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Anomalies in Partial Differential Equations

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Anomalies in Partial Differential Equations

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La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi agli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri ne' quali è scritto.

Egli è scritto in lingua matematica, e i caratteri sono triangoli, cerchi, ed altre figure geometriche, senza i quali mezzi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto.

G. Galilei, Il Saggiatore, VI, 232

Preface

The present volume is a collection of papers with a strong focus on recent results in the theory of PDEs, Harmonic Analysis and Time–Frequency Analysis. It addresses general theoretical issues such as linear models with low regular coefficients, qualitative properties of solutions to nonlinear models and models from applications as well.

The last decade has been marked by important breakthroughs in the study of well-posedness or local solvability for linear equations with low regular coefficients or the critical exponents in nonlinear evolution models. Here, we refer in particular to the results for blow-up phenomena or existence of global (in time) small data solutions. Moreover, applied models such as traffic flows, Einstein-Euler systems or stochastic PDEs are discussed, and, finally, recent results from Harmonic Analysis and Time-Frequency Analysis, such as the action of localizing operators in quasi-Banach settings and the description of wavefront sets, are considered.

The papers of the volume, written by leading experts in their respective fields, are expanded versions of talks given at the INDAM Workshop "Anomalies in Partial Differential Equations" held in September 2019 at the Istituto Nazionale di Alta Matematica, Dipartimento di Matematica "Guido Castelnuovo", Università di Roma "La Sapienza".

We wish to warmly thank all the contributors as well as the people who took part in the workshop.

We are grateful to the Istituto Nazionale di Alta Matematica "Francesco Severi" for having made possible the workshop through his administrative and financial support.

Bologna, Italy — Massimo Cicognani
Trieste, Italy — Daniele Del Santo
Bologna, Italy — Alberto Parmeggiani
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August 2020

Contents

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Semilinear p-Evolution Equations in Weighted Sobolev Spaces

Alessia Ascanelli and Marco Cappiello

To Massimo Cicognani and Michael Reissig in occasion of their 60-th birthday

Abstract We consider the initial value problem for a class of semilinear p-evolution equations with (t, x)-depending coefficients. Under suitable decay conditions for $|x| \to \infty$ on the imaginary part of the coefficients, we prove local in time well posedness of the Cauchy problem in suitable weighted Sobolev spaces.

Keywords p-evolution equations · Semilinear Cauchy problem · Nash-Moser theorem · Weighted Sobolev spaces · Pseudo-differential operators

1 Introduction

In the present paper we deal with the semilinear Cauchy problem

$$\begin{cases} P_u(D)u(t, x) = f(t, x), & (t, x) \in [0, T] \times \mathbb{R} \\ u(0, x) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{1}$$

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for the first order p-evolution operator

$$P_u(D)u = P(t,x,u(t,x),D_t,D_x)u := D_t u + a_p(t)D_x^p u + \sum_{j=0}^{p-1} a_j(t,x,u)D_x^j u \quad (2)$$

where $D = \frac{1}{i}\partial$, $p \in \mathbb{N}$, $p \geq 2$, $a_p \in C([0,T],\mathbb{R})$, a_j are for $0 \leq j \leq p-1$ continuous in time functions with values in $C^\infty(\mathbb{R}\times\mathbb{C})$, and moreover the functions $x \to a_j(t,x,w)$ are in $\mathscr{B}^\infty(\mathbb{R})$ (i.e. uniformly bounded together with all their derivatives).

For $p = 2$ our analysis will concern semilinear Schrödinger equations of the form

$$D_t u + D_x^2 u + a_1(t,x,u)D_x u + a_0(t,x,u) = f(t,x).$$

For $p = 3$, the most important model is represented by the Korteweg-de Vries equation describing the propagation of monodimensional waves of small amplitudes in waters of constant depth:

$$\partial_t u = \frac{3}{2}\sqrt{\frac{g}{h}}\partial_x\left(\frac{1}{2}u^2 + \frac{2}{3}\alpha u + \frac{1}{3}\sigma\partial_x^2 u\right),$$

that can be written in the form (1) as

$$D_t u + \frac{1}{2}\sqrt{\frac{g}{h}}\sigma D_x^3 u - \sqrt{\frac{g}{h}}\left(\alpha + \frac{3}{2}u\right)D_x u = 0.$$

Here u represents the wave elevation with respect to the water's surface, g is the gravity constant, h the (constant) level of water, α a fixed small constant and $\sigma = \frac{h^3}{3} - \frac{Th}{\rho g}$, with T the surface tension, ρ the density of the fluid. Assuming the level of water h depending on x, we are led to an operator with space-depending coefficients that can be applied to study the evolution of the wave when the depth of the seabed is variable, cf. [1].

Since a_p is real valued, the principal symbol (in the sense of Petrowski) of P, given by $\tau + a_p(t)\xi^p$, has the real characteristic root $\tau = -a_p(t)\xi^p$; by the Lax-Mizohata theorem, real characteristics are necessary for the existence of a unique solution in Sobolev spaces of the Cauchy problem (1) in a neighborhood of $t = 0$, for any $p \geq 1$. Moreover, whenever the lower order coefficients $a_j(t,x,w) \in \mathbb{C}$ for $0 \leq j \leq p-1$, decay conditions as $|x| \to \infty$ are necessary on the a_j for well-posedness in Sobolev spaces, see [6, 15] respectively for $p = 2$, p arbitrary.

Well-posedness for the Cauchy problem (1), (2) in $H^\infty(\mathbb{R}) = \cap_s H^s(\mathbb{R})$, where $H^s(\mathbb{R})$ is the usual Sobolev space on L^2, has been proved in the paper [1] under suitable decay conditions at infinity for the a_j, $0 \leq j \leq p-1$, relying on the linear results of [5]; in this paper, despite very precise decay assumptions on the coefficients, the authors have no information at all about the behavior at infinity of the solution.

In the last years, we started to study linear p-evolution equations in weighted Sobolev spaces, see [3, 4] and to state a relation between the behavior at infinity of the data and the one of the solution. Here we are interested to extend part of these results to the semilinear case, that is to give decay conditions on the coefficients of $P_u(D)$ that are sufficient for the local in time well-posedness of the Cauchy problem (1) in suitable weighted Sobolev spaces.

Namely, fixed $s_1, s_2 \in \mathbb{R}$, we define $H^{s_1,s_2}(\mathbb{R})$ as the space of all $u \in \mathscr{S}'(\mathbb{R})$ such that $\|u\|_{s_1,s_2} := \|\langle x\rangle^{s_2}\langle D\rangle^{s_1}u\|_{L^2} < \infty$ where we denote by $\langle D\rangle^{s_1}$ the Fourier multiplier with symbol $\langle\xi\rangle^{s_1} := (1+\xi^2)^{s_1/2}$. This space is a Hilbert space endowed with the inner product

$$\langle u, v\rangle_{s_1,s_2} := \langle\langle x\rangle^{s_2}\langle D\rangle^{s_1}u, \langle x\rangle^{s_2}\langle D\rangle^{s_1}v\rangle_{L^2}$$

which induces the norm $\|\cdot\|_{s_1,s_2}$. We have $H^{0,0}(\mathbb{R}) = L^2(\mathbb{R})$ and we shall denote the L^2 norm simply by $\|\cdot\|$. An equivalent norm on $H^{s_1,s_2}(\mathbb{R})$ is given by $|||u|||_{s_1,s_2} := \|\langle D\rangle^{s_1}\langle x\rangle^{s_2}u\|_{L^2}$. Notice that for $s_2 = 0$ we recapture the standard Sobolev spaces and that the obvious inclusions $H^{s_1,s_2}(\mathbb{R}) \subseteq H^{t_1,t_2}(\mathbb{R})$ for every $s_1 \geq t_1,\ s_2 \geq t_2$ hold. We also recall that $H^{s_1,s_2}(\mathbb{R})$ is an algebra with respect to multiplication for $s_1 > 1/2$ and $s_2 \geq 0$, cf. [2, Proposition 2.2]. For every given $s_1 \in \mathbb{R}$ (resp. $s_2 \in \mathbb{R}$) we define

$$H^{s_1,\infty}(\mathbb{R}) := \bigcap_{s_2\in\mathbb{R}} H^{s_1,s_2}(\mathbb{R}), \quad \text{resp.} \quad H^{\infty,s_2}(\mathbb{R}) := \bigcap_{s_1\in\mathbb{R}} H^{s_1,s_2}(\mathbb{R}).$$

We remark that $H^{s_1,\infty}(\mathbb{R})$ consists of functions with the same decay as the functions of $\mathscr{S}(\mathbb{R})$ but with a limited regularity, while $H^{\infty,s_2}(\mathbb{R})$ consists of functions in $H^{\infty}(\mathbb{R})$ with a prescribed decay as $|x| \to \infty$. As it will be shown in Sect. 2, these two spaces are graded Fréchet spaces endowed with the increasing families of seminorms

$$|u|_{s_1,k} := \max_{s_2\leq k}\|u\|_{s_1,s_2}, \quad \text{resp.} \quad |u|_{k,s_2} := \max_{s_1\leq k}\|u\|_{s_1,s_2}, \quad k \in \mathbb{N},$$

and they are tame (see Definition 1). Finally, we notice that

$$\bigcap_{s_1\in\mathbb{R}} H^{s_1,\infty}(\mathbb{R}) = \bigcap_{s_2\in\mathbb{R}} H^{\infty,s_2}(\mathbb{R}) = \mathscr{S}(\mathbb{R}). \tag{3}$$

The main result of the paper is the following.

Theorem 1 *Let $P(t, x, D_t, D_x)$ be an operator of the form (2). Assume that there exist a constant $C > 0$ and a function $\gamma : \mathbb{C} \to \mathbb{R}^+$ of class C^7 such that for all $(t, x, w) \in [0, T] \times \mathbb{R} \times \mathbb{C}$, $\beta, \delta \in \mathbb{N}$ the following conditions hold:*

$$a_p(t) \quad \text{is real valued and} \quad a_p(t) \neq 0, \qquad t \in [0, T]; \tag{4}$$

$$|\partial_w^\delta \partial_x^\beta \operatorname{Im} a_j(t,x,w)| \leq C\gamma(w)\langle x\rangle^{-\frac{j}{p-1}-|\beta|}, \quad 0 \leq j \leq p-1; \tag{5}$$

$$|\partial_w^\delta \partial_x^\beta \operatorname{Re} a_j(t,x,w)| \leq C\gamma(w)\langle x\rangle^{-|\beta|}, \quad 0 \leq j \leq p-1. \tag{6}$$

Then, for every given $s_2 \geq 3p-2$, the Cauchy problem (1) is well-posed locally in time in $H^{\infty,s_2}(\mathbb{R})$: namely for all $f \in C([0,T]; H^{\infty,s_2}(\mathbb{R}))$ and $u_0 \in H^{\infty,s_2}(\mathbb{R})$, there exists $0 < T^* \leq T$ and a unique solution $u \in C^1([0,T^*]; H^{\infty,s_2}(\mathbb{R}))$ of (1).

Remark 1 With respect to [1], in Theorem 1 from the decay at infinity of the data we can estimate the decay of the solution as $|x| \to \infty$. Indeed, by [1] we know that if the data are in H^∞ (and the decay conditions are satisfied), then the solution belongs to H^∞, too; Theorem 1 states that if the data are in H^{∞,s_2} for s_2 large enough, then also $u \in H^{\infty,s_2}$.

The idea of the proof of Theorem 1 is the following: to show the existence of a unique solution to the semilinear equation (1) in H^{∞,s_2}, we first linearize it, fixing a function $u \in C([0,T], H^{\infty,s_2}(\mathbb{R}))$ with $s_2 \in \mathbb{R}$ large enough, then we solve the linear Cauchy problem in the unknown $v(t,x)$

$$\begin{cases} P_u(D)v(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R} \\ v(0,x) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{7}$$

in $H^{\infty,s_2}(\mathbb{R})$; finally, inspired by [6], [10] and [12], we apply the Nash-Moser theorem to obtain the existence of a unique solution of (1) in the tame space $H^{\infty,s_2}(\mathbb{R})$. We remark that we cannot apply to the Cauchy problem (1), (2) a usual fixed point scheme in Banach spaces since the linearized problem (7) has a unique solution which presents a loss of regularity and/or a different behavior at infinity with respect to the data. Thus the problem (7) is not well posed in H^{s_1,s_2}; however it turns out to be well posed in $H^{\infty,s_2}(\mathbb{R})$ which is a tame Fréchet space, so there we can apply the Nash Moser theorem.

Remark 2 In the linear case treated in [3], as a consequence of the energy estimates in weighted Sobolev spaces, we also obtained that the Cauchy problem is well posed in $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}'(\mathbb{R})$. In the semilinear case, we are not able to prove in the same way well posedness in $\mathscr{S}(\mathbb{R})$. In fact, if the data of the problem are Schwartz functions, they belong in particular to $H^{\infty,s_2}(\mathbb{R})$ for every $s_2 > 0$, however, in the semilinear case, the upper bound T^* of the interval of existence of the solution may depend on s_2 and possibly tends to 0 when $s_2 \to +\infty$.

Remark 3 The techniques used in this paper may be adapted to study semilinear p-evolution equations in higher space dimension x at least in some particular cases as, for instance, Schrödinger-type equations ($p = 2$). For this type of equations, at least the linear theory is well established in general space dimension, cf. [8, 9, 16] and it could be easily applied to the analysis of the linearized Cauchy problem (7). We will treat this problem for general p-evolution equations in a future paper.

2 Preliminaries: SG-Calculus and Nash Moser Theorem

2.1 SG-Calculus

We recall here the definition and the main properties of the **SG** classes of pseudodifferential operators. In view of the purposes of this paper we shall state them for symbols defined on $\mathbb{R}^2$, but they have obvious extension in higher dimension. For this generalization and for more details on these classes we refer to [11, 19, 20]. Fixed $m_1, m_2 \in \mathbb{R}$, the space $\mathbf{SG}^{m_1,m_2}(\mathbb{R}^2)$ is the space of all functions $p(x,\xi) \in C^\infty(\mathbb{R}^2)$ satisfying the following estimates:

$$\sup_{(x,\xi)\in\mathbb{R}^2} \langle\xi\rangle^{-m_1+\alpha}\langle x\rangle^{-m_2+\beta}|\partial_\xi^\alpha\partial_x^\beta p(x,\xi)| < \infty \tag{8}$$

for every $\alpha, \beta \in \mathbb{N}$. We can associate to every $p \in \mathbf{SG}^{m_1,m_2}(\mathbb{R}^2)$ the pseudodifferential operator defined by

$$Pu(x) = p(x,D)u(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi)\,d\xi. \tag{9}$$

If $p \in \mathbf{SG}^{m_1,m_2}(\mathbb{R}^2)$, then the operator $p(x,D)$ is a linear continuous map from $\mathscr{S}(\mathbb{R})$ to $\mathscr{S}(\mathbb{R})$ and extends to a linear continuous map from $\mathscr{S}'(\mathbb{R})$ to $\mathscr{S}'(\mathbb{R})$ and from $H^{s_1,s_2}(\mathbb{R})$ to $H^{s_1-m_1,s_2-m_2}(\mathbb{R})$ for every $s_1, s_2 \in \mathbb{R}$. We also recall the following result concerning the composition and the adjoint of **SG** operators.

Proposition 1 *Let $p \in \mathbf{SG}^{m_1,m_2}(\mathbb{R}^2)$ and $q \in \mathbf{SG}^{m_1',m_2'}(\mathbb{R}^2)$. Then there exists a symbol $s \in \mathbf{SG}^{m_1+m_1',m_2+m_2'}(\mathbb{R}^2)$ such that $p(x,D)q(x,D) = s(x,D)+R$ where R is a smoothing operator $\mathscr{S}'(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$. Moreover, s has the following asymptotic expansion*

$$s(x,\xi) \sim \sum_\alpha \alpha!^{-1}\partial_\xi^\alpha p(x,\xi)D_x^\alpha q(x,\xi)$$

i.e. for every $N \geq 1$, we have

$$s(x,\xi) - \sum_{|\alpha|<N} \alpha!^{-1}\partial_\xi^\alpha p(x,\xi)D_x^\alpha q(x,\xi) \in \mathbf{SG}^{m_1+m_1'-N,m_2+m_2'-N}(\mathbb{R}^2).$$

Proposition 2 *Let $p \in \mathbf{SG}^{m_1,m_2}(\mathbb{R}^2)$ and let P^* be the L^2-adjoint of $p(x,D)$. Then there exists a symbol $p^* \in \mathbf{SG}^{m_1,m_2}(\mathbb{R}^2)$ such that $P^* = p^*(x,D)+R'$, where R' is a smoothing operator $\mathscr{S}'(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$. Moreover, p^* has the following asymptotic expansion*

$$p^*(x,\xi) \sim \sum_\alpha \alpha!^{-1}\partial_\xi^\alpha D_x^\alpha \overline{p(x,\xi)}$$

i.e. for every $N \geq 1$, *we have*

$$p^*(x,\xi) - \sum_{|\alpha|<N} \alpha!^{-1} \partial_\xi^\alpha D_x^\alpha \overline{p(x,\xi)} \in \boldsymbol{SG}^{m_1-N,m_2-N}(\mathbb{R}^2).$$

We will denote in the sequel by $S^m(\mathbb{R}^2), m \in \mathbb{R}$, the class of symbols $p(x,\xi) \in C^\infty(\mathbb{R}^2)$ satisfying

$$\sup_{(x,\xi)\in\mathbb{R}^2} \langle\xi\rangle^{-m+\alpha} |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| < \infty,$$

for every $\alpha, \beta \in \mathbb{N}$. We observe that the following inclusion holds

$$\mathbf{SG}^{m_1,m_2}(\mathbb{R}^2) \subset S^{m_1}(\mathbb{R}^2) \tag{10}$$

for every $m_1 \in \mathbb{R}, m_2 \leq 0$.

The following theorem has been proved in [3, Theorem 2.3], and provides an extension to pseudodifferential operators of **SG**-type of the well known sharp Gårding theorem.

Theorem 2 *Let* $m_1 \geq 0, m_2 \leq 0$, $a \in \boldsymbol{SG}^{m_1,m_2}(\mathbb{R}^2)$ *such that* $\operatorname{Re} a(x,\xi) \geq 0$ *if* $|\xi| \geq C$ *for some positive* C. *Then there exist pseudo-differential operators* $Q = q(x,D)$, $R = r(x,D)$ *and* $R_0 = r_0(x,D)$ *with symbols, respectively,* $q \in \boldsymbol{SG}^{m_1,m_2}(\mathbb{R}^2)$, $r \in \boldsymbol{SG}^{m_1-1,m_2}(\mathbb{R}^2)$ *and* $r_0 \in S^0(\mathbb{R}^2)$ *such that*

$$a(x,D) = q(x,D) + r(x,D) + r_0(x,D), \tag{11}$$

$$\operatorname{Re}\langle q(x,D)u,u\rangle \geq 0 \qquad \forall u \in \mathscr{S}(\mathbb{R}) \tag{12}$$

and

$$r(x,\xi) = \psi_1(\xi) D_x a(x,\xi) + \sum_{2\leq\alpha+\beta\leq 2m_1-1} \psi_{\alpha,\beta}(\xi) \partial_\xi^\alpha D_x^\beta a(x,\xi) \tag{13}$$

for some real valued functions $\psi_1, \psi_{\alpha,\beta}$ *with* $\psi_1 \in \boldsymbol{SG}^{-1,0}(\mathbb{R}^2)$ *and* $\psi_{\alpha,\beta} \in \boldsymbol{SG}^{\alpha-\beta/2,0}(\mathbb{R}^2)$ *depending only on* ξ.

We remark that the terms in (13) can be re-arranged so that we have

$$r(x,\xi) = \sum_{j=1}^{m-1} r_j(x,\xi), \tag{14}$$

$$r_j(x,\xi) = \begin{cases} \psi_1(\xi) D_x a(x,\xi) + \sum_{2\leq\alpha+\beta\leq 3} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta a(x,\xi), & j = m-1, \\ \sum_{2(m-j)\leq\alpha+\beta\leq 2(m-j)+1} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta a(x,\xi), & 1 \leq j \leq m-2. \end{cases} \tag{15}$$

We also remark that Theorem 2 implies the well-known sharp Gårding inequality

$$\mathrm{Re}\langle A(x, D_x)u, u\rangle \geq -c\|u\|^2_{(m-1)/2,0} \tag{16}$$

for some fixed constant $c > 0$ (cf. [17, Theorem 4.4]).

We recall here also the Fefferman-Phong inequality (cf. [13]):

Theorem 3 *Let $A(x, \xi) \in S^m(\mathbb{R}^2)$ with $A(x, \xi) \geq 0$. Then*

$$\mathrm{Re}\langle A(x, D_x)u, u\rangle \geq -c\|u\|^2_{(m-2)/2,0} \qquad \forall u \in H^{m,0} \tag{17}$$

for some $c > 0$.

We remark that, by Lerner and Morimoto [18], for $m = 2$ the constant c in (17) depends only on $\max\limits_{|\alpha|+|\beta|\leq 7} C_{\alpha,\beta}$ for $C_{\alpha,\beta} := \sup\limits_{x,\xi\in\mathbb{R}} |\partial_\xi^\alpha \partial_x^\beta A(x, \xi)|\langle\xi\rangle^{-2+\alpha}$.

2.2 Tame Fréchet Spaces and the Nash Moser Theorem

We recall here the notions of tame space, tame maps, and the statement of the Nash-Moser inversion theorem, see [14] for further details. Moreover, we show that, for every fixed $s_1, s_2 \in \mathbb{R}$, $H^{s_1,\infty}$ and H^{∞,s_2} are tame spaces.

A *graded* Fréchet space X is a Fréchet space endowed with a *grading*, i.e. an increasing sequence of semi-norms:

$$|x|_n \leq |x|_{n+1}, \qquad \forall n \in \mathbb{N}_0,\ x \in X.$$

Example 1 Given a Banach space B, consider the space $\Sigma(B)$ of all sequences $\{v_k\}_{k\in\mathbb{N}_0} \subset B$ such that

$$|\{v_k\}|_n := \sum_{k=0}^{+\infty} e^{nk}\|v_k\|_B < +\infty \qquad \forall n \in \mathbb{N}_0.$$

We have that $\Sigma(B)$ is a graded Fréchet space with the topology induced by the family of seminorms $|\cdot|_n$ (which is in fact a grading on $\Sigma(B)$).

We say that a linear map $L : X \to Y$ between two graded Fréchet spaces is a *tame linear map* if there exist $r, n_0 \in \mathbb{N}$ such that for every integer $n \geq n_0$ there exists a constant $C_n > 0$, depending only on n, s.t.

$$|Lx|_n \leq C_n|x|_{n+r} \qquad \forall x \in X. \tag{18}$$

The numbers n_0 and r are called respectively *base* and *degree* of the *tame estimate* (18).

Definition 1 A graded Fréchet space X is said to be *tame* if there exist a Banach space B and two tame linear maps $L_1 : X \to \Sigma(B)$ and $L_2 : \Sigma(B) \to X$ such that $L_2 \circ L_1$ is the identity on X.

Obviously, given a graded Fréchet space X and a tame space Y, if there exist two linear tame maps $L_1 : X \to Y$ and $L_2 : Y \to X$ such that $L_2 \circ L_1$ is the identity on X, then also X is a tame space.

Lemma 1 *The spaces* $H^{s_1,\infty}$ *and* H^{∞,s_2} *are tame.*

Proof We first recall that $H^\infty := \bigcap_{s\in\mathbb{R}} H^s$ endowed with the seminorms $|f|_n := \max_{s\le n} \|f\|_s$ for every $n \in \mathbb{N}$ is a tame Fréchet space, cf. [10]. Moreover the map $L : H^\infty \to H^{\infty,s_2}$ defined by $L(f) = \langle x\rangle^{-s_2} f$ is a tame isomorphism since for every $n = 0, 1, 2, \dots$ we have:

$$|L(f)|_{n,s_2} = \max_{s_1\le n} \|L(f)\|_{s_1,s_2} = \max_{s_1\le n} \|\langle x\rangle^{-s_2} f\|_{s_1,s_2}$$
$$\le C_n \max_{s_1\le n} |||\langle x\rangle^{-s_2} f|||_{s_1,s_2} = |f|_n$$

and

$$|f|_n = \max_{s_1\le n} \|f\|_{s_1} \le C'_n \max_{s_1\le n} \|\langle x\rangle^{-s_2} f\|_{s_1,s_2} = |L(f)|_{n,s_2}.$$

Thus, H^{∞,s_2} is a tame space. $H^{s_1,\infty}$ is also tame, since the Fourier transform $\mathscr{F}$ is an isomorphism between H^{s_1,s_2} and H^{s_2,s_1}, and $\|\mathscr{F}(f)\|_{s_2,s_1} = \|f\|_{s_1,s_2}$; by this, it is easy to prove that $\mathscr{F} : H^{s_1,\infty} \to H^{\infty,s_2}$ defines a tame map with tame inverse given by the inverse Fourier transform. □

Given now a nonlinear map $T : U \to Y$ where $U \subset X$ and X, Y are graded spaces, we say that T satisfies a *tame estimate* of degree r and base n_0 if for every integer $n \ge n_0$ there exists a constant $C_n > 0$ such that

$$|T(u)|_n \le C_n(1 + |u|_{n+r}) \qquad \forall u \in U. \tag{19}$$

We say that T is *tame* if it satisfies a tame estimate (19) in a neighborhood of each point $u \in U$ (with constants r, n_0 and C_n which may depend on the neighbourhood).

Notice that a linear map is tame if and only if it is a tame linear map.

Given a map $T : U \subset X \to Y$, we define the *Fréchet* derivative $DT(u)v$ of T at $u \in U$ in the direction $v \in X$ by

$$DT(u)v := \lim_{\epsilon\to 0} \frac{T(u+\epsilon v) - T(u)}{\epsilon}, \tag{20}$$

and we say that T is $C^1(U)$ if the limit (20) exists and the derivative $DT : U \times X \to Y$ is continuous. We can also define recursively the higher order Fréchet derivatives $D^n T : U \times X^n \to Y$ of T, cf. [14]; we say that T is $C^\infty(U)$ if all the Fréchet derivatives of T exist and are continuous. A *smooth tame* map $T : U \to Y$ defined on an open subset U of X is a C^∞ map such that $D^n T$ is tame for all $n \in \mathbb{N}_0$.

It is known that sums and compositions of smooth tame maps are smooth tame, and, moreover, linear and nonlinear partial differential operators and integration are smooth tame maps, see [14] for the proofs of these results. Finally we recall the statement of Nash-Moser inversion theorem in the tame Fréchet spaces category, which will be used in the sequel to approach the Cauchy problem (1).

Theorem 4 (Nash-Moser-Hamilton) *Let X, Y be tame spaces, U an open subset of X and $T : U \to Y$ a smooth tame map. If the equation $DT(u)v = h$ has a unique solution $v := S(u, h)$ for all $u \in U$ and $h \in Y$, and if $S : U \times Y \to X$ is smooth tame, then T is locally invertible and each local inverse is smooth tame.*

3 Well Posedness for the Linearized Cauchy Problem

The following theorem is the key to prove the main result of this paper. It deals with the linear Cauchy problem (7), and proves that if the data of (7) are chosen in the Sobolev space H^{s_1,s_2}, $s_1, s_2 \in \mathbb{R}$, then there exists a unique solution $v(t) \in H^{s_1-2\delta\eta(p-1),s_2-2\delta\epsilon}$ for some $\delta > 0$ and for every $0 \leq \epsilon, \eta \leq 1$ such that $\epsilon + \eta = 1$.

Theorem 5 *Under the assumptions of Theorem 1, there exists $\delta > 0$ such that for every $u \in C([0, T]; H^{3p-1,3p-2}(\mathbb{R}))$, $f \in C([0, T]; H^{s_1,s_2}(\mathbb{R}))$ and $u_0 \in H^{s_1,s_2}(\mathbb{R})$, there exists a unique solution v of (7) such that $v \in C^1([0, T]; H^{s_1-2\delta\eta(p-1),s_2-2\delta\epsilon}(\mathbb{R}))$ for every $\epsilon, \eta \in [0, 1]$ with $\epsilon+\eta = 1$. Moreover v satisfies the following energy estimate:*

$$\|v(t,\cdot)\|^2_{s_1-2\delta\eta(p-1),s_2-2\delta\epsilon} \leq C_{s_1,s_2,\gamma} e^{(1+\|u\|^{3p-2}_{3p-1,3p-2})t} \left(\|u_0\|^2_{s_1,s_2} + \int_0^t \|f(\tau,\cdot)\|^2_{s_1,s_2} d\tau \right) \forall t \in [0, T]. \tag{21}$$

Remark 4 Notice that the solution v presents the loss $2\delta\eta(p-1)$ in the first Sobolev index and the loss $2\delta\epsilon$ in the second one. In the case $s_2 = 0$, $\epsilon = 0$, $\eta = 1$ we recapture the result of [1, Theorem 2.1]. Moreover, in the linear case (i.e., if (7) does not depend on u), we can obtain either well-posedness with loss of $2\delta(p - 1)$ derivatives and no loss of decay (take $\eta = 1$ and $\epsilon = 0$), or the result of [3], that is well-posedness without loss of derivatives but with loss of decay 2δ (take $\eta = 0$ and $\epsilon = 1$). We can also obtain all the intermediate estimates. A similar result has been proved in [7], where intermediate estimates for Schrödinger equations ($p = 2$) have been proved in Gevrey classes.

The proof of Theorem 5 consists in choosing an appropriate and invertible change of variable

$$v(t, x) = e^\Lambda(x, D)w(t, x) \tag{22}$$

which transforms the Cauchy problem (7) into an equivalent Cauchy problem

$$\begin{cases} P_\Lambda(t,x,u(t,x),D_t,D_x)w(t,x) = f_\Lambda(t,x) & (t,x)\in[0,T]\times\mathbb{R} \\ w(0,x) = u_{0,\Lambda}(x) & x\in\mathbb{R} \end{cases} \tag{23}$$

for

$$P_\Lambda := (e^\Lambda)^{-1}Pe^\Lambda, \quad f_\Lambda := (e^\Lambda)^{-1}f, \quad u_{0,\Lambda} := (e^\Lambda)^{-1}u_0$$

which is well-posed in L^2 (and therefore in all the weighted Sobolev spaces H^{s_1,s_2}). By the energy estimate in H^{s_1,s_2} for the solution w to the Cauchy problem (23), we then deduce the energy estimate (21) from (22).

The operator Λ will be of the form

$$\Lambda(x,D) = \lambda_1(x,D) + \ldots + \lambda_{p-1}(x,D),$$

so

$$P_\Lambda := (e^{\lambda_1})^{-1}\cdots(e^{\lambda_{p-1}})^{-1}Pe^{\lambda_{p-1}}\cdots(e^{\lambda_1}),$$

$$f_\Lambda := (e^{\lambda_1})^{-1}\cdots(e^{\lambda_{p-1}})^{-1}f, \quad u_{0,\Lambda} := (e^{\lambda_1})^{-1}\cdots(e^{\lambda_{p-1}})^{-1}u_0.$$

We construct here below the transformation Λ and we point out its main properties in Proposition 3. Then we prove the invertibility of e^Λ in Proposition 4. In the subsequent Lemma 2 we show how to obtain the energy estimate (21) for the Cauchy problem (7) from the H^{s_1,s_2} energy estimate for the Cauchy problem (23). After that, in Lemma 5 we state the regularity with respect to x, u of the coefficients $a_j(t,x,u)$ of the linear operator (7), for $0\le j\le p-1$. This section ends with the proof of Theorem 5.

Definition 2 For every $k=1,\ldots,p-1$ we define the symbols

$$\lambda_{p-k}(x,\xi) := M_{p-k}\omega\left(\frac{\xi}{h}\right)\langle\xi\rangle_h^{-k+1}\int_0^x \langle y\rangle^{-\frac{p-k}{p-1}}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)dy, \tag{24}$$

where h and M_{p-k} are positive constants such that $h\ge 1$, $\omega\in C^\infty(\mathbb{R})$ is such that

$$\omega(y) = \begin{cases} 0 & |y|\le 1 \\ |y|^{p-1}/y^{p-1} & |y|\ge 2 \end{cases}, \tag{25}$$

and $\psi\in C_0^\infty(\mathbb{R})$ is such that $0\le\psi(y)\le 1$ for all $y\in\mathbb{R}$, $\psi(y)=1$ for $|y|\le\frac{1}{2}$, and $\psi(y)=0$ for $|y|\ge 1$.

Proposition 3 *There exists a constant $C > 0$ such that for every $(x, \xi) \in \mathbb{R}^2$ the following conditions hold:*

$$|\lambda_{p-1}(x,\xi)| \leq M_{p-1}\left(\log 2 + \epsilon \log\langle x\rangle + \eta(p-1)\log\langle \xi\rangle_h\right) \quad (26)$$
$$\forall \epsilon, \eta \in [0,1]\ \epsilon + \eta = 1;$$
$$|\lambda_{p-k}(x,\xi)| \leq CM_{p-k}, \quad 2 \leq k \leq p-1. \quad (27)$$

Moreover, for every α, β with $(\alpha, \beta) \neq (0, 0)$, there exists $C_{\alpha,\beta} > 0$ such that for $|\xi| > 2h$:

$$|\partial_\xi^\alpha \partial_x^\beta \lambda_{p-k}(x,\xi)| \leq C_{\alpha,\beta}\langle x\rangle^{-\beta}\langle \xi\rangle_h^{-\alpha}, \quad 1 \leq k \leq p-1. \quad (28)$$

Proof We only prove (26) and (27); the inequality (28) can be deduced as in the proof of [5, Lemma 2.1]. Let $E = \{(y, \xi) \in \mathbb{R}^2 : \langle y\rangle \leq \langle \xi\rangle_h^{p-1}\}$. If $x \in E$, $x > 0$, then by (24), integrating we have:

$$\begin{aligned}
|\lambda_{p-1}(x,\xi)| &\leq M_{p-1}\int_0^x \frac{1}{\sqrt{1+y^2}}dy \leq M_{p-1}\log\left(2\langle x\rangle\right) \\
&\leq M_{p-1}(\ln 2 + \log\langle x\rangle) \\
&\leq M_{p-1}(\ln 2 + \log\langle x\rangle^{\epsilon}\langle \xi\rangle_h^{\eta(p-1)}) \\
&\leq M_{p-1}(\ln 2 + \epsilon\log\langle x\rangle + \eta(p-1)\log\langle \xi\rangle_h)
\end{aligned}$$

for every $0 \leq \epsilon,\ \eta \leq 1, \epsilon + \eta = 1$. Analogously, if $x \notin E$ we get

$$\begin{aligned}
|\lambda_{p-1}(x,\xi)| &\leq M_{p-1}\int_0^{\sqrt{\langle \xi\rangle_h^{2(p-1)}-1}} \frac{1}{\sqrt{1+y^2}}dy \\
&\leq M_{p-1}\ln(2\langle \xi\rangle_h^{p-1}) \\
&\leq M_{p-1}(\ln 2 + \log\langle x\rangle^{\epsilon}\langle \xi\rangle_h^{\eta(p-1)}) \\
&\leq M_{p-1}(\ln 2 + \epsilon\log\langle x\rangle + \eta(p-1)\log\langle \xi\rangle_h),
\end{aligned}$$

using the fact that for $x \notin E$ we have $\langle \xi\rangle_h^{p-1} < \langle x\rangle$. Similar estimates can be obtained for $x < 0$. The estimate (27) can be proved by a similar argument. □

From Proposition 3 we obtain in particular that $e^{\pm\lambda_{p-1}} \in \mathbf{SG}^{M_{p-1}\eta(p-1),M_{p-1}\epsilon}$ for every $\epsilon, \eta \geq 0$ such that $\epsilon + \eta = 1$ whereas for $k = 2, \ldots, p-1$, we have $e^{\pm\lambda_{p-k}} \in \mathbf{SG}^{0,0}(\mathbb{R}^2) \subset S^0(\mathbb{R}^2)$.

Proposition 4 *For every* $k = 1, \ldots, p-1$, *let* λ_{p-k} *be defined by* (24). *There exists* $h_k \geq 1$ *such that for every* $h \geq h_k$ *the operator* $e^{\lambda_{p-k}}(x, D)$ *is invertible and*

$$\left(e^{\lambda_{p-k}}(x, D)\right)^{-1} = e^{-\lambda_{p-k}}(x, D)(I + r_{p-k}(x, D)), \tag{29}$$

where I *stands for the identity operator and* $r_{p-k}(x, D)$ *is a pseudodifferential operator with principal symbol*

$$r_{p-k,-k}(x, \xi) = \partial_\xi \lambda_{p-k}(x, \xi) D_x \lambda_{p-k}(x, \xi) \in \boldsymbol{SG}^{-k, -\frac{p-k}{p-1}}. \tag{30}$$

Proof We first observe that

$$e^{\lambda_{p-k}}(x, D) e^{-\lambda_{p-k}}(x, D) = I - \tilde{r}_{p-k}(x, D),$$

where $\tilde{r}_{p-k}$ has principal symbol $r_{p-k,-k}$ in (30). From (28) we have

$$|r_{p-k,-k}(x, \xi)| \leq C_k M_{p-k}^2 h^{-1},$$

and we similarly estimate the derivatives. We see that for h large enough, say $h \geq h_k$, the operator $I - \tilde{r}_{p-k}$ is invertible on L^2 with inverse given by the Neumann series

$$\sum_{j \geq 0} \tilde{r}_{p-k}^j = I + r_{p-k},$$

and the operator r_{p-k} has principal part (30). Thus,

$$e^{\lambda_{p-k}}(x, D) e^{-\lambda_{p-k}}(x, D)(I + r_{p-k}) = I,$$

and $e^{-\lambda_{p-k}}(x, D)(I + r_{p-k})$ is a right inverse of $e^{\lambda_{p-k}}(x, D)$. Similarly we can obtain that it is also a left inverse. □

Lemma 2 *If the Cauchy problem* (23) *is* H^{s_1,s_2} *well posed, and the energy estimate*

$$\|w\|_{s_1,s_2}^2 \leq C e^{(1+\|u\|_{3p-1,3p-2}^{3p-2})t} \left(\|u_{0,\Lambda}\|_{s_1,s_2}^2 + \int_0^t \|f_\Lambda(\tau)\|_{s_1,s_2}^2 d\tau \right) \tag{31}$$

holds for every $t \in [0, T]$, *then the Cauchy problem* (7) *admits a unique solution*

$$v \in C([0, T]; H^{s_1 - 2\delta\eta(p-1), s_2 - 2\delta\epsilon})$$

for every $\epsilon, \eta \in [0, 1]$ *with* $\epsilon + \eta = 1$ *which satisfies the energy estimate* (5).

Proof From Proposition 3 we know that

$$|\Lambda(x,\xi)| \leq M_{p-1}\left(\log 2 + \epsilon \log\langle x\rangle + \eta(p-1)\log\langle \xi\rangle_h\right) + \sum_{k=2}^{p-1} C_k M_{p-k}$$
$$\leq \delta\left(1 + \epsilon \log\langle x\rangle + \eta(p-1)\log\langle \xi\rangle_h\right)$$

with a positive constant δ depending on $M_1, \ldots, M_{p-1}$.This yelds

$$|e^{\pm\Lambda(x,\xi)}| \leq e^{\delta}\langle x\rangle^{\delta\epsilon}\langle \xi\rangle_h^{\delta\eta(p-1)},$$

and by the energy estimate (31) we get

$$\begin{aligned}
\|v\|^2_{s_1-2\delta\eta(p-1),s_2-2\delta\epsilon} &= \|e^{\Lambda}w\|^2_{s_1-2\delta\eta(p-1),s_2-2\delta\epsilon} \leq \|w\|^2_{s_1-\delta\eta(p-1),s_2-\delta\epsilon} \\
&\leq Ce^{(1+\|u\|^{3p-2}_{3p-1,3p-2})t}\left(\|u_{0,\Lambda}\|^2_{s_1-\delta\eta(p-1),s_2-\delta\epsilon} + \int_0^t \|f_\Lambda(\tau)\|^2_{s_1-\delta\eta(p-1),s_2-\delta\epsilon} d\tau\right) \\
&\leq Ce^{(1+\|u\|^{3p-2}_{3p-1,3p-2})t}\left(\|u_0\|^2_{s_1,s_2} + \int_0^t \|f(\tau)\|^2_{s_1,s_2} d\tau\right)
\end{aligned}$$

for every $t \in [0, T]$. □

The next Proposition 5 states the regularity with respect to x, u of the coefficients $a_j(t,x,\xi)$ of the linearized operator (7).

Proposition 5 *Under the assumptions* (5) *and* (6)*, there exists* $C' > 0$ *such that for every fixed* $u \in C([0,T]; H^{3p-1,3p-2}(\mathbb{R}))$ *the coefficients* $a_j(t,x,u(t,x))$ *of the operator* $P_u(D)$ *satisfy for every* $1 \leq j \leq p-1$, $(t,x) \in [0,T] \times \mathbb{R}$ *and* $\beta \leq 3p-2$*:*

$$|\partial_x^\beta \operatorname{Re} a_j(t,x,u(t,x))| \leq C'\gamma(u)(1 + \|u\|^\beta_{1+\beta,\beta})\langle x\rangle^{-\beta}, \tag{32}$$

$$|\partial_x^\beta \operatorname{Im} a_j(t,x,u(t,x))| \leq C'\gamma(u)(1 + \|u\|^\beta_{1+\beta,\beta})\langle x\rangle^{-\frac{j}{p-1}-\beta}. \tag{33}$$

Proof For every $\beta \geq 1$ and $1 \leq j \leq p-1$ we have

$$\partial_x^\beta(a_j(t,x,u)) = (\partial_x^\beta a_j)(t,x,u) + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2 \geq 1}} c_\beta \sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i \geq 1}} c_{q,r}(\partial_w^q \partial_x^{\beta_1} a_j)(t,x,u)(\partial_x^{r_1}u)\cdots(\partial_x^{r_q}u)$$

for some $c_\beta, c_{q,r} > 0$. By (6), using the relationship between geometric and arithmetic mean value and Sobolev inequality, this gives for every $\beta \leq 4(p-1)$:

$$\begin{aligned}
&|\partial_x^\beta(\mathrm{Re}\, a_j(t,x,u))| \\
&\quad \leq C\gamma(u)\langle x\rangle^{-\beta} + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2\geq 1}} c_{\beta_1,\beta_2} \sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i\geq 1}} C_{q,r_1,\ldots,r_q}\gamma(u)\langle x\rangle^{-\beta_1}|\partial_x^{r_1}u|\cdots|\partial_x^{r_q}u| \\
&\quad \leq C'\gamma(u)\langle x\rangle^{-\beta}\left(1+\sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2\geq 1}}\sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i\geq 1}} |\langle x\rangle^{r_1}\partial_x^{r_1}u|\cdots|\langle x\rangle^{r_q}\partial_x^{r_q}u|\right) \\
&\quad \leq C'\gamma(u)\langle x\rangle^{-\beta}\left(1+\sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_2\geq 1}}\sum_{\substack{r_1+\ldots+r_q=\beta_2 \\ r_i\geq 1}} \left(\frac{|\langle x\rangle^{r_1}\partial_x^{r_1}u|+\cdots+|\langle x\rangle^{r_q}\partial_x^{r_q}u|}{q}\right)^q\right) \\
&\quad \leq C''\gamma(u)(1+\|u\|_{1+\beta,\beta}^\beta)\langle x\rangle^{-\beta};
\end{aligned}$$

where we have used the fact that for every $1 \leq j \leq q$, $\beta \leq 3p-2$, we have

$$|\langle x\rangle^{r_j}\partial_x^{r_j}u| \leq C\|\langle x\rangle^{r_j}\partial_x^{r_j}u\|_{1,0} = \|u\|_{1+r_j,r_j} \leq \|u\|_{1+\beta,\beta} < \infty.$$

On the other hand, looking at $\mathrm{Im}\, a_j$ and using (5) instead of (6), the same computations give

$$|\partial_x^\beta(\mathrm{Im}\, a_j(t,x,u))| \leq C''\gamma(u)(1+\|u\|_{1+\beta,\beta}^\beta)\langle x\rangle^{-\frac{j}{p-1}-\beta}.$$

□

Remark 5 We observe that a conjugation of the type $(e^{\lambda_{p-k}})^{-1}T_j e^{\lambda_{p-k}}$ with λ_{p-k} given by (24) and $T_j \in \mathbf{SG}^{j,0}$, $j \geq k+1$ depending on γ_j derivatives of u, by Proposition 4 gives:

$$(e^{\lambda_{p-k}})^{-1}T_j e^{\lambda_{p-k}} = e^{-\lambda_{p-k}}\left(T_j + r_{p-k}T_j\right)e^{\lambda_{p-k}} \tag{34}$$

where the principal symbol of r_{p-k} is given by $\partial_\xi\lambda_{p-k}(x,\xi)D_x\lambda_{p-k}(x,\xi) \in \mathbf{SG}^{-k,-(p-k)/(p-1)}$. By the asymptotic expansion we get

$$\sigma\left(T_j + r_{p-k}T_j\right)(x,\xi) = T_j(x,\xi) + \sum_{\alpha=0}^{j-k-1}\frac{1}{\alpha!}\partial_\xi^\alpha r_{p-k}(x,\xi)D_x^\alpha T_j(x,\xi) + S_0(x,\xi)$$

with $S_0 \in \mathbf{SG}^{0,0}$. Since $\partial_\xi^\alpha r_{p-k} D_x^\alpha T_j \in \mathbf{SG}^{j-k-\alpha,-(p-k)/(p-1)-|\alpha|}$ and depends on $\gamma_j + \alpha$ derivatives of u, by re-ordering the sum we get

$$\sigma\left(T_j + r_{p-k}T_j\right)(x,\xi) = T_j(x,\xi) + \sum_{\ell=1}^{j-k} T_{j,\ell}(x,\xi) + T_0$$

with $T_{j,\ell} \in \mathbf{SG}^{\ell,-(p-k)/(p-1)-(j-k-\ell)}$ depending on $\gamma_j + j - k - \ell$ derivatives of u and on M_{p-k}, T_0 of order $(0, 0)$. Thus

$$(e^{\lambda_{p-k}})^{-1}\left(\sum_{j=0}^{p-1} T_j\right)e^{\lambda_{p-k}} = e^{-\lambda_{p-k}}\left(\sum_{j=0}^{p-1}(T_j + r_{p-k}T_j)\right)e^{\lambda_{p-k}}$$

and we have, modulo terms of order $(0, 0)$:

$$\begin{aligned}\sigma\left(\sum_{j=0}^{p-1}(T_j + r_{p-k}T_j)\right)(x,\xi) &= \sum_{j=1}^{p-1} T_j(x,\xi) + \sum_{j=1}^{p-1}\sum_{\ell=1}^{j-k} T_{j,\ell}(x,\xi)\\ &= \sum_{j=p-k}^{p-1} T_j + \sum_{j=1}^{p-k-1}\left(T_j + T_{j+k,j} + \ldots + T_{p-1,j}\right) = \sum_{j=1}^{p-1} T_j'\end{aligned}$$

with $T_j' = T_j$ for $j \ge p-k$, while for $j \le p-k-1$ $T_j' \in \mathbf{SG}^{j,0}$ as well as T_j but depend on $\max\{\gamma_{p-1} + p - 1 - k - j, \gamma_{p-2} + p - 2 - k - j, \ldots, \gamma_{j+k}\}$ derivatives of u and on the constant M_{p-k}.

