Young Walls and Equivariant Hilbert Schemes of Points in Type *D*

Ádám Gyenge

Abstract We give a combinatorial proof for a multivariable formula of the generating series of type D Young walls. Based on this we give a motivic refinement of a formula for the generating series of Euler characteristics of Hilbert schemes of points on the orbifold surface of type D.

Keywords Hilbert scheme of points · Young walls · Generating function

Subject Classifications Primary 14C05; Secondary 05E10

1 Introduction

In this paper we survey and refine some existing formulas expressing a connection between affine Lie algebras, Young diagram combinatorics and singularity theory as investigated in [9, 10]. This connection is in the context of Hilbert schemes of points on orbifold surface singularities.

Let $G \subset SL(2, \mathbb{C})$ be a finite subgroup. The equivariant Hilbert scheme $\mathrm{Hilb}([\mathbb{C}^2/G])$ is the moduli space of G-invariant finite colength subschemes of $\mathbb{C}^2$, the invariant part of $\mathrm{Hilb}(\mathbb{C}^2)$ under the lifted action of G. This space decomposes as

$$\mathrm{Hilb}([\mathbb{C}^2/G]) = \bigsqcup_{\rho \in \mathrm{Rep}(G)} \mathrm{Hilb}^{\rho}([\mathbb{C}^2/G])$$

where

$$\mathrm{Hilb}^{\rho}([\mathbb{C}^2/G]) = \{I \in \mathrm{Hilb}(\mathbb{C}^2)^G : H^0(\mathcal{O}_{\mathbb{C}^2}/I) \simeq_G \rho\}$$

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for any finite-dimensional representation $\rho \in \mathrm{Rep}(G)$ of G; here $\mathrm{Hilb}(\mathbb{C}^2)^G$ is the set of G-invariant ideals of $\mathbb{C}[x,y]$, and $\simeq_G$ means G-equivariant isomorphism. Being components of fixed point sets of a finite group acting on smooth quasiprojective varieties, the orbifold Hilbert schemes themselves are smooth and quasiprojective [3].

The topological Euler characteristics of the equivariant Hilbert scheme can be collected into a generating function. Let $\rho_0, \dots, \rho_n \in \mathrm{Rep}(G)$ denote the (isomorphism classes of) irreducible representations of G, with ρ_0 the trivial representation. The *orbifold generating series* of the orbifold $[\mathbb{C}^2/G]$ is

$$Z_{[\mathbb{C}^2/G]}(q_0, \dots, q_n) = \sum_{m_0,\dots,m_n=0}^{\infty} \chi\left(\mathrm{Hilb}^{m_0\rho_0+\dots+m_n\rho_n}([\mathbb{C}^2/G])\right) q_0^{m_0} \cdots q_n^{m_n}$$

where $\rho_0, \dots, \rho_n$ are the irreducible representations of G_Δ, and $q_0, \dots, q_n$ are formal variables.

Recall that finite subgroups of $SL(2,\mathbb{C})$ are the binary polyhedral groups. These are classified into three families: type A_n for $n \geq 1$ (binary cyclic group of an $(n+1)$-gon), type D_n for $n \geq 4$ (binary dihedral group of an n-gon) and type E_n for $n = 6, 7, 8$ (binary tetrahedral, binary octahedral and binary icosahedral groups respectively) [11]. To each such type there also corresponds a simply laced finite type root system. For such a root system Δ we will denote by G_Δ the corresponding finite subgroup of $SL(2,\mathbb{C})$ and by $\mathfrak{g}_\Delta$ the corresponding Lie algebra. Moreover, to each such finite type root system there also corresponds an affine Lie algebra $\tilde{\mathfrak{g}}_\Delta$ obtained as a central extension of the loop algebra of $\mathfrak{g}_\Delta$. The corresponding affine root system is denoted as $\widetilde{\Delta}$.

Let $\mathfrak{heis}$ be the infinite Heisenberg algebra, and let $\tilde{\mathfrak{g}}_\Delta \oplus_{\mathfrak{z}} \mathfrak{heis}$ be the Lie algebra that is obtained from the direct sum of $\tilde{\mathfrak{g}}_\Delta$ and $\mathfrak{heis}$ by identifying the centers of the two components. Let V_0 be the basic representation of $\tilde{\mathfrak{g}}_\Delta$. Let furthermore $\mathcal{F}$ be the standard Fock space representation of $\mathfrak{heis}$, having central charge 1. Then $V = V_0 \otimes \mathcal{F}$ is a representation of $\tilde{\mathfrak{g}}_\Delta \oplus_{\mathfrak{z}} \mathfrak{heis}$ that is called the *extended basic representation*. A distinguished basis of this representation was introduced by Kashiwara in the context of the associated quantum groups; this is known as the "crystal basis".

It is known that the equivariant Hilbert schemes $\mathrm{Hilb}^\rho([\mathbb{C}^2/\Gamma])$ for all finite dimensional representations ρ of G are Nakajima quiver varieties [16] associated with $\widetilde{\Delta}$, with dimension vector determined by ρ, and a specific stability condition (see [4, 15] for more details for type A). The results of [16] on the relation between the cohomology of quiver varieties and affine Lie algebras, specialized to this case, imply that the direct sum of all cohomology groups $H^*(\mathrm{Hilb}^\rho([\mathbb{C}^2/G]))$ is graded isomorphic to the extended basic representation V of the corresponding extended affine Lie algebra $\tilde{\mathfrak{g}}_\Delta \oplus_{\mathfrak{z}} \mathfrak{heis}$. This result combined with the Weyl-Kac character formula for the extended basic representation gives the following formula (see [10,

Appendix A]):

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n) = \left(\prod_{m=1}^{\infty}(1-q^m)^{-1}\right)^{n+1} \cdot \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{Z}^n} q_1^{m_1}\cdots q_n^{m_n}(q^{1/2})^{\mathbf{m}^\top\cdot C_\Delta\cdot\mathbf{m}} \tag{1.1}$$

where $q = \prod_{i=0}^n q_i^{d_i}$ with $d_i = \dim\rho_i$, and C_Δ is the finite type Cartan matrix corresponding to Δ.

