Real Seifert Forms, Hodge Numbers and Blanchfield Pairings

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Abstract In this survey article we present connections between Picard–Lefschetz invariants of isolated hypersurface singularities and Blanchfield forms for links. We emphasize the unifying role of Hermitian Variation Structures introduced by Némethi.

Keywords Seifert forms · Hodge numbers · Milnor fibration · Linking pairings · Blanchfield pairings

1 Introduction

Understanding a mathematical object via decomposing it into simple pieces is a very general procedure in mathematics, which can be seen in various branches and various fields. These procedures, often very different from each other, sometimes share common properties. In some cases, one mathematical object is defined in several fields and one procedure of decomposing is known under different names in different areas of mathematics.

The aim of this article is to show a bridge between real Blanchfield forms in knot theory and real Hermitian Variation Structures in singularity theory. In fact, we want to explain that these two apparently distant objects describe (almost) the same concept. Moreover, the methods for studying these two objects are very close. To be more specific, classification of simple Hermitian Variation Structures is an instance of a procedure known in algebraic geometry and algebraic topology as *dévissage*, which—in a vague sense—can be seen as a refinement of a primary decomposition of a torsion module over a PID. Dévissage is an important method of studying abstract linking forms, in particular, Blanchfield forms.

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These two points of view: the Hodge-theoretical one and the algebraic one, give possibility to apply methods of one field to answer questions that arise in another field. In this way, the first author and Némethi gave a proof of semicontinuity of a spectrum of a plane curve singularity [5] using Murasugi inequality of signatures. Conversely, the Hodge theoretic approach to Blanchfield forms, allows us to define Hodge-type invariants for links in S^3. Using these invariants we can quickly compute knot invariants based on a small piece of data: an exemplary calculation is shown in Example 4.1.

Another feature of Hodge-theoretical perspective is the formula for the Tristram–Levine signature, which we state in Proposition 4.7. This formula allows us to define the analog of the Tristram–Levine signature for twisted Blanchfield pairings, compare Definition 6.2. Many existing constructions of similar objects involve a choice of a matrix *representing* a pairing, see [7, Section 3.4]. However, finding a matrix representing given pairing, even for pairings over $\mathbb{C}[t, t^{-1}]$ is not a completely trivial task, see e.g. [7, Proposition 3.12]. The approach through Hodge numbers allows us to bypass this difficulty.

The structure of the paper is the following. In Sect. 2 we recall the basics of Picard–Lefschetz theory. This section serves as a motivation for introducing abstract Hermitian Variation Structures in Sect. 3. Section 4 recalls the construction of a Hermitian Variation Structure for general links in S^3. We also clarify several results of Keef, which were not completely correctly referred to in [6]. In Sect. 5 we give a definition of Blanchfield forms. We show that there is a correspondence between real Blanchfield forms and real Hermitian Variation Structures associated with the link. Moreover, the classification of the two objects is very similar.

In the last Sect. 6 we sketch the construction of twisted Blanchfield pairings and introduce Hodge numbers for such structures. We show how to recover the signature function from such a pairing. An example is given by Casson–Gordon signatures.

2 Milnor Fibration and Picard-Lefschetz Theory

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial map with $0 \in \mathbb{C}^{n+1}$ an isolated critical point.

Theorem 2.1 (Milnor's Fibration Theorem, See [23]) *For $\varepsilon > 0$ sufficiently small, the map $\Psi : S^{2n+1}_{\varepsilon} \setminus f^{-1}(0) \to S^1$ given by $\Psi(z) = \frac{f(z)}{\|f(z)\|}$ is a locally trivial fibration. The fiber $\Psi^{-1}(1)$ has the homotopy type of a wedge sum of some finite number of spheres S^n.*

The map Ψ can be explicitly described near the set $f^{-1}(0)$. Namely, choose a sufficiently small regular neighborhood N of $f^{-1}(0)$. It has a structure of a trivial D^2 bundle over $f^{-1}(0)$. Choose polar coordinates (r, ϕ) on $D^2 \setminus \{0\}$. For some choice of trivialization $N \cong f^{-1}(0) \times D^2$, the map Ψ is given by $\Psi(x, r, \phi) = \frac{1}{2\pi}\phi$ (here $x \in f^{-1}(0)$, r, ϕ are coordinates on the disk). See [35, Section 2.4.13] for more details.

Let F_t be the fiber $\Psi^{-1}(t)$. The *geometric monodromy* h_t is a diffeomorphism $h_t : F_1 \to F_t$, smoothly depending on t, which corresponds to the trivialization of the Milnor fibration on the arc of S^1 from 1 to t. Note that h_t is well-defined only up to homotopy.

Definition 2.1 The *homological monodromy* is the map $h : H_n(F_1; \mathbb{Z}) \to H_n(F_1; \mathbb{Z})$ induced by the monodromy.

Remark 2.1 The monodromy map in $N \setminus f^{-1}(0)$ can be defined to be $h_t(x, r, 0) = (x, r, 2\pi t)$. In particular, h_1 is the identity on $F_1 \cap (N \setminus f^{-1}(0))$.

The homological monodromy is not the only invariant that can be associated with the Milnor fibration. Let $\widetilde{F_1} = \overline{F_1 \setminus N}$. Then $\widetilde{F_1}$ is a manifold with boundary homotopy equivalent to F_1. The monodromy map h_1 takes $\widetilde{F_1}$ to $\widetilde{F_1}$ and it is the identity on $\partial\widetilde{F_1}$.

Consider now a relative cycle $\alpha \in H_n(\widetilde{F_1}, \partial\widetilde{F_1}; \mathbb{Z})$. The cycle $h_1(\alpha) - \alpha$ has no boundary, hence it is an absolute cycle.

Definition 2.2 The *variation map* var: $H_n(\widetilde{F_1}, \partial\widetilde{F_1}; \mathbb{Z}) \to H_n(\widetilde{F_1}; \mathbb{Z})$ is the map defined as $\operatorname{var}\alpha = h_1(\alpha) - \alpha$.

Remark 2.2 Poincaré–Lefschetz duality for F_1 implies that $H_n(\widetilde{F_1}, \partial\widetilde{F_1}; \mathbb{Z}) \cong \operatorname{Hom}(H_n(\widetilde{F_1}; \mathbb{Z}), \mathbb{Z})$. Therefore, the variation map can be regarded as a map from $H_n(F_1; \mathbb{Z})^*$ to $H_n(F_1; \mathbb{Z})$.