Remark 6 Similarly, a conjugation of the type $e^{-\lambda}T_k e^{\lambda}$, where $\lambda \in \mathbf{SG}^{0,0}$ and $T_k \in \mathbf{SG}^{k,0}$ depends on γ_k derivatives of u, gives, modulo terms of order $(0, 0)$, the operator

$$T_k + \sum_{\alpha=1}^{k-1}\frac{1}{\alpha!}\left(\partial_\xi^\alpha T_k\right)e^{-\lambda}D_x^\alpha e^{\lambda} + \sum_{\beta=1}^{k-1}\sum_{\alpha=0}^{k-\beta}\frac{1}{\alpha!\beta!}\partial_\xi^\beta e^{-\lambda}D_x^\beta\left(\partial_\xi^\alpha T_k D_x^\alpha e^{\lambda}\right);$$

at each level $1 \le j \le k-1$ we find, except for T_j itself, new terms of type $\partial_\xi^\beta e^{-\lambda}D_x^\beta\left(\partial_\xi^\alpha T_{j+\alpha+\beta}D_x^\alpha e^{\lambda}\right)$ with the same decay as T_j and depending on $\gamma_{j+\alpha+\beta} + \beta$ derivatives of u.

Proof of Theorem 5 First of all we observe that the assumption (4) implies that $a_p(t) \ge C_p$ for every $t \in [0, T]$ or $a_p(t) \le -C_p$ for every $t \in [0, T]$ for a positive constant C_p. We will prove the theorem under the first condition. If the second one holds the result remains valid with only modifications of signs in the proof.

Fixed u, we consider the linear operator

$$iP_u(t,x,u(t,x),D_t,D_x) = \partial_t + ia_p(t)D_x^p + \sum_{j=0}^{p-1} ia_j(t,x,u)D_x^j$$

with a_p satisfying (4) and a_j satisfying (32), (33) for every $1 \le j \le p-1$, and we apply for $h \ge h_1$ (see Proposition 4) the first conjugation $(e^{\lambda_{p-1}})^{-1} iP_u e^{\lambda_{p-1}}$, with λ_{p-1} in Definition 2 satisfying Proposition 3. Let us first notice that

$$\begin{aligned}(e^{\lambda_{p-1}})^{-1} iP_u e^{\lambda_{p-1}} &= \partial_t + e^{-\lambda_{p-1}}\left(ia_p(t)D_x^p + \sum_{j=0}^{p-1} ia_j(t,x,u)D_x^j\right)e^{\lambda_{p-1}} \\ &+ e^{-\lambda_{p-1}}\left(ir_{p-1}(x,D)a_p(t)D_x^p + \sum_{j=0}^{p-1} ir_{p-1}(x,D)a_j(t,x,u)D_x^j\right)e^{\lambda_{p-1}}\end{aligned}$$

and that the principal symbol of r_{p-1} is given by $\partial_\xi\lambda_{p-1}(x,\xi)D_x\lambda_{p-1}(x,\xi) \in \mathbf{SG}^{-1,-1}$. The composition $e^{-\lambda_{p-1}} ia_p\xi^p e^{\lambda_{p-1}}$ provides, among others, the term $-\partial_\xi\lambda_{p-1}(x,\xi)a_p\xi^p\partial_x\lambda_{p-1}(x,\xi) = -ia_p\xi^p r_{p-1,-1}(x,\xi)$ which cancels with the principal part of the symbol of $e^{-\lambda_{p-1}} ir_{p-1}a_p\xi^p e^{\lambda_{p-1}}$. Then, we notice that by Remark 5 we can write

$$\begin{aligned}(e^{\lambda_{p-1}})^{-1} iP_u e^{\lambda_{p-1}} &= \partial_t + e^{-\lambda_{p-1}}\left(ia_p(t)D_x^p + \sum_{j=0}^{p-1} ia'_j(t,x,u,D_x)\right)e^{\lambda_{p-1}} \\ &+ \mathrm{op}\left(ia_p\xi^p r_{p-1,-1}\right)(t,x,D)\end{aligned}$$

with new terms

$$a'_{p-1}(t,x,u,D_x) = a_{p-1}(t,x,u)D_x^{p-1}$$

and, for $0 \le j \le p-2$, $a'_j(t,x,u,D_x)$ is a pseudodifferential operator given by $a_j(t,x,u)D_x^j$ plus other terms of the same order. Namely, a'_j satisfy estimates of the form

$$\begin{aligned}&|\partial_\xi^\alpha\partial_x^\beta \mathrm{Re}\, a'_j(t,x,u(t,x),\xi)| \\ &\qquad \le C_{M_{p-1}}\gamma(u)(1+\|u\|_{p-1-j+\beta,p-2-j+\beta}^{p-2-j+\beta})\langle x\rangle^{-\beta}\langle\xi\rangle^{j-\alpha},\end{aligned} \tag{35}$$

$$\begin{aligned}&|\partial_\xi^\alpha\partial_x^\beta \mathrm{Im}\, a'_j(t,x,u(t,x),\xi)| \\ &\qquad \le C_{M_{p-1}}\gamma(u)(1+\|u\|_{p-1-j+\beta,p-2-j+\beta}^{p-2-j+\beta})\langle x\rangle^{-\frac{j}{p-1}-\beta}\langle\xi\rangle^{j-\alpha}.\end{aligned} \tag{36}$$

The asymptotic expansion gives

$$
\begin{aligned}
iP_1(t,x,u,D) &:= (e^{\lambda_{p-1}})^{-1} iP_u e^{\lambda_{p-1}} \qquad (37)\\
&= \partial_t + ia_p(t)D_x^p + ia_{p-1}(t,x,u)D_x^{p-1}\\
&+ \operatorname{op}\left(ipa_p\xi^{p-1}D_x\lambda_{p-1}\right)\\
&+ \sum_{\alpha=2}^{p-1}\sum_{\substack{\beta+\gamma=\alpha\\ \beta\le p}}\frac{1}{\beta!\gamma!}\operatorname{op}\left(a_p(t)\partial_\xi^\gamma e^{-\lambda_{p-1}}\cdot\partial_\xi^\beta\xi^p\cdot D_x^\alpha e^{\lambda_{p-1}}\right)\\
&+ \sum_{j=1}^{p-2} ia'_j(t,x,u,D_x)\\
&+ \sum_{j=1}^{p-1}\sum_{\alpha=1}^{j-1}\frac{1}{\alpha!}\operatorname{op}\left(e^{-\lambda_{p-1}}\partial_\xi^\alpha ia'_j D_x^\alpha e^{\lambda_{p-1}}\right)\\
&+ \sum_{j=1}^{p-1}\sum_{\beta=1}^{j-1}\sum_{\alpha=0}^{j-1-\beta}\sum_{\beta_1+\beta_2=\beta}\frac{1}{\alpha!\beta_1!\beta_2!}\operatorname{op}\left(\partial_\xi^\beta e^{-\lambda_{p-1}}D_x^{\beta_1}\partial_\xi^\alpha ia'_j D_x^{\alpha+\beta_2}e^{\lambda_{p-1}}\right)\\
&+ s_0(t,x,u,D)
\end{aligned}
$$

with a term s_0 of order $(0,0)$. Notice that, by (35), (36) and Remark 6, in (37) we find at each level $1 \le j \le p-2$, except for the original terms $a_j(t,x,u)D_x^j$, terms with decay at least of type $\langle x\rangle^{-1}$, depending at most on M_{p-1}, and depending at most on

$$
\gamma_{j+|\alpha|+|\beta|} + |\beta| = p-(j+|\alpha|+|\beta|)-1+|\beta| = p-j-|\alpha|-1 \le p-j-1
$$

derivatives of u, so that we get

$$
\begin{aligned}
iP_1 &= \partial_t + ia_p(t)D_x^p + ia_{p-1}(t,x,u)D_x^{p-1} \qquad (38)\\
&\quad +\operatorname{op}\left(ipa_p\xi^{p-1}D_x\lambda_{p-1}\right) + \sum_{j=1}^{p-2} ia''_j(t,x,u,D_x) + s_0(t,x,u,D)
\end{aligned}
$$

where the pseudodifferential operators a''_j are given by $a_jD_x^j$ plus other terms with the same behavior, namely a''_j still satisfy (35) and (36).

Now, let us focus on the term A_{p-1} of order $p-1$ with respect to ξ in (38). By (24) and (33), the choice of ω in (25), and (4) we get for every $|\xi| \ge 2h$:

$$
\begin{aligned}
\operatorname{Re} A_{p-1}(t,x,u,\xi) &:= \operatorname{Re}\left(ia_{p-1}(t,x,u)\xi^{p-1} + pa_p(t)\xi^{p-1}\partial_x\lambda_{p-1}(x,\xi)\right)\\
&= -\operatorname{Im} a_{p-1}(t,x,u)\xi^{p-1} + pa_p(t)\xi^{p-1}\partial_x\lambda_{p-1}(x,\xi)
\end{aligned}
$$

$$\begin{aligned}
&\geq -C'\gamma(u)\langle x\rangle^{-1}\langle\xi\rangle_h^{p-1} + M_{p-1}pa_p(t)|\xi|^{p-1}\langle x\rangle^{-1}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\\
&\geq \frac{\langle\xi\rangle_h^{p-1}}{\langle x\rangle}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\left(-C'\gamma(u) + M_{p-1}pC_p\left(\frac{2}{\sqrt{5}}\right)^{p-1}\right)\\
&- C'\gamma(u)\frac{\langle\xi\rangle_h^{p-1}}{\langle x\rangle}\left(1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right)\\
&\geq -2C'\gamma(u)
\end{aligned}$$

if we choose $M_{p-1} \geq \dfrac{C'\gamma(u)\sqrt{5}^{p-1}}{2^{p-1}pC_p}$, where we have also used the fact that $\dfrac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}} \geq \dfrac{1}{2}$ on the support of $1-\psi\left(\dfrac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)$ and $|\xi|^{p-1} \geq (2/\sqrt{5})^{p-1}\langle\xi\rangle_h^{p-1}$ for $|\xi| \geq 2h$. Being the symbol $\mathrm{Re}\, A_{p-1}(t,x,u,\xi) + 2C'\gamma(u)$ non negative, we can apply the sharp Gårding Theorem 2 and we obtain that there exist pseudodifferential operators $Q_{p-1}(t,x,u,D)$, $R_{p-1}(t,x,u,D)$, $R_{0,p-1}(t,x,u,D)$ with symbols

$$Q_{p-1}(t,x,u,\xi) \in \mathbf{SG}^{p-1,0}, \quad R_{p-1}(t,x,u,\xi) \in \mathbf{SG}^{p-2,0}, \quad R_{0,p-1}(t,x,u,\xi) \in S^0$$

such that

$$A_{p-1}(t,x,u,D) = Q_{p-1}(t,x,u,D) + iR_{p-1}(t,x,u,D) + R_{0,p-1}(t,x,u,D)$$

with

$$\mathrm{Re}\langle Q_{p-1}(t,x,u,D)h(t,x), h(t,x)\rangle \geq 0 \quad \forall h \in \mathscr{S}(\mathbb{R}),\ (t,x) \in [0,T]\times\mathbb{R}$$

and by (15)

$$R_{p-1}(t,x,u,\xi) = \sum_{j=1}^{p-2} R_{j,p-1}(t,x,u,\xi) \tag{39}$$

$$R_{p-2,p-1} = -i\left(\psi_1(\xi)D_xA_{p-1} + \sum_{2\leq\alpha+\beta\leq 3}\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-1}\right)$$

$$R_{j,p-1} = -i\sum_{2(p-1-j)\leq\alpha+\beta\leq 2(p-1-j)+1}\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-1}$$

for every $1 \leq j \leq p-3$, where ψ_1 and $\psi_{\alpha,\beta}$ are real valued symbols, $\psi_1(\xi) \in \mathbf{SG}^{-1,0}$ and $\psi_{\alpha,\beta}(\xi) \in \mathbf{SG}^{(\alpha-\beta)/2,0}$. We have so

$$iP_1 = \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D_x) + iR_{p-1}(t,x,u,D_x) + \sum_{j=1}^{p-2} ia_j''(t,x,u,D_x) + s_0(t,x,u,D_x).$$

We notice that, by (39), R_{p-1} adds to the terms a_j'' some new terms; whenever $\beta \neq 0$, these new terms have at least decay $\langle x\rangle^{-1}$, while for $\beta = 0$ we see that

$$\operatorname{Re}\left(-i\psi_{\alpha,0}(\xi)\,\partial_\xi^\alpha A_{p-1}(t,x,u,\xi)\right) = \psi_{\alpha,0}(\xi)\partial_\xi^\alpha \operatorname{Im} A_{p-1}(t,x,u,\xi) \in \mathbf{SG}^{p-1-\alpha/2,0} \subset \mathbf{SG}^{p-2,0}$$

can be added to $\operatorname{Re} a_j''$, while

$$\operatorname{Im}\left(-i\psi_{\alpha,0}(\xi)\,\partial_\xi^\alpha A_{p-1}(t,x,u,\xi)\right) = -\psi_{\alpha,0}(\xi)\partial_\xi^\alpha \operatorname{Re} A_{p-1}(t,x,u,\xi) \in \mathbf{SG}^{p-1-\alpha/2,-1} \subset \mathbf{SG}^{p-2,-\frac{p-2}{p-1}}$$

can be added to $\operatorname{Im} a_j''$. Again, by (39), we see that the largest number of x-derivatives of u appears when $\alpha = 0$, $\beta = 2(p-1-j)+1$ and we have

$$|\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-1}(t,x,u,\xi)| \leq C'\gamma(u)\left(1+\|u\|_{\beta+1,\beta}^{\beta}\right)\langle\xi\rangle^{p-1-\frac{\alpha+\beta}{2}}\langle x\rangle^{-\beta} \leq C'\gamma(u)\left(1+\|u\|_{2(p-j),2(p-j)-1}^{2(p-j)-1}\right)\langle\xi\rangle^{j}\langle x\rangle^{-1}$$

By these considerations, we understand that after the application of Theorem 2, we can write

$$iP_1 = \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D_x) + \sum_{j=1}^{p-2} ia_{j,1}(t,x,u,D_x) + s_1(t,x,u,D) \tag{40}$$

for a new operator s_1 with symbol in S^0, where $a_{j,1}$ are given by a_j'' plus other terms with the same order and decay, depending on $2(p-j)$ derivatives of u, this means that $a_{j,1}$ depend on $\max\{p-j-1, 2(p-j)\} = 2(p-j)$ derivatives of u. Summing up, for every $\beta \leq p-1$ (we need that $2(p-j)+\beta \leq 2(p-1)+\beta \leq 3p-1$) we have

$$|\partial_\xi^\alpha\partial_x^\beta \operatorname{Re} a_{j,1}(t,x,u(t,x),\xi)| \leq C_{M_{p-1}}\gamma(u)(1+\|u\|_{2(p-j)+\beta,2(p-j)-1+\beta}^{2(p-j)-1+\beta})\langle x\rangle^{-\beta}\langle\xi\rangle^{j-\alpha}, \tag{41}$$

$$|\partial_\xi^\alpha \partial_x^\beta \mathrm{Im}\, a_{j,1}(t,x,u(t,x),\xi)| \tag{42}$$
$$\le C_{M_{p-1}}\gamma(u)(1+\|u\|^{2(p-j)-1+\beta}_{2(p-j)+\beta,2(p-j)-1+\beta})\langle x\rangle^{-\frac{j}{p-1}-\beta}\langle\xi\rangle^{j-\alpha}.$$

Now, let us consider, for $h \ge \max\{h_1, h_2\}$ (see Proposition 4), the operator $(e^{\lambda_{p-2}})^{-1} iP_1 e^{\lambda_{p-2}}$, with λ_{p-2} in Definition 2 satisfying Proposition 3. We observe preliminarly that, since $e^{\pm\lambda_{p-2}} \in \mathbf{SG}^{0,0}(\mathbb{R}^2) \subset S^0(\mathbb{R}^2)$, then for the composition $(e^{\lambda_{p-2}})^{-1} s_1(t,x,u,D)e^{\lambda_{p-2}}$ we can use the symbolic calculus in the Hörmander class and obtain that $(e^{\lambda_{p-2}})^{-1} s_1(t,x,u,D)e^{\lambda_{p-2}}$ is again an operator with symbol in $S^0(\mathbb{R}^2)$. Moreover, since $\left(e^{\lambda_{p-2}}\right)^{-1} = e^{-\lambda_{p-2}}(I + r_{p-2})$ and the principal part of r_{p-2} has symbol $r_{p-2,-2}(x,\xi) = \partial_\xi \lambda_{p-2}(x,\xi) D_x \lambda_{p-2}(x,\xi)$ in $\mathbf{SG}^{-2,-\frac{p-2}{p-1}}$, by Remark 5 we obtain

$$(e^{\lambda_{p-2}})^{-1} iP_1 e^{\lambda_{p-2}} = \partial_t + \mathrm{op}\left(ia_p r_{p-2,-2}\right)$$
$$+e^{-\lambda_{p-2}}\left(ia_p(t)D_x^p + Q_{p-1}(t,x,u,D) + \sum_{j=0}^{p-2} ia'_{j,1}(t,x,u,D_x) + s_1(t,x,u,D)\right)e^{\lambda_{p-2}}$$

with $a'_{p-2,1}(t,x,u,D_x) = a_{p-2,1}(t,x,u,D_x)$ and, for $0 \le j \le p-3$, $a'_{j,1}(t,x,u,D_x)$ is given by $a_{j,1}(t,x,u,D_x)$ plus some new terms with the same order and decay as $a_{j,1}$ and depending on $\max\{\gamma_{p-1}+p-1-2-j,\ldots,\gamma_{p-\ell}+p-\ell-2-j,\ldots\gamma_{j+2}\} = \gamma_{j+2} = 2(p-j-2)$, because we have $\gamma_{p-\ell} = 2(p-(p-\ell)) = 2\ell$ for $1 \le \ell \le p-1$. The new terms contain a smaller number of derivatives with respect to (41) and (42). Thus for every $1 \le j \le p-2$ we have that $a'_{j,1}$ still satisfy (41) and (42) for a constant depending also on M_{p-2}; notice that the dependence on M_{p-2} is only at levels $1 \le j \le p-3$. The asymptotic expansion gives

$$\begin{aligned}
iP_2(t,x,u,D) :=\ & (e^{\lambda_{p-2}})^{-1} iP_1 e^{\lambda_{p-2}} \\
=\ & \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D) \qquad (43)\\
&+ ia_{p-2,1}(t,x,u,D_x) + \mathrm{op}\left(ipa_p\xi^{p-1}D_x\lambda_{p-2}\right)\\
&+ \sum_{\beta=2}^{p-1}\frac{1}{\beta!}\mathrm{op}\left(\partial_\xi^\beta(ia_p\xi^p e^{-\lambda_{p-2}})D_x^\beta\lambda_{p-2}\right)\\
&+ \sum_{j=1}^{p-3} ia'_{j,1}(t,x,u,D_x) + \sum_{\alpha=1}^{p-2}\frac{1}{\alpha!}\mathrm{op}\left(e^{-\lambda_{p-2}}\partial_\xi^\alpha Q_{p-1}D_x^\alpha e^{\lambda_{p-2}}\right)\\
&+ \sum_{\beta=1}^{p-2}\sum_{\alpha=0}^{p-2-\beta}\sum_{\beta_1+\beta_2=\beta}\frac{1}{\alpha!\beta_1!\beta_2!}\mathrm{op}\left(\partial_\xi^\beta e^{-\lambda_{p-2}}D_x^{\beta_1}\partial_\xi^\alpha Q_{p-1}D_x^{\alpha+\beta_2}e^{\lambda_{p-2}}\right)
\end{aligned}$$

$$+ \sum_{j=1}^{p-2}\sum_{\alpha=1}^{j-1}\frac{1}{\alpha!}\mathrm{op}\left(e^{-\lambda_{p-2}}\partial_\xi^\alpha i a'_{j,1}D_x^\alpha e^{\lambda_{p-2}}\right)$$
$$+ \sum_{j=1}^{p-2}\sum_{\beta=1}^{j-1}\sum_{\alpha=0}^{j-1-\beta}\sum_{\beta_1+\beta_2=\beta}\frac{1}{\alpha!\beta_1!\beta_2!}\mathrm{op}\left(\partial_\xi^\beta e^{-\lambda_{p-2}}D_x^{\beta_1}\partial_\xi^\alpha i a'_{j,1}D_x^{\alpha+\beta_2}e^{\lambda_{p-2}}\right)$$
$$+ s'_1(t,x,u,D)$$

with a new term $s'_1 \in S^0$. Let us now look at (43); by (41), (42), and using the estimate (28) wih $k = 2$, we find at each level $1 \le k \le p-3$, the original terms $a_{k,1}(t,x,u,D)$ plus terms which decay with respect to x at least like $\langle x\rangle^{-1}$, and possibly depending only on M_{p-1} and M_{p-2}; the largest number of derivatives with respect to u appears in

$$|\partial_\xi^\beta e^{-\lambda_{p-2}}D_x^{\beta_1}\partial_\xi^\alpha i a'_{j,1}D_x^{\alpha+\beta_2}e^{\lambda_{p-2}}|$$
$$\le C'_{M_{p-1},M_{p-2}}\gamma(u)(1+\|u\|^{2(p-j)+\beta-1}_{2(p-j)+\beta,2(p-j)+\beta-1})\langle\xi\rangle^{j-\alpha-\beta}\langle x\rangle^{-\alpha-\beta};$$

at the level $k = j-\alpha-\beta$ the largest number of x-derivatives of u appears when $\alpha = 0$ and $\beta = j-k$ and it is given by $2(p-j)+\beta = 2(p-k-\beta)+\beta = 2(p-k)-\beta \le 2(p-k)-1$. Thus, similarly as for (38), we get

$$iP_2 = \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D) + ia_{p-2,1}(t,x,u,D_x) \tag{44}$$
$$+ \mathrm{op}\left(ipa_p\xi^{p-1}D_x\lambda_{p-2}\right) + \sum_{j=1}^{p-3} ia''_{j,1}(t,x,u,D_x) + s'_1(t,x,u,D)$$

where $a''_{j,1}$ are given by a_j plus other terms of the same type, still satisfying (41), (42) but with a constant $C_{M_{p-1},M_{p-2}}$ depending on both M_{p-1} and M_{p-2}.

Now, let us focus on the term A_{p-2} of order $p-2$ with respect to ξ in (44). By (42), (24), the choice of ω in (25), and (4) we get for every $|\xi| \ge 2h$:

$$\mathrm{Re}\, A_{p-2}(t,x,u,\xi) := \mathrm{Re}\left(ia_{p-2,1}(t,x,u,\xi) + pa_p(t)\xi^{p-1}\partial_x\lambda_{p-2}(x,\xi)\right)$$
$$= -\mathrm{Im}\, a_{p-2,1}(t,x,u,\xi) + pa_p(t)\xi^{p-1}\partial_x\lambda_{p-2}(x,\xi)$$
$$\ge -C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right)\frac{\langle\xi\rangle_h^{p-2}}{\langle x\rangle^{\frac{p-2}{p-1}}} + M_{p-2}pa_p(t)|\xi|^{p-1}\langle x\rangle^{-\frac{p-2}{p-1}}\langle\xi\rangle_h^{-1}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)$$
$$\ge \frac{\langle\xi\rangle_h^{p-2}}{\langle x\rangle^{\frac{p-2}{p-1}}}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\left(-C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right) + M_{p-2}pC_p\left(\frac{2}{\sqrt{5}}\right)^{p-1}\right)$$

$$- C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right)\frac{\langle\xi\rangle_h^{p-2}}{\langle x\rangle^{\frac{p-2}{p-1}}}\left(1-\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right)$$
$$\geq -2C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right)$$

if we choose $M_{p-2} \geq \dfrac{C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right)\sqrt{5}^{p-1}}{2^{p-1}pC_p}$, and using again $\langle x\rangle/\langle\xi\rangle_h^{p-1} \geq 1/2$ on the support of $1-\psi(\langle x\rangle/\langle\xi\rangle_h^{p-1})$ and $|\xi|^p \geq (2/\sqrt{5})^{p-1}\langle\xi\rangle_h^{p-1}$ for $|\xi| \geq 2h$. We can so apply the sharp Gårding theorem to the symbol $A_{p-2}(t,x,u,\xi) + 2C_{M_{p-1}}\gamma(u)\left(1+\|u\|^3_{4,3}\right) \geq 0$ and we obtain that there exist pseudodifferential operators $Q_{p-2}(t,x,u,D)$, $R_{p-2}(t,x,u,D)$, $R_{0,p-2}(t,x,u,D)$ with symbols

$$Q_{p-2}(t,x,u,\xi)\in\mathbf{SG}^{p-2,0},\quad R_{p-2}(t,x,u,\xi)\in\mathbf{SG}^{p-3,0},\quad R_{0,p-2}(t,x,u,\xi)\in S^0$$

such that

$$A_{p-2}(t,x,u,D) = Q_{p-2}(t,x,u,D) + iR_{p-2}(t,x,u,D) + R_{0,p-2}(t,x,uD)$$

with

$$\mathrm{Re}\langle Q_{p-2}(t,x,u,D)h(t,x),h(t,x)\rangle \geq 0 \quad \forall h\in\mathscr{S}(\mathbb{R}),\ (t,x)\in[0,T]\times\mathbb{R}$$

and

$$R_{p-2} = \sum_{j=1}^{p-3} R_{j,p-2} \tag{45}$$

where

$$R_{p-3,p-2} = -i\left(\psi_1(\xi)D_xA_{p-2} + \sum_{2\leq\alpha+\beta\leq3}\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-2}\right)$$

and

$$R_{j,p-2} = -i\sum_{2(p-2-j)\leq\alpha+\beta\leq2(p-2-j)+1}\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-2},$$

for every $1\leq j\leq p-3$. We have so

$$iP_2 = \partial_t + ia_p(t,D) + Q_{p-1}(t,x,u,D_x) + Q_{p-2}(t,x,u,D_x)$$
$$+ iR_{p-2}(t,x,u,D_x) + \sum_{j=1}^{p-3} ia''_{j,1}(t,x,u,D_x) + s''_1(t,x,u,D_x).$$

Again, each $R_{j,p-2}$ adds to $a''_{j,1}$ new terms with the same order and decay as $a''_{j,1}$ (notice that the second application of Theorem 2 is needed only in the case $p \geq 3$ and in this case we have $5 \leq p+2$, so the term $\psi_1(\xi) D_x A_{p-2}(t,x,u,\xi)$ satisfies (41) and (42) with $j = p-3$ and a constant depending on M_{p-1}, M_{p-2}. The largest number of x-derivatives of u appears when $\alpha = 0$, $\beta = 2(p-2-j)+1$ and we have

$$|\psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta A_{p-2}(t,x,u,\xi)| \leq C_{M_{p-1}}\gamma(u)\left(1+\|u\|_{4+\beta,3+\beta}^{3+\beta}\right)\langle\xi\rangle^{p-2-\frac{\alpha+\beta}{2}}\langle x\rangle^{-\beta}$$
$$\leq C_{M_{p-1}}\gamma(u)\left(1+\|u\|_{2(p-j)+1,2(p-j)}^{2(p-j)}\right)\langle\xi\rangle^{j}\langle x\rangle^{-1}.$$

This means that, after the second application of the sharp Gårding theorem, we can write

$$iP_2 = \partial_t + ia_p(t,D) + Q_{p-1}(t,x,u,D_x) + Q_{p-2}(t,x,u,D_x) \tag{46}$$
$$+ \sum_{j=1}^{p-3} ia_{j,2}(t,x,u,D_x) + s_2(t,x,u,D)$$

for a new operator s_2 with symbol in S^0,where $a_{j,2}$ are given by $a_j D_x^j$ plus other terms with the same order and decay depending on $2(p-j)+1$ x-derivatives of u; thus $a_{j,2}$ depends on $\max\{2(p-j)+1, 2(p-j)\} = 2(p-j)+1$ x-derivatives of u. Summing up, for every $1 \leq j \leq p-3$ and for $\beta \leq p$ (we need that $2(p-j)+1+\beta \leq 2p-1+\beta \leq 3p-1$) we have

$$|\partial_\xi^\alpha\partial_x^\beta \mathrm{Re}\, a_{j,2}(t,x,u(t,x),\xi)| \tag{47}$$
$$\leq C_{M_{p-1},M_{p-2}}\gamma(u)(1+\|u\|_{2(p-j)+1+\beta,2(p-j)+\beta}^{2(p-j)+\beta})\langle x\rangle^{-\beta}\langle\xi\rangle^{j-\alpha},$$
$$|\partial_\xi^\alpha\partial_x^\beta \mathrm{Im}\, a_{j,2}(t,x,u(t,x),\xi)| \tag{48}$$
$$\leq C_{M_{p-1},M_{p-2}}\gamma(u)(1+\|u\|_{2(p-j)+1+\beta,2(p-j)+\beta}^{2(p-j)+\beta})\langle x\rangle^{-\frac{j}{p-1}-\beta}\langle\xi\rangle^{j-\alpha}.$$

We can proceed performing the next conjugations which follow the same argument as the second one. Arguing in this way, after $\ell = p-3$ applications of Theorem 2 we finally come for $h \geq \max\{h_1,\dots,h_{p-3}\}$ to

$$iP_{p-3} = (e^{\lambda_3})^{-1}\dots(e^{\lambda_{p-1}})^{-1}(iP)(e^{\lambda_{p-1}})\dots(e^{\lambda_3}) \tag{49}$$
$$= \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D_x) + \dots + Q_3(t,x,u,D_x) \tag{50}$$
$$+ ia_{2,p-3}(t,x,u,D_x) + ia_{1,p-3}(t,x,u,D_x) + s_{p-3}(t,x,u,D)$$

where, for every $1 \leq j \leq p-3$,

$$\mathrm{Re}\langle Q_{p-j}(t,x,u,D)h(t,x), h(t,x)\rangle \geq 0 \quad \forall h \in \mathscr{S}(\mathbb{R}),\ (t,x) \in [0,T] \times \mathbb{R}$$

and moreover for every $\beta \leq 7$

$$\begin{aligned} &|\partial_\xi^\alpha \partial_x^\beta \mathrm{Re}\, a_{2,p-3}(t,x,u,\xi)| \\ &\qquad \leq C_{M_{p-1},\ldots,M_3}\gamma(u)(1+\|u\|^{3p-9+\beta}_{3p-8+\beta,3p-9+\beta})\langle x\rangle^{-\beta}\langle \xi\rangle^{2-\alpha}, \end{aligned} \tag{51}$$

$$\begin{aligned} &|\partial_\xi^\alpha \partial_x^\beta \mathrm{Im}\, a_{2,p-3}(t,x,u,\xi)| \\ &\qquad \leq C_{M_{p-1},\ldots,M_3}\gamma(u)(1+\|u\|^{3p-9+\beta}_{3p-8+\beta,3p-9+\beta})\langle x\rangle^{-\frac{2}{p-1}-\beta}\langle \xi\rangle^{2-\alpha}, \end{aligned} \tag{52}$$

and for $\beta \leq 5$

$$\begin{aligned} &|\partial_\xi^\alpha \partial_x^\beta \mathrm{Re}\, a_{1,p-3}(t,x,u,\xi)| \\ &\qquad \leq C_{M_{p-1},\ldots,M_3}\gamma(u)(1+\|u\|^{3p-7+\beta}_{3p-6+\beta,3p-7+\beta})\langle x\rangle^{-\beta}\langle \xi\rangle^{1-\alpha}, \end{aligned} \tag{53}$$

$$\begin{aligned} &|\partial_\xi^\alpha \partial_x^\beta \mathrm{Im}\, a_{1,p-3}(t,x,u,\xi)| \\ &\qquad \leq C_{M_{p-1},\ldots,M_3}\gamma(u)(1+\|u\|^{3p-7+\beta}_{3p-6+\beta,3p-7+\beta})\langle x\rangle^{-\frac{1}{p-1}-\beta}\langle \xi\rangle^{1-\alpha}. \end{aligned} \tag{54}$$

Now, we define, for $h \geq \max\{h_1,\ldots,h_{p-2}\}$, $iP_{p-2}(t,x,u,D) := (e^{\lambda_2})^{-1} iP_{p-3}e^{\lambda_2}$ and we get

$$\begin{aligned} iP_{p-2} = \partial_t &+ ia_p(t)D_x^p + Q_{p-1}(t,x,u,D_x) + \ldots + Q_3(t,x,u,D_x) \\ &+ ia_{2,p-3}(t,x,u,D_x) + \mathrm{op}\left(ipa_p\xi^{p-1}D_x\lambda_2\right) \\ &+ ia''_{1,p-3}(t,x,u,D_x) + s'_{p-3}(t,x,u,D) \end{aligned} \tag{55}$$

where $a''_{1,p-3}$ are given by a_j plus other terms of the same type, still satisfying (53) and (54) but with a constant $C_{M_{p-1},\ldots,M_2}$ instead of $C_{M_{p-1},\ldots,M_3}$, and s'_{p-3} is still of order 0.

Now, as usual, by choosing $M_2 \geq C_{M_{p-1},\ldots,M_3}\gamma(u)\left(1+\|u\|^{3p-9}_{3p-8,3p-9}\right) \frac{\sqrt{5}^{p-1}}{2^{p-1}pC_p}$ we get

$$\begin{aligned} \mathrm{Re}\, A_2 :&= \mathrm{Re}\left(ia_{2,p-3}(t,x,u,D_x) + \mathrm{op}\left(pa_p\xi^{p-1}\partial_x\lambda_2\right)\right) \\ &\geq 2C_{M_{p-1},\ldots,M_3}\gamma(u)\left(1+\|u\|^{3p-9}_{3p-8,3p-9}\right). \end{aligned}$$

This time, since we are dealing with second order operators, we can apply the Fefferman-Phong inequality (see Theorem 3) to

$$\mathrm{Re}\, A_2 + 2C_{M_{p-1},\dots,M_3}\gamma(u)\left(1+\|u\|^{3p-9}_{3p-8,3p-9}\right)$$

and obtain

$$\mathrm{Re}\langle \mathrm{Re}\, A_2 h, h\rangle \geq -\left(c + 2C_{M_{p-1},\dots,M_3}\gamma(u)\left(1+\|u\|^{3p-9}_{3p-8,3p-9}\right)\right)\|h\|^2, \quad \forall h \in \mathscr{S}(\mathbb{R})$$

for a positive constant $c = c(u)$ depending on the derivatives $\partial_\xi^\alpha \partial_x^\beta$ with $|\alpha|+|\beta| \leq 7$ of the symbol $\mathrm{Re}\, A_2(t,x,u,\xi)+2C_{M_{p-1},\dots,M_3}\gamma(u)\left(1+\|u\|^{3p-9}_{3p-8,3p-9}\right)$. Since γ is of class C^7, we can now find a constant $C_\gamma > 0$ (depending also on $M_{p-1},\dots,M_3$) such that

$$\begin{aligned}\mathrm{Re}\langle \mathrm{Re}\, A_2 h, h\rangle &\geq -C_\gamma\left(1+\|u\|^{3p-9+7}_{3p-8+7,3p-9+7}\right)\|h\|^2\\ &= -C_\gamma\left(1+\|u\|^{3p-2}_{3p-1,3p-2}\right)\|h\|^2, \quad \forall h \in \mathscr{S}(\mathbb{R}).\end{aligned}$$

The advantage of the use of Fefferman-Phong inequality instead of Theorem 2 is that we avoid the remainder of that theorem, i.e. we save some derivatives of the fixed function u.

It now remains to treat the terms $i\,\mathrm{Im}\, A_2 = i\,\mathrm{Re}\, a_{2,p-3}$ and $i a''_{1,p-3}$ in (55). We split $i\,\mathrm{Re}\, a_{2,p-3}$ into its Hermitian and anti-Hermitian part:

$$i\,\mathrm{Im}\, A_2 = \frac{i\,\mathrm{Re}\, a_{2,p-3} + (i\,\mathrm{Re}\, a_{2,p-3})^*}{2} + \frac{i\,\mathrm{Re}\, a_{2,p-3} - (i\,\mathrm{Re}\, a_{2,p-3})^*}{2} =: H_1 + H_2,$$

and we have that

$$\mathrm{Re}\langle H_2 h, h\rangle = 0,$$

while

$$H_1 = -\frac{1}{2}\partial_\xi \partial_x \,\mathrm{Re}\, a_{2,p-3} \ (\mathrm{mod.}\ \mathbf{SG}^{0,0})$$

can be put together with $i a''_{1,p-3}$ since by (51) it satisfies (53). We get so

$$\begin{aligned} iP_{p-2} &= \partial_t + i a_p(t) D_x^p + Q_{p-1}(t,x,u,D_x) + \ldots + Q_3(t,x,u,D_x)\\ &\quad + \mathrm{Re}\, A_2(t,x,u,D_x) + H_2(t,x,u,D_x)\\ &\quad + i a_{1,p-2}(t,x,u,D_x) + s_{p-2}(t,x,u,D)\end{aligned}$$

with $ia_{1,p-2}$ still satisfying (53), (54) and $s_{p-2} \in S^0$. Finally, to treat the terms of order 1 with respect to ξ, we perform for $h \geq \max\{h_1, \ldots, h_{p-1}\}$ the last conjugation:

$$
\begin{aligned}
iP_\Lambda :&= (e^{\lambda_1})^{-1} iP_{p-2} e^{\lambda_1} \qquad (56)\\
&= \partial_t + ia_p(t)D_x^p + Q_{p-1}(t,x,u,D_x) + \ldots + Q_3(t,x,u,D_x)\\
&+ \operatorname{Re} A_2(t,x,u,D_x) + H_2(t,x,u,D_x)\\
&+ ia_{1,p-2}(t,x,u,D_x) + \operatorname{op}\left(ipa_p\xi^{p-1}D_x\lambda_1\right) + s'_{p-2}(t,x,u,D)
\end{aligned}
$$

with a new term $s'_{p-2} \in S^0$. Notice that the conjugation $e^{-\lambda_1}(\operatorname{Re} A_2 + H_2)\, e^{\lambda_1}$ gives $\operatorname{Re} A_2 + H_2$ plus a remainder of order $(0,0)$ whose principal part is given by

$$
\partial_\xi(\operatorname{Re} A_2 + H_2)\partial_x\lambda_1 - \partial_\xi\lambda_1 D_x(\operatorname{Re} A_2 + H_2) - \partial_\xi\lambda_1(\operatorname{Re} A_2 + H_2)D_x\lambda_1 \in \mathbf{SG}^{0,0}.
$$

As usual, by choosing $M_1 \geq C_{M_{p-1},\ldots,M_2}\gamma(u)\left(1 + \|u\|^{3p-7}_{3p-6,3p-7}\right)\sqrt{5}^{p-1}/\left(2^{p-1}pC_p\right)$ we get

$$
\begin{aligned}
\operatorname{Re} A_1 :&= \operatorname{Re}\left(ia_{1,p-2}(t,x,u,D_x) + \operatorname{op}\left(pa_p\xi^{p-1}\partial_x\lambda_1\right)\right)\\
&\geq 0 - 2C_{M_{p-1},\ldots,M_2}\gamma(u)\left(1 + \|u\|^{3p-7}_{3p-6,3p-7}\right).
\end{aligned}
$$

To the symbol $A_1(t,x,u,\xi)$ we can apply the sharp Gårding inequality (16) and we obtain

$$
\operatorname{Re}\langle A_1 h, h\rangle \geq -C'_\gamma(1 + \|u\|^{3p-7}_{3p-6,3p-7})\|h\| \qquad \forall h \in \mathscr{S}(\mathbb{R}).
$$

At this point we are finally ready to prove an energy estimate in L^2 for the Cauchy problem. We compute

$$
\begin{aligned}
\frac{d}{dt}\|w(t)\|^2 &= 2\operatorname{Re}\langle\partial_t w, w\rangle = 2\operatorname{Re}\langle iP_\Lambda w, w\rangle - 2\operatorname{Re}\langle ia_p w, w\rangle - \sum_{k=3}^{p-1} 2\operatorname{Re}\langle Q_k w, w\rangle\\
&- 2\operatorname{Re}\langle \operatorname{Re} A_2 w, w\rangle - 2\operatorname{Re}\langle H_2 w, w\rangle - 2\operatorname{Re}\langle A_1 w, w\rangle - 2\operatorname{Re}\langle s'_{p-2} w, w\rangle\\
&\leq C'_\gamma(1 + \|u\|^{3p-2}_{3p-1,3p-2})\left(\|P_\Lambda w\|^2 + \|w\|^2\right) \quad \forall w \in \mathscr{S}(\mathbb{R}).
\end{aligned}
$$

By Gronwall's Lemma we obtain

$$
\|w\|^2 \leq Ce^{(1+\|u\|^{3p-2}_{3p-1,3p-2})t}\left(\|w(0,\cdot)\|^2 + \int_0^t \|P_\Lambda w(\tau,\cdot)\|^2 d\tau\right)
$$

and, by standard arguments, the energy estimate (31). □

Remark 7 Notice that with respect to [1], by a different proof we can relax from $4p-3$ to $3p-1$ the number of derivatives of u needed to perform the computations in the linearized problem.

4 The Semilinear Problem

We now apply the energy estimates obtained in the previous section to prove the well posedness of the semilinear Cauchy problem (1). Fixed $s_2 \geq 3p-2$ and $T > 0$, we consider the space $X_T^{s_2} := C^1([0,T], H^{\infty,s_2}(\mathbb{R}))$ and the map $J : X_T^{s_2} \to X_T^{s_2}$ defined by

$$J(u) := u(t,x) - u_0(x) + i\int_0^t a_p(t)D_x^p u(s,x)\,ds$$
$$+i\sum_{j=0}^{p-1}\int_0^t a_j(s,x,u(s,x))D_x^j u(s,x)\,ds - i\int_0^t f(s,x)\,ds.$$

It is well known that the existence of a unique solution of (1) in $X_{T^*}^{s_2}$ for some $T^* \in (0,T]$ is equivalent to the existence of a unique solution in $X_{T^*}^{s_2}$ of the equation $Ju = 0$, cf. [1, 12]. We shall approach the latter problem via the Nash-Moser inversion theorem. As a direct consequence of Lemma 1, $X_T^{s_2}$ is a tame Fréchet space endowed with the family of seminorms

$$|g|_{n,s_2,T} = \sup_{[0,T]}(|g(t,\cdot)|_{n,s_2} + |D_t g(t,\cdot)|_{n,s_2}), \qquad n = 0,1,2,\ldots$$

The map J is smooth tame since it is defined in terms of sums and composition of integration and linear and nonlinear partial differential operators. In order to apply Nash-Moser theorem it is sufficient to prove that for every fixed $u, h \in X_T^{s_2}$, the equation $DJ(u)v = h$ has a unique solution $v = S(u,h) \in X_T^{s_2}$ and that the map

$$S : X_T^{s_2} \times X_T^{s_2} \to X_T^{s_2}, \quad (u,h) \to v = S(u,h) \tag{57}$$

is smooth tame.

Lemma 3 *For every $u, h \in X_T^{s_2}$, the equation $DJ(u)v = h$ admits a unique solution $v \in X_T^{s_2}$ satisfying for every $n \in \mathbb{N}$ the following estimate:*

$$|v(t,\cdot)|_{n,s_2}^2 \leq C_n(u)\left(|h(0,\cdot)|_{n+r,s_2}^2 + \int_0^t |D_t h(\tau,\cdot)|_{n+r,s_2}^2\,d\tau\right), \qquad t \in [0,T] \tag{58}$$

for every $r \geq 2\delta(p-1)$ with $C_n(u) = C_{n+2\delta(p-1),\gamma}\exp(1 + \|u\|_{3p-1,3p-2}^{3p-2})$.

Proof The proof follows the same argument as the proof of [1, Lemma 3.2], so we just sketch it. A direct computation of the Fréchet derivative of J gives

$$\begin{aligned} DJ(u)v &:= \lim_{\varepsilon\to 0}\frac{J(u+\varepsilon v)-J(u)}{\varepsilon} \\ &= v + i\int_0^t a_p(s)D_x^p v(s)\,ds + i\sum_{j=0}^{p-1}\int_0^t \tilde{a}_j(s,x,u)D_x^j v(s)\,ds, \end{aligned}$$

where

$$\tilde{a}_j(s,x,u) = \begin{cases} a_j(s,x,u) & 1\le j\le p-1 \\ a_0(s,x,u)+\sum_{h=0}^{p-1}\partial_w a_h(s,x,u)D_x^h u, & j=0. \end{cases}$$

Hence v is a solution of the equation $DJ(u)v = h$ if and only if it is a solution of the equation $J_{h_0,u,D_t h}(v) = 0$, where $h_0(x) := h(0,x)$ and for every $u, u_0, f \in X_T^{s_2}$ the map $J_{u_0,u,f} : X_T^{s_2} \to X_T^{s_2}$ is defined by

$$\begin{aligned} J_{u_0,u,f}(v) &:= v(t,x) - u_0(x) + i\int_0^t a_p(s)D_x v(s,x)\,ds \\ &\quad + i\sum_{j=0}^{p-1}\int_0^t \tilde{a}_j(s,x,u(s,x))D_x^j v(s,x)\,ds - i\int_0^t f(s,x)\,ds. \end{aligned}$$

On the other hand, v solves $J_{h_0,u,D_t h}(v) = 0$ if and only if it is a solution of the linear Cauchy problem

$$\begin{cases} \tilde{P}_u(D)v(t,x) = D_t h(t,x) \\ v(0,x) = h_0(x) \end{cases}, \tag{59}$$

where

$$\tilde{P}_u(D) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1}\tilde{a}_j(t,x,u)D_x^j.$$

Notice that $\tilde{a}_j(t,x,u)$ satisfy the same conditions as $a_j(t,x,u)$. Hence, we can apply Theorem 5 to (59), choosing $\eta = 1$ and $\epsilon = 0$. It follows that there exists $v \in X_T^{s_2}$ solution of (59) satisfying the estimate (58). This concludes the proof. □

Lemma 4 *The map S defined in* (57) *is smooth tame.*

Proof We observe that, fixed $(u_0, h_0) \in X_T^{s_2} \times X_T^{s_2}$, the constant $C_n(u)$ in the energy estimate (58) is bounded if u belongs to a bounded neighborhood of (u_0, h_0).

