Semilinear p-Evolution Equations in Weighted Sobolev Spaces

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To Massimo Cicognani and Michael Reissig in occasion of their 60-th birthday

Abstract We consider the initial value problem for a class of semilinear p-evolution equations with (t, x)-depending coefficients. Under suitable decay conditions for $|x| \to \infty$ on the imaginary part of the coefficients, we prove local in time well posedness of the Cauchy problem in suitable weighted Sobolev spaces.

Keywords p-evolution equations · Semilinear Cauchy problem · Nash-Moser theorem · Weighted Sobolev spaces · Pseudo-differential operators

1 Introduction

In the present paper we deal with the semilinear Cauchy problem

$$\begin{cases} P_u(D)u(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{1}$$

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for the first order p-evolution operator

$$P_u(D)u = P(t, x, u(t, x), D_t, D_x)u := D_t u + a_p(t) D_x^p u + \sum_{j=0}^{p-1} a_j(t, x, u) D_x^j u \quad (2)$$

where $D = \frac{1}{i}\partial$, $p \in \mathbb{N}$, $p \geq 2$, $a_p \in C([0, T], \mathbb{R})$, a_j are for $0 \leq j \leq p-1$ continuous in time functions with values in $C^\infty(\mathbb{R} \times \mathbb{C})$, and moreover the functions $x \to a_j(t, x, w)$ are in $\mathscr{B}^\infty(\mathbb{R})$ (i.e. uniformly bounded together with all their derivatives).

For $p = 2$ our analysis will concern semilinear Schrödinger equations of the form

$$D_t u + D_x^2 u + a_1(t, x, u) D_x u + a_0(t, x, u) = f(t, x).$$

For $p = 3$, the most important model is represented by the Korteweg-de Vries equation describing the propagation of monodimensional waves of small amplitudes in waters of constant depth:

$$\partial_t u = \frac{3}{2}\sqrt{\frac{g}{h}}\partial_x \left(\frac{1}{2}u^2 + \frac{2}{3}\alpha u + \frac{1}{3}\sigma \partial_x^2 u\right),$$

that can be written in the form (1) as

$$D_t u + \frac{1}{2}\sqrt{\frac{g}{h}}\sigma D_x^3 u - \sqrt{\frac{g}{h}}\left(\alpha + \frac{3}{2}u\right) D_x u = 0.$$

Here u represents the wave elevation with respect to the water's surface, g is the gravity constant, h the (constant) level of water, α a fixed small constant and $\sigma = \frac{h^3}{3} - \frac{Th}{\rho g}$, with T the surface tension, ρ the density of the fluid. Assuming the level of water h depending on x, we are led to an operator with space-depending coefficients that can be applied to study the evolution of the wave when the depth of the seabed is variable, cf. [1].

Since a_p is real valued, the principal symbol (in the sense of Petrowski) of P, given by $\tau + a_p(t)\xi^p$, has the real characteristic root $\tau = -a_p(t)\xi^p$; by the Lax-Mizohata theorem, real characteristics are necessary for the existence of a unique solution in Sobolev spaces of the Cauchy problem (1) in a neighborhood of $t = 0$, for any $p \geq 1$. Moreover, whenever the lower order coefficients $a_j(t, x, w) \in \mathbb{C}$ for $0 \leq j \leq p-1$, decay conditions as $|x| \to \infty$ are necessary on the a_j for well-posedness in Sobolev spaces, see [6, 15] respectively for $p = 2$, p arbitrary.

Well-posedness for the Cauchy problem (1), (2) in $H^\infty(\mathbb{R}) = \cap_s H^s(\mathbb{R})$, where $H^s(\mathbb{R})$ is the usual Sobolev space on L^2, has been proved in the paper [1] under suitable decay conditions at infinity for the a_j, $0 \leq j \leq p-1$, relying on the linear results of [5]; in this paper, despite very precise decay assumptions on the coefficients, the authors have no information at all about the behavior at infinity of the solution.

In the last years, we started to study linear p-evolution equations in weighted Sobolev spaces, see [3, 4] and to state a relation between the behavior at infinity of the data and the one of the solution. Here we are interested to extend part of these results to the semilinear case, that is to give decay conditions on the coefficients of $P_u(D)$ that are sufficient for the local in time well-posedness of the Cauchy problem (1) in suitable weighted Sobolev spaces.

Namely, fixed $s_1, s_2 \in \mathbb{R}$, we define $H^{s_1,s_2}(\mathbb{R})$ as the space of all $u \in \mathscr{S}'(\mathbb{R})$ such that $\|u\|_{s_1,s_2} := \|\langle x\rangle^{s_2}\langle D\rangle^{s_1}u\|_{L^2} < \infty$ where we denote by $\langle D\rangle^{s_1}$ the Fourier multiplier with symbol $\langle \xi\rangle^{s_1} := (1+\xi^2)^{s_1/2}$. This space is a Hilbert space endowed with the inner product

$$\langle u, v\rangle_{s_1,s_2} := \langle \langle x\rangle^{s_2}\langle D\rangle^{s_1}u, \langle x\rangle^{s_2}\langle D\rangle^{s_1}v\rangle_{L^2}$$

which induces the norm $\|\cdot\|_{s_1,s_2}$. We have $H^{0,0}(\mathbb{R}) = L^2(\mathbb{R})$ and we shall denote the L^2 norm simply by $\|\cdot\|$. An equivalent norm on $H^{s_1,s_2}(\mathbb{R})$ is given by $|||u|||_{s_1,s_2} := \|\langle D\rangle^{s_1}\langle x\rangle^{s_2}u\|_{L^2}$. Notice that for $s_2 = 0$ we recapture the standard Sobolev spaces and that the obvious inclusions $H^{s_1,s_2}(\mathbb{R}) \subseteq H^{t_1,t_2}(\mathbb{R})$ for every $s_1 \geq t_1,\ s_2 \geq t_2$ hold. We also recall that $H^{s_1,s_2}(\mathbb{R})$ is an algebra with respect to multiplication for $s_1 > 1/2$ and $s_2 \geq 0$, cf. [2, Proposition 2.2]. For every given $s_1 \in \mathbb{R}$ (resp. $s_2 \in \mathbb{R}$) we define

$$H^{s_1,\infty}(\mathbb{R}) := \bigcap_{s_2\in\mathbb{R}} H^{s_1,s_2}(\mathbb{R}), \quad \text{resp.} \quad H^{\infty,s_2}(\mathbb{R}) := \bigcap_{s_1\in\mathbb{R}} H^{s_1,s_2}(\mathbb{R}).$$

We remark that $H^{s_1,\infty}(\mathbb{R})$ consists of functions with the same decay as the functions of $\mathscr{S}(\mathbb{R})$ but with a limited regularity, while $H^{\infty,s_2}(\mathbb{R})$ consists of functions in $H^{\infty}(\mathbb{R})$ with a prescribed decay as $|x| \to \infty$. As it will be shown in Sect. 2, these two spaces are graded Fréchet spaces endowed with the increasing families of seminorms

$$|u|_{s_1,k} := \max_{s_2\leq k}\|u\|_{s_1,s_2}, \quad \text{resp.} \quad |u|_{k,s_2} := \max_{s_1\leq k}\|u\|_{s_1,s_2}, \quad k \in \mathbb{N},$$

and they are tame (see Definition 1). Finally, we notice that

$$\bigcap_{s_1\in\mathbb{R}} H^{s_1,\infty}(\mathbb{R}) = \bigcap_{s_2\in\mathbb{R}} H^{\infty,s_2}(\mathbb{R}) = \mathscr{S}(\mathbb{R}). \tag{3}$$

The main result of the paper is the following.

Theorem 1 *Let $P(t, x, D_t, D_x)$ be an operator of the form (2). Assume that there exist a constant $C > 0$ and a function $\gamma : \mathbb{C} \to \mathbb{R}^+$ of class C^7 such that for all $(t, x, w) \in [0, T] \times \mathbb{R} \times \mathbb{C}$, $\beta, \delta \in \mathbb{N}$ the following conditions hold:*

$$a_p(t) \quad \textit{is real valued and} \quad a_p(t) \neq 0, \qquad t \in [0, T]; \tag{4}$$

$$|\partial_w^\delta \partial_x^\beta \operatorname{Im} a_j(t,x,w)| \le C\gamma(w)\langle x\rangle^{-\frac{j}{p-1}-|\beta|}, \quad 0 \le j \le p-1; \tag{5}$$

$$|\partial_w^\delta \partial_x^\beta \operatorname{Re} a_j(t,x,w)| \le C\gamma(w)\langle x\rangle^{-|\beta|}, \quad 0 \le j \le p-1. \tag{6}$$

Then, for every given $s_2 \ge 3p-2$*, the Cauchy problem* (1) *is well-posed locally in time in* $H^{\infty,s_2}(\mathbb{R})$*: namely for all* $f \in C([0,T]; H^{\infty,s_2}(\mathbb{R}))$ *and* $u_0 \in H^{\infty,s_2}(\mathbb{R})$*, there exists* $0 < T^* \le T$ *and a unique solution* $u \in C^1([0,T^*]; H^{\infty,s_2}(\mathbb{R}))$ *of* (1)*.*

Remark 1 With respect to [1], in Theorem 1 from the decay at infinity of the data we can estimate the decay of the solution as $|x| \to \infty$. Indeed, by [1] we know that if the data are in H^∞ (and the decay conditions are satisfied), then the solution belongs to H^∞, too; Theorem 1 states that if the data are in H^{∞,s_2} for s_2 large enough, then also $u \in H^{\infty,s_2}$.

The idea of the proof of Theorem 1 is the following: to show the existence of a unique solution to the semilinear equation (1) in H^{∞,s_2}, we first linearize it, fixing a function $u \in C([0,T], H^{\infty,s_2}(\mathbb{R}))$ with $s_2 \in \mathbb{R}$ large enough, then we solve the linear Cauchy problem in the unknown $v(t,x)$

$$\begin{cases} P_u(D)v(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R} \\ v(0,x) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{7}$$

in $H^{\infty,s_2}(\mathbb{R})$; finally, inspired by [6], [10] and [12], we apply the Nash-Moser theorem to obtain the existence of a unique solution of (1) in the tame space $H^{\infty,s_2}(\mathbb{R})$. We remark that we cannot apply to the Cauchy problem (1), (2) a usual fixed point scheme in Banach spaces since the linearized problem (7) has a unique solution which presents a loss of regularity and/or a different behavior at infinity with respect to the data. Thus the problem (7) is not well posed in H^{s_1,s_2}; however it turns out to be well posed in $H^{\infty,s_2}(\mathbb{R})$ which is a tame Fréchet space, so there we can apply the Nash Moser theorem.

Remark 2 In the linear case treated in [3], as a consequence of the energy estimates in weighted Sobolev spaces, we also obtained that the Cauchy problem is well posed in $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}'(\mathbb{R})$. In the semilinear case, we are not able to prove in the same way well posedness in $\mathscr{S}(\mathbb{R})$. In fact, if the data of the problem are Schwartz functions, they belong in particular to $H^{\infty,s_2}(\mathbb{R})$ for every $s_2 > 0$, however, in the semilinear case, the upper bound T^* of the interval of existence of the solution may depend on s_2 and possibly tends to 0 when $s_2 \to +\infty$.

Remark 3 The techniques used in this paper may be adapted to study semilinear p-evolution equations in higher space dimension x at least in some particular cases as, for instance, Schrödinger-type equations ($p = 2$). For this type of equations, at least the linear theory is well established in general space dimension, cf. [8, 9, 16] and it could be easily applied to the analysis of the linearized Cauchy problem (7). We will treat this problem for general p-evolution equations in a future paper.

2 Preliminaries: SG-Calculus and Nash Moser Theorem

2.1 SG-Calculus

We recall here the definition and the main properties of the **SG** classes of pseudodifferential operators. In view of the purposes of this paper we shall state them for symbols defined on $\mathbb{R}^2$, but they have obvious extension in higher dimension. For this generalization and for more details on these classes we refer to [11, 19, 20]. Fixed $m_1, m_2 \in \mathbb{R}$, the space $\mathbf{SG}^{m_1,m_2}(\mathbb{R}^2)$ is the space of all functions $p(x,\xi) \in C^\infty(\mathbb{R}^2)$ satisfying the following estimates:

$$\sup_{(x,\xi)\in\mathbb{R}^2} \langle\xi\rangle^{-m_1+\alpha}\langle x\rangle^{-m_2+\beta}|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| < \infty \tag{8}$$

for every $\alpha, \beta \in \mathbb{N}$. We can associate to every $p \in \mathbf{SG}^{m_1,m_2}(\mathbb{R}^2)$ the pseudodifferential operator defined by

$$Pu(x) = p(x, D)u(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi)\, d\xi. \tag{9}$$

If $p \in \mathbf{SG}^{m_1,m_2}(\mathbb{R}^2)$, then the operator $p(x, D)$ is a linear continuous map from $\mathscr{S}(\mathbb{R})$ to $\mathscr{S}(\mathbb{R})$ and extends to a linear continuous map from $\mathscr{S}'(\mathbb{R})$ to $\mathscr{S}'(\mathbb{R})$ and from $H^{s_1,s_2}(\mathbb{R})$ to $H^{s_1-m_1,s_2-m_2}(\mathbb{R})$ for every $s_1, s_2 \in \mathbb{R}$. We also recall the following result concerning the composition and the adjoint of **SG** operators.

