \tilde{\Phi}_{\omega, \beta_1, \beta_2}^* \mathcal{C}_\lambda$ as defined in Sect. 3.1. This curve has, in general, three singular points at the vertices P_x, P_y, P_z . Recall that for $\zeta := \exp \frac{2i\pi}{3}$ an easy computation given in Corollary 3.5 shows that the fundamental group of $\mathbb{P}_\omega^2 \setminus \tilde{\mathcal{C}}_\zeta$ is abelian and hence the first cohomology group of any cyclic cover ramified along $\tilde{\mathcal{C}}_\zeta$ vanishes.

Fig. 2 A \mathbb{Q} -resolution of $(\tilde{\mathcal{C}}, P_z)$



In order to understand the maps $\pi^{(k)}$ and the corresponding modules of quasiadjunction $\mathcal{M}_{\tilde{\mathcal{C}}, P}^{(k)}$ described in Theorem 3.10 one needs to study the singular points of $\tilde{\mathcal{C}} := \tilde{\mathcal{C}}_1$ in \mathbb{P}_{ω}^2 . Recall that $\text{Sing } \tilde{\mathcal{C}} \supseteq \{P_x, P_y, P_z\}$. More precisely, we will restrict our attention to the case $\frac{k}{d} = \frac{5}{6}$. Since $d = 3(\alpha_1\alpha_2 + \alpha_3)$, the degrees of the curves involved in $\pi^{(k)}$ is

$$d_k = \frac{5d}{6} - (\alpha_1 + \alpha_2 + \alpha_3) = \frac{5\alpha_1\alpha_2 + 3\alpha_3}{2} - (\alpha_1 + \alpha_2).$$

Proposition 3.11 *A \mathbb{Q} -resolution of $(\tilde{\mathcal{C}}, P_z)$ has a dual graph with two vertices and its exceptional set is shown in Fig. 2. Then $\mathcal{M}_{\tilde{\mathcal{C}}, P_z}^{(k)}$, $k = \frac{5d}{6}$, is defined by the following conditions on germs $g \in \mathcal{O}_{\mathbb{P}_{\omega}^2, P_z}(d_k)$:*

$$\text{mult}_{E_{v_z}} \pi^* g \geq \frac{5\alpha_1\alpha_2 - 2(\alpha_1 + \alpha_2)}{2\alpha_3} + \frac{1}{2}, \quad \text{mult}_{E_w} \pi^* g \geq \frac{15\alpha_1\alpha_2 - 6(\alpha_1 + \alpha_2)}{2\alpha_3} + 2.$$

Proof The result is purely local, so one can assume \mathcal{C} is the cubic $zx^2 - (y+x)^3 = 0$ at the flex $[1 : -1 : 0]$. Then, the local equation of $\tilde{\mathcal{C}}$ at $[0 : 0 : 1]_{\omega}$, regarded as $[(0, 0) \in \frac{1}{\alpha_3}(\alpha_1, \alpha_2)]$, is $x^{\beta_1}y^{\beta_2+2\alpha_1} - (x^{\alpha_2} + y^{\alpha_1})^3 = 0$. Note that $\alpha_1\beta_1 + \alpha_2(\beta_2 + 2\alpha_1) = 3\alpha_1\alpha_2 + \alpha_3 > 3\alpha_1\alpha_2$. Hence the Newton polygon of this equation is a segment of slope $-\frac{\alpha_1}{\alpha_2}$, and we perform an (α_1, α_2) -blowing-up. Since we start from a cyclic point one chart of this blow-up is given by

$$(x, y) \mapsto (x^{\frac{\alpha_1}{\alpha_3}}, x^{\frac{\alpha_2}{\alpha_3}}y),$$

i.e., the total transform is $x^{\frac{3\alpha_1\alpha_2}{\alpha_3}}(xy^{\beta_2+2\alpha_1} - (1 + y^{\alpha_1})^3) = 0$. We denote this exceptional divisor as E_{v_z} . Hence $m_{v_z} = \frac{3\alpha_1\alpha_2}{\alpha_3}$ and after a change of coordinates the strict transform (through a smooth ambient point) has equation $x - y^3 = 0$.

One can check that the multiplicity of the relative canonical divisor is $\nu_{v_z} = \frac{\alpha_1 + \alpha_2}{\alpha_3}$. To complete the resolution, we perform a $(3, 1)$ -blow up, producing a new component E_w for which $m_w = 3\left(\frac{3\alpha_1\alpha_2}{\alpha_3} + 1\right)$ and $\nu_w = 3\frac{\alpha_1 + \alpha_2}{\alpha_3} + 1$.

By definition, the module of quasi-adjunction $\mathcal{M}_{\tilde{\mathcal{C}}, P_z}^{(k)}$ is a submodule of

$$\mathcal{O}_z(d_k) := \mathcal{O}_{\mathbb{P}_{\omega}^2, P_z}(d_k), \quad d_k = \frac{5\alpha_1\alpha_2 + 3\alpha_3}{2} - (\alpha_1 + \alpha_2).$$

given by the germs $g \in \mathcal{O}_z(d_k)$ satisfying

$$\begin{aligned} \text{mult}_{E_{v_z}} \pi^* g &> \frac{km_{v_z}}{d} - v_{v_z} = \frac{5\alpha_1\alpha_2 - 2(\alpha_1 + \alpha_2)}{2\alpha_3}, \\ \text{mult}_{E_w} \pi^* g &> \frac{km_w}{d} - v_w = \frac{15\alpha_1\alpha_2 + 3\alpha_3 - 6(\alpha_1 + \alpha_2)}{2\alpha_3}. \end{aligned} \tag{3.6}$$

Finally, note that the class of g imposes extra conditions, namely, if $H = V(h)$, $h \in \mathcal{O}_z(1)$, then $\text{mult}_{E_v} \pi^* \left(\frac{g}{h^{d_k}}\right)$ must be an integer for $v \in \{v_z, w\}$. Using (3.6) we can write $\text{mult}_{E_{v_z}} \pi^* g = \frac{5\alpha_1\alpha_2 - 2(\alpha_1 + \alpha_2)}{2\alpha_3} + \varepsilon_{v_z}$, for some $\varepsilon_{v_z} \in \mathbb{Q}_{>0}$. Hence,

$$\text{mult}_{E_{v_z}} \pi^* \left(\frac{g}{h^{d_k}}\right) = \frac{5\alpha_1\alpha_2 - 2(\alpha_1 + \alpha_2)}{2\alpha_3} + \varepsilon_{v_z} - \frac{d_k}{\alpha_3} = \varepsilon_{v_z} - \frac{3}{2}\alpha_3 \in \mathbb{Z}.$$

This implies $\varepsilon_{v_z} = \frac{1}{2} + n_{v_z}$, $n_{v_z} \in \mathbb{Z}_{\geq 0}$. Analogously for $v = w$ one obtains

$$\text{mult}_{E_w} \pi^* \left(\frac{g}{h^{d_k}}\right) = \frac{15\alpha_1\alpha_2 + 3\alpha_3 - 6(\alpha_1 + \alpha_2)}{2\alpha_3} + \varepsilon_w - 3\frac{d_k}{\alpha_3} = \varepsilon_w + \frac{3}{2} \in \mathbb{Z},$$

which implies $\varepsilon_w = \frac{1}{2} + n_w$, $n_w \in \mathbb{Z}_{\geq 0}$ and this ends the proof.

Proposition 3.12 *A \mathbb{Q} -resolution of $(\tilde{\mathcal{C}}, P_x)$ is obtained with one weighted blow-up. Then $\mathcal{M}_{\tilde{\mathcal{C}}, P_x}^{(k)}$, $k = \frac{5d}{6}$, is defined by the following condition on germs $g \in \mathcal{O}_{\mathbb{P}_{\omega}^2, P_x}(d_k)$:*

$$\text{mult}_{E_{v_x}} \pi^* g \geq \frac{3}{\gcd(3, \alpha_1)} \cdot \frac{\alpha_1 + 3\beta_2 - 2}{2\alpha_1} + 1$$

Proof We follow the same ideas as in the proof of Proposition 3.11. Locally we work with the cubic $xy^2 - (y + z)^3 = 0$ (this cubic has a flex at $[0 : 1 : -1]$). Then, the local equation of $\tilde{\mathcal{C}}$ at $[1 : 0 : 0]_{\omega}$, regarded as $[(0, 0)] \in \frac{1}{\alpha_1}(\alpha_2, \alpha_3) = \frac{1}{\alpha_1}(1, \beta_2)$, is $y^{\alpha_1}z^3 - (z + y^{\beta_2})^3 = 0$. We can change the coordinates (not affecting the action) where the equation becomes $y^{\alpha_1}(z - y^{\beta_2})^3 - z^3 = 0$. In these new coordinates the Newton polygon is non-degenerated and the singularity is resolved with a blowing-up with exceptional component E_{v_x} . Its weight is $(3, \alpha_1 + 3\beta_2)$ if $\gcd(3, \alpha_1) = 1$ and $(1, \frac{\alpha_1}{3} + \beta_2)$ otherwise.

The invariants are

$$m_{v_x} = 3 \frac{\alpha_1 + 3\beta_2}{\alpha_1}, \quad \nu_{v_x} = \frac{\alpha_1 + 3\beta_2 + 3}{\alpha_1}.$$

Let us compute the quasi-adjunction module $\mathcal{M}_{\mathcal{D}, P_x}^{(k)}$, as a submodule of $\mathcal{O}_x(\bar{d}_k) := \mathcal{O}_{\mathbb{P}_{\omega}^2, P_x}(\bar{d}_k)$, where \bar{d}_k is such that $\alpha_2 \bar{d}_k \equiv d_k \pmod{\alpha_1}$, which implies that $\bar{d}_k \equiv \frac{\alpha_1 + 3\beta_2 - 2}{2}$. The condition for a germ $g \in \mathcal{O}_x(\bar{d}_k)$ to be in $\mathcal{M}_{\mathcal{D}, P_x}^{(k)}$ is:

$$\text{mult}_{E_{v_x}} \pi^* g > 3 \frac{\alpha_1 + 3\beta_2 - 2}{2\alpha_1}.$$

As above, the restriction given by $g \in \mathcal{O}_x(\bar{d}_k)$ leads to

$$\text{mult}_{E_{v_x}} \left(\pi^* \frac{g}{h^{\bar{d}_k}} \right) = 3 \frac{\alpha_1 + 3\beta_2 - 2}{2\alpha_1} + \varepsilon_{v_x} - 3 \frac{\bar{d}_k}{\alpha_1} = \varepsilon_{v_x} \in \mathbb{Z}.$$

Hence, $\varepsilon_{v_x} \in \mathbb{Z}_{>0}$.

