On Tjurina Transform and Resolution of Determinantal Singularities

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Abstract Determinantal singularities are an important class of singularities, generalizing complete intersections, which recently have seen a large amount of interest. They are defined as preimage of $M^t_{m,n}$ the sets of matrices of rank less than t. The rank stratification of $M^t_{m,n}$ gives rise to some interesting structures on determinantal singularities. In this article we will focus on one of these, namely the *Tjurina transform*. We will show some properties of it, and discuss how it can or cannot be used to find resolutions of determinantal singularities.

Keywords Resolution of singularities · Determinantal singularities · Nash transformation · Tjurina transformation

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1 Introduction

Hypersurface singularities have in general been the starting point of singularity theory. They have some very good properties, one of the most important is the existence of the Milnor fibration [8]. The Milnor fibration makes it possible to define the Milnor number μ, which is a very important invariant. So a goal in singularity theory is to find more general families of singularities, for which it is possible to define the Milnor number. A classical example of a generalization, for which the Milnor number can be defined, is the isolated complete intersections. Determinantal singularities are a generalization of complete intersections. They are defined as the preimage of the set of $m \times n$ matrices of rank less than t under certain holomorphic maps. They have seen a lot of interest lately, including several different ways to define the Milnor number of certain classes of determinantal varieties by Ruas and

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Pereira [10], Damon and Pike [2] and Nuño Ballesteros et al. [9]. Moreover, Ebeling and Gusein-Zade defined the index of a 1-form [3], and their deformation theory has also been studied by Gaffney and Rangachev in [6].

In this article we study other aspects of determinantal singularities, not directly related to deformation theory, namely, transformations and resolutions. They played a very important role in [3], and the Tjurina transform, which will be one of our main subjects, was also studied for the case Cohen-Macaulay codimension 2 by Frühbis-Krüger and Zach in [5].

We first recall the Tjurina transform, Tjurina transpose transform and Nash transform for the model determinantal singularity in Sect. 3 as done in [3]. This will be our starting point for introducing the transformations for general determinantal singularities. We also explore how these transformations are related and how they are not, and give a description of their homotopy type. We introduce the Tjurina transform (and its transpose) for general determinantal singularities in Sect. 4, give some general properties, for example that the Tjurina transforms of most complete intersections are themselves complete intersections, and give some methods to find the Tjurina transform. In Sect. 5 we show that under some general assumptions the Tjurina transform or its dual is a complete intersection. This means that Tjurina transform cannot be used to provide resolutions in general, but in Sect. 6 we illustrate that by changing the determinantal type of the Tjurina transform of certain hypersurface singularities, we can continue the process of taking Tjurina transform, and in the end reach a resolution. Section 2 introduces determinantal singularities and notions of transformations used throughout the article.

2 Preliminaries

In this section we give the basic definitions and properties of determinantal varieties/singularities, and transformations we will need. We will in general follow the notation used in [3].

2.1 Determinantal Singularities

Let $M_{m,n}$ be the set of $m \times n$ matrices over $\mathbb{C}$. Then we define the *model determinantal variety of type* (m, n, t), denoted by $M^t_{m,n}$, for $1 \leq t \leq \min\{n, m\}$ to be the subset of $M_{m,n}$ consisting of matrices A of $\operatorname{rank}(A) < t$. $M^t_{m,n}$ has a natural structure of an irreducible algebraic variety, with defining equations given by requiring that the $t \times t$ minors have to vanish. The dimension of $M^t_{m,n}$ is $mn - (m - t + 1)(n - t + 1)$. The model determinantal variety is often called generic determinantal variety as for example in [10].

The singular set of $M^t_{m,n}$ is $M^{t-1}_{m,n}$ and the decomposition of $M^t_{m,n} = \bigcup_{i=1}^t (M^i_{m,n} \setminus M^{i-1}_{m,n})$, where $M^0_{m,n} := \emptyset$, is a Whitney stratification.

Let $F: U \subseteq \mathbb{C}^N \to M_{m,n}$ be a map with holomorphic entries. $X := F^{-1}(M^t_{m,n})$ is a *determinantal variety of type* (m, n, t) if $\text{codim}(X) = \text{codim}(M^t_{m,n}) = (m - t + 1)(n - t + 1)$. X has the structure of an analytic variety, with equations defined by the vanishing of the $t \times t$ minors of the matrix $F(x)$. We call this a variety as is custom in the treatment of determinantal singularities and singularity theory in general, even though X need not to be reduced or irreducible. The question if X is reduced or not is not important, since we will always consider X as a subset of $\mathbb{C}^N$ (later a subset of a complex manifold) equipped with the classical topology. In general in this article by a variety we mean a subset of $\mathbb{C}^N$ or a complex analytic manifold given locally as the zero set of a set of analytic equations with appropriate compatibility conditions. This means that we do not distinguish between a set given by non reduced equations, and the same set given by their reduced equations. We also do not make any assumptions on irreducibility. This is because we are interested in studying the classical topology of these sets which does not see whether the equations are reduced or not. Also even if we start with a reduced and irreducible equation, then many of the constructions we will make from them will not give reduced or irreducible equations.

The singular set of X includes $F^{-1}(M^{t-1}_{m,n})$. We make a decomposition $X = \bigcup_{i=1}^t X_i$, where $X_i := F^{-1}(M^i_{m,n} \setminus M^{i-1}_{m,n})$. Notice that even if $X = F^{-1}(M^t_{m,n})$ is an irreducible determinantal singularity which is given by reduced equations then $Y := F^{-1}(M^s_{m,n})$ for $s < t$ might not be irreducible, might not be a determinantal singularity or might be given by non reduced equations.

When we talk about the determinantal variety X, we do not just consider X as a variety in $\mathbb{C}^N$ but also the map $F: U \subseteq \mathbb{C}^N \to M_{m,n}$ used to define the variety. We will as is customary not include F in the notation and just write X, but one has to remember that the determinantal singularity also includes the map F. We will therefore also only consider two determinantal varieties X and X' equal if they are given by the same map.

We define *determinantal singularities* as germs of determinantal varieties, i.e. a germ of a space $(X, 0)$ defined as the preimage if of $M^t_{m,n}$ under a germ of a holomorphic map $F: (U, 0) \subseteq (\mathbb{C}^N, 0) \to M_{m,n}$.

Let $\text{GL}_n(\mathcal{O}_N)$ be the group of invertible $n \times n$ matrices with entries in $\mathcal{O}_N$ the sheaf of germs of holomorphic functions on $\mathbb{C}^N$. Let $\mathcal{H} := \text{GL}_m(\mathcal{O}_N) \times \text{GL}_n(\mathcal{O}_N)$ and $\mathcal{R}$ the group of analytic isomorphisms of $(\mathbb{C}^n, 0)$. Then the group $\mathcal{R} \times \mathcal{H}$ acts on map-germs $F: (U, 0) \subseteq (\mathbb{C}^N, 0) \to M_{m,n}$ by composition in the source and multiplication on the left and on the right in the target. We say that two determinantal singularities $(X, 0)$ and $(Y, 0)$ are equivalent (or G-equivalent), if their defining maps are in the same orbit of this action. This in particular implies that $(X, 0)$ is isomorphic to $(Y, 0)$ as germs of varieties.

If F is transverse to the stratum $M^i_{m,n} \setminus M^{i-1}_{m,n}$ at $F(x)$, then the singularity at x only depends upon $\text{rank}(F(x))$. We therefore call such a point *essentially nonsingular*. This naturally leads to the next definition.

Definition 2.1 Let $(X, 0)$ be a determinantal singularity defined by the map-germ F. Then $(X, 0)$ is an *essentially isolated determinantal singularity* (or EIDS for short) if there exists a neighbourhood $U \subset \mathbb{C}^N$ of the origin such that all points $x \in U \setminus \{0\}$ are essentially nonsingular.

An EIDS needs of course not be smooth, but the singularities away from $\{0\}$ are controlled, i.e. they only depend on the strata they belong to. An example of an EIDS is any isolated complete intersection given the type of a $(1, m, 1)$ (or $(m, 1, 1)$) determinantal singularity.

If $(X, 0)$ is a determinantal singularity of type (m, n, t) given by $F: U \subseteq \mathbb{C}^N \to M_{m,n}$ satisfying $F(0) \neq 0$ and $s := \operatorname{rank} F(0)$, then one can find another map $F': U' \subseteq \mathbb{C}^N \to M_{m-s,n-s}$ with $F'(0) = 0$ such that F' gives $(X, 0)$ the structure of a determinantal singularity of type $(m - s, n - s, t - s)$ where $U' \subseteq U$ are open neighbourhoods of the origin. This can be done by action on F by $\mathcal{H}$ to be of the form $\left(\begin{array}{c|c} \mathrm{id}_s & 0 \\ \hline 0 & F' \end{array}\right)$ in a neighbourhood of 0.

2.2 Transformations

Definition 2.2 Let X be a variety and $V \subset X$ a closed subvariety, then a *transformation of* (X, V) is a variety $\widetilde{X}$ together with a proper surjective analytic morphism $\pi: \widetilde{X} \to X$, such that $\pi: \pi^{-1}(X \setminus V) \to X \setminus V$ is an isomorphism and $\overline{\pi^{-1}(X \setminus V)} = \widetilde{X}$.

Here closure is the topological closure in the classical topology. The last requirement ensures that $\dim(\pi^{-1}(V)) < \dim(X)$.

This definition is sometimes also called a modification, but since we in this paper work with the Tjurina transform, we will use the word transform.

A resolution of $(X, SingX)$ is then just a transformation where $\widetilde{X}$ is smooth. We want to compare the different transformations, so we define a map between transformations as follows.

Definition 2.3 Let $f: T_1 \to T_2$ be a map between two different transformations of the same space and subspace $\pi_i: (T_i, E_i) \to (X, V)$. Then we call f a map of transformations if $\pi_1 = \pi_2 \circ f$. We call a map of transformation f an analytic morphism of transformations if it is an analytic morphism and an isomorphism of transformations, if it is an isomorphism of varieties.