Proposition 1 *Let $p \in \boldsymbol{SG}^{m_1,m_2}(\mathbb{R}^2)$ and $q \in \boldsymbol{SG}^{m_1',m_2'}(\mathbb{R}^2)$. Then there exists a symbol $s \in \boldsymbol{SG}^{m_1+m_1',m_2+m_2'}(\mathbb{R}^2)$ such that $p(x, D)q(x, D) = s(x, D)+R$ where R is a smoothing operator $\mathscr{S}'(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$. Moreover, s has the following asymptotic expansion*

$$s(x,\xi) \sim \sum_\alpha \alpha!^{-1}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi)$$

i.e. for every $N \geq 1$, we have

$$s(x,\xi) - \sum_{|\alpha|<N} \alpha!^{-1}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi) \in \boldsymbol{SG}^{m_1+m_1'-N,m_2+m_2'-N}(\mathbb{R}^2).$$

Proposition 2 *Let $p \in \boldsymbol{SG}^{m_1,m_2}(\mathbb{R}^2)$ and let P^* be the L^2-adjoint of $p(x, D)$. Then there exists a symbol $p^* \in \boldsymbol{SG}^{m_1,m_2}(\mathbb{R}^2)$ such that $P^* = p^*(x, D)+R'$, where R' is a smoothing operator $\mathscr{S}'(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$. Moreover, p^* has the following asymptotic expansion*

$$p^*(x,\xi) \sim \sum_\alpha \alpha!^{-1}\partial_\xi^\alpha D_x^\alpha \overline{p(x,\xi)}$$

i.e. for every $N \geq 1$, *we have*

$$p^*(x,\xi) - \sum_{|\alpha|<N} \alpha!^{-1} \partial_\xi^\alpha D_x^\alpha \overline{p(x,\xi)} \in \boldsymbol{SG}^{m_1-N, m_2-N}(\mathbb{R}^2).$$

We will denote in the sequel by $S^m(\mathbb{R}^2)$, $m \in \mathbb{R}$, the class of symbols $p(x,\xi) \in C^\infty(\mathbb{R}^2)$ satisfying

$$\sup_{(x,\xi)\in\mathbb{R}^2} \langle\xi\rangle^{-m+\alpha} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| < \infty,$$

for every $\alpha, \beta \in \mathbb{N}$. We observe that the following inclusion holds

$$\mathbf{SG}^{m_1,m_2}(\mathbb{R}^2) \subset S^{m_1}(\mathbb{R}^2) \tag{10}$$

for every $m_1 \in \mathbb{R}$, $m_2 \leq 0$.

The following theorem has been proved in [3, Theorem 2.3], and provides an extension to pseudodifferential operators of **SG**-type of the well known sharp Gårding theorem.

Theorem 2 *Let* $m_1 \geq 0, m_2 \leq 0$, $a \in \boldsymbol{SG}^{m_1,m_2}(\mathbb{R}^2)$ *such that* $\mathrm{Re}\, a(x,\xi) \geq 0$ *if* $|\xi| \geq C$ *for some positive* C. *Then there exist pseudo-differential operators* $Q = q(x,D)$, $R = r(x,D)$ *and* $R_0 = r_0(x,D)$ *with symbols, respectively,* $q \in \boldsymbol{SG}^{m_1,m_2}(\mathbb{R}^2)$, $r \in \boldsymbol{SG}^{m_1-1,m_2}(\mathbb{R}^2)$ *and* $r_0 \in S^0(\mathbb{R}^2)$ *such that*

$$a(x,D) = q(x,D) + r(x,D) + r_0(x,D), \tag{11}$$

$$\mathrm{Re}\langle q(x,D)u, u\rangle \geq 0 \qquad \forall u \in \mathscr{S}(\mathbb{R}) \tag{12}$$

and

$$r(x,\xi) = \psi_1(\xi) D_x a(x,\xi) + \sum_{2\leq\alpha+\beta\leq 2m_1-1} \psi_{\alpha,\beta}(\xi) \partial_\xi^\alpha D_x^\beta a(x,\xi) \tag{13}$$

for some real valued functions $\psi_1, \psi_{\alpha,\beta}$ *with* $\psi_1 \in \boldsymbol{SG}^{-1,0}(\mathbb{R}^2)$ *and* $\psi_{\alpha,\beta} \in \boldsymbol{SG}^{\alpha-\beta/2,0}(\mathbb{R}^2)$ *depending only on* ξ.

We remark that the terms in (13) can be re-arranged so that we have

$$r(x,\xi) = \sum_{j=1}^{m-1} r_j(x,\xi), \tag{14}$$

$$r_j(x,\xi) = \begin{cases} \psi_1(\xi) D_x a(x,\xi) + \sum\limits_{2\leq\alpha+\beta\leq 3} \psi_{\alpha,\beta}(\xi) \partial_\xi^\alpha D_x^\beta a(x,\xi), & j = m-1, \\ \sum\limits_{2(m-j)\leq\alpha+\beta\leq 2(m-j)+1} \psi_{\alpha,\beta}(\xi) \partial_\xi^\alpha D_x^\beta a(x,\xi), & 1 \leq j \leq m-2. \end{cases} \tag{15}$$

We also remark that Theorem 2 implies the well-known sharp Gårding inequality

$$\mathrm{Re}\langle A(x, D_x)u, u\rangle \geq -c\|u\|^2_{(m-1)/2,0} \tag{16}$$

for some fixed constant $c > 0$ (cf. [17, Theorem 4.4]).

We recall here also the Fefferman-Phong inequality (cf. [13]):

Theorem 3 *Let $A(x, \xi) \in S^m(\mathbb{R}^2)$ with $A(x, \xi) \geq 0$. Then*

$$\mathrm{Re}\langle A(x, D_x)u, u\rangle \geq -c\|u\|^2_{(m-2)/2,0} \qquad \forall u \in H^{m,0} \tag{17}$$

for some $c > 0$.

We remark that, by Lerner and Morimoto [18], for $m = 2$ the constant c in (17) depends only on $\max\limits_{|\alpha|+|\beta|\leq 7} C_{\alpha,\beta}$ for $C_{\alpha,\beta} := \sup\limits_{x,\xi\in\mathbb{R}} |\partial_\xi^\alpha \partial_x^\beta A(x, \xi)|\langle\xi\rangle^{-2+\alpha}$.

2.2 Tame Fréchet Spaces and the Nash Moser Theorem

We recall here the notions of tame space, tame maps, and the statement of the Nash-Moser inversion theorem, see [14] for further details. Moreover, we show that, for every fixed $s_1, s_2 \in \mathbb{R}$, $H^{s_1,\infty}$ and H^{∞,s_2} are tame spaces.

A *graded* Fréchet space X is a Fréchet space endowed with a *grading*, i.e. an increasing sequence of semi-norms:

$$|x|_n \leq |x|_{n+1}, \qquad \forall n \in \mathbb{N}_0,\ x \in X.$$

Example 1 Given a Banach space B, consider the space $\Sigma(B)$ of all sequences $\{v_k\}_{k\in\mathbb{N}_0} \subset B$ such that

$$|\{v_k\}|_n := \sum_{k=0}^{+\infty} e^{nk}\|v_k\|_B < +\infty \qquad \forall n \in \mathbb{N}_0.$$

We have that $\Sigma(B)$ is a graded Fréchet space with the topology induced by the family of seminorms $|\cdot|_n$ (which is in fact a grading on $\Sigma(B)$).

We say that a linear map $L : X \to Y$ between two graded Fréchet spaces is a *tame linear map* if there exist $r, n_0 \in \mathbb{N}$ such that for every integer $n \geq n_0$ there exists a constant $C_n > 0$, depending only on n, s.t.

$$|Lx|_n \leq C_n|x|_{n+r} \qquad \forall x \in X. \tag{18}$$

The numbers n_0 and r are called respectively *base* and *degree* of the *tame estimate* (18).

Definition 1 A graded Fréchet space X is said to be *tame* if there exist a Banach space B and two tame linear maps $L_1 : X \to \Sigma(B)$ and $L_2 : \Sigma(B) \to X$ such that $L_2 \circ L_1$ is the identity on X.

Obviously, given a graded Fréchet space X and a tame space Y, if there exist two linear tame maps $L_1 : X \to Y$ and $L_2 : Y \to X$ such that $L_2 \circ L_1$ is the identity on X, then also X is a tame space.

Lemma 1 *The spaces $H^{s_1,\infty}$ and H^{∞,s_2} are tame.*

Proof We first recall that $H^\infty := \bigcap_{s\in\mathbb{R}} H^s$ endowed with the seminorms $|f|_n := \max_{s\le n} \|f\|_s$ for every $n \in \mathbb{N}$ is a tame Fréchet space, cf. [10]. Moreover the map $L : H^\infty \to H^{\infty,s_2}$ defined by $L(f) = \langle x\rangle^{-s_2} f$ is a tame isomorphism since for every $n = 0, 1, 2, \dots$ we have:

$$|L(f)|_{n,s_2} = \max_{s_1\le n} \|L(f)\|_{s_1,s_2} = \max_{s_1\le n} \|\langle x\rangle^{-s_2} f\|_{s_1,s_2}$$
$$\le C_n \max_{s_1\le n} |||\langle x\rangle^{-s_2} f|||_{s_1,s_2} = |f|_n$$

and

$$|f|_n = \max_{s_1\le n} \|f\|_{s_1} \le C'_n \max_{s_1\le n} \|\langle x\rangle^{-s_2} f\|_{s_1,s_2} = |L(f)|_{n,s_2}.$$

Thus, H^{∞,s_2} is a tame space. $H^{s_1,\infty}$ is also tame, since the Fourier transform $\mathscr{F}$ is an isomorphism between H^{s_1,s_2} and H^{s_2,s_1}, and $\|\mathscr{F}(f)\|_{s_2,s_1} = \|f\|_{s_1,s_2}$; by this, it is easy to prove that $\mathscr{F} : H^{s_1,\infty} \to H^{\infty,s_2}$ defines a tame map with tame inverse given by the inverse Fourier transform. □

Given now a nonlinear map $T : U \to Y$ where $U \subset X$ and X, Y are graded spaces, we say that T satisfies a *tame estimate* of degree r and base n_0 if for every integer $n \ge n_0$ there exists a constant $C_n > 0$ such that

$$|T(u)|_n \le C_n(1 + |u|_{n+r}) \qquad \forall u \in U. \tag{19}$$

We say that T is *tame* if it satisfies a tame estimate (19) in a neighborhood of each point $u \in U$ (with constants r, n_0 and C_n which may depend on the neighbourhood).

Notice that a linear map is tame if and only if it is a tame linear map.

Given a map $T : U \subset X \to Y$, we define the *Fréchet* derivative $DT(u)v$ of T at $u \in U$ in the direction $v \in X$ by

$$DT(u)v := \lim_{\epsilon\to 0} \frac{T(u+\epsilon v) - T(u)}{\epsilon}, \tag{20}$$

and we say that T is $C^1(U)$ if the limit (20) exists and the derivative $DT : U \times X \to Y$ is continuous. We can also define recursively the higher order Fréchet derivatives $D^nT : U \times X^n \to Y$ of T, cf. [14]; we say that T is $C^\infty(U)$ if all the Fréchet derivatives of T exist and are continuous. A *smooth tame* map $T : U \to Y$ defined on an open subset U of X is a C^∞ map such that $D^n T$ is tame for all $n \in \mathbb{N}_0$.

It is known that sums and compositions of smooth tame maps are smooth tame, and, moreover, linear and nonlinear partial differential operators and integration are smooth tame maps, see [14] for the proofs of these results. Finally we recall the statement of Nash-Moser inversion theorem in the tame Fréchet spaces category, which will be used in the sequel to approach the Cauchy problem (1).

Theorem 4 (Nash-Moser-Hamilton) *Let X, Y be tame spaces, U an open subset of X and $T : U \to Y$ a smooth tame map. If the equation $DT(u)v = h$ has a unique solution $v := S(u, h)$ for all $u \in U$ and $h \in Y$, and if $S : U \times Y \to X$ is smooth tame, then T is locally invertible and each local inverse is smooth tame.*

3 Well Posedness for the Linearized Cauchy Problem

The following theorem is the key to prove the main result of this paper. It deals with the linear Cauchy problem (7), and proves that if the data of (7) are chosen in the Sobolev space H^{s_1,s_2}, $s_1, s_2 \in \mathbb{R}$, then there exists a unique solution $v(t) \in H^{s_1-2\delta\eta(p-1),s_2-2\delta\epsilon}$ for some $\delta > 0$ and for every $0 \le \epsilon, \eta \le 1$ such that $\epsilon + \eta = 1$.