Proposition 3.13 *Let $g(x, y, z)$ be a weighted homogeneous polynomial in $\ker \pi^{(k)}$ with $\deg_{\omega} g = \frac{5(\alpha_1\alpha_2 + \alpha_3) - 2(\alpha_1 + \alpha_2 + \alpha_3)}{2}$.*

Then, there is a weighted homogeneous polynomial f , $\deg_{\omega} f = \alpha_1\alpha_2 + \alpha_3$, such that $g(x, y, z) = x^{\frac{1}{2}(\alpha_2 + \beta_1 - 2)} y^{\frac{1}{2}(\alpha_1 + \beta_2 - 2)} f(x, y, z)$ and

$$\begin{aligned} \text{mult}_{E_{v_z}} \pi^* f(x, y, 1) &\geq \frac{\alpha_1\alpha_2}{\alpha_3}, & \text{mult}_{E_w} \pi^* f(x, y, 1) &\geq \frac{3\alpha_1\alpha_2}{\alpha_3} + \frac{1}{2}, \\ \text{mult}_{E_{v_x}} \pi^* f(1, y, z) &\geq \frac{3\beta_2}{\gcd(3, \alpha_1)\alpha_1} + 1, & \text{mult}_{E_{v_y}} \pi^* f(x, 1, z) &\geq \frac{3\beta_1}{\gcd(3, \alpha_2)\alpha_2} + 1. \end{aligned}$$

Proof The exponent of x^n as a factor of g is given by the maximal value $n \in \mathbb{Z}_{>0}$, such that the divisor $V(g) - nY$ is effective. Using the generalization of Noether's multiplicity Theorem in this context, see [8, Theorem 4.3(4)], and Proposition 3.11 one obtains

$$\begin{aligned} (V(g) \cdot Y)_{P_z} &\geq \frac{(\text{mult}_{E_{v_z}} \pi^* g) \cdot (\text{mult}_{E_{v_z}} \pi^* y)\alpha_3}{\alpha_1\alpha_2} \\ &\geq \frac{(5\alpha_1\alpha_2 - 2(\alpha_1 + \alpha_2))}{2\alpha_1\alpha_3} + \frac{1}{2\alpha_1} = \frac{1}{2\alpha_1\alpha_3} (5\alpha_1\alpha_2 - 2\alpha_1 - 2\alpha_2 + \alpha_3). \end{aligned}$$

Hence,

$$((V(g) - nY) \cdot Y)_{P_z} \geq \frac{1}{2\alpha_1\alpha_3} (5\alpha_1\alpha_2 - 2\alpha_1 - 2\alpha_2(1 + n) + \alpha_3).$$

Analogously, at P_x one can use Proposition 3.12 to obtain

$$(\mathbf{V}(g) \cdot Y)_{P_x} \geq \left(3 \frac{\alpha_1 + 3\beta_2 - 2}{2\alpha_1} + 1\right) \frac{1}{\alpha_1 + 3\beta_2} = \frac{5\alpha_1 + 9\beta_2 - 6}{2\alpha_1(\alpha_1 + 3\beta_2)},$$

regardless of the value of $\gcd(3, \alpha_1)$. Hence,

$$((\mathbf{V}(g) - nY) \cdot Y)_{P_x} \geq \frac{5\alpha_1 + 9\beta_2 - 6 - 6n}{2\alpha_1(\alpha_1 + 3\beta_2)}.$$

Then a global computation of the intersection multiplicity can be bounded by

$$\begin{aligned} ((\mathbf{V}(g) - nY) \cdot Y)_{\mathbb{P}_\omega^2} &\geq ((\mathbf{V}(g) - nY) \cdot Y)_{P_z} + ((\mathbf{V}(g) - nY) \cdot Y)_{P_x} \\ &\geq \frac{1}{2\alpha_1\alpha_3} (5\alpha_1\alpha_2 - 2\alpha_1 - 2\alpha_2(1+n) + \alpha_3) + \frac{5\alpha_1 + 9\beta_2 - 6 - 6n}{2\alpha_1(\alpha_1 + 3\beta_2)} \\ &= \frac{1}{2\alpha_1\alpha_3} (5(\alpha_1\alpha_2 + \alpha_3) - 2(\alpha_1 + \alpha_2 + \alpha_3) - 2n\alpha_2) - \frac{1}{\alpha_1} + \frac{5\alpha_1 + 9\beta_2 - 6 - 6n}{2\alpha_1(\alpha_1 + 3\beta_2)} \\ &= \frac{\deg_\omega(\mathbf{V}(g) - nY) \cdot \deg_\omega(Y)}{\alpha_1\alpha_2\alpha_3} + 3 \frac{\alpha_1 + \beta_2 - 2 - 2n}{2\alpha_1(\alpha_1 + 3\beta_2)}. \end{aligned}$$

By Bézout's Theorem for weighted projective planes, $n = \frac{1}{2}(\alpha_1 + \beta_2 - 2)$. The same calculation applies to the divisor X . This shows that $g = x^m y^n f(x, y, z)$, $m = \frac{1}{2}(\alpha_2 + \beta_1 - 2)$ where

$$\begin{aligned} \deg(f) &= \frac{1}{2} (5(\alpha_1\alpha_2 + \alpha_3) - 2(\alpha_1 + \alpha_2 + \alpha_3) - 2n\alpha_2 - 2m\alpha_1) \\ &= \frac{1}{2} (5(\alpha_1\alpha_2 + \alpha_3) - 2(\alpha_1 + \alpha_2 + \alpha_3) - (\alpha_1 + \beta_2 - 2)\alpha_2 - (\alpha_2 + \beta_1 - 2)\alpha_1) \\ &= \alpha_1\alpha_2 + \alpha_3. \end{aligned}$$

The last equality follows from $\alpha_1\beta_1 + \alpha_2\beta_2 = \alpha_1\alpha_2 + \alpha_3$.

The last part follows immediately from Propositions 3.11 and 3.12 and the additivity properties of the multiplicity.

The local algebraic information obtained in this section will help us effectively study the morphism $\pi^{(k)}$ described in Theorem 3.10. Let us use the notation introduced before Theorem 3.9 and at the beginning of this section, let us also denote by X_1 (resp. X_2) the cyclic cover of \mathbb{P}_ω^2 of order $d = 3(\alpha_1\alpha_2 + \alpha_3)$ ramified along $\tilde{\Phi}_1^*(C)$ (resp. $\tilde{\Phi}_2^*(C)$). Finally, denote by $L_i^{(k)}$ the invariant part of $H^1(X_i, \mathcal{O}_{X_i})$ with respect to the action of the monodromy by multiplication by $\exp \frac{2\pi i k}{d}$. Likewise, we denote by $\pi_i^{(k)}$ the map described in Theorem 3.10 for the curve $\tilde{\Phi}_i^*(C)$. The discussion above shows the following.

Proposition 3.14 *If the product $\alpha_1\alpha_2\alpha_3\beta_1\beta_2$ is odd and $\frac{k}{d} = \frac{5}{6}$, then $\dim \ker \pi_1^{(k)} = 0$ and $\dim \ker \pi_2^{(k)} = 1$.*

Proof By Proposition 3.13, the image by Φ_i of $V(f)$ is a line passing through the three flexes. The existence of this line for $\tilde{\Phi}_2^*(C)$ but not for $\tilde{\Phi}_1^*(C)$ ends the proof.

The machinery developed in this section allows one to give an alternative proof of Theorem 3.9 which is independent of fundamental group calculations.

Proof of Theorem 3.9 Since the curves $\tilde{\Phi}_1^*(C)$ and $\tilde{\Phi}_2^*(C)$ have the same combinatorics and the same local type of singularities, the target space for $\pi_1^{(k)}$ and $\pi_2^{(k)}$ are the same. Therefore Proposition 3.14 implies $\dim \operatorname{coker} \pi_1^{(k)} = 1 + \dim \operatorname{coker} \pi_2^{(k)}$ for $\frac{k}{d} = \frac{5}{6}$. By Theorem 3.10, $\dim \operatorname{coker} \pi_i^{(k)} = \dim L_i^{(k)}$ is a birational invariant of X_i and thus $X_1 \not\cong X_2$, which implies that the fundamental groups of $\mathbb{P}_\omega^2 \setminus \tilde{\Phi}_1^*(C)$ and $\mathbb{P}_\omega^2 \setminus \tilde{\Phi}_2^*(C)$ are not isomorphic and thus $(\tilde{\Phi}_1^*(C), \tilde{\Phi}_2^*(C))$ forms a Zariski pair.

4 Some Rational Cuspidal Curves on Weighted Projective Planes

The study of rational cuspidal curves in \mathbb{P}^2 is a classical subject. There is an extensive literature about them, and we recommend the beautiful paper [15] reviewing this topic, the most relevant conjectures, and bibliography. Two outstanding conjectures have been solved recently by Koras and Palka: the Nagata-Coolidge conjecture [19], that is, any rational cuspidal curve can be transported to a line via a Cremona transformation and such curves can have at most four singular points [20]. There is a strong knowledge of such curves in \mathbb{P}^2 which have helped for the solution of these conjectures and other important problems, like the semigroup conjecture in [15], which was proven in [10].

Only one rational cuspidal curve in \mathbb{P}^2 possesses four cusps: a quintic curve with singular locus $\mathbb{A}_6 + 3\mathbb{A}_2$. There are many of them with three singular points, see [16] for an infinite family. The simplest one is the cuspidal quartic with three ordinary cusps. The standard Cremona transformation is a way to produce this curve, namely, the standard Cremona transformation of a smooth conic with respect to three of its tangent lines produces a tricuspidal quartic. Note that the blowing-up at the vertices does not affect the curve, and the blowing-downs produce the three cusps.

4.1 Rational Cuspidal Curves via Weighted Cremona Transformations

In this section, we will study the strict transforms of the above tritangent conic using the inverse of the weighted Cremona transformations introduced in Sect. 2.4. As a first stage, let us compute their fundamental groups. As in Sect. 3, let us start with the arrangement of a smooth conic and three lines, giving a presentation which contains suitable meridians for all the components.

Let \mathcal{C} be a smooth conic and let X, Y, Z be three distinct tangent lines to \mathcal{C} . If the equations of the lines are $x = 0, y = 0, z = 0$, respectively, then the equation of \mathcal{C} (up to a suitable change of coordinates) is

$$x^2 + y^2 + z^2 - 2(yz + xz + xy) = 0.$$

The fundamental group of the complement of the smooth conic and three tangent lines is the Artin group of the triangle $T(4, 4, 2)$ (i.e. [9]). However, for our purposes, it is more suitable to use a presentation with a more geometrical interpretation. We present it here for completeness, but its proof is immediate using the classical Zariski-van Kampen method (as in [11]).

Proposition 4.1 *The fundamental group of $\mathbb{P}^2 \setminus (\mathcal{C} \cup X \cup Y \cup Z)$ is isomorphic to*

$$\langle c, \ell_x, \ell_y, \ell_z \mid [\ell_x, \ell_y] = [\ell_x, \ell_z] = [\ell_y^c, \ell_z] = \ell_y c \ell_x c \ell_z = 1 \rangle. \quad (4.1)$$

The element c is a meridian of \mathcal{C} , and ℓ_x, ℓ_y, ℓ_z are meridians of X, Y, Z , respectively. Moreover, (ℓ_x, ℓ_y) are meridians close to $[0 : 0 : 1]$, (ℓ_x, ℓ_z) are meridians close to $[0 : 1 : 0]$, and $(\ell_y^c = c^{-1} \ell_y c, \ell_z)$ are meridians close to $[0 : 0 : 1]$.

Considering $u := c \ell_z$, the above presentation of $\pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup X \cup Y \cup Z))$ can be alternatively written as:

$$\langle u, \ell_x, \ell_y, \ell_z \mid [\ell_x, \ell_y] = [\ell_x, \ell_z] = [\ell_y^u, \ell_z] = u \ell_y u \ell_x \ell_z^{-1} = 1 \rangle. \quad (4.2)$$

As in Sect. 3, fixing ω, β_1, β_2 , we consider the birational map Φ and we denote by $\tilde{\mathcal{C}}$ the strict transform of \mathcal{C} by Φ .