We can also define a bilinear form based on linking numbers of n-cycles in S^{2n+1}.

Definition 2.3 The *Seifert form* is the map $L : H_n(F_1, \mathbb{Z}) \times H_n(F_1, \mathbb{Z}) \to \mathbb{Z}$ given by $L(\alpha, \beta) = lk(\alpha, h_{\frac{1}{2}}\beta)$.

Here $\operatorname{lk}(A, B)$ is the generalized linking pairing of two disjoint n-cycles in S^{2n+1}. A classical definition deals first with the case when $H_n(B; \mathbb{Z}) \cong \mathbb{Z}$, e.g. B is a closed connected orientable n-dimensional manifold. In this case, the choice of isomorphism $H_n(B; \mathbb{Z}) \cong \mathbb{Z}$ (eg. given by choosing an orientation of B) gives, via Alexander duality, an isomorphism $H_n(S^{2n+1} \setminus B; \mathbb{Z}) \cong \mathbb{Z}$. Then, we define $\operatorname{lk}(A, B)$ as the class of A in $H_n(S^{2n+1} \setminus B; \mathbb{Z})$. The definition is later extended to the case when B is a sum of cycles with $H_n(B; \mathbb{Z}) \cong \mathbb{Z}$.

There are relations between the variation map, the Seifert form and the monodromy. These are usually called Picard–Lefschetz formulae. References include [35, Lemma 4.20] and [1].

Theorem 2.2 *The Seifert form, the variation map, the monodromy and the intersection form on $H_n(F_1; \mathbb{Z})$ are related by the following formulae:*

$$L(\operatorname{var} a, b) = \langle a, b\rangle$$

$$\langle a, b\rangle = -L(a, b) + (-1)^{n+1} L(b, a)$$

$$h = (-1)^{n+1} \operatorname{var}(\operatorname{var}^{-1})^*.$$

Here $\langle\cdot,\cdot\rangle$ *denotes the intersection form on* $H_n(F_1;\mathbb{Z})$.

Theorem 2.2 is a motivation to introduce Hermitian Variation Structures, which are the subject of the next section.

3 Hermitian Variation Structures and Their Classification

3.1 Abstract Definition

Let $\mathbb{F}$ be a field of characteristic zero. By $\bar{\cdot}$ we denote the involution of $\mathbb{F}$: if $\mathbb{F}=\mathbb{C}$, then it is a complex conjugation, if $\mathbb{F}=\mathbb{R},\mathbb{Q}$, then the involution is the identity. Set $\zeta=\pm 1$.

Definition 3.1 A *ζ-Hermitian variation structure* over $\mathbb{F}$ is a quadruple $(U;b,h,V)$ where

(HVS1) U is a finite dimensional vector space over $\mathbb{F}$;
(HVS2) $b:U\to U^*$ is a $\mathbb{F}$-linear endomorphism with $\overline{b^*\circ\theta}=\zeta b$, where $\theta: U\to U^{**}$ is the natural isomorphism;
(HVS3) $h:U\to U$ is b-orthogonal, that is $\overline{h}^*\circ b\circ h=b$;
(HVS4) $V:U^*\to U$ is a $\mathbb{F}$-linear endomorphism with $\overline{\theta^{-1}\circ V^*}=-\zeta V\circ\overline{h^*}$ and $V\circ b=h-I$.

The motivation is clearly Picard–Lefschetz theory. Suppose $f\colon(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ is a polynomial map as in Sect. 2. The following result is a direct consequence of Theorem 2.2.

Proposition 3.1 *Consider the quadruple* (U,b,h,V)*, where* $U=H_n(F_1;\mathbb{C})$*,* $b\colon H_n(F_1;\mathbb{C})\to H_n(F_1,\partial F_1;\mathbb{C})=\mathrm{Hom}_{\mathbb{C}}(H_n(F_1;\mathbb{C});\mathbb{C})$ *is the Poincaré–Lefschetz duality,* $h\colon U\to U$ *is the homological monodromy and* V *is the variation map. Then* (U,b,h,V) *is a Hermitian Variation Structure over* $\mathbb{C}$ *with* $\zeta=(-1)^n$.

Relations (HVS3) and (HVS4) suggest that having two of the three operators b,h and V we can recover the third one. This is true under some conditions, which we are now going to spell out.

Lemma 3.1

(a) If b *is an isomorphism then* $V=(h-I)b^{-1}$*. The HVS is determined by the triple* $(U;h,b)$
(b) If V *is an isomorphism then* $h=-\zeta V\overline{(\theta^{-1}\circ V^*)^{-1}}$ *and* $b=-V^{-1}-\zeta\overline{(\theta^{-1}\circ V^*)^{-1}}$*. So* V *determines the HVS.*

Definition 3.2 The HVS such that b is an isomorphism is called *nondegenerate*. If V is an isomorphism, we say that the HVS is *simple*.

3.2 Classification of HVS Over $\mathbb{C}$

In [25] Némethi provides a classification of simple HVS over $\mathbb{F} = \mathbb{C}$. This classification is based on a Jordan block decomposition of the operator h. Note that we do not usually assume that all the eigenvalues of the monodromy operator are roots of unity, as is the case of the HVS associated with isolated hypersurface singularities.

Following [25] we first list examples of HVS. Then we state the classification result. In the following we let J_k denote the k-dimensional matrix $\{c_{ij}\}$, with $c_{ij} = 1$ for $j = i, i+1$ and $c_{ij} = 0$ otherwise, that is, J_k is the single Jordan block of size k.

Example 3.1 Let $\nu \in \mathbb{C}^* \setminus S^1$ and $\ell \geq 1$. Define

$$\mathcal{V}_\nu^{2\ell} = \left(\mathbb{C}^{2\ell}; \begin{pmatrix} 0 & I \\ \zeta I & 0 \end{pmatrix}, \begin{pmatrix} \nu J_\ell & 0 \\ 0 & \frac{1}{\overline{\nu}} J_\ell^{*-1} \end{pmatrix}, \begin{pmatrix} 0 & \zeta(\nu J_\ell - I) \\ \frac{1}{\overline{\nu}} J_\ell^{*-1} - I & 0 \end{pmatrix}\right).$$

Then $\mathcal{V}_\nu^{2\ell}$ is a HVS. Furthermore, $\mathcal{V}_\nu^{2\ell}$ and $\mathcal{V}_{1/\overline{\nu}}^{2\ell}$ are isomorphic.

Before we state the next example, we need a simple lemma.