Theorem 5 *Under the assumptions of Theorem 1, there exists $\delta > 0$ such that for every $u \in C([0, T]; H^{3p-1,3p-2}(\mathbb{R}))$, $f \in C([0, T]; H^{s_1,s_2}(\mathbb{R}))$ and $u_0 \in H^{s_1,s_2}(\mathbb{R})$, there exists a unique solution v of (7) such that $v \in C^1([0, T]; H^{s_1-2\delta\eta(p-1),s_2-2\delta\epsilon}(\mathbb{R}))$ for every $\epsilon, \eta \in [0, 1]$ with $\epsilon+\eta = 1$. Moreover v satisfies the following energy estimate:*

$$\|v(t,\cdot)\|^2_{s_1-2\delta\eta(p-1),s_2-2\delta\epsilon} \le C_{s_1,s_2,\gamma} e^{(1+\|u\|^{3p-2}_{3p-1,3p-2})t} \left(\|u_0\|^2_{s_1,s_2} + \int_0^t \|f(\tau,\cdot)\|^2_{s_1,s_2}\, d\tau\right) \forall t \in [0, T]. \tag{21}$$

Remark 4 Notice that the solution v presents the loss $2\delta\eta(p-1)$ in the first Sobolev index and the loss $2\delta\epsilon$ in the second one. In the case $s_2 = 0$, $\epsilon = 0$, $\eta = 1$ we recapture the result of [1, Theorem 2.1]. Moreover, in the linear case (i.e., if (7) does not depend on u), we can obtain either well-posedness with loss of $2\delta(p-1)$ derivatives and no loss of decay (take $\eta = 1$ and $\epsilon = 0$), or the result of [3], that is well-posedness without loss of derivatives but with loss of decay 2δ (take $\eta = 0$ and $\epsilon = 1$). We can also obtain all the intermediate estimates. A similar result has been proved in [7], where intermediate estimates for Schrödinger equations ($p = 2$) have been proved in Gevrey classes.

The proof of Theorem 5 consists in choosing an appropriate and invertible change of variable

$$v(t,x) = e^{\Lambda}(x, D)w(t,x) \tag{22}$$

which transforms the Cauchy problem (7) into an equivalent Cauchy problem

$$\begin{cases} P_\Lambda(t,x,u(t,x),D_t,D_x)w(t,x) = f_\Lambda(t,x) & (t,x)\in[0,T]\times\mathbb{R} \\ w(0,x) = u_{0,\Lambda}(x) & x\in\mathbb{R} \end{cases} \tag{23}$$

for

$$P_\Lambda := (e^\Lambda)^{-1}Pe^\Lambda, \quad f_\Lambda := (e^\Lambda)^{-1}f, \quad u_{0,\Lambda} := (e^\Lambda)^{-1}u_0$$

which is well-posed in L^2 (and therefore in all the weighted Sobolev spaces H^{s_1,s_2}). By the energy estimate in H^{s_1,s_2} for the solution w to the Cauchy problem (23), we then deduce the energy estimate (21) from (22).

The operator Λ will be of the form

$$\Lambda(x,D) = \lambda_1(x,D) + \ldots + \lambda_{p-1}(x,D),$$

so

$$P_\Lambda := (e^{\lambda_1})^{-1}\cdots(e^{\lambda_{p-1}})^{-1}Pe^{\lambda_{p-1}}\cdots(e^{\lambda_1}),$$

$$f_\Lambda := (e^{\lambda_1})^{-1}\cdots(e^{\lambda_{p-1}})^{-1}f, \quad u_{0,\Lambda} := (e^{\lambda_1})^{-1}\cdots(e^{\lambda_{p-1}})^{-1}u_0.$$

We construct here below the transformation Λ and we point out its main properties in Proposition 3. Then we prove the invertibility of e^Λ in Proposition 4. In the subsequent Lemma 2 we show how to obtain the energy estimate (21) for the Cauchy problem (7) from the H^{s_1,s_2} energy estimate for the Cauchy problem (23). After that, in Lemma 5 we state the regularity with respect to x, u of the coefficients $a_j(t,x,u)$ of the linear operator (7), for $0\le j\le p-1$. This section ends with the proof of Theorem 5.

Definition 2 For every $k=1,\ldots,p-1$ we define the symbols

$$\lambda_{p-k}(x,\xi) := M_{p-k}\,\omega\left(\frac{\xi}{h}\right)\langle\xi\rangle_h^{-k+1}\int_0^x \langle y\rangle^{-\frac{p-k}{p-1}}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)dy, \tag{24}$$

where h and M_{p-k} are positive constants such that $h\ge 1$, $\omega\in C^\infty(\mathbb{R})$ is such that

$$\omega(y) = \begin{cases} 0 & |y|\le 1 \\ |y|^{p-1}/y^{p-1} & |y|\ge 2 \end{cases}, \tag{25}$$

and $\psi\in C_0^\infty(\mathbb{R})$ is such that $0\le\psi(y)\le 1$ for all $y\in\mathbb{R}$, $\psi(y)=1$ for $|y|\le\frac{1}{2}$, and $\psi(y)=0$ for $|y|\ge 1$.

Proposition 3 *There exists a constant $C > 0$ such that for every $(x, \xi) \in \mathbb{R}^2$ the following conditions hold:*

$$|\lambda_{p-1}(x,\xi)| \le M_{p-1}\left(\log 2 + \epsilon \log\langle x\rangle + \eta(p-1)\log\langle \xi\rangle_h\right) \tag{26}$$

$$\forall \epsilon, \eta \in [0,1]\ \epsilon + \eta = 1;$$

$$|\lambda_{p-k}(x,\xi)| \le CM_{p-k}, \quad 2 \le k \le p-1. \tag{27}$$

Moreover, for every α, β with $(\alpha, \beta) \neq (0,0)$, there exists $C_{\alpha,\beta} > 0$ such that for $|\xi| > 2h$:

$$|\partial_\xi^\alpha \partial_x^\beta \lambda_{p-k}(x,\xi)| \le C_{\alpha,\beta}\langle x\rangle^{-\beta}\langle \xi\rangle_h^{-\alpha}, \quad 1 \le k \le p-1. \tag{28}$$

Proof We only prove (26) and (27); the inequality (28) can be deduced as in the proof of [5, Lemma 2.1]. Let $E = \{(y,\xi) \in \mathbb{R}^2 : \langle y\rangle \le \langle \xi\rangle_h^{p-1}\}$. If $x \in E$, $x > 0$, then by (24), integrating we have:

$$\begin{aligned}
|\lambda_{p-1}(x,\xi)| &\le M_{p-1}\int_0^x \frac{1}{\sqrt{1+y^2}}dy \le M_{p-1}\log\left(2\langle x\rangle\right)\\
&\le M_{p-1}(\ln 2 + \log\langle x\rangle)\\
&\le M_{p-1}(\ln 2 + \log\langle x\rangle^{\epsilon}\langle \xi\rangle_h^{\eta(p-1)})\\
&\le M_{p-1}(\ln 2 + \epsilon\log\langle x\rangle + \eta(p-1)\log\langle \xi\rangle_h)
\end{aligned}$$

for every $0 \le \epsilon,\ \eta \le 1, \epsilon + \eta = 1$. Analogously, if $x \notin E$ we get

$$\begin{aligned}
|\lambda_{p-1}(x,\xi)| &\le M_{p-1}\int_0^{\sqrt{\langle \xi\rangle_h^{2(p-1)}-1}} \frac{1}{\sqrt{1+y^2}}dy\\
&\le M_{p-1}\ln(2\langle \xi\rangle_h^{p-1})\\
&\le M_{p-1}(\ln 2 + \log\langle x\rangle^{\epsilon}\langle \xi\rangle_h^{\eta(p-1)})\\
&\le M_{p-1}(\ln 2 + \epsilon\log\langle x\rangle + \eta(p-1)\log\langle \xi\rangle_h),
\end{aligned}$$

using the fact that for $x \notin E$ we have $\langle \xi\rangle_h^{p-1} < \langle x\rangle$. Similar estimates can be obtained for $x < 0$. The estimate (27) can be proved by a similar argument. □

From Proposition 3 we obtain in particular that $e^{\pm\lambda_{p-1}} \in \mathbf{SG}^{M_{p-1}\eta(p-1), M_{p-1}\epsilon}$ for every $\epsilon, \eta \ge 0$ such that $\epsilon + \eta = 1$ whereas for $k = 2, \ldots, p-1$, we have $e^{\pm\lambda_{p-k}} \in \mathbf{SG}^{0,0}(\mathbb{R}^2) \subset S^0(\mathbb{R}^2)$.

Proposition 4 *For every $k = 1, \dots, p-1$, let λ_{p-k} be defined by (24). There exists $h_k \geq 1$ such that for every $h \geq h_k$ the operator $e^{\lambda_{p-k}}(x, D)$ is invertible and*

$$\left(e^{\lambda_{p-k}}(x, D)\right)^{-1} = e^{-\lambda_{p-k}}(x, D)(I + r_{p-k}(x, D)), \tag{29}$$

where I stands for the identity operator and $r_{p-k}(x, D)$ is a pseudodifferential operator with principal symbol

$$r_{p-k,-k}(x, \xi) = \partial_\xi \lambda_{p-k}(x, \xi) D_x \lambda_{p-k}(x, \xi) \in \boldsymbol{SG}^{-k, -\frac{p-k}{p-1}}. \tag{30}$$

Proof We first observe that

$$e^{\lambda_{p-k}}(x, D) e^{-\lambda_{p-k}}(x, D) = I - \tilde{r}_{p-k}(x, D),$$

where $\tilde{r}_{p-k}$ has principal symbol $r_{p-k,-k}$ in (30). From (28) we have

$$|r_{p-k,-k}(x, \xi)| \leq C_k M_{p-k}^2 h^{-1},$$

and we similarly estimate the derivatives. We see that for h large enough, say $h \geq h_k$, the operator $I - \tilde{r}_{p-k}$ is invertible on L^2 with inverse given by the Neumann series

$$\sum_{j \geq 0} \tilde{r}_{p-k}^j = I + r_{p-k},$$

and the operator r_{p-k} has principal part (30). Thus,

$$e^{\lambda_{p-k}}(x, D) e^{-\lambda_{p-k}}(x, D)(I + r_{p-k}) = I,$$

and $e^{-\lambda_{p-k}}(x, D)(I + r_{p-k})$ is a right inverse of $e^{\lambda_{p-k}}(x, D)$. Similarly we can obtain that it is also a left inverse. □

Lemma 2 *If the Cauchy problem (23) is H^{s_1,s_2} well posed, and the energy estimate*

$$\|w\|_{s_1,s_2}^2 \leq C e^{(1+\|u\|_{3p-1,3p-2}^{3p-2})t} \left(\|u_{0,\Lambda}\|_{s_1,s_2}^2 + \int_0^t \|f_\Lambda(\tau)\|_{s_1,s_2}^2 d\tau \right) \tag{31}$$

holds for every $t \in [0, T]$, then the Cauchy problem (7) admits a unique solution

$$v \in C([0, T]; H^{s_1 - 2\delta\eta(p-1), s_2 - 2\delta\epsilon})$$

for every $\epsilon, \eta \in [0, 1]$ with $\epsilon + \eta = 1$ which satisfies the energy estimate (5).

Proof From Proposition 3 we know that

$$|\Lambda(x,\xi)| \leq M_{p-1}\left(\log 2 + \epsilon \log\langle x\rangle + \eta(p-1)\log\langle \xi\rangle_h\right) + \sum_{k=2}^{p-1} C_k M_{p-k}$$
$$\leq \delta\left(1 + \epsilon \log\langle x\rangle + \eta(p-1)\log\langle \xi\rangle_h\right)$$

with a positive constant δ depending on $M_1, \dots, M_{p-1}$.This yelds

$$|e^{\pm\Lambda(x,\xi)}| \leq e^{\delta}\langle x\rangle^{\delta\epsilon}\langle \xi\rangle_h^{\delta\eta(p-1)},$$

and by the energy estimate (31) we get

$$\|v\|^2_{s_1-2\delta\eta(p-1),s_2-2\delta\epsilon} = \|e^{\Lambda}w\|^2_{s_1-2\delta\eta(p-1),s_2-2\delta\epsilon} \leq \|w\|^2_{s_1-\delta\eta(p-1),s_2-\delta\epsilon}$$
$$\leq Ce^{(1+\|u\|^{3p-2}_{3p-1,3p-2})t}\left(\|u_{0,\Lambda}\|^2_{s_1-\delta\eta(p-1),s_2-\delta\epsilon} + \int_0^t \|f_\Lambda(\tau)\|^2_{s_1-\delta\eta(p-1),s_2-\delta\epsilon} d\tau\right)$$
$$\leq Ce^{(1+\|u\|^{3p-2}_{3p-1,3p-2})t}\left(\|u_0\|^2_{s_1,s_2} + \int_0^t \|f(\tau)\|^2_{s_1,s_2} d\tau\right)$$

for every $t \in [0, T]$. □

The next Proposition 5 states the regularity with respect to x, u of the coefficients $a_j(t,x,\xi)$ of the linearized operator (7).