Proposition 4.2 *Let $d := \gcd(\alpha_1 + 2\beta_2, \alpha_2 + 2\beta_1)$. Then $\pi_1(\mathbb{P}_\omega^2 \setminus \tilde{\mathcal{C}})$ is the semidirect product $(\mathbb{Z}/d)A \ltimes (\mathbb{Z}/2(\alpha_1\alpha_2 + \alpha_3))B$ where $BAB^{-1} = A^{-1}$. Hence the group has size $2d(\alpha_1\alpha_2 + \alpha_3)$ and it is abelian if and only if $d = 1$.*

Remark 4.3 Note that d is odd, since α_1, α_2 cannot be simultaneously even. Moreover,

$$\begin{aligned} \gcd(d, \alpha_1) &= \gcd(\alpha_1, \beta_2, \alpha_2 + 2\beta_1) = \gcd(\alpha_1, \alpha_2\beta_2, \alpha_2 + 2\beta_1) \\ &= \gcd(\alpha_1, \alpha_3, \alpha_2 + 2\beta_1) = 1 \end{aligned}$$

and analogously $\gcd(d, \alpha_2) = 1$. Note also

$$\gcd(d, \beta_1) = \gcd(d, \beta_1, \alpha_1 + 2\beta_2, \alpha_2) = 1,$$

and hence $\gcd(d, \beta_2) = 1$. The following congruences can easily be checked:

$$\alpha_1\beta_1 \equiv -2\beta_1\beta_2 \equiv \alpha_2\beta_2 \pmod{d}.$$

Moreover,

$$\begin{aligned} \gcd(d, \alpha_1\alpha_2 + \alpha_3) &= \gcd(d, \alpha_1\alpha_2 + \alpha_3, \alpha_2(\alpha_1 + 2\beta_2)) \\ &= \gcd(d, \alpha_1\alpha_2 + \alpha_3, \alpha_1(\alpha_2 - 2\beta_1)) \\ &= \gcd(d, \alpha_1\alpha_2 + \alpha_3, 2\alpha_1\beta_1) = 1. \end{aligned}$$

Proof of Proposition 4.2 The presentation of $\pi_1(\mathbb{P}_\omega^2 \setminus \tilde{\mathcal{C}})$ is obtained from (4.2) by adding the relations which *kill* the meridians of the exceptional divisors E_x, E_y, E_z :

$$\ell_x\ell_y = \ell_x^{\alpha_1}\ell_z^{\beta_2} = u^{-1}\ell_y^{\alpha_2}u\ell_z^{\beta_1} = 1. \tag{4.3}$$

Let us first check that the abelianization of this quotient is $\mathbb{Z}/2(\alpha_1\alpha_2 + \alpha_3)$. We will denote by $[\bullet]$ the class of \bullet in the abelianization. Note that $[\ell_z] = [u]^2$; moreover, using Bézout’s identity and the equations in (4.3), both $[\ell_x]$ and $[\ell_y]$ can be expressed in terms of $[\ell_z]$. Hence, the abelianization is cyclic. A presentation matrix in terms of the generators $[\ell_y], [u]$ is given by

$$\begin{pmatrix} \alpha_2 & 2\beta_1 \\ -\alpha_1 & 2\beta_2 \end{pmatrix}$$

whose determinant, $2(\alpha_1\alpha_2 + \alpha_3)$, is the size of the abelianization.

Let us study now the group itself. Note first that ℓ_y can be eliminated from (4.3) as $\ell_y = \ell_x^{-1}$. Let us check that u^2 is central. The last relation in (4.2) can be written as

$$u^2 = \ell_z\ell_x^{-1}(u^{-1}\ell_xu).$$

We deduce that u^2 commutes with ℓ_z , since it commutes with each factor; hence u^2 also commutes with $u\ell_zu^{-1}$. Also note that

$$\ell_x^{\alpha_1} = \ell_z^{-\beta_2}, \quad \ell_x^{\alpha_2} = u\ell_z^{\beta_1}u^{-1} \implies \ell_x = \ell_z^{-\hat{\alpha}_1\beta_2}u\ell_z^{-\hat{\alpha}_2\beta_1}u^{-1}.$$

Then u^2 commutes with ℓ_x and it is central.

Using the last relation in (4.2) ℓ_z can also be eliminated as $\ell_z = u\ell_x^{-1}u\ell_x$. The presentation of the group becomes:

$$\langle u, \ell_x \mid [\ell_x, u\ell_x^{-1}u] = [\ell_x^u, u\ell_x^{-1}u\ell_x] = \ell_x^{\alpha_1}(u\ell_x^{-1}u\ell_x)^{\beta_2} = u^{-1}\ell_x^{-q}u(u\ell_x^{-1}u\ell_x)^{\beta_1} = 1 \rangle$$

which can be further simplified using the centrality of u^2 :

$$\langle u, \ell_x \mid [\ell_x, u^2] = [\ell_x, u\ell_x u^{-1}] = u^{2\beta_2}\ell_x^{\alpha_1+\beta_2}(u\ell_x u^{-1})^{-\beta_2} = u^{2\beta_1}(u\ell_x u^{-1})^{-(\alpha_2+\beta_1)}\ell_x^{\beta_1} = 1 \rangle$$

The map $\pi_1(\mathbb{P}_\omega^2 \setminus \tilde{C}) \rightarrow \mathbb{Z}/2$ given by $u \mapsto 1$ and $\ell_x, \ell_z \mapsto 0$ is well defined. A presentation of its kernel K is obtained using Reidemeister-Schreier method. The generators are $X_0 := \ell_x, X_1 := u\ell_x u^{-1}$, and $U := u^2$. The first two relations imply that the group is abelian; the other relations yield:

$$\begin{aligned} u^{2\beta_2}\ell_x^{\alpha_1+\beta_2}(u\ell_x u^{-1})^{-\beta_2} = 1 &\implies U^{\beta_2}X_1^{\alpha_1+\beta_2}X_0^{-\beta_2} = U^{\beta_2}X_0^{\alpha_1+\beta_2}X_1^{-\beta_2} = 1, \\ u^{2\beta_1}(u\ell_x u^{-1})^{-(\alpha_2+\beta_1)}\ell_x^{\beta_1} = 1 &\implies U^{\beta_1}X_1^{-(\alpha_2+\beta_1)}X_0^{\beta_1} = U^{\beta_1}X_0^{-(\alpha_2+\beta_1)}X_1^{\beta_1} = 1. \end{aligned}$$

Let us express these relations in a matrix (the rows represent the relations and the columns stand for the generators). These relations become (recall $d = \gcd(\alpha_1 + 2\beta_2, \alpha_2 + 2\beta_1)$):

$$\begin{pmatrix} d & -d & 0 \\ -\beta_2 & \alpha_1 + \beta_2 & \beta_2 \\ \beta_1 & -(\alpha_2 + \beta_1) & \beta_1 \\ X_0 & X_1 & U \end{pmatrix} \sim \begin{pmatrix} d & 0 & 0 \\ -\beta_2 & \alpha_1 & \beta_2 \\ \beta_1 & -\alpha_2 & \beta_1 \\ X_0 X_1^{-1} & X_1 & U \end{pmatrix}.$$

The determinant of this matrix is $d(\alpha_1\alpha_2 + \alpha_3)$ which is the size of K ; the greatest common divisor of the 2-minors divides d and $2\alpha_1\beta_1$, i.e., it is 1. Hence, the group K is cyclic of order $d(\alpha_1\alpha_2 + \alpha_3)$. One also has that $D := X_0 X_1^{-1}$ is of order d . Considering the product of the second relation to the power α_2 and the third relation to the power α_1 one obtains

$$1 = D^{\alpha_1\beta_1 - \alpha_2\beta_2} U^{\alpha_1\alpha_2 + \alpha_3} = U^{\alpha_1\alpha_2 + \alpha_3}.$$

>From the order in the abelianization, we deduce that U is of order $\alpha_1\alpha_2 + \alpha_3$. Hence K is the direct product of the cyclic group of order d generated by D and the cyclic group of order $\alpha_1\alpha_2 + \alpha_3$ generated by U . The conjugation by u satisfies $uUu^{-1} = U$ and $uD u^{-1} = D^{-1}$. The result follows.

Embedded \mathbb{Q} -resolutions of the singularities of these curves can be computed as in Propositions 3.11 and 3.12, see Fig. 3.

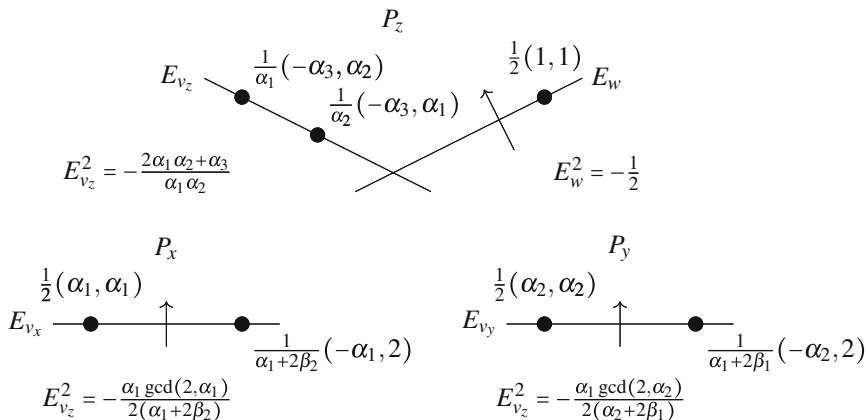


Fig. 3 Singularities of the rational cuspidal curves

4.2 Rational Cuspidal Curves via Weighted Kummer Covers

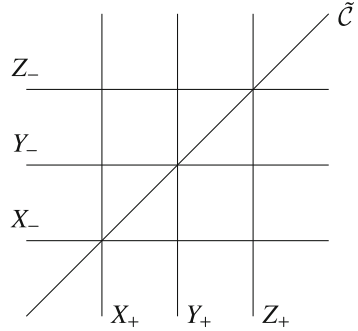
There is another simple way to produce rational cuspidal curves in weighted projective planes from this arrangement of curves. It is quite simple but it will be shown to be useful in the upcoming sections. Let d_1, d_2, d_3 be pairwise coprime integers and let $\omega := (e_1, e_2, e_3)$, where $e_i := d_j d_k$, $\{i, j, k\} = \{1, 2, 3\}$. Following Sect. 2.2 note that $\eta = (1, 1, 1)$ and thus there is an isomorphism $\mathbb{P}_\omega^2 \rightarrow \mathbb{P}^2$ given by $[x : y : z]_\omega \mapsto [x^{d_1} : y^{d_2} : z^{d_3}]$. This map gives a geometrical interpretation to the group

$$G_{d_1, d_2, d_3} := \pi_1(\mathbb{P}_\omega^2 \setminus (\mathcal{C} \cup X \cup Y \cup Z)) / \langle \ell_x^{d_1} = \ell_y^{d_2} = \ell_z^{d_3} = 1 \rangle.$$

as the orbifold group of \mathbb{P}_ω^2 with respect to the curve $\mathcal{C} \cup X \cup Y \cup Z$ and index $e(\mathcal{C}) = 0, n(X) = d_1, n(Y) = d_2, n(Z) = d_3$ as defined in [5]. Briefly, if X is a smooth projective surface, $D = D_1 \cup \dots \cup D_s$ is a normal crossing union of smooth hypersurfaces, and $n_i := n(D_i) \in \mathbb{Z}_{\geq 0}$, then one can define the orbifold fundamental group $\pi_1^{\text{orb}}(X)$ of X with respect to D with indices n_i as the quotient of the group $\pi_1(X \setminus D)$ by the normal subgroup generated by $\gamma_i^{n_i}$, where γ_i is a meridian of D_i . If D do not have normal crossings, then one resolves to a normal crossing divisor by blowing up points and defines the index at an exceptional divisor as the least common multiple of the indices of the components passing through the point (in this context we set $\text{lcm}(0, n) = 0$).

