No Loss of Derivatives for Hyperbolic Operators with Zygmund-Continuous Coefficients in Time

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To Massimo Cicognani and Michael Reissig on the occasion of their 60th birthday

Abstract We prove that, if the coefficients of an hyperbolic operator are Zygmund-continuous with respect to t and Lipschitz-continuous with respect to x, an energy estimate without loss of derivatives holds true. As a consequence, the Cauchy problem related to the hyperbolic operator is well-posed in Sobolev spaces.

Keywords Hyperbolic partial differential equations · Non-smooth coefficients · Energy estimates · Cauchy problem

1 Introduction

Consider the second order strictly hyperbolic operator

$$L = \partial_t^2 - \sum_{j,k=1}^{n} \partial_j(a_{jk}(t,x)\partial_k),$$

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where, for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$,

$$0 < \lambda_0 |\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(t, x) \xi_j \xi_k \leq \Lambda_0 |\xi|^2$$

and

$$a_{jk}(t, x) = a_{kj}(t, x).$$

It is well-known that, if the coefficients a_{jk} are Lipschitz-continuous in t and measurable in x, then the Cauchy problem related to L is well-posed in the energy space. In particular, a constant $C > 0$ exists, such that

$$\sup_{0 \leq t \leq T} (\|u(t, \cdot)\|_{H^1} + \|\partial_t u(t, \cdot)\|_{L^2})$$
$$\leq C(\|u(0, \cdot)\|_{H^1} + \|\partial_t u(0, \cdot)\|_{L^2} + \int_0^T \|Lu(s, \cdot)\|_{L^2} \, ds), \quad (1)$$

for all $u \in C([0, T]; H^1) \cap C^1([0, T]; L^2)$ with $Lu \in L^1([0, T]; L^2)$ (see [11, 12, Ch. IX]).

In this note we are interested in second order strictly hyperbolic operators having *non Lipschitz-continuous* coefficients with respect to time.

After the pioneering paper by Colombini, De Giorgi and Spagnolo [7], this topic has been widely studied. A result of particular interest has been obtained in [5], where it was proved that, if the coefficients are log-Lipschitz-continuous with respect to t and x, i.e. there exists $C > 0$ such that

$$\sup_{t,x} |a_{jk}(t + \tau, x + y) - a_{jk}(t, x)| \leq C(|\tau| + |y|)(1 + \log \frac{1}{|\tau| + |y|}),$$

then (1) is no more valid, but the following weaker energy estimate can be recovered:

$$\sup_{0 \leq t \leq T} (\|u(t, \cdot)\|_{H^{1-\theta-\beta t}} + \|\partial_t u(t, \cdot)\|_{H^{-\theta-\beta t}})$$
$$\leq C(\|u(0, \cdot)\|_{H^{1-\theta}} + \|\partial_t u(0, \cdot)\|_{H^{-\theta}} + \int_0^T \|Lu(s, \cdot)\|_{H^{-\theta-\beta s}} \, ds), \quad (2)$$

for some constants $C > 0$, $\beta > 0$ and for all $u \in C^2([0, T]; H^\infty)$ and $\theta \in]0, 1[$ (here and in the following $H^\infty = \bigcap_{s \in \mathbb{R}} H^s$). Remark that, while in (1) the norms of $u(t)$ and $\partial_t u(t)$ are estimated by the same norms of $u(0)$ and $\partial_t u(0)$, in (2) the Sobolev spaces in which $u(t)$ and $\partial_t u(t)$ are measured are different and bigger than the spaces in which initial data are, so the estimate is less effective. This phenomenon goes under the name of "loss of derivatives". We refer e.g. to the introductions of [8, 9] for more details and references about this problem.

Using a result obtained by Tarama in [16] (see also Remark 1 below), it is possible to prove that if the coefficients depend only on t and are Zygmund-continuous, i.e.

$$\sup_t |a_{jk}(t+\tau) + a_{jk}(t-\tau) - 2a_{jk}(t)| \, dt \le C_2 |\tau|, \tag{3}$$

then (1) is valid. Notice that the Zygmund assumption is weaker than the Lipschitz one. In [9], the authors proved that if the coefficients depend also on the space variable and verify an isotropic Zygmund assumption (i.e. they are Zygmund-continuous both in time and space variables), then the Cauchy problem is well-posed with no loss, but only in the space $H^{1/2} \times H^{-1/2}$. In particular, an estimate similar to (1) holds true, up to replacing the H^1 and L^2 norms respectively with the $H^{1/2}$ and $H^{-1/2}$ norms. See also Remark 2 below for more details.

The problem whether a Zygmund assumption both in time and space is still enough to recover well-posedness in general spaces $H^s \times H^{s-1}$ (and not only for $s = 1/2$) remains at present largely open. As a partial step in this direction, in this note we consider a stronger hypothesis with respect to the space variable: namely we prove that, if the coefficients are Zygmund-continuous with respect to t and Lipschitz-continuous with respect to x, then an estimate without loss of derivatives, similar to (1), holds true. Then, the Cauchy problem related to L is well-posed in any space $H^s \times H^{s-1}$, for all $s \in {]}0, 1]$.

Two are the main ingredients of the proof of our result. The first one is to resort to Tarama's idea of introducing a new type of energy associated to operator L: this new energy is equivalent to the classical energy, but it contains a lower order term, whose goal is to produce special algebraic cancellations, which reveal to be fundamental in the energy estimates. The second main ingredient, already introduced in [8] and [9], is the use of paradifferential calculus with parameters (see e.g. [13, 15]), in order to deal with coefficients depending also on x and having low regularity in that variable.

We conclude this introduction with a short overview of the paper. In the next section we fix our hypotheses and state our main result, see Theorem 1. In Sect. 3 we collect some elements of Littlewood-Paley theory, which are needed in the description of the functional classes where the coefficients belong to, and in the construction of paradifferential calculus with parameters. With those tools at hand, we tackle the proof of Theorem 1, which is carried out in Sect. 4.

2 Main Result

Given $T > 0$ and an integer $n \ge 1$, let L be the linear differential operator defined on $[0, T] \times \mathbb{R}^n$ by

$$Lu = \partial_t^2 u - \sum_{j,k=1}^n \partial_j(a_{jk}(t, x)\partial_k u), \tag{4}$$

where, for all $j, k = 1, \ldots, n$,

$$a_{jk}(t, x) = a_{kj}(t, x), \tag{5}$$

and there exist $\lambda_0, \Lambda_0 > 0$ such that

$$\lambda_0 |\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(t, x) \xi_j \xi_k \leq \Lambda_0 |\xi|^2, \tag{6}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$. Suppose moreover that there exist constants $C_0, C_1 > 0$ such that, for all $j, k = 1, \ldots, n$ and for all $\tau \in \mathbb{R}, y \in \mathbb{R}^n$,

$$\sup_{t,x} |a_{jk}(t + \tau, x) + a_{jk}(t - \tau, x) - 2a_{jk}(t, x)| \leq C_0 |\tau|, \tag{7}$$

$$\sup_{t,x} |a_{jk}(t, x + y) - a_{jk}(t, x)| \leq C_1 |y|. \tag{8}$$

We can now state the main result of this paper.

