

PDE Methods in Random Matrix Theory



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Abstract This article begins with a brief review of random matrix theory, followed by a discussion of how the large- N limit of random matrix models can be realized using operator algebras. I then explain the notion of “Brown measure,” which play the role of the eigenvalue distribution for operators in an operator algebra.

I then show how methods of partial differential equations can be used to compute Brown measures. I consider in detail the case of the circular law and then discuss more briefly the case of the free multiplicative Brownian motion, which was worked out recently by the author with Driver and Kemp.

1 Random Matrices

Random matrix theory consists of choosing an $N \times N$ matrix at random and looking at natural properties of that matrix, notably its eigenvalues. Typically, interesting results are obtained only for *large* random matrices, that is, in the limit as N tends to infinity. The subject began with the work of Wigner [43], who was studying energy levels in large atomic nuclei. The subject took on new life with the discovery that the eigenvalues of certain types of large random matrices resemble the energy levels of quantum chaotic systems—that is, quantum mechanical systems for which the underlying classical system is chaotic. (See, e.g., [20] or [39].) There is also a fascinating conjectural agreement, due to Montgomery [35], between the statistical behavior of zeros of the Riemann zeta function and the eigenvalues of random matrices. See also [30] or [6].

We will review briefly some standard results in the subject, which may be found in textbooks such as those by Tao [40] or Mehta [33].

Supported in part by a Simons Foundation Collaboration Grant for Mathematicians

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M. Th. Rassias (ed.), *Harmonic Analysis and Applications*, Springer Optimization and Its Applications 168, https://doi.org/10.1007/978-3-030-61887-2_5

1.1 The Gaussian Unitary Ensemble

The first example of a random matrix is the **Gaussian unitary ensemble** (GUE) introduced by Wigner [43]. Let H_N denote the real vector space of $N \times N$ Hermitian matrices, that is, those with $X^* = X$, where X^* is the conjugate transpose of X . We then consider a Gaussian measure on H_N given by

$$d_N e^{-N \operatorname{trace}(X^2)/2} dX, \quad X \in H_N, \quad (1)$$

where dX denotes the Lebesgue measure on H_N and where d_N is a normalizing constant. If X^N is a random matrix having this measure as its distribution, then the diagonal entries are normally distributed real random variables with mean zero and variance $1/N$. The off-diagonal entries are normally distributed complex random variables, again with mean zero and variance $1/N$. Finally, the entries are as independent as possible given that they are constrained to be Hermitian, meaning that the entries on and above the diagonal are independent (and then the entries below the diagonal are determined by those above the diagonal). The factor of N in the exponent in (1) is responsible for making the variance of the entries of order $1/N$. This scaling of the variances, in turn, guarantees that the eigenvalues of the random matrix X^N do not blow up as N tends to infinity.

In order to state the first main result of random matrix theory, we introduce the following notation.

Definition 1 For any $N \times N$ matrix X , the **empirical eigenvalue distribution** of X is the probability measure on \mathbb{C} given by

$$\frac{1}{N} \sum_{j=1}^N \lambda_j,$$

where $\{\lambda_1, \dots, \lambda_N\}$ are the eigenvalues of X , listed with their algebraic multiplicity.

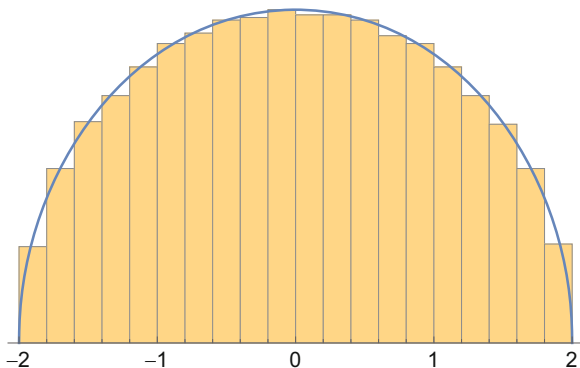
We now state **Wigner's semicircle law**.

Theorem 2 Let X^N be a sequence of independently chosen $N \times N$ random matrices, each chosen according to the probability distribution in (1). Then as $N \rightarrow \infty$, the empirical eigenvalue distribution of X^N converges almost surely in the weak topology to Wigner's semicircle law, namely, the measure supported on $[-2, 2]$ and given there by

$$\frac{1}{2\pi} \sqrt{4 - x^2} dx, \quad -2 \leq x \leq 2. \quad (2)$$

Figure 1 shows a simulation of the Gaussian unitary ensemble for $N = 2,000$, plotted against the semicircular density in (2). One notable aspect of Theorem 2 is that the limiting eigenvalue distribution (i.e., the semicircular measure in (2))

Fig. 1 A histogram of the eigenvalues of a GUE random variable with $N = 2,000$, plotted against a semicircular density



is *nonrandom*. That is to say, we are choosing a matrix at random, so that its eigenvalues are random, but in the large- N limit, the randomness in the bulk eigenvalue distribution disappears—it is always semicircular. Thus, if we were to select another GUE matrix with $N = 2,000$ and plot its eigenvalues, the histogram would (with high probability) look very much like the one in Figure 1.

It is important to note, however, that if one zooms in with a magnifying glass so that one can see the individual eigenvalues of a large GUE matrix, the randomness in the eigenvalues will persist. The behavior of these individual eigenvalues is of considerable interest, because they are supposed to resemble the energy levels of a “quantum chaotic system” (i.e., a quantum mechanical system whose classical counterpart is chaotic). Nevertheless, in this article, I will deal only with the bulk properties of the eigenvalues.

1.2 The Ginibre Ensemble

We now discuss the non-Hermitian counterpart to the Gaussian unitary ensemble, known as the **Ginibre ensemble** [15]. We let $M_N(\mathbb{C})$ denote the space of all $N \times N$ matrices, not necessarily Hermitian. We then make a measure on $M_N(\mathbb{C})$ using a formula similar to the Hermitian case:

$$f_N e^{-N\text{trace}(Z^*Z)} dZ, \quad Z \in M_N(\mathbb{C}), \tag{3}$$

where dZ denotes the Lebesgue measure on H_N and where f_N is a normalizing constant. In this case, the eigenvalues need not be real, and they follow the **circular law**.

Theorem 3 *Let Z^N be a sequence of independently chosen $N \times N$ random matrices, each chosen according to the probability distribution in (3). Then as $N \rightarrow \infty$, the empirical eigenvalue distribution of Z^N converges almost surely in the weak topology to the uniform measure on the unit disk.*

Fig. 2 A plot of the eigenvalues of a Ginibre matrix with $N = 2,000$

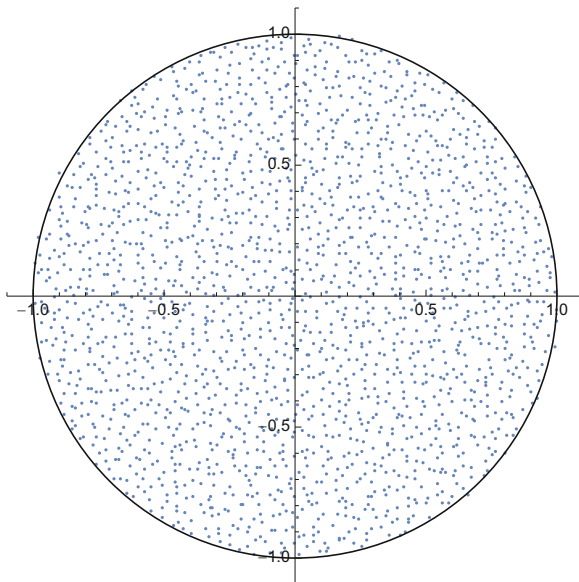


Figure 2 shows the eigenvalues of a random matrix chosen from the Ginibre ensemble with $N = 2,000$. As in the GUE case, the bulk eigenvalue distribution becomes deterministic in the large- N limit. As in the GUE case, one can also zoom in with a magnifying glass on the eigenvalues of a Ginibre matrix until the individual eigenvalues become visible, and the local behavior of these eigenvalues is an interesting problem—which will not be discussed in this article.

1.3 The Ginibre Brownian Motion

In this article, I will discuss a certain approach to analyzing the behavior of the eigenvalues in the Ginibre ensemble. The main purpose of this analysis is not so much to obtain the circular law, which can be proved by various other methods. The main purpose is rather to develop tools that can be used to study a more complex random matrix model in the group of *invertible* $N \times N$ matrices. The Ginibre case then represents a useful prototype for this more complicated problem.

It is then useful to introduce a time parameter into the description of the Ginibre ensemble, which we can do by studying the **Ginibre Brownian motion**. Specifically, in any finite-dimensional real inner product space V , there is a natural notion of Brownian motion. The Ginibre Brownian motion is obtained by taking V to be $M_N(\mathbb{C})$, viewed as a real vector space of dimension $2N^2$, and using the (real) inner product $\langle \cdot, \cdot \rangle_N$ given by

$$\langle X, Y \rangle_N := N \operatorname{Re}(\operatorname{trace}(X^*Y)).$$

We let C_t^N denote this Brownian motion, assumed to start at the origin.

At any one fixed time, the distribution of C_t^N is just the same as $\sqrt{t}Z^N$, where Z^N is distributed as the Ginibre ensemble. The *joint* distribution of the process C_t^N for various values of t is determined by the following property: For any collection of times $0 = t_0 < t_1 < t_2 < \dots < t_k$, the “increments”

$$C_{t_1}^N - C_{t_0}^N, C_{t_2}^N - C_{t_1}^N, \dots, C_{t_k}^N - C_{t_{k-1}}^N \tag{4}$$

are independent and distributed as $\sqrt{t_j - t_{j-1}}Z^N$.

2 Large- N Limits in Random Matrix Theory

Results in random matrix theory are typically expressed by first computing some quantity (e.g., the empirical eigenvalue distribution) associated to an $N \times N$ random matrix and then letting N tend to infinity. It is nevertheless interesting to ask whether there is some sort of limiting object that captures the large- N limit of the entire random matrix model. In this section, we discuss one common approach constructing such a limiting object.

2.1 Limit in *-Distribution

Suppose we have a matrix-valued random variable X , not necessarily normal. Then we can then speak about the *-moments of X , which are expressions like

$$\mathbb{E} \left\{ \frac{1}{N} \text{trace}(X^2(X^*)^3 X^4 X^*) \right\}.$$

Generally, suppose $p(a, b)$ is a polynomial in two noncommuting variables, that is, a linear combination of words involving products of a ’s and b ’s in all possible orders. We may then consider

$$\mathbb{E} \left\{ \frac{1}{N} \text{trace}[p(X, X^*)] \right\}.$$

If, as usual, we have a *family* X^N of $N \times N$ random matrices, we may consider the limits of such *-moments (if the limits exist):

$$\lim_{N \rightarrow \infty} \mathbb{E} \left\{ \frac{1}{N} \text{trace}[p(X^N, (X^N)^*)] \right\}. \tag{5}$$

2.2 Tracial von Neumann Algebras

Our goal is now to find some sort of limiting object that can encode *all* of the limits in (5). Specifically, we will try to find the following objects: (1) an operator algebra \mathcal{A} , (2) a “trace” $\tau : \mathcal{A} \rightarrow \mathbb{C}$, and (3) an element x of \mathcal{A} , such that for each polynomial p in two noncommuting variables, we have

$$\lim_{N \rightarrow \infty} \mathbb{E} \left\{ \frac{1}{N} \text{trace}[p(X^N, (X^N)^*)] \right\} = \tau[p(x, x^*)]. \quad (6)$$

We now explain in more detail what these objects should be. First, we generally take \mathcal{A} to be a von Neumann algebra, that is, an algebra of operators that contains the identity, is closed under taking adjoints, and is closed under taking weak operator limits. Second, the “trace” τ is not actually computed by taking the trace of elements of \mathcal{A} , which are typically not of trace class. Rather, τ is a linear functional that has properties similar to the properties of the *normalized* trace $\frac{1}{N} \text{trace}(\cdot)$ for matrices. Specifically, we require the following properties:

- $\tau(1) = 1$, where on the left-hand side, 1 denotes the identity operator,
- $\tau(a^*a) \geq 0$ with equality only if $a = 0$, and
- $\tau(ab) = \tau(ba)$, and
- τ should be continuous with respect to the weak-* topology on \mathcal{A} .

Last, x is a single element of \mathcal{A} .

We will refer to the pair (\mathcal{A}, τ) as a **tracial von Neumann algebra**. We will not discuss here the methods used for actually constructing interesting examples of tracial von Neumann algebras. Instead, we will simply accept as a known result that certain random matrix models admit large- N limits as operators in a tracial von Neumann algebra. (The interested reader may consult the work of Biane and Speicher [5], who use a Fock space construction to find tracial von Neumann algebras of the sort we will be using in this article.)

Let me emphasize that although X^N is a matrix-valued random variable, x is *not* an operator-valued random variable. Rather, x is a *single* operator in the operator algebra \mathcal{A} . This situation reflects a typical property of random matrix models, which we have already seen an example of in Sections 1.1 and 1.2, that certain random quantities become nonrandom in the large- N limit. In the present context, it is often the case that we have a stronger statement than (6), as follows: If we sample the X^N 's independently for different N 's, then with probability one, we will have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{trace}[p(X^N, (X^N)^*)] = \tau[p(x, x^*)].$$

That is to say, in many cases, the *random* quantity $\frac{1}{N} \text{trace}[p(X^N, (X^N)^*)]$ converges almost surely to the *single, deterministic* number $\tau[p(x, x^*)]$ as N tends to infinity.