Proposition 4.4 *The abelianization of the group G_{d_1, d_2, d_3} is $\mathbb{Z}/2d_1d_2d_3$. The group is abelian if and only if at least one of d_1, d_2, d_3 equals 1.*

Fig. 4 Double cover ramified along \mathcal{C}



Proof The computation for the abelianization is straightforward. Assume $d_3 = 1$, then

$$\begin{aligned} G_{d_1, d_2, 1} &= \langle \ell_x, u \mid [\ell_x, \underbrace{u\ell_x u}_{v^{-1}}] = 1, \ell_x^{d_1} = (u\ell_x u)^{d_2} = 1 \rangle \\ &= \langle \ell_x, v \mid [\ell_x, v^2] = 1, \ell_x^{d_1} = \ell_x^{d_2} v^{2d_2} = 1 \rangle. \end{aligned}$$

Using Bézout’s identity we can express ℓ_x in terms of v and the result follows.

For the case $(d_1, d_2, d_3) \neq (1, 1, 1)$, let us consider the double cover of \mathbb{P}^2 ramified along \mathcal{C} . It is $\mathbb{P}^1 \times \mathbb{P}^1$ and the preimage $\tilde{\mathcal{C}}$ of \mathcal{C} is the diagonal; the three lines are transformed in pairs of vertical-horizontal lines $X_{\pm}, Y_{\pm}, Z_{\pm}$ intersecting $\tilde{\mathcal{C}}$, see Fig. 4.

Note that the fundamental group of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus (\tilde{\mathcal{C}} \cup X_+ \cup X_- \cup Y_+ \cup Y_- \cup Z_+ \cup Z_-)$ is isomorphic to the fundamental group of the projective complement of Ceva’s arrangement. The kernel K_{d_1, d_2, d_3} of the map $G_{d_1, d_2, d_3} \rightarrow \mathbb{Z}/2, u \mapsto 1$ and the images of the other generators vanish, it is an index 2 subgroup of G_{d_1, d_2, d_3} and it is an orbifold fundamental group for the above configuration in $\mathbb{P}^1 \times \mathbb{P}^1$; the projection on each component induces orbifold morphisms onto $\mathbb{P}^1_{d_1, d_2, d_3}$, where $\mathbb{P}^1_{d_1, d_2, d_3}$ is an orbifold modeled on \mathbb{P}^1 , with three quotient points of order d_1, d_2, d_3 , respectively, and its orbifold fundamental group is isomorphic to $\langle \mu_1, \mu_2, \mu_3 \mid \mu_1^{d_1} = \mu_2^{d_2} = \mu_3^{d_3} = \mu_3 \mu_2 \mu_1 = 1 \rangle$, a triangle group. The combination of the two projections induces an epimorphism of K_{d_1, d_2, d_3} onto $\pi_1^{\text{orb}}(\mathbb{P}^1_{d_1, d_2, d_3})$. The triangle is hyperbolic if $\{d_1, d_2, d_3\} \neq \{2, 3, 5\}$ and hence its group is infinite. If $\{d_1, d_2, d_3\} = \{2, 3, 5\}$, the triangle group is the alternating group A_5 , with cardinal 60. Using GAP [34] we can compute the intersection of the kernels of the two projections (which is a subgroup of index 3600); its abelianization is \mathbb{Z}^{59} and the result follows. The computation can be checked in <https://github.com/enriqueartal/AnOrbifoldFundamentalGroup> using Sagemath [33] and Binder [30].

4.3 A rational Cuspidal Curve with Four Cusps

We present in this section a nice example in $\mathbb{P}^2_{(1,1,2)}$, which is a rational curve of degree 6 with 4 ordinary cusps – incidentally, note that 4 is the maximal number of singular points a rational cuspidal curve can have in \mathbb{P}^2 . Let us explain how to construct it via Cremona transformations. Let us start with a tricuspidal quartic \mathcal{C}_0 ; this curve, dual of the nodal cubic, has a bitangent line L . Let $P_0 \in \mathcal{C}_0 \cap L$; its blown-up produces a ruled surface Σ_1 , where the negative section E is the exceptional component and \mathcal{C}_1 , the strict transform of \mathcal{C}_0 has three cusps and one tangent fiber. Let us consider the Nagata transformation at $\mathcal{C}_1 \cap E$; the result is Σ_2 and the blow-down of the negative section produces $\mathbb{P}^2_{(1,1,2)}$. The strict transform \mathcal{C} of \mathcal{C}_1 is the desired curve.

Proposition 4.5 *The fundamental group $\pi_1(\mathbb{P}^2_{(1,1,2)} \setminus (\mathcal{C} \cup \{P_i\}))$ has a presentation*

$$\langle s, t, u \mid sts = tst, sus = usu, tut = utu, (stu)^2 = 1 \rangle.$$

Proof Following the construction it is the fundamental group of $\Sigma_2 \setminus (\mathcal{C}_2 \cup E_2)$ where \mathcal{C}_2 is the strict transform of \mathcal{C} and E_2 is the negative section. The Zariski-van Kampen method applied to the ruling yields the result.

4.4 Milnor Fibers

We have presented in Sect. 3 and in Sect. 4 several examples of irreducible quasi-projective curves such that their (maybe orbifold) fundamental groups are non-abelian. As a consequence their cones are quasi-homogeneous non-isolated surface singularities in \mathbb{C}^3 with non simply-connected Milnor fibers.

If $F \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial of degree d , an important topological invariant is its Milnor fiber. The Milnor fiber of a homogeneous singularity is a fiber of $F : \mathbb{C}^3 \setminus F^{-1}(0) \rightarrow \mathbb{C}^*$, say $F^{-1}(1)$. The restriction to $F^{-1}(1)$ of F of the natural map $\mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{P}^2$ is a d -cyclic cover onto the complement of the tangent cone \mathcal{C}_d in \mathbb{P}^2 , defined by an epimorphism $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_d) \rightarrow \mathbb{Z}/d$.

If ω is a weight and $F \in \mathbb{C}[x, y, z]$ is an ω -quasi-homogeneous polynomial of ω -degree d , its Milnor fiber $F = 1$ can also be recovered as a d -cyclic orbifold cover of $\mathbb{P}^2_\omega \setminus \mathcal{C}_d$ (the complement of the *tangent ω -quasi-cone*) defined by an epimorphism $\pi_1^{\text{orb}}(\mathbb{P}^2_\omega \setminus \mathcal{C}_d) \rightarrow \mathbb{Z}/d$. If the elements of ω are pairwise coprime and the vertices are in \mathcal{C}_d , then the notions of π_1 and π_1^{orb} coincide. If it is not the case, the notion of orbifold fundamental groups apply.

The curves obtained via the Cremona transformation provide homogeneous singularities whose topology is not complicated and such that the Milnor fiber

has non-trivial fundamental group. The following result is a direct consequence of Proposition 4.2.

Proposition 4.6 *Let $\omega = (\alpha_1, \alpha_2, \alpha_3)$ be pairwise coprime and let β_1, β_2 be such that $\alpha_1\alpha_2 + \alpha_3 = \alpha_1\beta_1 + \alpha_2\beta_2$. Let $S := \{F_{\omega, \beta_1, \beta_2}(x, y, z) = 0\}$, where*

$$F_{\omega, \beta_1, \beta_2}(x, y, z) = y^{2\alpha_1}z^2 + x^{2\alpha_2}z^2 + x^{2\beta_1}y^{2\beta_2} - 2z(x^{\alpha_2}y^{\alpha_1}z + x^{\alpha_2+\beta_1}y^{\alpha_1} + x^{\alpha_2}y^{\alpha_1+\beta_2})$$

defines a ω -homogeneous singularity. Then, the fundamental group of the Milnor fiber of S is cyclic of order $\gcd(\alpha_1 + 2\beta_2, \alpha_2 + 2\beta_1)$.

More complicated fundamental groups can be obtained by choosing the orbifold variant. Let (d_1, d_2, d_3) be a triple of pairwise coprime integers, $d_i > 1$, let $\omega = (d_2d_3, d_1d_3, d_1d_2)$ be a weight, and let

$$F_{d_1, d_2, d_3}(x, y, z) = x^{2d_1} + y^{2d_2} + z^{2d_3} - 2(x^{d_1}y^{d_2} + x^{d_1}z^{d_3} + y^{d_2}z^{d_3}).$$

As a direct consequence of Proposition 4.4 we obtain the following result.

Proposition 4.7 *The fundamental group of the Milnor fiber of $\{F_{d_1, d_2, d_3} = 0\}$ is infinite and non-abelian.*

5 Weighted Lê–Yomdin Surface Singularities

In this section we study the relationship between (weighted) projective plane curves and normal surface singularities whose link is a rational (or integral) homology sphere.

5.1 The Determinant of a Normal Surface Singularity

Let $(S, 0)$ be a germ of normal surface singularity and let K be its link. It is well known that K is a graph manifold whose plumbing decorated graph is the dual graph Γ of a simple normal crossing resolution. Each vertex v of Γ is decorated with two numbers (g_v, e_v) , where g_v is the genus of the corresponding irreducible component E_v and e_v is its self-intersection. Let A be the intersection matrix of the graph; recall that A is negative definite. In a natural way, A is also the presentation matrix of an abelian group yielding the following classical result.

Proposition 5.1 *The free part of $H_1(K; \mathbb{Z})$ has rank $2 \sum_v g_v + \text{Rank} H_1(\Gamma; \mathbb{Z})$. The torsion part is isomorphic to $\text{coker } A$ and, in particular, its cardinality is $\det(-A)$.*

As a direct consequence of this, the determinant $\det(-A)$ does not depend on the resolution. This justifies the definition of the determinant of a normal surface singularity.

Definition 5.2 The *determinant* $\det S$ of a normal surface singularity S is defined as $\det(-A)$, where A is the intersection matrix of any resolution of S .

As a consequence, one has the following combinatorial criteria to detect rational (resp. integral) homology sphere singularities, that is, surface singularities whose link is a rational (resp. integral) homology sphere.

Corollary 5.3 *The surface singularity S is a rational (resp. integral) homology sphere if and only if all g_v 's vanish and Γ is a tree (resp. and $\det S = 1$).*

5.2 Superisolated and $L\hat{e}$ -Yomdin Singularities

In [15], the authors relate hypersurface singularities whose link is a rational homology sphere with rational cuspidal curves using superisolated singularities. In our search for more examples of surface singularities whose link is a rational (or integral) homology sphere, a generalization of this method will be discussed here. For the sake of completeness we present a classical result.