At least in types A and D an even stronger statement can be obtained. In these cases the elements of the crystal basis are in bijection with certain combinatorial objects called Young walls of type Δ. The set of Young walls of type Δ will be denoted as $\mathcal{W}_\Delta$; these are endowed with an $n+1$ dimensional multi-weight: $\mathbf{wt}(\lambda) = (\mathrm{wt}_0(\lambda),\dots,\mathrm{wt}_n(\lambda))$. The multi-variable generating series of objects in $\mathcal{W}_\Delta$ is

$$F_\Delta(q_0,\dots,q_n) = \sum_{\lambda\in\mathcal{W}_\Delta}\mathbf{q}^{\mathbf{wt}(\lambda)}$$

where we used the multi-index notation

$$\mathbf{q}^{\mathbf{wt}(\lambda)} = \prod_{i=0}^{n} q_i^{\mathrm{wt}_i(\lambda)}.$$

Let Δ be of type A. It was observed first in [12] that there is a bijection

$$\mathcal{W}_\Delta \longleftrightarrow \mathcal{P}^{n+1}\times\mathbb{Z}^n \tag{1.2}$$

where $\mathcal{P}$ is the set of ordinary partitions and n is the rank of the root system. This serves as the starting point of the following enhancement of expression (1.1).

Theorem 1.1 ([4]) *Let Δ be of type A. Then*

1.

$$F_\Delta(q_0,\dots,q_n) = \left(\prod_{m=1}^{\infty}(1-q^m)^{-1}\right)^{n+1} \cdot \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{Z}^n} q_1^{m_1}\cdots q_n^{m_n}(q^{1/2})^{\mathbf{m}^\top\cdot C_\Delta\cdot\mathbf{m}}.$$

2. *There exist a locally closed decomposition of* $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$ *into strata indexed by the elements of* $\mathcal{W}_\Delta$*. Each stratum is isomorphic to an affine space.*
3. *In particular,*

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n) = F_\Delta(q_0,\dots,q_n).$$

This not just gives a new proof of (1.1) in type A, but it also shows that the combinatorics of Young walls is directly related to an explicit stratification of $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$. This relation is beneficial for example in motivic calculations (see e.g. Corollary 1.4 below). Although one can conclude Theorem 1.1 (3) from just formally comparing Eq. (1.1) with Theorem 1.1 (1), the above mentioned relation gives explanation for this coincidence on the level of the cells instead of just the homologies/Euler characteristics.

The main result of the paper is a complete combinatorial proof of the following statement.

Theorem 1.2 *Let Δ be of type D. Then*

$$F_\Delta(q_0,\dots,q_n) =$$
$$\left(\prod_{m=1}^{\infty}(1-q^m)^{-1}\right)^{n+1} \cdot \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{Z}^n} q_1^{m_1}\cdots q_n^{m_n}(q^{1/2})^{\mathbf{m}^\top\cdot C_\Delta\cdot\mathbf{m}}.$$

This result was already announced in [10], and the proof was sketched in [7]; we flesh out the details in Sects. 3–4 below. Again, the starting point will be a decomposition as in (1.2) for the type D case (see Proposition 3.2 below). Combining Theorem 1.2 with the next result one gets a complete analogue of Theorem 1.1 for type D, and hence also an alternative proof of (1.1).

Proposition 1.3 ([10, Theorem 4.1]) *Let Δ be of type D.*

1. *There exist a locally closed decomposition of* $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$ *into strata indexed by the elements of $\mathcal{W}_\Delta$. Each stratum is isomorphic to an affine space.*
2. *Moreover,*

$$Z_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n) = F_\Delta(q_0,\dots,q_n).$$

One can also consider a motivic enhancement of the series introduced above. Let $K_0(\mathrm{Var})$ be the Grothendieck ring of quasi-projective varieties over the complex numbers. The *motivic Hilbert zeta function* of the orbifold $[\mathbb{C}^2/G_\Delta]$ is

$$\mathcal{Z}_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n) = \sum_{m_0,\dots,m_n=0}^{\infty} [\mathrm{Hilb}^{m_0\rho_0+\dots+m_n\rho_n}([\mathbb{C}^2/G_\Delta])] q_0^{m_0}\cdots q_n^{m_n}.$$

Here $[X]$ denotes the class of X in $K_0(\mathrm{Var})$, and it is not to be confused with orbifold quotients. The series $\mathcal{Z}_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n)$ is an element in $K_0(Var)[[q_0,\dots,q_n]]$.

The combination of [2, Corollary 1.11] with Theorems 1.1 and 1.2 gives an explicit representation for the motivic Hilbert zeta function.

Corollary 1.4 *Let Δ be of type A or D.*

$$\mathcal{Z}_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n) = \left(\prod_{m=1}^{\infty}(1-\mathbb{L}^{m+1}q^m)^{-1}(1-\mathbb{L}^m q^m)^{-n}\right) \cdot \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{Z}^n} q_1^{m_1}\cdots q_n^{m_n}(q^{1/2})^{\mathbf{m}^\top\cdot C_\Delta\cdot\mathbf{m}}$$

where $\mathbb{L}=[\mathbb{A}^1]\in K_0(\mathrm{Var})$.

Once again, in type A this statement was proved in [4]. The above series has further specializations giving formulas for the Hodge polynomials and Poincaré polynomials of the equivariant Hilbert schemes.

Let $Y\subset\mathbb{C}^2$ be a closed subvariety invariant under the action of G_Δ. One can consider the moduli space $\mathrm{Hilb}([\mathbb{C}^2/G_\Delta],Y)\subset\mathrm{Hilb}([\mathbb{C}^2/G_\Delta])$ of points supported on Y. The corresponding motivic generating series is

$$\mathcal{Z}_{([\mathbb{C}^2/G_\Delta],Y)}(q_0,\dots,q_n) = \sum_{m_0,\dots,m_n=0}^{\infty}[\mathrm{Hilb}^{m_0\rho_0+\dots+m_n\rho_n}([\mathbb{C}^2/G_\Delta],Y)]q_0^{m_0}\cdots q_n^{m_n}.$$

The techniques of [6] imply that

$$\mathcal{Z}_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n) = \mathcal{Z}_{([\mathbb{C}^2/G_\Delta],Y)}(q_0,\dots,q_n)\cdot\mathcal{Z}_{[(\mathbb{C}^2\setminus Y)/G_\Delta]}(q_0,\dots,q_n).$$

This allows one to obtain further formulas from Corollary 1.4. For example,

$$\mathcal{Z}_{([\mathbb{C}^2/G_\Delta],0)}(q_0,\dots,q_n) = \frac{\mathcal{Z}_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n)}{\mathcal{Z}_{[(\mathbb{C}^2\setminus 0)/G_\Delta]}(q_0,\dots,q_n)}$$
$$= \frac{\mathcal{Z}_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n)}{\mathcal{Z}_{(\mathbb{C}^2\setminus 0)/G_\Delta}(q)} = \frac{\mathcal{Z}_{[\mathbb{C}^2/G_\Delta]}(q_0,\dots,q_n)}{\prod_{m=1}^{\infty}(1-\mathbb{L}^{m+1}q^m)^{-1}(1-\mathbb{L}^{m-1}q^m)} =$$
$$\left(\prod_{m=1}^{\infty}(1-\mathbb{L}^{m-1}q^m)^{-1}(1-\mathbb{L}^m q^m)^{-n}\right) \cdot \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{Z}^n} q_1^{m_1}\cdots q_n^{m_n}(q^{1/2})^{\mathbf{m}^\top\cdot C_\Delta\cdot\mathbf{m}},$$

where at the second equality we have used that G_Δ acts freely away from the origin, and at the third equality we have used the main result of [5] and that $[(\mathbb{C}^2\setminus 0)/G_\Delta] = [\mathbb{L}^2]-[pt]$ in $K_0(\mathrm{Var})$.