2.3 Free Independence

In random matrix theory, it is often convenient to construct random matrices as sums or products of other random matrices, which are frequently assumed to be independent of one another. The appropriate notion of independence in the large- N limit—that is, in a tracial von Neumann algebra—is the notion of “freeness” or “free independence.” This concept was introduced by Voiculescu [41, 42] and has become a powerful tool in random matrix theory. (See also the monographs [36] by Nica and Speicher and [34] by Mingo and Speicher.) Given an element a in a tracial von Neumann algebra (\mathcal{A}, τ) and a polynomial p , we may form the element $p(a)$. We also let $\dot{p}(a)$ denote the corresponding “centered” element, given by

$$\dot{p}(a) = p(a) - \tau(p(a))$$

We then say that elements a_1, \dots, a_k are **freely independent** (or, more concisely, **free**) if the following condition holds. Let j_1, \dots, j_n be any sequence of indices taken from $\{1, \dots, k\}$, with the property that j_l is distinct from j_{l+1} . Let p_{j_1}, \dots, p_{j_n} be any sequence p_{j_1}, \dots, p_{j_n} of polynomials. Then we should have

$$\tau(\dot{p}_{j_1}(a_{j_1})\dot{p}_{j_2}(a_{j_2}) \cdots \dot{p}_{j_n}(a_{j_n})) = 0.$$

Thus, for example, if a and b are freely independent, then

$$\tau[(a^2 - \tau(a^2))(b^2 - \tau(b^2))(a - \tau(a))] = 0.$$

The concept of freeness allows us, in principle, to disentangle traces of arbitrary words in freely independent elements, thereby reducing the computation to the traces of powers of individual elements. As an example, let us do a few computations with two freely independent elements a and b . We form the corresponding centered elements $a - \tau(a)$ and $b - \tau(b)$ and start applying the definition:

$$\begin{aligned} 0 &= \tau[(a - \tau(a))(b - \tau(b))] \\ &= \tau[ab] - \tau[\tau(a)b] - \tau[a\tau(b)] + \tau[\tau(a)\tau(b)] \\ &= \tau[ab] - \tau(a)\tau(b) - \tau(a)\tau(b) + \tau(a)\tau(b) \\ &= \tau[ab] - \tau(a)\tau(b), \end{aligned}$$

where we have used that scalars can be pulled outside the trace and that $\tau(1) = 1$. We conclude, then, that

$$\tau(ab) = \tau(a)\tau(b).$$

A similar computation shows that $\tau(a^2b) = \tau(a^2)\tau(b)$ and that $\tau(ab^2) = \tau(a)\tau(b^2)$.

The first really interesting case comes when we compute $\tau(abab)$. We start with

$$0 = \tau[(a - \tau(a))(b - \tau(b))(a - \tau(a))(b - \tau(b))]$$

and expand out the right-hand side as $\tau(abab)$ plus a sum of fifteen terms, all of which reduce to previously computed quantities. Sparing the reader the details of this computation, we find that

$$\tau(abab) = \tau(a^2)\tau(b)^2 + \tau(a)^2\tau(b^2) - \tau(a)^2\tau(b)^2.$$

Although the notion of free independence will not explicitly be used in the rest of this article, it is certainly a key concept that is always lurking in the background.

2.4 The Circular Brownian Motion

If Z^N is a Ginibre random matrix (Section 1.2), then the $*$ -moments of Z^N converge to those of a “circular element” c in a certain tracial von Neumann algebra (\mathcal{A}, τ) . The $*$ -moments of c can be computed in an efficient combinatorial way (e.g., Example 11.23 in [36]). We have, for example, $\tau(c^*c) = 1$ and $\tau(c^k) = 0$ for all positive integers k .

More generally, we can realize the large- N limit of the entire Ginibre Brownian motion C_t^N , for all $t > 0$, as a family of elements c_t in a tracial von Neumann algebra (\mathcal{A}, τ) . In the limit, the ordinary independence conditions for the increments of C_t^N (Section 1.3) is replaced by the free independence of the increments of c_t . That is, for all $0 = t_0 < t_1 < \dots < t_k$, the elements

$$c_{t_1} - c_{t_0}, c_{t_2} - c_{t_1}, \dots, c_{t_k} - c_{t_{k-1}}$$

are freely independent, in the sense described in the previous subsection. For any $t > 0$, the $*$ -distribution of c_t is the same as the $*$ -distribution of $\sqrt{t}c_1$.

3 Brown Measure

3.1 The Goal

Recall that if A is an $N \times N$ matrix with eigenvalues $\lambda_1, \dots, \lambda_N$, the empirical eigenvalue distribution μ_A of A is the probability measure on \mathbb{C} assigning mass $1/N$ to each eigenvalue:

$$\mu_A = \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j}.$$

Goal 4 Given an arbitrary element x in a tracial von Neumann algebra (\mathcal{A}, τ) , construct a probability measure μ_x on \mathbb{C} analogous to the empirical eigenvalue distribution of a matrix.

If $x \in \mathcal{A}$ is normal, then there is a standard way to construct such a measure. The spectral theorem allows us to construct a projection-valued measure γ_x [23, Section 10.3] associated to x . For each Borel set E , the projection $\gamma_x(E)$ will, again, belong to the von Neumann algebra \mathcal{A} , and we may therefore define

$$\mu_x(E) = \tau[\gamma_x(E)]. \quad (7)$$

We refer to μ_x as the **distribution** of x (relative to the trace τ). If x is not normal, we need a different construction—but one that we hope will agree with the above construction in the normal case.

3.2 A Motivating Computation

If A is an $N \times N$ matrix, define a function $s : \mathbb{C} \rightarrow \mathbb{R} \cup \{-\infty\}$ by

$$s(\lambda) = \log(|\det(A - \lambda)|^{2/N}),$$

where the logarithm takes the value $-\infty$ when $\det(A - \lambda) = 0$. Note that s is computed from the characteristic polynomial $\det(A - \lambda)$ of A . We can compute s in terms of its eigenvalues $\lambda_1, \dots, \lambda_N$ (taken with their algebraic multiplicity) as

$$s(\lambda) = \frac{2}{N} \sum_{j=1}^N \log |\lambda - \lambda_j|. \quad (8)$$

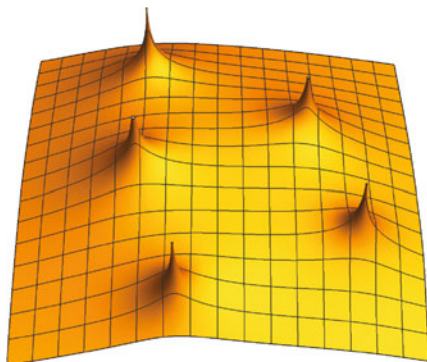
See Figure 3 for a plot of (the negative of) $s(\lambda)$.

We then recall that the function $\log |\lambda|$ is a multiple of the Green's function for the Laplacian on the plane, meaning that the function is harmonic away from the origin and that

$$\Delta \log |\lambda| = 2\pi \delta_0(\lambda),$$

where δ_0 is a δ -measure at the origin. Thus, if we take the Laplacian of $s(\lambda)$, with an appropriate normalizing factor, we get the following nice result.

Fig. 3 A plot of the function $-s(\lambda)$ for a matrix with five eigenvalues. The function is harmonic except at the singularities



Proposition 5 *The Laplacian, in the distribution sense, of the function $s(\lambda)$ in (8) satisfies*

$$\frac{1}{4\pi} \Delta s(\lambda) = \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j}(\lambda),$$

where δ_{λ_j} is a δ -measure at λ_j . That is to say, $\frac{1}{4\pi} \Delta s$ is the empirical eigenvalue distribution of A (Definition 1).

Recall that if B is a strictly positive self-adjoint matrix, then we can take the logarithm of B , which is the self-adjoint matrix obtained by keeping the eigenvectors of B fixed and taking the logarithm of the eigenvalues.

Proposition 6 *The function s in (8) can also be computed as*

$$s(\lambda) = \frac{1}{N} \text{trace}[\log((A - \lambda)^*(A - \lambda))] \tag{9}$$

or as

$$s(\lambda) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{N} \text{trace}[\log((A - \lambda)^*(A - \lambda) + \varepsilon)]. \tag{10}$$

Here the logarithm is the self-adjoint logarithm of a positive self-adjoint matrix.

Note that in (9), the logarithm is undefined when λ is an eigenvalue of A . In (10), inserting $\varepsilon > 0$ guarantees that the logarithm is well defined for all λ , but a singularity of $s(\lambda)$ at each eigenvalue still arises in the limit as ε approaches zero.

Proof An elementary result [24, Theorem 2.12] says that for any matrix X , we have $\det(e^X) = e^{\text{trace}(X)}$. If P is a strictly positive matrix, we may apply this result with $X = \log P$ (so that $e^X = P$) to get

$$\det(P) = e^{\text{trace}(X)}$$

or

$$\text{trace}(\log P) = \log[\det P].$$

Let us now apply this identity with $P = (A - \lambda)^*(A - \lambda)$, whenever λ is not an eigenvalue of A , to obtain

$$\begin{aligned} \frac{1}{N} \text{trace}[\log((A - \lambda)^*(A - \lambda))] &= \frac{1}{N} \log[\det((A - \lambda)^*(A - \lambda))] \\ &= \frac{1}{N} \log[\det(A - \lambda)^* \det(A - \lambda)] \\ &= \log(|\det(A - \lambda)|^{2/N}), \end{aligned}$$

where this last expression is the definition of $s(\lambda)$.

Continuity of the matrix logarithm then establishes (10). □

3.3 Definition and Basic Properties

To define the Brown measure of a general element x in a tracial von Neumann algebra (\mathcal{A}, τ) , we use the obvious generalization of (10). We refer to Brown's original paper [7] along with Chapter 11 of [34] for general references on the material in this section.

Theorem 7 *Let (\mathcal{A}, τ) be a tracial von Neumann algebra and let x be an arbitrary element of \mathcal{A} . Define*

$$S(\lambda, \varepsilon) = \tau[\log((x - \lambda)^*(x - \lambda) + \varepsilon)] \tag{11}$$

for all $\lambda \in \mathbb{C}$ and $\varepsilon > 0$. Then

$$s(\lambda) := \lim_{\varepsilon \rightarrow 0^+} S(\lambda, \varepsilon) \tag{12}$$

exists as an almost-everywhere-defined subharmonic function. Furthermore, the quantity

$$\frac{1}{4\pi} \Delta s, \tag{13}$$

where the Laplacian is computed in the distribution sense, is represented by a probability measure on the plane. We call this measure the Brown measure of x and denote it by μ_x .

The Brown measure of x is supported on the spectrum $\sigma(x)$ of x and has the property that

$$\int_{\sigma(x)} \lambda^k d\mu_x(\lambda) = \tau(x^k) \quad (14)$$

for all non-negative integers k .

See the original article [7] or Chapter 11 of the monograph [34] of Mingo and Speicher. We also note that the quantity $s(\lambda)$ is the logarithm of the **Fuglede–Kadison determinant** of $x - \lambda$; see [13, 14]. It is important to emphasize that, in general, the moment condition (14) *does not uniquely determine the measure* μ_x . After all, $\sigma(x)$ is an arbitrary nonempty compact subset of \mathbb{C} , which could, for example, be a closed disk. To uniquely determine the measure, we would need to know the value of $\int_{\sigma(x)} \lambda^k \bar{\lambda}^l d\mu_x(\lambda)$ for all non-negative integers k and l . There is not, however, any simple way to compute the value of $\int_{\sigma(x)} \lambda^k \bar{\lambda}^l d\mu_x(\lambda)$ in terms of the operator x . In particular, unless x is normal, this integral need not be equal to $\tau[x^k(x^*)^l]$. Thus, to compute the Brown measure of a general operator $x \in \mathcal{A}$, we actually have to work with the rather complicated definition in (11), (12), and (13).

We note two important special cases.

- Suppose \mathcal{A} is the space of all $N \times N$ matrices and τ is the normalized trace, $\tau[x] = \frac{1}{N} \text{trace}(x)$. Then the Brown measure of any $x \in \mathcal{A}$ is simply the empirical eigenvalue distribution of x , which puts mass $1/N$ at each eigenvalue of x .
- If x is normal, then the Brown measure μ_x of x agrees with the measure defined in (7) using the spectral theorem.

3.4 Brown Measure in Random Matrix Theory

Suppose one has a family of $N \times N$ random matrix models X^N and one wishes to determine the large- N limit of the empirical eigenvalue distribution of X^N . (Recall Definition 1.) One may naturally use the following three-step process.

Step 1. Construct a large- N limit of X^N as an operator x in a tracial von Neumann algebra (\mathcal{A}, τ) .

Step 2. Determine the Brown measure μ_x of x .

Step 3. Prove that the empirical eigenvalue distribution of X^N converges almost surely to μ_x as N tends to infinity.

It is important to emphasize that Step 3 in this process is not automatic. Indeed, this can be a difficult technical problem. Nevertheless, this article is concerned with exclusively with Step 2 in the process (in situations where Step 1 has been carried out). For Step 3, the main tool is the Hermitization method developed in Girko's pioneering paper [16] and further refined by Bai [1]. (Although neither of these authors explicitly uses the terminology of Brown measure, the idea is lurking there.)

There exist certain pathological examples where the limiting eigenvalue distribution does not coincide with the Brown measure. In light of a result of Śniady [38], we can say that such examples are associated with spectral instability, that is, matrices where a small change in the matrix produces a large change in the eigenvalues. Śniady shows that if we add to X^N a small amount of random Gaussian noise, then eigenvalues distribution of the perturbed matrices will converge to the Brown measure of the limiting object. (See also the papers [19] and [12], which obtain similar results by very different methods.) Thus, if the original random matrices X^N are somehow “stable,” adding this noise should not change the eigenvalues of X^N by much, and the eigenvalues of the original and perturbed matrices should be almost the same. In such a case, we should get convergence of the eigenvalues of X^N to the Brown measure of the limiting object.

The canonical example in which instability occurs is the case in which $X^N = \text{nil}_N$, the deterministic $N \times N$ matrix having 1s just above the diagonal and 0s elsewhere. Then of course nil_N is nilpotent, so all of its eigenvalues are zero. We note however that both $\text{nil}_N^* \text{nil}_N$ and $\text{nil}_N \text{nil}_N^*$ are diagonal matrices whose diagonal entries have $N - 1$ values of 1 and only a single value of 0. Thus, when N is large, nil_N is “almost unitary,” in the sense that $\text{nil}_N^* \text{nil}_N$ and $\text{nil}_N \text{nil}_N^*$ are close to the identity. Furthermore, for any positive integer k , we have that nil_N^k is again nilpotent, so that $\text{trace}[\text{nil}_N^k] = 0$. Using these observations, it is not hard to show that the limiting object is a “Haar unitary,” that is, a unitary element u of a tracial von Neumann algebra satisfying $\tau(u^k) = 0$ for all positive integers k . The Brown measure of a Haar unitary is the uniform probability measure on the unit circle, while of course the eigenvalue distribution X^N is entirely concentrated at the origin.

In Figure 4 we see that even under a quite small perturbation (adding 10^{-6} times a Ginibre matrix), the spectrum of the nilpotent matrix X^N changes quite a lot. After the perturbation, the spectrum clearly resembles a uniform distribution over the unit circle. In Figure 5, by contrast, we see that even under a much larger perturbation (adding 10^{-1} times a Ginibre matrix), the spectrum of a GUE matrix changes only slightly. (Note the vertical scale in Figure 5.)

3.5 The Case of the Circular Brownian Motion

We now record the Brown measure of the circular Brownian motion.