Proposition 5 *Under the assumptions* (5) *and* (6)*, there exists* $C' > 0$ *such that for every fixed* $u \in C([0,T]; H^{3p-1,3p-2}(\mathbb{R}))$ *the coefficients* $a_j(t,x,u(t,x))$ *of the operator* $P_u(D)$ *satisfy for every* $1 \leq j \leq p-1$, $(t,x) \in [0,T]\times\mathbb{R}$ *and* $\beta \leq 3p-2$*:*

$$|\partial_x^\beta \operatorname{Re} a_j(t,x,u(t,x))| \leq C'\gamma(u)(1 + \|u\|^\beta_{1+\beta,\beta})\langle x\rangle^{-\beta}, \tag{32}$$

$$|\partial_x^\beta \operatorname{Im} a_j(t,x,u(t,x))| \leq C'\gamma(u)(1 + \|u\|^\beta_{1+\beta,\beta})\langle x\rangle^{-\frac{j}{p-1}-\beta}. \tag{33}$$

Proof For every $\beta \geq 1$ and $1 \leq j \leq p-1$ we have

$$\partial_x^\beta(a_j(t,x,u)) = (\partial_x^\beta a_j)(t,x,u)$$
$$+ \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2\geq 1}} c_\beta \sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i\geq 1}} c_{q,r}(\partial_w^q \partial_x^{\beta_1} a_j)(t,x,u)(\partial_x^{r_1}u)\cdots(\partial_x^{r_q}u)$$

for some $c_\beta, c_{q,r} > 0$. By (6), using the relationship between geometric and arithmetic mean value and Sobolev inequality, this gives for every $\beta \le 4(p-1)$:

$$
\begin{aligned}
&|\partial_x^\beta(\mathrm{Re}\, a_j(t,x,u))| \\
&\quad \le C\gamma(u)\langle x\rangle^{-\beta} + \sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_2\ge 1}} c_{\beta_1,\beta_2} \sum_{\substack{r_1+\ldots+r_q=\beta_2\\ r_i\ge 1}} C_{q,r_1,\ldots,r_q}\gamma(u)\langle x\rangle^{-\beta_1}|\partial_x^{r_1}u|\cdots|\partial_x^{r_q}u| \\
&\quad \le C'\gamma(u)\langle x\rangle^{-\beta}\left(1+\sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_2\ge 1}}\sum_{\substack{r_1+\ldots+r_q=\beta_2\\ r_i\ge 1}} |\langle x\rangle^{r_1}\partial_x^{r_1}u|\cdots|\langle x\rangle^{r_q}\partial_x^{r_q}u|\right) \\
&\quad \le C'\gamma(u)\langle x\rangle^{-\beta}\left(1+\sum_{\substack{\beta_1+\beta_2=\beta\\ \beta_2\ge 1}}\sum_{\substack{r_1+\ldots+r_q=\beta_2\\ r_i\ge 1}} \left(\frac{|\langle x\rangle^{r_1}\partial_x^{r_1}u|+\cdots+|\langle x\rangle^{r_q}\partial_x^{r_q}u|}{q}\right)^q\right) \\
&\quad \le C''\gamma(u)(1+\|u\|_{1+\beta,\beta}^{\beta})\langle x\rangle^{-\beta};
\end{aligned}
$$

where we have used the fact that for every $1 \le j \le q$, $\beta \le 3p-2$, we have

$$|\langle x\rangle^{r_j}\partial_x^{r_j}u| \le C\|\langle x\rangle^{r_j}\partial_x^{r_j}u\|_{1,0} = \|u\|_{1+r_j,r_j} \le \|u\|_{1+\beta,\beta} < \infty.$$

On the other hand, looking at $\mathrm{Im}\, a_j$ and using (5) instead of (6), the same computations give

$$|\partial_x^\beta(\mathrm{Im}\, a_j(t,x,u))| \le C''\gamma(u)(1+\|u\|_{1+\beta,\beta}^{\beta})\langle x\rangle^{-\frac{j}{p-1}-\beta}.$$

□

Remark 5 We observe that a conjugation of the type $(e^{\lambda_{p-k}})^{-1}T_j e^{\lambda_{p-k}}$ with λ_{p-k} given by (24) and $T_j \in \mathbf{SG}^{j,0}$, $j \ge k+1$ depending on γ_j derivatives of u, by Proposition 4 gives:

$$(e^{\lambda_{p-k}})^{-1}T_j e^{\lambda_{p-k}} = e^{-\lambda_{p-k}}\left(T_j + r_{p-k}T_j\right)e^{\lambda_{p-k}} \tag{34}$$

where the principal symbol of r_{p-k} is given by $\partial_\xi\lambda_{p-k}(x,\xi)D_x\lambda_{p-k}(x,\xi) \in \mathbf{SG}^{-k,-(p-k)/(p-1)}$. By the asymptotic expansion we get

$$\sigma\left(T_j + r_{p-k}T_j\right)(x,\xi) = T_j(x,\xi) + \sum_{\alpha=0}^{j-k-1}\frac{1}{\alpha!}\partial_\xi^\alpha r_{p-k}(x,\xi)D_x^\alpha T_j(x,\xi) + S_0(x,\xi)$$

with $S_0 \in \mathbf{SG}^{0,0}$. Since $\partial_\xi^\alpha r_{p-k} D_x^\alpha T_j \in \mathbf{SG}^{j-k-\alpha,-(p-k)/(p-1)-|\alpha|}$ and depends on $\gamma_j + \alpha$ derivatives of u, by re-ordering the sum we get

$$\sigma\left(T_j + r_{p-k}T_j\right)(x,\xi) = T_j(x,\xi) + \sum_{\ell=1}^{j-k} T_{j,\ell}(x,\xi) + T_0$$

with $T_{j,\ell} \in \mathbf{SG}^{\ell,-(p-k)/(p-1)-(j-k-\ell)}$ depending on $\gamma_j + j - k - \ell$ derivatives of u and on M_{p-k}, T_0 of order $(0,0)$. Thus

$$(e^{\lambda_{p-k}})^{-1}\left(\sum_{j=0}^{p-1} T_j\right) e^{\lambda_{p-k}} = e^{-\lambda_{p-k}}\left(\sum_{j=0}^{p-1}(T_j + r_{p-k}T_j)\right) e^{\lambda_{p-k}}$$

and we have, modulo terms of order $(0,0)$:

$$\begin{aligned}\sigma\left(\sum_{j=0}^{p-1}(T_j + r_{p-k}T_j)\right)(x,\xi) &= \sum_{j=1}^{p-1} T_j(x,\xi) + \sum_{j=1}^{p-1}\sum_{\ell=1}^{j-k} T_{j,\ell}(x,\xi)\\ &= \sum_{j=p-k}^{p-1} T_j + \sum_{j=1}^{p-k-1}\left(T_j + T_{j+k,j} + \ldots + T_{p-1,j}\right) = \sum_{j=1}^{p-1} T'_j\end{aligned}$$

with $T'_j = T_j$ for $j \geq p-k$, while for $j \leq p-k-1$ $T'_j \in \mathbf{SG}^{j,0}$ as well as T_j but depend on $\max\{\gamma_{p-1} + p - 1 - k - j, \gamma_{p-2} + p - 2 - k - j, \ldots, \gamma_{j+k}\}$ derivatives of u and on the constant M_{p-k}.

Remark 6 Similarly, a conjugation of the type $e^{-\lambda}T_k e^{\lambda}$, where $\lambda \in \mathbf{SG}^{0,0}$ and $T_k \in \mathbf{SG}^{k,0}$ depends on γ_k derivatives of u, gives, modulo terms of order $(0,0)$, the operator

$$T_k + \sum_{\alpha=1}^{k-1}\frac{1}{\alpha!}\left(\partial_\xi^\alpha T_k\right) e^{-\lambda} D_x^\alpha e^{\lambda} + \sum_{\beta=1}^{k-1}\sum_{\alpha=0}^{k-\beta}\frac{1}{\alpha!\beta!}\partial_\xi^\beta e^{-\lambda} D_x^\beta\left(\partial_\xi^\alpha T_k D_x^\alpha e^{\lambda}\right);$$

at each level $1 \leq j \leq k-1$ we find, except for T_j itself, new terms of type $\partial_\xi^\beta e^{-\lambda} D_x^\beta\left(\partial_\xi^\alpha T_{j+\alpha+\beta} D_x^\alpha e^{\lambda}\right)$ with the same decay as T_j and depending on $\gamma_{j+\alpha+\beta} + \beta$ derivatives of u.

Proof of Theorem 5 First of all we observe that the assumption (4) implies that $a_p(t) \geq C_p$ for every $t \in [0,T]$ or $a_p(t) \leq -C_p$ for every $t \in [0,T]$ for a positive constant C_p. We will prove the theorem under the first condition. If the second one holds the result remains valid with only modifications of signs in the proof.

Fixed u, we consider the linear operator

$$iP_u(t,x,u(t,x),D_t,D_x) = \partial_t + ia_p(t)D_x^p + \sum_{j=0}^{p-1} ia_j(t,x,u)D_x^j$$

with a_p satisfying (4) and a_j satisfying (32), (33) for every $1 \le j \le p-1$, and we apply for $h \ge h_1$ (see Proposition 4) the first conjugation $(e^{\lambda_{p-1}})^{-1}iP_u e^{\lambda_{p-1}}$, with λ_{p-1} in Definition 2 satisfying Proposition 3. Let us first notice that

$$\begin{aligned}(e^{\lambda_{p-1}})^{-1}iP_u e^{\lambda_{p-1}} &= \partial_t + e^{-\lambda_{p-1}}\left(ia_p(t)D_x^p + \sum_{j=0}^{p-1} ia_j(t,x,u)D_x^j\right)e^{\lambda_{p-1}} \\ &+ e^{-\lambda_{p-1}}\left(ir_{p-1}(x,D)a_p(t)D_x^p + \sum_{j=0}^{p-1} ir_{p-1}(x,D)a_j(t,x,u)D_x^j\right)e^{\lambda_{p-1}}\end{aligned}$$

and that the principal symbol of r_{p-1} is given by $\partial_\xi\lambda_{p-1}(x,\xi)D_x\lambda_{p-1}(x,\xi) \in \mathbf{SG}^{-1,-1}$. The composition $e^{-\lambda_{p-1}}ia_p\xi^p e^{\lambda_{p-1}}$ provides, among others, the term $-\partial_\xi\lambda_{p-1}(x,\xi)a_p\xi^p\partial_x\lambda_{p-1}(x,\xi) = -ia_p\xi^p r_{p-1,-1}(x,\xi)$ which cancels with the principal part of the symbol of $e^{-\lambda_{p-1}}ir_{p-1}a_p\xi^p e^{\lambda_{p-1}}$. Then, we notice that by Remark 5 we can write

$$\begin{aligned}(e^{\lambda_{p-1}})^{-1}iP_u e^{\lambda_{p-1}} &= \partial_t + e^{-\lambda_{p-1}}\left(ia_p(t)D_x^p + \sum_{j=0}^{p-1} ia'_j(t,x,u,D_x)\right)e^{\lambda_{p-1}} \\ &+ \operatorname{op}\left(ia_p\xi^p r_{p-1,-1}\right)(t,x,D)\end{aligned}$$

with new terms

$$a'_{p-1}(t,x,u,D_x) = a_{p-1}(t,x,u)D_x^{p-1}$$

and, for $0 \le j \le p-2$, $a'_j(t,x,u,D_x)$ is a pseudodifferential operator given by $a_j(t,x,u)D_x^j$ plus other terms of the same order. Namely, a'_j satisfy estimates of the form

$$\begin{aligned}&|\partial_\xi^\alpha\partial_x^\beta \mathrm{Re}\, a'_j(t,x,u(t,x),\xi)| \\ &\qquad \le C_{M_{p-1}}\gamma(u)(1+\|u\|_{p-1-j+\beta,p-2-j+\beta}^{p-2-j+\beta})\langle x\rangle^{-\beta}\langle\xi\rangle^{j-\alpha},\end{aligned} \tag{35}$$

$$\begin{aligned}&|\partial_\xi^\alpha\partial_x^\beta \mathrm{Im}\, a'_j(t,x,u(t,x),\xi)| \\ &\qquad \le C_{M_{p-1}}\gamma(u)(1+\|u\|_{p-1-j+\beta,p-2-j+\beta}^{p-2-j+\beta})\langle x\rangle^{-\frac{j}{p-1}-\beta}\langle\xi\rangle^{j-\alpha}.\end{aligned} \tag{36}$$

The asymptotic expansion gives

$$\begin{aligned} iP_1(t,x,u,D) &:= (e^{\lambda_{p-1}})^{-1} iP_u e^{\lambda_{p-1}} \\ &= \partial_t + ia_p(t)D_x^p + ia_{p-1}(t,x,u)D_x^{p-1} \\ &+ \operatorname{op}\left(ipa_p\xi^{p-1}D_x\lambda_{p-1}\right) \\ &+ \sum_{\alpha=2}^{p-1}\sum_{\substack{\beta+\gamma=\alpha\\ \beta\le p}} \frac{1}{\beta!\gamma!}\operatorname{op}\left(a_p(t)\partial_\xi^\gamma e^{-\lambda_{p-1}}\cdot\partial_\xi^\beta\xi^p\cdot D_x^\alpha e^{\lambda_{p-1}}\right) \\ &+ \sum_{j=1}^{p-2} ia'_j(t,x,u,D_x) \\ &+ \sum_{j=1}^{p-1}\sum_{\alpha=1}^{j-1}\frac{1}{\alpha!}\operatorname{op}\left(e^{-\lambda_{p-1}}\partial_\xi^\alpha ia'_j D_x^\alpha e^{\lambda_{p-1}}\right) \\ &+ \sum_{j=1}^{p-1}\sum_{\beta=1}^{j-1}\sum_{\alpha=0}^{j-1-\beta}\sum_{\beta_1+\beta_2=\beta}\frac{1}{\alpha!\beta_1!\beta_2!}\operatorname{op}\left(\partial_\xi^\beta e^{-\lambda_{p-1}}D_x^{\beta_1}\partial_\xi^\alpha ia'_j D_x^{\alpha+\beta_2}e^{\lambda_{p-1}}\right) \\ &+ s_0(t,x,u,D) \end{aligned} \tag{37}$$

with a term s_0 of order $(0,0)$. Notice that, by (35), (36) and Remark 6, in (37) we find at each level $1\le j\le p-2$, except for the original terms $a_j(t,x,u)D_x^j$, terms with decay at least of type $\langle x\rangle^{-1}$, depending at most on M_{p-1}, and depending at most on

$$\gamma_{j+|\alpha|+|\beta|} + |\beta| = p-(j+|\alpha|+|\beta|)-1+|\beta| = p-j-|\alpha|-1 \le p-j-1$$

derivatives of u, so that we get

$$\begin{aligned} iP_1 &= \partial_t + ia_p(t)D_x^p + ia_{p-1}(t,x,u)D_x^{p-1} \\ &+\operatorname{op}\left(ipa_p\xi^{p-1}D_x\lambda_{p-1}\right) + \sum_{j=1}^{p-2} ia''_j(t,x,u,D_x) + s_0(t,x,u,D) \end{aligned} \tag{38}$$

where the pseudodifferential operators a''_j are given by $a_jD_x^j$ plus other terms with the same behavior, namely a''_j still satisfy (35) and (36).