Suppose that $\varDelta$ is of type D. Let $E \subset \mathbb{C}^2$ be the divisor defined by the ideal (xy). This is invariant under the action of $G_\varDelta$. Then

$$\begin{aligned}
\mathcal{Z}_{([\mathbb{C}^2/G_\varDelta],E)}(q_0,\dots,q_n) &= \mathcal{Z}_{([\mathbb{C}^2/G_\varDelta],0)}(q_0,\dots,q_n)\cdot \mathcal{Z}_{([\mathbb{C}^2/G_\varDelta],E\setminus 0)}(q_0,\dots,q_n)\\
&= \mathcal{Z}_{([\mathbb{C}^2/G_\varDelta],0)}(q_0,\dots,q_n)\cdot \mathcal{Z}_{(\mathbb{C}^2/G_\varDelta,(E\setminus 0)/G_\varDelta)}(q)\\
&= \mathcal{Z}_{([\mathbb{C}^2/G_\varDelta],0)}(q_0,\dots,q_n)\cdot \prod_{m=1}^{\infty}(1-\mathbb{L}^m q^m)^{-1}(1-\mathbb{L}^{m-1}q^m)\\
&= \left(\prod_{m=1}^{\infty}(1-\mathbb{L}^m q^m)^{-n-1}\right)\cdot \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{Z}^n} q_1^{m_1}\cdots q_n^{m_n}(q^{1/2})^{\mathbf{m}^\top\cdot C_\varDelta\cdot\mathbf{m}},
\end{aligned}$$

where again at the second equality we have used that $G_\varDelta$ acts freely away from the origin, and at the third equality we have used $[(E\setminus 0)/G_\varDelta]=[\mathbb{L}]-[pt]$ in $K_0(\mathrm{Var})$.

Corollary 1.5

1. *There exist a locally closed decomposition of* $\mathrm{Hilb}([\mathbb{C}^2/G_\varDelta],E)$ *into strata indexed by the elements of* $\mathcal{W}_\varDelta$*. Each stratum is isomorphic to an affine space.*
2. *The class in* $K_0(\mathrm{Var})$ *of the stratum* $\mathrm{Hilb}([\mathbb{C}^2/G_\varDelta],E)_Y$ *corresponding to a Young wall* $Y=(\lambda_1,\dots,\lambda_{n+1},\mathbf{m})\in\mathcal{W}_\varDelta\cong\mathcal{P}^{n+1}\times\mathbb{Z}^n$ *is*

$$[\mathrm{Hilb}([\mathbb{C}^2/G_\varDelta],E)_Y]=[\mathbb{L}]^{\sum_{i=1}^{n+1}|\lambda_i|},$$

where $|\lambda_i|=\sum_j\lambda_i^j$.

Proof The proof of Part (1) is very similar to that of [10, Theorem 4.1]. The divisor E is preserved by the diagonal torus action on $\mathbb{C}^2$ used in [10] for the stratification of $\mathrm{Hilb}([\mathbb{C}^2/G_\varDelta])$. It follows that the torus action on $\mathrm{Hilb}([\mathbb{C}^2/G_\varDelta],E)$ has the same fixed points as the torus action on $\mathrm{Hilb}([\mathbb{C}^2/G_\varDelta])$. By [10, Theorem 4.3],

$$\mathrm{Hilb}([\mathbb{C}^2/G_\varDelta])^{\mathbb{C}^*}=\bigsqcup_{Y\in\mathcal{W}_\varDelta}S_Y$$

where each S_Y is an affine space. Let $\mathrm{Hilb}([\mathbb{C}^2/G_\varDelta],E)_Y\subset\mathrm{Hilb}([\mathbb{C}^2/G_\varDelta],E)$ denote the locus of ideals which flow to S_Y under the torus action. Since $(E\setminus 0)/G_\varDelta\cong\mathbb{C}^*$, the Zariski locally trivial fibration $\mathrm{Hilb}([\mathbb{C}^2/G_\varDelta])_Y\to S_Y$ explored in [10, Theorem 4.1] restricts to a Zariski locally trivial fibration $\mathrm{Hilb}([\mathbb{C}^2/G_\varDelta],E)_Y\to S_Y$ with affine space fibers, and a compatible torus action on the fibers. By [1, Sect.3, Remarks] this fibration is an algebraic vector bundle over S_Y, and hence trivial (Serre–Quillen–Suslin).

Part (2) follows from Part (1) and the formula for $\mathcal{Z}_{([\mathbb{C}^2/G_\varDelta],E)}(q_0,\dots,q_n)$ above.

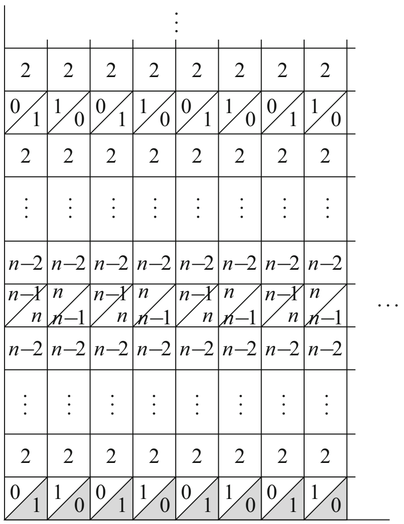
The aim of the current paper is twofold. First, we give an exposition about the combinatorics of the Young walls in type D. Second, we give a complete combinatorial proof of Theorem 1.2 in the type D case.

The structure of the rest of the paper is as follows. In Sect. 2 we review the combinatorics of the Young walls in type D. In Sect. 3 we introduce an associated combinatorial tool called the abacus. Using this we will calculate the generating series $F_\Delta(q_0, \ldots, q_n)$ of Young walls of type D and prove Theorem 1.2 in Sect. 4.

2 Young Walls of Type D_n

It is known that when $\Delta = A_n$, $n \geq 1$, the set of Young walls $\mathcal{W}_\Delta = \mathcal{P}$, the set of all Young diagrams/partitions equipped with the diagonal coloring (see [8]). We describe here the type D analogue of the set of diagonally colored partitions used in type A, following [13, 14].