Proposition 8 *For any $t > 0$, the Brown measure of c_t is the uniform probability measure on the disk of radius \sqrt{t} centered at the origin.*

Now, as we noted in Section 2.4, the $*$ -distribution of the circular Brownian motion at any time $t > 0$ is the same as the $*$ -distribution of $\sqrt{t}c_1$. Thus, the proposition will follow if we know that the Brown measure of a circular element c is the uniform probability measure on the unit disk. This result, in turn, is well known; see, for example, Section 11.6.3 of [34].

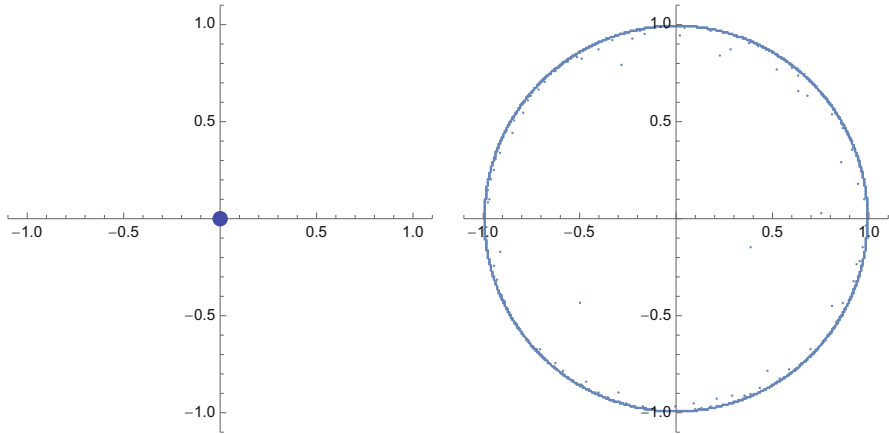


Fig. 4 Spectra of the nilpotent matrix nil_N (left) and of $\text{nil}_N + \varepsilon(\text{Ginibre})$ with $\varepsilon = 10^{-5}$ (right), with $N = 2,000$

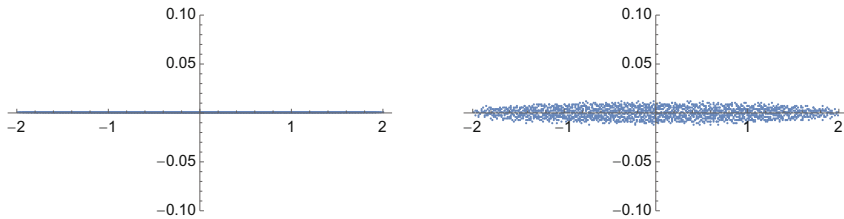


Fig. 5 Spectrum of a GUE matrix X (left) and $X + \varepsilon(\text{Ginibre})$ with $\varepsilon = 10^{-1}$ (right), with $N = 2,000$

4 PDE for the Circular Law

In this article, I present a different proof of Proposition 8 using the PDE method developed in [10]. The significance of this method is not so much that it gives another computation of the Brown measure of a circular element. Rather, it is a helpful warm-up case on the path to tackling the much more complicated problem in [10], namely, the computation of the Brown measure of the free multiplicative Brownian motion. In this section and the two that follow, I will show how the PDE method applies in the case of the circular Brownian motion. Then in the last section, I will describe the case of the free multiplicative Brownian motion.

The reader may also consult the recent preprint [29], which extends the results of [10] to case of the free multiplicative Brownian motion with arbitrary unitary initial distribution. Section 3 of this paper also analyzes the case of the free circular Brownian motion (with an arbitrary Hermitian initial distribution) using PDE methods.

We let c_t be the circular Brownian motion (Section 2.4). Then, following the construction of the Brown measure in Theorem 7, we define, for each $\lambda \in \mathbb{C}$, a function S^λ given by

$$S^\lambda(t, \varepsilon) = \tau[\log((c_t - \lambda)^*(c_t - \lambda) + \varepsilon)] \quad (15)$$

for all $t > 0$ and $\varepsilon > 0$. The Brown measure of c_t will then be obtained by letting ε tend to zero, taking the Laplacian with respect to λ , and dividing by 4π . Our first main result is that, for each λ , $S^\lambda(t, \varepsilon)$ satisfies a PDE in t and ε .

Theorem 9 *For each $\lambda \in \mathbb{C}$, the function S^λ satisfies the first-order, nonlinear differential equation*

$$\frac{\partial S^\lambda}{\partial t} = \varepsilon \left(\frac{\partial S^\lambda}{\partial \varepsilon} \right)^2 \quad (16)$$

subject to the initial condition

$$S^\lambda(0, \varepsilon) = \log(|\lambda|^2 + \varepsilon).$$

We now see the motivation for making λ a parameter rather than a variable for S : since λ does not appear in the PDE (16), we can think of solving the same equation for each different value of λ , with the dependence on λ entering only through the initial conditions.

On the other hand, we see that the regularization parameter ε plays a crucial role here as one of the variables in our PDE. Of course, we are ultimately interested in letting ε tend to zero, but since derivatives with respect to ε appear, we cannot merely set $\varepsilon = 0$ in the PDE.

Of course, the reader will point out that, formally, setting $\varepsilon = 0$ in (16) gives $\partial S^\lambda(t, 0)/\partial t = 0$, because of the leading factor of ε on the right-hand side. This conclusion, however, is not actually correct, because $\partial S^\lambda/\partial \varepsilon$ can blow up as ε approaches zero. Actually, it will turn out that $S^\lambda(t, 0)$ is independent of t when $|\lambda| > \sqrt{t}$, but not in general.

4.1 The Finite- N Equation

In this subsection, we give a heuristic argument for the PDE in Theorem 9. Although the argument is not rigorous as written, it should help explain what is going on. In particular, the computations that follow should make it clear why the PDE is only valid after taking the large- N limit.

4.1.1 The Result

We introduce a finite- N analog of the function S^λ in Theorem 9 and compute its time derivative. Let C_t^N denote the Ginibre Brownian motion introduced in Section 1.3.

Proposition 10 *For each N , let*

$$S^{\lambda,N}(t, \varepsilon) = \mathbb{E}\{\text{tr}[\log((C_t^N - \lambda)^*(C_t^N - \lambda) + \varepsilon)]\}.$$

Then we have the following results.

(1) *The time derivative of $S^{\lambda,N}$ may be computed as*

$$\frac{\partial S^{\lambda,N}}{\partial t} = \varepsilon \mathbb{E}\{\text{tr}[\text{tr}[\text{tr}[(C_t^N - \lambda)^*(C_t^N - \lambda) + \varepsilon]^{-1}]^2]\}. \quad (17)$$

(2) *We also have*

$$\frac{\partial}{\partial \varepsilon} \text{tr}[\log((C_t^N - \lambda)^*(C_t^N - \lambda) + \varepsilon)] = \text{tr}[\text{tr}[(C_t^N - \lambda)^*(C_t^N - \lambda) + \varepsilon]^{-1}]. \quad (18)$$

(3) *Therefore, if we set*

$$T^{\lambda,N} = \text{tr}[\text{tr}[(C_t^N - \lambda)^*(C_t^N - \lambda) + \varepsilon]^{-1}],$$

we may rewrite the formula for $\partial S^{\lambda,N}/\partial t$ as

$$\frac{\partial S^{\lambda,N}}{\partial t} = \varepsilon \left(\frac{\partial S^{\lambda,N}}{\partial \varepsilon} \right)^2 + \text{Cov}, \quad (19)$$

where Cov is a “covariance term” given by

$$\text{Cov} = \mathbb{E}\{(T^{\lambda,N})^2\} - (\mathbb{E}\{T^{\lambda,N}\})^2.$$

The key point to observe here is that in the formula (17) for $\partial S^{\lambda,N}/\partial t$, we have the *expectation value of the square* of a trace. On the other hand, if we computed $(\partial S^{\lambda,N}/\partial \varepsilon)^2$ by taking the expectation value of both sides of (18) and squaring, we would have the *square of the expectation value* of a trace. Thus, there is no PDE for $S^{\lambda,N}$ —we get an unavoidable covariance term on the right-hand side of (19).

On the other hand, the Ginibre Brownian motion C_t^N exhibits a **concentration phenomenon** for large N . Specifically, let us consider a family $\{Y^N\}$ of random variables of the form

$$Y^N = \text{tr}[\text{word in } C_t^N \text{ and } (C_t^N)^*].$$

(Thus, e.g., we might have $Y^N = \text{tr}[C_t^N (C_t^N)^* C_t^N (C_t^N)^*]$.) Then it is known that (1) the large- N limit of $\mathbb{E}\{Y^N\}$ exists, and (2) the variance of Y^N goes to zero. That is to say, when N is large, Y^N will be, with high probability, close to its expectation value. It then follows that $\mathbb{E}\{(Y^N)^2\}$ will be close to $(\mathbb{E}\{Y^N\})^2$. (This concentration phenomenon was established by Voiculescu in [42] for the analogous case of the ‘‘GUE Brownian motion.’’ The case of the Ginibre Brownian motion is similar.)

Now, although the quantity

$$((C_t^N - \lambda)^* (C_t^N - \lambda) + \varepsilon)^{-1}$$

is not a word in C_t^N and $(C_t^N)^*$, it is expressible—at least for large ε —as a power series in such words. It is therefore reasonable to expect—this is not a proof!—that the variance of X^N will go to zero as N goes to infinity and the covariance term in (19) will vanish in the limit.

4.1.2 Setting Up the Computation

We view $M_N(\mathbb{C})$ as a real vector space of dimension $2N^2$ and we use the following real-valued inner product $\langle \cdot, \cdot \rangle_N$:

$$\langle X, Y \rangle_N = N \text{Re}(\text{trace}(X^* Y)). \tag{20}$$

The distribution of C_t^N is the Gaussian measure of variance $t/2$ with respect to this inner product

$$d\gamma_t(C) = d_t e^{-\langle C, C \rangle / t} dC,$$

where d_t is a normalization constant and dC is the Lebesgue measure on $M_N(\mathbb{C})$. This measure is a **heat kernel measure**. If we let \mathbb{E}_t denote the expectation value with respect to γ_t , then we have, for any ‘‘nice’’ function,

$$\frac{d}{dt} \mathbb{E}_t\{f\} = \frac{1}{4} \mathbb{E}_t\{\Delta f\}, \tag{21}$$

where Δ is the Laplacian on $M_N(\mathbb{C})$ with respect to the inner product (20).

To compute more explicitly, we choose an orthonormal basis for $M_N(\mathbb{C})$ over \mathbb{R} consisting of X_1, \dots, X_{N^2} and Y_1, \dots, Y_{N^2} , where X_1, \dots, X_{N^2} are skew-Hermitian and where $Y_j = iX_j$. We then introduce the directional derivatives \tilde{X}_j and \tilde{Y}_j defined by

$$(\tilde{X}_j f)(a) = \left. \frac{d}{ds} f(a + sX_j) \right|_{s=0}; \quad (\tilde{Y}_j f)(Z) = \left. \frac{d}{ds} f(a + sY_j) \right|_{s=0}.$$

Then the Laplacian Δ is given by

$$\Delta = \sum_{j=1}^{N^2} \left((\tilde{X}_j)^2 + (\tilde{Y}_j)^2 \right).$$

We also introduce the corresponding complex derivatives, Z_j and \bar{Z}_j given by

$$\begin{aligned} Z_j &= \frac{1}{2}(\tilde{X}_j - i\tilde{Y}_j); \\ \bar{Z}_j &= \frac{1}{2}(\tilde{X}_j + i\tilde{Y}_j), \end{aligned}$$

which give

$$\frac{1}{4}\Delta = \sum_{j=1}^{N^2} \bar{Z}_j Z_j.$$

We now let C denote a matrix-valued variable ranging over $M_N(\mathbb{C})$. We may easily compute the following basic identities:

$$\begin{aligned} Z_j(C) &= X_j; & Z_j(C^*) &= 0; \\ \bar{Z}_j(C) &= 0; & \bar{Z}_j(C^*) &= -X_j. \end{aligned} \tag{22}$$

(Keep in mind that X_j is skew-Hermitian.) We will also need the following elementary but crucial identity

$$\sum_{j=1}^{N^2} X_j A X_j = -\text{tr}(A), \tag{23}$$

where $\text{tr}(\cdot)$ is the normalized trace, given by

$$\text{tr}(A) = \frac{1}{N} \text{trace}(A).$$

See, for example, Proposition 3.1 in [9]. When applied to function involving a normalized trace, this will produce second trace.

Finally, we need the following formulas for differentiating matrix-valued functions of a real variable:

$$\frac{d}{ds} A(s)^{-1} = -A(s)^{-1} \frac{dA}{ds} A(s)^{-1} \tag{24}$$

$$\frac{d}{ds} \text{tr}[\log A(s)] = \text{tr} \left[A(s)^{-1} \frac{dA}{ds} \right]. \tag{25}$$

The first of these is standard and can be proved by differentiating the identity $A(s)A(s)^{-1} = I$. The second identity is Lemma 1.1 in [7]; it is important to emphasize that this second identity does not hold as written without the trace. One may derive (25) by using an integral formula for the derivative of the logarithm *without* the trace (see, e.g., Equation (11.10) in [27]) and then using the cyclic invariance of the trace, at which point the integral can be computed explicitly.

4.1.3 Proof of Proposition 10

We continue to let \mathbb{E}_t denote the expectation value with respect to the measure γ_t , which is the distribution at time t of the Ginibre Brownian motion C_t^N , so that

$$S^{\lambda, N}(t, \varepsilon) = \mathbb{E}_t\{\text{tr}[\log((C - \lambda)^*(C - \lambda) + \varepsilon)]\},$$

where the variable C ranges over $M_N(\mathbb{C})$. We apply the derivative Z_j using (25) and (22), giving

$$Z_j S^{\lambda, N}(t, \varepsilon) = \mathbb{E}_t\{\text{tr}[\text{tr}[(C - \lambda)^*(C - \lambda) + \varepsilon]^{-1}(C - \lambda)^* X_j]\}.$$

We then apply the derivative \bar{Z}_j using (24) and (22), giving

$$\begin{aligned} \bar{Z}_j Z_j S^{\lambda, N}(t, \varepsilon) &= -\mathbb{E}_t\{\text{tr}[\text{tr}[(C - \lambda)^*(C - \lambda) + \varepsilon]^{-1} X_j^2]\} \\ &+ \mathbb{E}_t\{\text{tr}[\text{tr}[(C - \lambda)^*(C - \lambda) + \varepsilon]^{-1} X_j(C - \lambda)((C - \lambda)^*(C - \lambda) + \varepsilon)^{-1}(C - \lambda)^* X_j]\}. \end{aligned}$$

We now sum on j and apply the identity (23). After applying the heat equation (21) with $\Delta = \sum_j \bar{Z}_j Z_j$, we obtain

$$\begin{aligned} &\frac{d}{dt} S^{\lambda, N}(t, \varepsilon) \\ &= \sum_j \bar{Z}_j Z_j S^{\lambda, N}(t, \varepsilon) \\ &= \mathbb{E}_t\{\text{tr}[\text{tr}[(C - \lambda)^*(C - \lambda) + \varepsilon]^{-1}]\} - \mathbb{E}_t\{\text{tr}[\text{tr}[(C - \lambda)^*(C - \lambda) + \varepsilon]^{-1}] \times \\ &\quad \text{tr}[(C - \lambda)^*(C - \lambda)((C - \lambda)^*(C - \lambda) + \varepsilon)^{-1}]\}. \end{aligned} \tag{26}$$

But then

$$\begin{aligned} &(C - \lambda)^*(C - \lambda)((C - \lambda)^*(C - \lambda) + \varepsilon)^{-1} \\ &= ((C - \lambda)^*(C - \lambda) + \varepsilon - \varepsilon)((C - \lambda)^*(C - \lambda) + \varepsilon)^{-1} \\ &= 1 - \varepsilon((C - \lambda)^*(C - \lambda) + \varepsilon)^{-1}. \end{aligned}$$

Thus, there is a cancellation between the two terms on the right-hand side of (26), giving

$$\frac{\partial S^{\lambda, N}}{\partial t} = \varepsilon \mathbb{E}_t \{ (\text{tr} [((C - \lambda)^* (C - \lambda) + \varepsilon)^{-1}])^2 \},$$

as claimed in Point 1 of the proposition.