Theorem 1 *Under the previous hypotheses, for all fixed $\theta \in [0, 1[$, there exists a constant $C > 0$, depending only on θ and T, such that*

$$\sup_{0 \leq t \leq T} (\|u(t, \cdot)\|_{H^{1-\theta}} + \|\partial_t u(t, \cdot)\|_{H^{-\theta}})$$
$$\leq C(\|u(0, \cdot)\|_{H^{1-\theta}} + \|\partial_t u(0, \cdot)\|_{H^{-\theta}} + \int_0^T \|Lu(s, \cdot)\|_{H^{-\theta}} \, ds), \tag{9}$$

for all $u \in C^2([0, T], H^\infty(\mathbb{R}^n))$.

Some remarks are in order.

Remark 1 If the coefficients a_{jk} depend only on t, this result has been obtained by Tarama in [16], under the hypothesis that there exists a constant $C_2 > 0$ such that, for all $j, k = 1, \ldots, n$ and for all $\tau \in]0, T/2[$,

$$\int_\tau^{T-\tau} |a_{jk}(t + \tau) + a_{jk}(t - \tau) - 2a_{jk}(t)| \, dt \leq C_2 \tau. \tag{10}$$

Tarama's hypothesis is weaker than ours, but, when coefficients depend also on the space variable, it is customary to take a pointwise condition with respect to time, like in (7) above (see also [5, 6, 8, 9] in this respect). In particular, it is not clear at present whether or not the pointwise condition (7) can be relaxed to an integral one, similar to (10), in our framework.

Remark 2 If the hypoteses (7) and (8) are replaced by the weaker following one: there exists a constant $C_3 > 0$ such that, for all $j, k = 1, \ldots, n$ and for all $\tau \in \mathbb{R}$,

$y \in \mathbb{R}^n$,

$$\sup_{t,x} |a_{jk}(t + \tau, x + y) + a_{jk}(t - \tau, x - y) - 2a_{jk}(t, x)| \le C_3(|\tau| + |y|), \quad (11)$$

the estimate (9) has been proved, only in the case of $\theta = 1/2$, by the present authors and Métivier in [9].

Remark 3 Assume (7) and the following hypothesis: there exists a constant $C_4 > 0$ such that, for all $j, k = 1, \ldots, n$ and for all $y \in \mathbb{R}^n$ with $0 < |y| \le 1$,

$$\sup_{t,x} |a_{jk}(t, x + y) - a_{jk}(t, x)| \le C_4 |y| (1 + \log \frac{1}{|y|}). \quad (12)$$

As a consequence of a result of the present authors and Métivier in [8] (stated for coefficients which are actually log-Zygmund with respect to time), one gets that, for all fixed $\theta \in]0, 1[$, there exist a $\beta > 0$, a time $T' > 0$ and a constant $C > 0$ such that

$$\begin{aligned} \sup_{0 \le t \le T'} &(\|u(t, \cdot)\|_{H^{1-\theta-\beta t}} + \|\partial_t u(t, \cdot)\|_{H^{-\theta-\beta t}}) \\ &\le C(\|u(0, \cdot)\|_{H^{1-\theta}} + \|\partial_t u(0, \cdot)\|_{H^{-\theta}} + \int_0^{T'} \|Lu(s, \cdot)\|_{H^{-\theta-\beta s}} \, ds), \end{aligned} \quad (13)$$

for all $u \in C^2([0, T'], H^\infty(\mathbb{R}^n))$. The condition (12) is weaker than (8) but also (13) is weaker than (9): (13) has a loss of derivatives, while (9) performs no loss. In addition, observe that (13) holds only for $\theta \in]0, 1[$, while (9) holds also for $\theta = 0$.

3 Preliminary Results

We briefly list here some tools we will need in the proof of the main result. We follow closely the presentation of these topics given in [8] and [9].

3.1 Littlewood-Paley Decomposition

We will use the so called Littlewood-Paley theory. We refer to [2, 3, 14] and [1] for the details.

We start recalling Bernstein's inequalities.

Proposition 1 ([3, Lemma 2.2.1]) *Let $0 < r < R$. A constant C exists so that, for all nonnegative integer k, all $p, q \in [1, +\infty]$ with $p \le q$ and for all function $u \in L^p(\mathbb{R}^d)$, we have, for all $\lambda > 0$,*

(i) if $\operatorname{Supp}\hat{u} \subseteq B(0, \lambda R) = \{\xi \in \mathbb{R}^d \ : \ |\xi| \leq \lambda R\}$, *then*

$$\|\nabla^k u\|_{L^q} \leq C^{k+1} \lambda^{k+N(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p};$$

(ii) if $\operatorname{Supp}\hat{u} \subseteq C(0, \lambda r, \lambda R) = \{\xi \in \mathbb{R}^d \ : \ \lambda r \leq |\xi| \leq \lambda R\}$, *then*

$$C^{-k-1} \lambda^k \|u\|_{L^p} \leq \|\nabla^k u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}.$$

We introduce the dyadic decomposition. Let $\psi \in C^\infty([0, +\infty[, \mathbb{R})$ such that ψ is non-increasing and

$$\psi(t) = 1 \quad \text{for} \quad 0 \leq t \leq \frac{11}{10}, \qquad \psi(t) = 0 \quad \text{for} \quad t \geq \frac{19}{10}.$$

We set, for $\xi \in \mathbb{R}^d$,

$$\chi(\xi) = \psi(|\xi|), \qquad \varphi(\xi) = \chi(\xi) - \chi(2\xi). \tag{14}$$

We remark that the support of χ is contained in the ball $\{\xi \in \mathbb{R}^d \ : \ |\xi| \leq 2\}$, while the one of φ is contained in the annulus $\{\xi \in \mathbb{R}^d \ : \ 1/2 \leq |\xi| \leq 2\}$.

Given a tempered distribution u, the dyadic blocks are defined by

$$\Delta_0 u = \chi(D)u = \mathscr{F}^{-1}(\chi(\xi)\hat{u}(\xi)),$$

$$\Delta_j u = \varphi(2^{-j}D)u = \mathscr{F}^{-1}(\varphi(2^{-j}\xi)\hat{u}(\xi)) \quad \text{if} \quad j \geq 1,$$

where we have denoted by $\mathscr{F}^{-1}$ the inverse of the Fourier transform. We introduce also the operator

$$S_k u = \sum_{j=0}^{k} \Delta_j u = \mathscr{F}^{-1}(\chi(2^{-k}\xi)\hat{u}(\xi)).$$

It is well known the characterization of classical Sobolev spaces via Littlewood-Paley decomposition: for any $s \in \mathbb{R}$, $u \in \mathscr{S}'$ is in H^s if and only if, for all $j \in \mathbb{N}$, $\Delta_j u \in L^2$ and the series $\sum 2^{2js}\|\Delta_j u\|_{L^2}^2$ is convergent. Moreover, in such a case, there exists a constant $C_s > 1$ such that

$$\frac{1}{C_s} \sum_{j=0}^{+\infty} 2^{2js}\|\Delta_j u\|_{L^2}^2 \leq \|u\|_{H^s}^2 \leq C_s \sum_{j=0}^{+\infty} 2^{2js}\|\Delta_j u\|_{L^2}^2. \tag{15}$$

3.2 Lipschitz, Zygmund and Log-Lipschitz Functions

In this subsection, we give a description of some functional classes relevant in the study of hyperbolic Cauchy problems. Namely, via Littlewood-Paley analysis, we can characterise the spaces of Lipschitz, Zygmund and log-Lipschitz functions. We start by recalling their definitions.