Lemma 3.2 *Let $k \geq 1$ and $\zeta = \pm 1$. Up to a real positive scaling, there are precisely two non-degenerate matrices $b_\pm^k$ such that*

$$\overline{b_\pm^k}^* = \zeta b \text{ and } J_k^* b_\pm^k J_k = b_\pm^k.$$

The entries of $b_\pm^k$ satisfy $(b_\pm^k)_{i,j} = 0$ for $i + j \leq k$ and $b_{i,k+1-i} = (-1)^{i+1} b_{1,k}$. Moreover, $(b_\pm^k)_{1,k}$ is a power of i.

Convention 3.1 *By convention, we choose signs in such a way that* $(b_\pm^k)_{1,k} = \pm i^{-n^2-k+1}$, *where n is such that $\zeta = (-1)^n$.*

Using $b_\pm^k$ we can give an example of a HVS corresponding to the case $\mu \in S^1$.

Lemma 3.3 *Let $\mu \in S^1$ and $k \geq 1$ be an integer. Up to isomorphism, there are two non-degenerate HVS such that $h = \mu J_k$. These structures have $b = b_+^k$ and $b = b_-^k$, respectively.*

For these two structures we use the notation:

$$\mathcal{V}_\mu^k(\pm 1) = \left(\mathbb{C}^k; b_\pm^k, \mu J_k, (\mu J_k - I)(b_\pm^k)^{-1}\right).$$

These two structures are simple unless $\mu = 1$. For $\mu = 1$ we need another construction of a simple HVS.

Lemma 3.4 *Suppose $k \geq 2$. There are two degenerate HVS with $h = J_k$. These are:*

$$\widetilde{\mathcal{V}}_1^k(\pm 1) = \left(\mathbb{C}^k; \widetilde{b}_\pm, J_k, \widetilde{V}_\pm^k\right),$$

where

$$\widetilde{b}_\pm^k = \begin{pmatrix} 0 & 0 \\ 0 & b_\pm^{k-1} \end{pmatrix}$$

and $\widetilde{V}_\pm^k$ is uniquely determined by b and h. Moreover, $\widetilde{\mathcal{V}}_1^k(\pm 1)$ is simple.

While Lemma 3.4 deals with the case $k \geq 2$, there remains the case $k = 1$. Then, with $\mu = 1$, that is, $h = 1$, all possible structures can be enumerated explicitly. These are the following.

$$\mathcal{V}_1^1(\pm 1) = (\mathbb{C}, \pm i^{-n^2}, I, 0)$$
$$\widetilde{\mathcal{V}}_1^1(\pm 1) = (\mathbb{C}, 0, I, \pm i^{n^2+1})$$
$$\mathcal{T} = (\mathbb{C}, 0, I, 0).$$

In the above list, the structures $\mathcal{V}_1^1(\pm 1)$ and $\mathcal{T}$ are non-simple, and $\widetilde{\mathcal{V}}_1^1(\pm 1)$ are simple.

Concluding, for any $\mu \in S^1$ and in each dimension k, there are precisely two non-equivalent simple variation structures with $h = \mu J_k$. We use the following uniform notation for them:

$$\mathcal{W}_\mu^k(\pm 1) = \begin{cases} \mathcal{V}_\mu^k(\pm 1) & \text{if } \mu \neq 1 \\ \widetilde{\mathcal{V}}_1^k(\pm 1) & \text{if } \mu = 1. \end{cases} \tag{3.1}$$

The following result is one of the main results of [25].

Theorem 3.2 *A simple HVS is uniquely expressible as a sum of indecomposable ones up to ordering of summands and up to an isomorphism. The indecomposable pieces are*

$$\mathcal{W}_\mu^k(\pm 1) \ \textit{for } k \geq 1, \mu \in S^1$$
$$\mathcal{V}_\nu^{2\ell} \quad \textit{for } \ell \geq 1, 0 < |\nu| < 1.$$

Definition 3.3 Let $\mathcal{M}$ be a simple HVS. The *Hodge number* $p_\mu^k(\pm 1)$ for $\mu \in S^1$ is the number of times the structure $\mathcal{W}_\mu^k(\pm 1)$ enters $\mathcal{M}$ as a summand. The Hodge number q_ν^ℓ for $|\nu| \in (0, 1)$ is the number of times the structure $\mathcal{V}_\nu^{2\ell}$ enters $\mathcal{M}$ as a summand.

For an isolated hypersurface singularity, the whole 'Picard–Lefschetz package', that is, the monodromy, the variation map, the intersection form and the Seifert form, are defined over the integers. Passing to $\mathbb{C}$ in the definition of a Hermitian Variation Structure means that some information is lost. While we do not know how to recover the part coming from integer coefficients, the part of data coming from real coefficients is easy to see.

Suppose $\mathcal{M} = (U, b, h, V)$ is a HVS over $\mathbb{R}$. We construct a complexification of $\mathcal{M}$ by considering $\mathcal{M}_{\mathbb{C}} = (U \otimes \mathbb{C}, b \otimes \mathbb{C}, h \otimes \mathbb{C}, V \otimes \mathbb{C})$. Using Definition 3.3 we can associate Hodge numbers with $\mathcal{M}_{\mathbb{C}}$. The following result is implicit in [25], see also [6, Lemma 2.14].

Lemma 3.5 *The Hodge numbers of $\mathcal{M}$ satisfy*

$$p^k_{\overline{\mu}}(u) = p^k_{\mu}((-1)^{k+1+s}\zeta u) \text{ and } q^\ell_{\overline{\nu}} = q^\ell_{\nu}.$$

Here $s = 1$ if $\mu = 1$, otherwise $s = 0$.

The definition of a HVS is a generalization of the definition of Milnor's isometric structure [24]; compare also [26]. Lemma 3.1 implies that if the intersection form is an isomorphism, then the HVS is determined by the underlying isometric structure. Classification Theorem 3.2 shows, that the only simple degenerate HVS correspond to the eigenvalue $\mu = 1$. This is the main feature of the concept of a HVS: it allows us to deal with the case $\mu = 1$.

3.3 The Mod 2 Spectrum

The spectrum of an isolated hypersurface singularity was introduced by Steenbrink in [31]. It is an unordered s-tuple of rational numbers $a_1, \dots, a_s \in (0, n+1]$, where n is the dimension of the hypersurface and s is the Milnor number. The spectrum is one of the deepest invariants of hypersurface singularities. The definition of the spectrum involves the study of mixed Hodge structures associated with a singular point. We now show, following Némethi, that the mod 2 reduction (the tuple $a_1 \bmod 2, \dots, a_s \bmod 2$) of the spectrum can be recovered from Hodge numbers. In particular, for plane curve singularities, the whole spectrum is determined by the Hodge numbers.