3 Resolutions of the Model Determinantal Varieties

In [3] the authors introduce three different natural ways to resolve the model determinantal variety $M^t_{m,n}$. The first is the same as the Tjurina transform of $(M^t_{m,n}, M^{t-1}_{m,n})$ which was introduced by Tjurina in [12], and also used [13] and

[5]. Kempf also introduced the same transformation in his thesis [7] under the name *canonical desingularization*, and under that name it is for example used by Eisenbud in [4]. The Tjurina transform is defined as the following variety in $M_{m,n} \times \mathrm{Gr}(n-t+1, n)$:

$$\begin{aligned}\mathrm{Tjur}(M^t_{m,n}) :&= \{(A, W) \in M_{m,n} \times \mathrm{Gr}(n-t+1, n) \mid A(W) = 0\} \\ &= \{(A, W) \in M_{m,n} \times \mathrm{Gr}(n-t+1, n) \mid W \subseteq \ker(A)\}\end{aligned}$$

by considering $A \in M^t_{m,n}$ as a linear map $A\colon \mathbb{C}^n \to \mathbb{C}^m$. It is shown in [1], that this is a smooth variety. Let $\pi\colon \mathrm{Tjur}(M^t_{m,n}) \to M^t_{m,n}$ be the restriction of the projection to the first factor. Then over the regular part of $M^t_{m,n}$ we have that the map $A \to (A, \ker A)$ is an inverse to π, hence $\pi\colon \mathrm{Tjur}(M^t_{m,n}) \to M^t_{m,n}$ is a resolution. Corollary 3.3 in [5] shows that their definition gives the same as this one, their proof also works for general n, m and t.

The second resolution is as the Tjurina, but considering $A \in M^t_{m,n}$ as a linear map $A\colon \mathbb{C}^m \to \mathbb{C}^n$. This is of course the map given by the transpose of A, so we get the following:

$$\begin{aligned}\mathrm{Tjur}^T(M^t_{m,n}) :&= \{(A, W) \in M_{m,n} \times \mathrm{Gr}(m-t+1, m) \mid A^T(W) = 0\} \\ &= \{(A, W) \in M_{m,n} \times \mathrm{Gr}(m-t+1, m) \mid W \subseteq \ker(A^T)\}\end{aligned}$$

It is clear from the definition that this is also a smooth variety, the same proof as in the case of Tjurina transform works. If one chooses a Hermitian inner product on $\mathbb{C}^m$, then one gets that the relation $W \subseteq \ker(A^T)$ is equivalent to the relation $\mathrm{Im}(A) \subseteq \bar{W}^\perp$ where $V^\perp$ is the orthogonal complement with respect to the Hermitian inner product and $\bar{W}$ is the image of W under the real linear isomorphism given by complex conjugation. The choice of Hermitian inner product also gives an isomorphism of real algebraic varieties between $\mathrm{Gr}(m-t+1, m)$ and $\mathrm{Gr}(t-1, m)$ defined by sending V to $V^\perp$. Hence composing this with the real algebraic isomorphism induced on $\mathrm{Gr}(m-t+1, m)$ by complex conjugation gives an real isomorphism of $\mathrm{Gr}(m-t+1, m)$ and $\mathrm{Gr}(t-1, m)$ defined by sending W to $\bar{W}^\perp$. Using this we get that this transform is also real isomorphic to:

$$\mathrm{Tjur}^T(M^t_{m,n}) \cong \{(A, V) \in M_{m,n} \times \mathrm{Gr}(t-1, m) \mid \mathrm{Im}(A) \subseteq V\}. \tag{1}$$

This resolution is called the *dual canonical resolution* in [7].

The third resolution considered by Ebeling and Gusein-Zade is the Nash transform of $M^t_{m,n}$. In section 1 of [3] they show how to get the Nash transform which can be stated as the following proposition:

Proposition 3.1 *For a model determinantal variety the Nash transform is isomorphic to the following variety:*

$$\{(A, W_1, W_2) \in M_{m,n} \times \mathrm{Gr}(n-t+1, n) \times \mathrm{Gr}(t-1, m) \mid \ker(A) \supseteq W_1 \text{ and } \mathrm{Im}(A) \subseteq W_2\}.$$

It is only a sketch of a proof to this proposition that is given in [3], and we will below show that the two different spaces in the proposition are homeomorphic (remember we are using the classical topology, and not the Zariski topology).

Proof of Homeomorphism In [1] they show that for $A \in M^t_{m,n} \setminus M^{t-1}_{m,n}$, that is the regular points, $T_A M^t_{m,n} = \{B \in M_{m,n} \mid B\big(\ker(A)\big) \subseteq \mathrm{Im}(A)\}$. Consider the map $\alpha\colon \mathrm{Gr}(n-t+1,n)\times\mathrm{Gr}(t-1,m) \to \mathrm{Gr}(d^t_{m,n}, mn)$, where $d^t_{m,n} := mn-(m-t+1)(n-t+1) = \dim(M^t_{m,n})$, given by $\alpha(V,W) := \{B \in M_{m,n} \mid B(V) \subseteq W\}$. It is clear that $\alpha(V,W)$ is a linear subspace of $M_{m,n}$. To find the dimension of $\alpha(V,W)$ we will use the basis B_{ij} of $M_{m,n}$ defined given a basis v_j of $\mathbb{C}^n$ and a basis w_i of $\mathbb{C}^m$ as $B_{ij}(v_j) = w_i$ and $\mathrm{rank}(B_{ij}) = 1$. We choose a basis of $\mathbb{C}^n$ such that $V = \mathrm{Span}\{v_1,\dots,v_{n-t+1}\}$ and a basis of $\mathbb{C}^m$ such that the $W = \mathrm{Span}\{w_1,\dots,w_{t-1}\}$. Then $\alpha(V,W)$ is spanned by the B_{ij}'s that send one of the first $n-t+1$ basis vectors of C^n to one of the first $t-1$ basis vectors of $\mathbb{C}^m$, and the B_{ij}'s that send one of the last $t-1$ basis vectors of $\mathbb{C}^n$ to any basis vector of $\mathbb{C}^m$. This implies that $\dim\alpha(V,W) = (n-t+1)(t-1)+(t-1)m = d^t_{m.n}$. Hence $\alpha(V,W) \in \mathrm{Gr}(d^t_{m,n}, mn)$.

We will first show that α is injective. Assume that there exist two pairs (W_1,W_2) and (V_1,V_2) such that $\alpha(W_1,W_2) = \alpha(V_1,V_2)$. Assume that $W_1 \neq V_1$, let $v_1 \in V_1$ and $v_1 \notin W_1$, since $\dim(W_1) = \dim(V_1)$ such a v_1 exists, and choose $v_2 \notin V_2$. Define the linear map B as the map of rank 1 with $B(v_1) := v_2$. Then $B(W_1) = \{0\} \subseteq W_2$ and hence $B \in \alpha(W_1,W_2)$, but $B(V_1) = \mathrm{Span}\{v_2\} \nsubseteq V_2$, so $B \notin \alpha(V_1,V_2)$ and we have a contradiction. Assume now that there exist pairs (W_1,W_2) and (W_1,V_2) such that $\alpha(W_1,W_2) = \alpha(W_1,V_2)$. Assume that $W_2 \neq V_2$, choose $v_1 \in W_1$ and choose $v_2 \in V_2$ and $v_2 \notin W_2$, since $\dim(W_2) = \dim(V_2)$ such a v_2 exists. Define B as the linear map as the map of rank 1 with $B(v_1) := v_2$. Then $B(W_2) = \mathrm{Span}\{v_2\} \subseteq V_2$ so $B \in \alpha(W_1,V_2)$, but $\mathrm{Span}\{v_2\} \nsubseteq W_2$ so $B \notin \alpha(W_1,W_2)$ so we have a contradiction. This shows that α is injective.

Next we will show that α is continuous. Let $(V_i,W_i) \in \mathrm{Gr}(n-t+1,n) \times \mathrm{Gr}(t-1,m)$ be a convergent sequence and let $(V,W) := \lim(V_i,W_i)$. Let $\mathcal{B}_i := \alpha(V_i,W_i)$, and choose a convergent subsequence $\mathcal{B}'_i$ which exists because $\mathrm{Gr}(d^t_{m,n}, mn)$ is compact. Let $\mathcal{B} := \lim \mathcal{B}'_i$, choose $B \in \mathcal{B}$ and $B_i \in \mathcal{B}'_i$ a sequence of matrices converging to B. Choose $v \in V$ and $v_i \in V_i$ a sequence converging to v, set $w_j := B_j v_j$ for any j where B_j is defined. Now since B_j and v_j converge, w_j converges to $w := Bv$, but $w_j \in W_j$ and hence its limit is in W. So for all $v \in V$ and all $B \in \mathcal{B}$ $Bv \in W$, hence $\mathcal{B} \subset \alpha(V,W)$. But since $\dim\big(\mathcal{B}\big) = \dim\big(\alpha(V,W)\big)$ we have that $\mathcal{B} = \alpha(V,W)$. So any convergent subsequence of $\mathcal{B}_i$ converges to $\alpha(V,W)$, this implies that $\mathcal{B}_i$ converges to $\alpha(V,W)$ since $\mathrm{Gr}(d^t_{m,n}, mn)$ is compact. Therefore, $\lim\alpha(V_i,W_i) = \alpha(\lim(V_i,W_i))$ for all convergent sequences, hence α is continuous.

Since α is a continuous map from a compact Hausdorff space to a compact space it is closed, and since it is injective this implies it is an topological embedding (Closed Map Lemma).

Let $\beta\colon (M^t_{m,n} \setminus M^{t-1}_{m,n}) \to M_{m,n} \times \mathrm{Gr}(n-t+1,n) \times \mathrm{Gr}(t-1,m)$ be the map $\beta(A) = \big(A, \ker(A), \mathrm{Im}(A)\big)$. We define the map $\alpha'\colon M_{m,n} \times \mathrm{Gr}(n-t+1,n) \times$

$\mathrm{Gr}(t-1,m) \to M_{m,n} \times \mathrm{Gr}(d^t_{m,n}, mn)$ by $\alpha'(A,V,W) = (A, \alpha(V,W))$. Then $(\alpha' \circ \beta)(A) = (A, \mathcal{B})$, where

$$\mathcal{B} = \alpha\big(\ker(A), \mathrm{Im}(A)\big) = \{B \in M_{m,n} \mid B\big(\ker(A)\big) \subseteq \mathrm{Im}(A)\} = T_A M^t_{m,n}$$

So $\alpha' \circ \beta$ is the same as the Gauss map on the regular part of $M^t_{m,n}$. Then we have that $\mathrm{Nash}(M^t_{m,n}) = \overline{(\alpha' \circ \beta)(M^t_{m,n} \setminus M^{t-1}_{m,n})}$. Since α and hence α' is a closed topological embedding we have $\mathrm{Nash}(M^t_{m,n}) = \alpha'\Big(\overline{\beta(M^t_{m,n} \setminus M^{t-1}_{m,n})}\Big)$. Moreover, since α' is an embedding it follows that $\mathrm{Nash}(M^t_{m,n})$ is homeomorphic to $\overline{\beta(M^t_{m,n} \setminus M^{t-1}_{m,n})}$.