Now, let us focus on the term A_{p-1} of order $p-1$ with respect to ξ in (38). By (24) and (33), the choice of ω in (25), and (4) we get for every $|\xi|\ge 2h$:

$$\begin{aligned} \operatorname{Re} A_{p-1}(t,x,u,\xi) &:= \operatorname{Re}\left(ia_{p-1}(t,x,u)\xi^{p-1} + pa_p(t)\xi^{p-1}\partial_x\lambda_{p-1}(x,\xi)\right) \\ &= -\operatorname{Im} a_{p-1}(t,x,u)\xi^{p-1} + pa_p(t)\xi^{p-1}\partial_x\lambda_{p-1}(x,\xi) \end{aligned}$$

$$
\begin{aligned}
&\geq -C'\gamma(u)\langle x\rangle^{-1}\langle\xi\rangle_h^{p-1}+M_{p-1}pa_p(t)|\xi|^{p-1}\langle x\rangle^{-1}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\\
&\geq \frac{\langle\xi\rangle_h^{p-1}}{\langle x\rangle}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\left(-C'\gamma(u)+M_{p-1}pC_p\left(\frac{2}{\sqrt{5}}\right)^{p-1}\right)\\
&- C'\gamma(u)\frac{\langle\xi\rangle_h^{p-1}}{\langle x\rangle}\left(1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right)\\
&\geq -2C'\gamma(u)
\end{aligned}
$$

if we choose $M_{p-1} \geq \dfrac{C'\gamma(u)\sqrt{5}^{p-1}}{2^{p-1}pC_p}$, where we have also used the fact that $\dfrac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}} \geq \dfrac{1}{2}$ on the support of $1-\psi\left(\dfrac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)$ and $|\xi|^{p-1} \geq (2/\sqrt{5})^{p-1}\langle\xi\rangle_h^{p-1}$ for $|\xi| \geq 2h$. Being the symbol Re $A_{p-1}(t,x,u,\xi)+2C'\gamma(u)$ non negative, we can apply the sharp Gårding Theorem 2 and we obtain that there exist pseudodifferential operators $Q_{p-1}(t,x,u,D)$, $R_{p-1}(t,x,u,D)$, $R_{0,p-1}(t,x,u,D)$ with symbols

$$
Q_{p-1}(t,x,u,\xi)\in \mathbf{SG}^{p-1,0},\quad R_{p-1}(t,x,u,\xi)\in \mathbf{SG}^{p-2,0},\quad R_{0,p-1}(t,x,u,\xi)\in S^0
$$

such that

$$
A_{p-1}(t,x,u,D)=Q_{p-1}(t,x,u,D)+iR_{p-1}(t,x,u,D)+R_{0,p-1}(t,x,u,D)
$$

with

$$
\mathrm{Re}\langle Q_{p-1}(t,x,u,D)h(t,x),h(t,x)\rangle\geq 0\quad \forall h\in\mathscr{S}(\mathbb{R}),\ (t,x)\in[0,T]\times\mathbb{R}
$$

and by (15)

$$
R_{p-1}(t,x,u,\xi)=\sum_{j=1}^{p-2}R_{j,p-1}(t,x,u,\xi) \tag{39}
$$

$$
R_{p-2,p-1}=-i\left(\psi_1(\xi)D_xA_{p-1}+\sum_{2\leq\alpha+\beta\leq 3}\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-1}\right)
$$

$$
R_{j,p-1}=-i\sum_{2(p-1-j)\leq\alpha+\beta\leq 2(p-1-j)+1}\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-1}
$$

for every $1 \le j \le p-3$, where ψ_1 and $\psi_{\alpha,\beta}$ are real valued symbols, $\psi_1(\xi) \in \mathbf{SG}^{-1,0}$ and $\psi_{\alpha,\beta}(\xi) \in \mathbf{SG}^{(\alpha-\beta)/2,0}$. We have so

$$\begin{aligned} iP_1 &= \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D_x) \\ &\quad + iR_{p-1}(t,x,u,D_x) + \sum_{j=1}^{p-2} ia_j''(t,x,u,D_x) + s_0(t,x,u,D_x). \end{aligned}$$

We notice that, by (39), R_{p-1} adds to the terms a_j'' some new terms; whenever $\beta \neq 0$, these new terms have at least decay $\langle x\rangle^{-1}$, while for $\beta = 0$ we see that

$$\begin{aligned} &\operatorname{Re}\left(-i\psi_{\alpha,0}(\xi)\,\partial_\xi^\alpha A_{p-1}(t,x,u,\xi)\right) \\ &\qquad = \psi_{\alpha,0}(\xi)\partial_\xi^\alpha \operatorname{Im} A_{p-1}(t,x,u,\xi) \in \mathbf{SG}^{p-1-\alpha/2,0} \subset \mathbf{SG}^{p-2,0} \end{aligned}$$

can be added to $\operatorname{Re} a_j''$, while

$$\begin{aligned} &\operatorname{Im}\left(-i\psi_{\alpha,0}(\xi)\,\partial_\xi^\alpha A_{p-1}(t,x,u,\xi)\right) \\ &\qquad = -\psi_{\alpha,0}(\xi)\partial_\xi^\alpha \operatorname{Re} A_{p-1}(t,x,u,\xi) \in \mathbf{SG}^{p-1-\alpha/2,-1} \subset \mathbf{SG}^{p-2,-\frac{p-2}{p-1}} \end{aligned}$$

can be added to $\operatorname{Im} a_j''$. Again, by (39), we see that the largest number of x-derivatives of u appears when $\alpha = 0$, $\beta = 2(p-1-j)+1$ and we have

$$\begin{aligned} |\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-1}(t,x,u,\xi)| &\le C'\gamma(u)\left(1+\|u\|_{\beta+1,\beta}^{\beta}\right)\langle\xi\rangle^{p-1-\frac{\alpha+\beta}{2}}\langle x\rangle^{-\beta} \\ &\le C'\gamma(u)\left(1+\|u\|_{2(p-j),2(p-j)-1}^{2(p-j)-1}\right)\langle\xi\rangle^{j}\langle x\rangle^{-1} \end{aligned}$$

By these considerations, we understand that after the application of Theorem 2, we can write

$$iP_1 = \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D_x) + \sum_{j=1}^{p-2} ia_{j,1}(t,x,u,D_x) + s_1(t,x,u,D) \tag{40}$$

for a new operator s_1 with symbol in S^0, where $a_{j,1}$ are given by a_j'' plus other terms with the same order and decay, depending on $2(p-j)$ derivatives of u, this means that $a_{j,1}$ depend on $\max\{p-j-1, 2(p-j)\} = 2(p-j)$ derivatives of u. Summing up, for every $\beta \le p-1$ (we need that $2(p-j)+\beta \le 2(p-1)+\beta \le 3p-1$) we have

$$\begin{aligned} &|\partial_\xi^\alpha \partial_x^\beta \operatorname{Re} a_{j,1}(t,x,u(t,x),\xi)| \qquad (41) \\ &\qquad \le C_{M_{p-1}}\gamma(u)(1+\|u\|_{2(p-j)+\beta,2(p-j)-1+\beta}^{2(p-j)-1+\beta})\langle x\rangle^{-\beta}\langle\xi\rangle^{j-\alpha}, \end{aligned}$$

$$|\partial_\xi^\alpha \partial_x^\beta \mathrm{Im}\, a_{j,1}(t,x,u(t,x),\xi)| \qquad (42)$$
$$\leq C_{M_{p-1}}\gamma(u)(1+\|u\|^{2(p-j)-1+\beta}_{2(p-j)+\beta,2(p-j)-1+\beta})\langle x\rangle^{-\frac{j}{p-1}-\beta}\langle\xi\rangle^{j-\alpha}.$$

Now, let us consider, for $h \geq \max\{h_1, h_2\}$ (see Proposition 4), the operator $(e^{\lambda_{p-2}})^{-1} iP_1 e^{\lambda_{p-2}}$, with λ_{p-2} in Definition 2 satisfying Proposition 3. We observe preliminarly that, since $e^{\pm\lambda_{p-2}} \in \mathbf{SG}^{0,0}(\mathbb{R}^2) \subset S^0(\mathbb{R}^2)$, then for the composition $(e^{\lambda_{p-2}})^{-1}s_1(t,x,u,D)e^{\lambda_{p-2}}$ we can use the symbolic calculus in the Hörmander class and obtain that $(e^{\lambda_{p-2}})^{-1}s_1(t,x,u,D)e^{\lambda_{p-2}}$ is again an operator with symbol in $S^0(\mathbb{R}^2)$. Moreover, since $\left(e^{\lambda_{p-2}}\right)^{-1} = e^{-\lambda_{p-2}}(I + r_{p-2})$ and the principal part of r_{p-2} has symbol $r_{p-2,-2}(x,\xi) = \partial_\xi\lambda_{p-2}(x,\xi)D_x\lambda_{p-2}(x,\xi)$ in $\mathbf{SG}^{-2,-\frac{p-2}{p-1}}$, by Remark 5 we obtain

$$(e^{\lambda_{p-2}})^{-1} iP_1 e^{\lambda_{p-2}} = \partial_t + \mathrm{op}\left(ia_p r_{p-2,-2}\right)$$
$$+e^{-\lambda_{p-2}}\left(ia_p(t)D_x^p + Q_{p-1}(t,x,u,D) + \sum_{j=0}^{p-2} ia'_{j,1}(t,x,u,D_x) + s_1(t,x,u,D)\right)e^{\lambda_{p-2}}$$

with $a'_{p-2,1}(t,x,u,D_x) = a_{p-2,1}(t,x,u,D_x)$ and, for $0 \leq j \leq p-3$, $a'_{j,1}(t,x,u,D_x)$ is given by $a_{j,1}(t,x,u,D_x)$ plus some new terms with the same order and decay as $a_{j,1}$ and depending on $\max\{\gamma_{p-1}+p-1-2-j,\ldots,\gamma_{p-\ell}+p-\ell-2-j,\ldots\gamma_{j+2}\} = \gamma_{j+2} = 2(p-j-2)$, because we have $\gamma_{p-\ell} = 2(p-(p-\ell)) = 2\ell$ for $1 \leq \ell \leq p-1$. The new terms contain a smaller number of derivatives with respect to (41) and (42). Thus for every $1 \leq j \leq p-2$ we have that $a'_{j,1}$ still satisfy (41) and (42) for a constant depending also on M_{p-2}; notice that the dependence on M_{p-2} is only at levels $1 \leq j \leq p-3$. The asymptotic expansion gives

$$\begin{aligned}
iP_2(t,x,u,D) &:= (e^{\lambda_{p-2}})^{-1} iP_1 e^{\lambda_{p-2}} \\
&= \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D) \qquad (43)\\
&+ ia_{p-2,1}(t,x,u,D_x) + \mathrm{op}\left(ipa_p\xi^{p-1}D_x\lambda_{p-2}\right) \\
&+ \sum_{\beta=2}^{p-1}\frac{1}{\beta!}\mathrm{op}\left(\partial_\xi^\beta(ia_p\xi^p e^{-\lambda_{p-2}})D_x^\beta\lambda_{p-2}\right) \\
&+ \sum_{j=1}^{p-3} ia'_{j,1}(t,x,u,D_x) + \sum_{\alpha=1}^{p-2}\frac{1}{\alpha!}\mathrm{op}\left(e^{-\lambda_{p-2}}\partial_\xi^\alpha Q_{p-1}D_x^\alpha e^{\lambda_{p-2}}\right) \\
&+ \sum_{\beta=1}^{p-2}\sum_{\alpha=0}^{p-2-\beta}\sum_{\beta_1+\beta_2=\beta}\frac{1}{\alpha!\beta_1!\beta_2!}\mathrm{op}\left(\partial_\xi^\beta e^{-\lambda_{p-2}}D_x^{\beta_1}\partial_\xi^\alpha Q_{p-1}D_x^{\alpha+\beta_2}e^{\lambda_{p-2}}\right)
\end{aligned}$$