First we define the *Young wall pattern of type* D_n. This is the following infinite pattern, consisting of two types of blocks: half-blocks carrying possible labels $j \in \{0, 1, n-1, n\}$, and full blocks carrying possible labels $1 < j < n-1$:



A *Young wall*[1] *of type* D_n is a subset Y of the infinite Young wall pattern of type D_n, satisfying the following rules.

[1]In [13, 14], these arrangements are called *proper Young walls*. Since we will not meet any other Young wall, we will drop the adjective *proper* for brevity.

(YW1) Y contains all grey half-blocks, and a finite number of the white blocks and half-blocks.

(YW2) Y consists of continuous columns of blocks, with no block placed on top of a missing block or half-block.

(YW3) Except for the leftmost column, there are no free positions to the left of any block or half-block. Here the rows of half-blocks are thought of as two parallel rows; only half-blocks of the same orientation have to be present.

(YW4) A full column is a column with a full block or both half-blocks present at its top; then no two full columns have the same height.[2]

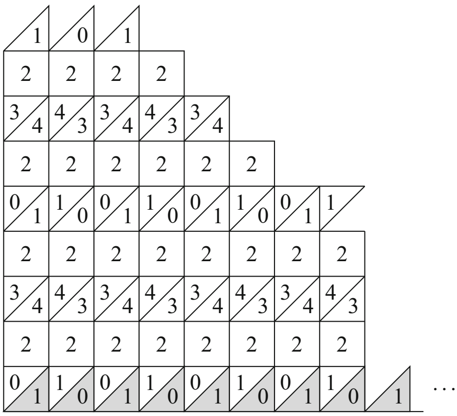
Let $\mathcal{W}_\Delta$ denote the set of all Young walls of type D_n. For any $Y \in \mathcal{W}_\Delta$ and label $j \in \{0, \ldots, n\}$ let $wt_j(Y)$ be the number of white half-blocks, respectively blocks, of label j. These are collected into the multi-weight vector $\mathbf{wt}(Y) = (wt_0(Y), \ldots, wt_n(Y))$. The total weight of Y is the sum

$$|Y| = \sum_{j=0}^{n} wt_j(Y),$$

and for the formal variables $q_0, \ldots, q_n$,

$$\mathbf{q}^{\mathbf{wt}(Y)} = \prod_{j=0}^{n} q_j^{wt_j(Y)}.$$

Example 2.1 The following is an example of a Young wall for $\Delta = D_4$:



^{2}This is the properness condition of [13].

3 Abacus Combinatorics

Recalling the Young wall rules (YW1)–(YW4), it is clear that every $Y \in \mathcal{W}_\Delta$ can be decomposed as $Y = Y_1 \sqcup Y_2$, where $Y_1 \in \mathcal{W}_\Delta$ has full columns only, and $Y_2 \in \mathcal{W}_\Delta$ has all its columns ending in a half-block. These conditions define two subsets $\mathcal{Z}^f_\Delta, \mathcal{Z}^h_\Delta \subset \mathcal{W}_\Delta$. Because of the Young wall rules, the pair (Y_1, Y_2) uniquely reconstructs Y, so we get a bijection

$$\mathcal{W}_\Delta \longleftrightarrow \mathcal{Z}^f_\Delta \times \mathcal{Z}^h_\Delta. \tag{3.1}$$

Given a Young wall $Y \in \mathcal{W}_\Delta$ of type D_n, let λ_k denote the number of blocks (full or half blocks both contributing 1) in the k-th vertical column. By the rules of Young walls, the resulting positive integers $\{\lambda_1, \dots, \lambda_r\}$ form a partition $\lambda = \lambda(Y)$ of weight equal to the total weight $|Y|$, with the additional property that its parts λ_k are distinct except when $\lambda_k \equiv 0 \mod (n-1)$. Corresponding to the decomposition (3.1), we get a decomposition $\lambda(Y) = \mu(Y) \sqcup \nu(Y)$. In $\mu(Y)$, no part is congruent to 0 modulo $(n-1)$, and there are no repetitions; all parts in $\nu(Y)$ are congruent to 0 modulo $(n-1)$ and repetitions are allowed. Note that the pair $(\mu(Y), \nu(Y))$ does almost, but not quite, encode Y, because of the ambiguity in the labels of half-blocks on top of non-complete columns.

We now introduce another combinatorial object, *the abacus of type D_n* [13, 14]. This is the arrangement of positive integers, called positions, in the following pattern:

$$\begin{array}{cccccc} 1 & \dots\ n-2 & n-1 & n & \dots\ 2n-3 & 2n-2 \\ 2n-1 & \dots\ 3n-4 & 3n-3 & 3n-2 & \dots\ 4n-5 & 4n-4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

For any integer $1 \le k \le 2n-2$, the set of positions in the k-th column of the abacus is the *k-th ruler*, denoted R_k. Several *beads* are placed on these rulers. For $k \not\equiv 0 \mod (n-1)$, the rulers R_k can only contain normal (uncolored) beads, with each position occupied by at most one bead. On the rulers R_{n-1} and R_{2n-2}, the beads are colored white and black. An arbitrary number of white or black beads can be put on each such position, but each position can only contain beads of the same color.

Given a type D_n Young wall $Y \in \mathcal{W}_\Delta$, let $\lambda = \mu \sqcup \nu$ be the corresponding partition with its decomposition. For each nonzero part ν_k of ν, set

$$n_k = \#\{1 \le j \le l(\mu) \mid \mu_j < \nu_k\}$$

to be the number of full columns shorter than a given non-full column. The abacus configuration of the Young wall Y is defined to be the set of beads placed at positions $\lambda_1, \dots, \lambda_r$. The beads at positions $\lambda_k = \mu_j$ are uncolored; the color of the bead at position $\lambda_k = \nu_l$ corresponding to a column C of Y is

$$\begin{cases} \text{white}, & \text{if the block at the top of } C \text{ is } ◿ \text{ and } n_l \text{ is even;} \\ \text{white}, & \text{if the block at the top of } C \text{ is } ◸ \text{ and } n_l \text{ is odd;} \\ \text{black}, & \text{if the block at the top of } C \text{ is } ◸ \text{ and } n_l \text{ is even;} \\ \text{black}, & \text{if the block at the top of } C \text{ is } ◿ \text{ and } n_l \text{ is odd.} \end{cases}$$

One can check that the abacus rules are satisfied, that all abacus configurations satisfying the above rules, with finitely many uncolored, black and white beads, can arise, and that the Young wall Y is uniquely determined by its abacus configuration.

Example 3.1 The abacus configuration associated with the Young wall of Example 2.1 is

R_1	R_2	R_3	R_4	R_5	R_6
1	2	3	4	5	$\textcircled{6}$
$\textcircled{7}$	$\textcircled{8}$	9	$\textcircled{10}$	$\textcircled{11}$	$\textcircled{12}^{3}$
13	14	15	16	17	18
	$\vdots$			$\vdots$	

The superscript at 12 indicates that there are 3 white beads at that position.