The last part of the proof is determining $\overline{\beta(M^t_{m,n} \setminus M^{t-1}_{m,n})}$. Now $\beta(M^t_{m,n} \setminus M^{t-1}_{m,n}) = \{(A, \ker A, \mathrm{Im}\, A) \in M_{m,n} \times \mathrm{Gr}(n-t+1,n) \times \mathrm{Gr}(t-1,m)\}$ and we want to show that the closure $\mathcal{N}$ is

$$\{(A,V,W) \in M_{m,n} \times \mathrm{Gr}(n-t+1,n) \times \mathrm{Gr}(t-1,m) \mid \ker(A) \supseteq V \text{ and } \mathrm{Im}(A) \subseteq W\}.$$

First assume that $(A,V,W) \in \overline{\beta(M^t_{m,n} \setminus M^{t-1}_{m,n})}$ is not in $\mathcal{N}$. This implies that that there is a $v \in V$ such that $Av \neq 0$ or a $v' \in \mathbb{C}^n$ such that $Av' \notin W$. In the first case let (A_i, V_i, W_i) be a sequence in $\overline{\beta(M^t_{m,n} \setminus M^{t-1}_{m,n})}$ converging to (A,V,W) and $v_i \in V_i$ a sequence converging to v, then $A_i v_i$ converges to Av but $A_i v_i = 0$ so this contradicts $Av \notin \mathcal{N}$. In the second case let (A'_i, V'_i, W'_i) be a sequence in $\overline{\beta(M^t_{m,n} \setminus M^{t-1}_{m,n})}$ converging to (A,V,W) and $v'_i \in V'_i$ a sequence converging to v', then $A'_i v'_i$ converges to Av' but $A_i v'_i \in W$ and hence $Av' \in W$ since W is closed, this gives a contradiction. Let $(A,V,W) \in \mathcal{N}$ and let $r = \mathrm{rank}\, A$. Now $V \subset \ker A$, so let $V' \subset \mathbb{C}^n$ be a subspace satisfying $V \oplus V' = \ker A$, and $\mathrm{Im}\, A \subset W$ so let $W' \subset \mathbb{C}^m$ be a subspace satisfying $\mathrm{Im}\, A \oplus W' = W$. Let A' be a matrix of rank $t-1-r$, such that $\ker A' \oplus V' = \mathbb{C}^n$ and $\mathrm{Im}\, A' = W'$, such a matrix exists since $\dim V' = \dim W' = t-1-r$. Set $A_i = A + \frac{1}{i}A'$ then $\ker A_i = \ker A \bigcap \ker \frac{1}{i}A' = \ker A \bigcap \ker A' = V$ and $\mathrm{Im}\, A_i = \mathrm{Im}\, A + \mathrm{Im}\, \frac{1}{i}A' = W$. Hence $(A_i, V_i, W_i) := (A_i, V, W)$ is a sequence in $\beta(M^t_{m,n} \setminus M^{t-1}_{m,n})$ converging to (A,V,W), so $\mathcal{N} \subseteq \overline{\beta(M^t_{m,n} \setminus M^{t-1}_{m,n})}$ which finishes the proof.

An important consequence of this is the following:

Corollary 3.1 $\mathrm{Nash}(M^t_{m,n})$ *is a complex manifold.*

Proof Using the description of $\mathrm{Nash}(M^t_{m,n})$ given in Proposition 3.1 we get that the projection to the two last factors $\mathrm{Gr}(n-t+1,n) \times \mathrm{Gr}(t-1,m)$ gives $\mathrm{Nash}(M^t_{m,n})$ the structure of the total space of a vector bundle over a complex manifold.

It follows from Definition 2.3 and Proposition 3.1, that we have a map of transformations $f\colon \mathrm{Nash}(M^t_{m,n}) \to \mathrm{Tjur}(M^t_{m,n})$ by setting $f(A,V,W) = (A,V)$ and a map of transformations $g\colon \mathrm{Nash}(M^t_{m,n}) \to \mathrm{Tjur}^T(M^t_{m,n})$ by setting $g(A,V,W) =$

(A, W) and using (1). These maps are never isomorphisms, as we will see later when we determine the homotopy type of these spaces.

Proposition 3.2 *There does not exist a continuous maps of transformations between* $\mathrm{Tjur}(M^t_{m,n})$ *and* $\mathrm{Tjur}^T(M^t_{m,n})$.

Proof We start by using (1) to identify $\mathrm{Tjur}^T(M^t_{m,n})$ with the set $\{(A, W) \in M_{m,n} \times \mathrm{Gr}(t-1, m) \mid \mathrm{Im}(A) \subseteq W\}$. Let $f\colon \mathrm{Tjur}(M^t_{m,n}) \to \mathrm{Tjur}^T(M^t_{m,n})$ be a map of transformations, this implies that over $\pi^{-1}(M^t_{m,n} \setminus M^{t-1}_{m,n})$ we have $f(A, \ker A) = (A, \mathrm{Im}\, A)$. Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathbb{C}^n$ and $\{y_1, \ldots, y_m\}$ be a basis for $\mathbb{C}^m$. Let A be the matrix in these bases of the map $A(a_1x_1 + \cdots + a_nx_n) = a_1y_1 + \cdots + a_{t-2}y_{t-2} + 0y_{t-1} + \cdots + 0y_m$, notice that there is at least 2 zeros at the end since $t \leq m$. Now rank $A = t-2$ and hence $A \in M^{t-1}_{m,n}$. Let $V = \mathrm{Span}\{x_t, \ldots, x_n\}$ then it is clear that $\ker A \supset V$.

We now define two different sequences of matrices A^1_s and A^2_s where $A^i_s \in M^t_{m,n}$. The first is defined as $A^1_s(a_1x_1+\cdots+a_nx_n) := a_1y_1+\cdots+a_{t-2}y_{t-2}+\frac{1}{s}a_{t-1}y_{t-1}+0y_t + \cdots + 0y_m$ and the second is defined as $A^2_s(a_1x_1 + \cdots + a_nx_n) := a_1y_1 + \cdots + a_{t-2}y_{t-2} + 0y_{t-1} + \frac{1}{s}a_{t-1}y_t + 0t_{t+1} + \cdots + 0y_m$. It is clear that $\ker A^i_s = V$ and $\lim_{s\to\infty}(A^i_s, V) = (A, V)$ for $i = 1, 2$. Since $A^i_s \in M^t_{m,n} \setminus M^{t-1}_{m,n}$ we get that $f(A^i_s, V) = (A^i_s, \mathrm{Im}\, A^i_s)$. Let $W_1 := \mathrm{Span}\{y_1, \ldots, y_{t-1}\} = \mathrm{Im}\, A^1_s$ and $W_2 := \mathrm{Span}\{y_1, \ldots, y_{t-2}, y_t\} = \mathrm{Im}\, A^2_s$. If f was continuous, then we would have that $f(A, W) = f(\lim_{s\to\infty}(A^i_s, V)) = \lim_{s\to\infty} f(A^i_s, V) = (A, W_i)$ for $i = 1, 2$. But $W_1 \neq W_2$ hence f cannot be continuous. The argument that there is no continuous map of transformations from $\mathrm{Tjur}^T\, M^t_{m,n}$ to $\mathrm{Tjur}\, M^t_{m,n}$ is similar.

Next we determine the homotopy type of the transformations, and the above shows that in the case $\mathrm{Tjur}(M^t_{m,n})$ and $\mathrm{Tjur}^T(M^t_{m,n})$ are homotopy equivalent they are not isomorphic as transformations. Even in the case $n = m$ where $\mathrm{Tjur}(M^t_{m,m})$ and $\mathrm{Tjur}^t(M^t_{m,m})$ are isomorphic as real varieties by the isomorphism given by $(A, W) \to (A, \bar{W}^{\perp})$, they are not isomorphic as transformations.

Proposition 3.3 *Let* $\pi\colon (T(M^t_{m,n}), E) \to (M^t_{m,n}, M^{t-1}_{m,n})$ *be one of the three transformations discussed above. Then* $T(M^t_{m,n})$ *deformation retracts onto* $\pi^{-1}(0)$.

This gives that $\mathrm{Nash}(M^t_{m,n}) \sim \mathrm{Gr}(n-t+1, n) \times \mathrm{Gr}(t-1, m)$, $\mathrm{Tjur}(M^t_{m,n}) \sim \mathrm{Gr}(n-t+1, n)$ and $\mathrm{Tjur}^T(M^t_{m,n}) \sim \mathrm{Gr}(t-1, m)$, where $\sim$ denotes homotopy equivalence.

Proof We will only show this for $\mathrm{Nash}(M^t_{m,n})$. The other proofs are similar. Define $F\colon \mathrm{Nash}(M^t_{m,n}) \times \mathbb{C} \to \mathrm{Nash}(M^t_{m,n})$ as $F(A, V, W, s) = f_s(A, V, W) = (sA, V, W)$, using the identification for the Nash transformation given by Proposition 3.1. The map is well defined since $(sA)(V) = s(A(V)) = 0$ and $\mathrm{Im}(sA) = \mathrm{Im}(A) \subset W$ if $s \neq 0$ and $\mathrm{Im}(sA) = \{0\} \subset W$ if $s = 0$. It is continuous since it is just scalar multiplication. Restrict the map to $s \in [0, 1]$. Then $f_1 = \mathrm{id}$, $f_s|_{\pi^{-1}(0)} = id|_{\pi^{-1}(0)}$ and $f_0(\mathrm{Nash}(M^t_{m,n})) = \pi^{-1}(0)$. Hence f_s is a deformation retraction, and $\mathrm{Nash}(M^t_{m,n})$ deformation retracts onto $\pi^{-1}(0)$.