Theorem 3.3 (See [25, Theorem 6.5]) *Let $p^k_\mu(u)$ be the Hodge numbers of an isolated hypersurface singularity in $\mathbb{C}^{n+1}$. For any $\alpha \in (0, 2) \setminus \{1\}$, the multiplicity of α in the mod 2 spectrum is equal to*

$$\sum_{k=1}^{\infty} \sum_{\epsilon=\pm 1} k p^{2k}_{\mu}(\epsilon) + \sum_{k=1}^{\infty} \sum_{\epsilon=\pm 1} (k + 1 - \epsilon \lfloor \alpha \rfloor) p^{2k+1}_{\mu}(\epsilon),$$

where $\mu = e^{2\pi i \alpha}$.

The integer part of the spectrum, i.e. the case $\alpha \in \{1, 2\}$ can be treated in a similar manner.

4 HVS for Knots and Links

From now on we assume that $\zeta = -1$, so we consider only (-1)-variation structures.

4.1 Three Results of Keef

The monodromy, the variation and the intersection form for an isolated hypersurface singularity are defined homologically. The construction does not involve any analytic structure, that is, we need only existence of a topological fibration of the complement of the link of singularity over S^1. Therefore, if we have any fibered link $L \subset S^3$, we can use the same approach as above to define a HVS for such link. With a choice of a basis of $H_1(F)$, where F is the fiber, the variation map is the inverse of the Seifert matrix.

The construction can be extended further: take a link with Seifert matrix S and associate to it a simple HVS with variation map S^{-1}. Now the Seifert matrix is defined only up to S-equivalence (see [16, Section 5.2]) and need not be invertible in general. We shall use results of Keef to show that every Seifert matrix is S-equivalent to a block sum of an invertible matrix and a zero matrix. This invertible matrix is well-defined up to rational congruence (for an analogous result for knots refer to [16, Theorem 12.2.9]). Therefore, a HVS for any link in S^3 is defined.

Hereafter, where we mean S-equivalence, we mean S-equivalence with rational coefficients. As shown in [33], not all the results carry over to the case of $\mathbb{Z}$.

Proposition 4.1 (See [19, Proposition 3.1]) *Any Seifert matrix S for a link L is S-equivalent over $\mathbb{Q}$ to a matrix S' which is a block sum of a zero matrix and an invertible matrix S_{in}.*

Proposition 4.2 (See [19, Theorem 3.5]) *Suppose $S = S_0 + S_{in}$ and $T = T_0 + T_{in}$ be two matrices over $\mathbb{Q}$, presented as block sums of a zero matrix (that is, S_0 and T_0) and an invertible matrix (that is, S_{in} and T_{in}). The matrices S and T are S-equivalent if and only if they are congruent over $\mathbb{Q}$. Furthermore, if S and T have the same size, then congruence of S and T is equivalent to congruence of S_{in} and T_{in}.*

Proposition 4.3 (See [19, Theorem 3.6]) *Two matrices S and T are S-equivalent over $\mathbb{Q}$ if and only if their Seifert systems are isomorphic.*

Here, a *Seifert system* relative to a square matrix S consists of the module $A_S = \mathbb{Q}[t, t^{-1}]/(tS - S^T)$ and a pairing on the torsion part of A_S as defined in [19, Section 2].

From these three results we deduce the following fact. This result was often used in [6], but actually its proof was never written down in detail.

Proposition 4.4 *Suppose S is S-equivalent to matrices S' and S'', which are both block sums of zero matrices S'_0 and S''_0 and S'_{in}, S''_{in}, such that S'_{in}, S''_{in} are non-degenerate. Then S'_{in} and S''_{in} are congruent over $\mathbb{Q}$.*

Proof As $\mathbb{Q}[t, t^{-1}]$ is a PID, the module $A_{S'} = A_{S''}$ decomposes as a direct sum of the free part and the torsion part. The sizes of S'_0 and S''_0 are equal to the rank over $\mathbb{Q}[t, t^{-1}]$ of the free part of the module.

Let TA denote the torsion-part of $A_{S'} = A_{S''}$. The order of TA is the degree of the polynomial $\det(tS'_{in} - {S'_{in}}^T) = \det(tS''_{in} - {S''_{in}}^T)$. As S'_{in} and S''_{in} are invertible, the degree of $\det(tS'_{in} - {S'_{in}}^T)$ is equal to the size of S'_{in}. Therefore, the sizes of S'_{in} and S''_{in} are equal. By Proposition 4.2, this shows that S'_{in} and S''_{in} are congruent over $\mathbb{Q}$.

Remark 4.1 One would be tempted to guess that given a matrix S, the size of S_0 is $\dim(\ker S \cap \ker S^T)$. Such remark was made in [5, Section 2.2] but it was nowhere used. In fact, it is false. For a counterexample, take

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One readily checks that $\ker S \cap \ker S^T = 0$ but S is S-equivalent to the matrix (0). So $\dim S_0 = 1$.

Definition 4.1 Let $L \subset S^3$ be a link with Seifert matrix S. Suppose S is S-equivalent to S', which is a block sum of a zero matrix and an invertible matrix S_{in}. The Hermitian Variation Structure for L is the Hermitian Variation Structure $\mathcal{M}(L)$ for which the variation operator is the inverse of S_{in}.

>From Proposition 4.4 we deduce the following result.

Corollary 4.1 *The Hermitian Variation Structure $\mathcal{M}(L)$ is independent on the S-equivalence class of the matrix S, i.e. it is an invariant of L.*

4.2 HVS for Links and Classical Invariants

Given the link $L \subset S^3$ and the HVS $\mathcal{M}(L)$ we define Hodge numbers for L. Denote them $p^k_\mu(\pm 1)$ and q^ℓ_ν. The Hodge numbers determine the one-variable Alexander polynomial of L over $\mathbb{R}$ and the signature function. To describe the relation in more detail, we introduce a family of polynomials.

$$
\begin{aligned}
&B_1(t) = (t-1), \quad B_{-1}(t) = (t+1) \\
&B_\mu(t) = (t-\mu)(1-\overline{\mu}t^{-1}) && \mu \in S^1,\ \operatorname{im}\mu > 0 \\
&B_\mu(t) = (t-\mu)(1-\mu^{-1}t^{-1}) && \mu \in \mathbb{R},\ 0 < |\mu| < 1 \\
&B_\mu(t) = (t-\mu)(t-\overline{\mu})(1-\mu^{-1}t^{-1})(1-\overline{\mu}^{-1}t^{-1}) && \mu \notin S^1 \cup \mathbb{R},\ 0 < |\mu| < 1.
\end{aligned}
\tag{4.1}
$$