Meanwhile, we may use again the identity (25) to compute

$$\frac{\partial}{\partial \varepsilon} \text{tr} [\log ((C_t^N - \lambda)^* (C_t - \lambda) + \varepsilon)]$$

to verify Point 2 3 then follows by simple algebra.

4.2 A Derivation Using Free Stochastic Calculus

4.2.1 Ordinary Stochastic Calculus

In this section, I will describe briefly how the PDE in Theorem 9 can be derived rigorously, using the tools of free stochastic calculus. We begin by recalling a little bit of ordinary stochastic calculus, for the ordinary, real-valued Brownian motion. To avoid notational conflicts, we will let ε_t denote Brownian motion in the real line. This is a random continuous path satisfying the properties proposed by Einstein in 1905, namely, that for any $0 = t_0 < t_1 < \dots < t_k$, the increments

$$x_{t_1} - x_{t_0}, x_{t_2} - x_{t_1}, \dots, x_{t_k} - x_{t_{k-1}}$$

should be independent normal random variables with mean zero and variance $t_j - t_{j-1}$. At a rigorous level, Brownian motion is described by the Wiener measure on the space of continuous paths.

It is a famous result that, with probability one, the path x_t is nowhere differentiable. This property has not, however, deterred people from developing a theory of “stochastic calculus” in which one can take the “differential” of x_t , denoted dx_t . (Since x_t is not differentiable, we should not attempt to rewrite this differential as $\frac{dx_t}{dt} dt$.) There is then a theory of “stochastic integrals,” in which one can compute, for example, integrals of the form

$$\int_a^b f(x_t) dx_t,$$

where f is some smooth function.

A key difference between ordinary and stochastic integration is that $(dx_t)^2$ is not negligible compared to dt . To understand this assertion, recall that the increments

of Brownian motion have variance $t_j - t_{j-1}$ —and therefore standard deviation $\sqrt{t_j - t_{j-1}}$. This means that in a short time interval Δt , the Brownian motion travels distance roughly Δt . Thus, if $\Delta x_t = x_{t+\Delta t} - x_t$, we may say that $(\Delta x_t)^2 \approx \Delta t$. Thus, if f is a smooth function, we may use a Taylor expansion to claim that

$$\begin{aligned} f(x_{t+\Delta t}) &\approx f(x_t) + f'(x_t)\Delta x_t + \frac{1}{2}f''(x_t)(\Delta x_t)^2 \\ &\approx f(x_t) + f'(x_t)\Delta x_t + \frac{1}{2}f''(x_t)\Delta t. \end{aligned}$$

We may express the preceding discussion in the heuristically by saying

$$(dx_t)^2 = dt.$$

Rigorously, this line of reasoning lies behind the famous Itô formula, which says that

$$df(x_t) = f'(x_t) dx_t + \frac{1}{2}f''(x_t) dt.$$

The formula means, more precisely, that (after integration)

$$f(x_b) - f(x_a) = \int_a^b f'(x_t) dx_t + \frac{1}{2} \int_a^b f''(x_t) dt,$$

where the first integral on the right-hand side is a stochastic integral and the second is an ordinary Riemann integral.

If we take, for example, $f(x) = x^2/2$, then we find that

$$\frac{1}{2}(x_b^2 - x_a^2) = \int_a^b x_t dx_t + \frac{1}{2}(b - a)$$

so that

$$\int_a^b x_t dx_t = \frac{1}{2}(x_b^2 - x_a^2) - \frac{1}{2}(b - a).$$

This formula differs from what we would get if x_t were smooth by the $b - a$ term on the right-hand side.

4.2.2 Free Stochastic Calculus

We now turn to the case of the circular Brownian motion c_t . Since c_t is a limit of ordinary Brownian motion in the space of $N \times N$ matrices, we expect that $(dc_t)^2$

will be non-negligible compared to dt . The rules are as follows; see [31, Lemma 2.5, Lemma 4.3]. Suppose g_t and h_t are processes “adapted to c_t ,” meaning that g_t and h_t belong to the von Neumann algebra generated by the operators c_s with $0 < s < t$. Then we have

$$dc_t g_t dc_t^* = dc_t^* g_t dc_t = \tau(g_t) dt \quad (27)$$

$$dc_t g_t dc_t = dc_t^* g_t dc_t^* = 0 \quad (28)$$

$$\tau(g_t dc_t h_t) = \tau(g_t dc_t^* h_t) = 0. \quad (29)$$

In addition, we have the following Itô product rule: if a_t^1, \dots, a_t^n are processes adapted to c_t , then

$$\begin{aligned} d(a_t^1 \cdots a_t^n) &= \sum_{j=1}^n (a_t^1 \cdots a_t^{j-1}) da_t^j (a_t^{j+1} \cdots a_t^n) \\ &+ \sum_{1 \leq j < k \leq n} (a_t^1 \cdots a_t^{j-1}) da_t^j (a_t^{j+1} \cdots a_t^{k-1}) da_t^k (a_t^{k+1} \cdots a_t^n). \end{aligned} \quad (30)$$

$$(31)$$

Finally, the differential “ d ” can be moved inside the trace τ .

Suppose, for example, we wish to compute $d\tau[c_t^* c_t]$. We start by applying the product rule in (30) and (31). But by (29), there will be no contribution from the first line (30) in the product rule. We then use the second line (31) of the product rule together with (27) to obtain

$$d\tau[c_t^* c_t] = \tau[dc_t^* dc_t] = \tau(1) dt = dt.$$

Thus,

$$\frac{d}{dt} \tau[c_t^* c_t] = 1.$$

Since, also, $c_0 = 0$, we find that $\tau[c_t^* c_t] = t$.

4.2.3 The Proof

In the proof that follows, the Itô formula (27) plays the same role as the identity (23) plays in the heuristic argument in Section 4.1. We begin with a lemma whose proof is an exercise in using the rules of free stochastic calculus.

Lemma 11 *For each $\lambda \in \mathbb{C}$, let us use the notation*

$$c_{t,\lambda} := c_t - \lambda.$$

Then for each positive integer n , we have

$$\frac{d}{dt} \tau[(c_{t,\lambda}^* c_{t,\lambda})^n] = n \sum_{l=0}^{n-1} \tau[(c_{t,\lambda}^* c_{t,\lambda})^j] \tau[(c_{t,\lambda} c_{t,\lambda}^*)^{n-j-1}]$$

Proof We first note that $dc_{t,\lambda} = dc_t$ and $dc_{t,\lambda}^* = dc_t^*$, since λ is a constant. We then compute $d\tau[(c_{t,\lambda}^* c_{t,\lambda})^n]$ by moving the d inside the trace and then applying the product rule in (30) and (31). By (29), the terms arising from (30) will not contribute. Furthermore, by (28), the only terms from (31) that contribute are those where one d goes on a factor of $c_{t,\lambda}$ and one goes on a factor of $c_{t,\lambda}^*$.

By choosing all possible factors of $c_{t,\lambda}$ and all possible factors of $c_{t,\lambda}^*$, we get n^2 terms. In each term, after putting the d inside the trace, we can cyclically permute the factors until, say, the $dc_{t,\lambda}$ factor is at the end. There are then only n distinct terms that occur, each of which occurs n times. By (27), each distinct term is computed as

$$\begin{aligned} & \tau[(c_{t,\lambda}^* c_{t,\lambda})^j dc_{t,\lambda}^* c_{t,\lambda} (c_{t,\lambda}^* c_{t,\lambda})^{n-j-2} c_{t,\lambda}^* dc_t] \\ &= \tau[c_{t,\lambda} (c_{t,\lambda}^* c_{t,\lambda})^{n-j-2} c_{t,\lambda}^*] \tau[(c_{t,\lambda}^* c_{t,\lambda})^j] dt \\ &= \tau[(c_{t,\lambda}^* c_{t,\lambda})^j] \tau[c_t c_t^* (c_{t,\lambda} c_{t,\lambda}^*)^{n-j-1}] dt. \end{aligned}$$

Since each distinct term occurs n times, we obtain

$$d\tau[(c_{t,\lambda}^* c_{t,\lambda})^n] = n \sum_{j=0}^{n-1} \tau[(c_{t,\lambda}^* c_{t,\lambda})^j] \tau[(c_{t,\lambda} c_{t,\lambda}^*)^{n-j-1}] dt,$$

which is equivalent to the claimed formula. □

We are now ready to give a rigorous argument for the PDE.

Proof of Theorem 9 We continue to use the notation $c_{t,\lambda} := c_t - \lambda$. We first compute, using the operator version of (25), that

$$\begin{aligned} \frac{\partial S}{\partial \varepsilon} &= \frac{\partial}{\partial \varepsilon} \tau[\log(c_{t,\lambda}^* c_{t,\lambda} + \varepsilon)] \\ &= \tau[(c_{t,\lambda}^* c_{t,\lambda} + \varepsilon)^{-1}]. \end{aligned} \tag{32}$$

We note that the definition of S in (15) actually makes sense for all $\varepsilon \in \mathbb{C}$ with $\text{Re}(\varepsilon) > 0$, using the standard branch of the logarithm function. We note that for $|\varepsilon| > |z|$, we have

$$\frac{1}{z + \varepsilon} = \frac{1}{\varepsilon \left(1 - \left(-\frac{z}{\varepsilon}\right)\right)}$$

$$= \frac{1}{\varepsilon} \left[1 - \frac{z}{\varepsilon} + \frac{z^2}{\varepsilon^2} - \frac{z^3}{\varepsilon^3} + \cdots \right]. \quad (33)$$

Integrating with respect to z gives

$$\log(z + \varepsilon) = \log \varepsilon + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{z}{\varepsilon} \right)^n.$$

Thus, for $|\varepsilon| > \|c_{t,\lambda}^* c_t\|$, we have

$$\tau[\log(c_{t,\lambda}^* c_t + \varepsilon)] = \log \varepsilon + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \varepsilon^n} \tau[(c_{t,\lambda}^* c_t)^n]. \quad (34)$$

Assume for the moment that it is permissible to differentiate (34) term by term with respect to t . Then by Lemma 11, we have

$$\frac{\partial S}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\varepsilon^n} \sum_{j=0}^{n-1} \tau[(c_{t,\lambda}^* c_t)^j] \tau[(c_{t,\lambda} c_{t,\lambda}^*)^{n-j-1}]. \quad (35)$$

Now, by [5, Proposition 3.2.3], the map $t \mapsto c_t$ is continuous in the operator norm topology; in particular, $\|c_t\|$ is a locally bounded function of t . From this observation, it is easy to see that the right-hand side of (35) converges locally uniformly in t . Thus, a standard result about interchange of limit and derivative (e.g., Theorem 7.17 in [37]) shows that the term-by-term differentiation is valid.

Now, in (35), we let $k = j$ and $l = n - j - 1$, so that $n = k + l + 1$. Then k and l go from 0 to ∞ , and we get

$$\frac{\partial S}{\partial t} = \varepsilon \left(\frac{1}{\varepsilon} \sum_{k=0}^{\infty} \frac{(-1)^k}{\varepsilon^k} \tau[(c_{t,\lambda}^* c_t)^k] \right) \left(\frac{1}{\varepsilon} \sum_{l=0}^{\infty} \frac{(-1)^l}{\varepsilon^l} \tau[(c_{t,\lambda} c_{t,\lambda}^*)^l] \right).$$

(We may check that the power of ε in the denominator is $k + l + 1 = n$ and that the power of -1 is $k + l = n - 1$.) Thus, moving the sums inside the traces and using (33), we obtain that

$$\frac{\partial S}{\partial t} = \varepsilon (\tau[(c_{t,\lambda}^* c_t + \varepsilon)^{-1}])^2, \quad (36)$$

which reduces to the claimed PDE for S , by (32).

We have now established the claimed formula for $\partial S / \partial t$ for ε in the right half-plane, provided $|\varepsilon|$ is sufficiently large, depending on t and λ . Since, also, $S(0, \lambda, \varepsilon) = \log(|\lambda - 1|^2 + \varepsilon)$, we have, for sufficiently large $|\varepsilon|$,

$$S(t, \lambda, \varepsilon) = \log(|\lambda - 1|^2 + \varepsilon) + \int_0^t \varepsilon \tau[(c_{s,\lambda}^* c_{s,\lambda} + \varepsilon)^{-1}] \tau[(c_{s,\lambda} c_{s,\lambda}^* + \varepsilon)^{-1}] ds. \tag{37}$$

We now claim that both sides of (37) are well-defined, holomorphic functions of ε , for ε in the right half-plane. This claim is easily established from the standard power-series representation of the inverse:

$$\begin{aligned} (A + \varepsilon + h)^{-1} &= (A + \varepsilon)^{-1} (1 + h(A + \varepsilon)^{-1})^{-1} \\ &= (A + \varepsilon)^{-1} \sum_{n=0}^{\infty} (-1)^n h^n (A + \varepsilon)^{-n}, \end{aligned}$$

and a similar power-series representation of the logarithm. Thus, (37) actually holds for all ε in the right half-plane. Differentiating with respect to t then establishes the desired formula (36) for dS/dt for all ε in the right half-plane. \square

5 Solving the Equation

5.1 The Hamilton–Jacobi Method

The PDE (16) in Theorem 9 is a first-order, nonlinear equation of Hamilton–Jacobi type. “Hamilton–Jacobi type” means that the right-hand side of the equation involves only ε and $\partial S/\partial \varepsilon$, and not S itself. The reader may consult Section 3.3 of the book [11] of Evans for general information about equations of this type. In this subsection, we describe the general version of this method. In the remainder of this section, we will then apply the general method to the PDE (16).