Definition 5.4 Let $(S, 0) \subset (\mathbb{C}^3, 0)$ be the germ of a hypersurface singularity with equation $F = f_d + f_{d+k} + \dots$, where the previous decomposition is the decomposition in homogeneous parts. Assume $f_d \neq 0, k > 0$. Let $\mathcal{C}_m := V_{\mathbb{P}^2}(f_m)$ denote the projective zero locus in \mathbb{P}^2 of the homogeneous polynomial f_m . We say that S is a $L\hat{e}$ -Yomdin singularity if $\text{Sing}(\mathcal{C}_d) \cap \mathcal{C}_{d+k} = \emptyset$. If $k = 1$, S is called a *superisolated* singularity.

Superisolated singularities were introduced by Luengo [24]: they can be resolved by one blow-up. In [25], the authors show that the link of a superisolated singularity is a rational homology sphere if and only if all the irreducible components of \mathcal{C}_d are cuspidal rational and if the curve is reducible they only intersect at one point. Besides the smooth case, no other one provides an integral homology sphere as can be deduced from the following result in [24]. We reproduce the proof since it will be generalized for other classes of singularities.

Proposition 5.5 ([24]) *Let S be a superisolated singularity with tangent cone \mathcal{C}_d of degree d . Let $\Pi : \widehat{\mathbb{C}^3} \rightarrow \mathbb{C}^3$ be the blow-up of $0 \in S \subset \mathbb{C}^3$ and $\pi : \hat{S} \rightarrow S$ the restriction of Π to the strict transform of S . If $E \cong \mathbb{P}^2$ is the exceptional divisor of Π , then the exceptional divisor of π is $\mathcal{C}_d = E \cap \hat{S}$.*

Moreover, if $\mathcal{C}_{d,1}, \dots, \mathcal{C}_{d,s}$ denote the irreducible components of \mathcal{C}_d and $\delta_i := \text{deg } \mathcal{C}_{d,i}$, then

$$(\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,i})_{\hat{S}} = -\delta_i(d - \delta_i + 1), \quad (\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,j})_{\hat{S},P} = (\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,j})_{\mathbb{P}^2,P}, \quad i \neq j, P \in \text{Sing } \mathcal{C}_d.$$

Proof Let us assume that $[0 : 0 : 1] \in \text{Sing } \mathcal{C}_d$. We can fix the usual chart of the blowing-up. Assume that $S = \{F = 0\}$, where $F = f_d + f_{d+1} + \dots$; in the chart $(x, y, z) \mapsto (xz, yz, z)$ and $E = \{z = 0\}$, $\hat{S} = \{f_d(x, y, 1) + z(f_{d+1}(x, y, 1) + \dots) = 0\}$, i.e., $\mathcal{C}_d = \{z = f_d(x, y, 1) = 0\}$. In the neighborhood of P , (E, \mathcal{C}_d) and (\hat{S}, \mathcal{C}_d) are isomorphic. We deduce that for $i \neq j$, $(\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,j})_{\hat{S}, P} = (\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,j})_{\mathbb{P}^2, P}$.

The surfaces E and \hat{S} are generically transversal, namely outside $\text{Sing } \mathcal{C}_d$. The Euler class $e(E) = -L$, where L is a line in E . Then,

$$(\mathcal{C}_d \cdot \mathcal{C}_{d,i})_{\hat{S}} = (e(E) \cdot \mathcal{C}_{d,i})_{\mathbb{P}^2} = -\delta_i.$$

Also

$$\begin{aligned} (\mathcal{C}_d \cdot \mathcal{C}_{d,i})_{\hat{S}} &= (\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,i})_{\hat{S}} + \sum_{j \neq i} (\mathcal{C}_{d,j} \cdot \mathcal{C}_{d,i})_{\hat{S}} = (\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,i})_{\hat{S}} + \sum_{j \neq i} (\mathcal{C}_{d,j} \cdot \mathcal{C}_{d,i})_{\mathbb{P}^2} = \\ &= (\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,i})_{\hat{S}} + \delta_i(d - \delta_i), \end{aligned}$$

and the result follows.

Although π is not necessarily a resolution with normal crossings, $\det S$ can be recovered from it using its intersection matrix; it is a classical result which will follow from a later proposition.

Corollary 5.6 *If S is as above, then*

$$\det S = (d + 1)^{s-1} \cdot \delta_1 \cdot \dots \cdot \delta_s.$$

In particular, if \mathcal{C}_d is irreducible, then $\det S = d$.

Proof By Proposition 5.5, the diagonal terms of the intersection matrix for π equal $-\delta_i(d - \delta_i + 1)$ and the non-diagonal terms are $\delta_i \cdot \delta_j$. Replacing the first row by the sum of all rows, one obtains $-(\delta_1, \dots, \delta_s)$. If we add the new first row multiplied by δ_i times the i th-row ($i > 1$), all the non-diagonal terms vanish and the diagonal term becomes $-\delta_i(d + 1)$.

For \hat{L}^e -Yomdin singularities we follow the same strategy. If $S = \{F = 0\}$ with $F = f_d + f_{d+k} + \dots$, and we keep the notation above, the main difference is that \hat{S} is no longer smooth, in general. If $P \in \text{Sing } \mathcal{C}_d$, then the local equation of \hat{S} at P is $z^k - f(x, y) = 0$ where $f(x, y) = 0$ is the local equation of \mathcal{C}_d at P . Intersection theory can be used also in normal surfaces, see [18, 28] for definitions and [8] for useful tips. As the following result shows the intersection form of a partial resolution is also useful.

Lemma 5.7 *Let $(S, 0)$ be a normal surface singularity and let $\pi : (X, D) \rightarrow (S, 0)$ be a proper birational morphism which is an isomorphism outside $D = \pi^{-1}(0)$ on*

the normal surface X . Let A be the intersection matrix for D . Then,

$$\det S = \det(-A) \prod_{P \in D} \det(X, P).$$

Proof Note first that the product in the formula is finite since only a finite number of singular points may arise. Let $\sigma : (Y, E) \rightarrow (X, D)$ be a resolution of the singularities of X . Let B the intersection matrix of E . Instead of expressing this matrix in terms of the irreducible components of E , we replace the strict transforms of the components of D by their total transforms.

Then, B is replaced by a matrix \tilde{B} , with the same determinant, which is a diagonal sum of A and the intersection matrices of the singular points. Then,

$$\det S = \det(-B) = \det(-\tilde{B}) = \det(-A) \prod_{P \in \text{Sing } X} \det(X, P).$$

Proposition 5.8 *Let S be a k -Lê-Yomdin singularity with tangent cone \mathcal{C}_d of degree d . With the notation of Proposition 5.5,*

$$(\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,i})_{\hat{S}} = -\frac{\delta_i(d - \delta_i + k)}{k}, (\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,j})_{\hat{S},P} = \frac{(\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,j})_{\mathbb{P}^2,P}}{k}, i \neq j, P \in \text{Sing } \mathcal{C}_d.$$

Proof We follow the guidelines of the proof of Proposition 5.5. Note that it is not true any more that in the neighborhood of $P \in \text{Sing } \mathcal{C}_d$ the germs (E, \mathcal{C}_d) and (\hat{S}, \mathcal{C}_d) are isomorphic. However, the projection $\rho(x, y, z) := (x, y)$ restricts to a $k : 1$ proper map $(\hat{S}, \mathcal{C}_d) \rightarrow (E, \mathcal{C}_d)$. Since $\pi^*(\pi_*\mathcal{C}_{d,i}) = k\mathcal{C}_{d,i}$ we have that for $i \neq j$

$$(\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,j})_{\hat{S},P} = \frac{1}{k^2}(\pi^*\pi_*\mathcal{C}_{d,i} \cdot \pi^*\pi_*\mathcal{C}_{d,j})_{\hat{S},P} = \frac{1}{k}(\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,j})_{\mathbb{P}^2,P}.$$

For the self-intersections we apply the same ideas:

$$(\mathcal{C}_d \cdot \mathcal{C}_{d,i})_{\hat{S}} = (e(E) \cdot \mathcal{C}_{d,i})_{\mathbb{P}^2} = -\delta_i$$

$$(\mathcal{C}_d \cdot \mathcal{C}_{d,i})_{\hat{S}} = (\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,i})_{\hat{S}} + \sum_{j \neq i} (\mathcal{C}_{d,j} \cdot \mathcal{C}_{d,i})_{\hat{S}} = (\mathcal{C}_{d,i} \cdot \mathcal{C}_{d,i})_{\hat{S}} + \frac{\delta_i(d - \delta_i)}{k},$$

and the result follows.

A similar proof to the one of Corollary 5.6 provides the following result.

Corollary 5.9 *If S is a k -Lê-Yomdin as above, then*

$$\det S = \delta_1 \cdot \dots \cdot \delta_s \cdot \left(\frac{d+k}{k}\right)^{s-1} \prod_{P \in \text{Sing } \mathcal{C}_d} \det S_{P,k},$$

where

$$S_{P,k} = \{z^k = f_P(x, y) \mid P \in \text{Sing } C_d\},$$

and $f_P(x, y) = 0$ is a local equation of C_d at P .

In particular, if C_d is smooth, then $\det S = d$.

Example 5.10 Let S_k be the singularity $\{z^k = x^2 + y^2\}$, then $\det S_k = k$. Denote by T_k the singularity $\{z^k = x^2 + y^3\}$, then we have

$$\det T_k = \begin{cases} 1 & \text{if } \gcd(k, 6) = 1, 6 \\ 3 & \text{if } \gcd(k, 6) = 2 \\ 4 & \text{if } \gcd(k, 6) = 3. \end{cases}$$

Note that T_k admits a \mathbb{Q} -resolution with only one exceptional divisor. This divisor has positive genus (equal to one) if and only if $\gcd(k, 6) = 6$.

We did not find in the literature a general formula for this determinant. >From the above computations and the periodicity properties of the Alexander invariants, the following statement may be true.

Conjecture 1 *Let $C : f(x, y) = 0$ be a germ of a reduced plane curve singularity, and let $S_k : z^k = f(x, y)$ be a cyclic germ of surface. Let N be the order of the semisimple factor of the monodromy of C . Then $\det S_k$ is a quasi-polynomial in k of period N .*

Proposition 5.11 *A k -Lê–Yomdin singularity with tangent cone C_d has as link a rational homology sphere if and only if C_d is a union of rational cuspidal curves with only one intersection point and the links of the k -cyclic singularities associated with the singular points of C_d have also a rational homology sphere as a link.*

The proof of this proposition is a direct consequence of the previous result. We have not proven that Lê–Yomdin singularities do not provide integral homology sphere links, mainly since we do not have a closed formula for the determinant of a cyclic singularity. Our experimentation leads to this conjecture.

Conjecture 2 *No k -Lê–Yomdin singularity $k > 1$ has an integral homology sphere link.*

5.3 Weighted Lê–Yomdin Singularities

We are going to generalize these families of singularities using weighted homogeneous curves. We use the notation ω, η , etc. introduced in Sect. 2.2. The following notion of weighted Lê–Yomdin singularity was introduced in [7].