Evidently, from (58) we have:

$$|v(t,\cdot)|^2_{n,s_2} \le C'_n |h|^2_{n+r,s_2,T} \qquad t \in [0,T]$$

for some $C'_n > 0$. Similarly, from the equation $\tilde{P}_u(D)v = D_t h$ we get

$$\begin{aligned}|D_t v(t,\cdot)|_{n,s_2} &\le |a_p(t)D^p v(t,\cdot)|_{n,s_2} + \sum_{j=0}^{p-1} |\tilde{a}_j(t,\cdot,u)D_x^j v(t,\cdot)|_{n,s_2} + |D_t h(t,\cdot)|_{n,s_2} \\ &\le C(|v(t,\cdot)|_{n+p,s_2} + |h|_{n,s_2,T})\end{aligned}$$

for some $C > 0$. Hence

$$|S(u,h)|_{n,s_2,T} = \sup_{t\in[0,T]} (|v|_{n,s_2} + |D_t v(t,\cdot)|_{n,s_2}) \le C_n |h|_{n+r',s_2,T} \le C_n |(u,h)|_{n+r',s_2,T}$$

for some $C_n > 0$ $r' \ge 2\delta(p-1) + p$. Then S is tame.

We now prove that DS is also a tame map. For $(u,h), (u_1,h_1) \in X_T^{s_2} \times X_T^{s_2}$ we have

$$DS(u,h)(u_1,h_1) = \lim_{\varepsilon\to 0} \frac{S(u+\varepsilon u_1, h+\varepsilon h_1) - S(u,h)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{v_\varepsilon - v}{\varepsilon},$$

where v is a solution of (59) and v_ε is the solution of

$$\begin{cases} \tilde{P}_{u+\varepsilon u_1}(D)v_\varepsilon = D_t(h+\varepsilon h_1) \\ v_\varepsilon(0,x) = h_0(x) + \varepsilon h_1(0,x) \end{cases}.$$

A direct computation shows that the function $w_\varepsilon = \frac{v_\varepsilon - v}{\varepsilon}$ solves the Cauchy problem

$$\begin{cases} \tilde{P}_{u+\varepsilon u_1} w_\varepsilon = f_\varepsilon \\ w_\varepsilon(0,x) = h_1(0,x) \end{cases},$$

where

$$f_\varepsilon = D_t h_1 - \sum_{j=0}^{p-1} \frac{\tilde{a}_j(t,x,u+\varepsilon u_1) - \tilde{a}_j(t,x,u)}{\varepsilon} D_x^j v.$$

We have the following: to prove that DS is tame it is sufficient to show that w_ε tends to some w_1 in $X_T^{s_2}$ for $\varepsilon \to 0$. Indeed, this would imply that w_1 solves the Cauchy problem

$$\begin{cases} \tilde{P}_u(D)w_1 = f_1 \\ w_1(0,x) = h_1(0,x) \end{cases}$$

where

$$f_1 := \lim_{\varepsilon\to 0} f_\varepsilon = D_t h_1 - \sum_{j=0}^{p-1} \partial_w \tilde{a}_j(t,x,u) u_1 D_x^j v$$

and so that w_1 will satisfy an energy estimate of the form

$$|w_1(t,\cdot)|^2_{n,s_2} \leq C_n(u)\left(|h_1(0,\cdot)|^2_{n+r,s_2} + \int_0^t |f_1(\tau,\cdot)|^2_{n+r,s_2}\, d\tau\right), \tag{60}$$

which would give, by the expression of f_1,

$$|w_1(t,\cdot)|_{n,s_2} \leq C_n'(u)(|h_1|_{n+r',s_2,T} + |h|_{n+r',s_2,T}), \qquad r' \geq 2r + p - 1$$

for (u,h) in a neighborhood of (u_0,h_0) and (u_1,h_1) in a neighborhood of some fixed $(\tilde{u}_1,\tilde{h}_1) \in X_T^{s_2} \times X_T^{s_2}$. Moreover, $D_t w_1$ would satisfy a similar estimate, and so w_1 is tame.

Let us so prove that w_ϵ converges in $X_T^{s_2}$ for $\epsilon \to 0$. Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ and consider the corresponding functions w_{ε_1} and w_{ε_2} which solve the Cauchy problems

$$\begin{cases} \tilde{P}_{u+\varepsilon_j u_1}(D) w_{\varepsilon_j} = f_{\varepsilon_j} \\ w_{\varepsilon_j}(0,x) = h_1(0,x) \end{cases}, \qquad j = 1,2.$$

Then, it is immediate to see that $w_{\varepsilon_1} - w_{\varepsilon_2}$ is a solution of

$$\begin{cases} \tilde{P}_{u+\varepsilon_1 u_1}(D)(w_{\varepsilon_1} - w_{\varepsilon_2}) = f_{\varepsilon_1} - f_{\varepsilon_2} + \displaystyle\sum_{j=0}^{p-1} (\tilde{a}_j(t,x,u+\varepsilon_2 u_1) - \tilde{a}_j(t,x,u+\varepsilon_1 u_1)) D_x^j w_{\varepsilon_2} \\ (w_{\varepsilon_1} - w_{\varepsilon_2})(0,x) = 0. \end{cases}$$

Then by the estimate (60) and the mean value theorem, we get

$$\begin{aligned} |w_{\varepsilon_1} - w_{\varepsilon_2}|_{n,s_2} \leq C_n(u+\varepsilon_1 u_1) \Bigg(&\sup_{t\in[0,T]} |f_{\varepsilon_1} - f_{\varepsilon_2}|_{n+r,s_2} \\ &+ \sum_{j=0}^{p-1} \sup_{t\in[0,T]} |\partial_w \tilde{a}_j(t,x,u_{1,2})(\varepsilon_1 - \varepsilon_2) u_1 D_x^j w_{\varepsilon_1}|_{n+r,s_2} \Bigg) \end{aligned}$$

for some constant $C_n(u+\varepsilon_1 u_1) > 0$ and for some $u_{1,2}$ between $u + \varepsilon_1 u_1$ and $u + \varepsilon_2 u_1$. Moreover, since H_{n+r,s_2} is an algebra, then

$$\begin{aligned} &|\partial_w a_j(t,x,u_{1,2})(\varepsilon_1 - \varepsilon_2)\, u_1 D_x^j w_{\varepsilon_2}|_{n+r,s_2} \\ &\qquad \leq |\partial_w a_j(t,x,u_{1,2})|_{n+r,s_2} |\varepsilon_1 - \varepsilon_2| |u_1|_{n+r,s_2} |w_{\varepsilon_2}|_{n+r+j,s_2}. \end{aligned}$$

Then, $|w_{\varepsilon_1} - w_{\varepsilon_2}|_{n,s_2}$ tends to 0 when $\varepsilon_1 \to \varepsilon_2 \to 0$ if (u, h) is in a neighoborhood of (u_0, h_0) and (u_1, h_1) is in a neighborhood of some fixed $(\tilde{u}_1, \tilde{h}_1) \in X_T^{s_2} \times X_T^{s_2}$. This shows that there exists a Cauchy sequence ε_j tending to 0 such that the corresponding function w_{ε_j} converges in $X_T^{s_2}$ and this implies that DS is tame.

Using the previous results we can prove by induction on m that

$$D^m S(u, h)(u_1, h_1) \cdots (u_m, h_m) = w^m$$

is a solution of the Cauchy problem

$$\begin{cases} \tilde{P}_u(D) w^m = f^m \\ w^m(0, x) = 0 \end{cases}$$

with

$$f^m := -\sum_{j=0}^{p-1} \partial_w \tilde{a}_j(t, x, u) u_m D_x^j w^{m-1} - \sum_{j=0}^{p-1} \partial_w^2 \tilde{a}_j(t, x, u) u_{m-1} u_m D_x^j w^{m-2}$$
$$- \cdots - \sum_{j=0}^{p-1} \partial_w^m \tilde{a}_j(t, x, u) u_1 \cdots u_{m-1} u_m D_x^j w^0,$$

$w_0 := v$, and satisfies, in a neighborhood of $(u, h), (u_1, h_1), \ldots (u_m, h_m)$ the estimate

$$|w^m|_{n,s_2,T} \leq C_n \sum_{j=0}^{m-1} |h_j|_{n+r(m),s_2,T}$$

for some $C_n > 0$ and some $r(m) \in \mathbb{N}$, where $h_0 := h$. The proof follows readily the argument in the proof of Lemma 3.3 in [1]. We leave the details to the reader. □

Proof of Theorem 1 We prove now the existence of a solution of the semilinear Cauchy problem (1) that is of the equation $Ju = 0$. We recall that $Ju = 0$ if and only if

$$u(t, x) = u_0(x) - i \int_0^t a_p(s) D_x^p u(s, x)\, ds \tag{61}$$
$$- i \sum_{j=0}^{p-1} \int_0^t a_j(s, x, u(s, x)) D_x^j u(s, x)\, ds + i \int_0^t f(s, x)\, ds.$$

By a linear approximation in t we get $u(t, x) = w(t, x) + o(t)$ for $t \to 0$ where

$$w(t, x) = u_0(x) - it \left(a_p(0) D_x^p u_0(x) + \sum_{j=0}^{p-1} a_j(0, x, u_0(x)) D_x^j u_0(x) - f(0, x) \right).$$

We also observe that, by the definition of J and w, we have:

$$\begin{aligned}
\partial_t (Jw(t, x)) &= \partial_t w + i a_p(t) D_x^p w + i \sum_{j=0}^{p-1} a_j(t, x, w) D_x^j w - i f(t, x) \\
&= i(a_p(t) - a_p(0)) D_x^p u_0 + i \sum_{j=0}^{p-1} \left(a_j(t, x, w) - a_j(0, x, u_0) \right) D_x^j u_0 \\
&\quad + t a_p(t) D_x^p \left[a_p(0) D_x^p u_0 + \sum_{j=0}^{p-1} a_j(0, x, u_0) D_x^j u_0 - f(0, x) \right] \\
&\quad + \sum_{j=0}^{p-1} t a_j(t, x, w) D_x^j \left[a_p(0) D_x^p u_0 + \sum_{k=0}^{p-1} a_k(0, x, u_0) D_x^k u_0 - f(0, x) \right] \\
&\quad + i(f(0, x) - f(t, x)).
\end{aligned}$$

From this it follows that

$$\begin{aligned}
|\partial_t Jw(t, \cdot)|_{n,s_2} &\leq \sup_{t\in[0,T]} |a_p(t) - a_p(0)| \cdot |u_0|_{n+p,s_2} \\
&+ \sum_{j=0}^{p-1} |[a_j(t, x, w) - a_j(0, x, u_0)] D_x^j u_0|_{n,s_2} + |f(0, x) - f(t, x)|_{n,s_2} \\
&+ t \sup_{t\in[0,T]} |a_p(t)| \cdot \left| a_p(0) D_x^p u_0 + \sum_{k=0}^{p-1} a_k(0, x, u_0) D_x^k u_0 - f(0, x) \right|_{n+p,s_2} \\
&+ t \sum_{j=0}^{p-1} \left| a_j(t, x, w) D_x^j \left[a_p(0) D_x^p u_0 + \sum_{k=0}^{p-1} a_k(0, x, u_0) D_x^k u_0 - f(0, x) \right] \right|_{n,s_2}.
\end{aligned} \tag{62}$$

Taking w in a sufficiently small neighborhood of u_0 and applying the mean value theorem to the right-hand side of (62) we obtain

$$|\partial_t Jw(t, \cdot)|_{n,s_2} \leq Ct \tag{63}$$

for a suitable constant $C = C(n, s_2, a_p, \ldots, a_0, u_0, f)$. Now, fixed $\varepsilon > 0$ we define

$$\phi_\varepsilon(t,x) = \int_0^t \rho\left(\frac{s}{\varepsilon}\right)(\partial_t Jw)(s,x)\,ds,$$

where $\rho \in C^\infty(\mathbb{R})$ such that $0 \le \rho \le 1$ and $\rho(s) = 0$ for $|s| \le 1$ and $\rho(s) = 1$ for $|s| \ge 2$. Notice that $\phi_\varepsilon = 0$ for $0 \le t \le \varepsilon$. Let U and V be neighborhoods of w and Jw respectively such that $J : U \to V$ is a bijection. We have that

$$\begin{aligned}
|Jw - \phi_\varepsilon|_{n,s_2} &= \left| \int_0^t \left(1 - \rho\left(\frac{s}{\varepsilon}\right)\right)(\partial_t Jw)(s,\cdot)\,ds \right|_{n,s_2} \\
&\le \int_0^{2\varepsilon} \left| \left(1 - \rho\left(\frac{s}{\varepsilon}\right)\right)(\partial_t Jw)(s,\cdot) \right|_{n,s_2} ds \\
&\le C \int_0^{2\varepsilon} s\,ds \le 2C\varepsilon^2,
\end{aligned}$$

where C is the same constant appearing in (63). Similarly we obtain that

$$|\partial_t(Jw - \phi_\varepsilon)|_{n,s_2} \le 2C\varepsilon.$$

Hence, taking $0 < \varepsilon < 1$ we conclude that

$$|Jw - \phi_\varepsilon|_{n,s_2,T} \le 2C\varepsilon.$$

If we choose ε sufficiently small, we have that $\phi_\varepsilon \in V$. Then there exists $u \in U$ such that $Ju = \phi_\varepsilon$. In particular we have $Ju = 0$ for $0 \le t \le \varepsilon$, that is u is a solution in $X_\varepsilon^{s_2}$ of the Cauchy problem. The uniqueness of the solution comes from standard arguments, cf. [1]. □

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Random-Field Solutions of Linear Parabolic Stochastic Partial Differential Equations with Polynomially Bounded Variable Coefficients

Alessia Ascanelli, Sandro Coriasco, and André Süß

To Massimo and Michael, on occasion of their 60th birthday

Abstract We study random-field solutions of a class of stochastic partial differential equations, involving operators with polynomially bounded coefficients. We consider linear equations under suitable parabolicity hypotheses, and we provide conditions on the initial data and on the stochastic term, namely, on the associated spectral measure, so that these kind of solutions exist in suitably chosen functional classes. We also give a regularity result for the expected value of these solutions.

Keywords Parabolic stochastic partial differential equations · Random-field solutions · Variable coefficients · Fundamental solution

1 Introduction

We consider linear stochastic partial differential equations (SPDEs in the sequel) of the general form

$$Lu(t, x) = [\partial_t + A(t)]u(t, x) = \gamma(t, x) + \sigma(t, x)\dot{\Xi}(t, x), \tag{1}$$

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where:

- $A(t)$ is a continuous family of linear partial differential operators that contain partial derivatives in space ($x \in \mathbb{R}^d$, $d \geq 1$),
- γ, σ are real-valued functions, subject to certain regularity conditions,
- Ξ is an $\mathcal{S}'(\mathbb{R}^d)$-valued Gaussian process, white in time and coloured in space, with correlation measure Γ and spectral measure $\mathfrak{M}$ (see Sect. 2 for a precise definition),
- u is an unknown stochastic process called *solution* of the SPDE.

To give meaning to (1) we rewrite it in its corresponding integral form and look for *mild solutions of* (1), that is, stochastic processes $u(t, x)$ satisfying

$$
\begin{aligned}
u(t,x) = v_0(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y)dyds \\
&+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)dyds,
\end{aligned} \tag{2}
$$

where:

- v_0 is a deterministic term, taking into account the initial condition;
- Λ is a suitable kernel, associated with the fundamental solution of the partial differential equation (PDE in the sequel) $Lu = [\partial_t + A(t)]u = 0$;
- the first integral in (2) is of deterministic type, while the second is a stochastic integral, and both are distributional integrals since $\Lambda(t, s, x, y)$ is, in general, a distribution with respect to the variables $(x, y) \in \mathbb{R}^{2d}$.

The kind of solution u we can construct for Eq. (1) depends on the approach we employ to make sense of the stochastic integral appearing in (2).

In the present paper we are looking for a *random-field solution* of (1), that is, we rely on the theory of stochastic integration with respect to a martingale measure developed in [8, 11, 21]. We are so going to define the stochastic integral in (2) through the martingale measure derived from the random noise $\dot{\Xi}$. Consequently, we are going to obtain a *random-field solution*, that is, a solution u defined as a map associating a random variable with each $(t, x) \in [0, T_0] \times \mathbb{R}^d$, where $T_0 > 0$ is the time horizon of the solution of the equation.

Recently, hyperbolic SPDEs involving operators with (t, x)-dependent coefficients have been studied. The existence of a random-field solution, first for linear operators with uniformly bounded coefficients [3], and subsequently for operators with polynomially bounded coefficients [6], has been shown. Moreover, the existence of a unique function-valued solution has been shown for semilinear hyperbolic SPDEs [5]. The main tools used for achieving this objective, namely, pseudodifferential and Fourier integral operators, come from microlocal analysis. To our knowledge, those are the first times that their full potential has been rigorously applied within the theory of random-field solutions to hyperbolic SPDEs.

Coming now to parabolic SPDEs, Dalang [11] studied random field solutions to parabolic equations with t-continuous coefficients of the form

$$\begin{aligned}\partial_t u(t,x) - \left(\sum_{i,j=1}^{n} a_{i,j}(t)\partial^2_{x_i x_j} + \sum_{i=1}^{n} b_i(t)\partial_{x_i} + c(t)\right) u(t,x) \\ = \gamma(u(t,x)) + \sigma(u(t,x))\dot{\Xi}(t,x)\end{aligned} \tag{3}$$

under the coercivity assumption

$$\sum_{i,j=1}^{n} a_{i,j}(t)\xi_i\xi_j \geq \epsilon|\xi|^2, \quad (t,\xi) \in [0,T] \times \mathbb{R}^d,$$

for some constant $\epsilon > 0$. He obtained a random field solution of (3) under the condition

$$\int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{1+|\xi|^2} d\xi < \infty.$$

Furthermore, Sanz-Solé and Vuillermont [19] proved the existence and uniqueness of a variational random-field solution to a class of initial-boundary value problems for second order parabolic equations with variable coefficients of the form

$$\partial_t u(t,x) = \operatorname{div}\left(k(t,x)\nabla u(t,x)\right) + \gamma(u(t,x)) + \sigma(u(t,x))W(t,x), \ (t,x) \in [0,T] \times D,$$

with D a sufficiently regular bounded domain in $\mathbb{R}^d$, k a positive definite symmetric matrix, $W(t,x)$ a Wiener process.

In the present paper we deal with the existence of a random-field solution to linear parabolic SPDEs of the form (1) with (t,x)-dependent coefficients admitting, at most, a polynomial growth as $|x| \to \infty$. More precisely, here we treat *parabolic equations* of arbitrary order $m, \mu > 0$ of the form (1), whose coefficients are defined on the whole space $\mathbb{R}^d$, that is

$$L = \partial_t + A(t), \qquad A(t)u(t,x) = \sum_{|\alpha|\leq\mu} a_\alpha(t,x)\,(D^\alpha_x u)(t,x), \tag{4}$$

$D = -i\partial$, where $\mu \geq 1$, $a_\alpha \in C([0,T], C^\infty(\mathbb{R}^d))$ for $|\alpha| \leq \mu$, and, for all $\beta \in \mathbb{N}^d_0 = (\mathbb{N} \cup \{0\})^d$, there exists a constant $C_{\alpha\beta} > 0$ such that

$$|\partial^\beta_x a_\alpha(t,x)| \leq C_{\alpha\beta}\langle x\rangle^{m-|\beta|},$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$, where $\langle x\rangle := (1+|x|^2)^{1/2}$. The parabolicity of L means that the parameter-dependent symbol $a(t,x,\xi)$ of the SG-operators family $A(t)$, defined here below, satisfies

$$a(t,x,\xi) := \sum_{|\alpha|\le\mu} a_\alpha(t,x)\xi^\alpha \ge C\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}, \tag{5}$$

with $C > 0$, $m \ge m' > 0$, $\mu \ge \mu' > 0$, that is, a is SG-hypoelliptic. Postponing to the next Sect. 3 the precise characterization, we give here an example.

Example 1 An example of a SG-parabolic operator L is given by the generalized SG-heat operator, defined for every $m, \mu \in \mathbb{N}\setminus\{0\}$ by

$$L = \partial_t + \langle x\rangle^{2m}\langle D\rangle^{2\mu}, \qquad x \in \mathbb{R}^d.$$

In this case $m = m'$, $\mu = \mu'$, that is, a is SG-elliptic.

We study SPDEs of the form (1), (4), (5), and we derive conditions on the right-hand side terms γ and σ, and on the spectral measure $\mathfrak{M}$ (hence, on $\dot\Xi$), such that there exists a random-field (mild) solution to the corresponding Cauchy problem.

As customary for the classes of the associated deterministic PDEs, we are interested in the present paper in both the smoothness, as well as the decay/growth at spatial infinity of the solutions. Here we also obtain an analog of such *global regularity* properties, employing suitable *weighted Sobolev spaces*, namely, the so-called Sobolev-Kato spaces $H^{z,\zeta}(\mathbb{R}^d)$, $z, \zeta \in \mathbb{R}$ defined by

$$H^{z,\zeta}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) \colon \|u\|_{z,\zeta} = \|\langle\cdot\rangle^z\langle D\rangle^\zeta u\|_{L^2} < \infty\}. \tag{6}$$

The results proved in this paper expand the theory developed in [3, 6] to the cases of operators L which are parabolic and whose coefficients are not uniformly bounded, and expand the results of [11] to the case of space-depending coefficients with polynomial growth and of higher order equations (there, second order operators are considered). Our main result reads as follows (see Sects. 3 and 4, and Theorem 6 below, for the precise definitions and statement).

Theorem *Let us consider the Cauchy problem*

$$\begin{cases} Lu(t,x) = \gamma(t,x) + \sigma(t,x)\dot\Xi(t,x), & (t,x) \in (0,T]\times\mathbb{R}^d,\\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{7}$$

for a SPDE associated with an SG-parabolic operator L of the form (4) *with $m \ge m' > 0$, $\mu \ge \mu' > 0$. Let $u_0 \in H^{z,\zeta}(\mathbb{R}^d)$, with $z \ge 0$ and $\zeta > d/2$, and assume that $\gamma \in C([0,T]; H^{z,\zeta}(\mathbb{R}^d))$, $\sigma \in C([0,T], H^{0,\zeta}(\mathbb{R}^d))$, $s \mapsto \mathcal{F}\sigma(s) = \nu_s \in L^2([0,T], \mathcal{M}_b(\mathbb{R}^d))$, where $\mathcal{M}_b(\mathbb{R}^d)$ is the space of complex-valued measures with* *finite total variation. Assume that one of the following conditions on the spectral measure* $\mathfrak{M}$*, associated with the random noise* $\dot{\Xi}$*, holds:*

(H0) *either, for every* $t \in [0, T]$*,*

$$\sup_{0\le s<t} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |e(t, s, x, \xi + \eta)|^2 \,\mathfrak{M}(d\xi) < \infty$$

and for every $0 \le s < t$

$$\lim_{h\downarrow 0} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t, s, x, \xi + \eta) - e(t, r, x, \xi + \eta)|^2 \,\mathfrak{M}(d\xi) = 0,$$

where $e(t, s)$ *is the (parameter-dependent) symbol of the fundamental solution of the operator* L*,*

(H1) *or*

$$\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) < \infty,$$

(H2) *or* $\mathfrak{M}$ *is absolutely continuous,* $|\nu_s|_{\mathrm{tv}} \in L^\infty(0, T)$*, and*

$$\int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{\langle\xi\rangle^{\mu'}} < \infty.$$

Then, there exists a random-field solution u *of* (7)*. Moreover, for any* $\kappa \in [0, 1)$*,*

$$\mathbb{E}[u] \in C([0, T], H^{z,\zeta}(\mathbb{R}^d)) \cap C^1([0, T], H^{z-m,\zeta-\mu}(\mathbb{R}^d)) \cap \\ \cap\, C^1((0, T], \mathcal{S}(\mathbb{R}^d)) \cap L^1([0, T], H^{z+\kappa m',\zeta+\kappa\mu'}(\mathbb{R}^d)),$$

and also $\partial_t \mathbb{E}[u] \in L^1([0, T], H^{z-m+\kappa m',\zeta-\mu+\kappa\mu'}(\mathbb{R}^d))$*,* $\kappa \in [0, 1)$*.*

Remark 1 Notice that we find, for general parabolic SPDEs with coefficients in (t, x), possibly polynomially growing as $|x| \to \infty$, in the case of an absolutely continuous spectral measure and $|\nu_s|_{\mathrm{tv}}$ bounded, the same condition given in [11] on the spectral measure, with $\mu = \mu' = 2$, see **(H2)**.

The main tools for proving the existence of random-field solutions to (1) will be pseudodifferential operators with symbols in the so-called SG classes. Such symbol classes have been introduced in the 1970s by H.O. Cordes (see, e.g. [9]) and C. Parenti [17] (see also R. Melrose [16]). The strategy to prove the main theorem consists of the following steps:

1. construction of the fundamental solution of L in (4), and then (formally) of the solution u to (7);
2. proof of the fact that v_0 and the stochastic and deterministic integrals, appearing in the (formal) expression (2) of u, are well-defined.

For point (1) we need, on one hand, to perform compositions between pseudodifferential operators, using the theory developed, e.g., in [9], and, on the other hand, the construction of the fundamental solution of parabolic operators in the SG environment. The latter can be achieved in analogy to the theory developed in [14, Chapter 7, §4], but here, in addition, we obtain more precise information about the order of the pseudodifferential operator family $E(t, s)$ that defines the fundamental solution of L. For point (2) we rely on (a variant of) results proved in [3].

With the aim of giving a presentation as self-contained as possible, for the convenience of the reader, we provide various preliminaries from the existing literature. The paper is organized as follows.

In Sect. 2 we recall some notions about stochastic integration with respect to martingale measures and the corresponding concept of random-field solution to a SPDE. Since, in contrast to the classical references [11, 21], here we have to deal with integrands of the form $\Lambda(t, s, x, y)\sigma(s, y)$ with (t, x) fixed, we directly present here the conditions that Λ and σ have to satisfy to let the stochastic integral with respect to a martingale measure in (2) be well-defined.

In Sect. 3 we first give a brief summary of the main tools, coming from microlocal analysis, that we use for the construction of the fundamental solution operator and of its kernel $\Lambda(t, s, x, y)$ (these results come mainly from [9, 15]). Subsequently, we perform the construction of the fundamental solution of the SG-parabolic operator L. To our best knowledge, compared with the previously existing literature, such construction for this operator class, which is essential to us to prove our main theorem, has not appeared elsewhere.

In Sect. 4 we focus on the parabolic SPDE (1), (4), (5), and prove our main theorem, under appropriate assumptions (see Theorem 6). Finally, we mention that the results illustrated in Sect. 4 about the structure of the kernel $\Lambda(t, s, x, y)$ appearing in (2) are employed also in [7], where we look for function-valued solutions to the semilinear parabolic SPDEs

$$Lu(t, x) = \gamma(t, x, u(t, x)) + \sigma(t, x, u(t, x))\dot{\Xi}(t, x) \tag{8}$$

associated with (1).

2 Stochastic Integration with Respect to a Martingale Measure

Let us consider a distribution-valued Gaussian process $\{\Xi(\phi);\ \phi \in \mathscr{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with mean zero and covariance functional given by

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} \big(\phi(t) * \tilde{\psi}(t)\big)(x)\, \Gamma(dx)dt, \tag{9}$$

where $\tilde{\psi}(t,x) := \overline{\psi(t,-x)}$, $*$ is the convolution operator and Γ is a nonnegative, nonnegative definite, tempered measure on $\mathbb{R}^d$ usually called *correlation measure*. Then [20, Chapter VII, Théorème XVIII] implies that there exists a nonnegative tempered measure $\mathfrak{M}$ on $\mathbb{R}^d$, usually called *spectral measure*, such that $\mathscr{F}\Gamma = \mathfrak{M}$, where $\mathscr{F}$ denotes the Fourier transform. By Parseval's identity, the right-hand side of (9) can be rewritten as

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} [\mathscr{F}\phi(t)](\xi) \cdot \overline{[\mathscr{F}\psi(t)](\xi)}\, \mathfrak{M}(d\xi)dt.$$

Definition 1 We call *(mild) random-field solution to* (1) an $L^2(\Omega)$-family of random variables $u(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}^d$, jointly measurable, satisfying the stochastic integral equation (2).

In this section we provide conditions to show that the stochastic integral in (2) is meaningful. This will be enough for our purposes, since the other two terms in (2) are deterministic, and will turn out to be well-defined by the theory of parabolic partial differential equations in our setting. For a complete set of conditions such that each term on the right-hand side of (2) is meaningful, when a general distribution Λ is involved, see [3].

We want to give a precise meaning to the stochastic integral in (2) by defining

$$\int_0^t\int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)dsdy := \int_0^t\int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)M(ds,dy), \quad (10)$$

where, on the right-hand side, we have a stochastic integral with respect to the martingale measure M related to Ξ. As explained in [12], by approximating indicator functions with $\mathscr{C}_0^\infty$-functions, the process Ξ can indeed be extended to a worthy martingale measure $M = (M_t(A);\ t \in \mathbb{R}_+, A \in \mathscr{B}_b(\mathbb{R}^d))$, where $\mathscr{B}_b(\mathbb{R}^d)$ denotes the bounded Borel subsets of $\mathbb{R}^d$. The stochastic integral with respect to the martingale measure M of stochastic processes f and g, indexed by $(t,x) \in [0,T] \times \mathbb{R}^d$ and satisfying suitable conditions, is constructed by steps (see [8, 11, 21]), starting from the class $\mathscr{E}$ of simple processes, and making use of the pre-inner product (defined for suitable f, g)

$$\begin{aligned}\langle f, g\rangle_0 &= \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^d} \big(f(s) * \tilde{g}(s)\big)(x)\,\Gamma(dx)ds\Big] \\ &= \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^d} [\mathscr{F}f(s)](\xi)\cdot\overline{[\mathscr{F}g(s)](\xi)}\,\mathfrak{M}(d\xi)ds\Big],\end{aligned} \quad (11)$$

with corresponding semi-norm $\|\cdot\|_0$. For a *simple process*

$$g(t,x;\omega) = \sum_{j=1}^m 1_{(a_j,b_j]}(t)1_{A_j}(x)X_j(\omega) \in \mathscr{E}$$

(with $m \in \mathbb{N}$, $0 \leq a_j < b_j \leq T$, $A_j \in \mathscr{B}_b(\mathbb{R}^d)$, X_j bounded, and $\mathscr{F}_{A_j}$-measurable random variable for all $1 \leq j \leq n$), the stochastic integral with respect to M is given by

$$(g \cdot M)_t := \sum_{j=1}^{m} \big(M_{t\wedge b_j}(A_j) - M_{t\wedge a_j}(A_j)\big) X_j,$$

where $x \wedge y := \min\{x, y\}$, and the fundamental isometry

$$\mathbb{E}\big[(g \cdot M)_t^2\big] = \|g\|_0^2 \tag{12}$$

holds for all $g \in \mathscr{E}$. The Hilbert space $\mathscr{P}_0$ of integrable stochastic processes is defined as the completion of $\mathscr{E}$ with respect to $\langle \cdot, \cdot \rangle_0$. On $\mathscr{P}_0$, the stochastic integral with respect to M is constructed as an $L^2(\Omega)$-limit of simple processes via the isometry (12). Moreover, by Lemma 2.2 in [18] we know that $\mathscr{P}_0 = L^2_p([0, T] \times \Omega, \mathscr{H})$, where here $L^2_p(\ldots)$ stands for the predictable stochastic processes in $L^2(\ldots)$ and $\mathscr{H}$ is the Hilbert space which is obtained by completing the Schwartz functions with respect to the inner product $\langle \cdot, \cdot \rangle_0$. Thus, $\mathscr{P}_0$ consists of predictable processes which may contain tempered distributions in the x-argument (whose Fourier transforms are functions, $\mathbb{P}$-almost surely).

Now, to give a meaning to the integral (10), we need to impose conditions on the distribution Λ and on the coefficient σ such that $\Lambda\sigma \in \mathscr{P}_0$. To this aim, we introduce the following space.

Definition 2 $\mathcal{S}'(\mathbb{R}^d)_\infty$ is the space of all the tempered distributions $T \in \mathcal{S}'(\mathbb{R}^d)$ such that, for every k, $\langle \cdot \rangle^k T$ is a bounded distribution on $\mathbb{R}^d$, i.e. it belongs to the dual space of $\{\varphi \in C^\infty(\mathbb{R}^d) | \forall \alpha \in \mathbb{N}^d \; \partial^\alpha \varphi \in L^1(\mathbb{R}^d)\}$.

It can be shown that $\mathcal{S}'(\mathbb{R}^d)_\infty = \mathscr{O}'_C(\mathbb{R}^d)$, where $\mathscr{O}'_C$ is the widest class of distributions such that the convolution with elements of $\mathcal{S}'$ is well-defined. A necessary and sufficient condition for $T \in \mathcal{S}'(\mathbb{R}^d)_\infty$, which is useful for us, is the following:

$$T \in \mathscr{O}'_C(\mathbb{R}^d) \iff \forall \chi \in \mathscr{C}_0^\infty(\mathbb{R}^d)\, T * \chi \in \mathcal{S}(\mathbb{R}^d). \tag{13}$$

For more details, see [20] and the recent paper [4].

In [3], sufficient conditions for the existence of the integral on the right-hand side of (10) have been given, in the case that σ depends on the spatial argument y, assuming that the spatial Fourier transform of the function σ is a complex-valued measure with finite total variation. Namely, we assume that, for all $s \in [0, T]$,

$$|\mathscr{F}\sigma(\cdot, s)| = |\mathscr{F}\sigma(\cdot, s)|(\mathbb{R}^d) = \sup_\pi \sum_{A\in\pi} |\mathscr{F}\sigma(\cdot, s)|(A) < \infty,$$

where π is any partition on $\mathbb{R}^d$ into measurable sets A, and the supremum is taken over all such partitions. Let, in the sequel, $\nu_s := \mathscr{F}\sigma(\cdot, s)$, and let $|\nu_s|_{\mathrm{tv}}$ denote its total variation. We summarize such conditions in the following theorem (see [2, 3, 5, 6] for details).

Theorem 1 *Let Δ_T be the simplex given by $0 \le t \le T$ and $0 \le s \le t$. Let, for $(t, s, x) \in \Delta_T \times \mathbb{R}^d$, $\Lambda(t, s, x)$ be a deterministic function with values in $\mathscr{S}'(\mathbb{R}^d)_\infty$, and let σ be a function in $L^2([0, T], C_b(\mathbb{R}^d))$, where C_b stands for the space of continuous and bounded functions, such that:*

(A1) *the function $(t, s, x, \xi) \mapsto [\mathscr{F}\Lambda(t, s, x)](\xi)$ is measurable, the function $s \mapsto \mathscr{F}\sigma(s) = \nu_s$ belongs to $L^2([0, T], \mathscr{M}_b(\mathbb{R}^d))$, and, for every $t \in [0, T]$,*

$$\int_0^t \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |[\mathscr{F}\Lambda(t, s, x)](\xi + \eta)|^2\, \mathfrak{M}(d\xi) \right) |\nu_s|^2_{\mathrm{tv}}\, ds < \infty; \tag{14}$$

(A2) *Λ and σ are as in (A1) and, for every $t \in [0, T]$,*

$$\lim_{h\downarrow 0} \int_0^t \chi_{[0,t-h)}(s) \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |[\mathscr{F}(\Lambda(t, s, x) - \Lambda(t, r, x))](\xi + \eta)|^2\, \mathfrak{M}(d\xi) \right) \times |\nu_s|^2_{\mathrm{tv}}\, ds = 0.$$

Then $\Lambda\sigma \in \mathscr{P}_0$. In particular, the stochastic integral on the right-hand side of (10) is well-defined and

$$\mathbb{E}\big[((\Lambda(t, \cdot, x, *)\sigma(\cdot, *)) \cdot M)^2_t\big] \le \le \int_0^t \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |[\mathscr{F}\Lambda(t, s, x)](\xi + \eta)|^2\, \mathfrak{M}(d\xi) \right) |\nu_s|^2_{\mathrm{tv}}\, ds.$$

Remark 2 In [3] conditions **(A1)** and **(A2)** are given in a slightly different way. Namely, an integral on $[0, T]$ appears there, in place of integrals on $[0, t]$ for every $t \in [0, T]$. Moreover, in (**A2**) a characteristic function naturally appears in the proof of Theorem 2.3 in [3]. The present formulation is actually the minimal requirement needed to prove that theorem, see the corresponding proof.

Remark 3 If $\sigma = \sigma(s)$, then $\mathscr{F}\sigma(s) = (2\pi)^d \sigma(s)\delta_0$, where δ_0 is the Dirac delta distribution with total variation 1. In such case, the necessary condition becomes $\int_0^T \sigma(s)^2 \int_{\mathbb{R}^d} |[\mathscr{F}\Lambda(t, s, x)](\xi)|^2\, \mathfrak{M}(d\xi) ds < \infty$, which is actually weaker than (14), in the sense that there is no supremum over η, and corresponds to the one given in [11, Example 9].

3 Microlocal Analysis and Fundamental Solution to Parabolic Equations with Polynomially Bounded Coefficients

3.1 Elements of the SG-Calculus

We recall here the basic definitions and facts about the so-called SG-calculus of pseudodifferential operators, through standard material appeared, e.g., in [5, 6], and elsewhere (sometimes with slightly different notational choices). In the sequel, we will often write $A \lesssim B$ when $|A| \le c \cdot |B|$, for a suitable constant $c > 0$.

The class $S^{m,\mu} = S^{m,\mu}(\mathbb{R}^d)$ of SG symbols of order $(m, \mu) \in \mathbb{R}^2$ is given by all the functions $a(x, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with the property that, for any multiindices $\alpha, \beta \in \mathbb{N}_0^d$, there exist constants $C_{\alpha\beta} > 0$ such that the conditions

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \le C_{\alpha\beta} \langle x\rangle^{m-|\alpha|} \langle\xi\rangle^{\mu-|\beta|}, \qquad (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{15}$$

hold. For $m, \mu \in \mathbb{R}$, $\ell \in \mathbb{N}_0$, $a \in S^{m,\mu}$, the quantities

$$\|a\|_\ell^{m,\mu} = \max_{|\alpha+\beta|\le\ell} \sup_{x,\xi\in\mathbb{R}^d} \langle x\rangle^{-m+|\alpha|} \langle\xi\rangle^{-\mu+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \tag{16}$$

are a family of seminorms, defining the Fréchet topology of $S^{m,\mu}$. The corresponding classes of pseudodifferential operators $\mathrm{Op}(S^{m,\mu}) = \mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))$ are given, for $a \in S^{m,\mu}(\mathbb{R}^d)$, $u \in \mathcal{S}(\mathbb{R}^d)$, by

$$(\mathrm{Op}(a)u)(x) = (a(\cdot, D)u)(x) = (2\pi)^{-d} \int e^{\mathrm{i}x\xi} a(x, \xi) \hat{u}(\xi) d\xi, \tag{17}$$

where $\hat{u}$ stands for the Fourier transform of u, extended by duality to $\mathcal{S}'(\mathbb{R}^d)$. The operators in (17) form a graded algebra with respect to composition, i.e.,

$$\mathrm{Op}(S^{m_1,\mu_1}) \circ \mathrm{Op}(S^{m_2,\mu_2}) \subseteq \mathrm{Op}(S^{m_1+m_2,\mu_1+\mu_2}).$$

The symbol $c \in S^{m_1+m_2,\mu_1+\mu_2}$ of the composed operator $\mathrm{Op}(a) \circ \mathrm{Op}(b)$, $a \in S^{m_1,\mu_1}$, $b \in S^{m_2,\mu_2}$, admits the asymptotic expansion

$$c(x, \xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha a(x, \xi)\, D_x^\alpha b(x, \xi), \tag{18}$$

which implies that the symbol c equals $a \cdot b$ modulo $S^{m_1+m_2-1,\mu_1+\mu_2-1}$.

The residual elements of the calculus are operators with symbols in

$$S^{-\infty,-\infty} = S^{-\infty,-\infty}(\mathbb{R}^d) = \bigcap_{(m,\mu)\in\mathbb{R}^2} S^{m,\mu}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^{2d}),$$

that is, those having kernel in $\mathcal{S}(\mathbb{R}^{2d})$, continuously mapping $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$. For any $a \in S^{m,\mu}$, $(m, \mu) \in \mathbb{R}^2$, $\mathrm{Op}(a)$ is a linear continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to itself, extending to a linear continuous operator from $\mathcal{S}'(\mathbb{R}^d)$ to itself, and from $H^{z,\zeta}(\mathbb{R}^d)$ to $H^{z-m,\zeta-\mu}(\mathbb{R}^d)$, where $H^{z,\zeta}(\mathbb{R}^d)$, $(z, \zeta) \in \mathbb{R}^2$, denotes the Sobolev-Kato (or *weighted Sobolev*) space defined in (6) with the naturally induced Hilbert norm. When $z \geq z'$ and $\zeta \geq \zeta'$, the continuous embedding $H^{z,\zeta} \hookrightarrow H^{z',\zeta'}$ holds true. It is compact when $z > z'$ and $\zeta > \zeta'$. Since $H^{z,\zeta} = \langle\cdot\rangle^z H^{0,\zeta} = \langle\cdot\rangle^z H^{\zeta}$, with H^{ζ} the usual Sobolev space of order $\zeta \in \mathbb{R}$, we find $\zeta > k + \dfrac{d}{2} \Rightarrow H^{z,\zeta} \hookrightarrow C^k$, $k \in \mathbb{N}_0$.

One also actually finds

$$\begin{aligned}
&\bigcap_{z,\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d) = H^{\infty,\infty}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d),\\
&\bigcup_{z,\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d) = H^{-\infty,-\infty}(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d),
\end{aligned} \tag{19}$$

as well as, for the space of *rapidly decreasing distributions*, see [4, 20],

$$\mathcal{S}'(\mathbb{R}^d)_\infty = \bigcap_{z\in\mathbb{R}} \bigcup_{\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d) = H^{+\infty,-\infty}(\mathbb{R}^d). \tag{20}$$

The continuity property of the elements of $\mathrm{Op}(S^{m,\mu})$ on the scale of spaces $H^{z,\zeta}(\mathbb{R}^d)$, $(m, \mu), (z, \zeta) \in \mathbb{R}^2$, is expressed more precisely in the next Theorem 2.

Theorem 2 *Let* $a \in S^{m,\mu}(\mathbb{R}^d)$, $(m, \mu) \in \mathbb{R}^2$. *Then, for any* $(z, \zeta) \in \mathbb{R}^2$, $\mathrm{Op}(a) \in \mathcal{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-m,\zeta-\mu}(\mathbb{R}^d))$, *and there exists a constant* $C > 0$, *depending only on* d, m, μ, z, ζ, *such that*

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-m,\zeta-\mu}(\mathbb{R}^d))} \leq C \|a\|^{m,\mu}_{\left[\frac{d}{2}\right]+1}, \tag{21}$$

where $[t]$ *denotes the integer part of* $t \in \mathbb{R}$ *and* $\mathcal{L}(X, Y)$ *stands for the space of linear and continuous maps from a space* X *to a space* Y.

Cordes introduced the class $\mathcal{O}(m, \mu)$ of the *operators of order* (m, μ) as follows, see, e.g., [9].

Definition 3 A linear continuous operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ belongs to the class $\mathcal{O}(m, \mu)$, of the operators of order $(m, \mu) \in \mathbb{R}^2$ if, for any $(z, \zeta) \in \mathbb{R}^2$, it extends to a linear continuous operator $A_{z,\zeta} : H^{z,\zeta}(\mathbb{R}^d) \to H^{z-m,\zeta-\mu}(\mathbb{R}^d)$. We also define

$$\mathcal{O}(\infty, \infty) = \bigcup_{(m,\mu)\in\mathbb{R}^2} \mathcal{O}(m, \mu), \quad \mathcal{O}(-\infty, -\infty) = \bigcap_{(m,\mu)\in\mathbb{R}^2} \mathcal{O}(m, \mu).$$

Remark 4

1. Trivially, any $A \in \mathcal{O}(m,\mu)$ admits a linear continuous extension $A_{\infty,\infty}\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. In fact, in view of (19), it is enough to set $A_{\infty,\infty}|_{H^{z,\zeta}(\mathbb{R}^d)} = A_{z,\zeta}$.
2. Theorem 2 implies $\mathrm{Op}(S^{m,\mu}(\mathbb{R}^d)) \subset \mathcal{O}(m,\mu)$, $(m,\mu) \in \mathbb{R}^2$.
3. $\mathcal{O}(\infty,\infty)$ and $\mathcal{O}(0,0)$ are algebras under operator multiplication, $\mathcal{O}(-\infty,-\infty)$ is an ideal of both $\mathcal{O}(\infty,\infty)$ and $\mathcal{O}(0,0)$, and $\mathcal{O}(m_1,\mu_1)\circ\mathcal{O}(m_2,\mu_2) \subset \mathcal{O}(m_1+m_2,\mu_1+\mu_2)$.

The following characterization of the class $\mathcal{O}(-\infty,-\infty)$ is often useful, see [9].

Theorem 3 *The class $\mathcal{O}(-\infty,-\infty)$ coincides with $\mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d))$ and with the class of smoothing operators, that is, the set of all the linear continuous operators $A\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. All of them coincide with the class of linear continuous operators A admitting a Schwartz kernel K_A belonging to $\mathcal{S}(\mathbb{R}^{2d})$.*

An operator $A = \mathrm{Op}(a)$ and its symbol $a \in S^{m,\mu}$ are called *elliptic* (or $S^{m,\mu}$-*elliptic*) if there exists $R \geq 0$ such that

$$C\langle x\rangle^m\langle\xi\rangle^\mu \leq |a(x,\xi)|, \qquad |x|+|\xi| \geq R,$$

for some constant $C > 0$. If $R = 0$, a^{-1} is everywhere well-defined and smooth, and $a^{-1} \in S^{-m,-\mu}$. If $R > 0$, then a^{-1} can be extended to the whole of $\mathbb{R}^{2d}$ so that the extension $\tilde{a}_{-1}$ satisfies $\tilde{a}_{-1} \in S^{-m,-\mu}$. An elliptic *SG* operator $A \in \mathrm{Op}(S^{m,\mu})$ admits a parametrix $A_{-1} \in \mathrm{Op}(S^{-m,-\mu})$ such that

$$A_{-1}A = I + R_1, \quad AA_{-1} = I + R_2,$$

for suitable $R_1, R_2 \in \mathrm{Op}(S^{-\infty,-\infty})$, where I denotes the identity operator. In such a case, A turns out to be a Fredholm operator on the scale of functional spaces $H^{z,\zeta}(\mathbb{R}^d)$, $(z,\zeta) \in \mathbb{R}^2$.

Proposition 1 *Let $A = \mathrm{Op}(a)$ be a SG pseudodifferential operator, with symbol $a \in S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu) \in \mathbb{R}^2$, and let K_A denote its Schwartz kernel. Then, the Fourier transform with respect to the second argument of K_A, $\mathcal{F}_{\cdot\mapsto\eta}K_A(x,\cdot)$, is given by*

$$\mathcal{F}_{\cdot\mapsto\eta}K_A(x,\cdot) = \mathrm{e}^{-\mathrm{i}x\cdot\eta}a(x,-\eta). \tag{22}$$

The proof of Proposition 1 can be found, e.g., in [9]. The next Lemma 1 is a special case of the similar, more general result for the kernel of *SG* Fourier integral operators proved, for instance, in [5]. We give its direct proof here, for the convenience of the reader.