The Hamilton–Jacobi method for analyzing solutions to equations of this type is a generalization of the method of characteristics. In the method of characteristics, one finds certain special curves along which the solution is constant. For a general equation of Hamilton–Jacobi type, the method of characteristics is not applicable. Nevertheless, we may hope to find certain special curves along which the solution varies in a simple way, allowing us to compute the solution along these curves in a more-or-less explicit way.

We now explain the representation formula for solutions of equations of Hamilton–Jacobi type. A self-contained proof of the following result is given as the proof of Proposition 6.3 in [10].

Proposition 12 *Fix a function $H(\mathbf{x}, \mathbf{p})$ defined for \mathbf{x} in an open set $U \subset \mathbb{R}^n$ and \mathbf{p} in \mathbb{R}^n . Consider a smooth function $S(t, \mathbf{x})$ on $[0, \infty) \times U$ satisfying*

$$\frac{\partial S}{\partial t} = -H(\mathbf{x}, \nabla_{\mathbf{x}} S) \tag{38}$$

for $\mathbf{x} \in U$ and $t > 0$. Now suppose $(\mathbf{x}(t), \mathbf{p}(t))$ is curve in $U \times \mathbb{R}^n$ satisfying Hamilton's equations:

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial p_j}(\mathbf{x}(t), \mathbf{p}(t)); \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial x_j}(\mathbf{x}(t), \mathbf{p}(t))$$

with initial conditions

$$\mathbf{x}(0) = \mathbf{x}_0; \quad \mathbf{p}(0) = \mathbf{p}_0 := (\nabla_{\mathbf{x}} S)(0, \mathbf{x}_0). \quad (39)$$

Then we have

$$S(t, \mathbf{x}(t)) = S(0, \mathbf{x}_0) - H(\mathbf{x}_0, \mathbf{p}_0) t + \int_0^t \mathbf{p}(s) \cdot \frac{d\mathbf{x}}{ds} ds \quad (40)$$

and

$$(\nabla_{\mathbf{x}} S)(t, \mathbf{x}(t)) = \mathbf{p}(t). \quad (41)$$

We emphasize that we are not using the Hamilton–Jacobi formula to *construct* a solution to the equation (38); rather, we are using the method to *analyze* a solution that is assumed ahead of time to exist. Suppose we want to use the method to compute (as explicitly as possible), the value of $S(t, \mathbf{x})$ for some fixed \mathbf{x} . We then need to try to choose the initial position \mathbf{x}_0 in (39)—which determines the initial momentum $\mathbf{p}_0 = (\nabla_{\mathbf{x}} S)(0, \mathbf{x}_0)$ —so that $\mathbf{x}(t) = \mathbf{x}$. We then use (40) to get an in-principle formula for $S(t, \mathbf{x}(t)) = S(t, \mathbf{x})$.

5.2 Solving the Equations

The equation for S^λ in Theorem 9 is of Hamilton–Jacobi form with $n = 1$, with Hamiltonian given by

$$H(\varepsilon, p) = -\varepsilon p^2. \quad (42)$$

Since $S^\lambda(t, \varepsilon)$ is only defined for $\varepsilon > 0$, we take open set U in Proposition 12 to be $(0, \infty)$. That is to say, the Hamilton–Jacobi formula (40) is only valid if the curve $\varepsilon(s)$ remains positive for $0 \leq s \leq t$.

Hamilton's equations for this Hamiltonian then take the explicit form

$$\frac{d\varepsilon}{dt} = \frac{\partial H}{\partial p} = -2\varepsilon p \quad (43)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial \varepsilon} = p^2. \quad (44)$$

Following the general method, we take an arbitrary initial position ε_0 , with the initial momentum p_0 given by

$$\begin{aligned}
 p_0 &= \left. \frac{\partial}{\partial \varepsilon} \log(|\lambda|^2 + \varepsilon) \right|_{\varepsilon=\varepsilon_0} \\
 &= \frac{1}{|\lambda|^2 + \varepsilon_0}.
 \end{aligned}
 \tag{45}$$

Theorem 13 *For any $\varepsilon_0 > 0$, the solution $(\varepsilon(t), p(t))$ to (43) and (44) with initial momentum $p_0 = 1/(|\lambda|^2 + \varepsilon_0)$ exists for $0 \leq t < |\lambda|^2 + \varepsilon_0$. On this time interval, we have*

$$\varepsilon(t) = \varepsilon_0 \left(1 - \frac{t}{|\lambda|^2 + \varepsilon_0} \right)^2.
 \tag{46}$$

The general Hamilton–Jacobi formula (40) then takes the form

$$\begin{aligned}
 &S^\lambda \left(t, \varepsilon_0 \left(1 - \frac{t}{|\lambda|^2 + \varepsilon_0} \right)^2 \right) \\
 &= \log(|\lambda|^2 + \varepsilon_0) - \frac{\varepsilon_0 t}{(|\lambda|^2 + \varepsilon_0)^2}, \quad 0 \leq t < |\lambda|^2 + \varepsilon_0.
 \end{aligned}
 \tag{47}$$

Proof Since the equation (44) for dp/dt does not involve $\varepsilon(t)$, we may easily solve it for $p(t)$ as

$$p(t) = \frac{p_0}{1 - p_0 t}.$$

We may then plug the formula for $p(t)$ into the equation (43) for $d\varepsilon/dt$, giving

$$\frac{d\varepsilon}{dt} = -2\varepsilon \frac{p_0}{1 - p_0 t}$$

so that

$$\frac{1}{\varepsilon} d\varepsilon = -2 \frac{p_0}{1 - p_0 t} dt.$$

Thus,

$$\log \varepsilon = 2 \log(p_0 t - 1) + c_1$$

so that

$$\varepsilon = c_2(1 - p_0 t)^2.$$

Plugging in $t = 0$ gives $c_2 = \varepsilon_0$. Recalling the expression (45) for p_0 gives the claimed formula for $\varepsilon(t)$.

Assuming $\varepsilon_0 > 0$, the solution to the system (43)–(44) continues to exist with $\varepsilon(t) > 0$ until $p(t)$ blows up, which occurs at time $t = 1/p_0 = |\lambda|^2 + \varepsilon_0$.

Finally, we work out the general Hamilton–Jacobi formula (40) in the case at hand. We note from (42) and (43) that $p(s) \frac{d\varepsilon}{ds} = -2\varepsilon(s)p(s)^2 = 2H(s)$. Since the Hamiltonian is always a conserved quantity in Hamilton’s equations, we find that

$$p(s) \frac{d\varepsilon}{ds} = 2H(0) = -2\varepsilon_0 p_0^2.$$

Thus, (40) reduces to

$$\begin{aligned} S^\lambda(t, \varepsilon(t)) &= S(0, \varepsilon_0) + H(0)t \\ &= \log(|\lambda|^2 + \varepsilon_0) - \varepsilon_0 p_0^2 t. \end{aligned}$$

Using the formula (45) for p_0 gives the claimed formula (47). □

6 Letting ε Tend to Zero

Recall that the Brown measure is obtained by first evaluating

$$s_t(\lambda) := \lim_{\varepsilon \rightarrow 0^+} S^\lambda(t, 0)$$

and then taking $1/(4\pi)$ times the Laplacian (in the distribution sense) of $s_t(\lambda)$. We record the result here and will derive it in the remainder of this section.

Theorem 14 *We have*

$$s_t(\lambda) = \begin{cases} \log(|\lambda|^2) & |\lambda| \geq \sqrt{t} \\ \log t - 1 + \frac{|\lambda|^2}{t} & |\lambda| < \sqrt{t} \end{cases}. \quad (48)$$

The Brown measure is then absolutely continuous with respect to the Lebesgue measure, with density $W_t(\lambda)$ given by

$$W_t(\lambda) = \begin{cases} 0 & |\lambda| \geq \sqrt{t} \\ \frac{1}{\pi t} & |\lambda| < \sqrt{t} \end{cases}. \quad (49)$$

That is to say, the Brown measure is the uniform probability measure on the disk of radius \sqrt{t} centered at the origin. The functions $s_t(\lambda)$ and $W_t(\lambda)$ are plotted for

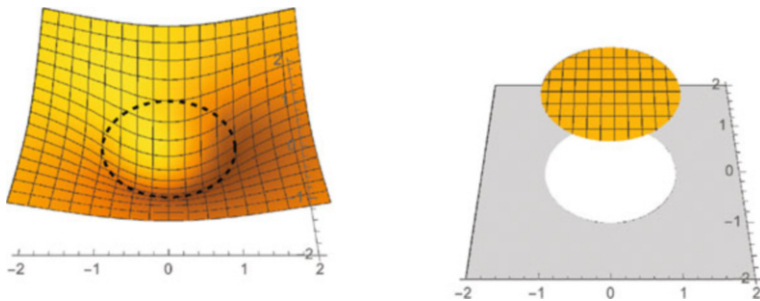


Fig. 6 Plot of $s_t(\lambda) := S^\lambda(t, 0^+)$ (left) and $\frac{1}{4\pi} \Delta s_t(\lambda)$ (right) for $t = 1$

$t = 1$ in Figure 6. On the left-hand side of the figure, the dashed line indicates the boundary of the unit disk.

6.1 Letting ϵ Tend to Zero: Outside the Disk

Our goal is to compute $s_t(\lambda) := \lim_{\epsilon \rightarrow 0^+} S^\lambda(t, \epsilon)$. Thus, in the Hamilton–Jacobi formalism, we want to try to choose ϵ_0 so that the quantity

$$\epsilon(t) = \epsilon_0 \left(1 - \frac{t}{|\lambda|^2 + \epsilon_0} \right)^2 \tag{50}$$

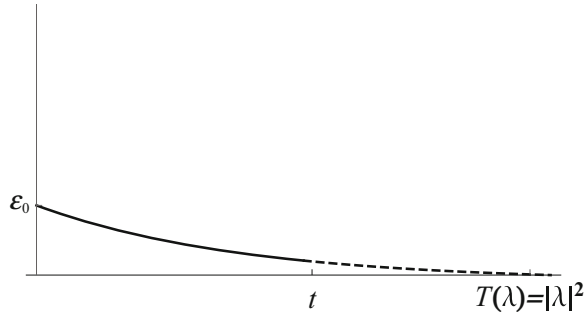
will be very close to zero. Since there is a factor of ϵ_0 on the right-hand side of the above formula, an obvious strategy is to take ϵ_0 itself very close to zero. There is, however, a potential difficulty with this strategy: If ϵ_0 is small, the lifetime of the solution may be smaller than the time t we are interested in. To see when the strategy works, we take the formula for the lifetime of the solution—namely, $|\lambda|^2 + \epsilon_0$ —and take the limit as ϵ_0 tends to zero.

Definition 15 For each $\lambda \in \mathbb{C}$, we define $T(\lambda)$ to be the lifetime of solutions to the system (43)–(44), in the limit as ϵ_0 approaches zero. Thus, explicitly,

$$\begin{aligned} T(\lambda) &= \lim_{\epsilon_0 \rightarrow 0^+} (|\lambda|^2 + \epsilon_0) \\ &= |\lambda|^2. \end{aligned}$$

Thus, if the time t we are interested in is larger than $T(\lambda) = |\lambda|^2$, our simple strategy of taking $\epsilon_0 \approx 0$ will not work. After all, if $t > T(\lambda)$ and $\epsilon_0 \approx 0$, then the lifetime of the path is less than t and the Hamilton–Jacobi formula (47) is not applicable. On the other hand, if the time t we are interested in is at most $T(\lambda) = |\lambda|^2$, the simple strategy does work. Figure 7 illustrates the situation.

Fig. 7 If ε_0 is small and positive, $\varepsilon(s)$ will remain small and positive up to time t , provided that $t \leq T(\lambda) = |\lambda|^2$



Conclusion 16 *The simple strategy of letting ε_0 approach zero works precisely when $t \leq T(\lambda) = |\lambda|^2$. Equivalently, the simple strategy works when $|\lambda| \geq \sqrt{t}$, that is, when λ is outside the open disk of radius \sqrt{t} centered at the origin.*

In the case that λ is outside the disk, we may then simply let ε_0 approach zero in the Hamilton–Jacobi formula, giving the following result.

Proposition 17 *Suppose $|\lambda| \geq \sqrt{t}$, that is, λ is outside the open disk of radius \sqrt{t} centered at 0. Then we may let ε_0 tend to zero in the Hamilton–Jacobi formula (47) to obtain*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} S^\lambda(t, \varepsilon) &= \lim_{\varepsilon_0 \rightarrow 0} \left(\log(|\lambda|^2 + \varepsilon_0) - \frac{\varepsilon_0 t}{(|\lambda|^2 + \varepsilon_0)^2} \right) \\ &= \log(|\lambda|^2). \end{aligned} \tag{51}$$

Since the right-hand side of (51) is harmonic, we conclude that

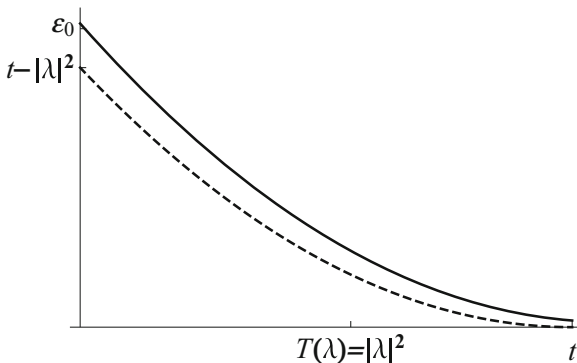
$$\Delta_{S_t}(\lambda) = 0, \quad |\lambda| > \sqrt{t}.$$

That is to say, the Brown measure of c_t is zero outside the disk of radius \sqrt{t} centered at 0.