Definition 1 A function $u \in L^\infty(\mathbb{R}^d)$ is a Lipschitz-continuous function if

$$|u|_{\text{Lip}} = \sup_{\substack{x,y \in \mathbb{R}^d, \\ y \neq 0}} \frac{|u(x+y) - u(x)|}{|y|} < +\infty,$$

u is a Zygmund-continuous function if

$$|u|_{\text{Zyg}} = \sup_{\substack{x,y \in \mathbb{R}^d, \\ y \neq 0}} \frac{|u(x+y) + u(x-y) - 2u(x)|}{|y|} < +\infty$$

and, finally, u is a log-Lipschitz-continuous function if

$$|u|_{\text{LL}} = \sup_{\substack{x,y \in \mathbb{R}^d, \\ 0 < |y| \leq 1}} \frac{|u(x+y) - u(x)|}{|y|(1 + \log \frac{1}{|y|})} < +\infty.$$

For $X \in \{\text{Lip}, \text{Zyg}, \text{LL}\}$, we define $\|u\|_X = \|u\|_{L^\infty} + |u|_X$.

Proposition 2 *Let $u \in L^\infty(\mathbb{R}^d)$. We have the following characterisation:*

$$u \in \text{Lip}(\mathbb{R}^d) \qquad \text{if and only if} \qquad \sup_j \|\nabla S_j u\|_{L^\infty} < +\infty, \tag{16}$$

$$u \in \text{Zyg}(\mathbb{R}^d) \qquad \text{if and only if} \qquad \sup_j 2^j \|\Delta_j u\|_{L^\infty} < +\infty, \tag{17}$$

$$u \in \text{LL}(\mathbb{R}^d) \qquad \text{if and only if} \qquad \sup_j \frac{\|\nabla S_j u\|_{L^\infty}}{j} < +\infty. \tag{18}$$

Proof The proof of (17) and (18) can be found in [3, Prop. 2.3.6] and [5, Prop. 3.3] respectively. We sketch the proof of (16), for the reader's convenience. Suppose $u \in \text{Lip}(\mathbb{R}^d)$. We have

$$\begin{aligned}
D_j(S_k u(x)) &= D_j(\mathscr{F}^{-1}(\chi(2^{-k}\xi)\hat{u}(\xi)))(x) \\
&= \mathscr{F}^{-1}(\xi_j \chi(2^{-k}\xi)\hat{u}(\xi))(x) \\
&= 2^k \mathscr{F}^{-1}(2^{-k}\xi_j \chi(2^{-k}\xi)\hat{u}(\xi))(x) \\
&= 2^k \int_{\mathbb{R}^d} \theta_j(2^k y) u(x-y) 2^{kd} dy
\end{aligned}$$

where $\theta_j(y) = \mathscr{F}^{-1}(\xi_j \chi(\xi))(y)$. From the fact that $\int_{\mathbb{R}^d} \theta_j(y)\,dy = 0$ we deduce that

$$|D_j(S_k u(x))| \leq 2^k |\int_{\mathbb{R}^d} \theta_j(2^k y)(u(x-y) - u(x))\, 2^{kd} dy|$$

$$\leq |u|_{\text{Lip}} \int_{\mathbb{R}^d} |\theta_j(z)||z|\,dz,$$

hence $\sup_j \|\nabla(S_j u)\|_{L^\infty} < C\,|u|_{\text{Lip}}$.

Conversely, let the second statement in (16) hold. Remarking that

$$D_j(\Delta_k u(x)) = \mathscr{F}^{-1}(\xi_j \varphi(2^{-k}\xi)\hat{u}(\xi))(x) = \mathscr{F}^{-1}(\xi_j(\chi(2^{-k}\xi) - \chi(2^{-k+1}\xi))\hat{u}(\xi))(x),$$

and, by Bernstein's inequalities,

$$|\Delta_k u(x)| \leq C2^{-k+1}(\|\nabla(S_k u)\|_{L^\infty} + \|\nabla(S_{k-1}u)\|_{L^\infty}),$$

we deduce that, for a new constant $C > 0$,

$$\|\Delta_k u\|_{L^\infty} \leq C2^{-k} \sup_j \|\nabla S_j u\|_{L^\infty}$$

for all $k \geq 0$. Then

$$|u(x+y) - u(x)| \leq |S_k u(x+y) - S_k u(x)| + |\sum_{h>k}(\Delta_h u(x+y) - \Delta_h u(x))|$$

$$\leq \|\nabla(S_k u)\|_{L^\infty}|y| + 2\sum_{h>k}\|\Delta_h u\|_{L^\infty}$$

$$\leq C \sup_j \|\nabla(S_j u)\|_{L^\infty} (|y| + 2^{-k}).$$

The conclusion follows from choosing k in such a way that $2^{-k} \leq |y|$. $\qquad\square$

Notice that, going along the lines of the previous proof, we have actually shown that there exists $C_d > 1$, depending only on d, such that, if $u \in \text{Lip}(\mathbb{R}^d)$ then

$$\frac{1}{C_d}|u|_{\text{Lip}} \leq \|\nabla S_j u\|_{L^\infty} \leq C_d\,|u|_{\text{Lip}}.$$

Proposition 3 ([3, Prop. 2.3.7])

$$\text{Lip}(\mathbb{R}^d) \subseteq \text{Zyg}(\mathbb{R}^d) \subseteq \text{LL}(\mathbb{R}^d).$$

In order to perform computations, we will need to smooth out our coefficients, because of their low regularity. To this end, let us fix an even function $\rho \in C_0^\infty(\mathbb{R})$

such that $0 \leq \rho \leq 1$, Supp $\rho \subseteq [-1, 1]$ and $\int_{\mathbb{R}} \rho(t)\,dt = 1$, and define $\rho_\varepsilon(t) = \frac{1}{\varepsilon}\rho(\frac{t}{\varepsilon})$. The following result holds true.

Proposition 4 ([9, Prop. 3.5]) *Let $u \in \mathrm{Zyg}(\mathbb{R})$. There exists $C > 0$ such that,*

$$|u_\varepsilon(t) - u(t)| \leq C|u|_{\mathrm{Zyg}}\,\varepsilon, \tag{19}$$

$$|u'_\varepsilon(t)| \leq C|u|_{\mathrm{Zyg}}\,(1 + \log\frac{1}{\varepsilon}), \tag{20}$$

$$|u''_\varepsilon(t)| \leq C|u|_{\mathrm{Zyg}}\,\frac{1}{\varepsilon}, \tag{21}$$

where, for $0 < \varepsilon \leq 1$,

$$u_\varepsilon(t) = (\rho_\varepsilon * u)(t) = \int_{\mathbb{R}} \rho_\varepsilon(t - s)u(s)\,ds. \tag{22}$$

3.3   Paradifferential Calculus with Parameters

Let us sketch here the paradifferential calculus depending on a parameter $\gamma \geq 1$. The interested reader can look at [15, Appendix B] (see also [13] and [6]).