Lemma 1 *Let $A = \mathrm{Op}(a)$ be a SG pseudodifferential operator with symbol $a \in S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu) \in \mathbb{R}^2$, and let K_A denote its Schwartz kernel. Then, for every $x \in \mathbb{R}^d$, $K_A(x,\cdot) \in \mathcal{S}'(\mathbb{R}^d)_\infty$. More precisely, we find $K_A \in C^\infty(\mathbb{R}^d, \mathcal{S}'(\mathbb{R}^d)_\infty)$.*

Proof Given a fixed $x \in \mathbb{R}^d$, by [4, Theorem 3.3], to see that $K_A(x,\cdot) \in \mathscr{S}'(\mathbb{R}^d)_\infty$ it suffices to show that for every $\chi \in \mathscr{D}(\mathbb{R}^d)$, $K_A(x,\cdot) * \chi \in \mathscr{S}(\mathbb{R}^d)$. We already know [20, p. 244/245] that $K_A(x,\cdot) * \chi$ is a C^∞ function of slow growth. Computing now its Fourier transform (in the distributional sense), using Proposition 1 we see that

$$\mathscr{F}_{\cdot\mapsto\eta}(K_A(x,\cdot) * \chi)(\eta) = \mathscr{F}_{\cdot\mapsto\eta} K_A(x,\cdot)\widehat{\chi}(\eta) = \mathrm{e}^{-\mathrm{i}x\cdot\eta} a(x,-\eta)\widehat{\chi}(\eta) \in \mathscr{S}(\mathbb{R}^d_\eta).$$

It follows that, for its inverse Fourier transform, $K_A(x,\cdot) * \chi \in \mathscr{S}(\mathbb{R}^d)$, too. Finally, the fact that the map

$$x \mapsto K_A(x,y) = \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,\xi) đ\xi$$

belongs to $C^\infty(\mathbb{R}^d, \mathscr{S}'(\mathbb{R}^d)_\infty)$ is a consequence of the general properties of oscillatory integrals, taking into account that $x \cdot \xi$ and $a(x,\xi)$ are smooth functions with respect to x. This completes the proof.

3.2 Construction of the Fundamental Solution of SG-Parabolic Operators

We work here with a class of operators with more general symbols than the (polynomial) ones appearing in (4). Namely, we consider operators of the form

$$L = \partial_t + A(t) = \partial_t + \mathrm{Op}(a(t)), \tag{23}$$

where, for $m, \mu > 0$, $A(t) = \mathrm{Op}(a(t))$ are SG pseudodifferential operators with parameter-dependent symbol $a \in C([0,T], S^{m,\mu}(\mathbb{R}^d))$. Notice that, of course, (4) is a special case of (23). The parabolicity condition on L is here expressed by means of the (SG-)hypoellipticity of $A(t)$, namely,

$$\begin{aligned} \exists C > 0 \quad & \mathrm{Re}\, a(t,x,\xi) \geq C\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}, \\ \forall \alpha,\beta \in \mathbb{N}^d\ \exists C_{\alpha\beta} > 0 \quad & \left|\frac{\partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)}{\mathrm{Re}\, a(t,x,\xi)}\right| \leq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}. \end{aligned} \tag{24}$$

where $0 < m' \leq m$, $0 < \mu' \leq \mu$, $t \in [0,T]$, $x, \xi \in \mathbb{R}^d$. $A(t)$ is (SG-)elliptic if $m = m'$, $\mu = \mu'$, see above. Elements of the microlocal analysis of SG-parabolic operators can be found in [9, 15]. As customary, $A(t)$, $t \in [0,T]$, is considered as an unbounded operator in L^2 with dense domain $H^{m,\mu}$ (see [9, Ch. 3, Sec. 3–4]; see also [15] for the spectral theory of corresponding self-adjoint elliptic operators).

Definition 4 We say that $L = \partial_t + \mathrm{Op}(a(t))$, $a \in C([0,T], S^{m,\mu}(\mathbb{R}^d))$ is $(SG$-) parabolic, with respect to m, m', μ, μ', $0 < m' \le m$, $0 < \mu' \le \mu$, if a satisfies the (SG-)hypoellipticity condition (24).

We now prove our first main result, namely, the existence of the fundamental solution operator of a SG-parabolic operator L.

Theorem 4 *Let $L = \partial_t + \mathrm{Op}(a(t))$, $a \in C([0,T], S^{m,\mu}(\mathbb{R}^d))$ be (SG-)parabolic, with respect to m, m', μ, μ', $0 < m' \le m$, $0 < \mu' \le \mu$. Then, L admits a fundamental solution operator $E(t,s)$, $0 \le s \le t \le T$, $0 \le s < T$, that is, an operator family $E(t,s) = \mathrm{Op}(e(t,s))$ with $e(\cdot, s) \in C((s,T], S^{0,0}(\mathbb{R}^d)) \cap C^1((s,T], S^{m,\mu}(\mathbb{R}^d))$, with the following properties:*

1. *E satisfies the equation*

$$LE(t,s) = 0, \quad 0 \le s < t \le T; \tag{25}$$

2. *the symbol family $e(t,s)$ satisfies*

$$e(t,s,x,\xi) \to 1 \text{ weakly in } S^{0,0}(\mathbb{R}^d) \text{ for } t \to s^+; \tag{26}$$

3. *writing $e(t,s)$ as*

$$e(t,s,x,\xi) = \exp\left(-\int_s^t a(\tau,x,\xi)\,d\tau\right) + r_0(t,s,x,\xi), \tag{27}$$

the symbol family $r_0(t,s)$ satisfies

$$r_0(t,s,x,\xi) \to 0 \text{ weakly in } S^{-1,-1}(\mathbb{R}^d) \text{ for } t \to s^+, \tag{28}$$

$$\left\{\frac{r_0(t,s,x,\xi)}{t-s}\right\}_{0\le s<t\le T} \text{ is a bounded set in } S^{m-1,\mu-1}(\mathbb{R}^d). \tag{29}$$

Remark 5

1. It is enough that (24) is satisfied for $|x| + |\xi| \ge R > 0$. In fact, if this is the case, there exists $M > 0$ such that $a_M(t,x,\xi) = a(t,x,\xi) + M$ satisfies (24) everywhere. Let then $E_M(t,s)$ be the fundamental solution of $L_M = \partial_t + \mathrm{Op}(a_M(t))$. Then, $E(t,s) = e^{M(t-s)}E_M(t,s)$ is the fundamental solution of L and

$$e^{M(t-s)}e^{-\int_s^t [a(\tau)+M]\,d\tau} = e^{-\int_s^t a(\tau)\,d\tau},$$

   so $E(t,s)$ has the properties stated in Theorem 4.
2. Similarly to the analogous result which holds true for parabolic operators defined by means of the Hörmander's symbols $S^m_{\rho,\delta}(\mathbb{R}^d)$, $0 \le \delta < \rho \le 1$, found in [14],

Theorem 4 holds true, with simple modifications, for the generalized class of SG-symbols $S^{m,\mu}_{r,\rho}(\mathbb{R}^d)$, $r, \rho \geq 0$, $r + \rho > 0$, considered, e.g., in [10].

The next Theorem 5 is an immediate consequence of Theorem 4, by a Duhamel's argument and the properties of the fundamental solution E.

Theorem 5 *Let* $u_0 \in H^{z,\zeta}(\mathbb{R}^d)$*,* $f \in C([0, T], H^{z,\zeta}(\mathbb{R}^d))$*,* $z, \zeta \geq 0$*, and* $L = \partial_t + A(t)$ *satisfy the same assumptions as in Theorem 4. Then, the Cauchy problem*

$$\begin{cases} Lu(t,x) = f(t,x), & (t,x) \in (s,T] \times \mathbb{R}^d, \\ u(s,x) = u_0(x), & x \in \mathbb{R}^d, s \in [0,T), \end{cases} \tag{30}$$

admits a solution given by

$$u(t,x) = E(t,s)u_0(x) + \int_s^t E(t,\tau)\, f(\tau,x)\, d\tau, \quad s \leq t \leq T, \tag{31}$$

with $E(t,s)$ *the fundamental solution operator obtained in Theorem 4. Moreover, such solution satisfies*

$$u \in C([s,T], H^{z,\zeta}(\mathbb{R}^d)) \cap C^1([s,T], H^{z-m,\zeta-\mu}(\mathbb{R}^d)).$$

Remark 6 Recall that the initial condition in (30) is understood as

$$\lim_{t \to s^+} u(t) = u_0 \quad \text{in } L^2(\mathbb{R}^d).$$

We prove Theorem 4 by extending to the SG setting the argument given in [14] for the analogous result in the $S^m_{\rho,\delta}$ setting. Similarly to the mentioned proof scheme, we rely on the next three technical lemmas, which are, essentially, consequences of the SG-calculus. In particular, the proof of Lemma 4 requires the properties of the multiproducts of SG pseudodifferential operators (see [1]). For the sake of brevity, we only sketch the corresponding arguments.

Lemma 2 *Assume that* $a \in C([0,T], S^{m,\mu}(\mathbb{R}^d))$ *satisfies (24),* $0 < m' \leq m$*,* $0 < \mu' \leq \mu$*,* $t \in [0,T]$*,* $x, \xi \in \mathbb{R}^d$*. Set*

$$e_0(t,s,x,\xi) = \exp\left(-\int_s^t a(\tau,x,\xi)\, d\tau\right),$$

and define inductively $\{e_j(t,s)\}_{j=1}^{\infty}$*,* $\{q_j(t,s)\}_{j=1}^{\infty}$*,* $0 \leq s \leq t \leq T$ *by*

$$q_j(t,s,x,\xi) = \sum_{k=0}^{j-1} \sum_{|\alpha|+k=j} \frac{1}{\alpha!} \partial_\xi^\alpha a(t,x,\xi) \cdot D_x^\alpha e_k(t,s,x,\xi), \quad j \geq 1, \tag{32}$$

and

$$\begin{cases} [\partial_t + a(t,x,\xi)]e_j(t,s,x,\xi) = -q_j(t,s,x,\xi), \\ \qquad\qquad\qquad e_j(s,s,x,\xi) = 0, \qquad\qquad j \ge 1. \end{cases} \tag{33}$$

Then, for any $\alpha, \beta \in \mathbb{N}^d$*, there exist* $C_{\alpha\beta}, C'_{\alpha\beta} > 0$ *such that*

$$|\partial_x^\alpha \partial_\xi^\beta e_j(t,s,x,\xi)| \le \begin{cases} C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle \xi\rangle^{-|\beta|}, & j \ge 0 \\ C'_{\alpha\beta}(t-s)\langle x\rangle^{m-|\alpha|-j}\langle \xi\rangle^{\mu-|\beta|-j}, & j + |\alpha+\beta| \ge 1. \end{cases} \tag{34}$$

The proof of Lemma 2 follows from an accurate usage of the trivial estimate $s^\kappa e^{-s} \le C_\kappa < \infty$ for every $s \ge 0$, with constants $C_\kappa > 0$, $\kappa \in [0, +\infty)$, and from condition (24). By explicitly writing

$$q_1(t,s,x,\xi) = -\sum_{j=1}^{n} \partial_{\xi_j} a(t,x,\xi)\, e_0(t,s,x,\xi) \int_s^t \partial_{x_j} a(\tau,x,\xi)\, d\tau,$$

observing that

$$|e_0(t,s,x,\xi)| = e^{-\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\, d\tau} \le e^{-C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} \le 1, \tag{35}$$

$$\begin{aligned} \left| e_0(t,s,x,\xi) \int_s^t \partial_{x_j} a(\tau,x,\xi)\, d\tau \right| &\lesssim e^{-\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\, d\tau} \int_s^t \langle x\rangle^{-1} \mathrm{Re}\, a(\tau,x,\xi)\, d\tau \\ &\le C_1 \langle x\rangle^{-1} \lesssim \langle x\rangle^{-1}, \end{aligned}$$

and similarly estimating derivatives, one can prove $q_1(t,s,x,\xi) \in SG^{m-1,\mu-1}$ (and, inductively, $q_j(t,s,x,\xi) \in SG^{m-j,\mu-j}$). Now, solving (33), it follows

$$e_j(t,s,x,\xi) = -e_0(t,s,x,\xi) \int_s^t \frac{q_j(\tau,s,x,\xi)}{e_0(\tau,s,x,\xi)}\, d\tau, \quad j \ge 1. \tag{36}$$

On one hand, we can estimate

$$|e_j(t,s,x,\xi)| \le \int_s^t |q_j(\tau,s,x,\xi)|\, d\tau \le C(t-s)\langle x\rangle^{m-j}\langle\xi\rangle^{\mu-j}, \quad j \ge 1.$$

On the other hand, by explicitly writing q_1 and using (24) twice, we get

$$\begin{aligned} |e_1(t,s,x,\xi)| &\lesssim \langle x\rangle^{-1}\langle\xi\rangle^{-1} e^{-\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\, d\tau} \left(\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\, d\tau \right)^2 \\ &\le C_2 \langle x\rangle^{-1}\langle\xi\rangle^{-1} \lesssim \langle x\rangle^{-1}\langle\xi\rangle^{-1}. \end{aligned}$$

Similar arguments work for the derivatives of e_j, $j \geq 2$, so that we can actually conclude $e_j(t, s, x, \xi) \in SG^{-j,-j}$, $j \geq 1$.

Lemma 3 *Let, for* $N \geq 1$*,*

$$E_N(t, s) = \sum_{j=0}^{N-1} \mathrm{Op}(e_j(t, s)),$$

and

$$R_N(t, s) = \mathrm{Op}(r_N(t, s)) = LE_N(t, s), \tag{37}$$

with L *from Theorem 4 and* $\{e_j(t, s)\}_{j=1}^{\infty}$ *from Lemma 2. Then,*

$$r_N(\cdot, s) \in C((s, T], S^{m-N,\mu-N}(\mathbb{R}^d)), \quad 0 \leq s < T, \tag{38}$$

$$\left\{ \frac{r_N(t, s)}{t - s} \right\}_{0 \leq s < t \leq T} \text{ is bounded in } S^{2m-N,2\mu-N}(\mathbb{R}^d). \tag{39}$$

The proof of Lemma 3 is straightforward, in view of Lemma 2. Indeed, by the SG-calculus, employing the asymptotic expansion of the symbol of $\mathrm{op}(a(t))E_j(t, s)$,

$$LE_N(t, s) = \sum_{j=0}^{N-1} \mathrm{op}(\partial_t e_j(t, s) + a(t)e_j(t, s))$$
$$+ \sum_{j=0}^{N-1} \sum_{|\alpha|=1}^{N-j} \frac{i^{|\alpha|}}{\alpha!} \mathrm{op}(D_\xi^\alpha a(t) D_x^\alpha e_j(t, s)) + \sum_{j=0}^{N-1} R_{N,j}(t, s),$$

with $r_{N,j}(t, s, x, \xi) \in SG^{m-N-1,\mu-N-1}$, since $e_j \in SG^{-j,-j}$ for every $j \geq 0$, and $r_{N,j}(t, s, x, \xi) \in SG^{2m-N-1,2\mu-N-1}$, for every $j \geq 1$, by the second inequality in (34). By the choice of q_j in (32) and by (33), we see that $LE_N(t, s) = \sum_{j=0}^{N-1} R_{N,j}(t, s) = R_N(t, s)$, and formulae (38) and (39) hold.

Lemma 4 *Let* $R_N(t, s) = \mathrm{Op}(r_N(t, s))$ *be defined by* (37)*, with*

$$N \geq 1 \textit{ such that } \max\{m, \mu\} - N \leq 0. \tag{40}$$

Define inductively the sequence of operator families $\{W_\nu(t,s)\}_{\nu=1}^{\infty} = \{\mathrm{Op}(w_\nu(t,s))\}_{\nu=1}^{\infty}$, $0 \leq s \leq t \leq T$*, by*

$$W_1(t, s) = -R_N(t, s) = -\mathrm{Op}(r_N(t, s)), \tag{41}$$

$$W_\nu(t, s) = \int_s^t W_1(t, \tau) \circ W_{\nu-1}(\tau, s)\, d\tau, \quad \nu \geq 2. \tag{42}$$

Then, for $l \geq 1$, $0 \leq s \leq t \leq T$,

$$\sum_{\nu=1}^{l} W_\nu(t,s) = -R_N(t,s) - \int_s^t R_N(t,\tau) \sum_{\nu=1}^{l-1} W_\nu(\tau,s)\, d\tau, \tag{43}$$

and, for any $\alpha, \beta \in \mathbb{N}^d$, there exist constants $A_{\alpha\beta}, A'_{\alpha\beta} > 0$ such that, for $0 \leq s \leq t \leq T$, $x, \xi \in \mathbb{R}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta w_\nu(t,s,x,\xi)| \leq (A_{\alpha\beta})^\nu \frac{(t-s)^{\nu-1}}{(\nu-1)!} \langle x\rangle^{m-N-|\alpha|} \langle \xi\rangle^{\mu-N-|\beta|}, \tag{44}$$

$$|\partial_x^\alpha \partial_\xi^\beta w_\nu(t,s,x,\xi)| \leq (A'_{\alpha\beta})^\nu \frac{(t-s)^{\nu}}{(\nu-1)!} \langle x\rangle^{2m-N-|\alpha|} \langle \xi\rangle^{2\mu-N-|\beta|}. \tag{45}$$

Formula (43) follows readily by definitions (41) and (42). To get (44) and (45) we need to write

$$W_\nu(t,s) = \int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-2}} W_1(t,t_1) \cdots W_1(t_{\nu-1},s) dt_{\nu-1} \cdots dt_1.$$

By the choice of N, we can look at $W_1(t,t_1)$ as an operator of order either $(m - N, \mu - N)$ or $(2m - N, 2\mu - N)$ according to (37) or (38), respectively, and we can look at $W_1(t_1,t_2), \ldots, W_1(t_{\nu-1},s)$ as operators of order $(0, 0)$. By integrating on the symplex $s \leq t_{\nu-1} \leq \cdots \leq t_1 \leq t$, formulae (44) and (45) follow.

Proof of Theorem 4 Lemma 4 implies that

$$W(t,s) = \sum_{\nu=1}^{\infty} W_\nu(t,s)$$

converges in the topology of $\mathrm{Op}(S^{m-N,\mu-N})$, since, by (44), $\sum_\nu w_\nu(t,s)$ converges in the topology of $S^{m-N,\mu-N}$, for any fixed N satisfying (40) and $0 \leq s \leq t \leq T$. With $E_N(t,s)$ from Lemma 3, define, for $0 \leq s < t \leq T$, $N \geq 1$, $\max\{m,\mu\} - N \leq 0$,

$$E(t,s) = E_N(t,s) + \int_s^t E_N(t,\tau) \circ W(\tau,s)\, d\tau. \tag{46}$$

Then, by (37),

$$\begin{aligned} LE(t,s) &= LE_N(t,s) + W(t,s) + \int_s^t [LE_N(t,\tau)] \circ W(\tau,s)\, d\tau \\ &= R_N(t,s) + W(t,s) + \int_s^t R_N(t,\tau) \circ W(\tau,s)\, d\tau. \end{aligned} \tag{47}$$

By letting $l \to +\infty$ in (43), we find, for any N satisfying (40),

$$W(t,s) = -R_N(t,s) - \int_s^t R_N(t,\tau) \circ W(\tau,s)\, d\tau,$$

so that, by (47), it follows $LE(t,s) = 0$, $0 \le s < t \le T$, as claimed. All the properties of the symbol $e(t,s)$ of the operator family $E(t,s)$ are then consequences of (46) and Lemmas 2, 3, and 4.

Remark 7 Clearly, by construction, $e(t,s)$ (and $E(t,s)$) are continuous also with respect to s, $0 \le s \le t \le T$ (see Lemmas 2, 3, and 4, and the proof of Theorem 4).

In the next Lemma 5, we obtain further estimates for the symbol family $e(t,s)$, showing that, actually, for $0 \le s < t \le T$, it gives rise to (a C^1 family of) operators in $\mathcal{O}(-\infty,-\infty)$. This, of course, cannot be extended by continuity up to $t = s$, but some L^1 regularity with respect to t, that we employ in Sect. 4, can still be achieved.

Lemma 5 *For every $j \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^d$, we have, for suitable constants $C'_{j\alpha\beta} > 0$,*

$$|\partial_x^\alpha \partial_\xi^\beta e_j(t,s,x,\xi)| \le C'_{j\alpha\beta} \sqrt{|e_0(t,s,x,\xi)|}\, \langle x\rangle^{-j-|\alpha|} \langle \xi\rangle^{-j-|\beta|}, \tag{48}$$

with $0 \le s \le t \le T$, $(x,\xi) \in \mathbb{R}^d$. Moreover, for every $j \in \mathbb{N}$, $0 \le s < T$, $e_j(\cdot,s) \in C^1((s,T], \mathcal{S}(\mathbb{R}^{2d}))$ and $e(\cdot,s) \in L^1([s,T], S^{-\kappa m',-\kappa\mu'}(\mathbb{R}^d))$, $\partial_t e(\cdot,s) \in L^1([s,T], S^{m-\kappa m',\mu-\kappa\mu'}(\mathbb{R}^d))$, for every $\kappa \in [0,1)$.

Proof From (35), for every $m', \mu' > 0$ we see that, for every $\kappa \in [0,1)$,

$$\begin{aligned} &|e_0(t,s,x,\xi)| \le \\ &\le e^{-C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} (C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'})^\kappa (C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'})^{-\kappa} \\ &\lesssim \frac{C_\kappa}{(t-s)^\kappa} \langle x\rangle^{-\kappa m'}\langle\xi\rangle^{-\kappa\mu'} \lesssim (t-s)^{-\kappa}\langle x\rangle^{-\kappa m'}\langle\xi\rangle^{-\kappa\mu'}, \end{aligned} \tag{49}$$

where C_κ is the upper bound of $s^\kappa e^{-s}$, $s \ge 0$, which gives $\langle x\rangle^{\kappa m'}\langle\xi\rangle^{\kappa\mu'} e_0(\cdot,s,x,\xi) \in L^1([s,T])$, and similarly for the derivatives with respect to x and ξ. By induction, (48) follows. Let us perform part of the induction step for $j = 1$, leaving the remaining details to the reader. We have:

$$|e_1(t,s,x,\xi)| \le \sum_{j=1}^d |e_0(t,s,x,\xi)| \left| \int_s^t \frac{\partial_{\xi_j} a(\tau,x,\xi) \cdot D_{x_j} e_0(\tau,s,x,\xi)}{e_0(\tau,s,x,\xi)}\, d\tau \right|$$

$$\le \sum_{j=1}^d |e_0(t,s,x,\xi)| \int_s^t |\partial_{\xi_j} a(\tau,x,\xi)| \cdot \left| \int_s^\tau D_{x_j} \mathrm{Re}\, a(r,x,\xi)\, dr \right| d\tau$$

$$\lesssim \langle\xi\rangle^{-1}\langle x\rangle^{-1} |e_0(t,s,x,\xi)| \int_s^t |\mathrm{Re}\, a(\tau,x,\xi)| \cdot \left(\int_s^\tau \mathrm{Re}\, a(r,x,\xi)\, dr \right) d\tau$$

$$\leq \langle\xi\rangle^{-1}\langle x\rangle^{-1}|e_0(t,s,x,\xi)|^{\frac{1}{2}}\left[|e_0(t,s,x,\xi)|\left(\int_s^t \operatorname{Re} a(\tau,x,\xi)\,d\tau\right)^4\right]^{\frac{1}{2}}$$

$$\leq \sqrt{C_4}\,\langle\xi\rangle^{-1}\langle x\rangle^{-1}|e_0(t,s,x,\xi)|^{\frac{1}{2}} \lesssim \langle\xi\rangle^{-1}\langle x\rangle^{-1}|e_0(t,s,x,\xi)|^{\frac{1}{2}},$$

with C_4 the upper bound of the function s^4e^{-s}, $s \geq 0$. This implies

$$|e_1(t,s,x,\xi)| \lesssim \sqrt{e_0(t,s,x,\xi)} \leq e^{-\frac{C}{2}(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} \lesssim (t-s)^{-\kappa}\langle x\rangle^{-\kappa m'}\langle\xi\rangle^{-\kappa\mu'},$$

for every $\kappa \in [0,1)$, and similar estimates hold for the derivatives of e_1, and for e_j, $j \geq 2$. From the definition of E_N in Lemma 3, we have $E_N(t,s) \in \operatorname{Op}(S^{-\kappa m',-\kappa\mu'})$. Again, reading $W(t,s)$ as an operator of order $(0,0)$, from Eq. (46) we now see that $E(\cdot,s) \in L^1([s,T],\operatorname{Op}(S^{-\kappa m',-\kappa\mu'}))$, that is, $e(\cdot,s) \in L^1([s,T];S^{-\kappa m',-\kappa\mu'})$. That $\partial_t e(\cdot,s) \in L^1([s,T],S^{m-\kappa m',\mu-\kappa\mu'})$ follows then by the result for $e(\cdot,s)$, recalling $\partial_t E(t,s) = -\operatorname{Op}(a(t))E(t,s)$, $0 \leq s < t \leq T$, by Theorem 4, and $a \in C([0,T],S^{m,\mu})$, by hypothesis.

Arguing similarly, using (24), (48), and (49), it follows, that, for all $j, M \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}^d$, there exists $C''_{jM\alpha\beta} > 0$ such that, for any $x, \xi \in \mathbb{R}^d$, $0 \leq s < t \leq T$,

$$|(\langle x\rangle\langle\xi\rangle)^M \partial_x^\alpha\partial_\xi^\beta e_j(t,s,x,\xi)| \leq C''_{jM\alpha\beta}(t-s)^{-\frac{M}{\min\{\{m',\mu'\}}},$$

and analogous estimates for $\partial_t e_j(t,s,x,\xi)$, which imply $e_j(\cdot,s) \in C^1((s,T],\mathcal{S}(\mathbb{R}^d))$, as claimed.

Corollary 1 *Under the same hypothesis of Theorem 5, the solution of the Cauchy problem (30) described there satisfies, for any $\kappa \in [0,1)$,*

$$u \in C([s,T],H^{z,\zeta}(\mathbb{R}^d)) \cap C^1([s,T],H^{z-m,\zeta-\mu}(\mathbb{R}^d)) \cap$$
$$\cap\, C^1((s,T],\mathcal{S}(\mathbb{R}^d)) \cap L^1([s,T],H^{z+\kappa m',\zeta+\kappa\mu'}(\mathbb{R}^d)).$$

It also satisfies $L^1([s,T],H^{z-m+\kappa m',\zeta-\mu+\kappa\mu'}(\mathbb{R}^d))$, $\kappa \in [0,1)$.

Proof The claim is an immediate consequence of Lemma 5 and Duhamel's formula (31) from Theorem 5, using (19) and Theorem 2.

4 Existence of a Random-Field Solution

In the next Theorem 6 we prove our second main result, the existence of a random-field solution of the SPDE (1), under the assumptions of (SG-)parabolicity for the operator L, see Definition 4. We consider, in the $L^2(\mathbb{R}^d)$ environment, the corresponding Cauchy problem

$$\begin{cases} Lu(t,x) = f(t,x) = \gamma(t,x) + \sigma(t,x)\dot{\Xi}(t,x), & (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{50}$$

with the aim of finding conditions on L, on the stochastic noise $\dot{\Xi}$, and on σ, γ, u_0, such that (50) admits a random-field solution. The conditions on the stochastic noise will be given on the spectral measure $\mathfrak{M}$ corresponding to the correlation measure Γ related to the noise $\dot{\Xi}$.

Theorem 6 *Let us consider the Cauchy problem* (50) *for a SPDE associated with a SG-parabolic operator L of the form* (23)*. Assume also, for the initial conditions, that $u_0 \in H^{z,\zeta}(\mathbb{R}^d)$, with $z \geq 0$ and $\zeta > d/2$. Furthermore, assume that $\gamma \in C([0,T]; H^{z,\zeta}(\mathbb{R}^d))$, $\sigma \in C([0,T], H^{0,\zeta}(\mathbb{R}^d))$, $s \mapsto \mathscr{F}\sigma(s) = \nu_s \in L^2([0,T], \mathscr{M}_b(\mathbb{R}^d))$. Assume that one of the following conditions on the spectral measure $\mathfrak{M}$, associated with the random noise $\dot{\Xi}$, hold true:*

(H0) *either, for every $t \in [0,T]$,*

$$\sup_{0 \leq s < t} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |e(t,s,x,\xi+\eta)|^2 \, \mathfrak{M}(d\xi) < \infty \tag{51}$$

and

$$\lim_{h \downarrow 0} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r \in (s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \, \mathfrak{M}(d\xi) = 0, \quad 0 \leq s < t, \tag{52}$$

where $e(t,s)$ is the (parameter-dependent) symbol of the fundamental solution of the operator L,

(H1) *or*

$$\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) < \infty, \tag{53}$$

(H2) *or $\mathfrak{M}$ is absolutely continuous, $|\nu_s|_{\mathrm{tv}} \in L^\infty(0,T)$, and*

$$\int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{\langle \xi \rangle^{\mu'}} < \infty. \tag{54}$$

Then, there exists a random-field solution u of (7)*. Moreover, for any $\kappa \in [0,1)$,*

$$\begin{aligned} \mathbb{E}[u] \in C([0,T], H^{z,\zeta}(\mathbb{R}^d)) \cap C^1([0,T], H^{z-m,\zeta-\mu}(\mathbb{R}^d)) \cap \\ \cap C^1((0,T], \mathscr{S}(\mathbb{R}^d)) \cap L^1([0,T], H^{z+\kappa m',\zeta+\kappa\mu'}(\mathbb{R}^d)). \end{aligned}$$

It also satisfies $\partial_t \mathbb{E}[u] \in L^1([0,T], H^{z-m+\kappa m',\zeta-\mu+\kappa\mu'}(\mathbb{R}^d))$, $\kappa \in [0,1)$.

Remark 8 The class of the stochastic noises which are admissible, if we want to obtain a random-field solution of the Cauchy problem for a SPDE through our method, is described by (51) and (52) for all SG-parabolic operators L, by (53) or (54) under some additional assumptions. Conditions (51), (53), and (54) can be understood as *compatibility conditions* between the noise and the equation.

Proof of Theorem 6 Let us insert $f(t,x) = \gamma(t,x) + \sigma(t,x)\dot{\Xi}(t,x)$ in (31), so that, formally,

$$\begin{aligned} u(t,x) = v_0(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y)\,dyds \\ &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)\,dyds \\ = v_0(t,x) &+ v_1(t,x) + v_2(t,x), \end{aligned} \tag{55}$$

where we indicated by $\Lambda(t,s)$ the Schwartz kernel of $E(t,s)$ and $v_0 = E(t,s)u_0$.

In view of the special structure of Λ (kernel of a smooth family of certain SG-pseudodifferential operators, as described in the previous section), the fact that the deterministic integral in (55) and v_0 are well-defined directly follows by the general theory of SG equations, under the assumptions on γ given in the statement of Theorem 6. By Theorem 5, recalling also Theorem 2 and (26), we find, for any $\kappa \in [0,1)$,

$$\begin{aligned} v_0 \in C([0,T], H^{z,\zeta}) \cap C^1([0,T], H^{z-m,\zeta-\mu}) \cap \\ \cap C^1((0,T], \mathscr{S}) \cap L^1([0,T], H^{z+\kappa m',\zeta+\kappa\mu'}) \subset C([0,T], L^2), \end{aligned}$$

which is a continuous function in $(t,x) \in [0,T] \times \mathbb{R}^d$. This implies that $v_0(t,x)$ is finite for every $(t,x) \in [0,T] \times \mathbb{R}^d$. Since $\gamma \in C([0,T], H^{z,\zeta})$, by the properties of $E(t,s)$ we find that v_1 is of the same regularity class of v_0, namely, it is a well-defined, continuous function in $(t,x) \in [0,T] \times \mathbb{R}^d$. For this term, since we also have $E(t,\cdot) \in L^1([0,T], \mathscr{O}(-\kappa m', -\kappa\mu'))$, we additionally find $v_1 \in C([0,T], H^{z+\kappa m',\zeta+\kappa\mu'})$. We can rewrite v_2 in (55) as

$$v_2(t,x) = \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)M(ds,dy),$$

where M is the martingale measure associated with the stochastic noise Ξ, as defined in Sect. 2. Then, we prove that conditions **(A1)**, **(A2)**, from Sect. 2 hold true, to achieve that such stochastic integral is well-defined. To this aim, we first observe that, by Proposition 1 and Theorem 4,

$$|\mathscr{F}_{y\mapsto\eta}\Lambda(t,s,x,\cdot)(\eta)|^2 = \left|\mathrm{e}^{-\mathrm{i}x\cdot\eta}e(t,s,x,-\eta)\right|^2 = |e(t,s,x,-\eta)|^2 \le C_{t,s}, \tag{56}$$

where $C_{t,s}$ can be chosen to be continuous in s and t, in view of the properties of $e(t,s)$, see Lemmas 2–5.

1. Using (56), we get that condition **(A1)**, with $\Lambda(t,s)$ being the Schwartz kernel of $E(t,s)$, is satisfied if for every $t \in [0,T]$

$$J = \int_0^t \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |e(t,s,x,\eta+\xi)|^2\, \mathfrak{M}(d\xi) \right) |\nu_s|^2_{\mathrm{tv}}\, ds < \infty.$$

If we assume the hypothesis **(H0)**, we find, by the assumptions on σ, for every $t \in [0,T]$,

$$J \le \left(\sup_{0\le s<t}\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |e(t,s,x,\eta+\xi)|^2\, \mathfrak{M}(d\xi) \right) \int_0^t |\nu_s|^2_{\mathrm{tv}}\, ds < \infty,$$

and **(A1)** holds true.

If we assume the hypothesis **(H1)**, we find, again by the assumptions on σ, taking into account that $e(t,s) \in S^{0,0}$, $0 \le s \le t \le T$,

$$J \lesssim \int_0^t \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) |\nu_s|^2_{\mathrm{tv}}\, ds = \left(\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) \int_0^t |\nu_s|^2_{\mathrm{tv}}\, ds < \infty,$$

showing that **(A1)** holds true as well in this second case.

Finally, if we assume the hypothesis **(H2)**, using the absolute continuity of $\mathfrak{M}$, the uniform boundedness of $|\nu_s|_{\mathrm{tv}}$, and Lemma 5, first we observe that (46) implies $e(t,s) = e_N(t,s) \mod C([s,T], S^{-\infty,-\infty})$, and compute, for any $M \ge \max\{m', \mu'\} > 0$, and a suitable $C_{t,s}$, continuous with respect to s, t, $0 \le s \le t \le T$,

$$\begin{aligned}
&\int_0^t\!\!\int_{\mathbb{R}^d} |e(t,s,x,\xi)|^2\, \mathfrak{M}(d\xi)\, |\nu_s|^2_{\mathrm{tv}}\, ds \\
&\quad\lesssim \int_0^t \int_{\mathbb{R}^d} \left[|e_N(t,s,x,\xi)|^2 \bmod C_{t,s}\cdot S^{-\infty,-\infty} \right] \mathfrak{M}(d\xi)\, ds \\
&\quad\lesssim \int_{\mathbb{R}^d} \int_0^t \left[e_0(t,s,x,\xi) + \frac{C_{t,s}}{(\langle x\rangle\langle\xi\rangle)^M} \right] ds\, \mathfrak{M}(d\xi) \\
&\quad\lesssim \int_{\mathbb{R}^d} \left[\frac{1 - e^{-Ct\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}}}{\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} + \frac{1}{(\langle x\rangle\langle\xi\rangle)^M} \right] \mathfrak{M}(d\xi) \\
&\quad\lesssim \int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{\langle\xi\rangle^{\mu'}} < \infty,
\end{aligned}$$

$$\Rightarrow J = \int_0^t \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |e(t,s,x,\xi+\eta)|^2 \,\mathfrak{M}(d\xi) \right) |\nu_s|_{\mathrm{tv}}^2 \, ds$$
$$= \int_0^t \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |e(t,s,x,\xi)|^2 \,\mathfrak{M}(d\xi) \right) |\nu_s|_{\mathrm{tv}}^2 \, ds$$
$$\lesssim \left(\int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{\langle\xi\rangle^{\mu'}} \right) \int_0^t |\nu_s|_{\mathrm{tv}}^2 \, ds < \infty,$$

proving that **(A1)** holds true also in this last case.

2. Using (56), we get that condition **(A2)**, with $\Lambda(t,s)$ being the Schwartz kernel of $E(t,s)$, is satisfied if

$$K = \lim_{h\downarrow 0} \int_0^t \chi_{[0,t-h)}(s) \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \,\mathfrak{M}(d\xi) \right) \times |\nu_s|_{\mathrm{tv}}^2 \, ds = 0.$$

If we assume the hypothesis **(H0)**, we find, by regularity of e with respect to (t,s), (52), the assumptions on σ, and, recalling (51), Lebesgue's Dominated Convergence Theorem, for every $t \in [0,T]$,

$$K = \lim_{h\downarrow 0} \int_0^t \chi_{[0,t-h)}(s) \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \,\mathfrak{M}(d\xi) \right) \times |\nu_s|_{\mathrm{tv}}^2 \, ds$$
$$= \int_0^t \lim_{h\downarrow 0} \chi_{[0,t-h)}(s) \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \,\mathfrak{M}(d\xi) \right) \times |\nu_s|_{\mathrm{tv}}^2 \, ds = 0,$$

and **(A2)** holds true.

If we assume the hypothesis **(H1)**, it suffices to show that

$$\sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \le C_{t,s,h}^2, \tag{57}$$

with $C_{t,s,h}$ a continuous function with respect to s, t, h, such that $C_{t,s,h} \to 0$ as $h \downarrow 0$ and $C_{t,s,h} \le C_T$ for every $h \in [0, t-s]$, $0 \le s < t \le T$. Indeed, since $e(t,s)$ is regular with respect to s and t, if (57) holds true we find, for $0 \le t \le T$,

$$
\begin{aligned}
&\int_0^t \chi_{[0,t-h)}(s) \\
&\quad \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \,\mathfrak{M}(d\xi) \right) \\
&\quad \times |\nu_s|^2_{\mathrm{tv}}\, ds \\
&\quad \le \int_0^t C^2_{t,s,h} \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) |\nu_s|^2_{\mathrm{tv}}\, ds = \left(\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) \int_0^t |\nu_s|^2_{\mathrm{tv}}\, C^2_{t,s,h}\, ds,
\end{aligned}
$$

which implies

$$
\begin{aligned}
0 &\le K \\
&= \lim_{h\downarrow 0} \int_0^t \chi_{[0,t-h)}(s) \\
&\qquad \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \,\mathfrak{M}(d\xi) \right) \\
&\qquad \times |\nu_s|^2_{\mathrm{tv}}\, ds \\
&\le \left(\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) \lim_{h\downarrow 0} \int_0^t |\nu_s|^2_{\mathrm{tv}}\, C^2_{t,s,h}\, ds \\
&= \left(\int_{\mathbb{R}^d} \mathfrak{M}(d\xi) \right) \int_0^t |\nu_s|^2_{\mathrm{tv}} \left(\lim_{h\downarrow 0} C^2_{t,s,h} \right) ds = 0,
\end{aligned}
$$

via Lebesgue's Dominated Convergence Theorem, showing that **(A2)** holds true as well in this second case. The proof of (57) is actually a simpler version of the analogous inequality proved in [5, 6], so we omit it here.

If we assume hypothesis **(H2)**, it suffices to show that

$$
\sup_{r\in(s,s+h)} |e_0(t,s,x,\xi) - e_0(t,r,x,\xi)|^2 \le C_{s,h}\, e^{-C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}}, \tag{58}
$$

where $C_{s,h}$ is a positive function, continuous with respect to h, s, $h \in [0, t-s]$, $0 \le s < t \le T$, and such that $C_{s,h} \to 0$ as $h \to 0$, while C is the constant which appears in (24). Indeed, if (58) holds true, writing as above $e(t,s) = e_N(t,s)$

mod $C([s,T], S^{-\infty,-\infty})$, choosing $M \geq \max\{m', \mu'\} > 0$, with $A_{t,s}$ a suitable continuous function of s, t, $0 \leq s \leq t \leq T$, we find, for $0 \leq s < t \leq T$,

$$\begin{aligned}
&\int_0^t \chi_{[0,t-h)}(s) \\
&\qquad \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \, \mathfrak{M}(d\xi) \right) \\
&\qquad \times |\nu_s|^2_{\mathrm{tv}} \, ds \\
&\leq \int_0^t \chi_{[0,t-h)}(s) \left(\int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi) - e(t,r,x,\xi)|^2 \, \mathfrak{M}(d\xi) \right) |\nu_s|^2_{\mathrm{tv}} \, ds \\
&\lesssim \int_0^t \chi_{[0,t-h)}(s) \\
&\qquad \times \left[\int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} \left(|e_0(t,s,x,\xi) - e_0(t,r,x,\xi)|^2 + \frac{|A_{t,s} - A_{t,r}|^2}{(\langle x\rangle\langle\xi\rangle)^{2M}} \right) \mathfrak{M}(d\xi) \right] \\
&\qquad \times |\nu_s|^2_{\mathrm{tv}} \, ds \\
&\lesssim \int_0^t \left[\int_{\mathbb{R}^d} \left(C_{s,h} e^{-C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} + \frac{B_{t,s,h}}{(\langle x\rangle\langle\xi\rangle)^{2M}} \right) \mathfrak{M}(d\xi) \right] ds \\
&\lesssim \widetilde{C}_{t,h} \int_{\mathbb{R}^d} \left[\int_0^t \left(e^{-C(t-s)\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} + \frac{1}{(\langle x\rangle\langle\xi\rangle)^{2M}} \right) ds \right] \mathfrak{M}(d\xi) \\
&\lesssim \widetilde{C}_{t,h} \int_{\mathbb{R}^d} \left(\frac{1 - e^{-Ct\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}}}{\langle x\rangle^{m'}\langle\xi\rangle^{\mu'}} + \frac{1}{(\langle x\rangle\langle\xi\rangle)^{2M}} \right) \mathfrak{M}(d\xi) \\
&\lesssim \widetilde{C}_{t,h} \int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{\langle\xi\rangle^{\mu'}},
\end{aligned}$$

where $\widetilde{C}_{t,h} = \max_{0\leq s\leq t}(C_{s,h} + B_{t,s,h})$, $\widetilde{C}_{t,h} \to 0$ for $h \downarrow 0$. This implies, by (54),

$$\begin{aligned}
&0 \leq K \\
&\quad = \lim_{h\downarrow 0} \int_0^t \chi_{[0,t-h)}(s) \\
&\qquad\qquad \times \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \sup_{r\in(s,s+h)} |e(t,s,x,\xi+\eta) - e(t,r,x,\xi+\eta)|^2 \, \mathfrak{M}(d\xi) \right)
\end{aligned}$$

$$\times\, |\nu_s|^2_{\mathrm{tv}}\, ds$$

$$\lesssim \left(\int_{\mathbb{R}^d} \frac{\mathfrak{M}(d\xi)}{\langle \xi \rangle^{\mu'}} \right) \lim_{h \downarrow 0} \widetilde{C}_{t,h} = 0,$$

proving that **(A2)** holds true also in this last case. Let us then show that (58) holds true. We have:

$$\begin{aligned}
&|e_0(t,s,x,\xi) - e_0(t,r,x,\xi)| \\
&\qquad = \left| e^{-\int_s^t a(\tau,x,\xi)\,d\tau} - e^{-\int_r^t a(\tau,x,\xi)\,d\tau} \right| \\
&\qquad = e^{-\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\,d\tau} \left| 1 - e^{\int_s^r a(\tau,x,\xi)\,d\tau} \right| \\
&\qquad \le e^{-\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\,d\tau} \int_s^r \mathrm{Re}\, a(\tau,x,\xi)\,d\tau \\
&\qquad \le e^{-\frac{1}{2}\int_s^t \mathrm{Re}\, a(\tau,x,\xi)\,d\tau} \left(e^{-\frac{1}{2}\int_s^r \mathrm{Re}\, a(\tau,x,\xi)\,d\tau} \int_s^r \mathrm{Re}\, a(\tau,x,\xi)\,d\tau \right) \\
&\qquad \le e^{-\frac{C}{2}(t-s)\langle x \rangle^{m'} \langle \xi \rangle^{\mu'}} C_{s,r}
\end{aligned}$$

with a function $C_{s,r}$, continuous in s, r and such that $C_{s,r} \le \sqrt{C_2}$, C_2 the supremum of $s^2 e^{-s}$, $s \ge 0$. This implies

$$\sup_{r \in (s,s+h)} |e_0(t,s,x,\xi) - e_0(t,r,x,\xi)|^2 \le C_{s,h}\, e^{-C(t-s)\langle x \rangle^{m'} \langle \xi \rangle^{\mu'}}$$

with $C_{s,h} = \sup_{r \in (s,s+h)} C^2_{s,r}$, which clearly has all the requested properties.

Summing up, v_2 in (55) is well-defined, as a stochastic integral with respect to the martingale measure canonically associated with $\mathfrak{M}$, under either one of the hypotheses **(H0)**, **(H1)**, or **(H2)**. Since $\mathbb{E}[v_2] = 0$, the regularity of $\mathbb{E}[u]$ is the same as the one of the solution of the associated deterministic Cauchy problem, described in Theorem 5. The proof is complete.

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The Non-isentropic Relativistic Euler System Written in a Symmetric Hyperbolic Form

Uwe Brauer and Lavi Karp

This paper is dedicated to our friend Michael Reissig

Abstract We cast the non-isentropic relativistic Euler system into a symmetric hyperbolic form. Such systems are very suited to treat initial value problems of hyperbolic type. We obtain this form by using the pressure p and not the density ρ as a variable. However, the system becomes degenerate when the pressure p approaches zero, and in these cases we regularise the system by replacing the pressure with an appropriate new matter variable, the Makino variable.

Keywords Non isentropic Euler equations · Symmetric hyperbolic systems · Entropy · Equation of state

1 Introduction

Existence and uniqueness theorems of a class of solutions have been proved for the non-relativistic compressible Euler equations for the isentropic case by Makino [13], and later for the non-isentropic case by Makino et al. [15].

The situation, however, for the relativistic compressible Euler equations is more involved. The equivalent to the result obtained by Makino [13], has been proven, for a restricted setting by Rendall, [17], which was later extended by the authors [2] and [1].

All those results had been obtained by casting, in one way or the other, the Euler equations into a symmetric-hyperbolic first-order system. Such systems had been

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introduced Friedrich in 1954 [11], and has been one of the most effective approaches to prove the well-posedness (existence, uniqueness, and continuity of the flow map) for these systems.

The non-isentropic case is more complicated. Speck [19] studied the Cauchy problem for the Nordström scalar gravitational field equation coupled to the non-isentropic relativistic Euler equations. He proved local existence, uniqueness and the continuity of the flow map, but since he claimed that the system could not be cast into symmetric hyperbolic form, he used Christodoulou's theory of the energy current [6] to obtain his results.