Definition 5.12 A hypersurface $(S, 0) := \{F = 0\}$ is an (ω, k) -weighted $L\hat{e}$ -Yomdin singularity if the following holds. Let $F := f_d + f_{d+k} + \dots$ be the decomposition in ω -weighted homogeneous forms, then $\text{Jac}(f_d) \cap \mathbf{V}(f_{d+k}) = \emptyset$.

In order to relate geometrically this definition with the definition of superisolated and $L\hat{e}$ -Yomdin singularities, let us consider the weighted blow-up $\Pi_\omega : \widehat{\mathbb{C}}_\omega^3 \rightarrow \mathbb{C}^3$. In Sect. 2.3 we have described a stratification of the exceptional divisor $E_\omega \cong \mathbb{P}_\omega^2 \cong \mathbb{P}_\eta^2$ according to the singularities of $\widehat{\mathbb{C}}_\omega^3$, see Proposition 2.3.

One needs to study the two curves $\mathcal{C}_d, \mathcal{C}_{d+k} \subset E_\omega$. In general, note that $f_d(x, y, z) = x^{\varepsilon_x} y^{\varepsilon_y} z^{\varepsilon_z} g(x^{d_1}, y^{d_2}, z^{d_3})$, where $\varepsilon_x, \varepsilon_y, \varepsilon_z \in \{0, 1\}$ and g is η -weighted homogeneous of degree $\frac{d - e_1\varepsilon_x - e_2\varepsilon_y - e_3\varepsilon_z}{d_1 d_2 d_3}$. If we see this curve in \mathbb{P}_η^2 its equation is $x^{\varepsilon_x} y^{\varepsilon_y} z^{\varepsilon_z} g(x, y, z) = 0$.

Proposition 5.13 Let $S = \{F = 0\}$ be an (ω, k) -weighted $L\hat{e}$ -Yomdin singularity with ω -quasi-tangent cone $\mathcal{C}_d = \{f_d = 0\}$. Let Π_ω be the ω -blow-up, $E_\omega \cong \mathbb{P}_\omega^2 \cong \mathbb{P}_\eta^2$ is the exceptional divisor and \hat{S} is the strict transform (and a partial resolution) of S . Recall the stratification of $E_\omega = \mathcal{P} \cup \mathcal{L} \cup \mathcal{T}$ as given in Notation 2.6. The structure of \hat{S} along $P \in \mathcal{C}_d = E_\omega \cap \hat{S}$ is as follows:

- (1) $P \in \mathcal{T}$.
 - (a) If $P \notin \text{Sing } \mathcal{C}_d$ then \hat{S} is smooth at P and $E_\omega \pitchfork_P \hat{S}$.
 - (b) If $P \in \text{Sing } \mathcal{C}_d$ then $P \notin \mathcal{C}_{d+k}$. There are local coordinates U, V, W centered at P such that $E_\omega = \{W = 0\}$, $\mathcal{C}_d = \{W = g(U, V) = 0\}$ and $\hat{S} = \{W^k = g(U, V)\}$; in particular \hat{S} is smooth at P if and only if $k = 1$ (but it is not transversal to E_ω).
- (2) $P \in \mathcal{L}_y$ (a similar statement holds for $\mathcal{L}_x, \mathcal{L}_z$).
 - (a) If \mathcal{C}_d is transversal to Y at P then $(\hat{S}, P) \cong \frac{1}{d_2}(e_2, -1)$. In the quotient ambient space $(\widehat{\mathbb{C}}_\omega^3, P)$ the situation is similar to (1a).
 - (b) If $(\mathcal{C}_d, P) = (Y, P)$ then (\hat{S}, P) is smooth. In the quotient ambient space $(\widehat{\mathbb{C}}_\omega^3, P)$ the situation is similar to (1a).
 - (c) If $\mathcal{C}_d \not\pitchfork_P Y$, i.e. the order of $f_d(x + t, y, 1)$ is > 1 ($P = [t : 0 : 1]$), then $P \notin \mathcal{C}_{d+k}$. The germ (\hat{S}, P) is isomorphic to $z^k = f_d(x + t, y, 1)$ in the threefold quotient singularity $\frac{1}{d_2}(0, e_2, -1)$, where $z = 0$ is the equation of E_ω .
- (3) $P = P_z$ (a similar statement holds for P_x, P_y).
 - (a) If \mathcal{C}_d is extremely quasi-smooth at P (i.e. the order of $f_d(x, y, 1)$ is 1) the situation is as in (1a) replacing the ambient smooth space by the threefold quotient singularity $\frac{1}{e_3}(e_1, e_2, -1)$. Let $h_1(x, y)$ be the linear part of $f_d(x, y, 1)$.
 - (i) If $h_1(x, y)$ is proportional to x , then $(\hat{S}, P) \cong \frac{1}{e_3}(e_1, -1)$.
 - (ii) If $h_1(x, y)$ is proportional to y , then $(\hat{S}, P) \cong \frac{1}{e_3}(e_2, -1)$.
 - (iii) Otherwise, $e_1 \equiv e_2 \pmod{e_3}$ and the above cases coincide.

- (b) If \mathcal{C}_d is not extremely quasi-smooth at P (i.e., the order of $f_d(x, y, 1)$ is > 1), then $P \notin \mathcal{C}_{d+k}$ and $d+k \equiv 0 \pmod{e_3}$. The germ (\hat{S}, P) is isomorphic to $z^k = f_d(x, y, 1)$ in the threefold quotient singularity $\frac{1}{e_3}(e_1, e_2, -1)$, where $z = 0$ is the equation of E_ω .

Proof The different parts of the statement will be particular cases of the following general situation. Assume $P = [0 : 0 : 1] \in \mathcal{C}_d = E_\omega \cap \hat{S}$ is a point of the strict transform \hat{S} of S on the exceptional divisor E_ω . The total transform of S is equal to $\hat{S} + dE_\omega$ and hence, its equation in the chart $\Psi_{\omega,3}$ is:

$$z^d (f_d(x, y, 1) + z^k \underbrace{(f_{d+k}(x, y, 1) + \dots)}_{q(x,y)}) = 0.$$

By hypothesis $f_d(0, 0, 1) = 0$, let us denote by $\ell(x, y)$ the linear part of $f_d(x, y, 1)$. The following conditions are immediate

$$\begin{cases} P \notin \mathcal{C}_{d+k} \iff q(x, y) \text{ is a unit,} \\ P \in \mathcal{C}_{d+k} \implies \ell(x, y) \neq 0. \end{cases}$$

We will consider (x_1, y_1, z_1) a change of coordinates where

$$(x_1, y_1, z_1) = \begin{cases} (x, y, zq(x, y)^{\frac{1}{k}}) & \text{if } \ell(x, y) = 0, \\ (\frac{1}{a}(f_d(x, y, 1) + z^k q(x, y)), y, z) & \text{if } \ell(x, y) = ax, \\ (x, \frac{1}{b}(f_d(x, y, 1) + z^k q(x, y) - ax), z) & \text{if } \ell(x, y) = ax + by, b \neq 0. \end{cases}$$

Note that the action of μ_{e_3} on (x_1, y_1, z_1) reads as in (x, y, z) . If $P \notin \mathcal{C}_{d+k}$ then $d+k \equiv 0 \pmod{e_3}$.

In these coordinates $E_\omega : z_1 = 0$ and $\mathcal{C}_d : W = g(x_1, y_1) = 0$, where

$$g(x_1, y_1) = \begin{cases} f_d(x_1, y_1, 1) & \text{if } \ell(x, y) = 0, \\ \ell(x_1, y_1) & \text{otherwise.} \end{cases}$$

The local equations for $dE_\omega + \hat{S}$ are $z_1^d(z_1^k + g) = 0$. If $\ell(x, y) \neq 0$, then both look like two surfaces in a quotient ambient space whose preimages in \mathbb{C}^3 are smooth and transversal.

The case $P \in \mathcal{T}$ locally corresponds to $\omega = (1, 1, 1)$. Note that (1a) implies $\ell(x_1, y_1) \neq 0$ whereas (1b) implies $\ell(x_1, y_1) = 0$. The case $P \in \mathcal{L}_y$ corresponds with the choice $\omega = (d_2, e_2, d_2)$ where (2a) and (2b) refers to $\ell \neq 0$ and (2c) refers to $\ell = 0$. Finally, $P = P_z$, corresponds to the choice $\omega = (e_1, e_2, e_3)$. In this case (3a) refers to the different cases of $\ell \neq 0$ and (3b) refers to $\ell = 0$.

The divisor C_d has an irreducible decomposition in $s + \varepsilon_x + \varepsilon_y + \varepsilon_z$ components $\varepsilon_x X + \varepsilon_y Y + \varepsilon_z Z + \tilde{C}_d$, where $\tilde{C}_d = \sum_{i=1}^s C_{d,i}$ and $\varepsilon_x, \varepsilon_y, \varepsilon_z \in \{0, 1\}$. Recall that $d_1 d_2 d_3$ divides $\deg C_{d,i}$ and we can write $\delta_i = d_1 d_2 d_3 \hat{\delta}_i$.

Consider the stratification of a weighted projective plane as above. We call a curve in a weighted projective plane *stratified smooth* if it is smooth, intersects the axes transversally and does not contain the vertices.

Proposition 5.14 *Let S be an (ω, k) -weighted $L\hat{e}$ -Yomdin singularity with quasi-tangent cone $C_d \subset \mathbb{P}_\omega^2$ of degree d . With the notation of Proposition 5.8:*

- (1) $(C_{d,i} \cdot C_{d,i})_{\hat{S}} = -\frac{\delta_i(d-\delta_i+k)}{ke_1e_2e_3} = -\frac{\hat{\delta}_i(d-\hat{\delta}_i+k)}{kd_1d_2d_3\alpha_1\alpha_2\alpha_3}$.
- (2) If $\varepsilon_x = 1$, then $(X \cdot X)_{\hat{S}} = -\frac{d_1^2e_1(d-e_1+k)}{ke_1e_2e_3} = -\frac{d_1^2(d-e_1+k)}{ke_2e_3} = -\frac{d-e_1+k}{kd_2d_3\alpha_2\alpha_3}$. Similar formulas hold for Y and Z .
- (3) If $i \neq j$, then $(C_{d,i} \cdot C_{d,j})_{\hat{S}} = \frac{\delta_i\delta_j}{ke_1e_2e_3} = \frac{\hat{\delta}_i\hat{\delta}_j}{k\alpha_1\alpha_2\alpha_3}$.
- (4) If $\varepsilon_x = 1$ then $(C_{d,i} \cdot X)_{\hat{S}} = \frac{d_1\delta_i}{ke_2e_3} = \frac{\hat{\delta}_i}{k\alpha_2\alpha_3}$. Similar formulas hold for Y and Z .
- (5) If $\varepsilon_x\varepsilon_y = 1$ $(X \cdot Y)_{\hat{S}, P_z} = \frac{d_1d_2}{ke_3} = \frac{1}{k\alpha_3}$. Similar formulas hold for the other pairs involving X, Y , and Z .