$$+ \sum_{j=1}^{p-2}\sum_{\alpha=1}^{j-1}\frac{1}{\alpha!}\mathrm{op}\left(e^{-\lambda_{p-2}}\partial_\xi^\alpha i a'_{j,1}D_x^\alpha e^{\lambda_{p-2}}\right)$$
$$+ \sum_{j=1}^{p-2}\sum_{\beta=1}^{j-1}\sum_{\alpha=0}^{j-1-\beta}\sum_{\beta_1+\beta_2=\beta}\frac{1}{\alpha!\beta_1!\beta_2!}\mathrm{op}\left(\partial_\xi^\beta e^{-\lambda_{p-2}}D_x^{\beta_1}\partial_\xi^\alpha i a'_{j,1}D_x^{\alpha+\beta_2}e^{\lambda_{p-2}}\right)$$
$$+ s'_1(t,x,u,D)$$

with a new term $s'_1 \in S^0$. Let us now look at (43); by (41), (42), and using the estimate (28) wih $k=2$, we find at each level $1 \le k \le p-3$, the original terms $a_{k,1}(t,x,u,D)$ plus terms which decay with respect to x at least like $\langle x\rangle^{-1}$, and possibly depending only on M_{p-1} and M_{p-2}; the largest number of derivatives with respect to u appears in

$$|\partial_\xi^\beta e^{-\lambda_{p-2}}D_x^{\beta_1}\partial_\xi^\alpha i a'_{j,1}D_x^{\alpha+\beta_2}e^{\lambda_{p-2}}|$$
$$\le C'_{M_{p-1},M_{p-2}}\gamma(u)(1+\|u\|^{2(p-j)+\beta-1}_{2(p-j)+\beta,2(p-j)+\beta-1})\langle\xi\rangle^{j-\alpha-\beta}\langle x\rangle^{-\alpha-\beta};$$

at the level $k=j-\alpha-\beta$ the largest number of x-derivatives of u appears when $\alpha=0$ and $\beta=j-k$ and it is given by $2(p-j)+\beta = 2(p-k-\beta)+\beta = 2(p-k)-\beta \le 2(p-k)-1$. Thus, similarly as for (38), we get

$$iP_2 = \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D) + ia_{p-2,1}(t,x,u,D_x) \tag{44}$$
$$+ \mathrm{op}\left(ipa_p\xi^{p-1}D_x\lambda_{p-2}\right) + \sum_{j=1}^{p-3} ia''_{j,1}(t,x,u,D_x) + s'_1(t,x,u,D)$$

where $a''_{j,1}$ are given by a_j plus other terms of the same type, still satisfying (41), (42) but with a constant $C_{M_{p-1},M_{p-2}}$ depending on both M_{p-1} and M_{p-2}.

Now, let us focus on the term A_{p-2} of order $p-2$ with respect to ξ in (44). By (42), (24), the choice of ω in (25), and (4) we get for every $|\xi| \ge 2h$:

$$\mathrm{Re}\,A_{p-2}(t,x,u,\xi) := \mathrm{Re}\left(ia_{p-2,1}(t,x,u,\xi) + pa_p(t)\xi^{p-1}\partial_x\lambda_{p-2}(x,\xi)\right)$$
$$= -\,\mathrm{Im}\,a_{p-2,1}(t,x,u,\xi) + pa_p(t)\xi^{p-1}\partial_x\lambda_{p-2}(x,\xi)$$
$$\ge -C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right)\frac{\langle\xi\rangle_h^{p-2}}{\langle x\rangle^{\frac{p-2}{p-1}}} + M_{p-2}pa_p(t)|\xi|^{p-1}\langle x\rangle^{-\frac{p-2}{p-1}}\langle\xi\rangle_h^{-1}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)$$
$$\ge \frac{\langle\xi\rangle_h^{p-2}}{\langle x\rangle^{\frac{p-2}{p-1}}}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\left(-C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right) + M_{p-2}pC_p\left(\frac{2}{\sqrt{5}}\right)^{p-1}\right)$$

$$- C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right)\frac{\langle\xi\rangle_h^{p-2}}{\langle x\rangle^{\frac{p-2}{p-1}}}\left(1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right)$$
$$\geq -2C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right)$$

if we choose $M_{p-2} \geq \dfrac{C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right)\sqrt{5}^{p-1}}{2^{p-1}pC_p}$, and using again $\langle x\rangle/\langle\xi\rangle_h^{p-1} \geq 1/2$ on the support of $1-\psi(\langle x\rangle/\langle\xi\rangle_h^{p-1})$ and $|\xi|^p \geq (2/\sqrt{5})^{p-1}\langle\xi\rangle_h^{p-1}$ for $|\xi| \geq 2h$. We can so apply the sharp Gårding theorem to the symbol $A_{p-2}(t,x,u,\xi) + 2C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right) \geq 0$ and we obtain that there exist pseudodifferential operators $Q_{p-2}(t,x,u,D)$, $R_{p-2}(t,x,u,D)$, $R_{0,p-2}(t,x,u,D)$ with symbols

$$Q_{p-2}(t,x,u,\xi)\in\mathbf{SG}^{p-2,0},\quad R_{p-2}(t,x,u,\xi)\in\mathbf{SG}^{p-3,0},\quad R_{0,p-2}(t,x,u,\xi)\in S^0$$

such that

$$A_{p-2}(t,x,u,D) = Q_{p-2}(t,x,u,D) + iR_{p-2}(t,x,u,D) + R_{0,p-2}(t,x,uD)$$

with

$$\mathrm{Re}\langle Q_{p-2}(t,x,u,D)h(t,x),h(t,x)\rangle \geq 0 \quad \forall h\in\mathscr{S}(\mathbb{R}),\ (t,x)\in[0,T]\times\mathbb{R}$$

and

$$R_{p-2} = \sum_{j=1}^{p-3} R_{j,p-2} \tag{45}$$

where

$$R_{p-3,p-2} = -i\left(\psi_1(\xi)D_xA_{p-2} + \sum_{2\leq\alpha+\beta\leq 3}\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-2}\right)$$

and

$$R_{j,p-2} = -i\sum_{2(p-2-j)\leq\alpha+\beta\leq 2(p-2-j)+1}\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-2},$$

for every $1\leq j\leq p-3$. We have so

$$iP_2 = \partial_t + ia_p(t,D) + Q_{p-1}(t,x,u,D_x) + Q_{p-2}(t,x,u,D_x)$$
$$+ iR_{p-2}(t,x,u,D_x) + \sum_{j=1}^{p-3} ia''_{j,1}(t,x,u,D_x) + s''_1(t,x,u,D_x).$$

Again, each $R_{j,p-2}$ adds to $a''_{j,1}$ new terms with the same order and decay as $a''_{j,1}$ (notice that the second application of Theorem 2 is needed only in the case $p \geq 3$ and in this case we have $5 \leq p+2$, so the term $\psi_1(\xi)D_x A_{p-2}(t,x,u,\xi)$ satisfies (41) and (42) with $j = p-3$ and a constant depending on M_{p-1}, M_{p-2}. The largest number of x-derivatives of u appears when $\alpha = 0$, $\beta = 2(p-2-j)+1$ and we have

$$|\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-2}(t,x,u,\xi)| \leq C_{M_{p-1}}\gamma(u)\left(1+\|u\|_{4+\beta,3+\beta}^{3+\beta}\right)\langle\xi\rangle^{p-2-\frac{\alpha+\beta}{2}}\langle x\rangle^{-\beta}$$
$$\leq C_{M_{p-1}}\gamma(u)\left(1+\|u\|_{2(p-j)+1,2(p-j)}^{2(p-j)}\right)\langle\xi\rangle^{j}\langle x\rangle^{-1}.$$

This means that, after the second application of the sharp Gårding theorem, we can write

$$iP_2 = \partial_t + ia_p(t,D) + Q_{p-1}(t,x,u,D_x) + Q_{p-2}(t,x,u,D_x) \tag{46}$$
$$+\sum_{j=1}^{p-3} ia_{j,2}(t,x,u,D_x) + s_2(t,x,u,D)$$

for a new operator s_2 with symbol in S^0,where $a_{j,2}$ are given by $a_j D_x^j$ plus other terms with the same order and decay depending on $2(p-j)+1$ x-derivatives of u; thus $a_{j,2}$ depends on $\max\{2(p-j)+1, 2(p-j)\} = 2(p-j)+1$ x-derivatives of u. Summing up, for every $1 \leq j \leq p-3$ and for $\beta \leq p$ (we need that $2(p-j)+1+\beta \leq 2p-1+\beta \leq 3p-1$) we have

$$|\partial_\xi^\alpha\partial_x^\beta \mathrm{Re}\, a_{j,2}(t,x,u(t,x),\xi)| \tag{47}$$
$$\leq C_{M_{p-1},M_{p-2}}\gamma(u)(1+\|u\|_{2(p-j)+1+\beta,2(p-j)+\beta}^{2(p-j)+\beta})\langle x\rangle^{-\beta}\langle\xi\rangle^{j-\alpha},$$

$$|\partial_\xi^\alpha\partial_x^\beta \mathrm{Im}\, a_{j,2}(t,x,u(t,x),\xi)| \tag{48}$$
$$\leq C_{M_{p-1},M_{p-2}}\gamma(u)(1+\|u\|_{2(p-j)+1+\beta,2(p-j)+\beta}^{2(p-j)+\beta})\langle x\rangle^{-\frac{j}{p-1}-\beta}\langle\xi\rangle^{j-\alpha}.$$

We can proceed performing the next conjugations which follow the same argument as the second one. Arguing in this way, after $\ell = p-3$ applications of Theorem 2 we finally come for $h \geq \max\{h_1,\ldots,h_{p-3}\}$ to

$$iP_{p-3} = (e^{\lambda_3})^{-1}\ldots(e^{\lambda_{p-1}})^{-1}(iP)(e^{\lambda_{p-1}})\ldots(e^{\lambda_3}) \tag{49}$$
$$= \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D_x) + \ldots + Q_3(t,x,u,D_x) \tag{50}$$
$$+\, ia_{2,p-3}(t,x,u,D_x) + ia_{1,p-3}(t,x,u,D_x) + s_{p-3}(t,x,u,D)$$

where, for every $1 \leq j \leq p-3$,

$$\mathrm{Re}\langle Q_{p-j}(t,x,u,D)h(t,x), h(t,x)\rangle \geq 0 \quad \forall h \in \mathscr{S}(\mathbb{R}),\ (t,x) \in [0,T] \times \mathbb{R}$$

and moreover for every $\beta \leq 7$

$$|\partial_\xi^\alpha \partial_x^\beta \mathrm{Re}\, a_{2,p-3}(t,x,u,\xi)| \leq C_{M_{p-1},\dots,M_3}\gamma(u)(1+\|u\|^{3p-9+\beta}_{3p-8+\beta,3p-9+\beta})\langle x\rangle^{-\beta}\langle\xi\rangle^{2-\alpha}, \tag{51}$$

$$|\partial_\xi^\alpha \partial_x^\beta \mathrm{Im}\, a_{2,p-3}(t,x,u,\xi)| \leq C_{M_{p-1},\dots,M_3}\gamma(u)(1+\|u\|^{3p-9+\beta}_{3p-8+\beta,3p-9+\beta})\langle x\rangle^{-\frac{2}{p-1}-\beta}\langle\xi\rangle^{2-\alpha}, \tag{52}$$

and for $\beta \leq 5$

$$|\partial_\xi^\alpha \partial_x^\beta \mathrm{Re}\, a_{1,p-3}(t,x,u,\xi)| \leq C_{M_{p-1},\dots,M_3}\gamma(u)(1+\|u\|^{3p-7+\beta}_{3p-6+\beta,3p-7+\beta})\langle x\rangle^{-\beta}\langle\xi\rangle^{1-\alpha}, \tag{53}$$

$$|\partial_\xi^\alpha \partial_x^\beta \mathrm{Im}\, a_{1,p-3}(t,x,u,\xi)| \leq C_{M_{p-1},\dots,M_3}\gamma(u)(1+\|u\|^{3p-7+\beta}_{3p-6+\beta,3p-7+\beta})\langle x\rangle^{-\frac{1}{p-1}-\beta}\langle\xi\rangle^{1-\alpha}. \tag{54}$$

Now, we define, for $h \geq \max\{h_1,\dots,h_{p-2}\}$, $iP_{p-2}(t,x,u,D) := (e^{\lambda_2})^{-1} iP_{p-3}e^{\lambda_2}$ and we get

$$\begin{aligned} iP_{p-2} = {} & \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D_x) + \ldots + Q_3(t,x,u,D_x) \\ & + ia_{2,p-3}(t,x,u,D_x) + \mathrm{op}\left(ipa_p\xi^{p-1}D_x\lambda_2\right) \\ & + ia''_{1,p-3}(t,x,u,D_x) + s'_{p-3}(t,x,u,D) \end{aligned} \tag{55}$$

where $a''_{1,p-3}$ are given by a_j plus other terms of the same type, still satisfying (53) and (54) but with a constant $C_{M_{p-1},\dots,M_2}$ instead of $C_{M_{p-1},\dots,M_3}$, and s'_{p-3} is still of order 0.