4 Transformations of General Determinantal Singularities

In this section we will introduce the transformations defined above for general determinantal varieties. We start by introducing the Tjurina transform. The Tjurina transform of a determinantal variety has been introduced in several places before for example in [1, 3, 12, 13] and [5]. They in general define the Tjurina transform of a determinantal variety X of type (m, n, t) given by $F\colon \mathbb{C}^N \to M_{m,n}$ as the fibre product $X \times_F \text{Tjur}(M^t_{m,n})$, which works very well in the cases they consider. But this definition gives the following problem in a more general setting: assume that $\dim(X) \leq (t-1)(n-t+1)$ and let $p\colon X \times_F \text{Tjur}(M^t_{m,n}) \to X$ be the projection to the first factor. Then $p^{-1}(0) \cong \text{Gr}(n-t+1, n)$, hence the exceptional fibre of p has dimension greater than or equal to the dimension of X. This means that the fibre product does not satisfy the conditions to be a transformation given in Definition 2.2.[1] We will give an alternative definition that does not have this problem. It should be said that in [3] and [5] they only consider the Tjurina transformation in situations where this does not happen, and that our definition agrees with theirs in these cases. We will see in Proposition 4.3 when the two definitions agree in general.

Definition 4.1 Let X be a determinantal variety of type (m, n, t) given by $F\colon \mathbb{C}^N \to M_{m,n}$ and assume that $\overline{X_t} = X$, define $B\colon X_t \to \text{Gr}(t-1, n)$ as the map that sends x into the row space of $F(x)$. Then we define the Tjurina transform $\text{Tjur}(X)$ of X as

$$\text{Tjur}(X) := \overline{\left\{\big(x, W\big) \in X_t \times \text{Gr}(t-1, n) \;\middle|\; W = B(x)\right\}} \subseteq X \times \text{Gr}(t-1, n),$$

where we again use the topological closure in the classical topology, and we define the map $\pi^{Tj}\colon \text{Tjur}(X) \to X$ as the projection to the first factor.

Remember as always we think of the determinantal variety X as the space X and the map F, hence as we just write X for the determinantal variety including the map F, we also write $\text{Tjur}(X)$ for the Tjurina transform which of course also depends of the map F.

The assumption that $\overline{X_t} = X$ is to avoid cases where there are irreducible components of X that do not give components of X_t. If $(X, 0)$ is an EIDS then $(X, 0)$ always satisfies this condition in a neighbourhood of the origin.

It is clear that this satisfies the conditions of Definition 2.2 to be a transformation of $(X, X_{<t})$ where $X_{<t} := \cup_{i=1}^{t-1} X_i = F^{-1}(M^{t-1}_{m,n})$, since $\pi^{Tj}|_{\text{Tjur}(X)\setminus(\pi_{Tj})^{-1}(X_t)}$ is the inverse of B, it is surjective because $\overline{X_t} = X$ and proper since all fibres are either points or closed subsets of $\text{Gr}(t-1, n)$ hence compact.

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[1]It is of course also possible that $p^{-1}(0)$ is a irreducible component of $X \times_F \text{Tjur}(M^t_{m,n})$ even without the condition on the dimensions, we will discuss this later in Proposition 4.3.

Notice that the choice of a Hermitian inner product on $\mathbb{C}^n$ gives a real linear isomorphism between the complex conjugate of the row space of $F(x)$ and $\ker\big(F(x)\big)$ and a real algebraic isomorphism $\mathrm{Gr}(t-1,n) \cong \mathrm{Gr}(n-t+1,n)$. Hence we get a real analytic isomorphism

$$\mathrm{Tjur}(X) \cong \overline{\Big\{\big(x,W\big) \in X_t \times \mathrm{Gr}(n-t+1,n) \mid W = \ker F(x)\Big\}} \subseteq X \times \mathrm{Gr}(n-t+1,n).$$

We use the row space in our definition, since it makes calculation easier as we see later.

Proposition 4.1 *Let* $(X,0)$ *be a determinantal singularity of type* $(m,n,1)$*, then* $(\mathrm{Tjur}(X),0) = (X,0)$.

Proof Since X is of type $(m,n,1)$ we have that $\mathrm{Tjur}(X) \subseteq X \times \mathrm{Gr}(0,n) = X$ and B is constant. The result then follows since $\mathrm{Tjur}(X) = \overline{X_1} = X$.

Notice that all determinantal singularities of type $(m,n,1)$ are local complete intersections, and that any local complete intersection can be given as a determinantal singularity of type $(m,n,1)$, in fact by a determinantal singularity of type $(m,1,1)$ of $(1,n,1)$. Hence the Proposition says that given the natural representation of a local complete intersection, then the Tjurina transform do not improve the singularity.

Hypersurfaces singularities can also some times be given as determinantal singularities of type (m,m,m), and we will later see some examples of hypersurfaces of type (m,m,m) for which the Tjurina transform is useful to simplify their singularities.

To study the local properties of the Tjurina transform closer we will use the following matrix charts on $\mathrm{Gr}(t-1,n)$. Let $I \subset \{1,\dots,n\}$ such that $\#I = t-1$. For each such $I = \{i_1,\dots,i_{t-1}\}$ let $a = (a_{ji})$ $j \in 1,\dots,t-1$ and $i \in \{1,\dots,n\} \setminus I$ be a $(t-1) \times (n-t+1)$ matrix with variables $a_{ji} \in \mathbb{C}$. We define a chart of $\mathrm{Gr}(t-1,n)$ by the $(t-1) \times n$ matrix $A_I(a)$ which consists of the columns C_i given as follows:

$$C_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{(t-1)i} \end{pmatrix} \text{ if } i \notin I, \text{ and } C_{i_l} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{where the 1 is in the } l^{th} \text{ entry.}$$

If we consider $a \in M_{t-1,n-t+1} = \mathbb{C}^{(t-1)(n-t+1)}$ then we can use $A_I(a)$ to define a map $\tilde{A}_I \colon \mathbb{C}^{(t-1)(n-t+1)} \to \mathrm{Gr}(t-1,n)$ by sending a to the row space of $A_I(a)$.

$\{\tilde{A}_I\}_I$ is a cover of $\mathrm{Gr}(t-1,n)$ by algebraic maps, and if $U_I = \mathrm{Im}(\tilde{A}_I)$ then the change of coordinates from $\tilde{A}_I^{-1}(U_I \cap U_J)$ to $\tilde{A}_J^{-1}(U_I \cap U_J)$ is given by $A_J^T A_I(a)$.

To see the row space of $F(x)$ in a given chart A_I, we construct the following $(m+t-1)\times n$ matrix:

$$\tilde{F}_I^{Tj}(x,a) := \begin{pmatrix} A_I(a) \\ F(x) \end{pmatrix}.$$

Then the row space of $F(x)$ is contained in $\tilde{A}_I(a)$ if and only if rank $\tilde{F}_I^{Tj}(x,a) = t-1$.

Let $\widetilde{\mathrm{Tjur}}_I(X) := (\tilde{F}_I^{Tj})^{-1}\big(M^t_{m+t-1,n}\big) \subset X \times \mathbb{C}^{(t-1)(n-t+1)}$, and $\tilde{\pi}_I^{Tj}\colon \widetilde{\mathrm{Tjur}}_I(X) \to X$ be the projection to the first factor. Then $\widetilde{\mathrm{Tjur}}_I(X)$ is the restriction of the fibre product $X \times_F \mathrm{Tjur}(M^t_{m,n})$ to the chart on $\mathbb{C}^N \times \mathrm{Gr}(t-1,n)$ given by I.

From the above construction we get $\mathrm{Tjur}_I(X) := \mathrm{Tjur}(X) \cap (X \times \mathrm{Im}\,\tilde{A}_I) \subset \widetilde{\mathrm{Tjur}}_I(X)$, but they are not necessarily equal. Notice that $\mathrm{Tjur}_I(X)$ is the restriction of the Tjurina transform $\mathrm{Tjur}(X)$ to the chart on $\mathbb{C}^N \times \mathrm{Gr}(t-1,n)$ given by I. $\mathrm{Tjur}_I(X)$ can be thought of as the strict transform of X in $\widetilde{\mathrm{Tjur}}_I(X)$. Moreover, $\widetilde{\mathrm{Tjur}}_I(X)$ is not necessarily a determinantal singularity. We have $(\pi_I^{Tj})^{-1}(X_t) = (\tilde{\pi}_I^{Tj})^{-1}(X_t)$. This implies that $\dim \widetilde{\mathrm{Tjur}}_I(X) = \max\big(\dim X, \dim(\tilde{\pi}_I^{Tj})^{-1}(X_{<t})\big)$. Now $\dim(\tilde{\pi}_I^{Tj})^{-1}(X_{<t})$ is the largest of the dimensions of the pullback of $X_1,\ldots,X_{t-1}$. Hence $(\tilde{\pi}_I^{Tj})^{-1}(X_s) \subset X \times \mathrm{Gr}(t-1,n)$ consists of the pairs (x,W) such that $x \in X_s$ and row space of $F(x)$ is a subset of W. We will denote the row space of $F(x)$ by R_xF. Since rank $F(x) = s-1$ we can write all such W as $W = R_xF + W_{F(x)}$ where $W_{F(x)}$ is $(t-s)$-dimensional subspace of the complement of $R_xF \subset \mathbb{C}^N$. Moreover, for any $t-s$ dimensional subspace V in the complement of $R_xF \subset \mathbb{C}^N$ we have $\mathrm{rank}(R_xF + V) = t-1$. Hence we get that $\{W \in \mathrm{Gr}(t-1,n) \mid R_xF \subset W\}$ is isomorphic to $\mathrm{Gr}(t-s, n-s+1)$. So we get that $\dim(\tilde{\pi}_I^{Tj})^{-1}(X_s) = \dim F^{-1}((M^s_{m,n} \setminus M^{s-1}_{m,n}) + \dim \mathrm{Gr}(t-s,n-s+1)$.

The above implies that $\dim \widetilde{\mathrm{Tjur}}_I(X) = \dim \mathrm{Tjur}_I(X) = \dim X$ if and only if

$$\begin{aligned}\dim(\tilde{\pi}_I^{Tj})^{-1}(X^s) &\le \dim X - \dim \mathrm{Gr}(t-s,n-s+1)\\ &= N-(m-t+1)(n-t+1)-(t-s)(n-t+1)\\ &= N-(m-s+1)(n-t+1)\end{aligned}$$

for all $s = 1,\ldots,t$. If X has an isolated singularity, this becomes $N \ge m(n-t+1)$.