Proof We follow the ideas in the proofs of Propositions 5.5 and 5.8 with some modifications. If $\omega \neq \eta$, the map $\pi_{\eta,\omega}^{-1} : \mathbb{P}_\eta^2 \rightarrow \mathbb{P}_\omega^2$ can be considered as the identity where \mathbb{P}_ω^2 is seen as $(\mathbb{P}_\eta^2)^{\text{orb}}$, where $(\pi_{\eta,\omega}^{-1})^*(X) = \frac{1}{d_1}X$, $(\pi_{\eta,\omega}^{-1})^*(Y) = \frac{1}{d_2}Y$, and $(\pi_{\eta,\omega}^{-1})^*(Z) = \frac{1}{d_3}Z$. Analogously, the abstract strict transform \hat{S} has a natural orbifold embedded structure $\hat{S}^{\text{orb}} \subset \widehat{\mathbb{C}}_\omega^3$ where the embedding $\pi : \hat{S} \rightarrow \hat{S}^{\text{orb}}$ has the same properties for X, Y, Z as $\pi_{\eta,\omega}^{-1}$ whenever X, Y, Z are contained in C_d .

The divisor $e(E_\omega)$ in $E_\omega \equiv \mathbb{P}_\omega^2$ has degree 1 and Bézout’s Theorem for the ω -projective plane states that the sum of the intersection numbers of two divisors equals the product of the degrees divided by $e_1e_2e_3$. Hence, we obtain the same formulas as in Proposition 5.8 with two differences: $e_1e_2e_3$ appears in the denominator and all the intersection numbers are considered in \hat{S}^{orb} .

When we consider the intersection numbers in \hat{S} , when X appears ($\varepsilon_x = 1$), the formulas must be multiplied by d_1 . A similar argument holds for Y, Z .

Corollary 5.15 *If S is an (ω, k) -weighted $L\hat{e}$ -Yomdin as above and A is the intersection matrix of the blowing-up, then*

$$\det S = d_1^{2\varepsilon_x} \cdot d_2^{2\varepsilon_y} \cdot d_3^{2\varepsilon_z} \cdot \delta_1 \cdot \dots \cdot \delta_s \cdot \left(\frac{d+k}{ke_1e_2e_3} \right)^{s-1} \prod_{P \in \hat{S}} \det(\hat{S}_{k,P}),$$

where $\hat{S}_{k,P}$ is the surface singularity at P as described in Proposition 5.13.

In particular, if C_d is stratified smooth, then $\det S = \frac{d}{e_1e_2e_3}$.

6 Normal Surface Singularities with Rational Homology Sphere Links

In this section we will use the results and strategies presented in Sect. 5 in order to exhibit examples of weighted $\widehat{\text{L}}\widehat{\text{e}}$ -Yomdin singularities whose links are rational homology spheres, generalizing the strategy in [15]. We will be using Proposition 5.11 in the context of weighted $\widehat{\text{L}}\widehat{\text{e}}$ -Yomdin singularities.

6.1 Brieskorn–Pham Singularities

We will interpret these singularities as $\widehat{\text{L}}\widehat{\text{e}}$ -Yomdin singularities and study their \mathbb{Q} -resolution graph. Consider $\omega_0 = (n_1, n_2, n_3)$ and the Brieskorn–Pham singularity $S = \{F_{\omega_0} = x^{n_1} + y^{n_2} + z^{n_3} = 0\} \subset (\mathbb{C}^3, 0)$, where n_1, n_2, n_3 are not assumed to be coprime.

Denote by $e := \text{gcd } \omega_0$, and $\alpha_k := \frac{1}{e} \text{gcd}(n_i, n_j)$, where $\{i, j, k\} = \{1, 2, 3\}$. Note that $d_i := \frac{n_i}{e\alpha_j\alpha_k} \in \mathbb{Z}_{>0}$ are pairwise coprime. If

$$\omega = (e_1, e_2, e_3) := \frac{1}{e^2\alpha_1\alpha_2\alpha_3} (n_2n_3, n_1n_3, n_1n_2) = (\alpha_1d_2d_3, \alpha_2d_1d_3, \alpha_3d_1d_2),$$

then $F_{\omega_0}(x, y, z)$ is an ω -weighted homogeneous polynomial of degree $d := \frac{n_1n_2n_3}{e^2\alpha_1\alpha_2\alpha_3}$ and hence S can be viewed as an (ω, k) -weighted $\widehat{\text{L}}\widehat{\text{e}}$ -Yomdin singularity for any $k \geq 1$. Following the general construction, $f_d = F_{\omega_0}(x, y, z) = g(x^{d_1}, y^{d_2}, z^{d_3}) = 0$ can be considered a curve in $\mathbb{P}_{\eta}^2 \cong \mathbb{P}_{\omega}^2$ for $\eta = (\alpha_1, \alpha_2, \alpha_3)$ of η -degree $d_{\eta} = e\alpha_1\alpha_2\alpha_3$ given by the equation $g(x, y, z) = x^{e\alpha_2\alpha_3} + x^{e\alpha_1\alpha_3} + z^{e\alpha_1\alpha_2} = 0$. Its genus is

$$\frac{d_{\eta}(d_{\eta} - |\eta|)}{2\alpha_1\alpha_2\alpha_3} + 1 = \frac{e^2\alpha_1\alpha_2\alpha_3 - e(\alpha_1 + \alpha_2 + \alpha_3) + 2}{2}.$$

Since the curve \mathcal{C}_d is transversal to the axes we obtain that the exceptional locus of \widehat{S} has (in the intersection with the axes) $e\alpha_i$ cyclic points of order d_i . The determinant of the singularity is

$$\frac{d}{(d_1d_2d_3)^2(\alpha_1\alpha_2\alpha_3)} (d_1^{\alpha_1} d_2^{\alpha_2} d_3^{\alpha_3})^e = ed_1^{e\alpha_1-1} d_2^{e\alpha_2-1} d_3^{e\alpha_3-1}.$$

As a consequence of this discussion one obtains the following.

Proposition 6.1 *The Brieskorn–Pham singularity $S = \{F_{\omega_0} = x^{n_1} + y^{n_2} + z^{n_3} = 0\} \subset (\mathbb{C}^3, 0)$ is a rational homology sphere singularity if and only if either $\alpha_1 = \alpha_2 = \alpha_3 = 1, e = 2$ or $\alpha_i = \alpha_j = e = 1$ for some $i \neq j$.*

Moreover, it is an integral homology sphere if and only if the exponents are pairwise coprime.

6.2 Examples Coming from Cremona Transformations and Kummer Covers

The purpose of this section is to provide more candidates to surface singularities with rational homology sphere links by applying the techniques used in Sect. 4. In particular, we will start with the strict transforms of the conic by the Cremona transformations.

In order to do so one needs $\omega := (\alpha_1, \alpha_2, \alpha_3)$ pairwise coprime, and $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$ such that $\alpha_1\alpha_2 + \alpha_3 = \alpha_1\beta_1 + \alpha_2\beta_2$. The weighted homogeneous polynomial

$$\begin{aligned} f_\omega(x, y, z) &= f(y^{\alpha_1}z, x^{\alpha_2}z, x^{\beta_1}y^{\beta_2}) \\ &= y^{2\alpha_1}z^2 + x^{2\alpha_2}z^2 + x^{2\beta_1}y^{2\beta_2} - 2z \left(x^{\alpha_2}y^{\alpha_1}z + x^{\beta_1}y^{\alpha_1+\beta_2} + x^{\alpha_2+\beta_1}y^{\beta_2} \right) \end{aligned}$$

has ω -degree $2(\alpha_1\alpha_2 + \alpha_3)$ and defines a rational curve in \mathbb{P}_ω^2 which is smooth outside the vertices. Assume for simplicity that $\alpha_1\alpha_2 + \alpha_3 < \alpha_1\alpha_2\alpha_3$. Hence for any generic quasi-homogeneous polynomial $g_\omega(x, y, z)$ of degree $2\alpha_1\alpha_2\alpha_3$, $F := f_\omega + g_\omega$ defines an (ω, k) -weighted Lê–Yomdin singularity, for $k = 2(\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2 - \alpha_3)$. A partial resolution of this singularity has an exceptional locus which is a rational curve with three singular points. In most cases the link of this singularity is a rational homology sphere. For simplicity, we will prove it in a special case.

Proposition 6.2 *With the previous notation, take $\alpha_1 = 1$, $\beta_1 = \alpha_3$, and $\beta_2 = 1$. Then, for any $\alpha_2, \alpha_3 > 1$ satisfying $\gcd(3, k) = \gcd(3, \alpha_2\alpha_3 - \alpha_2 - \alpha_3) = 1$ and a generic g_ω , the equation $\{F = f_\omega + g_\omega = 0\} \subset \mathbb{C}^3$ defines a surface singularity with a rational homology sphere link.*

Proof We study the strict transform of this singularity at P_x, P_y, P_z after an ω -weighted blow-up. At P_x , the ambient space is smooth and the strict transform has equation

$$0 = x^k + y^2z^2 + z^2 + y^2 - 2yz(z + y + 1) = x^k + (y_1 + z)^2z^2 + y_1^2 - 2(y_1 + z)z(2z + y_1)$$

if $y_1 = y - z$. This is topologically equivalent to $0 = x^k + y^2 + z^3$. Since $\gcd(3, \alpha_2\alpha_3 - \alpha_2 - \alpha_3) = 1$, by Proposition 6.1 the link of this singularity is a rational homology sphere.

At P_y , the ambient space is $\frac{1}{\alpha_2}(1, -1, \alpha_3)$ and the strict transform has equation

$$\begin{aligned} f_\omega(x, y, z) &= y^k + z^2 + x^{2\alpha_2}z^2 + x^{2\alpha_3} - 2z \left(x^{\alpha_2}z + x^{\alpha_3} + x^{\alpha_2+\alpha_3} \right) \\ &= y^k + z_1^2 - 2x^{\alpha_2+2\alpha_3} + \dots \end{aligned}$$

if $z_1 = z - x^{\alpha_3}$. This change of variable is compatible with the action; this equation defines a singularity in \mathbb{C}^3 whose link is a rational homology sphere, and so is the case in the quotient manifold.

By symmetry arguments, the same happens for P_z . Hence, F defines a singularity whose link is a rational homology sphere.

Let us use the orbifold approach. Given (d_1, d_2, d_3) pairwise coprime consider $\omega := (d_2d_3, d_1d_3, d_1d_2)$; the normalized η is $(1, 1, 1)$ and the isomorphism $\mathbb{P}_\omega^2 \rightarrow \mathbb{P}^2$ is given by $[x : y : z]_\omega \mapsto [x^{d_1} : y^{d_2} : z^{d_3}]$, see (2.1). This isomorphism can be seen as a *weighted Kummer cover* and the homogeneous polynomial

$$f_\omega(x, y, z) = f(x^{d_1}, y^{d_2}, z^{d_3}) = x^{2d_1} + y^{2d_2} + z^{2d_3} - 2 \left(y^{d_2} z^{d_3} + x^{d_1} z^{d_3} + x^{d_1} y^{d_2} \right)$$

of ω -degree $2d_1d_2d_3$, which defines a rational curve in $\mathbb{P}_\omega^2 \cong \mathbb{P}^2$ and it is tangent to the axes. In most cases the link of this singularity is a rational homology sphere. Let us study a special case.