Now, as usual, by choosing $M_2 \geq C_{M_{p-1},\dots,M_3}\gamma(u)\left(1+\|u\|^{3p-9}_{3p-8,3p-9}\right)\frac{\sqrt{5}^{p-1}}{2^{p-1}pC_p}$ we get

$$\begin{aligned} \mathrm{Re}\, A_2 := {} & \mathrm{Re}\left(ia_{2,p-3}(t,x,u,D_x) + \mathrm{op}\left(pa_p\xi^{p-1}\partial_x\lambda_2\right)\right) \\ \geq {} & 2C_{M_{p-1},\dots,M_3}\gamma(u)\left(1+\|u\|^{3p-9}_{3p-8,3p-9}\right). \end{aligned}$$

This time, since we are dealing with second order operators, we can apply the Fefferman-Phong inequality (see Theorem 3) to

$$\operatorname{Re} A_2 + 2C_{M_{p-1},\dots,M_3}\gamma(u)\left(1+\|u\|^{3p-9}_{3p-8,3p-9}\right)$$

and obtain

$$\operatorname{Re}\langle \operatorname{Re} A_2 h, h\rangle \geq -\left(c + 2C_{M_{p-1},\dots,M_3}\gamma(u)\left(1+\|u\|^{3p-9}_{3p-8,3p-9}\right)\right)\|h\|^2, \quad \forall h \in \mathscr{S}(\mathbb{R})$$

for a positive constant $c = c(u)$ depending on the derivatives $\partial_\xi^\alpha \partial_x^\beta$ with $|\alpha|+|\beta| \leq 7$ of the symbol $\operatorname{Re} A_2(t,x,u,\xi)+2C_{M_{p-1},\dots,M_3}\gamma(u)\left(1+\|u\|^{3p-9}_{3p-8,3p-9}\right)$. Since γ is of class C^7, we can now find a constant $C_\gamma > 0$ (depending also on $M_{p-1},\dots,M_3$) such that

$$\begin{aligned}\operatorname{Re}\langle \operatorname{Re} A_2 h, h\rangle &\geq -C_\gamma\left(1+\|u\|^{3p-9+7}_{3p-8+7,3p-9+7}\right)\|h\|^2\\ &= -C_\gamma\left(1+\|u\|^{3p-2}_{3p-1,3p-2}\right)\|h\|^2, \quad \forall h \in \mathscr{S}(\mathbb{R}).\end{aligned}$$

The advantage of the use of Fefferman-Phong inequality instead of Theorem 2 is that we avoid the remainder of that theorem, i.e. we save some derivatives of the fixed function u.

It now remains to treat the terms $i \operatorname{Im} A_2 = i \operatorname{Re} a_{2,p-3}$ and $i a''_{1,p-3}$ in (55). We split $i \operatorname{Re} a_{2,p-3}$ into its Hermitian and anti-Hermitian part:

$$i \operatorname{Im} A_2 = \frac{i \operatorname{Re} a_{2,p-3} + (i \operatorname{Re} a_{2,p-3})^*}{2} + \frac{i \operatorname{Re} a_{2,p-3} - (i \operatorname{Re} a_{2,p-3})^*}{2} =: H_1 + H_2,$$

and we have that

$$\operatorname{Re}\langle H_2 h, h\rangle = 0,$$

while

$$H_1 = -\frac{1}{2}\partial_\xi \partial_x \operatorname{Re} a_{2,p-3} \ (\text{mod. } \mathbf{SG}^{0,0})$$

can be put together with $i a''_{1,p-3}$ since by (51) it satisfies (53). We get so

$$\begin{aligned} iP_{p-2} = {} & \partial_t + i a_p(t) D_x^p + Q_{p-1}(t,x,u,D_x) + \ldots + Q_3(t,x,u,D_x)\\ & + \operatorname{Re} A_2(t,x,u,D_x) + H_2(t,x,u,D_x)\\ & + i a_{1,p-2}(t,x,u,D_x) + s_{p-2}(t,x,u,D)\end{aligned}$$

with $ia_{1,p-2}$ still satisfying (53), (54) and $s_{p-2} \in S^0$. Finally, to treat the terms of order 1 with respect to ξ, we perform for $h \geq \max\{h_1, \dots, h_{p-1}\}$ the last conjugation:

$$\begin{aligned} iP_\Lambda :&= (e^{\lambda_1})^{-1} iP_{p-2} e^{\lambda_1} \\ &= \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D_x) + \dots + Q_3(t,x,u,D_x) \\ &+ \operatorname{Re} A_2(t,x,u,D_x) + H_2(t,x,u,D_x) \\ &+ ia_{1,p-2}(t,x,u,D_x) + \operatorname{op}\left(ipa_p\xi^{p-1}D_x\lambda_1\right) + s'_{p-2}(t,x,u,D) \end{aligned} \tag{56}$$

with a new term $s'_{p-2} \in S^0$. Notice that the conjugation $e^{-\lambda_1}(\operatorname{Re} A_2 + H_2)\, e^{\lambda_1}$ gives $\operatorname{Re} A_2 + H_2$ plus a remainder of order $(0,0)$ whose principal part is given by

$$\partial_\xi(\operatorname{Re} A_2 + H_2)\partial_x\lambda_1 - \partial_\xi\lambda_1 D_x(\operatorname{Re} A_2 + H_2) - \partial_\xi\lambda_1(\operatorname{Re} A_2 + H_2)D_x\lambda_1 \in \mathbf{SG}^{0,0}.$$

As usual, by choosing $M_1 \geq C_{M_{p-1},\dots,M_2}\gamma(u)\left(1 + \|u\|^{3p-7}_{3p-6,3p-7}\right)\sqrt{5}^{p-1} / \left(2^{p-1}pC_p\right)$ we get

$$\begin{aligned} \operatorname{Re} A_1 :&= \operatorname{Re}\left(ia_{1,p-2}(t,x,u,D_x) + \operatorname{op}\left(pa_p\xi^{p-1}\partial_x\lambda_1\right)\right) \\ &\geq 0 - 2C_{M_{p-1},\dots,M_2}\gamma(u)\left(1 + \|u\|^{3p-7}_{3p-6,3p-7}\right). \end{aligned}$$

To the symbol $A_1(t,x,u,\xi)$ we can apply the sharp Gårding inequality (16) and we obtain

$$\operatorname{Re}\langle A_1 h, h\rangle \geq -C'_\gamma(1 + \|u\|^{3p-7}_{3p-6,3p-7})\|h\| \qquad \forall h \in \mathscr{S}(\mathbb{R}).$$

At this point we are finally ready to prove an energy estimate in L^2 for the Cauchy problem. We compute

$$\begin{aligned} \frac{d}{dt}\|w(t)\|^2 &= 2\operatorname{Re}\langle\partial_t w, w\rangle = 2\operatorname{Re}\langle iP_\Lambda w, w\rangle - 2\operatorname{Re}\langle ia_p w, w\rangle - \sum_{k=3}^{p-1} 2\operatorname{Re}\langle Q_k w, w\rangle \\ &- 2\operatorname{Re}\langle\operatorname{Re} A_2 w, w\rangle - 2\operatorname{Re}\langle H_2 w, w\rangle - 2\operatorname{Re}\langle A_1 w, w\rangle - 2\operatorname{Re}\langle s'_{p-2}w, w\rangle \\ &\leq C'_\gamma(1 + \|u\|^{3p-2}_{3p-1,3p-2})\left(\|P_\Lambda w\|^2 + \|w\|^2\right) \quad \forall w \in \mathscr{S}(\mathbb{R}). \end{aligned}$$

By Gronwall's Lemma we obtain

$$\|w\|^2 \leq Ce^{(1+\|u\|^{3p-2}_{3p-1,3p-2})t}\left(\|w(0,\cdot)\|^2 + \int_0^t \|P_\Lambda w(\tau,\cdot)\|^2 d\tau\right)$$

and, by standard arguments, the energy estimate (31). □

Remark 7 Notice that with respect to [1], by a different proof we can relax from $4p-3$ to $3p-1$ the number of derivatives of u needed to perform the computations in the linearized problem.

4 The Semilinear Problem

We now apply the energy estimates obtained in the previous section to prove the well posedness of the semilinear Cauchy problem (1). Fixed $s_2 \geq 3p-2$ and $T > 0$, we consider the space $X_T^{s_2} := C^1([0,T], H^{\infty,s_2}(\mathbb{R}))$ and the map $J : X_T^{s_2} \to X_T^{s_2}$ defined by

$$J(u) := u(t,x) - u_0(x) + i\int_0^t a_p(t) D_x^p u(s,x)\, ds$$
$$+ i\sum_{j=0}^{p-1}\int_0^t a_j(s,x,u(s,x)) D_x^j u(s,x)\, ds - i\int_0^t f(s,x)\, ds.$$

It is well known that the existence of a unique solution of (1) in $X_{T^*}^{s_2}$ for some $T^* \in (0,T]$ is equivalent to the existence of a unique solution in $X_{T^*}^{s_2}$ of the equation $Ju = 0$, cf. [1, 12]. We shall approach the latter problem via the Nash-Moser inversion theorem. As a direct consequence of Lemma 1, $X_T^{s_2}$ is a tame Fréchet space endowed with the family of seminorms

$$|g|_{n,s_2,T} = \sup_{[0,T]} (|g(t,\cdot)|_{n,s_2} + |D_t g(t,\cdot)|_{n,s_2}), \qquad n = 0, 1, 2, \ldots$$

The map J is smooth tame since it is defined in terms of sums and composition of integration and linear and nonlinear partial differential operators. In order to apply Nash-Moser theorem it is sufficient to prove that for every fixed $u, h \in X_T^{s_2}$, the equation $DJ(u)v = h$ has a unique solution $v = S(u,h) \in X_T^{s_2}$ and that the map

$$S : X_T^{s_2} \times X_T^{s_2} \to X_T^{s_2}, \quad (u,h) \to v = S(u,h) \tag{57}$$

is smooth tame.

Lemma 3 *For every $u, h \in X_T^{s_2}$, the equation $DJ(u)v = h$ admits a unique solution $v \in X_T^{s_2}$ satisfying for every $n \in \mathbb{N}$ the following estimate:*

$$|v(t,\cdot)|_{n,s_2}^2 \leq C_n(u)\left(|h(0,\cdot)|_{n+r,s_2}^2 + \int_0^t |D_t h(\tau,\cdot)|_{n+r,s_2}^2\, d\tau\right), \qquad t \in [0,T] \tag{58}$$

for every $r \geq 2\delta(p-1)$ with $C_n(u) = C_{n+2\delta(p-1),\gamma}\exp(1 + \|u\|_{3p-1,3p-2}^{3p-2})$.

Proof The proof follows the same argument as the proof of [1, Lemma 3.2], so we just sketch it. A direct computation of the Fréchet derivative of J gives

$$\begin{aligned} DJ(u)v &:= \lim_{\varepsilon\to 0} \frac{J(u+\varepsilon v) - J(u)}{\varepsilon} \\ &= v + i\int_0^t a_p(s) D_x^p v(s)\,ds + i\sum_{j=0}^{p-1}\int_0^t \tilde{a}_j(s,x,u) D_x^j v(s)\,ds, \end{aligned}$$

where

$$\tilde{a}_j(s,x,u) = \begin{cases} a_j(s,x,u) & 1 \le j \le p-1 \\ a_0(s,x,u) + \sum_{h=0}^{p-1} \partial_w a_h(s,x,u) D_x^h u, & j = 0. \end{cases}$$

Hence v is a solution of the equation $DJ(u)v = h$ if and only if it is a solution of the equation $J_{h_0,u,D_t h}(v) = 0$, where $h_0(x) := h(0,x)$ and for every $u, u_0, f \in X_T^{s_2}$ the map $J_{u_0,u,f} : X_T^{s_2} \to X_T^{s_2}$ is defined by

$$\begin{aligned} J_{u_0,u,f}(v) &:= v(t,x) - u_0(x) + i\int_0^t a_p(s) D_x v(s,x)\,ds \\ &\quad + i\sum_{j=0}^{p-1}\int_0^t \tilde{a}_j(s,x,u(s,x)) D_x^j v(s,x)\,ds - i\int_0^t f(s,x)\,ds. \end{aligned}$$

On the other hand, v solves $J_{h_0,u,D_t h}(v) = 0$ if and only if it is a solution of the linear Cauchy problem

$$\begin{cases} \tilde{P}_u(D)v(t,x) = D_t h(t,x) \\ v(0,x) = h_0(x) \end{cases}, \tag{59}$$

where

$$\tilde{P}_u(D) = D_t + a_p(t) D_x^p + \sum_{j=0}^{p-1} \tilde{a}_j(t,x,u) D_x^j.$$

Notice that $\tilde{a}_j(t,x,u)$ satisfy the same conditions as $a_j(t,x,u)$. Hence, we can apply Theorem 5 to (59), choosing $\eta = 1$ and $\epsilon = 0$. It follows that there exists $v \in X_T^{s_2}$ solution of (59) satisfying the estimate (58). This concludes the proof. □

Lemma 4 *The map S defined in* (57) *is smooth tame.*

Proof We observe that, fixed $(u_0, h_0) \in X_T^{s_2} \times X_T^{s_2}$, the constant $C_n(u)$ in the energy estimate (58) is bounded if u belongs to a bounded neighborhood of (u_0, h_0).