6.2 Letting ε Tend to Zero: Inside the Disk

We now turn to the case in which the time t we are interested in is greater than the small- ε_0 lifetime $T(\lambda)$ of the solutions to (43)–(44). This case corresponds to $t > T(\lambda)^2 = |\lambda|^2$, that is, $|\lambda| < \sqrt{t}$. We still want to choose ε_0 so that $\varepsilon(t)$ will approach zero, but we cannot let ε_0 tend to zero, or else the lifetime of the solution will be less than t . Instead, we allow the *second* factor in the formula (46) for $\varepsilon(t)$ to approach zero. To make this factor approach zero, we make $|\lambda|^2 + \varepsilon_0$ approach t , that is, ε_0 should approach $t - |\lambda|^2$. Note that since we are now assuming that

Fig. 8 If $|\lambda| < \sqrt{t}$ and we let ε_0 approach $t - |\lambda|^2$ from above, $\varepsilon(s)$ will remain positive until time t , and $\varepsilon(t)$ will approach zero



$|\lambda| < \sqrt{t}$, the quantity $t - |\lambda|^2$ is positive. This strategy is illustrated in Figure 8: When $\varepsilon_0 = t - |\lambda|^2$, we obtain $\varepsilon(t) = 0$, and if ε_0 approaches $t - |\lambda|^2$ from above, the value of $\varepsilon(t)$ approaches 0 from above.

Proposition 18 Suppose $|\lambda| \leq \sqrt{t}$, that is, λ is inside the closed disk of radius \sqrt{t} centered at 0. Then in the Hamilton–Jacobi formula (47), we may let ε_0 approach $t - |\lambda|^2$ from above, and we get

$$\lim_{\varepsilon \rightarrow 0^+} S^\lambda(t, \varepsilon) = \log t - 1 + \frac{|\lambda|^2}{t}, \quad |\lambda| \leq \sqrt{t}.$$

For $|\lambda| < \sqrt{t}$, we may then compute

$$\frac{1}{4\pi} \Delta s_t(\lambda) = \frac{1}{\pi t}.$$

Thus, inside the disk of radius \sqrt{t} , the Brown measure has a constant density of $1/(\pi t)$.

Proof We use the Hamilton–Jacobi formula (47). Since the lifetime of our solution is $|\lambda|^2 + \varepsilon_0$, if we let ε_0 approach $t - |\lambda|^2$ from above, the lifetime will always be at least t . In this limit, the formula (46) for $\varepsilon(t)$ approaches zero from above. Thus, we may take the limit $\varepsilon_0 \rightarrow (t - |\lambda|^2)^+$ in (47) to obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} S^\lambda(t, \varepsilon) &= \lim_{\varepsilon_0 \rightarrow (t - |\lambda|^2)^+} \left[\log(|\lambda|^2 + \varepsilon_0) - \frac{\varepsilon_0 t}{(|\lambda|^2 + \varepsilon_0)^2} \right] \\ &= \log t - \frac{(t - |\lambda|^2)t}{t^2}, \end{aligned}$$

which simplifies to the claimed formula. □

6.3 On the Boundary

Note that if $|\lambda|^2 = t$, both approaches are valid—and the two values of $s_t(\lambda) := \lim_{\varepsilon \rightarrow 0^+} S^\lambda(t, \varepsilon)$ agree, with a common value of $\log t = \log |\lambda|^2$. Furthermore, the radial derivatives of $s_t(\lambda)$ agree on the boundary: $2/r$ on the outside and $2r/t$ on the inside, which have a common value of $2/\sqrt{t}$ at $r = \sqrt{t}$. Of course, the angular derivatives of $s_t(\lambda)$ are identically zero, inside, outside, and on the boundary.

Since the first derivatives of s_t are continuous up to the boundary, we may take the distributional Laplacian by taking the ordinary Laplacian inside the disk and outside the disk and ignoring the boundary. (See the proof of Proposition 7.13 in [10].) Thus, we may compute the Laplacian of the two formulas in (48) to obtain the formula (49) for the Brown measure of c_t .

7 The Case of the Free Multiplicative Brownian Motion

7.1 Additive and Multiplicative Models

The standard GUE and Ginibre ensembles are given by Gaussian measures on the relevant space of matrices (Hermitian matrices for GUE and all matrices for the Ginibre ensemble). In light of the central limit theorem, these ensembles can be approximated by adding together large numbers of small, independent random matrices. We may therefore refer to these Gaussian ensembles as “additive” models.

It is natural to consider also “multiplicative” random matrix models, which can be approximated by *multiplying* together large numbers of independent matrices that are “small” in the multiplicative sense, that is, close to the identity. Specifically, if Z^{add} is a random matrix with a Gaussian distribution, we will consider a multiplicative version Z_t^{mult} , where the distribution of Z_t^{mult} may be approximated as

$$Z_t^{\text{mult}} \sim \prod_{j=1}^k \left(I + i\sqrt{\frac{t}{k}} Z_j^{\text{add}} - \frac{t}{k} \text{It}\hat{o} \right), \quad k \text{ large.} \quad (52)$$

Here t is a positive parameter, the Z_j^{add} s are independent copies of Z^{add} , and “It \hat{o} ” is an Itô correction term. This correction term is a fixed multiple of the identity, independent of t and k . (In the next paragraph, we will identify the Itô term in the main cases of interest.) Since the factors in (52) are independent and identically distributed, the order of the factors does not affect the distribution of the product.

The two main cases we will consider are those in which Z is distributed according to the Gaussian unitary ensemble or the Ginibre ensemble. In the case that Z is distributed according to the Gaussian unitary ensemble, the Itô term is $\text{It}\hat{o} = \frac{1}{2}I$. In this case, the resulting multiplicative model may be described as *Brownian motion*

in the unitary group $U(N)$, which we write as U_t^N . The Itô correction is essential in this case to ensure that Z_t^{mult} actually lives in the unitary group. In the case that Z is distributed according to the Ginibre ensemble, the Itô term is zero. In this case, the resulting multiplicative model may be described as *Brownian motion in the general linear group* $GL(N; \mathbb{C})$, which we write as B_t^N .

7.2 The Free Unitary and Free Multiplicative Brownian Motions

The large- N limits of the Brownian motions U_t^N and B_t^N were constructed by Biane [3]. The limits are the **free unitary Brownian motion** and the **free multiplicative Brownian motion**, respectively, which we write as u_t and b_t . The qualifier “free” indicates that the increments of these Brownian motions—computed in the multiplicative sense as $u_s^{-1}u_t$ or $b_s^{-1}b_t$ —are freely independent in the sense of Section 2.3. In the case of b_t , the convergence of B_t^N to b_t was conjectured by Biane [3] and proved by Kemp [31]. In both cases, we take the limiting object to be an element of a tracial von Neumann algebra (\mathcal{A}, τ) .

Since u_t is unitary, we do not need to use the machinery of Brown measure, but can rather use the spectral theorem as in (7) to compute the **distribution** of u_t , denoted ν_t . We emphasize that ν_t is, in fact, the Brown measure of u_t , but it easier to describe ν_t using the spectral theorem than to use the general Brown measure construction. The measure ν_t is a probability measure on the unit circle describing the large- N limit of Brownian motion in the unitary group $U(N)$. Biane computed the measure ν_t in [3] and established the following support result.

Theorem 19 *For $t < 4$, the measure ν_t is supported on a proper subset of the unit circle:*

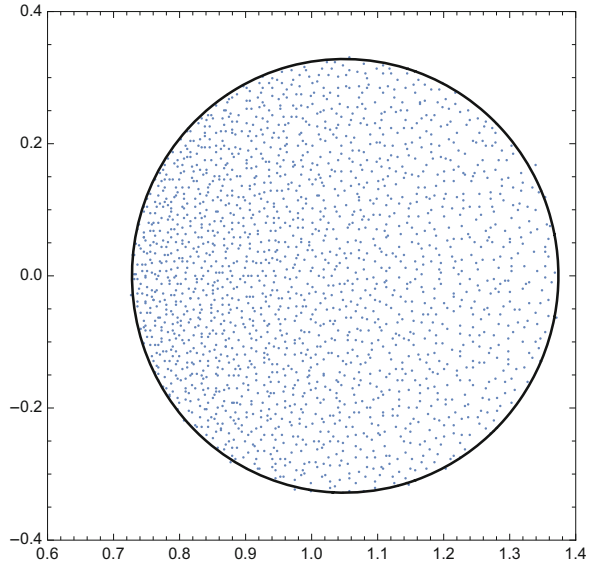
$$\text{supp}(\nu_t) = \left\{ e^{i\theta} \mid |\theta| \leq \frac{1}{2}\sqrt{t(4-t)} + \cos^{-1}\left(1 - \frac{t}{2}\right) \right\}, \quad t < 4.$$

By contrast, for all $t \geq 4$, the closed support of ν_t is the whole unit circle.

In the physics literature, the change in behavior of the support of ν_t at $t = 4$ is called a *topological phase transition*, indicating that the topology of $\text{supp}(\nu_t)$ changes from a closed interval to a circle.

The remainder of this article is devoted to recent results of the author with Driver and Kemp regarding the Brown measure of the free multiplicative Brownian motion b_t . We expect that the Brown measure of b_t will be the limiting empirical eigenvalue distribution of the Brownian motion B_t^N in the general linear group $GL(N; \mathbb{C})$. Now, when t is small, we may take $k = 1$ in (52), so that (since the Itô correction is zero in this case)

Fig. 9 The eigenvalues of B_t^N with $t = 0.1$ and $N = 2,000$



$$B_t^N \sim I + i\sqrt{\frac{t}{k}}Z, \quad t \text{ small.}$$

Thus, when t is small and N is large, the eigenvalues of B_t^N resemble a scaled and shifted version of the circular law. Specifically, the eigenvalue distribution should resemble a uniform distribution on the disk of radius \sqrt{t} centered at 1.

Figure 9 shows the eigenvalues of B_t^N with $t = 0.1$ and $N = 2,000$. The eigenvalue distribution bears a clear resemblance to the just-described picture, with $\sqrt{t} = \sqrt{0.1} \approx 0.316$. Nevertheless, we can already see some deviation from the small- t picture: The region into which the eigenvalues are clustering looks like a disk, but not quite centered at 1, while the distribution within the region is slightly higher at the left-hand side of the region than the right. Figures 10 and 11, meanwhile, show the eigenvalue distribution of B_t^N for several larger values of t . The region into which the eigenvalues cluster becomes more complicated as t increases, and the distribution of eigenvalues in the region becomes less and less uniform. We expect that the Brown measure of the limiting object b_t will be supported on the domain into which the eigenvalues are clustering.

7.3 The Domains Σ_t

We now describe certain domains Σ_t in the plane, as introduced by Biane in [4, pp. 273–274]. It will turn out that the Brown measure of b_t is supported on Σ_t . We use here a new the description of Σ_t , as given in Section 4 of [10]. For all nonzero $\lambda \in \mathbb{C}$, we define

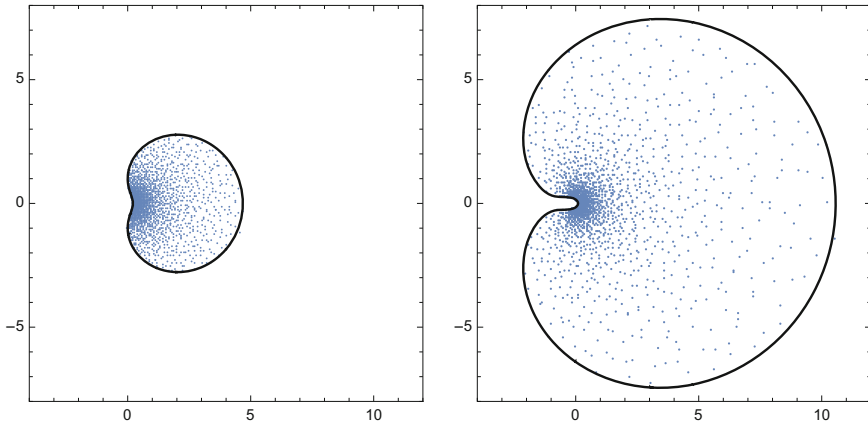


Fig. 10 Eigenvalues of B_t^N for $t = 2$ (left) and $t = 3.9$ (right), with $N = 2,000$

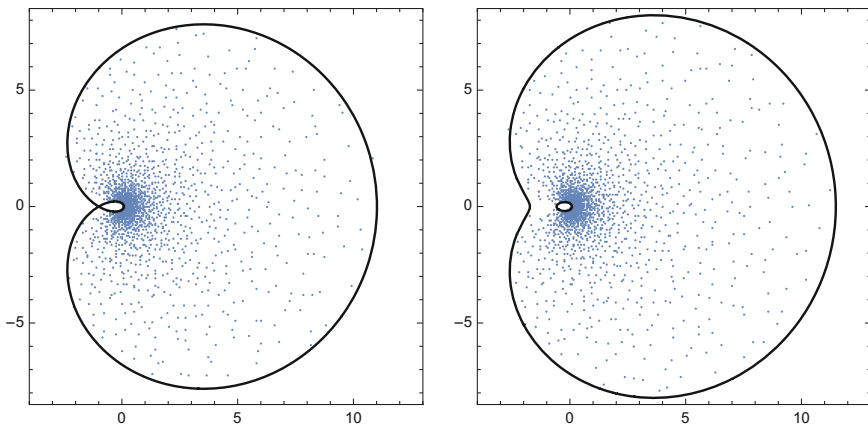


Fig. 11 Eigenvalues of B_t^N for $t = 4$ (left) and $t = 4.1$ (right), with $N = 2,000$

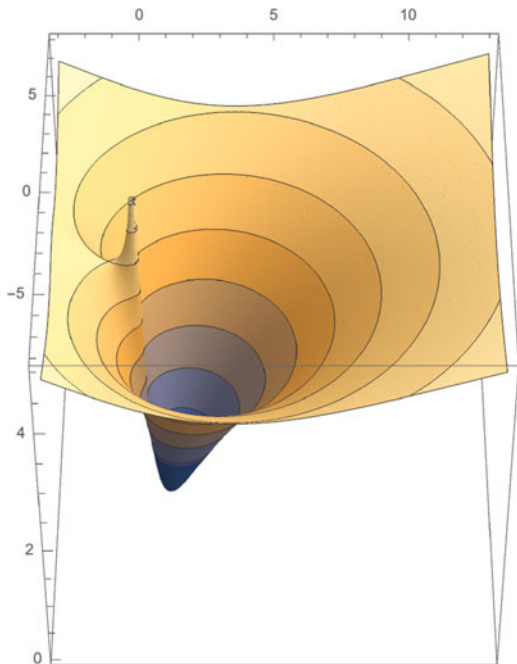
$$T(\lambda) = |\lambda - 1|^2 \frac{\log(|\lambda|^2)}{|\lambda|^2 - 1}. \tag{53}$$

If $|\lambda|^2 = 1$, we interpret $\log(|\lambda|^2)/(|\lambda|^2 - 1)$ as having the value 1 when $|\lambda|^2 = 1$, in accordance with the limit

$$\lim_{r \rightarrow 1} \frac{\log r}{r - 1} = 1.$$

See Figure 12 for a plot of this function.

Fig. 12 A plot of the function $T(\lambda)$. The function has a minimum at $\lambda = 1$, a saddle point at $\lambda = -1$, and a singularity at $\lambda = 0$



We then define the domains Σ_t as follows.