We now introduce certain distinguished Young walls of type D_n, and a method to obtain them with moving the beads on the abacus. On the Young wall side, define a *bar* to be a connected set of blocks and half-blocks, with each half-block occurring once and each block occurring twice. A Young wall $Y \in \mathcal{W}_\Delta$ will be called a *core* Young wall, if no bar can be removed from it without violating the Young wall rules. For an example of bar removal, see [13, Example 5.1(2)]. Let $\mathcal{C}_\Delta \subset \mathcal{W}_\Delta$ denote the set of all core Young walls.

Based on the calculations of [13, 14] the following result was obtained in [10, Proposition 7.2]. For completeness we include also its proof.

Proposition 3.2 *Given a Young wall $Y \in \mathcal{W}_\Delta$, any complete sequence of bar removals through Young walls results in the same core* core$(Y) \in \mathcal{C}_\Delta$*, defining a map of sets*

$$\mathrm{core}\colon \mathcal{W}_\Delta \to \mathcal{C}_\Delta.$$

The process can be described on the abacus, respects the decomposition (3.1)*, and results in a bijection*

$$\mathcal{W}_\Delta \longleftrightarrow \mathcal{P}^{n+1} \times \mathcal{C}_\Delta \tag{3.2}$$

where $\mathcal{P}$ is the set of ordinary partitions. Finally, there is also a bijection

$$\mathcal{C}_\Delta \longleftrightarrow \mathbb{Z}^n. \tag{3.3}$$

Proof Decompose Y into a pair of Young walls (Y_1, Y_2) as above. Let us first consider Y_1. On the corresponding rulers R_k, $k \not\equiv 0 \mod (n-1)$, the following steps correspond to bar removals [13, Lemma 5.2].

(B1) If b is a bead at position $s > 2n-2$, and there is no bead at position $(s-2n+2)$, then move b one position up and switch the color of the beads at positions k with $k \equiv 0 \mod (n-1)$, $s-2n+2 < k < s$.
(B2) If b and b' are beads at position s and $2n-2-s$ $(1 \le s \le n-2)$ respectively, then remove b and b' and switch the color of the beads at positions $k \equiv 0 \mod (n-1)$, $s < k < 2n-2-s$.

Performing these steps as long as possible results in a configuration of beads on the rulers R_k with $k \not\equiv 0 \mod (n-1)$ with no gaps from above, and for $1 \le s \le n-2$, beads on only one of R_s, R_{2n-2-s}. This final configuration can be uniquely described by an ordered set of integers $\{z_1, \ldots, z_{n-2}\}$, z_s being the number of beads on R_s minus the number of beads on R_{2n-2-s} [14, Remark 3.10(2)]. In the correspondence (3.3) this gives $\mathbb{Z}^{n-2}$. It turns out that the reduction steps in this part of the algorithm can be encoded by an $(n-2)$-tuple of ordinary partitions, with the summed weight of these partitions equal to the number of bars removed [13, Theorem 5.11(2)].

Let us turn to Y_2, represented on the rulers R_k, $k \equiv 0 \mod (n-1)$. On these rulers the following steps correspond to bar removals [14, Sections 3.2 and 3.3].

(B3) Let b be a bead at position $s \ge 2n-2$. If there is no bead at position $(s-n+1)$, and the beads at position $(s-2n+2)$ are of the same color as b, then shift b up to position $(s-2n+2)$.
(B4) If b and b' are beads at position $s \ge n-1$, then move them up to position $(s-n+1)$. If $s-n+1 > 0$ and this position already contains beads, then b and b' take that same color.

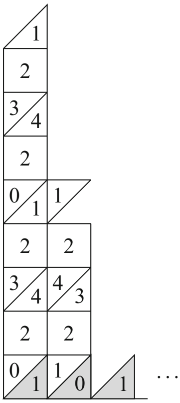
During these steps, there is a boundary condition: there is an imaginary position 0 in the rightmost column, which is considered to contain invisible white beads; placing a bead there means that this bead disappears from the abacus. It turns out that the reduction steps in this part of the algorithm can be described by a triple of ordinary partitions, again with the summed weight of these partitions equal to the number of bars removed [14, Proposition 3.6]. On the other hand, the final result can be encoded by a pair of ordinary partitions, or Young diagrams, which have the additional property of being a pyramid.

The different bar removal steps (B1)–(B4) construct the map c algorithmically and uniquely. The stated facts about parameterizing the steps prove the existence of the bijection (3.2). To complete the proof of (3.3), we only need to remark further that the set of ordinary Young diagrams having the shape of a pyramid is in bijection with the set of integers (see [14, Remark 3.10(2)]). This gives the remaining $\mathbb{Z}^2$ factor in the bijection (3.3).

Example 3.3 A possible sequence of bar removals on the abacus and Young wall of Examples 2.1 and 3.1 is as follows. Perform step (B1) on the beads at positions 7, 8, 10, 11. Perform step (B2) on the pairs of beads at positions (1,5) and (2,4). Perform step (B4) four times on two beads at position 12 by moving them consecutively to positions 9, 6 (where they take the color black), 3 and then 0 (which means removing them from the abacus). The resulting abacus configuration is then

R_1	R_2	R_3	R_4	R_5	R_6
1	2	3	4	5	6
7	8	9	10	11	12
⋮			⋮		

This configuration describes the following core Young wall:



4 Enumeration of Young Walls

We next determine the multi-weight of a Young wall Y in terms of the bijections (3.2)–(3.3). The quotient part is easy: the multi-weight of each bar is $(1, 1, 2, \dots, 2, 1, 1)$, so the $(n+1)$-tuple of partitions contributes a factor of

$$\left(\prod_{m=1}^{\infty}(1-q^m)^{-1}\right)^{n+1}.$$

Turning to cores, under the bijection $\mathcal{C}_\Delta \leftrightarrow \mathbb{Z}^n$, the total weight of a core Young wall $Y \in \mathcal{C}_\Delta$ corresponding to $(z_1, \ldots, z_n) \in \mathbb{Z}^n$ is calculated in [14, Remark 3.10]:

$$|Y| = \frac{1}{2}\sum_{i=1}^{n-2}\Big((2n-2)z_i^2 - (2n-2i-2)z_i\Big) + (n-1)\sum_{i=n-1}^{n}\Big(2z_i^2 + z_i\Big). \tag{4.1}$$

The next result gives a refinement of this formula for the multi-weight of Y.