Let $\gamma \geq 1$ and consider $\psi_\gamma \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with the following properties

(i) there exist $0 < \varepsilon_1 < \varepsilon_2 < 1$ such that

$$\psi_\gamma(\eta, \xi) = \begin{cases} 1 & \text{for} \quad |\eta| \leq \varepsilon_1(\gamma + |\xi|), \\ 0 & \text{for} \quad |\eta| \geq \varepsilon_2(\gamma + |\xi|); \end{cases} \tag{23}$$

(ii) for all $(\beta, \alpha) \in \mathbb{N}^d \times \mathbb{N}^d$, there exists $C_{\beta,\alpha} \geq 0$ such that

$$|\partial_\eta^\beta \partial_\xi^\alpha \psi_\gamma(\eta, \xi)| \leq C_{\beta,\alpha}(\gamma + |\xi|)^{-|\alpha|-|\beta|}. \tag{24}$$

The model for such a function will be

$$\psi_\gamma(\eta, \xi) = \chi(\frac{\eta}{2^\mu})\chi(\frac{\xi}{2^{\mu+3}}) + \sum_{k=\mu+1}^{+\infty} \chi(\frac{\eta}{2^k})\varphi(\frac{\xi}{2^{k+3}}), \tag{25}$$

where χ and φ are defined in (14) and μ is the integer part of $\log_2 \gamma$. With this setting, we have that the constants $\varepsilon_1, \varepsilon_2$ and $C_{\beta,\alpha}$ in (23) and (24) do not depend on γ.

To fix ideas, from now on we take ψ_γ as given in (25). Define now

$$G^{\psi_\gamma}(x, \xi) = (\mathscr{F}_\eta^{-1}\psi_\gamma)(x, \xi),$$

where $\mathscr{F}_\eta^{-1}\psi_\gamma$ is the inverse of the Fourier transform of ψ_γ with respect to the η variable.

Proposition 5 ([14, Lemma 5.1.7]) *For all $(\beta, \alpha) \in \mathbb{N}^d \times \mathbb{N}^d$, there exists $C_{\beta,\alpha}$, not depending on γ, such that*

$$\|\partial_x^\beta \partial_\xi^\alpha G^{\psi_\gamma}(\cdot, \xi)\|_{L^1(\mathbb{R}_x^d)} \le C_{\beta,\alpha}(\gamma + |\xi|)^{-|\alpha|+|\beta|}, \tag{26}$$

$$\| \, |\cdot| \, \partial_x^\beta \partial_\xi^\alpha G^{\psi_\gamma}(\cdot, \xi)\|_{L^1(\mathbb{R}_x^d)} \le C_{\beta,\alpha}(\gamma + |\xi|)^{-|\alpha|+|\beta|-1}. \tag{27}$$

Next, let $a \in L^\infty$. We associate to a the classical pseudodifferential symbol

$$\sigma_{a,\gamma}(x, \xi) = (\psi_\gamma(D_x, \xi)a)(x, \xi) = (G^{\psi_\gamma}(\cdot, \xi) * a)(x), \tag{28}$$

and define the paradifferential operator T_a^γ associated to a as the classical pseudodifferential operator associated to $\sigma_{a,\gamma}$ (from now on, to avoid cumbersome notations, we will write σ_a), i.e.

$$T_a^\gamma u(x) = \sigma_a(D_x)u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}_\xi^d} \sigma_a(x, \xi)\hat{u}(\xi) \, d\xi.$$

Remark that T_a^1 is the usual paraproduct operator

$$T_a^1 u = \sum_{k=0}^{+\infty} S_k a \Delta_{k+3}u,$$

while, in the general case,

$$T_a^\gamma u = S_{\mu-1}a S_{\mu+2}u + \sum_{k=\mu}^{+\infty} S_k a \Delta_{k+3}u. \tag{29}$$

with μ equal to the integer part of $\log_2 \gamma$.

In the following it will be useful to deal with Sobolev spaces depending on the parameter $\gamma \ge 1$.

Definition 2 Let $\gamma \ge 1$ and $s \in \mathbb{R}$. We denote by $H_\gamma^s(\mathbb{R}^d)$ the set of tempered distributions u such that

$$\|u\|_{H_\gamma^s}^2 = \int_{\mathbb{R}_\xi^d} (\gamma^2 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi < +\infty.$$

Let us remark that $H_\gamma^s = H^s$ and there exists $C_\gamma \geq 1$ such that, for all $u \in H^s$,

$$\frac{1}{C_\gamma} \|u\|_{H^s}^2 \leq \|u\|_{H_\gamma^s}^2 \leq C_\gamma \|u\|_{H^s}^2.$$

3.4 Low Regularity Symbols and Calculus

As in [8] and [9], it is important to deal with paradifferential operators having symbols with limited regularity in time and space.

Definition 3 A symbol of order m is a function $a(t, x, \xi, \gamma)$ which is locally bounded on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times [1, +\infty[$, of class C^∞ with respect to ξ such that, for all $\alpha \in \mathbb{N}^n$, there exists $C_\alpha > 0$ such that, for all (t, x, ξ, γ),

$$|\partial_\xi^\alpha a(t, x, \xi, \gamma)| \leq C_\alpha(\gamma + |\xi|)^{m-|\alpha|}. \tag{30}$$

We take now a symbol a of order $m \geq 0$, Zygmund-continuos with respect to t uniformly with respect to x and Lipschitz-continuous with respect to x uniformly with respect to t. We smooth out a with respect to time as done in (22), and call a_ε the smoothed symbol. We consider the classical symbol σ_{a_ε} obtained from a_ε via (28). In what follows, the variable t has to be thought of as a parameter.

Proposition 6 *Under the previous hypotheses, one has:*

$$|\partial_\xi^\alpha \sigma_{a_\varepsilon}(t, x, \xi, \gamma)| \qquad \leq C_\alpha(\gamma + |\xi|)^{m-|\alpha|},$$

$$|\partial_x^\beta \partial_\xi^\alpha \sigma_{a_\varepsilon}(t, x, \xi, \gamma)| \quad \leq C_{\beta,\alpha}(\gamma + |\xi|)^{m-|\alpha|+|\beta|-1},$$

$$|\partial_\xi^\alpha \sigma_{\partial_t a_\varepsilon}(t, x, \xi, \gamma)| \qquad \leq C_\alpha(\gamma + |\xi|)^{m-|\alpha|} \log(1 + \frac{1}{\varepsilon}),$$

$$|\partial_x^\beta \partial_\xi^\alpha \sigma_{\partial_t a_\varepsilon}(t, x, \xi, \gamma)| \leq C_{\beta,\alpha}(\gamma + |\xi|)^{m-|\alpha|+|\beta|-1} \frac{1}{\varepsilon},$$

$$|\partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}(t, x, \xi, \gamma)| \qquad \leq C_\alpha(\gamma + |\xi|)^{m-|\alpha|} \frac{1}{\varepsilon},$$

$$|\partial_x^\beta \partial_\xi^\alpha \sigma_{\partial_t^2 a_\varepsilon}(t, x, \xi, \gamma)| \leq C_{\beta,\alpha}(\gamma + |\xi|)^{m-|\alpha|+|\beta|-1} \frac{1}{\varepsilon^2},$$

where $|\beta| \geq 1$ and all the constants C_α and $C_{\beta,\alpha}$ do not depend on γ.