Proposition 6.3 *For any generic quasi-homogeneous polynomial $g_\omega(x, y, z)$ of degree $3d_1d_2d_3$, and d_1, d_2, d_3 odd numbers $\{F := f_\omega + g_\omega = 0\} \subset \mathbb{C}^3$ defines a surface singularity with a rational homology sphere link.*

Proof Note that $\{F = 0\}$ defines an (ω, k) -weighted Lê–Yomdin singularity, for $k = d_1d_2d_3$. A partial resolution of this singularity has an exceptional locus which is a rational curve with three singular points (corresponding to the tangencies). In most cases the link of this singularity is a rational homology sphere.

By symmetry reasons we study only the strict transform of this singularity at the tangency point with Y after an ω -weighted blow-up. After a change of coordinates the local equation of F is

$$\begin{aligned} 0 &= z^{d_1d_2d_3} + (x + 1)^{2d_1} + y^{2d_2} + 1 - 2 \left(y^{d_2} + (x + 1)^{d_1} + (x + 1)^{d_1} y^{d_2} \right) \\ &= z^{d_1d_2d_3} + d_1^2 x^2 - 2y^{d_2} + \dots \end{aligned}$$

and the ambient space is $\frac{1}{d_2}(0, d_1d_3, -1)$. Since d_1, d_2, d_3 are odd numbers, by Proposition 6.1 this equation defines a singularity in \mathbb{C}^3 whose link is a rational homology sphere, and so is the case in the quotient manifold.

6.3 Integral Homology Sphere Surface Singularities

Following ideas of the third named author, Veys, and Vos, we present an infinite family of normal surface singularities which are complete intersection in \mathbb{C}^4 and whose links are integral homology spheres. The splice diagram of this family is

precisely the one given in [29, p. 765] and the corresponding semigroup conditions are satisfied. The examples given here can be generalized to any dimension.

Let $n_0, n_1, n_2, n_3 \in \mathbb{Z}_{>0}$ and $b_{20}, b_{21}, b_{30}, b_{31}, b_{32} \in \mathbb{Z}_{\geq 0}$. Consider S the surface singularity in $(\mathbb{C}^4, 0)$ defined by

$$S = \{f_1 + f_2 = f_2 + f_3 = 0\} \subset (\mathbb{C}^4, 0), \text{ where } \begin{cases} f_1 = x_1^{n_1} - x_0^{n_0}, \\ f_2 = x_2^{n_2} - x_0^{b_{20}} x_1^{b_{21}}, \\ f_3 = x_3^{n_3} - x_0^{b_{30}} x_1^{b_{31}} x_2^{b_{32}}. \end{cases} \tag{6.1}$$

The purpose of this section is to show when the link of S is a rational homology sphere as well as to characterize when it is integral. The idea is to resolve S with \mathbb{Q} -normal crossings and apply Lemma 5.7 to compute $\det S$. In order to do so we consider the Cartier divisors of S defined by $Y = \{f_1 = 0\}$ and $H_i = \{x_i = 0\}$, $i = 0, 1, 2$. This family was recently studied in Vos’ PhD thesis in a more general context and we just briefly discuss here the construction of the partial resolution obtained in [27, section 5].

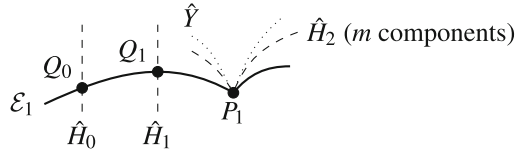
Theorem 6.4 *Let $S \subset (\mathbb{C}^4, 0)$ be the surface singularity defined above. Assume $n_0, n_1, n_2, n_3 \in \mathbb{Z}_{>0}$ are pairwise coprime, then S is a rational homology sphere. Moreover, in that case S is an integral homology sphere singularity if and only if $m := \gcd(n_3, b_{20}n_1 + b_{21}n_0) = 1$.*

Proof Let $\pi_1 : \widehat{\mathbb{C}}^4 \rightarrow \mathbb{C}^4$ be the weighted blow-up at the origin of \mathbb{C}^4 with weights $w_1 = (\frac{n}{n_0}, \frac{n}{n_1}, \frac{n}{n_2}, \frac{n}{n_3})$ where $n = n_0n_1n_2n_3$. The exceptional divisor of π_1 is the weighted projective variety $E_1 = \mathbb{P}^3_{w_1}$. The assumption on the integers n_i , $i = 0, \dots, 3$ being pairwise coprime implies that the exceptional divisor \mathcal{E}_1 of the restriction $\varphi_1 = \pi_1|_{\widehat{S}} : \widehat{S} \rightarrow S$ is a rational irreducible curve which contains three singular points of \widehat{S} , namely $Q_0 = \widehat{H}_0 \cap \mathcal{E}_1$, $Q_1 = \widehat{H}_1 \cap \mathcal{E}_1$, and $P_1 = \widehat{H}_2 \cap \mathcal{E}_1 = \widehat{Y} \cap \mathcal{E}_1$, see Fig. 5. The local type of the singularities at Q_0 and Q_1 are given by

$$Q_0 : \begin{cases} \widehat{S} = \frac{1}{n_0}(n_1n_2n_3, -1) \\ \mathcal{E}_1 : x_1 = 0, \widehat{H}_0^{\text{red}} : x_0 = 0, \end{cases} \quad Q_1 : \begin{cases} \widehat{S} = \frac{1}{n_1}(-1, n_0n_2n_3) \\ \mathcal{E}_1 : x_0 = 0, \widehat{H}_1^{\text{red}} : x_1 = 0. \end{cases}$$

Around P_1 the surface \widehat{S} can be described inside $\frac{1}{n_2n_3}(-1, n_0n_1n_3, n_0n_1n_2)$ as the set of zeros of $x_2^{n_2} - x_0^{b'_2} + x_3^{n_3} - x_0^{b'_3}x_2^{b_{32}} + (x_2^{n_2} - x_0^{b'_2})R'_2(x_0, x_2)$ where $b'_i = b_{i,0}\frac{n}{n_0} + \dots + b_{i,i-1}\frac{n}{n_{i-1}} - n$, $i = 2, 3$, and $R'_i(0, x_2) = 0$. Since the monomial with higher order will not play any role in the resolution of S , roughly speaking the

Fig. 5 First step of the \mathbb{Q} -resolution of S



situation at P_1 with variables $[(x_0, x_2, x_3)]$ can be thought of as

$$P_1 : \begin{cases} \hat{S} = \{x_0^{b'_2} + x_2^{n_2} + x_3^{n_3} = 0\} \subset \frac{1}{n_2 n_3}(-1, n_0 n_1 n_3, n_0 n_1 n_2) \\ \mathcal{E}_1 : x_0 = 0, \quad \hat{H}_2^{\text{red}} : x_2 = 0, \quad \hat{Y} : x_0^{b'_2} + x_2^{n_2} = 0. \end{cases} \quad (6.2)$$

The points Q_0 and Q_1 already have \mathbb{Q} -normal crossings, so one does not need to blow them up anymore. Consider the previous coordinates around P_1 and let π_2 be the blow-up at P_1 with weights $w_2 = (1, \frac{b'_2}{n_2}, \frac{b'_2}{n_3})$. The exceptional divisor of π_2 is $E_2 = \mathbb{P}^2_{w_2}/G$ where G is a cyclic group of order $n_2 n_3$ acting diagonally as in (6.2). The exceptional divisor \mathcal{E}_2 of the restriction $\varphi_2|_{\hat{S}} : \hat{S} \rightarrow \hat{S}$ is again a rational irreducible curve containing $2 + m$ cyclic quotient singular points of \hat{S} , namely $Q_{12} = \mathcal{E}_1 \cap \mathcal{E}_2$, $P_2 = \hat{Y} \cap \mathcal{E}_2$, and points $Q_{2j} \in \hat{H}_2 \cap \mathcal{E}_2$ (Fig. 6).

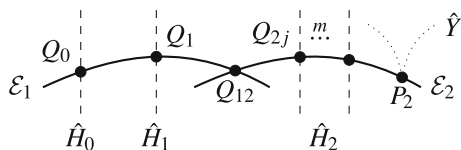
The composition $\varphi = \varphi_1 \circ \varphi_2 : \hat{S} \rightarrow S$ is a \mathbb{Q} -resolution of S and the order of the groups at Q_{12} , Q_{2j} , and P_2 are d , n_2 , and $\frac{n_3}{m}$, respectively. Since the \mathbb{Q} -resolution graph is a tree and the exceptional divisors are isomorphic to \mathbb{P}^1 the link of S is a rational homology sphere. In order to compute $\det S$ one needs to calculate the self-intersection numbers $\mathcal{E}_i^2 = -a_i$, $i = 1, 2$, which can be done by exploiting our information on the curve Y in the partial resolution of S . First, note that the intersection of \mathcal{E}_2 with \hat{Y} at P_2 is m . Second,

$$\varphi^* Y = \hat{Y} + N_1 \mathcal{E}_1 + N_2 \mathcal{E}_2$$

where $N_1 = n_0 n_1 n_2 n_3 = n$ and $N_2 = \frac{b'_2 + n}{m}$. Since $\mathcal{E}_i \cdot \varphi^* Y = 0$, $i = 1, 2$, one obtains that $a_1 = \frac{N_2}{N_1 d}$ and $a_2 = \frac{m + \frac{N_1}{d}}{N_2}$. Therefore the determinant of the intersection matrix is given by

$$\det(A) = \det \begin{pmatrix} -a_1 & \frac{1}{d} \\ \frac{1}{d} & -a_2 \end{pmatrix} = \frac{m}{N_1 d}.$$

Fig. 6 \mathbb{Q} -resolution of S



By Lemma 5.7 one has

$$\det S = \det(-A)n_0n_1dn_2^m \frac{n_3}{m} = n_2^{m-1}.$$

Therefore by Corollary 5.3 the link of S is an integral homology sphere if and only if $\det S = 1$, or equivalently, $m = 1$ as claimed.

Remark 6.5 If the exponents n_i 's are not pairwise coprime, then $\mathcal{E}_1 = \bigsqcup_j \mathcal{E}_{1j}$ has $n_{23} = \gcd(n_2, n_3)$ irreducible components and \mathcal{E}_2 is irreducible. They have genus

$$g(\mathcal{E}_{1j}) = \frac{1}{2} \left(\frac{n_{123}}{n_{23}} - 1 \right) \left(\frac{n_{023}}{n_{23}} - 1 \right) \quad \text{and} \quad g(\mathcal{E}_2) = \frac{1}{2}(n_{23} - 1)(m - 1),$$

where $m = \gcd(n_3, b)$ with $b = b_{20}n_1 + b_{21}n_0$. The determinant of S can be rewritten as

$$\det S = \left(\frac{b}{m} \right)^{n_{23}-1} \left(\frac{N_1}{\alpha} \right)^{n_{123}-n_{23}} \left(\frac{N_1}{\beta} \right)^{n_{023}-n_{23}} \left(\frac{n_2}{n_{23}} \right)^{m-1}$$

where $N_1 = \text{lcm}(n_0, n_1, n_2, n_3)$, $\alpha = \text{lcm}(n_1, n_2, n_3)$, $\beta = \text{lcm}(n_0, n_2, n_3)$, and $n_{ijk} = \frac{n_i n_j n_k}{\text{lcm}(n_i, n_j, n_k)}$. From here it can easily be shown that the link of S is an integral homology sphere if and only if $\gcd(n_i, n_j) = 1, i \neq j$, and $m = 1$. The details are left to the reader.

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