Theorem 4.1 *Let $q = q_0q_1q_2^2 \ldots q_{n-2}^2 q_{n-1}q_n$, corresponding to a single bar.*

(1) *Composing the bijection (3.3) with an appropriate $\mathbb{Z}$-change of coordinates in the lattice $\mathbb{Z}^n$, the multi-weight of a core Young wall $Y \in \mathcal{C}_\Delta$ corresponding to an element $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ can be expressed as*

$$q_1^{m_1} \cdot \cdots \cdot q_n^{m_n} (q^{1/2})^{\mathbf{m}^\top \cdot C \cdot \mathbf{m}},$$

where C is the Cartan matrix of type D_n.

(2) *The multi-weight generating series*

$$F_\Delta(q_0, \ldots, q_n) = \sum_{Y \in \mathcal{W}_\Delta} \mathbf{q}^{\mathbf{wt}(Y)}$$

of Young walls for Δ of type D_n can be written as

$$F_\Delta(q_0, \ldots, q_n) = \frac{\displaystyle\sum_{\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{Z}^n}^{\infty} q_1^{m_1} \cdot \cdots \cdot q_n^{m_n} (q^{1/2})^{\mathbf{m}^\top \cdot C \cdot \mathbf{m}}}{\displaystyle\prod_{m=1}^{\infty}(1-q^m)^{n+1}}.$$

(3) *The following identity is satisfied between the coordinates $(m_1, \ldots, m_n)$ and $(z_1, \ldots, z_n)$ on $\mathbb{Z}^n$:*

$$\sum_{i=1}^{n} m_i = -\sum_{i=1}^{n-2}(n-1-i)z_i - (n-1)c(z_{n-1}+z_n) - (n-1)b.$$

Here $z_1 + \cdots + z_{n-2} = 2a - b$ for integers $a \in \mathbb{Z}$, $b \in \{0, 1\}$, and $c = 2b - 1 \in \{-1, 1\}$.

Statement (2) clearly follows from (1) and the discussion preceding Theorem 4.1. Statement (3) is used to achieve additional results in [10].

Let us write $z_I = \sum_{i\in I} z_i$ for $I \subseteq \{1, \ldots, n-2\}$. Each such number decomposes uniquely as $z_I = 2a_I - b_I$, where $a_I \in \mathbb{Z}$ and $b_I \in \{0, 1\}$. Let us introduce also

$c_I = 2b_I - 1 \in \{-1, 1\}$. We will make use of the relations

$$a_I = \sum_{i \in I} a_i - \sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}} b_{i_1} b_{i_2} + \sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}, i_3 \in I \setminus \{i_1, i_2\}} 2 b_{i_1} b_{i_2} b_{i_3} - \dots ,$$

$$b_I = \sum_{i \in I} b_i - \sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}} 2 b_{i_1} b_{i_2} + \sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}, i_3 \in I \setminus \{i_1, i_2\}} 4 b_{i_1} b_{i_2} b_{i_3} - \dots .$$

To simplify notations let us introduce

$$r_I := a_I - \sum_{i \in I} a_i = - \sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}} b_{i_1} b_{i_2} + \sum_{i_1 \in I, i_2 \in I \setminus \{i_1\}, i_3 \in I \setminus \{i_1, i_2\}} 2 b_{i_1} b_{i_2} b_{i_3} - \dots .$$

Using these notations the colored refinement of the weight formula (4.1) is the following.

Lemma 4.2 *Given a core Young wall $Y \in \mathcal{C}_\Delta$ corresponding to $(z_i) \in \mathbb{Z}^n$ in the bijection of* (3.3)*, its content is given by the formula*

$$\mathbf{q}^{\mathbf{wt}(Y)} =$$

$$q_1^{-\sum_{i=1}^{n-2} b_i} q_2^{-2a_1 - \sum_{i=2}^{n-2} b_i} \dots q_{n-2}^{-\sum_{i=1}^{n-3} 2a_i - b_{n-2}} (q_0 q_1^{-1} q_{n-1} q_n)^{-\sum_{i=1}^{n-2} a_i} (q_0 q_1^{-1})^{a_{1\dots n-2}}$$

$$\cdot q^{\frac{1}{2} \sum_{i=1}^{n-2} (z_i^2 + b_i) + z_{n-1}^2 + z_n^2}$$

$$\cdot (q^{b_{1\dots n-2}} (q_1^{-1} \dots q_{n-2}^{-1} q_{n-1}^{-1})^{c_{1\dots n-2}})^{z_{n-1}} (q^{b_{1\dots n-2}} (q_1^{-1} \dots q_{n-2}^{-1} q_n^{-1})^{c_{1\dots n-2}})^{z_n}.$$

When forgetting the coloring a straightforward check shows that Lemma 4.2 gives back (4.1). Notice also that $z_i^2 + b_i = 4a_i^2 - 4a_i b_i + 2b_i$ is always an even number, so the exponents are always integers.

Proof of Lemma 4.2 Suppose that we restrict our attention to blocks of color i by substituting $q_j = 1$ for $j \neq i$. Clearly,

$$wt_i(Y) \leq \sum_j wt_j(Y) = |Y|$$

where $|Y|$ is the total weight of Y. This inequality is true for each $0 \leq i \leq n$, and $|Y|$ is a linear combination of the parameters $\{z_i\}_{1 \leq i \leq n}$, their squares and a constant. It follows from the definition of the $\{z_i\}_{1 \leq i \leq n}$ that each wt_i is a convex, increasing function of them. These imply that, when considered over the reals, each wt_i are at most quadratically growing, convex analytic functions of $\{z_i\}_{1 \leq i \leq n}$. As a consequence, each wt_i is again a linear combination of constants, the parameters $\{z_i\}_{1 \leq i \leq n}$ and their products. Hence, it is enough to check that the claimed formula is correct in two cases:

1. when any of the z_i's is set to a given number and the others are fixed to 0; and
2. when all of the parameters are fixed to 0 except for an arbitrary pair z_i and z_j, $i \neq j$.

First, consider that $z_i \neq 0$ for a fixed i, and $z_j = 0$ in case $j \neq i$.

(a) When $1 \leq i \leq n-2$, then the colored weight of the corresponding core Young wall is

$$(q_1 \dots q_i)^{-b_i} (q_{i+1}^2 \dots q_{n-2}^2 q_{n-1} q_n)^{-a_i} q^{2a_i^2 - 2a_i b_i + b_i} .$$

(b) When $i \in \{n-1, n\}$, then the associated core Young wall has colored weight

$$q^{z_i^2} (q_1 q_2 \dots q_{n-2} q_i)^{z_i} .$$

Both of these follow from (4.1) and its proof in [14] by taking into account the colors of the blocks in the pattern.