Definition 20 For each $t > 0$, we define

$$\Sigma_t = \{\lambda \in \mathbb{C} \mid T(\lambda) < t\}.$$

Several examples of these domains were plotted already in Figures 9, 10, and 11. The domain Σ_t is simply connected for $t \leq 4$ and doubly connected for $t > 4$. The change in behavior at $t = 4$ occurs because T has a saddle point at $\lambda = -1$ and because $T(-1) = 4$. We note that a change in the topology of the region occurs at $t = 4$, which is the same value of t at which the topology of the support of Biane’s measure changes (Theorem 19).

7.4 The Support of the Brown Measure of b_t

As we have noted, the domains Σ_t were introduced by Biane in [4]. Two subsequent works in the physics literature, the article [18] by Gudowska-Nowak, Janik, Jurkiewicz, and Nowak and the article [32] by Lohmayer, Neuberger, and Wettig then argued, using nonrigorous methods, that the eigenvalues of B_t^N should concentrate into Σ_t for large N . The first rigorous result in this direction was

obtained by the author with Kemp [26]; we prove that the Brown measure of b_t is supported on the closure of Σ_t .

Now, we have already noted that Σ_t is simply connected for $t \leq 4$ but doubly connected for $t > 4$. Thus, the support of the Brown measure of the free *multiplicative* Brownian motion undergoes a “topological phase transition” at precisely the same value of the time parameter as the distribution of the free *unitary* Brownian motion (Theorem 19).

The methods of [26] explain this apparent coincidence, using the “free Hall transform” \mathcal{G}_t of Biane [4]. Biane constructed this transform using methods of free probability as an infinite-dimensional analog of the Segal–Bargmann transform for $U(N)$, which was developed by the author in [21]. More specifically, Biane’s definition \mathcal{G}_t draws on the stochastic interpretation of the transform in [21] given by Gross and Malliavin [17]. Biane conjectured (with an outline of a proof) that \mathcal{G}_t is actually the large- N limit of the transform in [21]. This conjecture was then verified by independent works of Cébron [8] and the author with Driver and Kemp [9]. (See also the expository article [25].)

Recall from Section 7.2 that the distribution of the free unitary Brownian motion is Biane’s measure ν_t on the unit circle, the support of which is described in Theorem 19. A key ingredient in [26] is the function f_t given by

$$f_t(\lambda) = \lambda e^{\frac{t}{2} \frac{1+\lambda}{1-\lambda}}. \tag{54}$$

This function maps the complement of the closure of Σ_t conformally to the complement of the support of Biane’s measure:

$$f_t : \mathbb{C} \setminus \overline{\Sigma}_t \rightarrow \mathbb{C} \setminus \text{supp}(\nu_t). \tag{55}$$

(This map f_t will also play a role in the results of Section 7.5; see Theorem 23.)

The key computation in [26] is that for λ outside $\overline{\Sigma}_t$, we have

$$\mathcal{G}_t^{-1} \left(\frac{1}{z - \lambda} \right) = \frac{f_t(\lambda)}{\lambda} \frac{1}{u - f_t(\lambda)}, \quad \lambda \notin \overline{\Sigma}_t. \tag{56}$$

See Theorem 6.8 in [26]. Properties of the free Hall transform then imply that for λ outside $\overline{\Sigma}_t$, the operator $b_t - \lambda$ has an inverse. Indeed, the noncommutative L^2 norm of $(b_t - \lambda)^{-1}$ equals to the norm in $L^2(S^1, \nu_t)$ of the function on the right-hand side of (56). This norm, in turn, is finite because $f_t(\lambda)$ is outside the support of ν_t whenever λ is outside $\overline{\Sigma}_t$. The existence of an inverse to $b_t - \lambda$ then shows that λ must be outside the support of μ_{b_t} .

An interesting aspect of the paper [26] is that we not only *compute* the support of μ_{b_t} but also that we *connect* it to the support of Biane’s measure ν_t , using the transform \mathcal{G}_t and the conformal map f_t .

We note, however, that none of the papers [18, 32], or [26] says anything about the distribution of μ_{b_t} within Σ_t ; they are only concerned with identifying the region Σ_t . The actual computation of μ_{b_t} (not just its support) was done in [10].

7.5 The Brown Measure of b_t

We now describe the main results of [10]. Many of these results have been extended by Ho and Zhong [29] to the case of the free multiplicative Brownian motion with an arbitrary unitary initial distribution.

The first key result in [10] is the following formula for the Brown measure of b_t (Theorem 2.2 of [10]).

Theorem 21 *For each $t > 0$, the Brown measure μ_{b_t} is zero outside the closure of the region Σ_t . In the region Σ_t , the Brown measure has a density W_t with respect to Lebesgue measure. This density has the following special form in polar coordinates:*

$$W_t(r, \theta) = \frac{1}{r^2} w_t(\theta), \quad r e^{i\theta} \in \Sigma_t,$$

for some positive continuous function w_t . The function w_t is determined entirely by the geometry of the domain and is given as

$$w_t(\theta) = \frac{1}{4\pi} \left(\frac{2}{t} + \frac{\partial}{\partial \theta} \frac{2r_t(\theta) \sin \theta}{r_t(\theta)^2 + 1 - 2r_t(\theta) \cos \theta} \right),$$

where $r_t(\theta)$ is the “outer radius” of the region Σ_t at angle θ .

See Figure 13 for the definition of $r_t(\theta)$, Figure 14 for plots of the function $w_t(\theta)$, and Figure 15 for a plot of W_t . The simple explicit dependence of W_t on r is a major surprise of our analysis. See Corollary 22 for a notable consequence of the form of W_t .

Using implicit differentiation, it is possible to compute $dr_t(\theta)/d\theta$ explicitly as a function of $r_t(\theta)$. This computation yields the following formula for w_t , which does not involve differentiation:

$$w_t(\theta) = \frac{1}{2\pi t} \omega(r_t(\theta), \theta),$$

where

$$\omega(r, \theta) = 1 + h(r) \frac{\alpha(r) \cos \theta + \beta(r)}{\beta(r) \cos \theta + \alpha(r)}, \quad (57)$$

and

Fig. 13 The quantity $r_t(\theta)$ is the larger of the two radii at which the ray of angle θ intersects the boundary of Σ_t

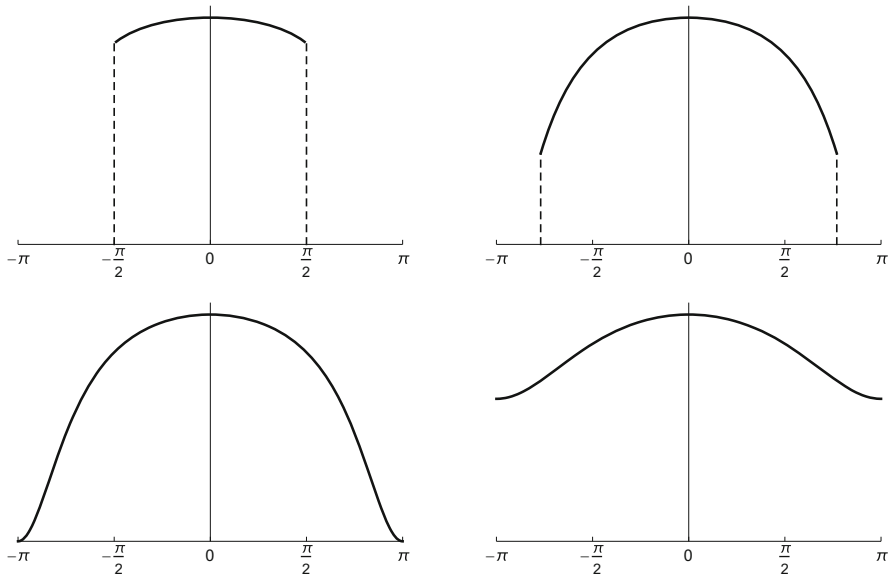
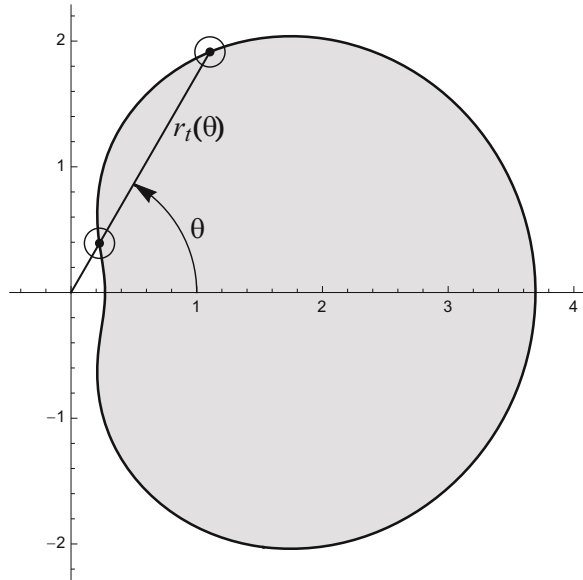


Fig. 14 Plots of $w_t(\theta)$ for $t = 2, 3.5, 4,$ and 7

$$h(r) = r \frac{\log(r^2)}{r^2 - 1}; \quad \alpha(r) = r^2 + 1 - 2rh(r); \quad \beta(r) = (r^2 + 1)h(r) - 2r.$$

See Proposition 2.3 in [10].

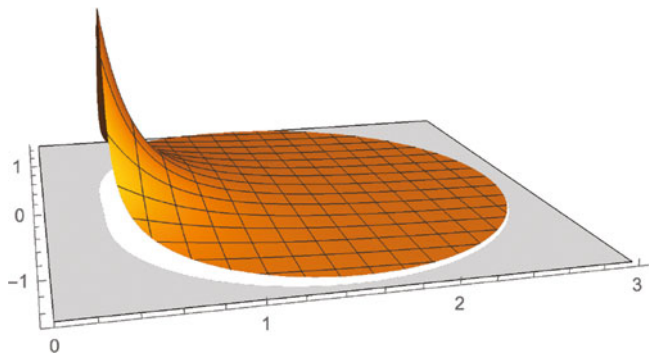


Fig. 15 Plot of the density W_t for $t = 1$

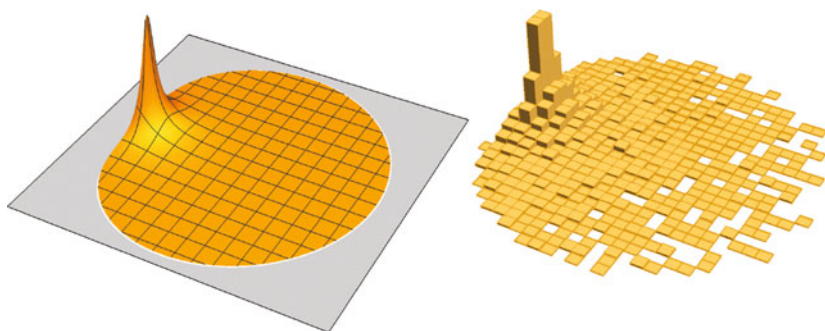


Fig. 16 The density W_t (left) and a histogram of the eigenvalues of B_t^N (right), for $t = 1$ and $N = 2,000$

We expect that the Brown measure of b_t will coincide with the limiting empirical eigenvalue distribution of the Brownian motion B_t^N in $GL(N; \mathbb{C})$. This expectation is supported by simulations; see Figure 16.

We note that the Brown measure (inside Σ_t) can also be written as

$$\begin{aligned} d\mu_{b_t} &= \frac{1}{r^2} w_t(\theta) r dr d\theta \\ &= w_t(\theta) \frac{1}{r} dr d\theta \\ &= w_t(\theta) d \log r d\theta. \end{aligned}$$

Since the complex logarithm is given by $\log(re^{i\theta}) = \log r + i\theta$, we obtain the following consequence of Theorem 21.

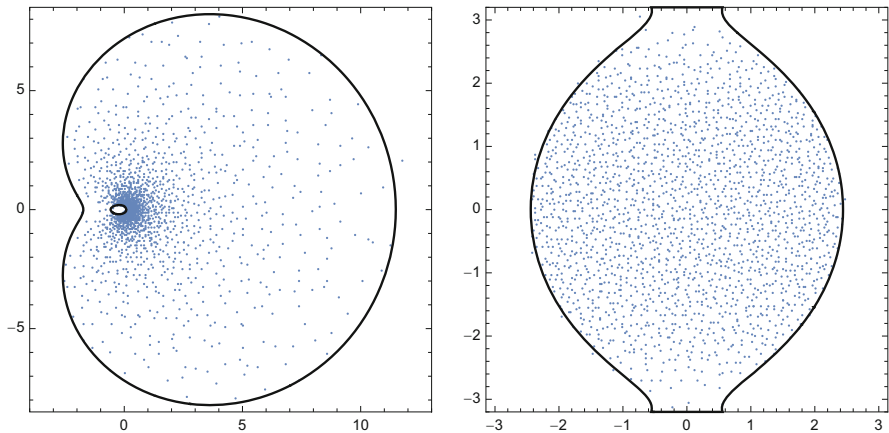


Fig. 17 The eigenvalues of B_t^N for $t = 4.1$ and $N = 2,000$ (left) and the logarithms thereof (right). The density of points on the right-hand side of the figure is approximately constant in the horizontal direction

Corollary 22 *The push-forward of the Brown measure μ_{b_t} under the complex logarithm has density that is constant in the horizontal direction and given by w_t in the vertical direction.*

In light of this corollary, we expect that for large N , the logarithms of the eigenvalues of B_t^N should be approximately uniformly distributed in the horizontal direction. This expectation is confirmed by simulations, as in Figure 17.