Choquet-Bruhat studied the Cauchy problem for both, the isentropic and the non-isentropic, Einstein–Euler system, using Leray hyperbolic systems [8]. Moreover, she also used a different method relying upon Leray-Ohya hyperbolic systems, see [3] and [4]. A different approach was proposed by Friedrich [9], with the motivation to treat free initial boundary problems. So he was able to write the relativistic Euler equations in Lagrangian coordinates as a symmetric hyperbolic system by differentiating the equations in an appropriate manner. This leads to a system with constraint equations, whose propagation needs to be shown separately. The advantage of his system is the fact that it is more suited to deal with initial free-boundary problems since in Lagrangian coordinates the boundary is fixed.

Disconzi used Friedrich's approach to derive local existence and uniqueness of classical solutions for the non-isentropic Einstein–Euler system [7], using uniformly local Sobolev spaces, assuming the density to be strictly positive and a smooth equation of state. Another approach for the non-isentropic relativistic Euler equations was presented by Walton [20], however, no local existence and uniqueness system is known using this approach.

The purpose of these notes is to generalize our approach as provided in [2] and to present the non-isentropic relativistic Euler equations as a symmetric hyperbolic system, which would enable as to prove similar local existence and uniqueness theorem, therefore removing some of the restrictions posed in the results of [7].

2 The Relativistic Euler Equations with Entropy

We now briefly introduce the notion of a relativistic perfect, but and non-isentropic fluid. For more information and the thermodynamical background see for example [4, 5, 10]. We consider the fluid in a prescribed Lorentzian manifold $(\mathscr{M}, g_{\alpha\beta})$, $\alpha, \beta = 0, 1, 2, 3$, and we chose units such that the speed of light $c = 1$. For a perfect fluid, the energy-momentum tensor takes the following form

$$T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + pg^{\alpha\beta}, \tag{1}$$

where ϵ is the proper energy density of the fluid, p is the pressure, and u^α is the four-velocity, which is subject to the normalization constraint

$$g_{\alpha\beta}u^\alpha u^\beta = -1. \tag{2}$$

The relativistic Euler equations for a perfect fluid are (see e.g. [5])

$$\nabla_\alpha T^{\alpha\beta} = 0 \qquad (\beta = 0, 1, 2, 3) \tag{3}$$

$$\nabla_\alpha (n u^\alpha) = 0, \tag{4}$$

where n is the *proper number density* and ∇_α denotes the covariant derivative induced by the spacetime metric $g_{\alpha\beta}$. As we will discuss in Sect. 3.2, the projection $u_\beta \nabla_\alpha T^{\alpha\beta} = 0$ leads to the energy equation

$$u^\nu \nabla_\nu \epsilon + (\epsilon + p)\nabla_\nu u^\nu = 0. \tag{5}$$

A non-isentropic fluid contains a thermodynamic variable s that represents the *Entropy*, and satisfies the following thermodynamic relation, called Gibbs relation, [4]

$$T ds = d\left(\frac{\epsilon}{n}\right) + p d\left(\frac{1}{n}\right), \tag{6}$$

where T denotes the temperature. As it was proven by Pichon [16], the energy equation (5), the rest-mass conservation equation (4) and the Gibbs relation (6) imply the following relation for the entropy

$$u^\alpha \nabla_\alpha s = 0, \tag{7}$$

which just expresses the fact that it is conserved along the fluid lines.

The equation of state specifies the relations between the number density n, entropy s, and the mass density ϵ. We assume an equation of state is given by a nonnegative function

$$\epsilon = \epsilon(n, s), \qquad n, s \geq 0. \tag{8}$$

From laws of thermodynamics (see e.g. [10]) it follows that the pressure is given by

$$p = n\frac{\partial \epsilon}{\partial n} - \epsilon, \tag{9}$$

and the speed of sound is given by

$$\sigma^2 = \frac{\partial p}{\partial \epsilon} = \frac{\frac{\partial p}{\partial n}}{\frac{\partial \epsilon}{\partial n}}. \tag{10}$$

A fundamental thermodynamic assumption is that the right-hand side of (10) is positive, hence we require that

$$\frac{\partial \epsilon}{\partial n} > 0, \qquad \frac{\partial p}{\partial n} > 0. \tag{11}$$

Another requirement is that $\sigma < 1$, which means that the sound speed is always less than the speed of light.

2.1 Energy Conditions

The General Relativity literature refers to three types of energy conditions (see e.g. [4]). The energy-momentum tensor $T^{\alpha\beta}$ satisfies:

1. The *weak energy condition*, if $T_{\alpha\beta}X^\alpha X^\beta \geq 0$ for all timelike vectors X^α.
2. The *strong energy condition*, if $[T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}]X^\alpha X^\beta \geq 0$ for all timelike vectors X^α, where $T = g_{\mu\nu}T^{\mu\nu}$.
3. The *dominant energy condition*, if $-T^\alpha_\beta X^\beta$ is timelike future-directed vector for all X^α future-directed timelike vector.

Whenever $\epsilon \geq 0$ and $p \geq 0$, the perfect fluid satisfies the weak and strong energy conditions. If $\epsilon \geq p$, then it satisfies also the dominant energy condition, see [4]. We shall see that the examples below meet all the three energy conditions.

2.2 Examples of an Equation of State for the Non-isentropic Relativistic Euler Equations

A typical non-isentropic equation of state is given by (see also [12])

$$\epsilon = n + \frac{A(s)}{\gamma - 1}n^\gamma, \tag{12}$$

where $1 < \gamma < 2$ and $A(s)$ is a positive function. Equation (9) implies that $p = A(s)n^\gamma$, and from (10) we can compute the speed of sound as follows,

$$\sigma^2 = \frac{\gamma(\gamma-1)A(s)n^{\gamma-1}}{(\gamma-1) + \gamma A(s)n^{\gamma-1}}. \tag{13}$$

As a function of n, the speed of sound σ is increasing and tends to $\sqrt{\gamma - 1}$ as n tends to infinity. Hence the speed of sound is less than the speed of light. The equation of state (12) also satisfies the dominant energy condition, since

$$\epsilon - p = n + \frac{(2-\gamma)A(s)n^\gamma}{\gamma - 1} \geq 0. \tag{14}$$

Another example is a polytropic equation of state with index $\gamma = \frac{4}{3}$. We follow the convention of Choquet–Bruhat [4], here

$$p = \frac{K}{3}\left(\frac{3s}{4K}\right)^{\frac{4}{3}} n^{\frac{4}{3}} \qquad \text{and} \quad \epsilon = 3p + n, \tag{15}$$

where K is a positive constant. We see that $\frac{\partial \epsilon}{\partial n} = \frac{4K}{3}\left(\frac{3s}{4K}\right)^{\frac{4}{3}} n^{\frac{1}{3}} + 1 = \frac{p+\epsilon}{n}$, hence (9) is fulfilled. We also note that

$$p = n + K\left(\frac{3s}{4K}\right)^{\frac{4}{3}} n^{\frac{4}{3}},$$

and hence it is a particular case of the equation of state (12). So this equation of state also satisfies the dominant energy condition.

3 The Non-isentropic Equations in Symmetric Hyperbolic Form

The equation of state (8) and the explicit formula of the pressure (9) allows us to express the pressure p as a function of n and s, which leads to consider $U = (n, u^\alpha, s)$, $\alpha = 0, 1, 2, 3$ as the unknowns for the relativistic Euler equations (3) and (4).

However, such an equation of state implies also that $\nabla_\alpha p = \frac{\partial p}{\partial n}\nabla_\alpha n + \frac{\partial p}{\partial s}\nabla_\alpha s$, which destroys the symmetry of the corresponding matrices and makes it almost impossible to cast the relativistic Euler equations in symmetric hyperbolic form. The same problem occurs for the non-relativistic case, and there the solution consists in using the pressure p as a matter variable instead of the density n.

That is why we take a similar approach here for the relativistic equations and cast the equations in symmetric hyperbolic form. Moreover, the resulting system is a more convenient starting point to introduce the regularizing Makino variable.

3.1 Symmetric Hyperbolic Systems

We recall the definition of symmetric hyperbolic systems.

Definition (Symmetric Hyperbolic System) A first order quasi-linear $k \times k$ system is *symmetric hyperbolic system* in a region $G \subset \mathbb{R}^k$, if it is of the form

$$L[U] = A^\alpha(U)\partial_\alpha U + B(U) = 0, \tag{16}$$

where the matrices $A^\alpha(U)$ are symmetric and for every arbitrary $U \in G$, and there exists a covector ξ_α such that

$$\xi_\alpha A^\alpha(U) \tag{17}$$

is positive definite. The covectors ξ_α for which (17) is positive definite, are called *spacelike with respect to Eq.* (16).

Remark 1 In most applications, and in particular, for initial value problems, it is essential that $A^0(U)$ is positive definite, and then system (16) takes the form

$$A^0(U)\partial_t U = \sum_{i=1}^{3} A^i(U)\partial_{x^i} U + B(U). \tag{18}$$

To derive Eq. (16) in the above form requires to show that $(1, 0, 0, 0)$ is spacelike with respect to the equation. Under the assumption that the speed of sound is less than one, we shall prove that the covector $(1, 0, 0, 0)$ belongs the future sound cone, and hence it is spacelike with respect to Eq. (16).

3.2 Fluid Decomposition

First, we apply the well known fluid decomposition (see for example [2]) to Eq. (3). We project $\nabla_\nu T^{\nu\beta}$ along the flow lines u^ν, by $u_\beta \nabla_\nu T^{\nu\beta}$, and on the orthogonal subspace to the flow lines $\mathcal{O}$, by $P_{\alpha\beta}\nabla_\nu T^{\nu\beta}$, where

$$P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta. \tag{19}$$

These projections result in

$$u^\nu \nabla_\nu \epsilon + (\epsilon + p)\nabla_\nu u^\nu = 0 \tag{20}$$

$$(\epsilon + p)P_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^\nu{}_\alpha \nabla_\nu p = 0, \tag{21}$$

which together with the continuity equation (4) form a system of equations. As we already pointed out the energy equation (20), together with the continuity equation (4) and the thermodynamical relation (6) imply the conservation of the entropy (7). Moreover, we will also need that fact, that thanks to Eq. (11), we can express n as a function of p. All these considerations allow us to consider the following system of equations:

$$u^\nu \nabla_\nu n + n\nabla_\nu u^\nu = 0 \tag{22}$$

$$(\epsilon + p)P_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^\nu{}_\alpha \nabla_\nu p = 0 \tag{23}$$

$$u^\alpha \nabla_\alpha s = 0. \tag{24}$$

3.3 Modification of the Fluid Decomposed System

In order to obtain a symmetric hyperbolic system we modify the coupled Eqs. (22)–(24) the following way. The normalisation condition (2) implies that

$$u_\beta u^\nu \nabla_\nu u^\beta = 0. \tag{25}$$

So we add $nu_\beta u^\nu \nabla_\nu u^\beta = 0$ to Eq. (22), $u_\alpha u_\beta u^\nu \nabla_\nu u^\beta = 0$ to (23) and we obtain finally that

$$u^\nu \nabla_\nu n + n P^\nu{}_\beta \nabla_\nu u^\beta = 0 \tag{26}$$

$$(\epsilon + p)\Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^\nu{}_\alpha \nabla_\nu p = 0, \tag{27}$$

where

$$\Gamma_{\alpha\beta} = P_{\alpha\beta} + u_\alpha u_\beta = g_{\alpha\beta} + 2u_\alpha u_\beta \tag{28}$$

is a reflection with respect to the hyperplane $\mathcal{O}$.

We now use the equation of state (8) and (9), which allow us to express p as a function of n and s, that is, $p = p(n, s)$. Hence,

$$\nabla_\nu p = \frac{\partial p}{\partial n} \nabla_\nu n + \frac{\partial p}{\partial s} \nabla_\nu s, \tag{29}$$

and by the conservation of the entropy (7), we conclude that

$$u^\nu \nabla_\nu p = \frac{\partial p}{\partial n} u^\nu \nabla_\nu n + \frac{\partial p}{\partial s} u^\nu \nabla_\nu s = \frac{\partial p}{\partial n} u^\nu \nabla_\nu n. \tag{30}$$

So we finally obtain the system

$$u^\nu \nabla_\nu p + n \frac{\partial p}{\partial n} P^\nu{}_\beta \nabla_\nu u^\beta = 0 \tag{31}$$

$$(\epsilon + p)\Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^\nu{}_\alpha \nabla_\nu p = 0 \tag{32}$$

$$u^\alpha \nabla_\alpha s = 0. \tag{33}$$

Remark 2 (The Pressure as a Matter Variable) The idea of using the pressure as a matter variable instead of the density is widely used in the non-relativistic case, see for example [18]. In the relativistic case, Guo and Tahvildar-Zadeh [12] presented the following system for the variables (p, u^α, s)

$$\frac{1}{(\epsilon + p)\sigma} u^\nu \partial_\nu p + \sigma \partial_\nu u^\nu = 0 \tag{34}$$

$$\sigma P^{\mu\nu} \partial_\nu p + (\epsilon + p)\sigma u^\nu \partial_\nu u^\mu = 0 \tag{35}$$

$$u^\nu \partial_\nu s = 0. \tag{36}$$

It should be pointed out, that this system, however, is not symmetric hyperbolic as it can be easily checked.

3.4 Symmetric Hyperbolic Form

We now write system (31)–(33) in matrix form

$$\begin{pmatrix} u^\nu & n\frac{\partial p}{\partial n}{P^\nu}_\beta & 0 \\ {P^\nu}_\alpha & (\epsilon + p)\Gamma_{\alpha\beta}u^\nu & 0 \\ 0 & 0 & u^\nu \end{pmatrix} \nabla_\nu \begin{pmatrix} p \\ u^\alpha \\ s \end{pmatrix} = 0. \tag{37}$$

These matrices are not symmetric, but they can be cast into a symmetric form by choosing an appropriate multiplier, for example, we multiply the second row of the matrices by $n\frac{\partial p}{\partial n}$, and then we obtain

$$\begin{pmatrix} u^\nu & n\frac{\partial p}{\partial n}{P^\nu}_\beta & 0 \\ n\frac{\partial p}{\partial n}{P^\nu}_\alpha & n\frac{\partial p}{\partial n}(\epsilon + p)\Gamma_{\alpha\beta}u^\nu & 0 \\ 0 & 0 & u^\nu \end{pmatrix} \nabla_\nu \begin{pmatrix} p \\ u^\alpha \\ s \end{pmatrix} = 0, \tag{38}$$

which are symmetric matrices.

In fact, it turns out that system (38) is a symmetric hyperbolic system. The following theorem gives a precise statement.

Theorem *Let ϵ in (8) be nonnegative density function, the pressure p be defined by (9) and assume conditions (11). Then the relativistic Euler equations (3)–(4) coupled with the constraint (2) can be written as a symmetric hyperbolic system. Moreover, under the assumption that the speed of sound is less than the speed of light, the matrix A^0 is positive definite and therefore the relativistic Euler equations (3)–(4) form are symmetric hyperbolic system as specified in Eq. (18).*

Proof To show that the system (38) is symmetric hyperbolic we need to show that $\xi_\alpha A^\alpha(U)$ is positive definite for some covectors ξ_α. For that we slightly rewrite system (38). Using Eqs. (9) and (10) we see that

$$n\frac{\partial p}{\partial n} = \frac{\partial p}{\partial \epsilon} n \frac{\partial \epsilon}{\partial n} = \sigma^2(\epsilon + p), \tag{39}$$

hence (38) is equivalent to system

$$\begin{pmatrix} u^\nu & \sigma^2\,(\epsilon + p)\,{P^\nu}_\beta & 0 \\ \sigma^2\,(\epsilon + p)\,{P^\nu}_\alpha & \sigma^2\,(\epsilon + p)^2\,\Gamma_{\alpha\beta}u^\nu & 0 \\ 0 & 0 & u^\nu \end{pmatrix} \nabla_\nu \begin{pmatrix} p \\ u^\alpha \\ s \end{pmatrix} = 0. \tag{40}$$

Now we compute the principal symbol of system (40). For each $\xi_\alpha \in T_x^*V$ the principal symbol is a linear map from $\mathbb{R} \times E_x$ to $\mathbb{R} \times F_x$, where E_x is a fiber in T_xV and F_x is a fiber in the cotangent space T_x^*V. In local coordinates $\nabla_\nu = \partial_\nu + \Gamma$, where $\Gamma = \Gamma(g^{\gamma\delta}, \partial g_{\alpha\beta})$ denotes the Christoffel symbols, hence the principal symbol of system (40) is

$$\xi_\nu A^\nu = \left(\begin{array}{c|c|c} (u^\nu\xi_\nu) & \sigma^2\,(p+\epsilon)\,P^\nu{}_\beta\xi_\nu & 0 \\ \hline \sigma^2\,(p+\epsilon)\,P^\nu{}_\alpha\xi_\nu & \sigma^2\,(p+\epsilon)\,(u^\nu\xi_\nu)\Gamma_{\alpha\beta} & 0 \\ \hline 0 & 0 & (u^\nu\xi_\nu) \end{array}\right). \tag{41}$$

The characteristics are the set of covectors ξ_ν for which $(\xi_\nu A^\nu)$ is not an isomorphism. Hence the characteristics are the zeros of

$$Q(\xi) \stackrel{\text{def}}{=} \det(\xi_\nu A^\nu). \tag{42}$$

The geometric advantages of fluid decomposition are the following. The operators in the blocks of the matrix (41) are the projection $P^\nu{}_\alpha$, on the hyperplane $\mathcal{O}$ that is orthogonal to the flow lines, and the reflection $\Gamma_{\alpha\beta}$, with respect to the same hyperplane. Therefore, the following relations hold:

$$\Gamma^{\alpha\gamma}\Gamma_{\gamma\beta} = \delta_\beta{}^\alpha, \qquad \Gamma^{\alpha\gamma}P_\gamma{}^\nu = P^{\alpha\nu} \qquad \text{and} \qquad P_\beta{}^\alpha P_\alpha{}^\nu = P^\nu{}_\beta,$$

which yields

$$\begin{aligned} &\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \Gamma^{\alpha\gamma} & 0 \\ \hline 0 & 0 & 1 \end{array}\right)(\xi_\nu A^\nu) \\ &= \left(\begin{array}{c|c|c} (u^\nu\xi_\nu) & \sigma^2\,(p+\epsilon)\,P^\nu{}_\beta\xi_\nu & 0 \\ \hline \sigma^2\,(p+\epsilon)\,P^{\alpha\nu}\xi_\nu & \sigma^2\,(p+\epsilon)\,(u^\nu\xi_\nu)\left(\delta^\alpha_\beta\right) & 0 \\ \hline 0 & 0 & (u^\nu\xi_\nu) \end{array}\right). \end{aligned} \tag{43}$$

It is now fairly easy to calculate the determinant of the right-hand side of (43) and we have

$$\begin{aligned} &\det\left(\begin{array}{c|c|c} (u^\nu\xi_\nu) & \sigma^2\,(p+\epsilon)\,P^\nu{}_\beta\xi_\nu & 0 \\ \hline \sigma^2\,(p+\epsilon)\,P^{\alpha\nu}\xi_\nu & \sigma^2\,(p+\epsilon)\,(u^\nu\xi_\nu)\left(\delta^\alpha_\beta\right) & 0 \\ \hline 0 & 0 & (u^\nu\xi_\nu) \end{array}\right) \\ &= \sigma^2\,(p+\epsilon)^2\,(u^\nu\xi_\nu)^4\left\{(u^\nu\xi_\nu)^2 - \sigma^2 P^{\alpha\nu}\xi_\nu P^\nu_\alpha\xi_\nu\right\}. \end{aligned}$$

Since P^{α}_{β} is a projection,

$$P^{\alpha\nu}\xi_{\nu}P^{\nu}_{\alpha}\xi_{\nu} = g^{\nu\beta}\xi_{\nu}P^{\alpha}_{\beta}P^{\nu}_{\alpha}\xi_{\nu} = g^{\nu\beta}\xi_{\nu}P^{\nu}{}_{\beta}\xi_{\nu} = P^{\nu}{}_{\beta}\xi_{\nu}\xi^{\beta},$$

and since Γ^{γ}_{β} is a reflection,

$$\det\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \Gamma^{\alpha\gamma} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \det\left(g^{\alpha\beta}\Gamma^{\gamma}_{\beta}\right) = -\left(\det\left(g_{\alpha\beta}\right)\right)^{-1} > 0. \tag{44}$$

Consequently,

$$Q(\xi) = \det(\xi_{\nu}A^{\nu}) = -\sigma^2\,(p+\epsilon)^2\det(g_{\alpha\beta})(u^{\nu}\xi_{\nu})^4\left\{(u^{\nu}\xi_{\nu})^2 - \sigma^2 P^{\alpha}{}_{\beta}\xi_{\alpha}\xi^{\beta}\right\} \tag{45}$$

and therefore the characteristic covectors are determined by two simple equations:

$$\xi_{\nu}u^{\nu} = 0 \tag{46}$$

$$(\xi_{\nu}u^{\nu})^2 - \sigma^2 P^{\alpha}{}_{\beta}\xi_{\alpha}\xi^{\beta} = 0. \tag{47}$$

Remark 3 The characteristics conormal cone is a union of two hypersurfaces in T^*_xV. One of these hypersurfaces is given by the condition (46) and it is a three dimensional hyperplane $\mathcal{O}$ with the normal u^{α}. The other hypersurface is given by the condition (47) and forms a three-dimensional cone, the so-called, *sound cone*.

Let us now consider the timelike vector u_{ν} and insert the covector $-u_{\nu}$ into the principal symbol (41), since $P_{\beta}{}^{\nu}u_{\nu} = 0$,

$$-u_{\nu}A^{\nu} = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \sigma^2(p+\epsilon)\Gamma_{\alpha\beta} & 0 \\ \hline 0 & 0 & 1 \end{array}\right),$$

and hence $-u_{\nu}A^{\nu}$ is a positive definite matrix . Indeed, $\Gamma_{\alpha\beta}$ is a reflection with respect to a hyperplane having a timelike normal, and as in (44) we see that $\det(\Gamma_{\alpha\beta}) > 0$. Hence, $-u_{\nu}$ is a spacelike covector with respect to the hydrodynamical equations (40). Herewith, we have shown relatively elegant and elementary that the relativistic hydrodynamical equations are symmetric hyperbolic.

We want now to show that A^0 is positive definite. To do that it suffices to show that the covector $\zeta_{\nu} = (1, 0, 0, 0)$ is also spacelike with respect to the system (40). Since $P^{\alpha}{}_{\beta}u_{\alpha} = 0$, the covector $-u_{\nu}$ belongs to the sound cone

$$(\xi_{\nu}u^{\nu})^2 - \sigma^2 P^{\alpha}{}_{\beta}\xi_{\alpha}\xi^{\beta} > 0. \tag{48}$$

Inserting $\zeta_\nu = (1, 0, 0, 0)$ the right-hand side of (48), yields

$$(u^0)^2(1-\sigma^2) - \sigma^2 g^{00}. \tag{49}$$

Under the assumption sound velocity is less than the speed of light, that is $\sigma^2 = \frac{\partial p}{\partial \epsilon} < c^2 = 1$, we conclude that (49) is positive, and hence $\zeta_\nu = (1, 0, 0, 0)$ also belongs to the sound cone (48). Hence, the vector $-u_\nu$ can be continuously deformed to ζ_ν while condition (48) holds along the deformation path. Consequently, the determinant of (45) remains positive under this process and hence $\zeta_\nu A^\nu = A^0$ is also positive definite.

4 Symmetrization and Regularization

In the case of a physical vacuum, that is, if the density or the pressure vanish in certain regions, or fall-off at infinity, the symmetrization we obtained in Sect. 3 breaks down. The reason for this can be seen easily by inspecting the matrix $A^0(U)$ which is no longer uniformly positive definite if the pressure approaches zero. Makino symmetrised and regularised the Euler–Poisson system by introducing a new nonlinear matter variable $w = M(\rho)$ [13], so that the matrix $A^0(U)$ remains uniformly positive even for $\rho = 0$. Later Makino generalised his regularisation to the non isentropic Euler-Poisson system [14], starting with a system for (p, u^α, s). We follow this strategy but, naturally, have to modify it due to the more complicated character of our equations.

So, we start with system (31)–(33)

$$u^\nu \nabla_\nu p + n\frac{\partial p}{\partial n} P^\nu{}_\beta \nabla_\nu u^\beta = 0 \tag{50}$$

$$(\epsilon + p)\Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^\nu{}_\alpha \nabla_\nu p = 0 \tag{51}$$

$$u^\alpha \nabla_\alpha s = 0. \tag{52}$$

and replace p by $w = w(p)$. Then we multiply Eq. (50) by $\kappa^2(w, s)\frac{\partial w}{\partial p}$ where κ is a positive function we specify later in order to simplify our calculations. Moreover, we divide Eq. (51) by $(\epsilon + p)$, then Eqs. (50) and (51) written in matrix form, take the following form

$$\begin{pmatrix} \kappa^2 u^\nu & \kappa^2 n \frac{\partial p}{\partial n}\frac{\partial w}{\partial p} P^\nu{}_\beta & 0 \\ \frac{1}{(\epsilon+p)}\frac{\partial p}{\partial w} P^\nu{}_\alpha & \Gamma_{\alpha\beta} u^\nu & 0 \\ 0 & 0 & u^\nu \end{pmatrix} \nabla_\nu \begin{pmatrix} w \\ u^\alpha \\ s \end{pmatrix} = 0. \tag{53}$$

The matrices (53) are symmetric provided that

$$\kappa^2 n \frac{\partial w}{\partial n} = \kappa^2 n \frac{\partial w}{\partial p}\frac{\partial p}{\partial n} = \frac{1}{\epsilon + p}\frac{\partial p}{\partial w}, \tag{54}$$

which results in

$$w = \int \frac{1}{\kappa}\left(\frac{1}{(\epsilon + p)\, n}\right)^{\frac{1}{2}}\left(\frac{\partial n}{\partial p}\right)^{\frac{1}{2}} dp. \tag{55}$$

We will now, in the subsection below, calculate an explicit form of this new variable using the equation of state (12) presented in Sect. 2.2.

4.1 The Makino Variable for the Equation of State (12)

For this equation of state we easily compute

$$\epsilon + p = n + \frac{1}{\gamma - 1}A(s)n^{\gamma} + p = n + \frac{\gamma}{\gamma - 1}p, \tag{56}$$

$$n\frac{\partial p}{\partial n} = \gamma p \tag{57}$$

and

$$n = A^{-\frac{1}{\gamma}}(s)p^{\frac{1}{\gamma}}. \tag{58}$$

This allows us to calculate

$$\frac{1}{(\epsilon + p)\, n\frac{\partial p}{\partial n}} = \frac{1}{\left(n + \frac{\gamma}{\gamma-1}p\right)p\gamma} = \frac{1}{\gamma}\frac{1}{np + \frac{\gamma}{\gamma-1}p^2}$$
$$= \frac{1}{\gamma}\frac{1}{A^{-\frac{1}{\gamma}}(s)p^{1+\frac{1}{\gamma}} + \frac{\gamma}{\gamma-1}p^2} = \frac{1}{\gamma}\left(\frac{1}{A^{-\frac{1}{\gamma}}(s) + \frac{\gamma}{\gamma-1}p^{1-\frac{1}{\gamma}}}\right)\frac{1}{p^{1+\frac{1}{\gamma}}}.$$

Keeping in mind the symmetry condition (54), we see that setting

$$\kappa^2 = \left(\left(\frac{2\gamma}{\gamma - 1}\right)^2 \frac{1}{\gamma}\frac{1}{A^{-\frac{1}{\gamma}}(s) + \frac{\gamma}{\gamma-1}p^{\frac{\gamma-1}{\gamma}}}\right), \tag{59}$$

implies that $\frac{\partial w}{\partial p} = \frac{\gamma-1}{2} p^{-\frac{\gamma-1}{2\gamma}}$, which leads to

$$w = p^{\frac{\gamma-1}{2\gamma}} \tag{60}$$

and

$$\kappa^2(w, s) = \left(\left(\frac{2\gamma}{\gamma - 1} \right)^2 \frac{1}{\gamma} \frac{1}{A^{-\frac{1}{\gamma}}(s) + \frac{\gamma}{\gamma-1} w^2} \right). \tag{61}$$

So we conclude the relativistic Euler equations (3)–(4) coupled with the constraint (2) can be written in the form

$$\begin{pmatrix} \kappa^2 u^\nu & \kappa^2 \frac{\gamma(\gamma-1)}{2} w P^\nu{}_\beta & 0 \\ \kappa^2 \frac{\gamma(\gamma-1)}{2} w P^\nu{}_\alpha & \Gamma_{\alpha\beta} u^\nu & 0 \\ 0 & 0 & u^\nu \end{pmatrix} \nabla_\nu \begin{pmatrix} w \\ u^\alpha \\ s \end{pmatrix} = 0, \tag{62}$$

which is symmetric and regular when p, or equivalently w approaches zero.

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Blow-up Result for a Semilinear Wave Equation with a Nonlinear Memory Term

Wenhui Chen and Alessandro Palmieri

Abstract In this note, we study the blow-up dynamic of a semilinear Cauchy problem for the wave equation with a nonlinear memory term. More precisely, we consider as memory term the Riemann-Liouville fractional integral of order $1-\gamma$ of the p power of the solution, where $\gamma \in (0, 1)$. We prove two blow-up results by using an iteration argument. In the subcritical case we show the blow-up in finite time of the space average of a local in time solution, under certain integral sign assumptions for the initial data. In the result for the limit case, we refine this approach by considering a weighted average of a local solution instead and applying the so-called slicing method.

Keywords Semilinear wave equation · Nonlinear memory term · Riemann-Liouville fractional integral · Generalized Strauss exponent · Blow-up · Iteration argument

1 Introduction

In this paper, we investigate the blow-up dynamic for local in time solutions to the semilinear wave equation with the Riemann-Liouville fractional integral of order $1-\gamma$ of the p power of the solution as nonlinear term

$$\begin{cases} u_{tt} - \Delta u = N_{\gamma,p}(u) & x \in \mathbb{R}^n,\ t \in (0,T), \\ u(0,x) = \varepsilon u_0(x) & x \in \mathbb{R}^n, \\ u_t(0,x) = \varepsilon u_1(x) & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

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where

$$N_{\gamma,p}(u)(t,x) \doteq c_\gamma \int_0^t (t-s)^{-\gamma} |u(s,x)|^p \, \mathrm{d}s, \qquad c_\gamma \doteq 1/\Gamma(1-\gamma), \tag{2}$$

and $p > 1$, $\gamma \in (0,1)$, $\varepsilon > 0$ is a parameter describing the size of initial data and Γ denotes the Euler integral of the second kind.

For the sake of brevity we shall refer hereafter to the nonlinearity $N_{\gamma,p}(u)$ in (2) as nonlinear memory term and, vice versa, whenever we mention in what follows a nonlinear memory term we mean the nonlinearity in (2).

Over the last decade several papers have been devoted to the study of semilinear evolution model with the nonlinear term of memory type as in (2). In the pioneering paper [4] the authors determine the critical exponent for the semilinear heat equation with nonlinear memory term. Afterwards, this kind of result has been generalized for fractional (either in space or in time) heat equations [13, 21, 39] and for weakly coupled system of heat equations [12, 26, 37].

Another evolution equation, which has already been studied with nonlinear memory term on the right-hand side, is the classical damped wave equation (cf. [2, 3, 7, 10]). Moreover, we recall that the structural damped wave equation and the beam equation have been investigated in the case of a nonlinear memory term in [6] and [8], respectively.

Finally, we mention that the semilinear wave equation with nonlinear memory term has been considered in the case of bounded domains in [11] and in the case of initial-boundary value problem (and in space dimension 1) in [23]. So far, up to the knowledge of the authors, no satisfactory result has been obtained for the semilinear wave equation with nonlinear memory term in the whole space. For this reason, we shall determine two blow-up results for the Cauchy problem (1).

By a slight abuse of terminology, we shall refer to the two different cases in which we are able to prove the blow-up of the solution as to the subcritical case and to the critical case, respectively.

Recalling that

$$\lim_{\gamma \to 1^-} c_\gamma s_+^{-\gamma} = \delta_0(s) \quad \text{in the sense of distributions, where } s_+^{-\gamma} \doteq \begin{cases} s^{-\gamma} & \text{if } s > 0, \\ 0 & \text{if } s < 0, \end{cases}$$

it would be suitable to find in the blow-up results an upper bound $p_0(n,\gamma)$ for the exponent p in (2) that satisfies formally

$$\lim_{\gamma \to 1^-} p_0(n,\gamma) = p_{\mathrm{Str}}(n), \tag{3}$$

where $p_{\mathrm{Str}}(n)$ denotes the Strauss exponent, i.e. the critical exponent for the semilinear wave equation with power nonlinearity $|u|^p$, whose analytic expression can be derived from the quadratic equation $\frac{n-1}{2} p^2 - \frac{n+1}{2} p - 1 = 0$ for $n \geqslant 2$ (in the one spatial dimensional case, we put $p_{\mathrm{Str}}(1) = \infty$). For the formulation and proof of Strauss' conjecture on the critical exponent for the semilinear wave equation with power nonlinearity we refer to classical works [14–16, 18–20, 25, 30–32, 35, 38, 43, 44] (moreover, for the sharp lifespan estimates in the subcritical and critical case we quote [9, 17, 22, 24, 33, 34, 40–42, 45]).

Let us introduce the following quadratic equation:

$$\frac{n-1}{2}\,p^2-\left(\frac{n+1}{2}+1-\gamma\right)p-1=0, \tag{4}$$

where $\gamma \in (0,1)$ and $p > 1$. Then, for any $n \geqslant 2$ we denote by $p_0(n,\gamma)$ the positive root of the above equation, that is,

$$p_0(n,\gamma) \doteq \frac{n+3-2\gamma+\sqrt{n^2+(14-4\gamma)n+4\gamma(\gamma-3)+1}}{2(n-1)}.$$

Moreover, for $n = 1$ we set formally $p_0(1,\gamma) = \infty$ for any $\gamma \in (0,1)$. This exponent $p_0(n,\gamma)$ is the upper bound for p, below which we shall prove the blow-up results. Let us point out explicitly that according to this choice of $p_0(n,\gamma)$, the formal limit relation (3) is always fulfilled.

Therefore, goal of this paper is to show the blow-up in finite time of local in time solutions to (1) in the case $1 < p \leqslant p_0(n,\gamma)$, provided that the initial data satisfy certain integral sign assumptions and regardless of the size of the Cauchy data. Our approach is quite standard; in fact, we will study the blow-up dynamic of the spatial average of a local in time solution by determining a sequence of lower bound estimates for this time-dependent functional via an iteration procedure. Let us stress that in the critical case (that is, for $p = p_0(n,\gamma)$ and $n \geqslant 2$), this standard approach with the spatial average is no longer successful and it has to be refined by working with a weighted space average instead. More specifically, we shall employ the approach recently introduced in [36]. As byproducts of the iteration arguments we will obtain upper bound estimates for the lifespan of the solution.

1.1 Main Results

Before stating the main results, we introduce the notion of energy solutions to the Cauchy problem (1) that we are going to use in our results.

Definition 1 Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. We say that

$$u \in \mathscr{C}\big([0,T), H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([0,T), L^2(\mathbb{R}^n)\big) \text{ such that } N_{\gamma,p}(u) \in L^1_{\mathrm{loc}}([0,T)\times\mathbb{R}^n)$$

is an energy solution of (1) on $[0, T)$ if u fulfills $u(0, \cdot) = \varepsilon u_0$ in $H^1(\mathbb{R}^n)$ and the integral relation

$$\begin{aligned}
&\int_{\mathbb{R}^n} \partial_t u(t, x)\, \psi(t, x)\, \mathrm{d}x - \varepsilon \int_{\mathbb{R}^n} u_1(x)\, \psi(0, x)\, \mathrm{d}x \\
&\qquad + \int_0^t \int_{\mathbb{R}^n} \left(\nabla u(s, x) \cdot \nabla \psi(s, x) - \partial_t u(s, x)\, \psi_s(s, x)\right) \mathrm{d}x\, \mathrm{d}s \\
&\quad = c_\gamma \int_0^t \int_{\mathbb{R}^n} \psi(s, x) \int_0^s (s-\tau)^{-\gamma} |u(\tau, x)|^p\, \mathrm{d}\tau\, \mathrm{d}x\, \mathrm{d}s
\end{aligned} \tag{5}$$

for any $\psi \in \mathscr{C}_0^\infty\big([0, T) \times \mathbb{R}^n\big)$ and any $t \in [0, T)$.

After a further step of integration by parts in (5), one has

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\psi(t, x)\, \partial_t u(t, x) - \psi_s(t, x)\, u(t, x)\right) \mathrm{d}x - \varepsilon \int_{\mathbb{R}^n} \left(\psi(0, x)\, u_1(x) - \psi_s(0, x)\, u_0(x)\right) \mathrm{d}x \\
&\qquad + \int_0^t \int_{\mathbb{R}^n} \left(\psi_{ss}(s, x) - \Delta \psi(s, x)\right) u(s, x)\, \mathrm{d}x\, \mathrm{d}s \\
&\quad = c_\gamma \int_0^t \int_{\mathbb{R}^n} \psi(s, x) \int_0^s (s-\tau)^{-\gamma} |u(\tau, x)|^p\, \mathrm{d}\tau\, \mathrm{d}x\, \mathrm{d}s.
\end{aligned} \tag{6}$$

for any $\psi \in \mathscr{C}_0^\infty\big([0, T) \times \mathbb{R}^n\big)$ and any $t \in [0, T)$.

Let us state now our first result in the subcritical case.

Theorem 1 *Let us consider $p > 1$ such that*

$$\begin{cases} p < \infty & \textit{if } n = 1, \\ p < p_0(n, \gamma) & \textit{if } n \geqslant 2. \end{cases}$$

Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ be nonnegative and compactly supported functions with supports contained in B_R for some $R > 0$ such that u_0 is not identically zero. Let

$$u \in \mathscr{C}\big([0, T), H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([0, T), L^2(\mathbb{R}^n)\big) \quad \textit{such that } N_{\gamma,p}(u) \in L^1_{\mathrm{loc}}([0, T) \times \mathbb{R}^n)$$

be an energy solution on $[0, T)$ to (1) according to Definition 1 with lifespan $T = T(\varepsilon)$ such that

$$\operatorname{supp} u(t, \cdot) \subset B_{R+t} \quad \textit{for any } t \in (0, T). \tag{7}$$

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, \gamma, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the energy solution u blows up in finite time. Furthermore, the upper *bound estimate for the lifespan*

$$T(\varepsilon) \leqslant C\varepsilon^{-\frac{2p(p-1)}{\Upsilon(p,n,\gamma)}}$$

holds, where C is a positive constant independent of ε and

$$\Upsilon(p, n, \gamma) \doteq 2 + (n + 1 + 2(1 - \gamma))p - (n - 1)p^2. \tag{8}$$

In the next result, we examine the critical case.

Theorem 2 *Let $n \geqslant 2$ and $p = p_0(n, \gamma)$. Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ be nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$. Let*

$$u \in \mathscr{C}\big([0, T), H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([0, T), L^2(\mathbb{R}^n)\big) \quad \text{such that } N_{\gamma,p}(u) \in L^1_{\text{loc}}([0, T) \times \mathbb{R}^n)$$

be an energy solution on $[0, T)$ to (1) according to Definition 1 with lifespan $T = T(\varepsilon)$ and satisfying (7). Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, \gamma, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the energy solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan

$$T(\varepsilon) \leqslant \exp\left(C\varepsilon^{-p(p-1)}\right)$$

holds, where C is a positive constant independent of ε.

Notation
We give some notations to be used in this paper. We write $f \lesssim g$ when there exists a positive constant C such that $f \leqslant Cg$. We denote $g \lesssim f \lesssim g$ by $f \approx g$. Moreover, B_R denotes the ball around the origin with radius R in $\mathbb{R}^n$.

2 Subcritical Case: Proof of Theorem 1

Let us introduce the time-dependent functional

$$U(t) \doteq \int_{\mathbb{R}^n} u(t, x)\, \mathrm{d}x.$$

We can choose ψ such that $\psi = 1$ over $\{(s, x) \in [0, t] \times \mathbb{R}^n : |x| \leqslant R + s\}$. Then, using this test function in (5), it results

$$\int_{\mathbb{R}^n} u_t(t, x)\, \mathrm{d}x - \varepsilon \int_{\mathbb{R}^n} u_1(x)\, \mathrm{d}x = c_\gamma \int_0^t \int_{\mathbb{R}^n} \int_0^s (s - \tau)^{-\gamma} |u(\tau, x)|^p\, \mathrm{d}\tau\, \mathrm{d}x\, \mathrm{d}s,$$

that is,

$$U'(t) = U'(0) + c_\gamma \int_0^t \int_{\mathbb{R}^n} \int_0^s (s-\tau)^{-\gamma} |u(\tau,x)|^p \, \mathrm{d}\tau \, \mathrm{d}x \, \mathrm{d}s. \tag{9}$$

Hence, integrating the above relation over $[0, t]$, we get

$$\begin{aligned} U(t) &= U(0) + U'(0)\, t + c_\gamma \int_0^t \int_0^s \int_{\mathbb{R}^n} \int_0^\tau (\tau-\sigma)^{-\gamma} |u(\sigma,x)|^p \, \mathrm{d}\sigma \, \mathrm{d}x \, \mathrm{d}\tau \, \mathrm{d}s \\ &\geqslant c_\gamma \int_0^t \int_0^s \int_0^\tau (\tau-\sigma)^{-\gamma} \int_{\mathbb{R}^n} |u(\sigma,x)|^p \, \mathrm{d}x \, \mathrm{d}\sigma \, \mathrm{d}\tau \, \mathrm{d}s \geqslant 0, \end{aligned}$$

where the nonnegativity of u_0 and u_1 is applied.
The use of Hölder's inequality, as well as (7), implies

$$\int_{\mathbb{R}^n} |u(\sigma,x)|^p \, \mathrm{d}x \geqslant C(R+\sigma)^{-n(p-1)} (U(\sigma))^p,$$

which leads to

$$U(t) \geqslant C c_\gamma \int_0^t \int_0^s \int_0^\tau (\tau-\sigma)^{-\gamma} (R+\sigma)^{-n(p-1)} (U(\sigma))^p \, \mathrm{d}\sigma \, \mathrm{d}\tau \, \mathrm{d}s. \tag{10}$$

Our proof of Theorem 1 is based on an iteration procedure which provides us a sequence of lower bounds for the functional U. This sequence of lower bounds will be determined iteratively by applying the iteration frame (10).

With the aim of deriving a first lower bound estimate for functional $U(t)$, we follow [38] and we introduce the function

$$\Phi(x) \doteq \begin{cases} \mathrm{e}^x + \mathrm{e}^{-x} & \text{if } n = 1, \\ \int_{\mathbb{S}^{n-1}} \mathrm{e}^{x\cdot\omega} \mathrm{d}\sigma_\omega & \text{if } n \geqslant 2. \end{cases} \tag{11}$$

The function Φ is a positive smooth function and satisfies the remarkable properties

$$\begin{aligned} &\Delta\Phi = \Phi, \\ &\Phi(x) \sim |x|^{-\frac{n-1}{2}} \mathrm{e}^x \qquad \text{as } |x| \to \infty. \end{aligned}$$

If we introduce the function with separate variables $\Psi = \Psi(t,x) = \mathrm{e}^{-t}\Phi(x)$, clearly, the function Ψ is a solution to the wave equation $\Psi_{tt} - \Delta\Psi = 0$.

Furthermore, we introduce the auxiliary functional

$$U_0(t) \doteq \int_{\mathbb{R}^n} u(t,x)\Psi(t,x) \, \mathrm{d}x.$$

Differentiating with respect to t Eq. (9), we obtain

$$U''(t) = c_\gamma \int_0^t (t-s)^{-\gamma} \int_{\mathbb{R}^n} |u(s,x)|^p \, dx \, ds.$$

Therefore, by applying Hölder's inequality to $U_0(s)$, one finds

$$\int_{\mathbb{R}^n} |u(s,x)|^p \, dx \geqslant |U_0(s)|^p \left(\int_{B_{R+s}} |\Psi(s,x)|^{\frac{p}{p-1}} \, dx \right)^{-(p-1)}. \tag{12}$$

So, if we determine a lower bound estimate for $U_0(s)$, then, the previous inequality provides a lower bound for $\int_{\mathbb{R}^n} |u(s,x)|^p \, dx$ in turn.

According to [38] the time-dependent functional U_0 satisfies

$$U_0(t) \geqslant \tfrac{\varepsilon}{2}(1-e^{-2t}) \int_{\mathbb{R}^n} (u_0(x)+u_1(x)) \, \Phi(x) \, dx + \varepsilon \, e^{-2t} \int_{\mathbb{R}^n} u_0(x)\Phi(x) \, dx \geqslant \widetilde{C} \, \varepsilon$$

for any $t \geqslant 0$ with a suitable constant $\widetilde{C} > 0$ depending on u_0 and u_1, where we applied our assumption that u_0 is nonnegative and not identically 0.

Indeed, in [38, Lemma 2.2] the only condition on the nonlinearity that is actually used is the nonnegativity, which holds trivially also for our nonlinear memory term. For a detailed proof of the lower bound estimate for U_0 see also [27, Lemma 4.3.6], for example.

Additionally, by the asymptotic behavior of Ψ, it is known that the inequality

$$\int_{B_{R+s}} |\Psi(s,x)|^{\frac{p}{p-1}} \, dx \leqslant \widetilde{K}(R+s)^{(n-1)(1-p'/2)}$$

holds for some positive constant $\widetilde{K} > 0$, cf. [38, Estimate (2.5)]. So, from (12) we have

$$\int_{\mathbb{R}^n} |u(s,x)|^p \, dx \geqslant C_0 \varepsilon^p (R+s)^{n-1-\frac{n-1}{2}p} \qquad \text{for any } s \geqslant 0, \tag{13}$$

where $C_0 = \widetilde{C}^p \widetilde{K}^{1-p}$, and, consequently,

$$\begin{aligned} U''(t) &\geqslant C_0 c_\gamma \varepsilon^p \int_0^t (t-s)^{-\gamma} (R+s)^{(n-1)(1-p/2)} \, ds \\ &\geqslant C_0 c_\gamma \varepsilon^p \int_0^t (t-s)^{-\gamma} (R+s)^{-\frac{n-1}{2}p} s^{n-1} \, ds \\ &\geqslant \frac{C_0 c_\gamma \varepsilon^p}{n} (R+t)^{-\frac{n-1}{2}p} t^{n-\gamma} \end{aligned}$$

for any $t \geqslant 0$. By integrating the above inequality twice, we get for U the lower bound estimate

$$
\begin{aligned}
U(t) &\geqslant U(0) + U'(0)\,t + \frac{C_0 c_\gamma \varepsilon^p}{n} \int_0^t \int_0^s (R+\tau)^{-\frac{n-1}{2}p} \tau^{n-\gamma}\, \mathrm{d}\tau\, \mathrm{d}s \\
&\geqslant \frac{C_0 c_\gamma \varepsilon^p}{n(n-\gamma+1)(n-\gamma+2)} (R+t)^{-\frac{n-1}{2}p} t^{n+2-\gamma}
\end{aligned}
$$

for any $t \geqslant 0$. In other words, we have

$$
U(t) \geqslant K_0 (R+t)^{-\alpha_0} t^{\beta_0} \quad \text{for any } t \geqslant 0, \tag{14}
$$

where the multiplicative constant is defined by

$$
K_0 \doteq \frac{C_0 c_\gamma \varepsilon^p}{n(n-\gamma+1)(n-\gamma+2)}
$$

and the exponents are

$$
\alpha_0 \doteq \frac{n-1}{2} p \quad \text{and} \quad \beta_0 \doteq n+2-\gamma.
$$

In the next step, we will derive a sequence of lower bounds of U by using the iteration frame (10). To be specific, we will show that

$$
U(t) \geqslant K_j (R+t)^{-\alpha_j} t^{\beta_j} \quad \text{for any } t \geqslant 0, \tag{15}
$$

where $\{K_j\}_{j\in\mathbb{N}}$, $\{\alpha_j\}_{j\in\mathbb{N}}$ and $\{\beta_j\}_{j\in\mathbb{N}}$ are sequences of nonnegative real numbers that will be specified later.