Proposition 4.2 *If* $\dim \widetilde{\mathrm{Tjur}}_I(X) = \dim X$ *then* $\widetilde{\mathrm{Tjur}}_I(X)$ *is a determinantal variety.*

Proof We just need to check if $\operatorname{codim}\widetilde{\mathrm{Tjur}}_I(X) = \operatorname{codim} M^t_{m+t-1,n} = (m+t-1-t+1)(n-t+1) = m(n-t+1)$. But $\operatorname{codim}\widetilde{\mathrm{Tjur}}_I(X) = \operatorname{codim}\mathrm{Tjur}_I(X) = \operatorname{codim} X+(t-1)(n-t+1) = (m-t+1)(n-t+1)+(t-1)(n+t-1) = m(n-t+1)$.

In this case we get that $\widetilde{\mathrm{Tjur}}_I(X)$ is a determinantal variety of type $(m+t-1, n, t)$. But rank $\tilde{F}_I^{Tj}(0,0) = t-1$, so one can find another matrix $F'_I(x,a)$ defining $\widetilde{\mathrm{Tjur}}_I(X)$ such that $F'_I(0,0) = 0$ and this is a determinantal variety of type $(m+t-1-(t-1), n-(t-1), t-(t-1)) = (m, n-t+1, 1)$. Since $\operatorname{codim}\widetilde{\mathrm{Tjur}}_I(X) = m(n-t+1)$ we get that $\widetilde{\mathrm{Tjur}}_I(X)$ is a complete intersection. We will later show how to explicitly find $F'_I(x,a)$ also in the case $\dim\widetilde{\mathrm{Tjur}}_I(X) \neq \dim X$.

We can also use this to determine when $\mathrm{Tjur}_I(X)$ and $\widetilde{\mathrm{Tjur}}_I(X)$ are equal. Notice that $\widetilde{\mathrm{Tjur}}_I(X) = \big(X \times_F \mathrm{Tjur}(M^t_{m,n})\big) \cap \big(X \times \operatorname{Im}\tilde{A}_I\big)$, hence the next proposition also answers the question, when is our definition of Tjurina transform the same as the one used by other authors. Remember that we earlier defined $X_s := F^{-1}(M^s_{m,n} \setminus M^{s-1}_{m,n})$.

Proposition 4.3 *Let X be a determinantal variety. Then $\widetilde{\mathrm{Tjur}}_I(X) = \mathrm{Tjur}_I(X)$ if and only if $\dim X_s < N-(m-s+1)(n-t+1)$ for all $s \in 1,\dots,t-1$. Furthermore, if $(X,0)$ is an EIDS, then $\widetilde{\mathrm{Tjur}}_I(X) = \mathrm{Tjur}_I(X)$ if and only if $\dim X_1 < N-m(n-t+1)$.*

Proof Since $\mathrm{Tjur}(X)$ is a transformation, we have that $\dim \pi^{Tj}(X^{t-1}) < \dim X$. Then the above calculations of the dimensions of the fibres give the inequalities, and we get the only if direction.

So assume that the inequalities are satisfied, this implies $\dim\mathrm{Tjur}_I(X) = \dim\widetilde{\mathrm{Tjur}}_I(X)$ and $\dim(\tilde{\pi}_I^{Tj})^{-1}(X^{t-1}) < \dim X$. Now $\mathrm{Tjur}_I(X)$ is a union of irreducible components of $\widetilde{\mathrm{Tjur}}_I(X)$, and each irreducible component of $\mathrm{Tjur}_I(X)$ is not a proper subvariety of any irreducible variety of the same dimension, since they are closed. This implies that if $\widetilde{\mathrm{Tjur}}_I(X) \neq \mathrm{Tjur}_I(X)$ then there exists another irreducible component $V \subseteq \widetilde{\mathrm{Tjur}}_I(X)$ not contained in $\mathrm{Tjur}_I(X)$. But since $\widetilde{\mathrm{Tjur}}_I(X)$ is a complete intersection it is equidimensional, and hence $\dim V = \dim\mathrm{Tjur}_I(X)$. Since $(\pi_I^{Tj})^{-1}(X_t) = (\tilde{\pi}_I^{Tj})^{-1}(X_t)$ we have that $V \subset (\tilde{\pi}_I^{Tj})^{-1}(X^{t-1})$, but this is a contradiction since $\dim V > \dim \tilde{\pi}_I^{Tj}(X^{t-1})$.

For the case of EIDS remember that if $(X,0)$ is an EIDS, $1 < s < t$ and $X_s \neq \emptyset$ then $\operatorname{codim} X_s = (m-s+1)(n-s+1)$. Hence the inequality of the first part of the theorem is satisfied for all $1 < s < t$ and therefore one only needs that $\dim X_1 < N-(m-1+1)(n-t+1)$ to get the conclusion $\widetilde{\mathrm{Tjur}}_I(X) = \mathrm{Tjur}_I(X)$.

If X is a determinantal singularity and we assume that $X = \overline{X_t}$ then $\dim X_t = N-(m-t+1)(n-t+1)$. But remember even if $X_s \neq \emptyset$ for some $s < t$ then $\overline{X_s}$ needs not be a determinantal singularity, and hence $\dim X_s$ can be larger than $N-(m-s+1)(n-s+1)$ as the following example shows. Hence the assumptions of Proposition 4.3 need not be satisfied.

Example 4.1 Let $X := F^{-1}(M^3_{3,3})$ be the determinantal singularity given by the matrix:

$$F(x, y, z) := \begin{pmatrix} x & 0 & z \\ y & x & 0 \\ 0 & y & x \end{pmatrix}.$$

We have that X is the variety given by the equation $x^3 + y^2 z = 0$, which have the z-axis as its singular set. X is determinantal since $\operatorname{codim} X = 1 = \operatorname{codim} M^3_{3,3}$. Now X_3 is X minus the z-axis, X_2 is the z-axis minus the origin and X_1 is the origin. This implies that the $\operatorname{codim} X_2 = 2$ but $\operatorname{codim} M^3_{3,3} = 4$, so X do not satisfy Proposition 4.3.

We now want to give an explicit method to find $F'_I(x, a)$. Let $I = \{i_1, \ldots, i_{t-1}\} \subset \{1, \ldots, n\}$ as before. Now by adding columns of the form $-a_{ji} C_{i_j}$ to the i'th column, for all $i \notin I$ and all $j = 1, \ldots, t-1$, we get a matrix which has $t-1$ linearly independent rows R_{i_j} of the form $R_{i_j} = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is the i_j entry. To this matrix we then add rows of the form $-f_{li_j}(x) R_{i_j}$ to the $l+t-1$'th row for $l = 1, \ldots, m$ and $j = 1, \ldots, t-1$. We now have a matrix $\bar{F}_I(x, a)$ consisting of the following columns:

$$\bar{F}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_{1i}(x) - \sum_{j=1}^{t-1} a_{ji} f_{1i_j}(x) \\ \vdots \\ f_{mi}(x) - \sum_{j=1}^{t-1} a_{ji} f_{mi_j}(x) \end{pmatrix} \text{ if } i \notin I, \text{ and } \bar{F}_{i_l} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where the 1 in $\bar{F}_{i_l}$ is on the l^{th} entry. The $t \times t$ minors of $\bar{F}_I(x, a)$ still defines $\widetilde{\mathrm{Tjur}}_I(X)$. Notice that we can choose special minors $\Delta_{l,i}$, with $l \in \{1, \ldots, m\}$ and $i \notin I$, where each row and each column have a single non zero entry, which is 1 except for the li'th entry which is $f_{li}(x) - \sum_{j=1}^{t-1} a_{ji} f_{li_j}(x)$. This implies that $\widetilde{\mathrm{Tjur}}_I(X)$ is defined by the $(n-t+1)m$ equations $f_{li}(x) - \sum_{j=1}^{t-1} a_{ji} f_{li_j}(x) = 0$. Hence it is defined by the 1×1 minors of the matrix $m \times (n-t+1)$ matrix $F'_I(x, a)$ with columns:

$$F'_i = \begin{pmatrix} f_{1i}(x) - \sum_{j=1}^{t-1} a_{ji} f_{1i_j}(x) \\ \vdots \\ f_{mi}(x) - \sum_{j=1}^{t-1} a_{ji} f_{mi_j}(x) \end{pmatrix} \text{ if } i \notin I.$$

This still does not imply that $\widetilde{\text{Tjur}}_I(X)$ is a determinantal variety, since the codimension might not be right. Even if $\widetilde{\text{Tjur}}_I(X)$ is a determinantal variety, it might have components of maximal dimension which is contained in $(\widetilde{\pi}^{Tj})^{-1}(X_{<t})$ and hence $\widetilde{\text{Tjur}}_I(X) \neq \text{Tjur}_I(X)$, as we will see in the next examples.

Example 4.2 Let X be the irreducible determinantal variety of type (2, 3, 2) defined by the following matrix

$$F_1(x, y, z, w) := \begin{pmatrix} w^l & y & x \\ z & w & y^k \end{pmatrix},$$

with $k, l > 2$. Then $\widetilde{\text{Tjur}}_I(X)$ is a determinantal variety for all I. Let us start by looking in the chart defined by $I = \{1\}$.

$$F'_{\{1\}}(x, y, z, w, a_2, a_3) := \begin{pmatrix} y - a_2 w^l & x - a_3 w^l \\ w - a_2 z & y^k - a_3 z \end{pmatrix}.$$

The equations $y - a_2 w^l = 0$, $x - a_3 w^l = 0$ and $w - a_2 z = 0$ all just give the variables x, y and w as functions of z, a_2 and a_3. Using these equations the last equation becomes $a_2^{k(l+1)} z^{kl} - a_3 z = 0$ which shows that $\widetilde{\text{Tjur}}_I(X)$ has two irreducible components. The first given by $\{x = y = z = w = 0\}$ which is the fibre over the origin. The second irreducible component, which is $\text{Tjur}_I(X)$, is given by the equations $y - a_2 w^l = 0$, $x - a_3 w^l = 0$, $w - a_2 z = 0$ and $a_2^{k(l+1)} z^{k(l-1)} - a_3 = 0$ and is hence smooth.