Proof We have

$$\sigma_{a_\varepsilon}(t, x, \xi, \gamma) = (G^{\psi_\gamma}(\cdot, \xi) * a_\varepsilon(t, \cdot, \xi, \gamma))(x),$$

so that the first inequality follows from (26) and (30).

Next, we remark that

$$\int \partial_{x_j} G^{\psi_\gamma}(x,\xi)\, dx = \int \mathcal{F}_\eta^{-1}(\eta_j \psi_\gamma(\eta,\xi))(z)\, dz = (\eta_j \psi(\eta,\xi))_{|\eta=0} = 0. \tag{31}$$

Consequently, using also (27),

$$|\partial_{x_j} \sigma_{a_\varepsilon}(t,x,\xi,\gamma)| = |\int \partial_{y_j} G^{\psi_\gamma}(y,\xi)(a_\varepsilon(t,x-y,\xi,\gamma) - a_\varepsilon(t,x,\xi,\gamma))\, dy|,$$

$$\leq C \int |\partial_{y_j} G^{\psi_\gamma}(y,\xi)|\, |y|\, dy\, (\gamma + |\xi|)^m,$$

$$\leq C(\gamma + |\xi|)^m.$$

The other cases of the second inequality can be proved similarly.

The third inequality is again a consequence of (26), keeping in mind (20). It is in fact possible to prove that

$$|\partial_\xi^\alpha \partial_t a_\varepsilon(t,x,\xi,\gamma)| \leq C_\alpha(1 + \log \frac{1}{\varepsilon})(\gamma + |\xi|)^{m-|\alpha|}.$$

Next, considering again (31), we have

$$\partial_{x_j} \sigma_{\partial_t a_\varepsilon}(t,x,\xi,\gamma)$$

$$= \int_{\mathbb{R}_y^n} \partial_{y_j} G^{\psi_\gamma}(y,\xi)(\partial_t a_\varepsilon(t,x-y,\xi,\gamma) - \partial_t a_\varepsilon(t,x,\xi,\gamma))\, dy,$$

$$\leq \int_{\mathbb{R}_y^n} \partial_{y_j} G^{\psi_\gamma}(y,\xi) \int_{\mathbb{R}_s} \frac{1}{\varepsilon^2} \rho'(\frac{t-s}{\varepsilon})(a(s,x-y,\xi,\gamma) - a(s,x,\xi,\gamma))\, ds\, dy$$

$$\leq \int_{\mathbb{R}_s} \frac{1}{\varepsilon^2} \rho'(\frac{t-s}{\varepsilon}) \int_{\mathbb{R}_y^n} \partial_{y_j} G^{\psi_\gamma}(y,\xi)(a(s,x-y,\xi,\gamma) - a(s,x,\xi,\gamma))\, dy\, ds.$$

so that the fourth inequality easily follows.

The last two inequalities are obtained in similar way, using also (21).   □

To end this section it is worthy to recall some results on symbolic calculus. Again details can be found in [8, 9] and [15, Appendix B].

Proposition 7 ([8, Prop. 3.19])

(i) *Let a be a symbol of order m (see Def. 3). Suppose that a is L^∞ in the x variable. If we set*

$$T_a u(x) = \sigma_a(D_x)u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}_\xi^d} \sigma_a(x,\xi,\gamma)\hat{u}(\xi)\, d\xi,$$

then T_a maps H_γ^s into H_γ^{s-m} continuously.

(ii) *Let a and b be two symbols of order m and m′ respectively. Suppose that a and b are* Lip *in the x variable. Then*

$$T_a \circ T_b = T_{ab} + R,$$

and R maps H_γ^s into $H_\gamma^{s-m-m'+1}$ continuously.

(iii) *Let a be a symbol of order m which is* Lip *in the x variable. Then, denoting by T_a^* the L^2-adjoint operator of T_a,*

$$T_a^* = T_{\overline{a}} + R,$$

and R maps H_γ^s into H_γ^{s-m+1} continuously.

(iv) *Let a be a symbol of order m which is* Lip *in the x variable. Suppose*

$$\operatorname{Re} a(x, \xi, \gamma) \geq \lambda_0 (\gamma + |\xi|)^m.$$

with $\lambda_0 > 0$. Then there exists $\gamma_0 \geq 1$, depending only on $\|a\|_{\mathrm{Lip}}$ and λ_0, such that, for all $\gamma \geq \gamma_0$ and for all $u \in H^\infty$,

$$\operatorname{Re} (T_a u, u)_{L^2} \geq \frac{\lambda_0}{2} \|u\|_{H_\gamma^{m/2}}^2.$$

4   Proof of Theorem 1

Also for the proof of the main result, we will closely follow the strategy implemented in [8] and [9].

4.1   Approximate Energy

First of all we regularize the coefficients a_{jk} with respect to t via (22) and we obtain $a_{jk,\varepsilon}$. We consider the 0-th order symbol

$$\alpha_\varepsilon(t, x, \xi, \gamma) = (\gamma^2 + |\xi|^2)^{-\frac{1}{2}} (\gamma^2 + \sum_{j,k} a_{jk,\varepsilon}(t, x)\xi_j \xi_k)^{\frac{1}{2}}.$$

We fix

$$\varepsilon = 2^{-\nu},$$

and we write α_ν and $a_{jk,\,\nu}$ instead of $\alpha_{2-\nu}$ and $a_{jk,\,2-\nu}$ respectively. From Prop. 7, point *iv)*, we have that there exists $\gamma \geq 1$ such that, for all $w \in H^\infty$,

$$\|T^\gamma_{\alpha_\nu^{-1/2}} w\|_{L^2} \geq \frac{\lambda_0}{2} \|w\|_{L^2} \qquad \text{and} \qquad \|T^\gamma_{\alpha_\nu^{1/2}(\gamma^2+|\xi|^2)^{1/2}} w\|_{L^2} \geq \frac{\lambda_0}{2} \|w\|_{H^1_\gamma},$$

where λ_0 has been defined in (6). We remark that γ depends only on λ_0 and $sup_{j,k}\|a_{jk}\|_{\mathrm{Lip}}$, in particular γ does not depend on ν. We fix such a γ (this means also that μ is fixed in (29)) and from now on we will omit to write it when denoting the operator T and the Sobolev spaces H^s.