Obviously, we already proved (15) for $j=0$. Therefore, in order to prove (15) for all $j \in \mathbb{N}$ by using an inductive argument, it remains to show the induction step.

Plugging (15) in the iteration frame (10), we derive

$$
\begin{aligned}
U(t) &\geqslant C c_\gamma K_j^p \int_0^t \int_0^s \int_0^\tau (\tau-\sigma)^{-\gamma} (R+\sigma)^{-n(p-1)-p\alpha_j} \sigma^{p\beta_j}\, \mathrm{d}\sigma\, \mathrm{d}\tau\, \mathrm{d}s \\
&\geqslant C c_\gamma K_j^p (R+t)^{-n(p-1)-p\alpha_j} t^{-\gamma} \int_0^t \int_0^s \int_0^\tau \sigma^{p\beta_j}\, \mathrm{d}\sigma\, \mathrm{d}\tau\, \mathrm{d}s \\
&\geqslant \frac{C c_\gamma K_j^p}{(p\beta_j+1)(p\beta_j+2)(p\beta_j+3)} (R+t)^{-n(p-1)-p\alpha_j} t^{p\beta_j+3-\gamma}
\end{aligned}
$$

for all $t \geqslant 0$. Thus, we showed (15) for $j+1$, provided that the recursive relations

$$K_{j+1} \doteq \frac{C c_\gamma K_j^p}{(p\beta_j+1)(p\beta_j+2)(p\beta_j+3)}, \ \alpha_{j+1} \doteq n(p-1)+p\alpha_j, \ \beta_{j+1} \doteq p\beta_j+3-\gamma$$

are satisfied.

For what follows it is useful to determine a suitable estimate from below of K_j. For this purpose, we have to determine first the explicit representation for α_j and β_j. From the relation $\alpha_j = n(p-1)+p\alpha_{j-1}$ and $\beta_j = p\beta_{j-1}+3-\gamma$, we deduce

$$\alpha_j = p^j\alpha_0 + n(p-1)\left(1+p+\cdots+p^{j-1}\right) = (\alpha_0+n)p^j - n, \tag{16}$$

$$\beta_j = p^j\beta_0 + (3-\gamma)\left(1+p+\cdots+p^{j-1}\right) = \left(\tfrac{\gamma-3}{1-p}+\beta_0\right)p^j - \tfrac{\gamma-3}{1-p}. \tag{17}$$

Thus,

$$\begin{aligned}(p\beta_{j-1}+1)(p\beta_{j-1}+2)(p\beta_{j-1}+3) &\leqslant (p\beta_{j-1}+2)^3 = (\beta_j+\gamma-1)^3 \\ &\leqslant \beta_j^3 \leqslant \left(\tfrac{\gamma-3}{1-p}+\beta_0\right)^3 p^{3j},\end{aligned}$$

where we used $\gamma \in (0,1)$. It follows that

$$K_j \geqslant \underbrace{\tfrac{C}{\Gamma(1-\gamma)}\left(\tfrac{\gamma-3}{1-p}+\beta_0\right)^{-3}}_{\doteq D} p^{-3j}K_{j-1}^p = D\,p^{-3j}K_{j-1}^p \quad \text{for any } j \in \mathbb{N}.$$

Applying the logarithmic function to both sides of the inequality $K_j \geqslant Dp^{-3j}K_{j-1}^p$ and using iteratively the resulting inequality, we derive

$$\begin{aligned}\log K_j &\geqslant p^j \log K_0 - 3\left(\sum_{k=0}^{j-1}(j-k)p^k\right)\log p + \left(\sum_{k=0}^{j-1}p^k\right)\log D \\ &\geqslant p^j\left(\log K_0 - \frac{3p\log p}{(p-1)^2} + \frac{\log D}{p-1}\right) + \frac{3j\log p}{p-1} + \frac{3p\log p}{(p-1)^2} - \frac{\log D}{p-1}\end{aligned}$$

for any $j \in \mathbb{N}$, where the identity

$$\sum_{k=0}^{j-1}(j-k)p^k = \frac{1}{p-1}\left(\frac{p^{j+1}-p}{p-1} - j\right) \tag{18}$$

is used. Let $j = j_0(n, \gamma, p) \in \mathbb{N}$ be the smallest nonnegative integer such that

$$j_0 \geqslant \frac{\log D}{3 \log p} - \frac{p}{p-1}.$$

Therefore, for any $j \geqslant j_0$ the inequality holds

$$\begin{aligned} \log K_j &\geqslant p^j \left(\log K_0 - \frac{3p \log p}{(p-1)^2} + \frac{\log D}{p-1} \right) \\ &= p^j \log \left(p^{-3p/(p-1)^2} D^{1/(p-1)} K_0 \right) = p^j \log \left(E_0 \varepsilon^p \right) \end{aligned} \tag{19}$$

for a suitable constant $E_0 = E_0(n, \gamma, p) > 0$.

If we combine with (15), (16), (17) and (19), we get

$$\begin{aligned} U(t) &\geqslant \exp \left(p^j \log(E_0 \varepsilon^p) \right) (R+t)^{-\alpha_j} t^{\beta_j} \\ &= \exp \left(p^j \left(\log(E_0 \varepsilon^p) - (\alpha_0 + n) \log(R+t) + \left(\tfrac{\gamma-3}{1-p} + \beta_0 \right) \log t \right) \right) (R+t)^n t^{\frac{3-\gamma}{1-p}} \end{aligned}$$

for any $j \geqslant j_0$ and any $t \geqslant 0$.

Finally, since for $t \geqslant R$ it holds $\log(t+R) \leqslant \log(2t)$, from the previous inequality we have

$$U(t) \geqslant \exp \left(p^j \log \left(E_0 \varepsilon^p 2^{-(\alpha_0+n)} t^{\frac{\gamma-3}{1-p} + \beta_0 - (\alpha_0+n)} \right) \right) (R+t)^n t^{\frac{3-\gamma}{1-p}} \tag{20}$$

for any $j \geqslant j_0$. The exponent of t in the exponential term in the last inequality is

$$\tfrac{\gamma-3}{1-p} + \beta_0 - (\alpha_0 + n) = \tfrac{1}{2(p-1)} \left(2 + (n+3-2\gamma)p - (n-1)p^2 \right) = \tfrac{\Upsilon(p,n,\gamma)}{2(p-1)},$$

where $\Upsilon(p, n, \gamma)$ is defined in (8). So, for $p > 1$ when $n = 1$ and $1 < p < p_0(n, \gamma)$ when $n \geqslant 2$, the exponent for t in the exponential term of (20) is positive. Let us fix $\varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, \gamma, R) > 0$ such that

$$\varepsilon_0^{-\frac{2p(p-1)}{\Upsilon(p,n,\gamma)}} \geqslant E_1 R, \quad \text{where } E_1 \doteq \left(2^{-(\alpha_0+n)} E_0 \right)^{\frac{2(p-1)}{\Upsilon(p,n,\gamma)}}.$$

Thus, for any $\varepsilon \in (0, \varepsilon_0]$ and $t > E_1^{-1} \varepsilon^{-\frac{2p(p-1)}{\Upsilon(p,n,\gamma)}} \geqslant R$, it holds

$$\log \left(\varepsilon^p \, 2^{-(\alpha_0+n)} E_0 \, t^{\frac{\Upsilon(p,n,\gamma)}{2(p-1)}} \right) > 0.$$

Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > E_1 \varepsilon^{-\frac{2p(p-1)}{\Upsilon(p,n,\gamma)}}$ letting $j \to \infty$ in (20) we may observe that the lower bound for $U(t)$ blows up. So, U may not be finite for this t as well. This proves that u is not globally in time defined and, in particular, the lifespan of the local (in time) solution u can be estimated by

$$T(\varepsilon) \lesssim \varepsilon^{-\frac{2p(p-1)}{\Upsilon(p,n,\gamma)}}.$$

All in all, the proof of Theorem 1 is complete.

3 Critical Case: Proof of Theorem 2

3.1 Auxiliary Functions

Let us recall the definition of a pair of auxiliary functions from [36], which are necessary in order to introduce the time-dependent functional that will be considered for the iteration argument in the critical case $p = p_0(n, \gamma)$.

Let $r > -1$ be a real parameter. Then, we introduce the functions

$$\xi_r(t, x) \doteq \int_0^{\lambda_0} \mathrm{e}^{-\lambda(t+R)} \cosh(\lambda t)\, \Phi(\lambda x)\, \lambda^r \, \mathrm{d}\lambda, \tag{21}$$

$$\eta_r(t, s, x) \doteq \int_0^{\lambda_0} \mathrm{e}^{-\lambda(t+R)} \frac{\sinh(\lambda(t-s))}{\lambda(t-s)}\, \Phi(\lambda x)\, \lambda^r \, \mathrm{d}\lambda, \tag{22}$$

where λ_0 is a fixed positive parameter and Φ is defined by (11).

Some useful properties of ξ_r and η_r are stated in the following lemma, whose proof can be found in [36, Lemma 3.1].

Lemma 1 *Let $n \geqslant 2$ and $\lambda_0 > 0$. Then, the following properties hold:*

(i) *if $r > -1$, $|x| \leqslant R$ and $t \geqslant 0$, then,*

$$\xi_r(t, x) \geqslant A_0,$$
$$\eta_r(t, 0, x) \geqslant B_0 \langle t \rangle^{-1};$$

(ii) *if $r > -1$, $|x| \leqslant s + R$ and $t > s \geqslant 0$, then,*

$$\eta_r(t, s, x) \geqslant B_1 \langle t \rangle^{-1} \langle s \rangle^{-r};$$

(iii) *if $r > \frac{n-3}{2}$, $|x| \leqslant t + R$ and $t > 0$, then,*

$$\eta_r(t, t, x) \leqslant B_2 \langle t \rangle^{-\frac{n-1}{2}} \langle t - |x| \rangle^{\frac{n-3}{2} - r}.$$

Here A_0 and B_k, with $k = 0, 1, 2$, are positive constants depending only on λ_0, r and R and we denote $\langle y \rangle \doteq 3 + |y|$.

Remark 1 Although in [36] the previous lemma is stated by assuming $r > 0$ in (i) and (ii), the proof provided in that paper holds true for any $r > -1$ as well.

Proposition 1 *Let $n \geqslant 2$ and $r > -1$. Assume that $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ are nonnegative, nontrivial and compactly supported in B_R functions. Let u be an energy solution to (1) on $[0, T)$ according to Definition 1 satisfying (7). Then, the following integral identity holds:*

$$\int_{\mathbb{R}^n} u(t,x)\,\eta_r(t,t,x)\,\mathrm{d}x = \varepsilon \int_{\mathbb{R}^n} u_0(x)\,\xi_r(t,x)\,\mathrm{d}x + \varepsilon t \int_{\mathbb{R}^n} u_1(x)\,\eta_r(t,0,x)\,\mathrm{d}x + c_\gamma \int_0^t (t-s) \int_0^s (s-\sigma)^{-\gamma} \int_{\mathbb{R}^n} |u(\sigma,x)|^p\,\eta_r(t,s,x)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}s, \tag{23}$$

for any $t \in (0, T)$, where ξ_r and η_r are defined in (21) and (22), respectively.

Proof According to (7) $u(t, \cdot)$ has compact support contained in B_{R+t} for any $t \geqslant 0$. Therefore, we may employ (6) for a noncompactly supported test function. So, we choose as test function

$$\psi = \psi(s,x) = \lambda^{-1}\sinh(\lambda(t-s))\Phi(\lambda x),$$

where Φ is defined by (11). As Φ is an eigenfunction of the Laplace operator and the function $y(t, s; \lambda) = \lambda^{-1}\sinh(\lambda(t - s))$ solves the parameter dependent ODE

$$(\partial_s^2 - \lambda^2)y(t,s;\lambda) = 0$$

with final conditions $y(t, t; \lambda) = 0$ and $\partial_s y(t, t; \lambda) = -1$, we get that ψ solves the free wave equation $\psi_{ss} - \Delta\psi = 0$ and satisfies

$$\psi(t,x) = 0, \qquad \psi(0,x) = \lambda^{-1}\sinh(\lambda t)\Phi(\lambda x),$$
$$\psi_s(t,x) = -\Phi(\lambda x), \qquad \psi_s(0,x) = -\cosh(\lambda t)\Phi(\lambda x).$$

Let us prove (23). Employing in (6) the above defined ψ and its properties, we get

$$\int_{\mathbb{R}^n} u(t,x)\,\Phi(\lambda x)\,\mathrm{d}x = \varepsilon\cosh(\lambda t)\int_{\mathbb{R}^n} u_0(x)\,\Phi(\lambda x)\,\mathrm{d}x + \varepsilon\frac{\sinh(\lambda t)}{\lambda}\int_{\mathbb{R}^n} u_1(x)\,\Phi(\lambda x)\,\mathrm{d}x + c_\gamma \int_0^t \frac{\sinh(\lambda(t-s))}{\lambda} \int_{\mathbb{R}^n} \int_0^s (s-\sigma)^{-\gamma}|u(\sigma,x)|^p\,\mathrm{d}\sigma\,\Phi(\lambda x)\,\mathrm{d}x\,\mathrm{d}s.$$

Multiplying both sides of the last equality by $\mathrm{e}^{-\lambda(t+R)}\lambda^r$, integrating with respect to λ over $[0, \lambda_0]$ and applying Tonelli's theorem, we get finally (23).

3.2 Iteration Frame and First Lower Bound Estimate

Hereafter until the end of Sect. 3, we shall assume that u_0, u_1 satisfy the assumptions from the statement of Theorem 2. Let u be an energy solution of (1) on $[0, T)$. We introduce the following time-dependent functional:

$$\mathcal{U}(t) \doteq \int_{\mathbb{R}^n} u(t,x)\,\eta_r(t,t,x)\,\mathrm{d}x, \tag{24}$$

where

$$r \doteq \frac{n-1}{2} - \frac{1}{p}.$$

From Proposition 1 it follows immediately the positiveness of the functional $\mathcal{U}$.

The next step is to derive an integral inequalities involving $\mathcal{U}$ both in the left and in the right-hand side, which will set the iteration frame for the iteration procedure.

Proposition 2 *Let $\mathcal{U}$ be the functional defined by* (24)*. Then, there exist positive constants C depending on $n, p, \gamma, \lambda_0, R$ such that the estimate*

$$\mathcal{U}(t) \geqslant C\langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}}\int_0^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{(n-1)(1-\frac{p}{2})}\frac{(\mathcal{U}(\sigma))^p}{(\log\langle\sigma\rangle)^{(p-1)}}\,\mathrm{d}\sigma\,\mathrm{d}s \tag{25}$$

holds for any $t \geqslant 0$.

Proof For the proof of this proposition we follow the main ideas of Proposition 4.2 in [36]. Applying Hölder's inequality and the support property for $u(\sigma,\cdot)$, we obtain

$$\mathcal{U}(\sigma) \leqslant \left(\int_{\mathbb{R}^n} |u(\sigma,x)|^p\eta_r(t,s,x)\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_{\sigma+R}}\frac{\eta_r(\sigma,\sigma,x)^{p'}}{\eta_r(t,s,x)^{\frac{p'}{p}}}\,\mathrm{d}x\right)^{\frac{1}{p'}}. \tag{26}$$

We begin with the estimate of the second factor on the right hand side in the last inequality.

By (ii) and (iii) in Lemma 1 (note that, according to our choice of r, both $r > \frac{n-3}{2}$ and $r > -1$ are always fulfilled), since $|x| \leqslant \sigma + R$ implies $|x| \leqslant s + R$ for any

$\sigma \in [0, s]$, we obtain

$$\int_{B_{\sigma+R}} \frac{\eta_r(\sigma,\sigma,x)^{p'}}{\eta_r(t,s,x)^{\frac{p'}{p}}}\,\mathrm{d}x \lesssim \langle t\rangle^{\frac{p'}{p}}\langle s\rangle^{\frac{p'}{p}r}\langle\sigma\rangle^{-\frac{n-1}{2}p'}\int_{B_{\sigma+R}}\langle\sigma-|x|\rangle^{(\frac{n-3}{2}-r)p'}\,\mathrm{d}x$$
$$\lesssim \langle t\rangle^{\frac{1}{p-1}}\langle s\rangle^{\frac{r}{p-1}}\langle\sigma\rangle^{-\frac{n-1}{2}p'}\int_{B_{\sigma+R}}\langle\sigma-|x|\rangle^{-1}\,\mathrm{d}x$$
$$\lesssim \langle t\rangle^{\frac{1}{p-1}}\langle s\rangle^{\frac{r}{p-1}}\langle\sigma\rangle^{-\frac{n-1}{2}p'+n-1}\log\langle\sigma\rangle,$$

where in the second step we used the definition of r. Combining (23), (26) and the previous estimate, we find

$$\mathcal{U}(t) \gtrsim \int_0^t (t-s)\int_0^s (s-\sigma)^{-\gamma}\int_{\mathbb{R}^n}|u(\sigma,x)|^p\,\eta_r(t,s,x)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}s$$
$$\gtrsim \int_0^t (t-s)\int_0^s (s-\sigma)^{-\gamma}\langle t\rangle^{-1}\langle s\rangle^{-r}\langle\sigma\rangle^{\frac{n-1}{2}p-(n-1)(p-1)}\frac{(\mathcal{U}(\sigma))^p}{(\log\langle\sigma\rangle)^{(p-1)}}\,\mathrm{d}\sigma\,\mathrm{d}s$$

which is exactly (25).

Proposition 3 *Let us assume $p = p_0(n,\gamma)$. Let $\mathcal{U}$ be the functional defined by (24). Then, there exist a positive constant M depending on $n, p, \gamma, \lambda_0, R, u_0, u_1$ such that*

$$\mathcal{U}(t) \geqslant M\varepsilon^p \log(2t/3) \tag{27}$$

holds for any $t \geqslant 3/2$.

Proof We start by noticing that (13) may be rewritten as

$$\int_{\mathbb{R}^n}|u(\sigma,x)|^p\,\mathrm{d}x \geqslant C_0\varepsilon^p\langle\sigma\rangle^{n-1-\frac{n-1}{2}p} \qquad \text{for any } \sigma \geqslant 1, \tag{28}$$

up to a modification of the multiplicative constant. By using (23), Lemma 1 (ii) and (28), we get

$$\mathcal{U}(t) \gtrsim \int_0^t (t-s)\int_0^s (s-\sigma)^{-\gamma}\int_{\mathbb{R}^n}|u(\sigma,x)|^p\,\eta_r(t,s,x)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}s$$
$$\gtrsim \langle t\rangle^{-1}\int_0^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}}\int_0^s (s-\sigma)^{-\gamma}\int_{\mathbb{R}^n}|u(\sigma,x)|^p\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}s$$
$$\gtrsim \varepsilon^p\langle t\rangle^{-1}\int_1^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}}\int_1^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{n-1-\frac{n-1}{2}p}\,\mathrm{d}\sigma\,\mathrm{d}s.$$

Therefore, for $t \geqslant 1$ by shrinking the domain of integration we find

$$\begin{aligned}\mathcal{U}(t) &\gtrsim \varepsilon^p \langle t\rangle^{-1} \int_1^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}} \int_{s/2}^s (s-\sigma)^{-\gamma} \langle\sigma\rangle^{n-1-\frac{n-1}{2}p} \,\mathrm{d}\sigma\,\mathrm{d}s \\ &\gtrsim \varepsilon^p \langle t\rangle^{-1} \int_1^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}+n-1-\frac{n-1}{2}p} s^{1-\gamma}\,\mathrm{d}s \\ &\gtrsim \varepsilon^p \langle t\rangle^{-1} \int_1^t (t-s)\langle s\rangle^{-\frac{n-1}{2}p+\frac{n-1}{2}+1-\gamma+\frac{1}{p}}\,\mathrm{d}s.\end{aligned}$$

Since $p = p_0(n,\gamma)$, from (4) we get

$$-\tfrac{n-1}{2}p + \tfrac{n-1}{2} + 1 - \gamma + \tfrac{1}{p} = -1. \tag{29}$$

Hence, for $t \geqslant 3/2$ it follows

$$\begin{aligned}\mathcal{U}(t) &\gtrsim \varepsilon^p \langle t\rangle^{-1} \int_1^t (t-s)\langle s\rangle^{-1}\,\mathrm{d}s \gtrsim \varepsilon^p \langle t\rangle^{-1} \int_1^t \frac{t-s}{s}\,\mathrm{d}s = \varepsilon^p \langle t\rangle^{-1} \int_1^t \log s\,\mathrm{d}s \\ &\gtrsim \varepsilon^p (3t)^{-1} \int_{2t/3}^t \log s\,\mathrm{d}s \gtrsim \varepsilon^p \log(2t/3).\end{aligned}$$

This completes the proof.

In this subsection we determined the iteration frame (25) for the functional $\mathcal{U}$ and a first lower bound estimate (27) for $\mathcal{U}$ containing a logarithmic factors. In the next subsection we are going to prove a sequence of lower bound estimates for $\mathcal{U}$ by using the so-called slicing procedure, which has been introduced for the first time in [1]. More specifically, we will follow the main ideas of [5, 28, 29] concerning the slicing procedure.

3.3 *Iteration Argument via Slicing Method*

Let us introduce the sequence $\{\ell_j\}_{j\in\mathbb{N}}$, where $\ell_j \doteq 2 - 2^{-(j+1)}$. The goal is to prove the following sequence of lower bound estimates for the functional $\mathcal{U}$

$$\mathcal{U}(t) \geqslant M_j (\log\langle t\rangle)^{-b_j} \left(\log\left(\frac{t}{\ell_{2j}}\right)\right)^{a_j} \qquad \text{for } t \geqslant \ell_{2j} \text{ and for any } j \in \mathbb{N}, \tag{30}$$

where $\{M_j\}_{j\in\mathbb{N}}$, $\{a_j\}_{j\in\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}$ are sequences of nonnegative real numbers that we shall determine recursively throughout the iteration procedure. For $j = 0$

we have already shown that (30) is true thanks to Proposition 3 with

$$M_0 \doteq M\varepsilon^p, \quad a_0 \doteq 1 \quad \text{and} \quad b_0 \doteq 0.$$

We are going to prove the validity of (30) for any $j \in \mathbb{N}$ by using an inductive proof. As we have already pointed out the validity of the base case, it remains to prove the inductive step. Let us assume that (30) holds for $j \geqslant 1$, we want to prove it now for $j+1$. Plugging (30) for j in (25), one finds

$$\begin{aligned}
\mathcal{U}(t) &\geqslant C M_j^p \langle t\rangle^{-1} \int_{\ell_{2j}}^t (t-s)\langle s\rangle^{-r} \int_{\ell_{2j}}^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{(n-1)(1-\frac{p}{2})} \frac{\left(\log\left(\frac{\sigma}{\ell_{2j}}\right)\right)^{a_j p}}{(\log\langle\sigma\rangle)^{(p-1)+b_j p}}\, \mathrm{d}\sigma\, \mathrm{d}s \\
&\geqslant C M_j^p \,(\log\langle t\rangle)^{-(p-1)-b_j p}\, \langle t\rangle^{-1} \\
&\quad \times \int_{\ell_{2j}}^t (t-s)\langle s\rangle^{-\frac{n-1}{2}+\frac{1}{p}-\frac{n-1}{2}p} \int_{\ell_{2j}}^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{n-1} \left(\log\left(\frac{\sigma}{\ell_{2j}}\right)\right)^{a_j p} \mathrm{d}\sigma\, \mathrm{d}s
\end{aligned}$$

for $t \geqslant \ell_{2j+2}$. For $s \geqslant \ell_{2j+1}$, the σ-integral in the last line can be estimated in the following way:

$$\begin{aligned}
&\int_{\ell_{2j}}^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{n-1} \left(\log\left(\frac{\sigma}{\ell_{2j}}\right)\right)^{a_j p} \mathrm{d}\sigma \\
&\qquad \geqslant \int_{\frac{\ell_{2j}s}{\ell_{2j+1}}}^s (s-\sigma)^{-\gamma}\sigma^{n-1} \left(\log\left(\frac{\sigma}{\ell_{2j}}\right)\right)^{a_j p} \mathrm{d}\sigma \\
&\qquad \geqslant \left(\frac{\ell_{2j}}{\ell_{2j+1}}\right)^{n-1} s^{n-1} \left(\log\left(\frac{s}{\ell_{2j+1}}\right)\right)^{a_j p} \int_{\frac{\ell_{2j}s}{\ell_{2j+1}}}^s (s-\sigma)^{-\gamma}\, \mathrm{d}\sigma \\
&\qquad \geqslant \frac{1}{1-\gamma}\left(\frac{\ell_{2j}}{\ell_{2j+1}}\right)^{n-1} \left(1-\frac{\ell_{2j}}{\ell_{2j+1}}\right)^{1-\gamma} s^{n-\gamma} \left(\log\left(\frac{s}{\ell_{2j+1}}\right)\right)^{a_j p}.
\end{aligned}$$

Using the inequalities $2\ell_{2j} > \ell_{2j+1}$ and $1-\ell_{2j}/\ell_{2j+1} > 2^{-(2j+3)}$ and the estimate $4s \geqslant \langle s\rangle$ for any $s \geqslant 1$, it follows

$$\begin{aligned}
&\int_{\ell_{2j}}^s (s-\sigma)^{-\gamma}\langle\sigma\rangle^{n-1} \left(\log\left(\frac{\sigma}{\ell_{2j}}\right)\right)^{a_j p} \mathrm{d}\sigma \\
&\qquad \geqslant \frac{1}{1-\gamma} 2^{-2(1-\gamma)j-3n-2+5\gamma} \langle s\rangle^{n-\gamma} \left(\log\left(\frac{s}{\ell_{2j+1}}\right)\right)^{a_j p}.
\end{aligned}$$

So, combining the lower bound estimate for the σ-integral with the lower bound estimate for $\mathcal{U}(t)$ and using again (29), for $t \geqslant \ell_{2j+2}$ it holds

$$\begin{aligned}
\mathcal{U}(t) &\geqslant \widehat{C}\, 2^{-2(1-\gamma)j} M_j^p \,(\log\langle t\rangle)^{-(p-1)-b_j p}\, \langle t\rangle^{-1} \\
&\quad \times \int_{\ell_{2j+1}}^t (t-s)\langle s\rangle^{-\frac{n-1}{2}p+\frac{n-1}{2}+1-\gamma+\frac{1}{p}} \left(\log\left(\frac{s}{\ell_{2j+1}}\right)\right)^{a_j p} \mathrm{d}s
\end{aligned}$$

$$\geqslant \widehat{C}\, 2^{-2(1-\gamma)j} M_j^p \, (\log\langle t\rangle)^{-(p-1)-b_j p} \, \langle t\rangle^{-1} \int_{\ell_{2j+1}}^t (t-s)\langle s\rangle^{-1} \left(\log\left(\tfrac{s}{\ell_{2j+1}}\right)\right)^{a_j p} \mathrm{d}s$$

$$\geqslant 2^{-2}\widehat{C}\, 2^{-2(1-\gamma)j} M_j^p \, (\log\langle t\rangle)^{-(p-1)-b_j p} \, \langle t\rangle^{-1} \int_{\ell_{2j+1}}^t \tfrac{t-s}{s} \left(\log\left(\tfrac{s}{\ell_{2j+1}}\right)\right)^{a_j p} \mathrm{d}s,$$

where $\widehat{C} \doteq C(1-\gamma)^{-1}2^{-3n-2+5\gamma}$. Integration by parts and a further shrinking of the domain of integration lead to

$$\mathscr{U}(t) \geqslant \frac{2^{-2}\widehat{C}\, M_j^p}{2^{2(1-\gamma)j}(a_j p+1)} (\log\langle t\rangle)^{-(p-1)-b_j p} \, \langle t\rangle^{-1} \int_{\ell_{2j+1}}^t \left(\log\left(\tfrac{s}{\ell_{2j+1}}\right)\right)^{a_j p+1} \mathrm{d}s$$

$$\geqslant \frac{2^{-2}\widehat{C}\, M_j^p}{2^{2(1-\gamma)j}(a_j p+1)} (\log\langle t\rangle)^{-(p-1)-b_j p} \, \langle t\rangle^{-1} \int_{\frac{\ell_{2j+1}t}{\ell_{2j+2}}}^t \left(\log\left(\tfrac{s}{\ell_{2j+1}}\right)\right)^{a_j p+1} \mathrm{d}s$$

$$\geqslant \frac{2^{-2}\widehat{C}\, M_j^p}{2^{2(1-\gamma)j}(a_j p+1)} \left(1-\tfrac{\ell_{2j+1}}{\ell_{2j+2}}\right) (\log\langle t\rangle)^{-(p-1)-b_j p} \, \langle t\rangle^{-1} t \left(\log\left(\tfrac{t}{\ell_{2j+2}}\right)\right)^{a_j p+1}$$

$$\geqslant 2^{-8}\widehat{C}\,(a_j p+1)^{-1} 2^{-2(2-\gamma)j} M_j^p \, (\log\langle t\rangle)^{-(p-1)-b_j p} \left(\log\left(\tfrac{t}{\ell_{2j+2}}\right)\right)^{a_j p+1}$$

for $t \geqslant \ell_{2j+2}$. Also, we proved (30) for $j+1$ provided that

$$M_{j+1} \doteq 2^{-8}\widehat{C}\,(a_j p+1)^{-1} 2^{-2(2-\gamma)j} M_j^p, \quad a_{j+1} \doteq a_j p+1, \quad b_{j+1} \doteq (p-1)+b_j p.$$

Next we determine a suitable lower bound for the term M_j. For this purpose, we provide the explicit representations of the exponents a_j and b_j. By using recursively the relations $a_j = 1 + pa_{j-1}$ and $b_j = (p-1) + pb_{j-1}$ and the initial exponents $a_0 = 1$, $b_0 = 0$, we get

$$a_j = a_0 p^j + \sum_{k=0}^{j-1} p^k = \tfrac{p^{j+1}-1}{p-1} \quad \text{and} \quad b_j = p^j b_0 + (p-1)\sum_{k=0}^{j-1} p^k = p^j - 1. \tag{31}$$

In particular, $a_{j-1}p + 1 = a_j \leqslant p^{j+1}/(p-1)$ implies that

$$M_j \geqslant \widehat{D}\,(2^{2(2-\gamma)}p)^{-j} M_{j-1}^p \tag{32}$$

for any $j \geqslant 1$, where $\widehat{D} \doteq 2^{-8+2(2-\gamma)}\widehat{C}(p-1)/p$. Applying the logarithmic function to both sides of (32) and using iteratively the resulting inequality, we obtain

$$\log M_j \geqslant p \log M_{j-1} - j \log\left(2^{2(2-\gamma)}p\right) + \log \widehat{D}$$

$$\geqslant p^j \log M_0 - \left(\sum_{k=0}^{j-1}(j-k)p^k\right) \log\left(2^{2(2-\gamma)}p\right) + \left(\sum_{k=0}^{j-1} p^k\right) \log \widehat{D}$$

$$= p^j \left(\log M_0 - \frac{p \log\left(2^{2(2-\gamma)} p\right)}{(p-1)^2} + \frac{\log \widehat{D}}{p-1} \right)$$
$$+ \left(\frac{j}{p-1} + \frac{p}{(p-1)^2} \right) \log\left(2^{2(2-\gamma)} p\right) - \frac{\log \widehat{D}}{p-1},$$

where we used the identity (18). Let us define $j_1 = j_1(n, p, \gamma)$ as the smallest nonnegative integer such that

$$j_1 \geqslant \frac{\log \widehat{D}}{\log\left(2^{2(2-\gamma)} p\right)} - \frac{p}{p-1}.$$

Then, for any $j \geqslant j_1$ we may estimate

$$\log M_j \geqslant p^j \left(\log M_0 - \frac{p \log\left(2^{2(2-\gamma)} p\right)}{(p-1)^2} + \frac{\log \widehat{D}}{p-1} \right) = p^j \log(L_0 \varepsilon^p), \tag{33}$$

where $L_0 \doteq M\left(2^{2(2-\gamma)} p\right)^{-p/(p-1)^2} \widehat{D}^{1/(p-1)}$.

Combining (30), (31) and (33), we arrive at

$$\mathscr{U}(t) \geqslant \exp\left(p^j \log(L_0 \varepsilon^p)\right) (\log\langle t\rangle)^{-p^j+1} (\log(t/2))^{(p^{j+1}-1)/(p-1)}$$
$$= \exp\left(p^j \log\left(L_0 \varepsilon^p (\log\langle t\rangle)^{-1} (\log(t/2))^{p/(p-1)}\right)\right) \log\langle t\rangle (\log(t/2))^{-1/(p-1)}$$

for $t \geqslant 2$ and any $j \geqslant j_1$.

For $t \geqslant 4$ the inequalities

$$\log\langle t\rangle \leqslant \log(2t) \leqslant 2 \log t \quad \text{and} \quad \log(t/2) \geqslant 2^{-1} \log t$$

hold true, so,

$$\mathscr{U}(t) \geqslant \exp\left(p^j \log\left(2^{-(2p-1)/(p-1)} L_0 \varepsilon^p (\log t)^{1/(p-1)}\right)\right) \log\langle t\rangle (\log(t/2))^{-1/(p-1)} \tag{34}$$

for $t \geqslant 4$ and any $j \geqslant j_1$. Let us denote $J(t, \varepsilon) \doteq 2^{-(2p-1)/(p-1)} L_0 \varepsilon^p (\log t)^{1/(p-1)}$. We can choose $\varepsilon_0 = \varepsilon_0(n, p, \gamma, \lambda_0, R, u_0, u_1)$ sufficiently small so that

$$\exp\left(2^{1-2p} L_0^{1-p} \varepsilon_0^{-p(p-1)}\right) \geqslant 4.$$

Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t > \exp\left(2^{2p-1} L_0^{1-p} \varepsilon^{-p(p-1)}\right)$ we get $t \geqslant 4$ and $J(t, \varepsilon) > 1$.

Therefore, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t > \exp\left(2^{2p-1} L_0^{1-p} \varepsilon^{-p(p-1)}\right)$ taking the limit as $j \to \infty$ in (34) we see that the lower bound for $\mathscr{U}(t)$ blows up; so, $\mathscr{U}(t)$ may not be finite. Thus, we proved that $\mathscr{U}(t)$ blows up in finite time and, furthermore, we have shown the upper bound estimate for the lifespan

$$T(\varepsilon) \leqslant \exp\left(2^{2p-1} L_0^{1-p} \varepsilon^{-p(p-1)}\right).$$

This completes the proof of Theorem 2.

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An Introduction to Barenblatt Solutions for Anisotropic p-Laplace Equations

Simone Ciani and Vincenzo Vespri

To celebrate the 60th genethliac of Massimo Cicognani and Michael Reissig.

Abstract We introduce Fundamental solutions of Barenblatt type for the equation

$$u_t = \sum_{i=1}^{N} \left(|u_{x_i}|^{p_i-2} u_{x_i} \right)_{x_i}, \qquad p_i > 2 \quad \forall i = 1, .., N, \qquad \text{on} \quad \Sigma_T = \mathbb{R}^N \times [0, T], \tag{1}$$

and we prove their importance for the regularity properties of the solutions.

Keywords Degenerate orthotropic parabolic equations · p-Laplace · Anisotropic · Barenblatt fundamental solution · Self-similarity

1 Introduction

Consider the Cauchy problem

$$\begin{cases} u_t = \operatorname{div} A(x, u, Du), & \text{in} \quad \Sigma_T = \mathbb{R}^N \times (0, T), \\ u(x, 0) = M\delta(x), \end{cases} \tag{2}$$

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where $M > 0$, initial datum is the Dirac function $\delta(x)$, the field $A : \Sigma_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is only measurable and has an anisotropic behavior

$$\begin{cases} A_i(x,s,z)z_i \geq \Lambda^*|z_i|^{p_i} \\ |A_i(x,s,z)| \leq \Lambda_*|z_i|^{p_i-1}, \end{cases} \tag{3}$$

for some constants $\Lambda^*, \Lambda_* > 0$ and $p_i > 2$ for any $i \in \{1, .., N\}$. We recall that when all p_is are greater than 2 the equation is called degenerate. In order to have the existence of solutions, we require the following monotonicity property to the field A:

$$[A(x,s,\xi) - A(x,s,\zeta)] \cdot [\xi - \zeta] > 0, \qquad \forall \quad \xi \neq \zeta \quad \text{in} \quad \mathbb{R}^N. \tag{4}$$

When $p_i \equiv p$ Eq. (2) is named the orthotropic p-Laplace, and has nevertheless a different behavior from the classic p-Laplace, as its principal part evolves in a way dictated only by the growth in the i-th direction. The problem (2) reflects the modeling of many materials that reveal different diffusion rates along different directions, such as liquid crystals, wood or earth's crust (see [26]). Moreover, as shown in [14] the solution to this equation have finite speed of propagation. Note that this is a more reasonable assumption than the usual infinite-speed typical of heat equation, for most of the physical phenomena.

1.1 The Open Problem of Regularity

The strong nonlinear character and in particular the anisotropy which is prescribed by Eq. (2) has proved to be a hard challenge from the regularity point of view. The main difference with standard non linear regularity theory is the growth (3) of the operator A, usually referred to as *non standard growth* (see [1, 5]). This opens the way to a new class of function spaces, called anisotropic Sobolev spaces (see next Section), and whose study is still open and challenging. Even in the elliptic case, the regularity theory for such equations requires a bound on the sparseness of the powers p_i. For instance in the general case the weak solution can be unbounded, as proved in [16, 20]. However, the boundedness of solutions was proved in [5] under the assumption that

$$\overline{p} < N, \quad \max\{p_1, .., p_N\} < \overline{p}^*, \tag{5}$$

where

$$\overline{p} := \left(\frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \right)^{-1}, \qquad \overline{p}^* := \frac{N\overline{p}}{N - \overline{p}}. \tag{6}$$

Regularity properties are proved only on strong assumptions on the regularity of the coefficients (see [15, 21, 22]). Even in the elliptic case, when the coefficients are rough, Hölder continuity remains still nowadays an open problem. Indeed, continuity conditioned to boundedness has been proved in [13] by means of intrinsic scaling method, but with a condition of stability on the exponents p_i which is only qualitative. Removability of singularities has been considered in [28]. We refer to [15] and [24] for a complete survey on the subject and related bibliography.

1.2 Aim of the Note

We will consider the homogeneous prototype problem

$$\begin{cases} u_t = \sum_{i=1}^N \left(|u_{x_i}|^{p_i - 2} u_{x_i} \right)_{x_i}, & \text{in} \quad \Sigma_T = \mathbb{R}^N \times (0, T), \\ u(x, 0) = \delta_o. \end{cases}$$

The purpose of this note is to show the importance of a Barenblatt Fundamental solution $\mathcal{B}$ to this equation, paralleling the construction of Fundamental solutions for the p-Laplace equation. We will show a fundamental connection between the previous equation and a particular Fokker-Planck equation, as proved for the porous medium equation by Carrillo and Toscani [7]. The achievement of such Fundamental solution would provide important tools for the study of regularity of parabolic anisotropic problems as (2). As we will see in the sequel, the problem is more delicate than in the isotropic case, because of the lack of radial solutions. In the isotropic case the adoption of radial symmetry brings the equation, set in a proper scale, to a solvable ODE. In the doubly nonlinear case, a non-explicit Barenblatt Fundamental solution has been found with this approach in [23], using a Leray-Schauder technique. Also in mathematical physics, the use of radial solution is usual. For instance this strategy can be used for the Navier Stokes equation (see [17]). In our case, as already stated, the anisotropy does not allow the use of radial solutions, and this fact compels us to look for new ideas.

2 Preliminaries

2.1 Self-Similar Fundamental solutions, Motivations and Historical Perspectives

The issue of finding Fundamental solutions to elliptic and parabolic equations is one of paramount importance in the study of linear elliptic and parabolic equations (see [11]). In nonlinear theory their role is not so evident, and yet the epithet

“Fundamental” is iconic, because representation in terms of kernels usually fails. But they are a tool of extraordinary importance in the existence and regularity theory as well as very important to describe the asymptotic behaviour, that’s why the name Fundamental Solutions is deserved. Much more information about techniques to be employed, sharp-condition examples and counterexamples can be extracted from the knowledge of a Fundamental solution. A typical example is the Barenblatt Fundamental Solution

$$\mathcal{B}(x,t) = t^{-\frac{N}{\lambda}} \left\{ 1 - \gamma_p \left(\frac{|x|}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}}, \quad t > 0,$$

for the p-Laplace equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad \text{in} \quad [0,T] \times \mathbb{R}^N, \quad p > 1. \tag{7}$$

These special solutions can be used to reveal a gap between the elliptic theory and the corresponding parabolic one for p-Laplace type equations. Indeed solutions to

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad u \in W^{1,p}_{loc}(\Omega), \quad p > 1, \tag{8}$$

do obey to a Harnack inequality (see [27]), while the corresponding solutions to the parabolic version of (8) do not in general. We show this briefly. Let (x_0, t_0) be a point of the boundary of the support of $\mathcal{B}$, the free boundary $\{t = |x|^{\lambda}\}$, and let $\rho > 0$. The ball $B_\rho(x_0)$ intersects at the time level $t_0 - \rho^p$ the support of $x \to \mathcal{B}(x, t_0 - \rho^p)$ in an open set, hence

$$\mathcal{B}(x_0, t_0) = 0, \quad \text{but} \quad \sup_{B_\rho(x_0)} \mathcal{B}(x, t_0 - \rho^p) > 0.$$

Generalizing the classical heat equation to nonlinear versions, another chief example in evolution theories is the Porous Medium Equation

$$u_t - \Delta(u^m) = 0, \quad m > 1. \tag{9}$$

This equation, introduced in the last century in connection with a number of physical applications, has been extensively studied (see the monograph [31]) in parallel to the p-Laplace as another prototype of nonlinear diffusive evolution equation, with interest also in the geometry of free boundaries. Fundamental solutions were discovered in 1950’s by Zeldovich and Kompanyeets in [32] and Barenblatt [2], and later a complete description has been brought by Pattle in [25]. The discovery of these explicit solutions, usually called Barenblatt solutions since then, has been the starting point of the rigorous mathematical theory that has been gradually developed since then.

The surprising relation between existence and uniqueness of Fundamental solutions and precise asymptotic behaviour relies on the existence of a scaling group under whose action the solutions to the equation are invariant. This implies that a Fundamental solution is self-similar: this is what we call a *Barenblatt solution*. Self-similarity has big relevance for the understanding of Fundamental processes in mathematics and physics, as described in [4]. Self-similar phenomena got in mathematical physics quite early, perhaps with the famous work of Fourier in 1822 on the analytical theory of heat conduction. In this memoir he performed a construction of a *source-type* solution

$$u(x,t) = \frac{A}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right), \quad \text{for} \quad f(\zeta) = e^{-\frac{\zeta^2}{A}},\ A > 0,$$

to the heat conduction equation

$$u_t = \Delta u. \tag{10}$$

Subsequently the phenomena under consideration and their mathematical models became increasingly complicated and very often nonlinear. To obtain self-similar solutions was considered a success in the pre-computer era. Indeed, the construction of such solutions always reduces the problem to solving the boundary value problems for an ODE, which is a substantial simplification, as we will see in [3]. Furthermore, in 'self-similar' coordinates (as $u\sqrt{t}$, $x/\sqrt{t}$ for (10)), self-similar phenomena become time independent. This enlightens a certain type of stabilization. Thus during the pre-computer era, the achievement of a self-similar solution was the only way to understand the qualitative features of the phenomena, and the exponents of the independent variables x, t in self-similar variables were obtained often by dimensional analysis. Dimensional analysis is merely a simple sequence of rules based on the Fundamental covariance principle of physics: all physical laws can be represented in a form which is equally valid for all observers.

The very idea of self-similarity is connected with the *group of transformations* of solutions (see [3]). These groups are already present in the differential equations of the process and are determined by the dimensions of the variables appearing in them: the transformations of the units of time, length, mass, etc. are the simplest examples. This kind of self-similarity is obtained by power laws with exponents that are simple fractions defined in an elementary way from dimensional considerations. Such a course of argument has led to results of immense and permanent importance, as the theory of turbulence and the Reynolds number, of linear and nonlinear heat propagation from a point source, and of a point explosion. Moreover it has enlightened the way toward to a nonlinear theory developed by DiBenedetto [10] with the nowadays well-known method of intrinsic scaling (see also [29]).

The Group of Transformations for the p-Laplace Equation

Let us examine the group of transformations under scaling of the p-Laplace equation

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

We apply the following dilation in all variables

$$u' = Ku, \qquad x' = Lx, \qquad t' = Tt,$$

and impose that the function u' so defined

$$u'(x', t') = Ku\left(\frac{x'}{L}, \frac{t'}{T}\right), \tag{11}$$

is again a solution to the p-Laplace equation above. Then by the simple calculations

$$u_{t'} = \frac{K}{T}u_t\left(\frac{x'}{L}, \frac{t'}{T}\right), \quad |\nabla_{x'}u'| = \frac{K}{L}|\nabla u|$$

we arrive to the conclusion that u' is a solution to the p-Laplace equation if and only if

$$TK^{p-2} = L^p.$$

So we obtain a two-parametric transformation group $\mathcal{T}(L, T)$ acting on the set of solutions of the p-Laplace equation:

$$(\mathcal{T}u)(x, t) = \left(\frac{L^p}{T}\right)^{\frac{1}{p-2}} u\left(\frac{x}{L}, \frac{t}{T}\right). \tag{12}$$

and we can conclude the following Lemma.

Lemma 1 *If u is a solution to the p-Laplace equation in a certain class of solutions $\mathcal{S}$ which is closed under dilation in x, t, u, then $(\mathcal{T}u)$ given by (12) is again a solution to the equation in the same class $\mathcal{S}$.*

Those special solutions that are themselves invariant under the scaling group are called *self similar-solutions*: this means that $(\mathcal{T}u)(x, t) = u(x, t)$ for all (x, t) in the domain of definition, which has to be itself scale-invariant.