The (Laurent) polynomials B_μ for $\mu \notin \{1, -1\}$ are characterized by the property that they have real coefficients, they are symmetric ($B_\mu(t) = B_\mu(t^{-1})$) and they cannot be presented as products of real symmetric polynomials. Moreover, these are (up to multiplication by t) the characteristic polynomials of the monodromy operators associated with HVS $\mathcal{W}^k_\mu$. With notation (4.1) we obtain (see [7, Section 4.1]):

Proposition 4.5 *Let L be a knot. Then—up to multiplication by a unit in* $\mathbb{R}[t, t^{-1}]$*—the Alexander polynomial of L is equal to*

$$\Delta_L(t) = \prod_{\substack{\mu \in S^1 \\ \operatorname{im}\mu \geq 0}} \prod_{\substack{k \geq 1 \\ u = \pm 1}} B_\mu(t)^{p^k_\mu(u)} \cdot \prod_{\substack{0<|\nu|<1 \\ \operatorname{im}\nu \geq 0}} \prod_{\ell \geq 1} B_\nu(t)^{q^\ell_\nu}. \tag{4.2}$$

Another result gives the minimal number of generators of the Alexander module of a knot L over $\mathbb{R}[t, t^{-1}]$; see [6, Section 4.3].

Proposition 4.6 *Suppose* Δ_L *is not identically zero. The minimal number of generators of the Alexander module over* $\mathbb{R}[t, t^{-1}]$ *is equal to*

$$\max\left(\max_{\mu \in S^1} \sum_{k,u} p^k_\mu(u),\ \max_{0<|\nu|<1} \sum_\ell q^\ell_\nu \right).$$

The Hodge numbers of a link determine its Tristram–Levine signature. Recall that for a link L, the Tristram–Levine signature $\sigma_L(z)$ is the signature of the Hermitian matrix $(1-z)S + (1-\overline{z})S^T$, where S is s Seifert matrix for L; see the recent survey of Conway [11] for the definition, properties and recent applications of signatures.

We will now show that the Hodge numbers determine the Tristram–Levine signature. The following two results can be deduced from [6, Proposition 4.14], see also [7, Section 5].

Proposition 4.7 *Let L be a link and* $z_0 = e^{ix} \in S^1$ *(*$x \in (0, \pi)$*) be such that* z_0 *is not a zero of the Alexander polynomial of L. Then*

$$\sigma_L(z_0) = -\sum_{y \in [0,x)} \sum_{\substack{u \in \{-1,1\} \\ k\ odd}} u p^k_{e^{iy}}(u) + \sum_{y \in (x,\pi)} \sum_{\substack{u \in \{-1,1\} \\ k\ odd}} u p^k_{e^{iy}}(u). \tag{4.3}$$

Proposition 4.8 *With notation of Proposition 4.7 if z_0 is a root of the Alexander polynomial, then:*

$$\sigma_L(z_0) - \frac{1}{2}\left(\lim_{t\to 0^+} \sigma_L(e^{it}z) + \sigma_L(e^{-it}z)\right) = \sum_{\substack{u\in\{-1,1\}\\ k\ even}} u p^k_{z_0}(u). \tag{4.4}$$

The bottomline of Propositions 4.7 and 4.8 is that the Hodge numbers $p^k_\mu(u)$ with k odd determine the values of signature functions outside of roots of the Alexander polynomial, while the Hodge numbers with k even determine the jumps at the roots. Note that the jumps at the roots of the signature function (i.e. the left-hand side of (4.4)) are not concordance invariants.

Propositions 4.5, 4.7, 4.8 and 4.6 can be used to determine the Hodge numbers directly, without referring to explicit study of the Jordan block decomposition.

Example 4.1 Let $K = 8_{20}$. It is known that K is slice. From [10] we read off that $\Delta_K = (t-\mu)^2(t-\overline{\mu})^2$ for $\mu = \frac{1}{2}(1+i\sqrt{3})$. Moreover, the Nakanishi index (the minimal number of generators of the Alexander module of K) is 1. This data alone will determine all the Hodge numbers and—up to sign—also the signature function.

Namely, by Proposition 4.5 we know that for all $\lambda \neq \mu$, $k \geq 1$ and $u \in \{-1, 1\}$, we must have $p^k_\lambda(u) = 0$. Otherwise the Alexander polynomial for K has a root a λ.

As the Alexander polynomial of K has a double root at μ, we infer that $\sum_{k,u} p^k_\mu(u) = 2$. As $p^k_\mu(u) \geq 0$ for all k, u, we have essentially two possibilities.

- $p^1_\mu(+1) + p^1_\mu(-1) = 2$;
- for some $u \in \{-1, 1\}$, $p^2_\mu(u) = 0$ and for $(k, w) \neq (2, u)$ we have $p^k_\mu(w) = 0$.

Now the Nakanishi index of K is 1; by Proposition 4.6 the first case does not occur. Therefore $p^2_\mu(u) = 1$ and without extra data we cannot determine the sign u.

We conclude from Propositions 4.7 and 4.8 that the Tristram–Levine signature of K is zero except for μ and $\overline{\mu}$, where it attains the value u.

Note that the maximum absolute value of the signature function is a lower bound for the unknotting number; see [11, Theorem 2.6] or [4, Theorem 4.1]. In particular, the n-fold connected sum of K, nK has unknotting number at least $n/2$.

Remark 4.2 Finding bounds for the unknotting number of smoothly slice knots is a notoriously difficult problem, because most known invariants that bound the unknotting number, are actually bounds for the 4-genus.

4.3 Signatures, HVS and Semicontinuity of the Spectrum

Hodge numbers can be used to provide the relation between the signature of the link of singularity and the mod 2 spectrum. For simplicity, we state the result for curve singularities in $\mathbb{C}^2$.

Theorem 4.1 (See [6, Corollary 4.15]) *Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ define an isolated singularity with link L and spectrum Sp. Suppose $x \in (0,1)$ is such that neither x nor $x+1$ belong to the spectrum. Then*

$$\sigma_L(e^{2\pi i x}) = -\#Sp \cap (x, x+1) + \#Sp \setminus [x, x+1].$$

Theorem 4.1 can be regarded as a generalization of Litherland's formula expressing the signature of a torus knot in terms of the number elements in $Sp_{p,q} \cap (x, x+1)$, where $Sp_{p,q} = \{\frac{i}{p} + \frac{j}{q},\ 1 \leq i < p,\ 1 \leq j < q\}$ is the spectrum of singularity $x^p - y^q = 0$; see [21].