We conclude this section by describing a remarkable connection between the Brown measure μ_{b_t} and the distribution ν_t of the free unitary Brownian motion. Recall the holomorphic function f_t in (54) and (55). This map takes the boundary of Σ_t to the unit circle. We may then define a map

$$\Phi_t : \overline{\Sigma}_t \rightarrow S^1$$

by requiring (a) that Φ_t should agree with f_t on the boundary of Σ_t and (b) that Φ_t should be constant along each radial segment inside $\overline{\Sigma}_t$, as in Figure 18. (This specification makes sense because f_t has the same value at the two boundary points on each radial segment.) We then have the following result, which may be summarized by saying that *the distribution ν_t of free unitary Brownian motion is a “shadow” of the Brown measure of b_t .*

Theorem 23 *The push-forward of the Brown measure of b_t under the map Φ_t is Biane’s measure ν_t on S^1 . Indeed, the Brown measure of b_t is the unique measure μ on $\overline{\Sigma}_t$ with the following two properties: (1) the push-forward of μ by Φ_t is ν_t , and (2) μ is absolutely continuous with respect to Lebesgue measure with a density W having the form*

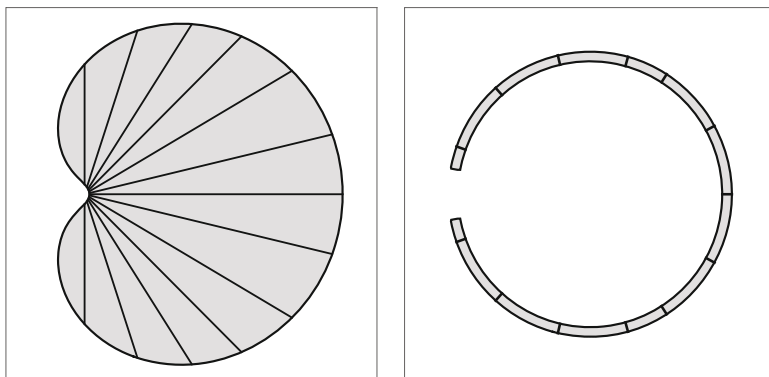
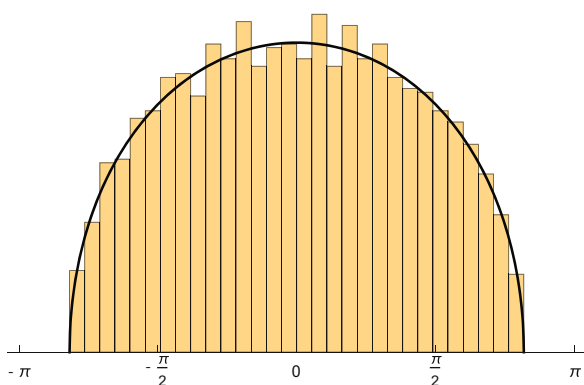


Fig. 18 The map Φ_t maps $\overline{\Sigma}_t$ to the unit circle by mapping each radial segment in $\overline{\Sigma}_t$ to a single point in S^1

Fig. 19 The eigenvalues of B_t^N , mapped to the unit circle by Φ_t , plotted against the density of Biane’s measure ν_t . Shown for $t = 2$ and $N = 2,000$



$$W(r, \theta) = \frac{1}{r^2}g(\theta)$$

in polar coordinates, for some continuous function g .

This result is Proposition 2.6 in [10]. Figure 19 shows the eigenvalues for B_t^N after applying the map Φ_t , plotted against the density of Biane’s measure ν_t . We emphasize that we have computed the eigenvalues of the Brownian B_t^N motion in $GL(N; \mathbb{C})$ (in the two-dimensional region Σ_t) and then mapped these points to the unit circle. The resulting histogram, however, looks precisely like a histogram of the eigenvalues of the Brownian motion in $U(N)$.

7.6 The PDE and Its Solution

We conclude this article by briefly outlining the methods used to obtain the results in the previous subsection.

7.6.1 The PDE

Following the definition of the Brown measure in Theorem 7, we consider the function

$$S(t, \lambda, \varepsilon) := \tau[\log((b_t - \lambda)^*(b_t - \lambda) + \varepsilon)]. \tag{58}$$

We then record the following result [10, Theorem 2.8].

Theorem 24 *The function S in (58) satisfies the following PDE:*

$$\frac{\partial S}{\partial t} = \varepsilon \frac{\partial S}{\partial \varepsilon} \left(1 + (|\lambda|^2 - \varepsilon) \frac{\partial S}{\partial \varepsilon} - a \frac{\partial S}{\partial a} - b \frac{\partial S}{\partial b} \right), \quad \lambda = a + ib, \tag{59}$$

with the initial condition

$$S(0, \lambda, \varepsilon) = \log(|\lambda - 1|^2 + \varepsilon). \tag{60}$$

Recall that in the case of the circular Brownian motion (the PDE in Theorem 9), the complex number λ enters only into the initial condition and not into the PDE itself. By contrast, the right-hand side of the PDE (59) involves differentiation with respect to the real and imaginary parts of λ .

On the other hand, the PDE (59) is again of Hamilton–Jacobi type. Thus, following the general Hamilton–Jacobi method in Section 5.1, we define a Hamiltonian function H from (the negative of) the right-hand side of (59), replacing each derivative of S by a corresponding momentum variable:

$$H(a, b, \varepsilon, p_a, p_b, p_\varepsilon) = -\varepsilon p_\varepsilon (1 + (a^2 + b^2) p_\varepsilon - \varepsilon p_\varepsilon - a p_a - b p_b). \tag{61}$$

We then consider Hamilton’s equations for this Hamiltonian:

$$\begin{aligned} \frac{da}{dt} &= \frac{\partial H}{\partial p_a}; & \frac{db}{dt} &= \frac{\partial H}{\partial p_b}; & \frac{d\varepsilon}{dt} &= \frac{\partial H}{\partial p_\varepsilon}; \\ \frac{dp_a}{dt} &= -\frac{\partial H}{\partial a}; & \frac{dp_b}{dt} &= -\frac{\partial H}{\partial b}; & \frac{dp_\varepsilon}{dt} &= -\frac{\partial H}{\partial \varepsilon}. \end{aligned} \tag{62}$$

Then, after a bit of simplification, the general Hamilton–Jacobi formula in (40) then takes the form

$$\begin{aligned}
S(t, \lambda(t), \varepsilon(t)) &= \log(|\lambda_0 - 1|^2 + \varepsilon_0) - \frac{\varepsilon_0 t}{(|\lambda_0 - 1|^2 + \varepsilon_0)^2} \\
&\quad + \log |\lambda(t)| - \log |\lambda_0|. \tag{63}
\end{aligned}$$

(See Theorem 6.2 in [10].)

The analysis in [10] then proceeds along broadly similar lines to those in Sections 5 and 6. The main structural difference is that because λ is now a variable in the PDE, the ODE's in (62) now involve both x and λ and the associated momenta. (That is to say, the vector \mathbf{x} in Proposition 12 is equal to $(\lambda, \varepsilon) \in \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$.) The first key result is that the system of ODE's associated to (59) can be solved explicitly; see Section 6.3 of [10]. Solving the ODE's gives an implicit formula for the solution to (59) with the initial conditions (60).

We then evaluate the solution in the limit as ε tends to zero. We follow the strategy in Section 6. Given a time t and a complex number λ , we attempt to choose initial conditions ε_0 and λ_0 so that $\varepsilon(t)$ will be very close to zero and $\lambda(t)$ will equal λ . (Recall that the initial momenta in the system of ODE's are determined by the positions by (39).)

7.6.2 Outside the Domain

As in the case of the circular Brownian motion, we use different approaches for λ outside Σ_t and for λ in Σ_t . For λ outside Σ_t , we allow the initial condition ε_0 in the ODE's to approach zero. As it turns out, when ε_0 is small and positive, $\varepsilon(t)$ remains small and positive for as long as the solution to the system exists. Furthermore, when ε_0 is small and positive, $\lambda(t)$ is approximately constant. Thus, our strategy will be to take $\varepsilon_0 \approx 0$ and $\lambda_0 \approx \lambda$.

A key result is the following.

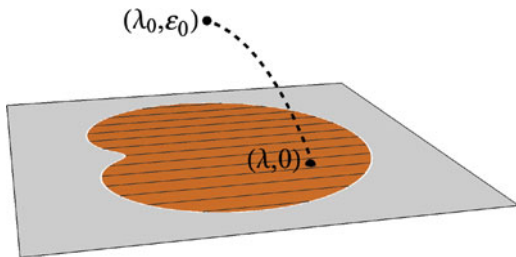
Proposition 25 *In the limit as ε_0 tends to zero, the lifetime of the solution to (62) with initial conditions λ_0 and ε_0 —and initial moment determined by (39)—approaches $T(\lambda_0)$, where T is the same function (53) that enters into the definition of the domain Σ_t .*

This result is Proposition 6.13 in [10]. Thus, the strategy in the previous paragraph will work—meaning that the solution continues to exist up to time t —provided that $T(\lambda_0) \approx T(\lambda)$ is greater than t . The condition for success of the strategy is, therefore, $T(\lambda) > t$. In light of the characterization of Σ_t in Definition 20, we make have the following conclusion.

Conclusion 26 *The simple strategy of taking $\varepsilon_0 \approx 0$ and $\lambda_0 \approx \lambda$ is successful precisely if $T(\lambda) > t$ or, equivalently, if λ is outside $\overline{\Sigma}_t$.*

When this strategy works, we obtain a simple expression for $\lim_{\varepsilon \rightarrow 0^+} S(t, \lambda, \varepsilon)$, by letting ε_0 approach zero and λ_0 approach λ in (63). Since $\lambda(t)$ approaches λ in this limit [10, Proposition 6.11], we find that

Fig. 20 For each λ in Σ_t , there exists $\varepsilon_0 > 0$ and $\lambda_0 \in \Sigma_t$ such that with these initial conditions, we have $\varepsilon(t) = 0$ and $\lambda(t) = \lambda$



$$\lim_{\varepsilon \rightarrow 0^+} S(t, \lambda, \varepsilon) = \log(|\lambda - 1|^2), \quad \lambda \notin \overline{\Sigma}_t. \tag{64}$$

This function is harmonic (except at $\lambda = 1$, which is always in the domain Σ_t), so we conclude that *the Brown measure of b_t is zero outside $\overline{\Sigma}_t$* . See Section 7.2 in [10] for more details.

7.6.3 Inside the Domain

For λ inside Σ_t , the simple approach in the previous subsection does not work, because when λ is outside Σ_t and ε_0 is small, the solutions to the ODE’s (62) will cease to exist prior to time t (Proposition 25). Instead, we must prove a “surjectivity” result: For each $t > 0$ and $\lambda \in \Sigma_t$, there exist—in principle— $\lambda_0 \in \mathbb{C}$ and $\varepsilon_0 > 0$ giving $\lambda(t) = \lambda$ and $\varepsilon(t) = 0$. See Figure 20. Actually the proof shows that λ_0 again belongs to the domain Σ_t ; see Section 6.5 in [10].

We then make use of the second Hamilton–Jacobi formula (41), which allows us to compute the derivatives of S directly, without having to attempt to differentiate the formula (63) for S . Working in logarithmic polar coordinates, $\rho = \log |\lambda|$ and $\theta = \arg \lambda$, we find an amazingly simple expression for the quantity

$$\frac{\partial s_t}{\partial \rho} = \lim_{\varepsilon \rightarrow 0^+} \frac{\partial S}{\partial \rho}(t, \lambda, \varepsilon),$$

inside Σ_t , namely,

$$\frac{\partial s_t}{\partial \rho} = \frac{2\rho}{t} + 1, \quad \lambda \in \Sigma_t. \tag{65}$$

(See Corollary 7.6 in [10].) This result is obtained using a certain constant of motion of the system of ODE’s, namely, the quantity

$$\Psi = \varepsilon p_\varepsilon + \frac{1}{2}(ap_a + bp_b)$$

in [10, Proposition 6.5].

If we evaluate this constant of motion at a time t when $\varepsilon(t) = 0$, the $\varepsilon p_\varepsilon$ term vanishes. But if $\varepsilon(t) = 0$, the second Hamilton–Jacobi formula (41) tells us that

$$\left(a \frac{\partial S}{\partial a} + b \frac{\partial S}{\partial b} \right) (t, \lambda(t), 0) = a(t) p_a(t) + b(t) p_b(t).$$

Furthermore, $a \frac{\partial S}{\partial a} + b \frac{\partial S}{\partial b}$ is just $\partial S / \partial \rho$, computed in rectangular coordinates. A bit of algebraic manipulation yields an explicit formula for $a \frac{\partial S}{\partial a} + b \frac{\partial S}{\partial b}$, as in [10, Theorem 6.7], explaining the formula (65). To complete the proof (65), it still remains to address certain regularity issues of $S(t, \lambda, \varepsilon)$ near $\varepsilon > 0$, as in Section 7.3 of [10].

Once (65) is established, we note that the formula for $\partial s_t / \partial \rho$ in (65) is independent of θ . It follows that

$$\frac{\partial}{\partial \rho} \frac{\partial s_t}{\partial \theta} = \frac{\partial}{\partial \theta} \frac{\partial s_t}{\partial \rho} = 0,$$

that is, that $\partial s_t / \partial \theta$ is independent of ρ inside Σ_t . Writing the Laplacian in logarithmic polar coordinates, we then find that

$$\begin{aligned} \Delta s_t(\lambda) &= \frac{1}{r^2} \left(\frac{\partial^2 s_t}{\partial \rho^2} + \frac{\partial^2 s_t}{\partial \theta^2} \right) \\ &= \frac{1}{r^2} \left(\frac{2}{t} + \frac{\partial}{\partial \theta} \left(\frac{\partial s_t}{\partial \theta} \right) \right), \quad \lambda \in \Sigma_t, \end{aligned} \quad (66)$$

where $2/t$ term in the expression comes from differentiating (65) with respect to ρ . Since $\partial s_t / \partial \theta$ is independent of ρ , we can understand the structure of the formula in Theorem 21.

The last step in the proof of Theorem 21 is to compute $\partial s_t / \partial \theta$. Since $\partial s_t / \partial \theta$ is independent of ρ —or, equivalently, independent of $r = |\lambda|$ —inside Σ_t , the value of $\partial s_t / \partial \theta$ at a point λ in Σ_t is the same as its value as we approach the boundary of Σ_t along the radial segment through λ . We show that $\partial s_t / \partial \theta$ is continuous over the whole complex plane, even at the boundary of Σ_t . (See Section 7.4 of [10].) Thus, on the boundary of Σ_t , the function $\partial s_t / \partial \theta$ will agree with the angular derivative of $\log(|\lambda - 1|^2)$, namely

$$\begin{aligned} \frac{\partial}{\partial \theta} \log(|\lambda - 1|^2) &= \frac{2 \operatorname{Im} \lambda}{|\lambda - 1|^2} \\ &= \frac{2r \sin \theta}{r^2 + 1 - 2r \cos \theta}. \end{aligned} \quad (67)$$

Thus, to compute $\partial s_t / \partial \theta$ at a point λ in Σ_t , we simply evaluate (67) at either of the two points where the radial segment through λ intersects $\partial \Sigma_t$. (We get the same value at either point.)

One such boundary point is the point with argument $\theta = \arg \lambda$ and radius $r_t(\theta)$, as in Figure 13. Thus, inside Σ_t , we have

$$\frac{\partial s_t}{\partial \theta} = \frac{2r_t(\theta) \sin \theta}{r_t(\theta)^2 + 1 - 2r_t(\theta) \cos \theta}.$$

Plugging this expression into (66) gives the claimed formula in Theorem 21.

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