We consider $u \in C^2([0, T], H^\infty)$. We have

$$\partial_t^2 u = \sum_{j,k} \partial_j(a_{jk}(t, x)\partial_k u) + Lu = \sum_{j,k} \partial_j(T_{a_{jk}} \partial_k u) + \tilde{L}u,$$

where

$$\tilde{L}u = Lu + \sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u).$$

We apply the operator Δ_ν and we obtain

$$\partial_t^2 u_\nu = \sum_{j,k} \partial_j(T_{a_{jk}} \partial_k u_\nu) + \sum_{j,k} \partial_j([\Delta_\nu, T_{a_{jk}}]\partial_k u) + (\tilde{L}u)_\nu,$$

where $u_\nu = \Delta_\nu u$, $(\tilde{L}u)_\nu = \Delta_\nu(\tilde{L}u)$ and $[\Delta_\nu, T_{a_{jk}}]$ is the commutator between the localization operator Δ_ν and the paramultiplication operator $T_{a_{jk}}$.

We set

$$v_\nu(t, x) = T_{\alpha_\nu^{-1/2}} \partial_t u_\nu - T_{\partial_t(\alpha_\nu^{-1/2})} u_\nu,$$

$$w_\nu(t, x) = T_{\alpha_\nu^{1/2}(\gamma^2+|\xi|^2)^{1/2}} u_\nu,$$

$$z_\nu(t, x) = u_\nu,$$

and we define the approximate energy associated to the ν-th component as

$$e_\nu(t) = \|v_\nu(t, \cdot)\|_{L^2}^2 + \|w_\nu(t, \cdot)\|_{L^2}^2 + \|z_\nu(t, \cdot)\|_{L^2}^2.$$

We fix $\theta \in [0,\ 1[$ and we define the total energy

$$E_\theta(t) = \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t).$$

We remark that, as a consequence of Bernstein's inequalities,

$$\|w_\nu\|_{L^2}^2 \sim \|\nabla u_\nu\|_{L^2}^2 \sim 2^{2\nu}\|u_\nu\|_{L^2}^2.$$

Moreover, from (20) and, again, Bernstein's inequalities,

$$\|T_{\partial_t(\alpha_\nu^{-\frac{1}{2}})} u_\nu\|_{L^2} \le C(\nu+1)\|u_\nu\|_{L^2} \le C'\|w_\nu\|_{L^2},$$

so that

$$
\begin{aligned}
\|\partial_t u_\nu\|_{L^2} &\le C\|T_{\alpha_\nu^{-1/2}}u_\nu\|_{L^2} \\
&\le C(\|v_\nu\|_{L^2} + \|T_{\partial_t(\alpha_\nu^{-1/2})}u_\nu\|_{L^2}) \\
&\le C(e_\nu(t))^{\frac{1}{2}}.
\end{aligned}
\tag{32}
$$

We deduce that there exist constants C_θ and C'_θ, depending only on θ, such that

$$(E_\theta(0))^{\frac{1}{2}} \le C_\theta(\|\partial_t u(0)\|_{H^{-\theta}} + \|u(0)\|_{H^{1-\theta}}),$$

$$(E_\theta(t))^{\frac{1}{2}} \ge C'_\theta(\|\partial_t u(t)\|_{H^{-\theta}} + \|u(t)\|_{H^{1-\theta}}).$$

4.2   Time Derivative of the Approximate Energy

We want to estimate the time derivative of e_ν.

Since

$$\partial_t v_\nu = T_{\alpha_\nu^{-1/2}}\partial_t^2 u_\nu - T_{\partial_t^2(\alpha_\nu^{-1/2})}u_\nu,$$

we deduce

$$
\begin{aligned}
&\frac{d}{dt}\|v_\nu(t)\|_{L^2}^2 \\
&= 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}}\partial_t^2 u_\nu\big)_{L^2} - 2\,\mathrm{Re}\big(v_\nu, T_{\partial_t^2(\alpha_\nu^{-1/2})}u_\nu\big)_{L^2} \\
&= -2\,\mathrm{Re}\big(v_\nu, T_{\partial_t^2(\alpha_\nu^{-1/2})}u_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}}\partial_j(T_{a_{jk}}\partial_k u_\nu)\big)_{L^2} \\
&\quad + 2\,\mathrm{Re}\big(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}}\partial_j([\Delta_\nu, T_{a_{jk}}]\partial_k u)\big)_{L^2} + 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}}(\tilde{L}u)_\nu\big)_{L^2}.
\end{aligned}
$$

We have

$$\left| 2\operatorname{Re}\bigl(v_\nu, T_{\alpha_\nu^{-1/2}}(\tilde L u)_\nu\bigr)_{L^2}\right| \le C(e_\nu)^{\frac{1}{2}}\,\|(\tilde L u)_\nu\|_{L^2},$$

and, from the fifth inequality in Prop. 6,

$$\left| 2\operatorname{Re}\bigl(v_\nu, T_{\partial_t^2(\alpha_\nu^{-1/2})}u_\nu\bigr)_{L^2}\right| \le C\,\|v_\nu\|_{L^2}\,2^\nu\|u_\nu\|_{L^2} \le C\,e_\nu(t).$$

Therefore, we obtain

$$
\begin{aligned}
\frac{d}{dt}\|v_\nu(t)\|_{L^2}^2 = {}& 2\operatorname{Re}\Bigl(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}}\partial_j(T_{a_{jk}}\partial_k u_\nu)\Bigr)_{L^2} \\
& + 2\operatorname{Re}\Bigl(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}}\partial_j([\Delta_\nu,\, T_{a_{jk}}]\partial_k u)\Bigr)_{L^2} \qquad (33)\\
& + 2\operatorname{Re}\bigl(v_\nu, T_{\alpha_\nu^{-1/2}}(\tilde L u)_\nu\bigr)_{L^2} + Q_1,
\end{aligned}
$$

with $|Q_1| \le Ce_\nu(t)$.

Next

$$\partial_t w_\nu = T_{\partial_t(\alpha_\nu^{1/2})(\gamma^2+|\xi|^2)^{1/2}}u_\nu + T_{\alpha_\nu^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\partial_t u_\nu,$$