Spectrum of singularity is semicontinuous under deformation of singularities. While stating the result of Steenbrink and Varchenko [32, 34] is beyond the scope of this survey, we note that in [5], Murasugi inequality for signatures of links was used to obtain semicontinuity results.

5 Blanchfield Forms

We now pass to defining Blanchfield forms. In some sense, Blanchfield forms generalize Hermitian Variation Structures, although the connection might be hard to observe at first. We restrict to the case of knots, referring to [15] for the case of links. First, we need to set up some conventions. Suppose R be a ring with involution (usually we consider $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ with trivial involution or $R = \mathbb{C}$ with complex conjugation). The ring $R[t, t^{-1}]$ has an involution given by $\overline{\sum a_j t^j} = \sum \overline{a_j} t^{-j}$.

5.1 Definitions

Let $K \subset S^3$ be a knot. Let $X = S^3 \setminus K$. By Alexander duality $H_1(X;\mathbb{Z}) = \mathbb{Z}$. Hurewicz theorem implies the existence of a surjetion $\pi_1(X) \to \mathbb{Z}$. We call the cover of X corresponding to this surjection the *universal abelian cover* of X. We denote it by $\widetilde{X}$. The first homology group $H_1(\widetilde{X};\mathbb{Z})$ has a structure of $\mathbb{Z}[t, t^{-1}]$-module, with multiplication by t being induced by the action of the deck transformation on $\widetilde{X}$. This module is called the *Alexander module* of K. Usually it is denoted by $H_1(X; \mathbb{Z}[t, t^{-1}])$; in Sect. 5 we will denote it by H.

Blanchfield [2] defined a bilinear pairing $H \times H \to \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]$. He also proved that it is Hermitian and non-degenerate. The pairing is nowadays called the *Blanchfield pairing* of K. Instead of going through the definition of the form, we will show how the pairing is computed.

Theorem 5.1 *Let K be a knot and let S be a Seifert matrix for K, assume the size of S is n. Denote $\Lambda = \mathbb{Z}[t, t^{-1}]$. Then $H = \Lambda^n/(tS - S^T)\Lambda^n$ and with this identification the Blanchfield pairing is $(x, y) \mapsto x^T(t-1)(S - tS^T)^{-1}\overline{y} \in \mathbb{Q}(t)/\Lambda$.*

Remark 5.1 There is some confusion in the literature about the correct statement of Theorem 5.1. We refer the reader to [13], where various possibilities are discussed and some commonly appearing mistakes are corrected.

Theorem 5.1 shows that a Seifert matrix of K determines the Blanchfield pairing. The reverse implication is also true; see e.g. [30, 33].

Theorem 5.2 *The S-equivalence class of a Seifert matrix of a knot K is determined by the Blanchfield form.*

The importance of a Blanchfield form in knot theory justifies the following abstract definition.

Definition 5.1 Let R be an integral domain with (possibly trivial) involution. Let Ω be the field of fractions of R.

A *linking form* over R is a pair (M, λ), where M is a torsion R-module and $\lambda\colon M \times M \to \Omega/R$ is a non-singular sesquilinear pairing. Here 'non-singular' means that the map $M \to \overline{\mathrm{Hom}_R(M, \Omega/R)}$ induced by λ is an isomorphism.

We refer to Ranicki's books [28] and [29] for a detailed study of abstract linking forms and their properties.

5.2 Blanchfield Pairing Over $\mathbb{R}[t, t^{-1}]$

We will now study classification of Blanchfield pairings over $\mathbb{R}[t, t^{-1}]$. As in Sect. 3.2 we will first give some examples and then, based on these examples, we state the classification result. First we deal with the case $\mu \in S^1$.

Definition 5.2 Let $\mu \in S^1$, $\operatorname{im}\mu > 0$. Let $k > 0$, $\epsilon \in \{-1, 1\}$. The hermitian form $\mathfrak{e}(\mu, k, \epsilon)$ is the pair (M, λ), where

$$M = \mathbb{R}[t, t^{-1}]/B_\lambda(t)^k$$
$$\lambda(x, y) = \frac{\epsilon x\overline{y}}{B_\mu(t)^k}.$$

The second definition is for $\mu \notin S^1$.

Definition 5.3 Suppose $\nu \in \mathbb{C}$, $\operatorname{im}\nu \geq 0$ and $0 < |\nu| < 1$. For $\ell > 0$ we define the hermitian form $\mathfrak{f}(\nu, \ell)$ as the pair (M, λ), where

$$M = \mathbb{R}[t, t^{-1}]/B_\lambda(t)^\ell$$
$$\lambda(x, y) = \frac{x\overline{y}}{B_\nu(t)^\ell}.$$

Note that Definitions 5.2 and 5.3 do not cover the case $\mu = \pm 1$. These two cases are special, because $B_{\pm 1}(t)$ is not symmetric, but they do not occur in knot case, because ± 1 is never a root of the Alexander polynomial of a knot.

The following result goes back at least to Milnor, see [24, Theorem 3.3]. We present the statement from [3], see also [7].

Theorem 5.3 *Suppose* (M, λ) *is a non-degenerate linking form over* $\mathbb{R}[t, t^{-1}]$ *such that the multiplication by* $(t \pm 1)$ *is an isomorphism of* M*. Then* (M, λ) *decomposes into a finite sum:*

$$(M, \lambda) = \bigoplus_{i \in I} \mathfrak{e}(\mu_i, k_i, \epsilon_i) \oplus \bigoplus_{j \in J} \mathfrak{f}(\nu_j, \ell_j), \tag{5.1}$$

where $\mu_i \in S^1$*,* $0 < |\nu_j| < 1$*, and* $\operatorname{im} \mu_i > 0$*,* $\operatorname{im} \nu_j \geq 0$ *Such a decomposition is unique up to permuting summands.*

Theorem 5.3 motivates the following definition.

Definition 5.4 Let (M, λ) be as in the statement of Theorem 5.3. The number $e^k_\mu(\epsilon)$ (respectively f^ℓ_ν) is the number of times the form $\mathfrak{e}(\mu, k, \epsilon)$ (respectively $\mathfrak{f}(\nu, \ell)$) enters (M, λ) as a direct summand.