Now let us look closer on the equations in the chart defined by $I = \{2\}$.

$$F'_{\{2\}}(x, y, z, w, a_1, a_3) := \begin{pmatrix} w^l - a_1 y & x - a_3 y \\ z - a_1 w & y^k - a_3 w \end{pmatrix}.$$

Notice that the equations $x - a_3 y = 0$ and $z - a_1 w = 0$ define x and z as holomorphic functions of the other variables, and give embeddings of a $\mathbb{C}^4$ into $\mathbb{C}^6$. Now if we multiply the equations $y^k - a_3 w = 0$ and $w^l - a_1 y = 0$ we get:

$$\begin{aligned} 0 &= (y^k - a_3 w)(w^l - a_1 y) = y^k w^l - a_1 y^{k+1} - a_3 w^{l+1} + a_1 a_3 y w \\ &= y^k w^l - a_1 a_3 y w - a_1 a_3 y w + a_1 a_3 y w = y w (y^{k-1} w^{l-1} - a_1 a_3). \end{aligned}$$

Hence we see that $\widetilde{\text{Tjur}}_I(X)$ is not irreducible. $y = 0$ and $w = 0$ both define the fibre $(\tilde{\pi}_I^{Tj})^{-1}(0)$ which is two dimensional and therefore cannot be a subset of $\text{Tjur}_I(X)$. Therefore, $\text{Tjur}_I(X)$ is given by the equations $y^{k-1} w^{l-1} - a_1 a_3 = 0$, $w^l - a_1 y = 0$ and $y^k - a_3 w = 0$. Hence it can be given as a determinantal variety of the same type as X given by the matrix

$$\begin{pmatrix} w^{l-1} & y & a_3 \\ a_1 & w & y^{k-1} \end{pmatrix}.$$

The case of the last chart $I = \{3\}$ is similar to that of the first chart, hence in that chart we also have that $\mathrm{Tjur}_I(X)$ is smooth. In all charts we have that $\widetilde{\mathrm{Tjur}}_I(X) \neq \mathrm{Tjur}_I(X)$.

Example 4.3 Let $X \subset \mathbb{C}^4$ be the determinantal variety of type $(3, 2, 2)$ given by

$$F_2(x, y, z, w) := \begin{pmatrix} w^l & z \\ y & w \\ x & y^k \end{pmatrix},$$

with $k, l > 2$. Then $\widetilde{\mathrm{Tjur}}_I(X)$ is given in the two charts $I = \{1\}, \{2\}$ by the matrices

$$F'_{\{1\}}(x, y, z, w, a_1) := \begin{pmatrix} z - a_1 w^l \\ w - a_1 y \\ y^k - a_1 x \end{pmatrix} \text{ and } F'_{\{2\}}(x, y, z, w, a_2) := \begin{pmatrix} w^l - a_2 z \\ y - a_2 w \\ x - a_2 y^k \end{pmatrix}$$

In this case we see that $\widetilde{\mathrm{Tjur}}_I(X) = \mathrm{Tjur}_I(X)$, and hence the Tjurina transform of X is a complete intersection.

Notice that the underlying varieties in Examples 4.2 and 4.3 are the same, it is just their representations as determinantal varieties which are different. In fact the difference is that $F_1(x, y, z, w) = F_2(x, y, z, w)^T$. In Example 4.2 we get that $\widetilde{\mathrm{Tjur}}_I(X) \neq \mathrm{Tjur}(X)$ and in Example 4.3 that $\widetilde{\mathrm{Tjur}}_I(X) = \mathrm{Tjur}(X)$. This does not contradict Proposition 4.3 since in Example 4.2 the inequality $\dim X_s < N - (m - s + 1)(n - t + 1)$ is not satisfied for $s = 1$ since $N - (m - 1 + 1)(n - t + 1) = 4 - 3(3 - 2 + 1) = 0$. In Example 4.3 the inequality is satisfied and we get that $\widetilde{\mathrm{Tjur}}_I(X) = \mathrm{Tjur}(X)$.

Let us define $\mathrm{Tjur}^T(X)$.

Definition 4.2 Let X be a determinantal variety of type (m, n, t) given by $F \colon \mathbb{C}^N \to M_{m,n}$ such that $\overline{X_t} = X$, define $C \colon X_t \to \mathrm{Gr}(t - 1, m)$ as the map that sends x into the column space of $F(x)$. Then we define $\mathrm{Tjur}^T(X)$ of X as

$$\mathrm{Tjur}^T(X) = \overline{\left\{ \big(x, W\big) \in X_t \times \mathrm{Gr}(t - 1, m) \;\middle|\; W = C(x) \right\}} \subseteq X \times \mathrm{Gr}(t - 1, m),$$

and we define the map $\pi^{Tj^T} \colon \mathrm{Tjur}^T(X) \to X$ as the projection to the first factor.

This definition gives us that $\mathrm{Tjur}^T(X) = \mathrm{Tjur}(X^T)$, where X^T is X but defined as a determinantal singularity by $F^T \colon \mathbb{C}^N \to M_{n,m}$. This means that we can define $\widetilde{\mathrm{Tjur}}_I^T(X)$ as for $\mathrm{Tjur}(X)$, either by setting $\widetilde{\mathrm{Tjur}}_I^T(X) = \widetilde{\mathrm{Tjur}}_I(X^T)$ or by defining it using $\bar{F}_I^T(x, a) := \big(F(x) \big| A_I^T(a)\big)$, where I now is a subset of $1, \ldots, m$.

This immediately gives us the following results.

Proposition 4.4 $\widetilde{\mathrm{Tjur}}_I^T(X)$ *is a determinantal variety if and only if* $\dim X_s \leq N - (m - t + 1)(n - s + 1)$ *for all* $s \in 1, \ldots, t$.

Proposition 4.5 $\widetilde{\text{Tjur}}_I^T(X) = \text{Tjur}_I^T(X)$ *if and only if* $\dim X_s < N-(m-t+1)(n-s+1)$ *for all* $s \in 1, \ldots, t-1$.

Notice that this definition of $\text{Tjur}^T(M^t_{m,n})$ is the same as the one we gave earlier, since the column space of a matrix is the same as its image.

The next example shows that just like the blow-up and the Nash transform, the Tjurina transform of a normal variety needs not be normal and that the dimension of the singular set can increase under the Tjurina transform.

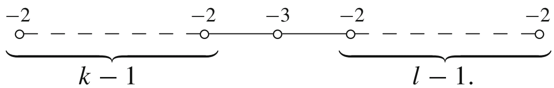
*Example 4.4 (*Tjur(X) *Might Have Singular Locus of Larger Dimension that* X*)* Let X be the hypersurface with an isolated singularity at the origin given by $z^2 - x^4 - x^2y^3 - x^2y^5 - y^8 = 0$. It can be given as a determinantal variety of type $(2, 2, 2)$ by the matrix $\begin{pmatrix} z & x^2+y^3 \\ x^2+y^5 & z \end{pmatrix}$. We get that the Tjurina transform is given by the following matrices

$$F'_{\{1\}}(x, y, z, a_2) = \begin{pmatrix} x^2 + y^3 - a_2 z \\ z - a_2(x^2 + y^5) \end{pmatrix} \text{ and } F'_{\{2\}}(x, y, z, a_1) = \begin{pmatrix} z - a_1(x^2 + y^3) \\ x^2 + y^5 - a_1 z \end{pmatrix}.$$

In the first chart we can, by a change of coordinates, see that we have the hypersurface $x^2 + y^3 - a_2^2(x^2 + y^5) = 0$, which has all of the a_2-axis as its singular set. In the same way the second chart gives us the hypersurface $x^2 + y^5 - a_1^2(x^2 + y^3) = 0$, which has the a_1-axis as its singular set. Hence Tjur(X) has singularities of codimension 1, and is, therefore, not normal. It also illustrates that the singular set of Tjur(X) might have larger dimension than the singular set of X.

Notice in general that if X is a determinantal singularity given by a matrix F such that all entries in F have orders ≥ 2 then the singular set of Tjur(X) contains the full fibre over the common zero locus of the entries of F.

We saw in Sect. 3 that $\text{Nash}(M^t_{m,n}) \cong \text{Tjur}(M^t_{m,n}) \times_{M^t_{m,n}} \text{Tjur}^T(M^t_{m,n})$ where the isomorphism is real algebraic. Is this then true in general? Is $\text{Nash}(X) \cong \text{Tjur}(X) \times_X \text{Tjur}^T(X)$? The answer is unfortunately no as we can see in the following. Let X be the determinantal singularity defined in Example 4.2. There we saw that the exceptional divisor of Tjur(X) consists of two irreducible components. In Example 4.3 we got that the exceptional divisor of $\text{Tjur}^T(X)$ is a single irreducible curve. Hence the exceptional divisor of $\text{Tjur}(X) \times_X \text{Tjur}^T(X)$ consists of three irreducible curves. But in [12] Tjurina shows that $(X, 0)$ is a minimal surface singularity with the following minimal dual resolution graph.



Following the work of Spivakovsky [11] the irreducible components of the exceptional divisor of the normalized Nash transform of a surface singularity correspond to the irreducible components of the exceptional divisor of the minimal resolution intersecting the strict transform of the polar curve of a generic plane projection. By Theorem 5.4 in Chapter III of [11] we find that the polar of a generic plane projection of X intersects the exceptional divisor in two different components. This implies that the exceptional divisor of $\mathrm{Nash}(X)$ has at most two components, since the number of components cannot decrease under normalization. Hence $\mathrm{Nash}(X)$ and $\mathrm{Tjur}(X) \times_X \mathrm{Tjur}^T(X)$ have non isomorphic exceptional divisors, and can, therefore, not be isomorphic as transformations.

5 When Is the Tjurina Transform a Complete Intersection

In Lemma 5.2 of their article [5] Frühbis-Krüger and Zach find conditions under which the Tjurina transforms of Cohen-Macaulay codimension 2 singularities in $\mathbb{C}^5$ only have isolated complete intersection singularities. In this section we give some general condition on when the Tjurina transform of an EIDS is a local complete intersection.

If $(X, 0)$ is an EIDS, remember that it means that F is transverse to all strata of $M^t_{m,n}$ in a punctured neighbourhood of the origin, then we get the following result concerning the Tjurina transform.