so that

$$
\begin{aligned}
& \frac{d}{dt}\|w_\nu(t)\|_{L^2}^2 \\
& = 2\operatorname{Re}\bigl(T_{\partial_t(\alpha_\nu^{1/2})(\gamma^2+|\xi|^2)^{1/2}}u_\nu, w_\nu\bigr)_{L^2} + 2\operatorname{Re}\bigl(T_{\alpha_\nu^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\partial_t u_\nu, w_\nu\bigr)_{L^2} \\
& = 2\operatorname{Re}\bigl(T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}}T_{-\partial_t(\alpha_\nu^{-1/2})}u_\nu, w_\nu\bigr)_{L^2} + 2\operatorname{Re}\bigl(R_1 u_\nu, w_\nu\bigr)_{L^2} \\
& \quad + 2\operatorname{Re}\bigl(T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}}T_{\alpha_\nu^{-1/2}}\partial_t u_\nu, w_\nu\bigr)_{L^2} + 2\operatorname{Re}\bigl(R_2 u_\nu, w_\nu\bigr)_{L^2} \\
& = 2\operatorname{Re}\bigl(v_\nu, T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}}w_\nu\bigr)_{L^2} + 2\operatorname{Re}\bigl(v_\nu, R_3 w_\nu\bigr)_{L^2} \\
& \quad + 2\operatorname{Re}\bigl(R_1 u_\nu, w_\nu\bigr)_{L^2} + 2\operatorname{Re}\bigl(R_2 u_\nu, w_\nu\bigr)_{L^2} \\
& = 2\operatorname{Re}\bigl(v_\nu, T_{\alpha_\nu^{-1/2}}T_{\alpha_\nu^{3/2}(\gamma^2+|\xi|^2)^{1/2}}w_\nu\bigr)_{L^2} + 2\operatorname{Re}\bigl(v_\nu, R_4 w_\nu\bigr)_{L^2} \\
& \quad + 2\operatorname{Re}\bigl(v_\nu, R_3 w_\nu\bigr)_{L^2} + 2\operatorname{Re}\bigl(R_1 u_\nu, w_\nu\bigr)_{L^2} + 2\operatorname{Re}\bigl(R_2 u_\nu, w_\nu\bigr)_{L^2} \\
& = 2\operatorname{Re}\bigl(v_\nu, T_{\alpha_\nu^{-1/2}}T_{\alpha_\nu^2(\gamma^2+|\xi|^2)}u_\nu\bigr)_{L^2} \\
& \quad + 2\operatorname{Re}\bigl(v_\nu, T_{\alpha_\nu^{-1/2}}R_5 u_\nu\bigr)_{L^2} + 2\operatorname{Re}\bigl(v_\nu, R_4 w_\nu\bigr)_{L^2} \\
& \quad + 2\operatorname{Re}\bigl(v_\nu, R_3 w_\nu\bigr)_{L^2} + 2\operatorname{Re}\bigl(R_1 u_\nu, w_\nu\bigr)_{L^2} + 2\operatorname{Re}\bigl(R_2 u_\nu, w_\nu\bigr)_{L^2}.
\end{aligned}
$$

It is a straightforward computation, from the results of symbolic calculus recalled in Prop. 7, to verify that all the operators R_1, R_2, R_3, R_4 and R_5 are 0-th order operators. Consequently,

$$\frac{d}{dt}\|w_\nu(t)\|_{L^2}^2 = 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}}T_{\alpha_\nu^2(\gamma^2+|\xi|^2)}u_\nu\big)_{L^2} + Q_2, \tag{34}$$

with $|Q_2| \le Ce_\nu(t)$.

Finally, from (32),

$$\frac{d}{dt}\|z_\nu(t)\|_{L^2}^2 \le |2\,\mathrm{Re}\big(u_\nu, \partial_t u_\nu\big)_{L^2}| \le Ce_\nu(t). \tag{35}$$

Now we pair the first term in right hand side of (33) with the first term in right hand side of (34). We obtain

$$|2\,\mathrm{Re}\big(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}}\partial_j(T_{a_{jk}}\partial_k u_\nu)\big)_{L^2} + 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}}T_{\alpha_\nu^2(\gamma^2+|\xi|^2)}u_\nu\big)_{L^2}|$$

$$\le C\,\|v_\nu\|_{L^2}\,\|\zeta_\nu\|_{L^2},$$

where

$$\begin{aligned}
\zeta_\nu &= T_{\alpha_\nu^2(\gamma^2+|\xi|^2)}u_\nu + \sum_{j,k}\partial_j(T_{a_{jk}}\partial_k u_\nu) \\
&= T_{\gamma^2+\sum_{j,k} a_{jk,\nu}\xi_j\xi_k}u_\nu + \sum_{j,k}\partial_j(T_{a_{jk}}\partial_k u_\nu) \\
&= T_{\gamma^2}u_\nu + \sum_{j,k}(T_{a_{jk,\nu}\xi_j\xi_k}u_\nu + T_{\partial_j a_{jk}}\partial_k u_\nu - T_{a_{jk}\xi_j\xi_k}u_\nu).
\end{aligned}$$

We have

$$\|\sum_{j,k} T_{\partial_j a_{jk}}\partial_k u_\nu\|_{L^2} \le C\sup_{j,k}\|a_{jk}\|_{\mathrm{Lip}}\|\nabla u_\nu\|_{L^2} \le C(e_\nu(t))^{\frac{1}{2}},$$

and, from Bernstein's inequalities and (19),

$$\|\sum_{j,k} T_{(a_{jk,\nu}-a_{jk})\xi_j\xi_k}u_\nu\|_{L^2} \le C\sup_{j,k}\|a_{jk}\|_{\mathrm{Lip}}\,2^{-\nu}\,\|\nabla^2 u_\nu\|_{L^2} \le C(e_\nu(t))^{\frac{1}{2}}.$$

From this we deduce

$$\|\zeta_\nu\|_{L^2} \le C(e_\nu(t))^{\frac{1}{2}}.$$

Summing up, from (33), (34) and (32) we get

$$
\begin{aligned}
\frac{d}{dt} e_v(t) \leq\ & C_1 e_v(t) + C_2 (e_v(t))^{\frac{1}{2}} \, \|(\tilde{L}u)_v\|_{L^2} \\
& + |2\operatorname{Re}\big(v_v, \sum_{j,k} T_{\alpha_v^{-1/2}} \partial_j ([\Delta_v,\, T_{a_{jk}}] \partial_k u)\big)_{L^2}|.
\end{aligned}
\tag{36}
$$

4.3 Commutator Estimate

We want to estimate

$$
|\sum_{j,k} 2\operatorname{Re}\big(v_v,\, T_{\alpha_v^{-1/2}} \partial_j ([\Delta_v,\, T_{a_{jk}}] \partial_k u)\big)_{L^2}|.
$$

We remark that

$$
\begin{aligned}
[\Delta_v,\, T_{a_{jk}}]w &= \Delta_v (S_{\mu-1} a_{jk} \, S_{\mu+2} w) + \Delta_v \Big(\sum_{h=\mu}^{+\infty} S_h a_{jk} \, \Delta_{h+3} w\Big) \\
&\quad - S_{\mu-1} a_{jk} \, S_{\mu+2} (\Delta_v w) - \sum_{h=\mu}^{+\infty} S_h a_{jk} \, \Delta_{h+3} (\Delta_v w) \\
&= \Delta_v (S_{\mu-1} a_{jk} \, S_{\mu+2} w) - S_{\mu-1} a_{jk} \, \Delta_v (S_{\mu+2} w) \\
&\quad + \sum_{h=\mu}^{+\infty} \Delta_v (S_h a_{jk} \, \Delta_{h+3} w) - \sum_{h=\mu}^{+\infty} S_h a_{jk} \, \Delta_v (\Delta_{h+3} w) \\
&= [\Delta_v,\, S_{\mu-1} a_{jk}] S_{\mu+2} w + \sum_{h=\mu}^{+\infty} [\Delta_v,\, S_h a_{jk}] \Delta_{h+3} w,
\end{aligned}
$$

where we recall that μ is a fixed constant (depending on γ, which has been chosen at the beginning of Sect. 4.1). Hence we have

$$
\begin{aligned}
\partial_j & ([\Delta_v T_{a_{jk}}] \partial_k u) \\
&= \partial_j ([\Delta_v,\, S_{\mu-1} a_{jk}] \partial_k (S_{\mu+2} u)) + \partial_j \Big(\sum_{h=\mu}^{+\infty} [\Delta_v,\, S_h a_{jk}] \partial_k (\Delta_{h+3} u)\Big).
\end{aligned}
$$