5.3 Variation Operators and Linking Forms

Let $\mathcal{M}$ be a simple HVS over $\mathbb{R}$ with variation operator V with $\zeta = -1$. Let $S = V^{-1}$. Motivated by Theorem 5.1 define the pairing (M, λ) by

$$M = \mathbb{R}[t, t^{-1}]^n / (tS - S^T)\mathbb{R}[t, t^{-1}]^n, \;\; \lambda(x, y) = x^T (t-1)(S - tS^T)^{-1}\overline{y}. \tag{5.2}$$

We call this form the *linking form associated to* $\mathcal{M}$. We have the following result.

Proposition 5.1 *Let* $\mu \in S^1$, $\operatorname{im} \mu > 0$*. Suppose* $\mathcal{M} = \mathcal{V}^k_\mu(\epsilon) \oplus \mathcal{V}^k_{\overline{\mu}}((-1)^k \epsilon)$*. Then, the linking form associated with* $\mathcal{M}$ *is equal to* $\mathfrak{e}(\mu, k, \epsilon)$*.*

Proof The statement is well-known to the experts. The underlying $\mathbb{R}[t, t^{-1}]$-modules are clearly isomorphic and the sign ϵ is determined by comparing appropriate signatures, see [17, 18] and also Conway's survey [11, Section 4.2].

It is instructive to give an elementary proof of Proposition 5.1 in case $k = 1$. The method of computing the sign of a non-degenerate pairing over $\mathbb{R}[t, t^{-1}]/B_\mu(t)^k$ is as follows. Take an element $v \in \mathbb{R}[t, t^{-1}]/B_\mu(t)^k$ and compute $\lambda(v, v) = q/B_\mu(t)^k$. If q is coprime with B_μ, then the sign of $q(\mu)$ (this is clearly a real number) is precisely the sign of $\mathfrak{e}(\mu, k, \epsilon)$. A proof of the last statement follows quickly from the proof of [3, Proposition 4.2].

We will first compute the Seifert matrix S and then $\lambda(v, v)$ via (5.2). From Lemma 3.2 we have $b^1_\epsilon = -\epsilon i$. Therefore, the variation operator associated with $\mathcal{V}^1_\mu(\epsilon)$ is $\epsilon i(\mu - 1)$. The variation operator corresponding to $\mathcal{V}^1_\mu(\epsilon) \oplus \mathcal{V}^1_{\overline{\mu}}(-\epsilon)$ is thus equal to

$$V = \epsilon \begin{pmatrix} i(\mu - 1) & 0 \\ 0 & -i(\overline{\mu} - 1) \end{pmatrix}.$$

Hence

$$S = V^{-1} = \frac{-i\epsilon}{|\mu - 1|^2} \begin{pmatrix} \overline{\xi} & 0 \\ 0 & \xi \end{pmatrix},$$

where $\xi = i(\mu - 1)$. Write in polar coordinates $\xi = r\cos\phi + ir\sin\phi$. Then, S is congruent to the matrix

$$S = \frac{\epsilon}{r} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix},$$

The module $\mathbb{R}[t, t^{-1}]/B_\mu(t)$ is isomorphic to the module $\mathbb{R}[t, t^{-1}]^2/(tS - S^T)\mathbb{R}[t, t^{-1}]^2$.

Since $\det(S - tS^T) = tB_\mu(t)$, we have for any $v \in \mathbb{R}[t, t^{-1}]^2$:

$$\lambda(v, v) = v^T(t-1)(S - tS^T)^{-1}v = v^T \frac{(t-1)\epsilon r}{tB_\mu(t)} \begin{pmatrix} (1-t)\cos\phi & -(1+t)\sin\phi \\ (1+t)\sin\phi & (1-t)\cos\phi \end{pmatrix} v$$

Now take the vector $v = (1, 0)$ and consider its class in $\mathbb{R}[t, t^{-1}]^2/(tS - S^T)$, which we denote by the same letter. We obtain

$$\lambda(v, v) = \frac{\epsilon(t - 2 + t^{-1})r\cos\phi}{B_\mu(t)}.$$

Now the sign of $2 - \mu - \overline{\mu}$ is positive. To see the sign of $\cos\phi$ we note that $\operatorname{im}\mu > 0$, hence $\mu - 1$ is in the second quadrant, so $i(\mu - 1)$ is in the third one, thus $\cos\phi$ is negative.

Remark 5.2 An analog of Proposition 5.1 for $\mu \notin S^1$ is trivial, because the pairing is determined by the underlying module structure.

The following result is an easy consequence of Proposition 5.1.

Theorem 5.4 *There is an equality* $p^k_\mu(\epsilon) = e^k_\mu(\epsilon)$, $q^\ell_\nu = f^\ell_\nu$.

6 Twisted Blanchfield Forms and Applications

One of the features of the Hodge-theoretic point of view on Blanchfield pairings is that we can define signature-type invariants of pairings on torsion $\mathbb{R}[t,t^{-1}]$-modules, which do not necessarily come from Seifert matrices. In particular, we can easily define signature-type invariants for twisted Blanchfield pairings. This includes for instance so-called Casson-Gordon signatures.

6.1 Construction of Twisted Pairings

We begin with a general construction. For a 3-manifold X we consider its universal cover $\widetilde{X}$. This space is acted upon by $\pi_1(X)$. With $C_*(\widetilde{X})$ denoting the singular chain complex of $\widetilde{X}$, we can regard $C_*(\widetilde{X})$ as a left module over $\mathbb{Z}[\pi_1(X)]$. Suppose that M is a $(R, \mathbb{Z}[\pi_1(X)])$-module for some ring R (by this we mean a left R-module and a right $\mathbb{Z}[\pi_1(X)$-module). We define $C_*(X; M) = M \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\widetilde{X})$. This chain complex of left R-modules is called the *twisted chain complex* of X. Its homology is called the *twisted homology* of X; see [7, Section 6.1], [20].

A special instance of this operation is when we consider a representation $\beta\colon \pi_1(X) \to GL_d(R)$ for some ring R with involution and some integer $d > 0$. The space R^d has a structure of right $\mathbb{Z}[\pi_1(X)]$-module: an action of $\gamma \in \pi_1(X)$ is the multiplication the vector in R^d by $\beta(\gamma)$ from the right. Taking $M = R^d$ and repeating the construction from the paragraph above, we obtain the twisted chain complex $C_*(X; R^d_\beta)$ (we write the subscript β) to stress that this is a twisted chain complex).