Second, assume that z_i and z_j are nonzero, but everything else is zero. Then the total weight is not the product of the two individual weights, but some correction term has to be introduced. The particular cases are:

(a) $1 \leq i, j \leq n-2$. There can only be a difference in the numbers of q_0's and q_1's which comes from the fact that in the first row there are only half blocks with 0's in the odd columns and 1's in the even columns. Exactly $-r_{ij}$ blocks change color from 0 to 1 when both z_i and z_j are nonzero compared to when one of them is zero. In general, this gives the correction term $(q_0 q_1^{-1})^{r_{1\dots n-2}} = (q_0 q_1^{-1})^{a_{1\dots n-2} - \sum_{i=1}^{n-2} a_i}$.
(b) $1 \leq i \leq n-2$, $j \in \{n-1, n\}$. For the same reason as in the previous case the parity of z_i modifies the colored weight of the contribution of z_j, but not the total weight of it. If z_i is even, then the linear term of the contribution of z_j is $q_1 q_2 \dots q_{n-2} q_j$. In the odd case it is $q_0 q_2 \dots q_{n-2} q_{\kappa(j)}$. This is encoded in the correction term $(q^{b_{1\dots n-2}} (q_1^{-1} \dots q_{n-2}^{-1} q_j^{-1})^{c_{1\dots n-2}})^{z_j}$ where $j \to \kappa(j)$ swaps $n-1$ and n.
(c) $i = n-1$, $j = n$. z_{n-1} and z_n count into the total colored weight completely independently, so no correction term is needed.

Putting everything together gives the claimed formula for the colored weight of an arbitrary core Young wall.

Now we turn to the proof of Theorem 4.1. After recollecting the terms in the formula of Lemma 4.2 it becomes

$$q_1^{-b_{1\dots n-2} - c_{1\dots n-2}(z_{n-1}+z_n)} \prod_{i=2}^{n-2} q_i^{-2a_{1\dots i-1} + c_{1\dots i-1} b_{i\dots n-2} - c_{1\dots n-2}(z_{n-1}+z_n)}$$
$$\cdot q_{n-1}^{-a_{1\dots n-2} - c_{1\dots n-2} z_{n-1}} q_n^{-a_{1\dots n-2} - c_{1\dots n-2} z_n}$$
$$\cdot q^{\sum_{i=1}^{n-2}(2a_i^2 - 2a_i b_i + b_i) + b_{1\dots n-2} z_{n-1} + z_{n-1}^2 + b_{1\dots n-2} z_n + z_n^2 + r_{1\dots n-2}}$$

Let us define the following series of integers:

$$\begin{aligned} m_1 &= -b_{1\ldots n-2} - c_{1\ldots n-2}(z_{n-1} + z_n)\ , \\ m_2 &= -2a_1 + c_1 b_{2\ldots n-2} - c_{1\ldots n-2}(z_{n-1} + z_n)\ , \\ &\vdots \\ m_{n-2} &= -2a_{1\ldots n-3} + c_{1\ldots n-3} b_{n-2} - c_{1\ldots n-2}(z_{n-1} + z_n)\ , \\ m_{n-1} &= -a_{1\ldots n-2} - c_{1\ldots n-2} z_{n-1}\ , \\ m_n &= -a_{1\ldots n-2} - c_{1\ldots n-2} z_n\ . \end{aligned}$$

It is an easy and enlightening task to verify that the map

$$\mathbb{Z}^n \to \mathbb{Z}^n, \quad (z_1, \ldots, z_n) \mapsto (m_1, \ldots, m_n)$$

is a bijection, which is left to the reader.

Proof of Theorem 4.1 (1) One has to check that

$$\sum_{i=1}^{n} m_i^2 - m_1 m_2 - m_2 m_3 - \cdots - m_{n-2}(m_{n-1} + m_n) =$$
$$= \sum_{i=1}^{n-2} (2a_i^2 - 2a_i b_i + b_i) + b_{1\ldots n-2} z_{n-1} + z_{n-1}^2 + b_{1\ldots n-2} z_n + z_n^2 + r_{1\ldots n-2}\ .$$

The terms containing z_{n-1} or z_n on the left hand side are

$$\begin{aligned} &(n-2)(z_{n-1} + z_n)^2 + z_{n-1}^2 + z_n^2 - (n-3)(z_{n-1} + z_n)^2 - z_{n-1}^2 - z_n^2 - 2z_{n-1}z_n \\ &+ \left(2b_{1\ldots n-2} + \sum_{i=1}^{n-3} 2(2a_{1\ldots i} - c_{1\ldots i} b_{i+1\ldots n-2}) + 2a_{1\ldots n-2}\right) c_{1\ldots n-2}(z_{n-1} + z_n) \\ &- \left(b_{1\ldots n-2} + \sum_{i=1}^{n-3} 2(2a_{1\ldots i} - c_{1\ldots i} b_{i+1\ldots n-2}) + 2a_{1\ldots n-2}\right) c_{1\ldots n-2}(z_{n-1} + z_n) \\ &= b_{1\ldots n-2} z_{n-1} + z_{n-1}^2 + b_{1\ldots n-2} z_n + z_n^2\ , \end{aligned}$$

since $b_{1\ldots n-2} c_{1\ldots n-2} = b_{1\ldots n-2}$.

The terms containing neither z_{n-1} nor z_n on the left hand side are

$$b_{1\ldots n-2} + \sum_{i=1}^{n-3}(2a_{1\ldots i} - c_{1\ldots i}b_{i+1\ldots n-2})^2 + 2a^2_{1\ldots n-2} - b_{1\ldots n-2}(2a_1 - c_1b_{2\ldots n-2})$$
$$- \sum_{i=1}^{n-4}(2a_{1\ldots i} - c_{1\ldots i}b_{i+1\ldots n-2})(2a_{1\ldots i+1} - c_{1\ldots i+1}b_{i+2\ldots n-2})$$
$$-2(2a_{1\ldots n-3} - c_{1\ldots n-3}b_{n-2})a_{1\ldots n-2} \ .$$

Lemma 4.3

$$2a_{1\ldots i} - c_{1\ldots i}b_{i+1\ldots n-2} = \sum_{j=1}^{i}(2a_j - b_j) + b_{1\ldots n-2} \ ,$$

Proof

$$\begin{aligned}
&2a_{1\ldots i} - c_{1\ldots i}b_{i+1\ldots n-2}\\
&= 2a_{1\ldots i-1} + 2a_i - 2b_{1\ldots i-1}b_i + c_{1\ldots i-1}c_ib_{i+1\ldots n-2}\\
&= 2a_{1\ldots i-1} + 2a_i - 2b_{1\ldots i-1}b_i + 2c_{1\ldots i-1}b_ib_{i+1\ldots n-2} - c_{1\ldots i-1}b_{i+1\ldots n-2}\\
&= 2a_{1\ldots i-1} + 2a_i - b_i - c_{1\ldots i-1}(b_{i+1\ldots n-2} + b_i - 2b_ib_{i+1\ldots n-2})\\
&= 2a_{1\ldots i-1} - c_{1\ldots i-1}b_{i\ldots n-2} + 2a_i - b_i \ ,
\end{aligned}$$

and then use induction.