Consider first

$$
\partial_j ([\Delta_v,\, S_{\mu-1} a_{jk}] \partial_k (S_{\mu+2} u)).
$$

The support of the Fourier transform of $[\Delta_\nu, \, S_{\mu-1}a_{jk}] \, \partial_k(S_{\mu+2}u)$ is contained in $\{|\xi| \le 2^{\mu+4}\}$ and $[\Delta_\nu, \, S_{\mu-1}a_{jk}] \, \partial_k(S_{\mu+2}u)$ is identically 0 if $\nu \ge \mu + 5$. From Bernstein's inequalities and [4, Th. 35] we deduce that

$$\|\partial_j([\Delta_\nu, \, S_{\mu-1}a_{jk}] \, \partial_k(S_{\mu+2}u))\|_{L^2} \le C \, 2^\mu \sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}} \, \|S_{\mu+2}u\|_{L^2}.$$

We have

$$\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} | \sum_{j,k} 2 \, \mathrm{Re}(v_\nu, \, T_{\alpha_\nu^{-1/2}}\partial_j([\Delta_\nu, \, S_{\mu-1}a_{jk}] \, \partial_k(S_{\mu+2}u)))_{L^2}|$$

$$\le C \, 2^\mu \sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}} \sum_{\nu=0}^{\mu+4} 2^{-2\nu\theta} \|v_\nu\|_{L^2} (\sum_{h=0}^{\mu+2} \|u_h\|_{L^2})$$

$$\le C \, 2^{\mu+(\mu+4)\theta} \sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}} \sum_{\nu=0}^{\mu+4} 2^{-\nu\theta} \|v_\nu\|_{L^2} \sum_{h=0}^{\mu+4} 2^{-h\theta} \|u_h\|_{L^2}$$

$$\le C \sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}} \sum_{h=0}^{\mu+4} 2^{-2\nu\theta} e_\nu(t).$$

Consider then

$$\partial_j(\sum_{h=\mu}^{+\infty} [\Delta_\nu, \, S_h a_{jk}] \, \partial_k(\Delta_{h+3}u)).$$

Looking at the support of the Fourier transform, it is possible to see that

$$[\Delta_\nu, \, S_h a_{jk}] \, \partial_k(\Delta_{h+3}u)$$

is identically 0 if $|h + 3 - \nu| \ge 3$. As a consequence, the sum over h is reduced to at most 5 terms: $\partial_j([\Delta_\nu, \, S_{\nu-5}a_{jk}] \, \partial_k(\Delta_{\nu-2}u)), \dots, \partial_j([\Delta_\nu, \, S_{\nu-1}a_{jk}] \, \partial_k(\Delta_{\nu+2}u))$. Each of these terms has the support of the Fourier transform contained in the ball $\{|\xi| \le 2^{\nu+4}\}$.

We consider the term $\partial_j([\Delta_\nu, \, S_{\nu-3}a_{jk}] \, \partial_k(\Delta_\nu u))$: for the other terms the estimate will be similar. Again by Bernstein's inequalities and [4, Th. 35] we infer

$$\|\partial_j([\Delta_\nu, \, S_{\nu-3}a_{jk}] \, \partial_k(\Delta_\nu u))\|_{L^2} \le C \, 2^\nu \sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}} \, \|\Delta_\nu u\|_{L^2},$$

and then

$$|\sum_{j,k} 2 \, \mathrm{Re}(v_\nu, \, T_{\alpha_\nu^{-1/2}}\partial_j(\sum_{h=\mu}^{+\infty} [\Delta_\nu, \, S_h a_{jk}] \, \partial_k(\Delta_{h+3}u)))_{L^2}|$$

$$\le C \sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}} (e_{\nu-2}(t) + e_{\nu-1}(t) + e_\nu(t) + e_{\nu+1}(t) + e_{\nu+2}(t)).$$

Thus we have

$$\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} | \sum_{j,k} 2 \operatorname{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}} \partial_j \big(\sum_{h=\mu}^{+\infty} [\Delta_\nu,\ S_h a_{jk}]\, \partial_k(\Delta_{h+3}u))\big)\big)_{L^2}|$$

$$\leq C \sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}} \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t).$$

As a conclusion

$$\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} | \sum_{j,k} 2 \operatorname{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}} \partial_j ([\Delta_\nu,\ T_{a_{jk}}]\partial_k u)\big)_{L^2}| \leq C_3 \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t), \qquad (37)$$

where C_3 depends on γ, θ and $\sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}}$.

4.4   Final Estimate

From (36) and (37) we obtain

$$\frac{d}{dt} E_\theta(t) \leq (C_1 + C_3) \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t) + C_2 \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} (e_\nu(t))^{\frac{1}{2}} \|(\tilde{L}u(t))_\nu\|_{L^2}$$

$$\leq (C_1 + C_3) \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t) + C_2 \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} (e_\nu(t))^{\frac{1}{2}} \|(Lu(t))_\nu\|_{L^2}$$

$$+ C_2 \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} (e_\nu(t))^{\frac{1}{2}} \|\big(\sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u)\big)_\nu\|_{L^2}.$$

We have

$$\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} (e_\nu(t))^{\frac{1}{2}} \|\big(\sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u)\big)_\nu\|_{L^2}$$

$$\leq \Big(\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t)\Big)^{\frac{1}{2}} \Big(\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} \|\big(\sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u)\big)_\nu\|_{L^2}^2\Big)^{\frac{1}{2}}.$$

From (15) we deduce

$$\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} \|\big(\sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u)\big)_\nu\|_{L^2}^2 \leq C \, \|\sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u)\|_{H^{-\theta}}^2$$

Now, using [10, Prop. 3.5] in the case $\theta \in {]}0, 1{[}$ and [14, Th. 5.2.8] in the case $\theta = 0$,

$$\| \sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u) \|_{H^{-\theta}}^2 \leq C (\sup_{j,k} \|a_{jk}\|_{\text{Lip}}) \|u(t)\|_{H^{1-\theta}},$$

so that

$$\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} (e_\nu(t))^{\frac{1}{2}} \| \big(\sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u) \big)_\nu \|_{L^2} \leq C_4 E_\theta(t),$$

and finally

$$\frac{d}{dt} E_\theta(t) \leq C(E_\theta(t) + (E_\theta(t))^{\frac{1}{2}} \|Lu(t)\|_{H^{-\theta}}).$$

The energy estimate (9) easily follows from this last inequality and the Grönwall Lemma.

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