Proposition 5.1 *Let $(X, 0) \subset \mathbb{C}^N$ be an EIDS of type (m, n, t), then* $\mathrm{Tjur}(X)$ *is a local complete intersection if $N - m(n - t + 1) > \dim X_1$ and* $\mathrm{Tjur}^T(X)$ *is a local complete intersection if $N - n(m - t + 1) > \dim X_1$.*

Proof To show that $\mathrm{Tjur}(X)$ is a local complete intersection, it is enough to show that $\widetilde{\mathrm{Tjur}_I(X)}$ is a complete intersection for all I. To do this we show that $\mathrm{Tjur}_I(X) = \widetilde{\mathrm{Tjur}}_I(X)$. Since $(X, 0)$ is EIDS then by Proposition 4.3 we just need that $\dim X_1 < N - m(n - t + 1)$ which follows from the assumption. So $\mathrm{Tjur}_I(X) = \widetilde{\mathrm{Tjur}}_I(X)$ and $\widetilde{\mathrm{Tjur}}_I(X)$ is a complete intersection. Hence $\mathrm{Tjur}(X)$ is a local complete intersection.

The proof for $\mathrm{Tjur}^T(X)$ is similar, just exchange n and m.

We are in different situations if $X_1 = \{0\}$ or if $X_1 \neq \{0\}$. Let us first give the following theorem that takes care of the second case.

Theorem 5.1 *Let $(X, 0)$ be ab EIDS and assume that $X_1 \neq \{0\}$. Then* $\mathrm{Tjur}(X)$ *and* $\mathrm{Tjur}^T(X)$ *are both local complete intersections.*

Proof Assume that X is defined by $F : \mathbb{C}^N \to M_{m,n}$. Since $(X, 0)$ is an EIDS there exist an open neighbourhood of the origin U such that for $x \in (X_1 \setminus \{0\}) \cap U$ we have that F is transverse to the strata $M^1_{m,n}$ at x. But this implies that F is a submersion at x because $M^1_{m,n} = \{0\}$. Hence there is an open neighbourhood in $\mathbb{C}^N$ of $(X_1 \setminus \{0\}) \cap U$ on which F is a submersion. Then the Submersion Theorem gives that $(X_1 \setminus \{0\}) \cap U$ is a smooth manifold of dimension $N - mn$. Adding the origin to $(X_1 \setminus \{0\}) \cap U$ does not change the dimension (but might make it singular), hence $\dim X_1 = N - mn$.

Proposition 5.1 is then satisfied for both $\mathrm{Tjur}(X)$ and $\mathrm{Tjur}^T(X)$ since $mn > m(n-t+1)$ and $mn > n(m-t+1)$ for all $1 < t \leq \min\{m, n\}$. If $t = 1$ the result follows from Proposition 4.1.

If $X_1 = \{0\}$ then the equations to determine whether $\mathrm{Tjur}(X)$ is a local complete intersection becomes $\dim X_1 = 0 < N - m(n-t+1)$ or $m(n-t+1) < N$ and likewise $\mathrm{Tjur}^T(X)$ is a local complete intersection if $n(m-t+1) < N$. The assumption on N can be replaced by an assumption on t and the strata of X as seen in the next proposition.

Proposition 5.2 *Let* $(X, 0)$ *be an EIDS of type* (m, n, t)*, where* $t \geq 3$*,* $X_1 = \{0\}$ *and* $X_2 \neq \emptyset$*. Then at least one of* $\mathrm{Tjur}(X)$ *and* $\mathrm{Tjur}^T(X)$ *is a local complete intersection.*

Proof First notice that since $t \geq 3$ one of the following two inequalities holds $n - 1 < m(t-2)$ or $m - 1 < n(t-2)$. We will first show that if the first equation holds, then $\mathrm{Tjur}(X)$ is a local complete intersection.

Assume that $n - 1 < m(t-2)$. To show that $\mathrm{Tjur}(X)$ is a complete intersection, we just need to show that $0 < N - m(n-t+1)$. Now $0 < \dim X_2 = N - (m-1)(n-1) = N - mn + m + n - 1 < N - mn + m + m(t-2) = N - m(n-t+1)$ by the assumption $X_2 \neq \emptyset$. So $\mathrm{Tjur}(X)$ is a local complete intersection.

If $m - 1 < n(t-2)$, then the same argument with exchanging m and n shows that $\mathrm{Tjur}^T(X)$ is a local complete intersection.

As we saw in Examples 4.2 and 4.3, this proposition can still hold if $t < 3$, but next we will give an example with $t = 2$ where we have $\mathrm{Tjur}_I(X) \neq \widetilde{\mathrm{Tjur}}_I(X)$ and $\mathrm{Tjur}^T_J(X) \neq \widetilde{\mathrm{Tjur}}^T_J(X)$ for all I, J. But in the example, both $\mathrm{Tjur}(X)$ and $\mathrm{Tjur}^T(X)$ are complete local intersections.

Example 5.1 Let $X \subset \mathbb{C}^3$ be the determinantal variety of type $(3, 2, 2)$ given by

$$F(x, y, z, w) := \begin{pmatrix} z & y & x^{k-3} \\ 0 & x & y \end{pmatrix}.$$

For $k > 4$. Then $\widetilde{\mathrm{Tjur}}_I(X)$ is given in the three charts $I = \{1\}, \{2\}, \{3\}$ as follows. In the first chart the matrix is

$$F'_{\{1\}}(x, y, z, a_2, a_3) := \begin{pmatrix} y - a_2 z & x^{k-3} - a_3 z \\ x & y - a_3 x \end{pmatrix}.$$

We see that $\widetilde{\mathrm{Tjur}}_{\{1\}}(X)$ is the fibre over 0 (given by $x = y = z = 0$) union the z-axis (given by $x = y = a_2 = a_3 = 0$), so we get that $\mathrm{Tjur}_{\{1\}}(X)$ is the z-axis.

In the second chart we get

$$F'_{\{2\}}(x, y, z, w, a_1, a_3) := \begin{pmatrix} z - a_1 y & x^{k-3} - a_3 y \\ -a_1 x & y - a_3 x \end{pmatrix}.$$

Here we see that $\widetilde{\mathrm{Tjur}}_{\{2\}}(X)$ is the fibre over 0 (given by $x = y = z = 0$) union the curve singularity given by $x^{k-4} - a_3^2 = 0$, $y = a_3 x$ and $a_1 = z = 0$. Hence $\mathrm{Tjur}_{\{2\}}(X)$ is an A_{k-5} plane curve singularity embedded in $\mathbb{C}^5$.

In the last chart we get

$$F'_{\{3\}}(x, y, z, w, a_1, a_2) := \begin{pmatrix} z - a_1 x^{k-3} & y - a_2 x^{k-3} \\ -a_1 y & x - a_2 y \end{pmatrix}.$$

Now we see that $\widetilde{\mathrm{Tjur}}_{\{3\}}(X)$ is the fibre over 0 (given by $x = y = z = 0$) union the curve given by $1 - a_2^2 x^{k-4} = 0$, $y = a_2 x^{k-3}$ and $a_1 = z = 0$. Hence $\mathrm{Tjur}_{\{3\}}(X)$ is a smooth curve in this chart.

So $\mathrm{Tjur}(X)$ is a line disjoint union an A_{k-5} curve, and the fibre over 0 is 2 dimensional.

If we calculate $\widetilde{\mathrm{Tjur}}_I^T(X)$ in the charts $\{1\}$ and $\{2\}$, we get

$$F'_{\{1\}}(x, y, z, a_2) = \begin{pmatrix} -a_2 z \\ x - a_2 y \\ y - a_2 x^{k-3} \end{pmatrix} \text{ and } F'_{\{2\}}(x, y, z, w, a_1) = \begin{pmatrix} z \\ y - a_1 x \\ x^{k-3} - a_1 y \end{pmatrix}.$$

We see that in the first chart we have a line union the fibre over 0 and in the second chart we have an A_{k-5} curve singularity union the fibre over zero.

So in this case we have that $\mathrm{Tjur}(X)$ and $\mathrm{Tjur}^T(X)$ are the same, a line disjoint union an A_{k-5}. Notice that in this case $\mathrm{Tjur}(X)$ is also a local complete intersection. Now X is the union of a line l and an A_{k-3} singularity intersecting at the origin. We see that the transformation has separated the line and the singularity, and improved the singularity i.e. what was before an A_{k-3} singularity is now an A_{k-5} singularity.

In Theorem 5.1 we saw that $X_1 \neq \{0\}$ both $\mathrm{Tjur}(X)$ and $\mathrm{Tjur}^T(X)$ are local complete intersections and in Proposition 5.2 we saw that if $t \geq 3$ then one of $\mathrm{Tjur}(X)$ or $\mathrm{Tjur}^T(X)$ is a local complete intersection. The case $t = 1$ is not interesting, because in this case $\mathrm{Tjur}(X) = \mathrm{Tjur}^T(X) = X$ and X is a complete intersection. The next proposition will explain the case $t = 2$.

Proposition 5.3 *Let* $(X, 0)$ *be an EIDS of type* $(m, n, 2)$ *with* $X_1 = \{0\}$*, then one of* $\mathrm{Tjur}(X)$ *or* $\mathrm{Tjur}^T(X)$ *is a local complete intersection if* $\min(n, m) \leq \dim X$.

Proof To prove that $\mathrm{Tjur}(X)$ is a complete intersection we just need to see that $0 = \dim X_1 < N - m(n - t + 1) = N - m(n - 1)$ by Proposition 5.1. But $(m-t+1)(n-t+1) = (m-1)(n-1) = \operatorname{codim} X$, hence $N = (m-1)(n-1) + \dim X$. Then the inequality becomes $0 < (m - 1)(n - 1) + dimX - m(n - 1)$. Hence $\mathrm{Tjur}(X) = \widetilde{\mathrm{Tjur}}(X)$ and hence a complete intersection if $n - 1 < \dim X$. The case $\mathrm{Tjur}^T(X) = \widetilde{\mathrm{Tjur}^T}(X)$ is gotten by exchanging n and m.

These results are only in one direction, because what we really prove is that if the inequalities are satisfied, then $\mathrm{Tjur}(X) = \widetilde{\mathrm{Tjur}}(X)$ or $\mathrm{Tjur}^T(X) = \widetilde{\mathrm{Tjur}^T}(X)$.

But Tjur(X) or Tjur$^T(X)$ can still be local complete intersections, even if this is not true, as we saw in Example 5.1.

6 Using Tjurina Transform to Resolve Hypersurface Singularities

In the previous section we saw that very often the Tjurina transform is a complete intersection of type $(m, n, 1)$, which means that one cannot get a resolution by using only the Tjurina transform because of Proposition 4.1. Notice also that in several of the examples Tjur(X) is normal, so using only Tjurina transform and normalizations will also not produce a resolution. In the next example we will look at the case of the A_n surface singularities and see that the Tjurina transform in some cases can be used to achieve a resolution.