Applying Lemma 4.3 and the last intermediate expression in its proof to the terms considered above, they simplify to

$$\begin{aligned}
&b_{1\ldots n-2} + \sum_{i=1}^{n-3}(2a_{1\ldots i} - c_{1\ldots i}b_{i+1\ldots n-2})^2 + 2a^2_{1\ldots n-2} - b_{1\ldots n-2}(2a_1 - c_1b_{2\ldots n-2})\\
&- \sum_{i=1}^{n-4}(2a_{1\ldots i} - c_{1\ldots i}b_{i+1\ldots n-2})(2a_{1\ldots i} - c_{1\ldots i}b_{i+1\ldots n-2} + 2a_{i+1} - b_{i+1})\\
&-2(2a_{1\ldots n-3} - c_{1\ldots n-3}b_{n-2})a_{1\ldots n-2}\\
&= b_{1\ldots n-2} + (2a_{1\ldots n-3} - c_{1\ldots n-3}b_{n-2})^2 + 2a^2_{1\ldots n-2} - b_{1\ldots n-2}(2a_1 - c_1b_{2\ldots n-2})\\
&- \sum_{i=1}^{n-4}(2a_{1\ldots i} - c_{1\ldots i}b_{i+1\ldots n-2})(2a_{i+1} - b_{i+1}) - 2(2a_{1\ldots n-3} - c_{1\ldots n-3}b_{n-2})a_{1\ldots n-2}\\
&= b_{1\ldots n-2} + (2a_{1\ldots n-3} - c_{1\ldots n-3}b_{n-2})(2a_{1\ldots n-3} - c_{1\ldots n-3}b_{n-2} - 2a_{1\ldots n-2})
\end{aligned}$$

$$+2a_{1\dots n-2}^2 - b_{1\dots n-2}(2a_1 - c_1 b_{2\dots n-2}) - \sum_{i=1}^{n-4}(2a_{1\dots i} - c_{1\dots i} b_{i+1\dots n-2})(2a_{i+1} - b_{i+1})$$

$$= b_{1\dots n-2} + 2a_{1\dots n-2}^2 - b_{1\dots n-2}(2a_1 - c_1 b_{2\dots n-2})$$

$$- \sum_{i=1}^{n-3}(2a_{1\dots i} - c_{1\dots i} b_{i+1\dots n-2})(2a_{i+1} - b_{i+1})$$

$$= 2a_{1\dots n-2}^2 - b_{1\dots n-2}(2a_1 - b_1) - \sum_{i=1}^{n-3}\left(\sum_{j=1}^{i}(2a_j - b_j) + b_{1\dots n-2}\right)(2a_{i+1} - b_{i+1}) \ .$$

Let us denote this expression temporarily as s_{n-2}. Taking into account that

$$a_{1\dots n-2} = a_{1\dots n-3} + a_{n-2} - b_{1\dots n-3} b_{n-2} \ ,$$

$$b_{1\dots n-2} = b_{1\dots n-3} + b_{n-2} - 2b_{1\dots n-3} b_{n-2} \ ,$$

s_{n-2} can be rewritten as

$$2a_{1\dots n-3}^2 + 2a_{n-2}^2 + 2b_{1\dots n-3} b_{n-2} + 4a_{1\dots n-3} a_{n-2}$$

$$-4a_{1\dots n-3} b_{1\dots n-3} b_{n-2} - 4a_{n-2} b_{1\dots n-3} b_{n-2}$$

$$-(b_{1\dots n-3} + b_{n-2} - 2b_{1\dots n-3} b_{n-2})(2a_1 - b_1)$$

$$- \sum_{i=1}^{n-4}\left(\sum_{j=1}^{i}(2a_j - b_j) + b_{1\dots n-3}\right)(2a_{i+1} - b_{i+1})$$

$$- \sum_{i=1}^{n-3}(b_{n-2} - 2b_{1\dots n-3} b_{n-2})(2a_{i+1} - b_{i+1})$$

$$- \left(\sum_{j=1}^{n-3}(2a_j - b_j) + b_{1\dots n-3}\right)(2a_{n-2} - b_{n-2})$$

$$= s_{n-3} + 2a_{n-2}^2 + 2b_{1\dots n-3} b_{n-2} + 4a_{1\dots n-3} a_{n-2}$$

$$-4a_{1\dots n-3} b_{1\dots n-3} b_{n-2} - 4a_{n-2} b_{1\dots n-3} b_{n-2}$$

$$-(b_{n-2} - 2b_{1\dots n-3} b_{n-2})\left(\sum_{i=1}^{n-2} 2a_i - b_i\right)$$

$$- \left(\sum_{j=1}^{n-3}(2a_j - b_j) + b_{1\dots n-3}\right)(2a_{n-2} - b_{n-2})$$

$$= s_{n-3} + 2a_{n-2}^2 - 2a_{n-2}b_{n-2} + b_{n-2} + 2a_{n-2}\left(2a_{1\dots n-3} - \sum_{j=1}^{n-3}(2a_j - b_j)\right)$$
$$+ b_{1\dots n-3}b_{n-2} - 2a_{n-2}b_{1\dots n-3}$$
$$= s_{n-3} + 2a_{n-2}^2 - 2a_{n-2}b_{n-2} + b_{n-2} - b_{1\dots n-3}b_{n-2}\ ,$$

where at the last equality the identity

$$\sum_{j=1}^{n-3}(2a_j - b_j) = z_{1\dots n-3} = 2a_{1\dots n-3} - b_{1\dots n-3}$$

was used.

It can be checked that $s_1 = 2a_1^2 - 2a_1b_1 + b_1$, so induction shows that

$$s_{n-2} = \sum_{i=1}^{n-2}(2a_i^2 - 2a_ib_i + b_i + b_{1\dots i-1}b_i)\ .$$

It remains to show that

$$\sum_{i=2}^{n-2} b_{1\dots i-1}b_i = r_{1\dots n-2}\ ,$$

which requires another induction argument, and is left to the reader.

(3) Apply Lemma 4.3 on

$$\sum_{i=1}^{n} m_i = -b_{1\dots n-2}$$
$$-\sum_{i=1}^{n-2}(2a_{1\dots i-1} - c_{1\dots i-1}b_{i\dots n-2}) - 2a_{1\dots n-2} - (n-1)c_{1\dots n-2}(z_{n-1} + z_n).$$

□

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