Evidently, from (58) we have:

$$|v(t,\cdot)|^2_{n,s_2} \le C'_n |h|^2_{n+r,s_2,T} \qquad t \in [0,T]$$

for some $C'_n > 0$. Similarly, from the equation $\tilde{P}_u(D)v = D_t h$ we get

$$\begin{aligned}|D_t v(t,\cdot)|_{n,s_2} &\le |a_p(t)D^p v(t,\cdot)|_{n,s_2} + \sum_{j=0}^{p-1} |\tilde{a}_j(t,\cdot,u)D_x^j v(t,\cdot)|_{n,s_2} + |D_t h(t,\cdot)|_{n,s_2} \\ &\le C(|v(t,\cdot)|_{n+p,s_2} + |h|_{n,s_2,T})\end{aligned}$$

for some $C > 0$. Hence

$$|S(u,h)|_{n,s_2,T} = \sup_{t\in[0,T]} (|v|_{n,s_2} + |D_t v(t,\cdot)|_{n,s_2}) \le C_n |h|_{n+r',s_2,T} \le C_n |(u,h)|_{n+r',s_2,T}$$

for some $C_n > 0$ $r' \ge 2\delta(p-1)+p$. Then S is tame.

We now prove that DS is also a tame map. For $(u,h),(u_1,h_1) \in X_T^{s_2} \times X_T^{s_2}$ we have

$$DS(u,h)(u_1,h_1) = \lim_{\varepsilon\to 0} \frac{S(u+\varepsilon u_1, h+\varepsilon h_1) - S(u,h)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{v_\varepsilon - v}{\varepsilon},$$

where v is a solution of (59) and v_ε is the solution of

$$\begin{cases} \tilde{P}_{u+\varepsilon u_1}(D)v_\varepsilon = D_t(h+\varepsilon h_1) \\ v_\varepsilon(0,x) = h_0(x) + \varepsilon h_1(0,x) \end{cases}.$$

A direct computation shows that the function $w_\varepsilon = \frac{v_\varepsilon - v}{\varepsilon}$ solves the Cauchy problem

$$\begin{cases} \tilde{P}_{u+\varepsilon u_1} w_\varepsilon = f_\varepsilon \\ w_\varepsilon(0,x) = h_1(0,x) \end{cases},$$

where

$$f_\varepsilon = D_t h_1 - \sum_{j=0}^{p-1} \frac{\tilde{a}_j(t,x,u+\varepsilon u_1) - \tilde{a}_j(t,x,u)}{\varepsilon} D_x^j v.$$

We have the following: to prove that DS is tame it is sufficient to show that w_ε tends to some w_1 in $X_T^{s_2}$ for $\varepsilon \to 0$. Indeed, this would imply that w_1 solves the Cauchy problem

$$\begin{cases} \tilde{P}_u(D)w_1 = f_1 \\ w_1(0,x) = h_1(0,x) \end{cases}$$

where

$$f_1 := \lim_{\varepsilon\to 0} f_\varepsilon = D_t h_1 - \sum_{j=0}^{p-1} \partial_w \tilde{a}_j(t,x,u) u_1 D_x^j v$$

and so that w_1 will satisfy an energy estimate of the form

$$|w_1(t,\cdot)|^2_{n,s_2} \le C_n(u)\left(|h_1(0,\cdot)|^2_{n+r,s_2} + \int_0^t |f_1(\tau,\cdot)|^2_{n+r,s_2}\,d\tau\right), \qquad (60)$$

which would give, by the expression of f_1,

$$|w_1(t,\cdot)|_{n,s_2} \le C_n'(u)(|h_1|_{n+r',s_2,T} + |h|_{n+r',s_2,T}), \qquad r' \ge 2r+p-1$$

for (u,h) in a neighborhood of (u_0,h_0) and (u_1,h_1) in a neighborhood of some fixed $(\tilde{u}_1,\tilde{h}_1) \in X_T^{s_2} \times X_T^{s_2}$. Moreover, $D_t w_1$ would satisfy a similar estimate, and so w_1 is tame.

Let us so prove that w_ϵ converges in $X_T^{s_2}$ for $\epsilon \to 0$. Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ and consider the corresponding functions w_{ε_1} and w_{ε_2} which solve the Cauchy problems

$$\begin{cases} \tilde{P}_{u+\varepsilon_j u_1}(D) w_{\varepsilon_j} = f_{\varepsilon_j} \\ w_{\varepsilon_j}(0,x) = h_1(0,x) \end{cases}, \qquad j = 1,2.$$

Then, it is immediate to see that $w_{\varepsilon_1} - w_{\varepsilon_2}$ is a solution of

$$\begin{cases} \tilde{P}_{u+\varepsilon_1 u_1}(D)(w_{\varepsilon_1} - w_{\varepsilon_2}) = f_{\varepsilon_1} - f_{\varepsilon_2} + \sum_{j=0}^{p-1} (\tilde{a}_j(t,x,u+\varepsilon_2 u_1) - \tilde{a}_j(t,x,u+\varepsilon_1 u_1)) D_x^j w_{\varepsilon_2} \\ (w_{\varepsilon_1} - w_{\varepsilon_2})(0,x) = 0. \end{cases}$$

Then by the estimate (60) and the mean value theorem, we get

$$\begin{aligned} |w_{\varepsilon_1} - w_{\varepsilon_2}|_{n,s_2} \le C_n(u+\varepsilon_1 u_1) \Bigg(& \sup_{t\in[0,T]} |f_{\varepsilon_1} - f_{\varepsilon_2}|_{n+r,s_2} \\ & + \sum_{j=0}^{p-1} \sup_{t\in[0,T]} |\partial_w \tilde{a}_j(t,x,u_{1,2})(\varepsilon_1 - \varepsilon_2) u_1 D_x^j w_{\varepsilon_1}|_{n+r,s_2} \Bigg) \end{aligned}$$

for some constant $C_n(u+\varepsilon_1 u_1) > 0$ and for some $u_{1,2}$ between $u+\varepsilon_1 u_1$ and $u+\varepsilon_2 u_1$. Moreover, since H_{n+r,s_2} is an algebra, then

$$\begin{aligned} &|\partial_w a_j(t,x,u_{1,2})(\varepsilon_1 - \varepsilon_2)\, u_1 D_x^j w_{\varepsilon_2}|_{n+r,s_2} \\ &\qquad \le |\partial_w a_j(t,x,u_{1,2})|_{n+r,s_2} |\varepsilon_1 - \varepsilon_2| |u_1|_{n+r,s_2} |w_{\varepsilon_2}|_{n+r+j,s_2}. \end{aligned}$$

Then, $|w_{\varepsilon_1} - w_{\varepsilon_2}|_{n,s_2}$ tends to 0 when $\varepsilon_1 \to \varepsilon_2 \to 0$ if (u, h) is in a neighoborhood of (u_0, h_0) and (u_1, h_1) is in a neighborhood of some fixed $(\tilde{u}_1, \tilde{h}_1) \in X_T^{s_2} \times X_T^{s_2}$. This shows that there exists a Cauchy sequence ε_j tending to 0 such that the corresponding function w_{ε_j} converges in $X_T^{s_2}$ and this implies that DS is tame.

Using the previous results we can prove by induction on m that

$$D^m S(u, h)(u_1, h_1) \cdots (u_m, h_m) = w^m$$

is a solution of the Cauchy problem

$$\begin{cases} \tilde{P}_u(D)w^m = f^m \\ w^m(0, x) = 0 \end{cases}$$

with

$$f^m := -\sum_{j=0}^{p-1} \partial_w \tilde{a}_j(t, x, u) u_m D_x^j w^{m-1} - \sum_{j=0}^{p-1} \partial_w^2 \tilde{a}_j(t, x, u) u_{m-1} u_m D_x^j w^{m-2}$$
$$- \cdots - \sum_{j=0}^{p-1} \partial_w^m \tilde{a}_j(t, x, u) u_1 \cdots u_{m-1} u_m D_x^j w^0,$$

$w_0 := v$, and satisfies, in a neighborhood of $(u, h), (u_1, h_1), \ldots (u_m, h_m)$ the estimate

$$|w^m|_{n,s_2,T} \le C_n \sum_{j=0}^{m-1} |h_j|_{n+r(m),s_2,T}$$

for some $C_n > 0$ and some $r(m) \in \mathbb{N}$, where $h_0 := h$. The proof follows readily the argument in the proof of Lemma 3.3 in [1]. We leave the details to the reader. □

Proof of Theorem 1 We prove now the existence of a solution of the semilinear Cauchy problem (1) that is of the equation $Ju = 0$. We recall that $Ju = 0$ if and only if

$$u(t, x) = u_0(x) - i \int_0^t a_p(s) D_x^p u(s, x)\, ds \tag{61}$$
$$- i \sum_{j=0}^{p-1} \int_0^t a_j(s, x, u(s, x)) D_x^j u(s, x)\, ds + i \int_0^t f(s, x)\, ds.$$

By a linear approximation in t we get $u(t,x) = w(t,x) + o(t)$ for $t \to 0$ where

$$w(t,x) = u_0(x) - it\left(a_p(0)D_x^p u_0(x) + \sum_{j=0}^{p-1} a_j(0,x,u_0(x))D_x^j u_0(x) - f(0,x)\right).$$

We also observe that, by the definition of J and w, we have:

$$\begin{aligned}
\partial_t(Jw(t,x)) &= \partial_t w + ia_p(t)D_x^p w + i\sum_{j=0}^{p-1} a_j(t,x,w)D_x^j w - if(t,x) \\
&= i(a_p(t) - a_p(0))D_x^p u_0 + i\sum_{j=0}^{p-1}\left(a_j(t,x,w) - a_j(0,x,u_0)\right)D_x^j u_0 \\
&\quad + ta_p(t)D_x^p\left[a_p(0)D_x^p u_0 + \sum_{j=0}^{p-1} a_j(0,x,u_0)D_x^j u_0 - f(0,x)\right] \\
&\quad + \sum_{j=0}^{p-1} ta_j(t,x,w)D_x^j\left[a_p(0)D_x^p u_0 + \sum_{k=0}^{p-1} a_k(0,x,u_0)D_x^k u_0 - f(0,x)\right] \\
&\quad + i(f(0,x) - f(t,x)).
\end{aligned}$$

From this it follows that

$$\begin{aligned}
|\partial_t Jw(t,\cdot)|_{n,s_2} &\le \sup_{t\in[0,T]} |a_p(t) - a_p(0)| \cdot |u_0|_{n+p,s_2} \\
&\quad + \sum_{j=0}^{p-1} |[a_j(t,x,w) - a_j(0,x,u_0)]D_x^j u_0|_{n,s_2} + |f(0,x) - f(t,x)|_{n,s_2} \\
&\quad + t\sup_{t\in[0,T]} |a_p(t)| \cdot \left|a_p(0)D_x^p u_0 + \sum_{k=0}^{p-1} a_k(0,x,u_0)D_x^k u_0 - f(0,x)\right|_{n+p,s_2} \\
&\quad + t\sum_{j=0}^{p-1}\left|a_j(t,x,w)D_x^j\left[a_p(0)D_x^p u_0 + \sum_{k=0}^{p-1} a_k(0,x,u_0)D_x^k u_0 - f(0,x)\right]\right|_{n,s_2}.
\end{aligned} \tag{62}$$

Taking w in a sufficiently small neighborhood of u_0 and applying the mean value theorem to the right-hand side of (62) we obtain

$$|\partial_t Jw(t,\cdot)|_{n,s_2} \le Ct \tag{63}$$

for a suitable constant $C = C(n, s_2, a_p, \dots, a_0, u_0, f)$. Now, fixed $\varepsilon > 0$ we define

$$\phi_\varepsilon(t, x) = \int_0^t \rho\left(\frac{s}{\varepsilon}\right)(\partial_t Jw)(s, x)\, ds,$$

where $\rho \in C^\infty(\mathbb{R})$ such that $0 \le \rho \le 1$ and $\rho(s) = 0$ for $|s| \le 1$ and $\rho(s) = 1$ for $|s| \ge 2$. Notice that $\phi_\varepsilon = 0$ for $0 \le t \le \varepsilon$. Let U and V be neighborhoods of w and Jw respectively such that $J : U \to V$ is a bijection. We have that

$$\begin{aligned} |Jw - \phi_\varepsilon|_{n,s_2} &= \left| \int_0^t \left(1 - \rho\left(\frac{s}{\varepsilon}\right)\right)(\partial_t Jw)(s, \cdot)\, ds \right|_{n,s_2} \\ &\le \int_0^{2\varepsilon} \left| \left(1 - \rho\left(\frac{s}{\varepsilon}\right)\right)(\partial_t Jw)(s, \cdot) \right|_{n,s_2} ds \\ &\le C \int_0^{2\varepsilon} s\, ds \le 2C\varepsilon^2, \end{aligned}$$

where C is the same constant appearing in (63). Similarly we obtain that

$$|\partial_t(Jw - \phi_\varepsilon)|_{n,s_2} \le 2C\varepsilon.$$

Hence, taking $0 < \varepsilon < 1$ we conclude that

$$|Jw - \phi_\varepsilon|_{n,s_2,T} \le 2C\varepsilon.$$

If we choose ε sufficiently small, we have that $\phi_\varepsilon \in V$. Then there exists $u \in U$ such that $Ju = \phi_\varepsilon$. In particular we have $Ju = 0$ for $0 \le t \le \varepsilon$, that is u is a solution in $X_\varepsilon^{s_2}$ of the Cauchy problem. The uniqueness of the solution comes from standard arguments, cf. [1]. □

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