Example 6.1 (A_n Singularities) In this example we show how different representations of the simple A_n singularity can lead to different Tjurina transforms.

First we can represent A_n as a determinantal singularity of type $(1, 1, 1)$, then the Tjurina transform of A_n is just A_n itself, by Proposition 4.1. But we can also represent A_n as the determinantal singularity of type $(2, 2, 2)$ defined by:

$$F(x, y, z) = \begin{pmatrix} x & z^l \\ z^{n-l+1} & y \end{pmatrix},$$

where $0 < l \leq n$. In this case we get that the Tjurina transform is given by:

$$F'_{\{1\}}(x, y, z, a_2) = \begin{pmatrix} z^l - a_2 x \\ y - a_2 z^{n-l+1} \end{pmatrix} \text{ and } F'_{\{2\}}(x, y, z, a_1) = \begin{pmatrix} x - a_1 z^l \\ z^{n-l+1} - a_1 y \end{pmatrix}.$$

So we see that Tjur(A_n) using these representations has an A_{l-1} and an A_{n-l} singularity, so we have simplified the singularity. It is clear that by writing these new A_m singularities as determinantal singularities of type $(2, 2, 2)$, we can apply the Tjurina transform again to simplify the singularity. By repeatedly doing this we can resolve the A_n singularity.

As we can see in Example 6.1 the Tjurina transform depends not only on the singularity type of X but we can also get different transforms if we have different matrix presentations of the same type.

In the next example we will show how to obtain a resolution through repeated Tjurina transforms changing the determinantal type and matrix presentation. By this we mean that if the Tjurina transform gives us a complete intersection of the form $(m, n, 1)$, which by change of coordinates locally can be seen as a hypersurface, we will then write this hypersurface as a determinantal singularity of type (t, t, t).

Example 6.2 (E_7 Singularity) The simple surface singularity E_7 can be defined by the equation $y^2 + x(x^2 + z^3) = 0$. This can be seen as the determinantal singularity of type $(2, 2, 2)$ given by the following matrix: $\begin{pmatrix} y & x^2+z^3 \\ -x & y \end{pmatrix}$. We then perform the Tjurina transform and get:

$$F'_{\{1\}}(x, y, z, a_2) = \begin{pmatrix} x^2 + z^3 - a_2 y \\ y + a_2 x \end{pmatrix} \text{ and } F'_{\{2\}}(x, y, z, a_1) = \begin{pmatrix} y - a_1(x^2 + z^3) \\ -x - a_1 y \end{pmatrix}.$$

By changing coordinates we see that $F'_{\{1\}}$ is equivalent to the hypersurface $x^2 + z^3 + w^2 x = 0$, which has a singular point at $(0, 0, 0)$, and $F'_{\{2\}}$ is equivalent to the hypersurface $x + v^2(x^2 + z^3) = 0$ which is non singular.

So we will continue working in the first chart, and we will denote this singularity $\mathrm{Tjur}(E_7)$. The exceptional divisor $E_1 = (\pi^{Tj})^{-1}(0)$ is given by $x = z = 0$. We now write $\mathrm{Tjur}(E_7)$ as the matrix $\begin{pmatrix} x & -z^2 \\ z & x+w^2 \end{pmatrix}$ and perform the Tjurina transform.

$$F'_{\{1\}}(x, z, w, a_2) = \begin{pmatrix} -z^2 - a_2 x \\ x + w^2 - a_2 z \end{pmatrix} \text{ and } F'_{\{2\}}(x, z, w, a_1) = \begin{pmatrix} x + a_1 z^2 \\ z - a_1(x + w^2) \end{pmatrix}.$$

The first chart is equivalent to the hypersurface $yw^2 - z^2 - y^2 z = 0$ which has a singularity at $(0, 0, 0)$, and the second chart is equivalent to $v(w^2 - vz^2) - z = 0$ which is smooth. The exceptional divisor consist of two components, the strict transform of the exceptional divisor from before (which we still denote by E_1) is given by $z = y = 0$ and a new component E_2 given by $x = w = 0$. They intersect each other in the singular point.

We will continue in the first chart and denote this singularity by $\mathrm{Tjur}^2(E_7)$. It can be given by the matrix $\begin{pmatrix} y & -z \\ z & w^2-yz \end{pmatrix}$ as a determinantal singularity of type $(2, 2, 2)$. Its Tjurina transform is given by

$$F'_{\{1\}}(y, z, w, a_2) = \begin{pmatrix} -z - a_2 y \\ w^2 - yz - a_2 z \end{pmatrix} \text{ and } F'_{\{2\}}(y, z, w, a_1) = \begin{pmatrix} y + a_1 z \\ z - a_1(w^2 - yz) \end{pmatrix}.$$

In the first chart we have the hypersurface $xy^2 + w^2 + x^2 y = 0$ which has $(0, 0, 0)$ as its only singular point. The second chart is $z + v(w^2 - vz^2) = 0$ which is smooth. The exceptional divisor consist of E_1 given by $z = v = 0$ (so it only exists in the second chart), E_2 given by $x = w = 0$ and the new E_3 given by $y = w = 0$. E_1 and E_2 do not meet, but E_3 intersects them both, E_1 in a smooth point and E_2 in the singular point.

We present the singularity $\mathrm{Tjur}^3(E_7)$ as the matrix $\begin{pmatrix} xy & w \\ -w & x+y \end{pmatrix}$. Its Tjurina transform is then given by

$$F'_{\{1\}}(x, y, w, a_2) = \begin{pmatrix} w - a_2 xy \\ x + y + a_2 w \end{pmatrix} \text{ and } F'_{\{2\}}(x, y, w, a_1) = \begin{pmatrix} xy - a_1 w \\ -w - a_1(x + y) \end{pmatrix}.$$

In the first chart we have the hypersurface $x + y + v^2xy = 0$ which is smooth. The second chart gives the hypersurface singularity $xy + z^2(x + y) = 0$, which has a singular point at $(0, 0, 0)$. E_1 does not exist in these charts, but intersects E_3 in a smooth point in the other charts. E_2 is given by $x = z = 0$, E_3 is given by $y = z = 0$ and the new E_4 is given by $x = y = 0$. E_2, E_3 and E_4 intersect each other in the singular point.

Next we can present the singularity $\text{Tjur}^4(E_7)$ by the matrix $\begin{pmatrix} x & z(x+y) \\ -z & y \end{pmatrix}$. Its Tjurina transform is then given by

$$F'_{\{1\}}(x, y, z, a_2) = \begin{pmatrix} z(x+y) - a_2x \\ y + a_2z \end{pmatrix} \text{ and } F'_{\{2\}}(x, y, z, a_1) = \begin{pmatrix} x - a_1z(x+y) \\ -z - a_1y \end{pmatrix}.$$

The first chart gives the hypersurface $zx - wx - wz^2 = 0$ which has a singular point at $(0, 0, 0)$, and the second chart gives $x + v^2y(x + y) = 0$ which is smooth. The exceptional divisor consists of E_2 given by $z = v = 0$ so not in the chart that contains the singularity, E_3 given by $z = w = 0$, E_4 given by $x = w = 0$ and E_5 given by $x = z = 0$. E_2 intersects E_5 in a smooth point, E_3, E_4 and E_5 intersect each other in the singular point, and E_3 intersects E_1 in a smooth point outside these charts.

We can present $\text{Tjur}^5(E_7)$ by the matrix $\begin{pmatrix} z & x \\ w & x-wz \end{pmatrix}$. In this case its Tjurina transform is given by

$$F'_{\{1\}}(x, z, w, a_2) = \begin{pmatrix} x - a_2z \\ x - wz - a_2w \end{pmatrix} \text{ and } F'_{\{2\}}(x, z, w, a_1) = \begin{pmatrix} z - a_1x \\ w - a_1(x - wz) \end{pmatrix}.$$

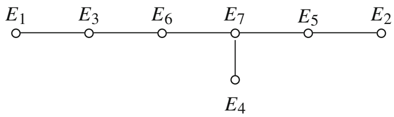
The first chart gives the hypersurface $yz - wz + yw = 0$ which has a singularity at $(0, 0, 0)$, and the second chart gives the smooth hypersurface $w - vx - v^2wx = 0$. The exceptional divisor consists of E_1 and E_2 that do not appear in any of these charts, E_3 given by $z = v = 0$ (so only appearing in the second chart), E_4 given by $w = y = 0$, E_5 given by $z = y = 0$ and E_6 given by $w = z = 0$. E_3 intersects E_1 and E_6 in different smooth points, E_2 intersects E_5 in a smooth point, E_4, E_5 and E_6 intersect each other in the singular point.

For $\text{Tjur}^6(E_7)$ we use the matrix $\begin{pmatrix} y & w \\ z & z+w \end{pmatrix}$. We get that its Tjurina transform is given by

$$F'_{\{1\}}(y, z, w, a_2) = \begin{pmatrix} w - a_2y \\ z + w - a_2z \end{pmatrix} \text{ and } F'_{\{2\}}(y, z, w, a_1) = \begin{pmatrix} y - a_1w \\ z - a_1(z + w) \end{pmatrix}.$$

The first chart gives the smooth hypersurface $z + xy - xz = 0$, and the second chart gives $z - vz - y = 0$ which is also smooth. So we have reached a resolution of E_7. The exceptional divisor consist of $E_1, \dots, E_7$, where only $E_4 \dots, E_7$ appear in the last two charts. E_4 is given by $y = x - 1 = 0$, E_5 is given by $z = v = 0$, E_6 is given by $z = x = 0$ and E_7 is given by $z = y = 0$. E_7 intersects E_4, E_5 and E_6 in three different smooth points, E_2 intersects E_5 in a smooth point, and E_3 intersects

E_1 and E_6 in two different smooth points. If we represent the exceptional divisor by a dual resolution graph (where vertices represent the curves and edges represent the intersection points) we get:



which is indeed the E_7 graph.

One can also use this method to produce resolutions of the D_n and E_6 singularities, and probably many more. But it is not always possible to use this method. For example the E_8 given by $x^2 + y^3 + z^5 = 0$ cannot be written as the determinant of a 2×2 matrix which is 0 at the origin of $\mathbb{C}^3$, nor can it be written as the determinant of a larger matrix such the value at the origin is 0. If the value at the origin is not zero, then the Tjurina transform does not improve the singularity, it only changes variables.

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