Let us specify our situation more. Restrict to the case X is a closed 3-manifold (the case of manifolds with boundary has also been studied, but there are more technical details). Suppose $R = \mathbb{F}[t,t^{-1}]$ for some field $\mathbb{F}$ and $\beta\colon \pi_1(X) \to GL_d(R)$ is a *unitary* representation.

If the twisted homology group $H_1(X; \mathbb{F}[t,t^{-1}]^d)$ is $\mathbb{F}[t,t^{-1}]$-torsion, then one can define a hermitian non-singular pairing

$$H_1(X; \mathbb{F}[t,t^{-1}]^d_\beta) \times H_1(X; \mathbb{F}[t,t^{-1}]^d_\beta) \to \mathbb{F}(t)/\mathbb{F}[t,t^{-1}];$$

see [7, 22, 27]. This pairing is usually called the *twisted Blanchfield pairing*.

6.2 Twisted Hodge Numbers and Twisted Signatures

We specify now to the situation, when $\mathbb{F} = \mathbb{R}$ and $X = M(K)$, the zero-framed surgery on a knot K. Let $\beta\colon \pi_1(X) \to GL_d(\mathbb{R}[t,t^{-1}])$ be a unitary representation such that $H_1(X; \mathbb{R}[t,t^{-1}]^d_\beta)$ is $\mathbb{R}[t,t^{-1}]$-torsion. Assume furthermore that

$H_1(X; \mathbb{R}[t, t^{-1}]^d_\beta)$ has no $(t \pm 1)$-torsion. Then the twisted Blanchfield pairing is defined and by Theorem 5.3 above, it decomposes as a sum of $\mathfrak{e}(\mu, k, \epsilon)$ and $\mathfrak{f}(\nu, \ell)$.

Definition 6.1 The *twisted Hodge number* $p^k_\mu(\epsilon)_\beta$ and $f^\ell_{\nu,\beta}$ is the number of times the summand $\mathfrak{e}(\mu, k, \epsilon)$, respectively $\mathfrak{f}(\nu, \ell)$ enters the decomposition (5.1).

Having defined twisted Hodge numbers, we can define twisted signatures via an analog of (4.3).

Definition 6.2 Suppose $\mu = e^{2\pi i x}$, $x \in (0, 1/2)$. The function

$$\mu \mapsto \sigma_\beta(\mu) = \sum_{\substack{k \text{ odd} \\ \epsilon=\pm 1}} \left(p^k_\mu(\epsilon)_\beta + 2 \sum_{y \in (0,x)} p^k_{e^{2\pi i y}}(\epsilon)_\beta \right)$$

is called the *twisted signature function*. The function is extended via $\sigma_\beta(\overline{\mu}) = \sigma_\beta(\mu)$.

There is a subtle difference between Definition 6.2 and Proposition 4.7. The classical result, Proposition 4.7, sums contributions of the Hodge numbers in a range including 0. Therefore it is perfectly possible that the signature function is equal to 1 for all values close to 1. This is the case for example for the Hopf link.

Definition 6.2 sums over y in an open interval $(0, x)$, so the previous behavior is impossible. This is not merely a technical issue: it seems difficult to extend the definition of twisted signature to get a meaningful contribution of $\mu = 1$.

6.3 A Few Words on Case $\mathbb{F} = \mathbb{C}$

The construction of Hodge numbers via classification of linking pairings can be done over $\mathbb{C}[t, t^{-1}]$. We can define $\mathfrak{e}(\mu, k, \epsilon)$ for $\mu \in S^1$, and $\mathfrak{f}(\mu, k)$ for $0 < |\mu| < 1$. The underlying module structure is $\mathbb{C}[t, t^{-1}]/(t - \mu)^k$. However, the specific construction seems to be harder than in case over $\mathbb{R}$; see [7, Section 2]. Once this technical difficulty is overcome, we can define twisted Hodge numbers and twisted signatures essentially via Definitions 6.1 and 6.2.

An important instance of twisted signatures over $\mathbb{C}[t, t^{-1}]$ are signatures defined from Casson–Gordon invariants introduced by Casson and Gordon, see [8, 9]. In short, let K be a knot and let n be an integer. Consider the n-fold cyclic branched cover $L_n(K)$. Let m be a prime power coprime with n. For any non-trivial homomorphism $\chi : H_1(L_n(K); \mathbb{Z}) \to \mathbb{Z}_m$ we can construct a unitary representation $\pi_1(M(K)) \to GL_n(\mathbb{C}[t, t^{-1}])$. The concrete formula for the representation is beyond the scope of this article, we refer to [7, Section 8.1] The signature associated to this representation via Definition 6.2 is called a *Casson-Gordon signature* $\sigma_{\chi,m} : S^1 \to \mathbb{Z}$. Casson–Gordon sliceness obstruction can be translated into vanishing of some Casson–Gordon signatures. The following result is stated in [7, Theorem 8.8, Corollary 8.16] as a corollary of a result of Miller and Powell [22].

Theorem 6.1 *Let K be a topologically slice knot. Then for any prime power n, there exists a metabolizer P of the linking form on $H_1(L_n(K); \mathbb{Z})$ such that for any prime power q^a and any non-trivial homomorphism $\chi: H_1(L_n(K); \mathbb{Z}_{q^a}) \to \mathbb{Z}$ vanishing on P, there is $b \geq a$ such that σ_{χ,q^b} is zero almost everywhere on S^1.*

The main feature of Theorem 6.1 is computability. Miller and Powell [22] gave an algorithm to compute the twisted Blanchfield pairing using Fox differential calculus. The methods of [7], which we presented in this article, allow us to compute the Casson-Gordon signatures. As an application [7] and later [12] could prove non-sliceness of some linear combinations of iterated torus knots, generalizing previous results of Hedden et al. [14].

6.4 A Closing Remark

The two decomposition results: the classification of HVS of Theorem 3.2 and the classification of real Blanchfield forms in Theorem 5.3 share many properties. There are some differences, which we now want to summarize.

The classification of HVS deals much more efficiently with the case $\mu = 1$, because of the special definition of a simple HVS for $\mu = 1$. The presence of $(t-1)$-torsion modules in the theory of linking forms is a source o notorious technical difficulties.

The classification of Blanchfield forms is more general and is usually much easier to generalize. The construction of twisted Blanchfield pairings is a straightforward generalization of the construction of the classical pairing. Also, in many classification results, it is more convenient to have a single object (a pairing), than a quadruple of objects.

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