Suppose now that we have an important information, such as (27) or conservation of mass. We want to use some of the free parameters to force $\mathcal{T}$ to preserve this important behaviour of the orbit. Analytically it consists in imposing a new relation between two independent parameters, as K and L for instance, and in reducing the transformation to a one-parameter family of scaled functions. Thus we set

$$K = L^{-\chi}, \tag{13}$$

and consequently

$$K = T^{-\alpha}, \qquad L = T^{\beta},$$

with α, β, χ linked by conserving the equation:

$$\alpha(p,\chi) = \frac{\chi}{\chi(p-2)+2}, \qquad \beta(p,\chi) = \frac{1}{\chi(p-2)+2}, \quad \text{unless} \quad \chi = \frac{-2}{(p-2)}.$$

Observing that $\chi = \alpha/\beta$, the equation changes into

$$(\mathcal{T}u)(x,t) = T^{-\alpha}u(x/T^{\beta}, t/T), \tag{14}$$

where α, β are linked by $\alpha(p-2)+\beta = 1$. The condition of preserving the initial mass is

$$\int_{\mathbb{R}^N} K u_0\left(\frac{x}{L}\right)dx = \int_{\mathbb{R}^N} (\mathcal{T}u_0)(x)dx = \int_{\mathbb{R}^N} u_0(x)dx \tag{15}$$

which obliges $KL^N = 1$, so that the one parameter family $\mathcal{T}$ will be given by

$$\alpha = \frac{N}{N(p-2)+2}, \qquad \beta = \frac{1}{N(p-2)+2}, \quad p > 2. \tag{16}$$

Observe the formula for the transformation of the initial data (which obviously must satisfy the same transformation) must be

$$(\mathcal{T}u_0)(x) = T^{-\frac{N}{\lambda}}u_0\left(\frac{x}{T^{\frac{1}{\lambda}}}\right), \qquad \lambda = N(p-2)+p. \tag{17}$$

In the case of Barenblatt Fundamental solution (24) the couple (x,t) is fixed as a single variable so that

$$u(x,t) = t^{-\alpha}u(xt^{-\beta}, 1) = t^{-\alpha}F(xt^{-\beta}), \tag{18}$$

where $F(\eta) = u(\eta, 1)$ is the *profile* of the solution.

Remark 1 A complete theory of existence and uniqueness for the main equation would allow us to obtain self-similar solutions almost for free. Indeed we can consider the solution to the Cauchy problem for scale invariant data, and then use uniqueness to show that this must be self-similar. Let the initial data for instance be of the form

$$u'(x) = \frac{G(\xi)}{|x|^{\chi}}, \quad \chi \in \mathbb{R}, \; \xi = \frac{x}{|x|}, \quad \text{and} \quad G : \mathbb{S}^{N-1} \to \mathbb{R}.$$

Let us suppose that we are able to solve with uniqueness the Cauchy problem for our equation with this initial data, say the solution is u. We produce another solution to the same equation by $\mathcal{T}(u)$ given by (11) and if $K = L^{-\chi}$ then the transformed initial data is the same one:

$$(\mathcal{T}u)(x,0) = KG(\xi)\left|\frac{x}{L}\right|^{-\chi} = u(x,0)$$

and so u and $\mathcal{T}(u)$ solve the same Cauchy problem and u is self-similar.

2.2 Notation and Settings

Given $\mathbf{p} := (p_1, .., p_N)$, $\mathbf{p} > 1$ with the usual meaning, we assume that the harmonic mean is smaller than the dimension of the space variables

$$\overline{p} := \left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}\right)^{-1} < N, \tag{19}$$

and we define the Sobolev exponent of the harmonic mean $\overline{p}$,

$$\overline{p}^* := \frac{N\overline{p}}{N-\overline{p}}. \tag{20}$$

We will suppose without loss of generality along this note that the p_is are ordered increasingly. Next we introduce the natural parabolic anisotropic spaces. Given $T > 0$ and a bounded open set $\Omega \subset \mathbb{R}$ we define

$$W_o^{1,\mathbf{p}}(\Omega) := \{u \in W_o^{1,1}(\Omega) |\, D_i u \in L^{p_i}(\Omega)\}$$

$$W_{loc}^{1,\mathbf{p}}(\Omega) := \{u \in L_{loc}^{1}(\Omega) |\, D_i u \in L_{loc}^{p_i}(\Omega)\}$$

$$L^{\mathbf{p}}(0,T; W_o^{1,\mathbf{p}}(\Omega)) := \{u \in L^1(0,T; W_o^{1,1}(\Omega)) |\, D_i u \in L^{p_i}(0,T; L_{loc}^{p_i}(\Omega))\}$$

$$L_{loc}^{\mathbf{p}}(0,T; W_o^{1,\mathbf{p}}(\Omega)) := \{u \in L_{loc}^1(0,T; W_o^{1,1}(\Omega)) |\, D_i u \in L_{loc}^{p_i}(0,T; L_{loc}^{p_i}(\Omega))\}$$

Now let A be a measurable vector field satisfying the growth conditions (3). By a *local weak solution* of

$$u_t = \operatorname{div} A(x, u, Du), \quad (x, t) \in \Sigma_T,$$

we understand a function $u \in C^0_{loc}(0, T; L^2_{loc}(\mathbb{R}^N)) \cap L^{\mathbf{p}}_{loc}(0, T; W^{1,\mathbf{p}}(\mathbb{R}^N))$ such that for all $0 < t_1 < t_2 < T$ and any test function $\varphi \in C^\infty_{loc}(0, T; C^\infty_o(\mathbb{R}^N))$ satisfies

$$\int u\varphi\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int (-u\, \varphi_t + A(x, u, Du) \cdot D\varphi)\, dxdt = 0, \tag{21}$$

where the integral is assumed to be in $\mathbb{R}^N$ when no domain has been specified. By a density and approximation argument this actually holds for any test function of the kind $\varphi \in W^{1,2}_{loc}(0, T; L^2_{loc}(\mathbb{R}^n)) \cap L^{\mathbf{p}}_{loc}(0, T; W^{1,\mathbf{p}}_o(\Omega))$ for any semirectangular domain $\Omega \subset\subset \mathbb{R}^N$ (see [18] for a discussion on anisotropic embeddings and semirectangular domains).

Remark 2 We further give the definition of solution to the prototype equation (1) with L^1 initial data, to be used during the development of our work.

A measurable function $(x, t) \to u(x, t)$ defined in Σ_T is a *weak solution* to the Cauchy Problem (2) *with L^1 initial data* if for every bounded open set $\Omega \subset \mathbb{R}$, if

$$u \in C(0, T; L^1(\Omega)) \cap L^{\mathbf{p}}(0, T; W^{1,\mathbf{p}}(\Omega)), \quad \text{and}$$

$$\begin{aligned}\int_\Omega u(x, t)\varphi(x, t)dx + \int_0^t \int_\Omega \{-u\varphi_t + \sum_{i=1}^N |D_i u|^{p_i-2} D_i u D_i \varphi\} dx d\tau \\ = \int_\Omega u_0(x)\varphi(x, 0)dx,\end{aligned} \tag{22}$$

for all $0 < t < T$ and all test functions $\varphi \in C^\infty(0, T; C^\infty_o(\Omega))$.
Weak subsolutions (resp. supersolutions) are defined as above except that in (22) equality is replaced by $\leq$ (resp. $\geq$) and test functions $\varphi \geq 0$ are taken to be nonnegative.

3 A Self-Similar Solution to the p-Laplace Equation

Consider the equation

$$\begin{cases} u \in C_{loc}(0, T; L^2_{loc}(\mathbb{R}^N)) \cap L^p_{loc}(0, T; W^{1,p}_{loc}(\mathbb{R}^N)), \\ u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad \text{in} \quad \Sigma_T = \mathbb{R}^N \times (0, T). \end{cases} \tag{23}$$

In this case we recover the classic p-Laplace equation, and we can write explicitly its self-similarity source-solution since the work of Barenblatt [2] as

$$\mathcal{B}(x,t) = t^{-\frac{N}{\lambda}} \left\{ 1 - \gamma_p \left(\frac{|x|}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}}, \quad t > 0 \tag{24}$$

with

$$\lambda = N(p-2) + p, \qquad \gamma_p = \left(\frac{1}{\lambda} \right)^{\frac{1}{p-1}} \frac{p-2}{p}. \tag{25}$$

We observe that $\mathcal{B}$ satisfies the self-similar transformation (18). This function $\mathcal{B}$ solves the Cauchy problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, & \text{in} \quad \mathbb{R}^N \times (0,\infty), \\ \mathcal{B}(\cdot, 0) = M\delta_o, \end{cases} \tag{26}$$

where δ_o is the Dirac measure concentrated at the origin and for every $t > 0$ the mass $M = \|\mathcal{B}(\cdot,t)\|_{L^1(\mathbb{R}^N)}$ is conserved. The initial datum is taken in the sense of measures, which is, for every $\varphi \in C_o(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \mathcal{B}(x,t)\varphi\, dx \to M\varphi(0), \quad \text{as} \quad t \downarrow 0.$$

For $t > 0$ and every $\rho > 0$ we have the important bound

$$\|\mathcal{B}(\cdot,t)\|_{L^\infty(K_\rho)} = t^{-\frac{N}{\lambda}}, \tag{27}$$

being K_ρ the cube of edge ρ. The explicit function $\mathcal{B}$ is classically named Fundamental solution in literature, because it converges pointwise in Σ_T to the heat kernel $\Gamma(x,t)$ when p approaches 2,

$$\mathcal{B}(x,t) \to (4\pi)^{N/2}\Gamma(x,t) = \frac{1}{t^{N/2}} e^{-\frac{|x|^2}{4t}}, \quad \text{if} \quad p \downarrow 2,$$

but the name does not refer to the kernel property i.e. solutions to (23) are not representable as convolutions of $\mathcal{B}$ with initial data. Nevertheless all non-negative solutions to (23) behave as $t \downarrow 0$ like the Fundamental solution $\mathcal{B}$, and as $|x| \to \infty$ they grow no faster than $|x|^{p/(p-2)}$. Barenblatt Fundamental solutions $\mathcal{B}$ are useful, together with the comparison principle, for proving an intrinsic Harnack estimate (see further Sect. 5), uniqueness in existence with L^1 data (as in [19]), and more generally to understand the behavior of solutions from the point of view of the physics. In this way, a suitable revisiting of the linear theory had been shaped to face nonlinear equations as the p-Laplace. It is possible to build Barenblatt Fundamental solutions centered in $\bar{x}$ with initial datum at a time $\bar{t}$ in the following way

$$\mathcal{B}_{k,\rho}(x,t,\bar{x},\bar{t}) = \frac{k\rho^N}{S^{\frac{N}{\lambda}}(t)} \left\{ 1 - \left(\frac{|x-\bar{x}|}{S^{\frac{1}{\lambda}}(t)} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}}, \qquad \lambda = N(p-2)+p, \tag{28}$$

with

$$S(t) = \lambda \left(\frac{p}{p-2} \right)^{p-1} k^{p-2} \rho^{N(p-2)} (t-\bar{t}) + \rho^\lambda. \tag{29}$$

These functions enjoy the following important properties.

1. They are weak solutions to (23) in $\mathbb{R}^N \times \{t > \bar{t}\}$.
2. If we fix $t = \bar{t}$ then $B_{k,\rho} \equiv 0$ for all $x \in \left(\mathbb{R}^N - B_\rho(\bar{x}) \right)$ and for $t > \bar{t}$ the function $x \to B_{k,\rho}$ vanishes, in a C^1 fashion, across the boundary of the ball $\{|x-\bar{x}| < S^{\frac{1}{\lambda}}(t)\}$.
   Their support evolves compactly:
   $$\text{supp}\left(B_{k,\rho}(x,t,\bar{x},\bar{t}) \right) = \left\{ |x-\bar{x}| \le S^{\frac{1}{\lambda}}(t) \right\} \times [\bar{t}, t^*], \tag{30}$$
   thus
   $$\text{supp}\left(B_{k,\rho}(x,t,\bar{x},\bar{t}) \right) \subseteq B_{S^{1/\lambda}(t^*)}(\bar{x}) \times [\bar{t}, t^*]. \tag{31}$$
3. They are bounded for fixed ρ and $k \in \mathbb{R}^+$:
   $$B_{k,\rho}(x,t,\bar{x},\bar{t}) \le k, \qquad x \in \mathbb{R}^N. \tag{32}$$

In the sequel when no explicit formula for a solution as (28) (as in (1)), we will refer to a Barenblatt Fundamental Solution as a function (resp. to (1)) satisfying properties analogous to 1–3 above.

3.1 The Construction of $\mathcal{B}$: Reduction to an Isotropic Fokker-Planck Equation

As far as we know if we look for a Barenblatt Fundamental solution as $\mathcal{B}$, we have to impose the condition (27), because this is the behaviour that non-negative solutions to the p-Laplace Cauchy problem with the right decay of the initial datum do satisfy (see [10] Theorem 4.5). This motivates us to apply the following (formal) transformations to Eq. (23) and

$$\begin{cases} u(x,t) = t^{-\frac{N}{\lambda}} v(xt^{\alpha},t) = v(y,t), \\ y = xt^{\alpha}, \qquad \alpha = -\frac{1}{\lambda}, \end{cases} \Rightarrow \begin{cases} u_x = t^{-\frac{N}{\lambda}} v_y y_x = t^{\alpha-\frac{N}{\lambda}} v_y, \\ \frac{\partial}{\partial x} = t^{\alpha}\frac{\partial}{\partial y}. \end{cases} \tag{33}$$

Remark 3 We notice that the applied transformation does not belong to the group of transformations (12), so we expect that Eq. (23) turns into another one. This is what is called in [30] the continuous rescaling: as the change of variables (33) belongs to the transformation group only for the fixed time $t = 1$, source-type solutions transform into stationary profiles of the transformed equation.

By direct calculation we obtain

$$u_t = -\frac{N}{\lambda} t^{-\frac{N}{\lambda}-1} v + t^{-\frac{N}{\lambda}} \left[\sum_{i=1}^{N} v_{y_i} (y_i)_t + v_t \right] =$$
$$-\frac{N}{\lambda} t^{-\frac{N}{\lambda}-1} v + t^{-\frac{N}{\lambda}} \left[\nabla_y v \cdot \frac{\alpha y}{t} + v_t \right]$$

and

$$\nabla_x u = t^{\alpha-\frac{N}{\lambda}} \nabla_y v. \tag{34}$$

We set

$$\tilde{v}(y,\tilde{t}) = \tilde{v}(y, \ln(t)) = v(y,t), \quad \Rightarrow \quad \tilde{v}_t = \tilde{v}_{\tilde{t}} t^{-1} = v_t \tag{35}$$

and Eq. (23) becomes, by multiplying it for $t^{\frac{N}{\lambda}+1}$

$$\tilde{v}_{\tilde{t}} = \frac{N}{\lambda} v - \frac{N}{\lambda} \nabla_y \tilde{v} \cdot y + t^{\alpha} \nabla_y \cdot \left[t^{(\alpha-\frac{N}{\lambda})(p-1)} |\nabla_y \tilde{v}|^{p-2} \nabla_y \tilde{v} \right] t^{\frac{N}{\lambda}+1} =$$
$$\frac{N}{\lambda} v - \frac{N}{\lambda} \nabla_y \tilde{v} \cdot y + \nabla_y \cdot \left[|\nabla_y \tilde{v}|^{p-2} \nabla_y \tilde{v} \right] t^{\alpha+(\alpha-\frac{N}{\lambda})(p-1)+\frac{N}{\lambda}+1} =$$
$$\frac{N}{\lambda} v - \frac{N}{\lambda} \nabla_y \tilde{v} \cdot y + \nabla_y \cdot \left[|\nabla_y \tilde{v}|^{p-2} \nabla_y \tilde{v} \right],$$

being $\alpha = -\frac{1}{\lambda}$. So we obtain the isotropic Fokker-Planck equation

$$\tilde{v}_{\tilde{t}} = \nabla_y \cdot \left(|\nabla_y \tilde{v}|^{p-2} \nabla_y \tilde{v} + \frac{y\tilde{v}}{\lambda} \right). \tag{36}$$

3.2 Barenblatt Solution Solves the Isotropic Fokker Planck Equation

Consider the Barenblatt function $\mathcal{B}(x,t)$, with explicitly scaled space variables

$$\mathcal{B}(x,t) = t^{-\frac{N}{\lambda}} \left\{ 1 - \gamma_p \left(\sqrt{\sum_{i=1}^{N} \left(\frac{x_i}{t^{\frac{1}{\lambda}}} \right)^2} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}}. \tag{37}$$

We claim that $\mathcal{B}$ solves the stationary version of (36), by taking the flux to be zero, i.e.

$$|\nabla_y \tilde{v}|^{p-2} \nabla_y \tilde{v} + \frac{y\tilde{v}}{\lambda} = 0.$$

We have, by setting $y_i = x_i t^{-\frac{1}{\lambda}}$, that

$$\mathcal{B}(y,t) = t^{-\frac{N}{\lambda}} \left\{ 1 - \gamma_p |y|^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}} = t^{-\frac{N}{\lambda}} \left\{ 1 - \gamma_p \left(\frac{\sqrt{\sum_{i=1}^{N} x_i^2}}{t^{\frac{1}{\lambda}}} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}} = \mathcal{B}(x,t)$$

and thus the function

$$\mathcal{C}(y,t) = \left\{ 1 - \gamma_p |y|^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}}$$

is independent from t, and

$$\begin{aligned} \nabla_y \mathcal{C} = &- \gamma_p \left(\frac{p}{p-2} \right) \left\{ 1 - \gamma_p |y|^{\frac{p}{p-1}} \right\}_+^{\frac{1}{p-2}} |y|^{\frac{2-p}{p-1}} y = \\ &= -\gamma_p \left(\frac{p}{p-2} \right) \mathcal{C}^{\frac{1}{p-1}} |y|^{\frac{2-p}{p-1}} y. \end{aligned}$$

Thus by calculation we have that $\mathcal{C}(y) = t^{\frac{N}{\lambda}} \mathcal{B}(y,t)$ solves the zero flux equation

$$\begin{aligned} &|\nabla_y \mathcal{C}|^{p-2} \nabla_y \mathcal{C} + \frac{y\mathcal{C}}{\lambda} = \\ &\quad \left[\gamma_p \left(\frac{p}{p-2} \right) \mathcal{C}^{\frac{1}{p-1}} \right]^{p-2} |y|^{\frac{2-p}{p-1}(p-2)} |y|^{p-2} \left[-\gamma_p \left(\frac{p}{p-1} \right) \mathcal{C}^{\frac{1}{p-1}} |y|^{\frac{2-p}{p-1}} y \right] + \frac{y\mathcal{C}}{\lambda} = \\ &\quad \mathcal{C} \left[\frac{1}{\lambda} - \gamma_p \left(\frac{p}{p-2} \right)^{p-1} \right] y = 0, \quad \text{for} \quad \gamma = \left(\frac{p-2}{p} \right)^{p-1} \frac{1}{\lambda}. \end{aligned}$$

Consequently, so does $\mathcal{B}(x, t)$. Now we show that the converse reasoning holds too, in order to show how the whole calculation is in fact reduced to a ODE solution.

3.3 Function $\mathcal{C}$ Solves a Particular ODE

Consider

$$\mathcal{C}(\eta) = \left\{1 - \gamma_p \eta^{\frac{p}{p-1}}\right\}_+^{\frac{p-1}{p-2}} = \mathcal{C}(|y|), \quad \eta > 0. \tag{38}$$

In $0 \le \eta < \left(\frac{1}{\gamma_p}\right)^{\frac{p-1}{p}}$ we have

$$\mathcal{C}(\eta)^{\frac{p-2}{p-1}} = 1 - \gamma_p \eta^{\frac{p}{p-1}}.$$

We derive the equation to obtain

$$\left(\frac{p-1}{p-2}\right)\mathcal{C}(\eta)^{-\frac{1}{p-1}}\mathcal{C}'(\eta)d\eta = -\left(\frac{p-2}{p}\right)\frac{1}{\lambda^{1/(p-1)}}\left(\frac{p}{p-1}\right)\eta^{\frac{1}{p-1}}d\eta.$$

Now, we manipulate the equation with $\mathcal{C}'(\eta) \le 0$, because

$$\mathcal{C}'(\eta) = \left(\frac{p-1}{p-2}\right)\left\{1 - \gamma_p \eta^{\frac{p}{p-1}}\right\}_+^{-\frac{1}{p-2}}\left(-\gamma\left(\frac{p}{p-1}\right)\eta^{\frac{1}{p-1}}\right) \le 0$$

so that

$$\left(\frac{(-\mathcal{C}'(\eta))^{p-1}}{\mathcal{C}(\eta)}\right)^{\frac{1}{p-1}} = \left(\frac{\eta}{\lambda}\right)^{\frac{1}{p-1}}$$

and so the desired mono-dimensional Fokker-Planck equation is obtained

$$|\mathcal{C}'(\eta)|^{p-2}\mathcal{C}'(\eta) + \frac{\eta\mathcal{C}(\eta)}{\lambda} = 0. \tag{39}$$

If one reads conversely from the end to the beginning of these calculations, it is clear how to arrive to a solution to the isotropic Fokker Planck equation (36) by imposing radial symmetry.

4 Solving the Isotropic Cauchy Problem with Measure Data

Suppose now that we are not able to solve by radial symmetry the isotropic Fokker-Planck equation (36). If we look for a solution to (26) that exhibits the properties (30)–(32), we may adopt the following strategy. First we find a general solution u to (26) with datum the Dirac measure δ_o, we show that it is positive by the maximum principle, and then we use the transformation (33) to get a solution w to (36). Observe that a comparison principle for subsolutions to the p-Laplace equation can be transported to a comparison principle for subsolutions to the isotropic Fokker-Planck equation. But we need a solution to the stationary Fokker-Planck equation to recover the self-similarity (see Remark 3), so that we can control the behavior for all times by scaling, and we gain for free the correct evolution of its support. More generally speaking, if the initial data in (22) is given by

$$u_0(\cdot, 0) = \mu, \tag{40}$$

where μ is a σ—finite Borel measure in $\mathbb{R}^N$, then we say that u is a *weak solution* of (22) *with measura data* if for every bounded open set $\Omega \subset \mathbb{R}^N$ and $\forall t \in (0, T)$, u satisfies the above integral equality (22) with the right-hand side replaced by

$$\int_\Omega \varphi(x, 0) d\mu,$$

$\forall \varphi \in C^1(\overline{\Omega_T})$ such that $x \to \varphi(x, t)$ is compactly supported in $\Omega\, \forall t \in [0, T]$.

In the pioneering work [12] for the isotropic p-Laplace, the authors consider a way of measuring the growth of a function $f \in L^1_{loc}(\mathbb{R}^N)$ as $|x| \to \infty$ by setting

$$|\|f\||_r := \sup_{\rho \geq r} \rho^{-\lambda/(p-2)} \int_{B_\rho} |f| dx, \quad r > 0, \quad \lambda = N(p-2) + p.$$

Note that if $f \in L^1(\mathbb{R}^N)$ then $|\|f\||_r < \infty$, $\forall r > 0$. Similarly, if μ is a σ-finite Borel measure in $\mathbb{R}^N$, we set

$$|\|\mu\||_r := \sup_{\rho \geq r} \rho^{-\lambda/(p-2)} \int_{B_\rho} |d\mu|,$$

where $|d\mu|$ is the variation of μ.

In that Fundamental work, the authors demonstrate the existence of a weak solution to the problem (22) in its isotropic configuration, within $\Sigma_T = \Sigma_T(\mu)$, where

$$T(\mu) = \begin{cases} C_0(N, p)\Big[\lim_{r\to\infty} |\|\mu\||_r \Big]^{(2-p)}, & \text{if} \quad \lim_{r\to\infty} |\|\mu\||_r > 0 \\ +\infty & \text{if} \quad \lim_{r\to\infty} |\|\mu\||_r = 0. \end{cases} \tag{41}$$

So the existence is proved in a cylindrical domain whose last time T is dictated by the behavior at infinity of the initial measure μ. The method relies on suitable estimates and compactness, which permit a standard limiting process. Indeed, given a σ—finite Borel measure μ in $\mathbb{R}^N$ satisfying $|\|\mu\||_r < \infty$ for some $r > 0$, there exists a sequence of regular functions $\{u_{0,n}\}_{n\in\mathbb{N}} \in C_o^\infty(\mathbb{R}^N)$ such that $\forall\varphi \in C_o(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} u_{0,n}\varphi\, dx \to \int_{\mathbb{R}^N} \varphi d\mu, \quad \& \quad |\|u_{0,n}\||_r \to |\|\mu\||_r, \quad r > 0.$$

The Cauchy Problem

$$\begin{cases} u_t - \operatorname{div}(|Du|^{p-2}Du) = 0 \quad \text{in} \quad \Sigma_T, \quad p > 2, \\ u(\cdot, 0) = u_{0,n}. \end{cases} \tag{42}$$

has a unique solution u_n, global in time (see [6]). Next, the authors prove the following estimates, for all $0 < t < T_r(\mu) := C_0[\||\mu\||_r]^{(2-p)}$, $\forall\rho \geq r > 0$:

$$\||u(\cdot, t)\||_r \leq C_1(N, p)\||\mu\||_r, \tag{43}$$

$$\|u(\cdot, t)||_{L^\infty(B_\rho)} \leq C_2(N, p)t^{-N/\lambda}\rho^{p/(p-2)}\||\mu\||_r^{p/\lambda}, \tag{44}$$

$$\|Du(\cdot, t)||_{L^\infty(B_\rho)} \leq C_3(N, p)t^{-(N+1)/\lambda}\rho^{2/(p-2)}\||\mu\||_r^{2/\lambda}, \tag{45}$$

$$\int_0^t \int_\Omega |Du|^q dxd\tau \leq C_4(N, P, \epsilon, \operatorname{diam}\Omega)\, \||\mu\||_r^{C_5(N,p,\epsilon)}, \quad q = p - (N+\epsilon)/(N+1), \tag{46}$$

and in particular with $\epsilon = 1$ we obtain

$$\int_0^t \int_{B_\rho} |Du|^{p-1}dxd\tau \leq C_5(N.p)t^{1/\lambda}\rho^{1+\lambda/(p-2)}\, \||\mu\||_r^{1+(p-2)/\lambda} \tag{47}$$

Moreover the function $(x, t) \to Du(x, t)$ is Hölder continuous in $\overline{\Omega} \times [\eta, T(\mu) - \eta]$, $0 < \eta < T(\mu)$, with Hölder constants and exponents depending upon $N, p, C_1, .., C_4$, $\operatorname{diam}\Omega, \eta, |\|\mu\||_r$. It can be shown that their estimates are sharp, by means of Barenblatt solutions. Finally, the estimates above (43)–(45) with a monotonicity property as (4), permit to pass to the limit in the approximating problems (42).

5 An Application of $\mathcal{B}$ to Intrinsic Harnack Estimates

In this section we outline the importance of the construction of a Barenblatt Fundamental solution for the aim of proving regularity. Indeed the rough idea is that once that we have a solution of (23) whose support and positivity can be easily manipulated, by means of a comparison argument is possible to expand the positivity set of a whatever solution that is bigger than the Fundamental one in the parabolic boundary. More precisely we will review the proof of the following Theorem of [10].

Theorem 1 *Let u be a non-negative weak solution of Eq.* (23) *in $\Omega_T = \Omega \times [0, T]$ where $\Omega \subset \mathbb{R}^N$ bounded open set. Fix a point $(x_0, t_0) \in \Omega_T$ and assume $u(x_0, t_0) > 0$. There exist constants $\gamma > 1$ and $C > 1$, depending only on N, p, such that*

$$u(x_0, t_0) \leq \gamma \inf_{B_\rho(x_0)} u(\cdot, t_0 + \theta), \quad \theta = \frac{C\rho^p}{[u(x_0, t_0)]^{p-2}}, \tag{48}$$

provided the cylinder

$$Q_{4\rho}(\theta) = \{|x - x_0| < 4\rho\} \times \{t_0 - 4\theta, t_0 + 4\theta\} \tag{49}$$

is contained in Ω_T.

Remark 4 As we can see, the geometry is intrinsically defined by the value of the solution in (x_0, t_0). This brings to light a difficulty in exposition, as a priori weak solutions to (23) are not meant to be well defined in every point. Nonetheless by standard regularity theory we know that local weak solutions to (23) are locally Hölder continuous, and so they are well defined pointwise as elements of $C(0, T; W^{1,p}_{loc}(\Omega))$.

Remark 5 The constants γ and C in previous Theorem tend to infinity as p tend to infinity, but they are stable as $p \downarrow 2$ in the following meaning

$$\lim_{p\downarrow 2} \gamma(N, p) = \gamma(N, p), \quad \text{and} \quad \lim_{p\downarrow 2} C(N, p) = C(N, p). \tag{50}$$

5.1 Outline of the Proof of Theorem 1

For the sake of conciseness ad to the aim of highlighting the importance of Barenblatt Fundamental solutions, we will demonstrate only the case when p is not too close to 2. The proof for $p \in (2, 5/2]$ uses local comparison functions built especially to do the same job of $\mathcal{B}$, being subsolutions of (23) and observing the same ordering imposed by the following Lemma.

Lemma 2 *Let u, v be two solutions of (23) in $\Omega_T = \Omega \times [0, T]$ such that $u, v \in C(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)) \cap C(\overline{\Omega_T})$. If $u \geq v$ in the parabolic boundary of Ω_T, then $u \geq v$ in Ω_T.*

STEP 1. *Transforming the problem by scaling.*
Let $(x_0, t_0) \in \Omega_T$, $\rho > 0$ to be fixed a posteriori, assume that $u(x_0, t_0) > 0$ and for a constant C to be determined later let $Q_{4\rho}$ be the box

$$Q_{4\rho} = \{|x - x_0| < 4\rho\} \times \left\{t_0 - \frac{4C\rho^p}{[u(x_0, t_0)]^{p-2}}, t_0 + \frac{4C\rho^p}{[u(x_0, t_0)]^{p-2}}\right\}. \tag{51}$$

Now introduce the change of variables

$$\Phi(x, t) = \left(\frac{x - x_0}{\rho}, \frac{(t - t_0)[u(x_o, t_0)]^{p-2}}{\rho^p}\right), \qquad \Phi(Q_{4\rho}) = B_4 \times (-4C, 4C) =: Q \tag{52}$$

Let us denote again with x, t the new variables $\Phi(x, t)$, and observe that the function

$$v(x, t) = \frac{1}{u(x_0, t_0)} u\left(x_0 + \rho x, \frac{t\rho^p}{[u(x_0, t_0]^{p-2}}\right), \tag{53}$$

is a bounded non-negative solution to the Cauchy problem

$$\begin{cases} v_t - \operatorname{div}(|Dv|^{p-2} Dv) = 0, & (x, t) \in Q \\ v(0, 0) = 1. \end{cases} \tag{54}$$

Theorem 1 will be proved, as shown by a simple converse rescaling, if we are able to find constants $\gamma_o \in (0, 1]$, $C > 1$ depending only upon N, p holding the inequality

$$\inf_{B_1} v(x, C) \geq \gamma_o. \tag{55}$$

The constant γ_o defined successively in (62) tends to zero as $p \downarrow 2$.
STEP 2. *Finding qualitatively a point where v equals a power-like function of time.*
We consider the family of nested and expanding boxes

$$Q_\tau = \{|x| < \tau\} \times (-\tau^p, 0], \qquad \tau \in (0, 1] \tag{56}$$

and for each of these boxes we consider the numbers

$$M_\tau = \sup_{Q_\tau} v, \qquad N_\tau = (1 - \tau)^{-b}, \tag{57}$$

where the number $b > 0$ will be suitably defined later to render quantitative the following estimate. As $M_0 = N_0$ and considering that M_τ remains a bounded function of τ (because v is a *bounded* solution) while $N_\tau \to +\infty$ when τ tends to 1, we can choose a number τ_o to be the largest root of the equation

$$M_\tau = N_\tau .$$

This implies by construction

$$\sup_{Q_\tau} v \le N_\tau, \qquad \forall \tau > \tau_o. \tag{58}$$

Since v is continuous in Q there exists at least one point $(\bar{x}, \bar{t}) \in \overline{Q_{\tau_o}}$ such that

$$v(\bar{x}, \bar{t}) = N_{\tau_o} = (1 - \tau_o)^{-b}. \tag{59}$$

STEP 3. *Ordering v and $(1-\tau_o)^{-b}$ within a small ball centered in $\bar{x}$.*
Let

$$R = \frac{1-\tau_o}{2},$$

and consider the cylinder $[(\bar{x}, \bar{t}) + Q(R^p, R)] = \{|x - \bar{x}| < R\} \times \{\bar{t} - R^p, \bar{t}\}$. As $\tau_o \in (0, 1]$ we have the inclusion $[(\bar{x}, \bar{t}) + Q(R^p, R)] \subset Q_{\frac{1+\tau_o}{2}}$ which implies

$$\sup_{[(\bar{x},\bar{t})+Q(R^p,R)]} v \le N_{\frac{1+\tau_o}{2}} = 2^b(1-\tau_o)^{-b} =: \omega.$$

Now we use Hölder continuity of the function v in the fashion of Proposition 3.1 of Chap. III of [10], choosing $b > 0$ so large that the starting one of the family of shrinking cylinders is contained in $[(\bar{x}, \bar{t}) + Q(R^p, R)]$. Hence there exist $\gamma > 1$ and $a, \varepsilon_o \in (0, 1)$ such that for all $r \in (0, R]$ we have

$$\begin{aligned} \operatorname*{osc}_{[(\bar{x},\bar{t})+Q(R^p,R)]} v(\cdot, \bar{t}) &\le \gamma(\omega + R^{\varepsilon_o})\left(\frac{r}{R}\right)^a \\ &\le 2^{b+1}\gamma(1-\tau_o)^{-b}\left(\frac{r}{R}\right)^a \end{aligned} \tag{60}$$

We let $r = \sigma R$ and we choose σ so small that for all $\{|x - \bar{x}| < \sigma R\}$ we obtain

$$\begin{aligned} v(x, \bar{t}) &\ge v(\bar{x}, \bar{t}) - 2^{b+1}\gamma(1-\tau_o)^{-b}\sigma^a \\ &(1 - 2^{b+1})\gamma\sigma^a)(1-\tau_o)^{-b} \\ &\frac{1}{2}(1-\tau_o)^{-b}, \quad \forall\{|x - \bar{x}|\} < \sigma R, \quad R = \frac{1}{2}(1-\tau_o) \end{aligned} \tag{61}$$

STEP 5. *Expansion of the positivity set and conclusion.*
In this last step we will choose the constants $b > 1$ and $C > 1$ so that the qualitative largeness of $v(\cdot, \bar{t})$ in the small ball $B_{\sigma R}(\bar{x})$ turns into a quantitative bound below over the full sphere B_1 at some later time level C. This will be carried on by means of the comparison with the functions $\mathcal{B}_{k,\rho}$ defined in (28) by

$$\mathcal{B}_{k,\rho}(x,t,\bar{x},\bar{t}) = \frac{k\rho^N}{S^{\frac{N}{\lambda}}(t)}\left\{1 - \left(\frac{|x-\bar{x}|}{S^{\frac{1}{\lambda}}(t)}\right)^{\frac{p}{p-1}}\right\}_+^{\frac{p-1}{p-2}},$$

$$S(t) = \lambda\left(\frac{p}{p-2}\right)^{p-1} k^{p-2}\rho^{N(p-2)}(t-\bar{t}) + \rho^\lambda.$$

Indeed, we choose appropriately

$$k = \frac{1}{2}(1-\tau_o)^{-b}, \qquad \rho = \sigma R,$$

and we observe that at the time level $t = C$ the support of $\mathcal{B}_{k,\rho}(\cdot, C, \bar{x}, \bar{t})$ is the ball

$$|x-\bar{x}|^\lambda < S(t) = \{d\gamma^{p-2}(1-\tau_o)^{(N-b)/(p-2)}(C-\bar{t}) + (\sigma R)^\lambda)\}$$

for

$$\gamma(N,b) = \frac{1}{2}\left(\frac{\sigma}{2}\right)^N, \quad \text{and} \quad d = \lambda\left(\frac{p}{p-2}\right)^{p-1}.$$

Now choose

$$b = N, \qquad C = \frac{3^\lambda}{d\gamma^{p-2}}, \tag{62}$$

so that the support of $\mathcal{B}_{k,\rho}(\cdot, C, \bar{x}, \bar{t})$ contains B_2 and we can use the comparison principle with v as we have in B_ρ

$$v(\cdot,\bar{t}) \geq \frac{1}{2}(1-\tau_o)^{-N} = k \geq B_{k,\rho}(\cdot,\bar{t}). \tag{63}$$

Thence

$$\begin{aligned}\inf_{x\in B_1} v(x,C) &\geq \inf_{x\in B_1} B_{k,\rho}(x,C,\bar{x},\bar{t}) \\ &\geq 2^{-(1+2N/\lambda)}\left(\frac{\sigma}{2}\right)^N\left\{1-\left(\frac{2}{3}\right)^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p-2}} =: \gamma_o,\end{aligned} \tag{64}$$

and the proof is concluded.

6 Looking for a Barenblatt-Type Solution to (1)

In this section we calculate the right exponents for the transformation of Eq. (1) into an anisotropic Fokker-Planck equation. Next we observe that the impossibility of using radial solutions does not allow us to obtain an ODE from the Fokker-Planck equation. Finally we show a strategy to find a non-explicit Barenblatt Fundamental solution.

Remark 6 Observe initially that we can construct a source-type solution, but that unfortunately has not a compact support. Indeed, consider the following solution to (1). Let $i \in \{1, .., N\}$ and

$$f_i(x_i, t, T_i) = \kappa_i \left(\frac{|x_i|^{p_i}}{(T_i - t)} \right)^{\frac{1}{p_i - 2}}, \quad \kappa_i = \kappa_i(p_i) > 0, \quad p_i > 2, \tag{65}$$

be solutions of the equations

$$u_t - (|u_{x_i}|^{p_i - 2} u_{x_i})_{x_i} = 0, \quad x_i \in \mathbb{R}, \quad t > 0. \tag{66}$$

Then the function

$$\begin{aligned} \mathcal{F}(x, t) &= \sum_{i=1}^{N} f_i(x_i, t, T_i) \\ &= \sum_{i=1}^{N} \kappa_i \left(\frac{|x_i|^{p_i}}{(T_i - t)} \right)^{\frac{1}{p_i - 2}} \end{aligned} \tag{67}$$

solves the prototype equation (1). The same can be done by choosing $f_i \equiv \mathcal{B}_i$ the mono-dimensional Barenblatt solutions solving (66). These functions reveal some of the physical aspects of Eq. (1): for instance they can be used to show that the lifetime of solutions is dictated by the largest exponent p_N in the case of large initial mass (see Remark 3 in [8]). Unfortunately solutions so-built do not have a compactly supported evolution and we cannot use them to expand the positivity by comparison as done in Sect. 5.

6.1 Finite Speed of Propagation

Consider the Cauchy problem

$$\begin{cases} u_t = \operatorname{div}(\mathbf{A}(t, x, u, \nabla u)), & \text{in} \quad \Sigma_T = \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) \in L^2(\mathbb{R}^N), \end{cases} \tag{68}$$

where $\mathbf{A}(t, x, u, \nabla u) = (A_i(x, t, u, \nabla u))_{i=1,..,N}$ is a Caratheodory vector field satisfying the growth conditions (3). In [14] the authors proved the following decay properties, that will be useful to us to intercept the right exponents in the scaling transformation leading to the Fokker-Planck equation for solutions to (1).

Theorem 2 *Suppose that $p_i > 2$ for all $i \in \{1, .., N\}$. Let u be a local weak solution to* (68) *in Σ_T under the growth conditions* (3) *with*

$$u_0 \in L^2(\mathbb{R}^N), \qquad \emptyset \neq supp(u_0) \subseteq [-R_0, R_0]^N \tag{69}$$

Then there is a solution $\tilde{u} \neq 0$ such that

$$supp(\tilde{u}(\cdot, t)) \subseteq \prod_{i=1}^{N} [-R_j(t), R_j(t)], \tag{70}$$

for any $t < T$, where

$$R_j(t) = 2R_0 + Ct^{\frac{N(\bar{p}-p_j)+\bar{p}}{\lambda p_j}} ||u_0||_1^{\frac{\bar{p}}{p_j}\frac{p_j-2}{\lambda}}, \qquad \lambda = N(\bar{p} - 2) + \bar{p}. \tag{71}$$

Moreover, they proved the following L^∞-L^1 estimates of the decay for the solution.

Theorem 3 *Let $\bar{p} < N$ and let $u \in \cap_{i=1}^N \mathcal{L}^{p_i}(\Sigma_T)$ solve* (68) *for $u_0 \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Then if $p_i > 2$, $\forall i = 1, .., N$ the following estimate holds true for any $\tau \in [0, T]$*

$$||u(\cdot, t)||_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\frac{N}{\lambda}} ||u_0||_{L^1(\mathbb{R}^N)}^{\frac{\bar{p}}{\lambda}}. \tag{72}$$

6.2 The Anisotropic Fokker-Planck Equation

We consider a similar continuous transformation as (17), owing the choice of the right exponent to the decay of a solution to (68), and we perform the following formal calculations.

$$u(x, t) = t^{-\beta} v\Big(x_1 t^{\alpha_1}, \dots, x_N t^{\alpha_N}, t\Big) = t^{-\beta} v(y_1, .., y_N, t), \qquad \begin{cases} y_i = x_i t^{\alpha_i}, \\ \frac{\partial}{\partial x_i} = t^{\alpha_i} \frac{\partial}{\partial y_i}. \end{cases} \tag{73}$$

We calculate formally

$$u_t = -\beta t^{-\beta-1} v + t^{-\beta} \Big[\sum_{i=1}^{N} \Big(\frac{\partial}{\partial y_i} v \Big) \frac{\partial y_i}{\partial t} + v_t \Big] = -\beta t^{-\beta-1} v + t^{-\beta} \sum_{i=1}^{N} \Big(\frac{\partial}{\partial y_i} v \Big) \Big[\frac{\alpha_i x_i t^{\alpha_i}}{t} \Big] + t^{-\beta} v_t,$$

being

$$\frac{\partial}{\partial x_i} u = t^{\alpha_i - \beta} \frac{\partial}{\partial y_i} v.$$

We substitute these into (1) to get

$$-\beta t^{-\beta-1} v + t^{-\beta} \sum_{i=1}^{N} \frac{\alpha_i y_i}{t} \left(\frac{\partial}{\partial y_i} v \right) + t^{-\beta} v_t = \sum_{i=1}^{N} t^{\alpha_i} \frac{\partial}{\partial y_i} \left(t^{(\alpha_i - \beta)(p_i - 1)} \left| \frac{\partial}{\partial y_i} v \right|^{p_i - 2} \frac{\partial}{\partial y_i} v \right).$$

Re-ordering and multiplying each term for $t^{\beta+1}$ we get

$$t v_t = \beta v - \sum_{i=1}^{N} \alpha_i y_i \frac{\partial}{\partial y_i} v + \sum_{i=1}^{N} t^{(\alpha_i - \beta)(p_i - 1) + \alpha_i + \beta + 1} \frac{\partial}{\partial y_i} \left(\left| \frac{\partial}{\partial y_i} v \right|^{p_i - 2} \frac{\partial}{\partial y_i} v \right) =$$

$$\beta v + \sum_{i=1}^{N} \alpha_i v + \sum_{i=1}^{N} \frac{\partial}{\partial y_i} \left[\left(\left| \frac{\partial}{\partial y_i} v \right|^{p_i - 2} \frac{\partial}{\partial y_i} v \right) - \alpha_i y_i v \right],$$

by choosing

$$(\alpha_i - \beta)(p_i - 1) + \alpha_i + \beta + 1 = 0,$$

which means

$$\alpha_i = \beta - \frac{1 + 2\beta}{p_i} < 0. \tag{74}$$

This is an Euler equation. So, by redefining $v(y, t) = w(y, \ln(t))$ Eq. (1) becomes the non-homogeneous Fokker-Planck equation

$$w_t = \left(\beta + \sum_{i=1}^{N} \alpha_i \right) w + \sum_{i=1}^{N} \frac{\partial}{\partial y_i} \left[\left(\left| \frac{\partial}{\partial y_i} w \right|^{p_i - 2} \frac{\partial}{\partial y_i} w \right) - \alpha_i y_i w \right]. \tag{75}$$

If, according to (72), we consider

$$\beta = \frac{N}{N(\bar{p} - 2) + \bar{p}}, \tag{76}$$

then the equation reduces to

$$w_t = \sum_{i=1}^{N} \frac{\partial}{\partial y_i} \left[\left(\left| \frac{\partial}{\partial y_i} w \right|^{p_i - 2} \frac{\partial}{\partial y_i} w \right) - \alpha_i y_i w \right]. \tag{77}$$

Remark 7 Equation (77) conserves the $L^1(\Omega)$-norm in time.

Moreover, a solution to the stationary version of (77) would give us the wanted Barenblatt Fundamental solution to (1).

This anisotropic Fokker-Planck type equation is deeply different from its isotropic counterpart (36). Anisotropy does not permit the identification of a single variable ODE as in (39), and this is physically evident and due to the lack of radial symmetry of the diffusion process in consideration: there is no homogeneous flux here to be vanished. Moreover the steady equation

$$\sum_{i=1}^{N} \frac{\partial}{\partial y_i}\Bigg[\bigg(\Big|\frac{\partial}{\partial y_i} w\Big|^{p_i-2}\frac{\partial}{\partial y_i} w\bigg) - \alpha_i y_i w\Bigg], \quad \text{in} \quad \Omega \subset \mathbb{R}^N, \tag{78}$$

is not a variational one i.e. it is not known if it can be written as the Euler Lagrange equation of an energy functional. Moreover, its monotonicity and coercivity properties suffer heavily the second term influence relatively to the length in the i-th direction of the medium Ω. These considerations leading to the difficulty of an explicit formula as in the previous case (24), the existence and the main properties characterizing a Barenblatt Fundamental solution may be derived by the simpler original equation (1) and then defining a suitable function which solves the steady Fokker-Planck equation (78). This would ensure that the solution to (1) found has the properties of Theorem 2, which characterize a Barenblatt Fundamental Solution.

6.3 On the Solvability of the Anisotropic Cauchy Problem with Measure Initial Data

We consider the prototype problem with measure initial data, i.e

$$\begin{cases} u_t - \sum_{i=1}^N (|u_{x_i}|^{p_i-2}u_{x_i})_{x_i} = 0, & (x,t) \in \mathbb{R}^N \times [0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{79}$$

We begin the study of a weak solution to (79) i.e. a function $u \in C(0,T; L^1(\mathbb{R}^N)) \cap L^{\mathbf{p}}(0,T; W^{1,\mathbf{p}}(\mathbb{R}^N))$ such that for each open bounded $\Omega \subset \mathbb{R}^N$ and for all $t \in [0,T)$ satisfies for all test function $\varphi(x,t) \in W^{1,\infty}([0,T,L^\infty(\Omega)) \cap L^\infty([0,T], W_o^{1,\infty}(\Omega))$ the equality

$$\int_\Omega u\varphi(x,t)dx + \sum_{i=1}^N \int_0^t \int_\Omega |u_{x_i}|^{p_i-2}u_{x_i}\varphi_{x_i}dxd\tau = \int_\Omega \varphi(x,0)du_0 + \int_0^t \int_\Omega u\varphi_\tau(x,\tau)dxd\tau. \tag{80}$$

This has been done in [8, 9] for more general doubly nonlinear anisotropic equations. We recall the notation $\lambda = N(\overline{p} - 2) + \overline{p}$. In [8] the authors prove a generalised version of the following a priori estimates.

Theorem 4 *Consider the problem* (79) *with* $2 < p_i \leq \bar{p}\left(1 + \frac{1}{N}\right)$, $u_0(x) \geq 0$ *and*

$$|||u_0|||_r := \sup_{\rho \geq r} \rho^{-\frac{\lambda}{N}} \int_{B_\rho} u_0(x)dx < \infty, \quad r > 0, \tag{81}$$

being

$$B_\rho := \left\{ x \in \mathbb{R}^N \, | \, |x_i| \leq \frac{\rho^{\frac{\bar{p}(p_i-2)}{p_i(\bar{p}-2)}}}{2} \right\}.$$

Define by monotonicity $M_\infty := \lim_{r\to\infty} |||u_0|||_r$ *and for a* $\gamma > 0$ *to be specified later*

$$T_* := \begin{cases} \infty, & \textit{if} \quad M_\infty = 0, \\ \left(\frac{M_\infty}{\gamma}\right)^{\frac{N(\overline{p}-p_N)+\overline{p}}{\overline{p}(p_N-2)}}, & \textit{if} \quad M_\infty \geq \gamma, \\ \left(\frac{M_\infty}{\gamma}\right)^{\frac{N(\overline{p}-p_1)+\overline{p}}{\overline{p}(p_1-2)}}, & \textit{if} \quad M_\infty < \gamma. \end{cases} \tag{82}$$

Then there exists a positive constant $\gamma(p_i, N)$ *such that every nonnegative weak solution to* (79) *defined on* $[0, T_*]$ *must satisfy the following estimates for all* $t, \bar{t} \in (0, T_*)$*:*

$$|||u(\cdot, t)|||_r \leq C|||u_0|||_r, \tag{83}$$

$$\|u(\cdot, t)\|_{L^\infty(B_r)} \leq C r^{\frac{\overline{p}}{N}} t^{-\frac{N}{\lambda}} |||u_0|||_r^{\frac{\overline{p}}{\lambda}}, \tag{84}$$

$$\sum_{i=1}^N \int_0^t \int_{B_r} |u_{x_i}|^{p_i-1} dx d\tau < C(r, t), \tag{85}$$

$$\sum_{i=1}^N \int_{\bar{t}}^t \int_{B_r} |u_{x_i}|^{p_i} dx d\tau < C(r, t, \bar{t}). \tag{86}$$

Remark 8 For $p_i = p$, $\forall i = 1, .., N$ estimates (83), (84), (85), (86) and the number $T_* > 0$ do coincide with the ones of Sect. 4 for the isotropic equation found in [12]. Secondly, it is interesting to observe that the lifetime of the solution is determined by the largest exponent p_N in case of large initial mass $\|u_0\|\|_r$ while it is determined by the smaller p_1 in case of a modest initial mass.

7 Future Strategy and Conclusion

In this note we have proven the strong connection between the Barenblatt Fundamental solution and the solutions to the stationary equation (78). We have shown the existence of solutions to (77) thanks to a recent result in [9]. However, this is not enough to use this result to prove regularity results. Indeed, we can invoke the previous Theorem to find a solution u to (1). We already know that there exists a solution of u that satisfies the growths (70), (72). But what is missing, to repeat the same ideas of Sect. 1, is a nice description *from below* of the support of u. The aim of our next papers is to carry on a deep analysis of the interplay between these two equations and to develop the necessary tools for deriving regularity results and Harnack inequalities for nonnegative solutions to (1).

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No Loss of Derivatives for Hyperbolic Operators with Zygmund-Continuous Coefficients in Time

Ferruccio Colombini, Daniele Del Santo, and Francesco Fanelli

To Massimo Cicognani and Michael Reissig on the occasion of their 60th birthday

Abstract We prove that, if the coefficients of an hyperbolic operator are Zygmund-continuous with respect to t and Lipschitz-continuous with respect to x, an energy estimate without loss of derivatives holds true. As a consequence, the Cauchy problem related to the hyperbolic operator is well-posed in Sobolev spaces.

Keywords Hyperbolic partial differential equations · Non-smooth coefficients · Energy estimates · Cauchy problem

1 Introduction

Consider the second order strictly hyperbolic operator

$$L = \partial_t^2 - \sum_{j,k=1}^{n} \partial_j (a_{jk}(t,x)\partial_k),$$

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where, for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$,

$$0 < \lambda_0 |\xi|^2 \le \sum_{j,k=1}^{n} a_{jk}(t, x)\xi_j \xi_k \le \Lambda_0 |\xi|^2$$

and

$$a_{jk}(t, x) = a_{kj}(t, x).$$

It is well-known that, if the coefficients a_{jk} are Lipschitz-continuous in t and measurable in x, then the Cauchy problem related to L is well-posed in the energy space. In particular, a constant $C > 0$ exists, such that

$$\begin{aligned} \sup_{0 \le t \le T} (\|u(t, \cdot)\|_{H^1} + \|\partial_t u(t, \cdot)\|_{L^2}) \\ \le C(\|u(0, \cdot)\|_{H^1} + \|\partial_t u(0, \cdot)\|_{L^2} + \int_0^T \|Lu(s, \cdot)\|_{L^2}\, ds), \end{aligned} \tag{1}$$

for all $u \in C([0, T]; H^1) \cap C^1([0, T]; L^2)$ with $Lu \in L^1([0, T]; L^2)$ (see [11, 12, Ch. IX]).

In this note we are interested in second order strictly hyperbolic operators having *non Lipschitz-continuous* coefficients with respect to time.

After the pioneering paper by Colombini, De Giorgi and Spagnolo [7], this topic has been widely studied. A result of particular interest has been obtained in [5], where it was proved that, if the coefficients are log-Lipschitz-continuous with respect to t and x, i.e. there exists $C > 0$ such that

$$\sup_{t,x} |a_{jk}(t + \tau, x + y) - a_{jk}(t, x)| \le C(|\tau| + |y|)(1 + \log \frac{1}{|\tau| + |y|}),$$

then (1) is no more valid, but the following weaker energy estimate can be recovered:

$$\begin{aligned} \sup_{0 \le t \le T} (\|u(t, \cdot)\|_{H^{1-\theta-\beta t}} + \|\partial_t u(t, \cdot)\|_{H^{-\theta-\beta t}}) \\ \le C(\|u(0, \cdot)\|_{H^{1-\theta}} + \|\partial_t u(0, \cdot)\|_{H^{-\theta}} + \int_0^T \|Lu(s, \cdot)\|_{H^{-\theta-\beta s}}\, ds), \end{aligned} \tag{2}$$

for some constants $C > 0$, $\beta > 0$ and for all $u \in C^2([0, T]; H^\infty)$ and $\theta \in\,]0, 1[$ (here and in the following $H^\infty = \bigcap_{s \in \mathbb{R}} H^s$). Remark that, while in (1) the norms of $u(t)$ and $\partial_t u(t)$ are estimated by the same norms of $u(0)$ and $\partial_t u(0)$, in (2) the Sobolev spaces in which $u(t)$ and $\partial_t u(t)$ are measured are different and bigger than the spaces in which initial data are, so the estimate is less effective. This phenomenon goes under the name of "loss of derivatives". We refer e.g. to the introductions of [8, 9] for more details and references about this problem.

Using a result obtained by Tarama in [16] (see also Remark 1 below), it is possible to prove that if the coefficients depend only on t and are Zygmund-continuous, i.e.

$$\sup_t |a_{jk}(t+\tau) + a_{jk}(t-\tau) - 2a_{jk}(t)|\, dt \le C_2|\tau|, \tag{3}$$

then (1) is valid. Notice that the Zygmund assumption is weaker than the Lipschitz one. In [9], the authors proved that if the coefficients depend also on the space variable and verify an isotropic Zygmund assumption (i.e. they are Zygmund-continuous both in time and space variables), then the Cauchy problem is well-posed with no loss, but only in the space $H^{1/2} \times H^{-1/2}$. In particular, an estimate similar to (1) holds true, up to replacing the H^1 and L^2 norms respectively with the $H^{1/2}$ and $H^{-1/2}$ norms. See also Remark 2 below for more details.

The problem whether a Zygmund assumption both in time and space is still enough to recover well-posedness in general spaces $H^s \times H^{s-1}$ (and not only for $s = 1/2$) remains at present largely open. As a partial step in this direction, in this note we consider a stronger hypothesis with respect to the space variable: namely we prove that, if the coefficients are Zygmund-continuous with respect to t and Lipschitz-continuous with respect to x, then an estimate without loss of derivatives, similar to (1), holds true. Then, the Cauchy problem related to L is well-posed in any space $H^s \times H^{s-1}$, for all $s \in]0, 1]$.

Two are the main ingredients of the proof of our result. The first one is to resort to Tarama's idea of introducing a new type of energy associated to operator L: this new energy is equivalent to the classical energy, but it contains a lower order term, whose goal is to produce special algebraic cancellations, which reveal to be fundamental in the energy estimates. The second main ingredient, already introduced in [8] and [9], is the use of paradifferential calculus with parameters (see e.g. [13, 15]), in order to deal with coefficients depending also on x and having low regularity in that variable.

We conclude this introduction with a short overview of the paper. In the next section we fix our hypotheses and state our main result, see Theorem 1. In Sect. 3 we collect some elements of Littlewood-Paley theory, which are needed in the description of the functional classes where the coefficients belong to, and in the construction of paradifferential calculus with parameters. With those tools at hand, we tackle the proof of Theorem 1, which is carried out in Sect. 4.

2 Main Result

Given $T > 0$ and an integer $n \ge 1$, let L be the linear differential operator defined on $[0, T] \times \mathbb{R}^n$ by

$$Lu = \partial_t^2 u - \sum_{j,k=1}^{n} \partial_j (a_{jk}(t, x)\partial_k u), \tag{4}$$

where, for all $j,\ k = 1, \ldots, n$,

$$a_{jk}(t, x) = a_{kj}(t, x), \tag{5}$$

and there exist $\lambda_0,\ \Lambda_0 > 0$ such that

$$\lambda_0|\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(t, x)\xi_j\xi_k \leq \Lambda_0|\xi|^2, \tag{6}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$. Suppose moreover that there exist constants $C_0,\ C_1 > 0$ such that, for all $j,\ k = 1, \ldots, n$ and for all $\tau \in \mathbb{R}$, $y \in \mathbb{R}^n$,

$$\sup_{t,x} |a_{jk}(t + \tau, x) + a_{jk}(t - \tau, x) - 2a_{jk}(t, x)| \leq C_0|\tau|, \tag{7}$$

$$\sup_{t,x} |a_{jk}(t, x + y) - a_{jk}(t, x)| \leq C_1|y|. \tag{8}$$

We can now state the main result of this paper.

Theorem 1 *Under the previous hypotheses, for all fixed* $\theta \in [0, 1[$*, there exists a constant* $C > 0$*, depending only on* θ *and* T*, such that*

$$\begin{aligned} \sup_{0\leq t\leq T} (\|u(t, \cdot)\|_{H^{1-\theta}} + \|\partial_t u(t, \cdot)\|_{H^{-\theta}}) \\ \leq C(\|u(0, \cdot)\|_{H^{1-\theta}} + \|\partial_t u(0, \cdot)\|_{H^{-\theta}} + \int_0^T \|Lu(s, \cdot)\|_{H^{-\theta}}\, ds), \end{aligned} \tag{9}$$

for all $u \in C^2([0, T], H^\infty(\mathbb{R}^n))$.

Some remarks are in order.

Remark 1 If the coefficients a_{jk} depend only on t, this result has been obtained by Tarama in [16], under the hypothesis that there exists a constant $C_2 > 0$ such that, for all $j,\ k = 1, \ldots, n$ and for all $\tau \in]0, T/2[$,

$$\int_\tau^{T-\tau} |a_{jk}(t + \tau) + a_{jk}(t - \tau) - 2a_{jk}(t)|\, dt \leq C_2\tau. \tag{10}$$

Tarama's hypothesis is weaker than ours, but, when coefficients depend also on the space variable, it is customary to take a pointwise condition with respect to time, like in (7) above (see also [5, 6, 8, 9] in this respect). In particular, it is not clear at present whether or not the pointwise condition (7) can be relaxed to an integral one, similar to (10), in our framework.

Remark 2 If the hypoteses (7) and (8) are replaced by the weaker following one: there exists a constant $C_3 > 0$ such that, for all $j,\ k = 1, \ldots, n$ and for all $\tau \in \mathbb{R}$,

$y \in \mathbb{R}^n$,

$$\sup_{t,x} |a_{jk}(t+\tau, x+y) + a_{jk}(t-\tau, x-y) - 2a_{jk}(t,x)| \leq C_3(|\tau| + |y|), \tag{11}$$

the estimate (9) has been proved, only in the case of $\theta = 1/2$, by the present authors and Métivier in [9].

Remark 3 Assume (7) and the following hypothesis: there exists a constant $C_4 > 0$ such that, for all $j,\ k = 1, \ldots, n$ and for all $y \in \mathbb{R}^n$ with $0 < |y| \leq 1$,

$$\sup_{t,x} |a_{jk}(t, x+y) - a_{jk}(t,x)| \leq C_4 |y| (1 + \log \frac{1}{|y|}). \tag{12}$$

As a consequence of a result of the present authors and Métivier in [8] (stated for coefficients which are actually log-Zygmund with respect to time), one gets that, for all fixed $\theta \in]0, 1[$, there exist a $\beta > 0$, a time $T' > 0$ and a constant $C > 0$ such that

$$\begin{aligned} \sup_{0 \leq t \leq T'} & (\|u(t,\cdot)\|_{H^{1-\theta-\beta t}} + \|\partial_t u(t,\cdot)\|_{H^{-\theta-\beta t}}) \\ & \leq C(\|u(0,\cdot)\|_{H^{1-\theta}} + \|\partial_t u(0,\cdot)\|_{H^{-\theta}} + \int_0^{T'} \|Lu(s,\cdot)\|_{H^{-\theta-\beta s}}\, ds), \end{aligned} \tag{13}$$

for all $u \in C^2([0, T'], H^\infty(\mathbb{R}^n))$. The condition (12) is weaker than (8) but also (13) is weaker than (9): (13) has a loss of derivatives, while (9) performs no loss. In addition, observe that (13) holds only for $\theta \in]0, 1[$, while (9) holds also for $\theta = 0$.

3 Preliminary Results

We briefly list here some tools we will need in the proof of the main result. We follow closely the presentation of these topics given in [8] and [9].

3.1 Littlewood-Paley Decomposition

We will use the so called Littlewood-Paley theory. We refer to [2, 3, 14] and [1] for the details.

We start recalling Bernstein's inequalities.

Proposition 1 ([3, Lemma 2.2.1]) *Let $0 < r < R$. A constant C exists so that, for all nonnegative integer k, all $p,\ q \in [1, +\infty]$ with $p \leq q$ and for all function $u \in L^p(\mathbb{R}^d)$, we have, for all $\lambda > 0$,* *(i) if* $\operatorname{Supp}\hat{u} \subseteq B(0, \lambda R) = \{\xi \in \mathbb{R}^d \ : \ |\xi| \leq \lambda R\}$*, then*

$$\|\nabla^k u\|_{L^q} \leq C^{k+1} \lambda^{k+N(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p};$$

(ii) if $\operatorname{Supp}\hat{u} \subseteq C(0, \lambda r, \lambda R) = \{\xi \in \mathbb{R}^d \ : \ \lambda r \leq |\xi| \leq \lambda R\}$*, then*

$$C^{-k-1} \lambda^k \|u\|_{L^p} \leq \|\nabla^k u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}.$$

We introduce the dyadic decomposition. Let $\psi \in C^\infty([0, +\infty[, \mathbb{R})$ such that ψ is non-increasing and

$$\psi(t) = 1 \quad \text{for} \quad 0 \leq t \leq \frac{11}{10}, \qquad \psi(t) = 0 \quad \text{for} \quad t \geq \frac{19}{10}.$$

We set, for $\xi \in \mathbb{R}^d$,

$$\chi(\xi) = \psi(|\xi|), \qquad \varphi(\xi) = \chi(\xi) - \chi(2\xi). \tag{14}$$

We remark that the support of χ is contained in the ball $\{\xi \in \mathbb{R}^d \ : \ |\xi| \leq 2\}$, while the one of φ is contained in the annulus $\{\xi \in \mathbb{R}^d \ : \ 1/2 \leq |\xi| \leq 2\}$.

Given a tempered distribution u, the dyadic blocks are defined by

$$\Delta_0 u = \chi(D) u = \mathscr{F}^{-1}(\chi(\xi)\hat{u}(\xi)),$$

$$\Delta_j u = \varphi(2^{-j} D) u = \mathscr{F}^{-1}(\varphi(2^{-j}\xi)\hat{u}(\xi)) \quad \text{if} \quad j \geq 1,$$

where we have denoted by $\mathscr{F}^{-1}$ the inverse of the Fourier transform. We introduce also the operator

$$S_k u = \sum_{j=0}^{k} \Delta_j u = \mathscr{F}^{-1}(\chi(2^{-k}\xi)\hat{u}(\xi)).$$

It is well known the characterization of classical Sobolev spaces via Littlewood-Paley decomposition: for any $s \in \mathbb{R}$, $u \in \mathcal{S}'$ is in H^s if and only if, for all $j \in \mathbb{N}$, $\Delta_j u \in L^2$ and the series $\sum 2^{2js} \|\Delta_j u\|_{L^2}^2$ is convergent. Moreover, in such a case, there exists a constant $C_s > 1$ such that

$$\frac{1}{C_s} \sum_{j=0}^{+\infty} 2^{2js} \|\Delta_j u\|_{L^2}^2 \leq \|u\|_{H^s}^2 \leq C_s \sum_{j=0}^{+\infty} 2^{2js} \|\Delta_j u\|_{L^2}^2. \tag{15}$$

3.2 Lipschitz, Zygmund and Log-Lipschitz Functions

In this subsection, we give a description of some functional classes relevant in the study of hyperbolic Cauchy problems. Namely, via Littlewood-Paley analysis, we can characterise the spaces of Lipschitz, Zygmund and log-Lipschitz functions. We start by recalling their definitions.

Definition 1 A function $u \in L^\infty(\mathbb{R}^d)$ is a Lipschitz-continuous function if

$$|u|_{\mathrm{Lip}} = \sup_{\substack{x,y\in\mathbb{R}^d,\\ y\neq 0}} \frac{|u(x+y)-u(x)|}{|y|} < +\infty,$$

u is a Zygmund-continuous function if

$$|u|_{\mathrm{Zyg}} = \sup_{\substack{x,y\in\mathbb{R}^d,\\ y\neq 0}} \frac{|u(x+y)+u(x-y)-2u(x)|}{|y|} < +\infty$$

and, finally, u is a log-Lipschitz-continuous function if

$$|u|_{\mathrm{LL}} = \sup_{\substack{x,y\in\mathbb{R}^d,\\ 0<|y|\leq 1}} \frac{|u(x+y)-u(x)|}{|y|(1+\log\frac{1}{|y|})} < +\infty.$$

For $X \in \{\mathrm{Lip}\,,\ \mathrm{Zyg}\,,\ \mathrm{LL}\}$, we define $\|u\|_{\mathrm{X}} = \|u\|_{L^\infty} + |u|_{\mathrm{X}}$.

Proposition 2 *Let $u \in L^\infty(\mathbb{R}^d)$. We have the following characterisation:*

$$u \in \mathrm{Lip}(\mathbb{R}^d) \qquad \textit{if and only if} \qquad \sup_j \|\nabla S_j u\|_{L^\infty} < +\infty, \tag{16}$$

$$u \in \mathrm{Zyg}(\mathbb{R}^d) \qquad \textit{if and only if} \qquad \sup_j 2^j \|\Delta_j u\|_{L^\infty} < +\infty, \tag{17}$$

$$u \in \mathrm{LL}(\mathbb{R}^d) \qquad \textit{if and only if} \qquad \sup_j \frac{\|\nabla S_j u\|_{L^\infty}}{j} < +\infty. \tag{18}$$

Proof The proof of (17) and (18) can be found in [3, Prop. 2.3.6] and [5, Prop. 3.3] respectively. We sketch the proof of (16), for the reader's convenience. Suppose $u \in \mathrm{Lip}(\mathbb{R}^d)$. We have

$$\begin{aligned} D_j(S_k u(x)) &= D_j(\mathscr{F}^{-1}(\chi(2^{-k}\xi)\hat{u}(\xi)))(x) \\ &= \mathscr{F}^{-1}(\xi_j \chi(2^{-k}\xi)\hat{u}(\xi))(x) \\ &= 2^k \mathscr{F}^{-1}(2^{-k}\xi_j \chi(2^{-k}\xi)\hat{u}(\xi))(x) \\ &= 2^k \int_{\mathbb{R}^d} \theta_j(2^k y) u(x-y)\, 2^{kd} dy \end{aligned}$$

where $\theta_j(y) = \mathscr{F}^{-1}(\xi_j \chi(\xi))(y)$. From the fact that $\int_{\mathbb{R}^d} \theta_j(y)\, dy = 0$ we deduce that

$$\begin{aligned} |D_j(S_k u(x))| &\leq 2^k \Big| \int_{\mathbb{R}^d} \theta_j(2^k y)(u(x-y) - u(x))\, 2^{kd} dy \Big| \\ &\leq |u|_{\mathrm{Lip}} \int_{\mathbb{R}^d} |\theta_j(z)||z|\, dz, \end{aligned}$$

hence $\sup_j \left\| \nabla(S_j u) \right\|_{L^\infty} < C\, |u|_{\mathrm{Lip}}$.

Conversely, let the second statement in (16) hold. Remarking that

$$D_j(\Delta_k u(x)) = \mathscr{F}^{-1}(\xi_j \varphi(2^{-k}\xi)\hat{u}(\xi))(x) = \mathscr{F}^{-1}(\xi_j (\chi(2^{-k}\xi) - \chi(2^{-k+1}\xi))\hat{u}(\xi))(x),$$

and, by Bernstein's inequalities,

$$|\Delta_k u(x)| \leq C 2^{-k+1} (\|\nabla(S_k u)\|_{L^\infty} + \|\nabla(S_{k-1} u)\|_{L^\infty}),$$

we deduce that, for a new constant $C > 0$,

$$\|\Delta_k u\|_{L^\infty} \leq C 2^{-k} \sup_j \left\| \nabla S_j u \right\|_{L^\infty}$$

for all $k \geq 0$. Then

$$\begin{aligned} |u(x+y) - u(x)| &\leq |S_k u(x+y) - S_k u(x)| + \Big| \sum_{h>k} (\Delta_h u(x+y) - \Delta_h u(x)) \Big| \\ &\leq \|\nabla(S_k u)\|_{L^\infty} |y| + 2 \sum_{h>k} \|\Delta_h u\|_{L^\infty} \\ &\leq C \sup_j \left\| \nabla(S_j u) \right\|_{L^\infty} (|y| + 2^{-k}). \end{aligned}$$

The conclusion follows from choosing k in such a way that $2^{-k} \leq |y|$. □

Notice that, going along the lines of the previous proof, we have actually shown that there exists $C_d > 1$, depending only on d, such that, if $u \in \mathrm{Lip}(\mathbb{R}^d)$ then

$$\frac{1}{C_d} |u|_{\mathrm{Lip}} \leq \|\nabla S_j u\|_{L^\infty} \leq C_d\, |u|_{\mathrm{Lip}}.$$

Proposition 3 ([3, Prop. 2.3.7])

$$\mathrm{Lip}(\mathbb{R}^d) \subseteq \mathrm{Zyg}(\mathbb{R}^d) \subseteq \mathrm{LL}(\mathbb{R}^d).$$

In order to perform computations, we will need to smooth out our coefficients, because of their low regularity. To this end, let us fix an even function $\rho \in C_0^\infty(\mathbb{R})$

such that $0 \leq \rho \leq 1$, $\operatorname{Supp}\rho \subseteq [-1,\ 1]$ and $\int_{\mathbb{R}} \rho(t)\,dt = 1$, and define $\rho_\varepsilon(t) = \frac{1}{\varepsilon}\rho(\frac{t}{\varepsilon})$. The following result holds true.

Proposition 4 ([9, Prop. 3.5]) *Let* $u \in \mathrm{Zyg}(\mathbb{R})$. *There exists* $C > 0$ *such that,*

$$|u_\varepsilon(t) - u(t)| \leq C|u|_{\mathrm{Zyg}}\,\varepsilon, \tag{19}$$

$$|u'_\varepsilon(t)| \leq C|u|_{\mathrm{Zyg}}\,(1 + \log\frac{1}{\varepsilon}), \tag{20}$$

$$|u''_\varepsilon(t)| \leq C|u|_{\mathrm{Zyg}}\,\frac{1}{\varepsilon}, \tag{21}$$

where, for $0 < \varepsilon \leq 1$,

$$u_\varepsilon(t) = (\rho_\varepsilon * u)(t) = \int_{\mathbb{R}} \rho_\varepsilon(t-s)u(s)\,ds. \tag{22}$$

3.3 *Paradifferential Calculus with Parameters*

Let us sketch here the paradifferential calculus depending on a parameter $\gamma \geq 1$. The interested reader can look at [15, Appendix B] (see also [13] and [6]).

Let $\gamma \geq 1$ and consider $\psi_\gamma \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with the following properties

(i) there exist $0 < \varepsilon_1 < \varepsilon_2 < 1$ such that

$$\psi_\gamma(\eta, \xi) = \begin{cases} 1 & \text{for} \quad |\eta| \leq \varepsilon_1(\gamma + |\xi|), \\ 0 & \text{for} \quad |\eta| \geq \varepsilon_2(\gamma + |\xi|); \end{cases} \tag{23}$$

(ii) for all $(\beta, \alpha) \in \mathbb{N}^d \times \mathbb{N}^d$, there exists $C_{\beta,\alpha} \geq 0$ such that

$$|\partial_\eta^\beta \partial_\xi^\alpha \psi_\gamma(\eta, \xi)| \leq C_{\beta,\alpha}(\gamma + |\xi|)^{-|\alpha|-|\beta|}. \tag{24}$$

The model for such a function will be

$$\psi_\gamma(\eta, \xi) = \chi(\frac{\eta}{2^\mu})\chi(\frac{\xi}{2^{\mu+3}}) + \sum_{k=\mu+1}^{+\infty} \chi(\frac{\eta}{2^k})\varphi(\frac{\xi}{2^{k+3}}), \tag{25}$$

where χ and φ are defined in (14) and μ is the integer part of $\log_2 \gamma$. With this setting, we have that the constants ε_1, ε_2 and $C_{\beta,\alpha}$ in (23) and (24) do not depend on γ.

To fix ideas, from now on we take ψ_γ as given in (25). Define now

$$G^{\psi_\gamma}(x,\xi) = (\mathscr{F}_\eta^{-1}\psi_\gamma)(x,\xi),$$

where $\mathscr{F}_\eta^{-1}\psi_\gamma$ is the inverse of the Fourier transform of ψ_γ with respect to the η variable.

Proposition 5 ([14, Lemma 5.1.7]) *For all $(\beta,\alpha) \in \mathbb{N}^d \times \mathbb{N}^d$, there exists $C_{\beta,\alpha}$, not depending on γ, such that*

$$\|\partial_x^\beta \partial_\xi^\alpha G^{\psi_\gamma}(\cdot,\xi)\|_{L^1(\mathbb{R}_x^d)} \le C_{\beta,\alpha}(\gamma + |\xi|)^{-|\alpha|+|\beta|}, \tag{26}$$

$$\|\,|\cdot|\,\partial_x^\beta \partial_\xi^\alpha G^{\psi_\gamma}(\cdot,\xi)\|_{L^1(\mathbb{R}_x^d)} \le C_{\beta,\alpha}(\gamma + |\xi|)^{-|\alpha|+|\beta|-1}. \tag{27}$$

Next, let $a \in L^\infty$. We associate to a the classical pseudodifferential symbol

$$\sigma_{a,\gamma}(x,\xi) = (\psi_\gamma(D_x,\xi)a)(x,\xi) = (G^{\psi_\gamma}(\cdot,\xi) * a)(x), \tag{28}$$

and define the paradifferential operator T_a^γ associated to a as the classical pseudodifferential operator associated to $\sigma_{a,\gamma}$ (from now on, to avoid cumbersome notations, we will write σ_a), i.e.

$$T_a^\gamma u(x) = \sigma_a(D_x)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}_\xi^d} \sigma_a(x,\xi)\hat{u}(\xi)\,d\xi.$$

Remark that T_a^1 is the usual paraproduct operator

$$T_a^1 u = \sum_{k=0}^{+\infty} S_k a \Delta_{k+3} u,$$

while, in the general case,

$$T_a^\gamma u = S_{\mu-1} a S_{\mu+2} u + \sum_{k=\mu}^{+\infty} S_k a \Delta_{k+3} u. \tag{29}$$

with μ equal to the integer part of $\log_2 \gamma$.

In the following it will be useful to deal with Sobolev spaces depending on the parameter $\gamma \ge 1$.

Definition 2 Let $\gamma \ge 1$ and $s \in \mathbb{R}$. We denote by $H_\gamma^s(\mathbb{R}^d)$ the set of tempered distributions u such that

$$\|u\|_{H_\gamma^s}^2 = \int_{\mathbb{R}_\xi^d} (\gamma^2 + |\xi|^2)^s |\hat{u}(\xi)|^2\,d\xi < +\infty.$$

Let us remark that $H^s_\gamma = H^s$ and there exists $C_\gamma \geq 1$ such that, for all $u \in H^s$,

$$\frac{1}{C_\gamma} \|u\|^2_{H^s} \leq \|u\|^2_{H^s_\gamma} \leq C_\gamma \|u\|^2_{H^s}.$$

3.4 Low Regularity Symbols and Calculus

As in [8] and [9], it is important to deal with paradifferential operators having symbols with limited regularity in time and space.

Definition 3 A symbol of order m is a function $a(t, x, \xi, \gamma)$ which is locally bounded on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times [1, +\infty[$, of class C^∞ with respect to ξ such that, for all $\alpha \in \mathbb{N}^n$, there exists $C_\alpha > 0$ such that, for all (t, x, ξ, γ),

$$|\partial^\alpha_\xi a(t, x, \xi, \gamma)| \leq C_\alpha (\gamma + |\xi|)^{m-|\alpha|}. \tag{30}$$

We take now a symbol a of order $m \geq 0$, Zygmund-continuos with respect to t uniformly with respect to x and Lipschitz-continuous with respect to x uniformly with respect to t. We smooth out a with respect to time as done in (22), and call a_ε the smoothed symbol. We consider the classical symbol σ_{a_ε} obtained from a_ε via (28). In what follows, the variable t has to be thought of as a parameter.

Proposition 6 *Under the previous hypotheses, one has:*

$$\begin{aligned}
|\partial^\alpha_\xi \sigma_{a_\varepsilon}(t, x, \xi, \gamma)| &\leq C_\alpha (\gamma + |\xi|)^{m-|\alpha|}, \\
|\partial^\beta_x \partial^\alpha_\xi \sigma_{a_\varepsilon}(t, x, \xi, \gamma)| &\leq C_{\beta,\alpha} (\gamma + |\xi|)^{m-|\alpha|+|\beta|-1}, \\
|\partial^\alpha_\xi \sigma_{\partial_t a_\varepsilon}(t, x, \xi, \gamma)| &\leq C_\alpha (\gamma + |\xi|)^{m-|\alpha|} \log(1 + \frac{1}{\varepsilon}), \\
|\partial^\beta_x \partial^\alpha_\xi \sigma_{\partial_t a_\varepsilon}(t, x, \xi, \gamma)| &\leq C_{\beta,\alpha} (\gamma + |\xi|)^{m-|\alpha|+|\beta|-1} \frac{1}{\varepsilon}, \\
|\partial^\alpha_\xi \sigma_{\partial^2_t a_\varepsilon}(t, x, \xi, \gamma)| &\leq C_\alpha (\gamma + |\xi|)^{m-|\alpha|} \frac{1}{\varepsilon}, \\
|\partial^\beta_x \partial^\alpha_\xi \sigma_{\partial^2_t a_\varepsilon}(t, x, \xi, \gamma)| &\leq C_{\beta,\alpha} (\gamma + |\xi|)^{m-|\alpha|+|\beta|-1} \frac{1}{\varepsilon^2},
\end{aligned}$$

where $|\beta| \geq 1$ *and all the constants* C_α *and* $C_{\beta,\alpha}$ *do not depend on* γ.

Proof We have

$$\sigma_{a_\varepsilon}(t, x, \xi, \gamma) = (G^{\psi_\gamma}(\cdot, \xi) * a_\varepsilon(t, \cdot, \xi, \gamma))(x),$$

so that the first inequality follows from (26) and (30).

Next, we remark that

$$\int \partial_{x_j} G^{\psi_\gamma}(x,\xi)\,dx = \int \mathscr{F}_\eta^{-1}(\eta_j \psi_\gamma(\eta,\xi))(z)\,dz = (\eta_j \psi(\eta,\xi))_{|\eta=0} = 0. \tag{31}$$

Consequently, using also (27),

$$\begin{aligned} |\partial_{x_j}\sigma_{a_\varepsilon}(t,x,\xi,\gamma)| &= |\int \partial_{y_j} G^{\psi_\gamma}(y,\xi)(a_\varepsilon(t,x-y,\xi,\gamma) - a_\varepsilon(t,x,\xi,\gamma))\,dy|,\\ &\le C\int |\partial_{y_j} G^{\psi_\gamma}(y,\xi)|\,|y|\,dy\;(\gamma+|\xi|)^m,\\ &\le C(\gamma+|\xi|)^m. \end{aligned}$$

The other cases of the second inequality can be proved similarly.

The third inequality is again a consequence of (26), keeping in mind (20). It is in fact possible to prove that

$$|\partial_\xi^\alpha \partial_t a_\varepsilon(t,x,\xi,\gamma)| \le C_\alpha (1+\log\frac{1}{\varepsilon})(\gamma+|\xi|)^{m-|\alpha|}.$$

Next, considering again (31), we have

$$\begin{aligned} &\partial_{x_j}\sigma_{\partial_t a_\varepsilon}(t,x,\xi,\gamma)\\ &= \int_{\mathbb{R}^n_y} \partial_{y_j} G^{\psi_\gamma}(y,\xi)(\partial_t a_\varepsilon(t,x-y,\xi,\gamma) - \partial_t a_\varepsilon(t,x,\xi,\gamma))\,dy,\\ &\le \int_{\mathbb{R}^n_y} \partial_{y_j} G^{\psi_\gamma}(y,\xi) \int_{\mathbb{R}_s} \frac{1}{\varepsilon^2}\rho'(\frac{t-s}{\varepsilon})(a(s,x-y,\xi,\gamma) - a(s,x,\xi,\gamma))\,ds\,dy\\ &\le \int_{\mathbb{R}_s} \frac{1}{\varepsilon^2}\rho'(\frac{t-s}{\varepsilon}) \int_{\mathbb{R}^n_y} \partial_{y_j} G^{\psi_\gamma}(y,\xi)(a(s,x-y,\xi,\gamma) - a(s,x,\xi,\gamma))\,dy\,ds. \end{aligned}$$

so that the fourth inequality easily follows.

The last two inequalities are obtained in similar way, using also (21). □

To end this section it is worthy to recall some results on symbolic calculus. Again details can be found in [8, 9] and [15, Appendix B].

Proposition 7 ([8, Prop. 3.19])

(i) Let a be a symbol of order m (see Def. 3). Suppose that a is L^∞ in the x variable. If we set

$$T_a u(x) = \sigma_a(D_x)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d_\xi} \sigma_a(x,\xi,\gamma)\hat{u}(\xi)\,d\xi,$$

then T_a maps H^s_γ into H^{s-m}_γ continuously.

(ii) *Let a and b be two symbols of order m and m′ respectively. Suppose that a and b are* Lip *in the x variable. Then*

$$T_a \circ T_b = T_{ab} + R,$$

and R maps H^s_γ *into* $H^{s-m-m'+1}_\gamma$ *continuously.*

(iii) *Let a be a symbol of order m which is* Lip *in the x variable. Then, denoting by* T_a^* *the* L^2*-adjoint operator of* T_a*,*

$$T_a^* = T_{\overline{a}} + R,$$

and R maps H^s_γ *into* H^{s-m+1}_γ *continuously.*

(iv) *Let a be a symbol of order m which is* Lip *in the x variable. Suppose*

$$\mathrm{Re}\ a(x, \xi, \gamma) \geq \lambda_0(\gamma + |\xi|)^m.$$

with $\lambda_0 > 0$*. Then there exists* $\gamma_0 \geq 1$*, depending only on* $\|a\|_{\mathrm{Lip}}$ *and* λ_0*, such that, for all* $\gamma \geq \gamma_0$ *and for all* $u \in H^\infty$*,*

$$\mathrm{Re}\ (T_a u, u)_{L^2} \geq \frac{\lambda_0}{2} \|u\|^2_{H^{m/2}_\gamma}.$$

4 Proof of Theorem 1

Also for the proof of the main result, we will closely follow the strategy implemented in [8] and [9].

4.1 Approximate Energy

First of all we regularize the coefficients a_{jk} with respect to t via (22) and we obtain $a_{jk,\varepsilon}$. We consider the 0-th order symbol

$$\alpha_\varepsilon(t, x, \xi, \gamma) = (\gamma^2 + |\xi|^2)^{-\frac{1}{2}}(\gamma^2 + \sum_{j,k} a_{jk,\varepsilon}(t, x)\xi_j\xi_k)^{\frac{1}{2}}.$$

We fix

$$\varepsilon = 2^{-\nu},$$

and we write α_ν and $a_{jk,\,\nu}$ instead of $\alpha_{2^{-\nu}}$ and $a_{jk,\,2^{-\nu}}$ respectively. From Prop. 7, point *iv)*, we have that there exists $\gamma \geq 1$ such that, for all $w \in H^\infty$,

$$\|T^\gamma_{\alpha_\nu^{-1/2}} w\|_{L^2} \geq \frac{\lambda_0}{2}\|w\|_{L^2} \qquad \text{and} \qquad \|T^\gamma_{\alpha_\nu^{1/2}(\gamma^2+|\xi|^2)^{1/2}} w\|_{L^2} \geq \frac{\lambda_0}{2}\|w\|_{H^1_\gamma},$$

where λ_0 has been defined in (6). We remark that γ depends only on λ_0 and $sup_{j,k}\|a_{jk}\|_{\text{Lip}}$, in particular γ does not depend on ν. We fix such a γ (this means also that μ is fixed in (29)) and from now on we will omit to write it when denoting the operator T and the Sobolev spaces H^s.

We consider $u \in C^2([0,T], H^\infty)$. We have

$$\partial_t^2 u = \sum_{j,k} \partial_j(a_{jk}(t,x)\partial_k u) + Lu = \sum_{j,k} \partial_j(T_{a_{jk}}\partial_k u) + \tilde{L}u,$$

where

$$\tilde{L}u = Lu + \sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u).$$

We apply the operator Δ_ν and we obtain

$$\partial_t^2 u_\nu = \sum_{j,k} \partial_j(T_{a_{jk}}\partial_k u_\nu) + \sum_{j,k} \partial_j([\Delta_\nu, T_{a_{jk}}]\partial_k u) + (\tilde{L}u)_\nu,$$

where $u_\nu = \Delta_\nu u$, $(\tilde{L}u)_\nu = \Delta_\nu(\tilde{L}u)$ and $[\Delta_\nu, T_{a_{jk}}]$ is the commutator between the localization operator Δ_ν and the paramultiplication operator $T_{a_{jk}}$.

We set

$$v_\nu(t,x) = T_{\alpha_\nu^{-1/2}}\partial_t u_\nu - T_{\partial_t(\alpha_\nu^{-1/2})} u_\nu,$$

$$w_\nu(t,x) = T_{\alpha_\nu^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\, u_\nu,$$

$$z_\nu(t,x) = u_\nu,$$

and we define the approximate energy associated to the ν-th component as

$$e_\nu(t) = \|v_\nu(t,\cdot)\|^2_{L^2} + \|w_\nu(t,\cdot)\|^2_{L^2} + \|z_\nu(t,\cdot)\|^2_{L^2}.$$

We fix $\theta \in [0,\ 1[$ and we define the total energy

$$E_\theta(t) = \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t).$$

We remark that, as a consequence of Bernstein's inequalities,

$$\|w_\nu\|^2_{L^2} \sim \|\nabla u_\nu\|^2_{L^2} \sim 2^{2\nu}\|u_\nu\|^2_{L^2}.$$

Moreover, from (20) and, again, Bernstein's inequalities,

$$\|T_{\partial_t(\alpha_\nu^{-\frac{1}{2}})} u_\nu\|_{L^2} \le C(\nu+1)\|u_\nu\|_{L^2} \le C'\|w_\nu\|_{L^2},$$

so that

$$\begin{aligned} \|\partial_t u_\nu\|_{L^2} &\le C\|T_{\alpha_\nu^{-1/2}} u_\nu\|_{L^2} \\ &\le C(\|v_\nu\|_{L^2} + \|T_{\partial_t(\alpha_\nu^{-1/2})} u_\nu\|_{L^2}) \\ &\le C(e_\nu(t))^{\frac{1}{2}}. \end{aligned} \tag{32}$$

We deduce that there exist constants C_θ and C'_θ, depending only on θ, such that

$$(E_\theta(0))^{\frac{1}{2}} \le C_\theta(\|\partial_t u(0)\|_{H^{-\theta}} + \|u(0)\|_{H^{1-\theta}}),$$
$$(E_\theta(t))^{\frac{1}{2}} \ge C'_\theta(\|\partial_t u(t)\|_{H^{-\theta}} + \|u(t)\|_{H^{1-\theta}}).$$

4.2 Time Derivative of the Approximate Energy

We want to estimate the time derivative of e_ν.

Since

$$\partial_t v_\nu = T_{\alpha_\nu^{-1/2}} \partial_t^2 u_\nu - T_{\partial_t^2(\alpha_\nu^{-1/2})} u_\nu,$$

we deduce

$$\begin{aligned} &\frac{d}{dt}\|v_\nu(t)\|^2_{L^2} \\ &= 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}}\partial_t^2 u_\nu\big)_{L^2} - 2\,\mathrm{Re}\big(v_\nu, T_{\partial_t^2(\alpha_\nu^{-1/2})} u_\nu\big)_{L^2} \\ &= -2\,\mathrm{Re}\big(v_\nu, T_{\partial_t^2(\alpha_\nu^{-1/2})} u_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}}\partial_j(T_{a_{jk}}\partial_k u_\nu)\big)_{L^2} \\ &\quad + 2\,\mathrm{Re}\big(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}}\partial_j([\Delta_\nu,\ T_{a_{jk}}]\partial_k u)\big)_{L^2} + 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}}(\tilde{L}u)_\nu\big)_{L^2}. \end{aligned}$$

We have

$$\left|2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}}(\tilde{L}u)_\nu\big)_{L^2}\right| \le C(e_\nu)^{\frac{1}{2}}\,\|(\tilde{L}u)_\nu\|_{L^2},$$

and, from the fifth inequality in Prop. 6,

$$\left|2\,\mathrm{Re}\big(v_\nu, T_{\partial_t^2(\alpha_\nu^{-1/2})}u_\nu\big)_{L^2}\right| \le C\,\|v_\nu\|_{L^2}\,2^\nu\|u_\nu\|_{L^2} \le C\,e_\nu(t).$$

Therefore, we obtain

$$\begin{aligned}\frac{d}{dt}\|v_\nu(t)\|_{L^2}^2 = 2\,\mathrm{Re}\Big(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}}\partial_j(T_{a_{jk}}\partial_k u_\nu)\Big)_{L^2}\\ + 2\,\mathrm{Re}\Big(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}}\partial_j([\Delta_\nu,\ T_{a_{jk}}]\partial_k u)\Big)_{L^2}\\ + 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}}(\tilde{L}u)_\nu\big)_{L^2} + Q_1,\end{aligned} \tag{33}$$

with $|Q_1| \le Ce_\nu(t)$.

Next

$$\partial_t w_\nu = T_{\partial_t(\alpha_\nu^{1/2})(\gamma^2+|\xi|^2)^{1/2}}u_\nu + T_{\alpha_\nu^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\partial_t u_\nu,$$

so that

$$\begin{aligned}&\frac{d}{dt}\|w_\nu(t)\|_{L^2}^2\\
&= 2\,\mathrm{Re}\big(T_{\partial_t(\alpha_\nu^{1/2})(\gamma^2+|\xi|^2)^{1/2}}u_\nu, w_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(T_{\alpha_\nu^{1/2}(\gamma^2+|\xi|^2)^{1/2}}\partial_t u_\nu, w_\nu\big)_{L^2}\\
&= 2\,\mathrm{Re}\big(T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}}T_{-\partial_t(\alpha_\nu^{-1/2})}u_\nu, w_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(R_1u_\nu, w_\nu\big)_{L^2}\\
&\quad + 2\,\mathrm{Re}\big(T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}}T_{\alpha_\nu^{-1/2}}\partial_t u_\nu, w_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(R_2u_\nu, w_\nu\big)_{L^2}\\
&= 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}}w_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(v_\nu, R_3w_\nu\big)_{L^2}\\
&\quad + 2\,\mathrm{Re}\big(R_1u_\nu, w_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(R_2u_\nu, w_\nu\big)_{L^2}\\
&= 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}}T_{\alpha_\nu^{3/2}(\gamma^2+|\xi|^2)^{1/2}}w_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(v_\nu, R_4w_\nu\big)_{L^2}\\
&\quad + 2\,\mathrm{Re}\big(v_\nu, R_3w_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(R_1u_\nu, w_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(R_2u_\nu, w_\nu\big)_{L^2}\\
&= 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}}T_{\alpha_\nu^2(\gamma^2+|\xi|^2)}u_\nu\big)_{L^2}\\
&\quad + 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}}R_5u_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(v_\nu, R_4w_\nu\big)_{L^2}\\
&\quad + 2\,\mathrm{Re}\big(v_\nu, R_3w_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(R_1u_\nu, w_\nu\big)_{L^2} + 2\,\mathrm{Re}\big(R_2u_\nu, w_\nu\big)_{L^2}.\end{aligned}$$

It is a straightforward computation, from the results of symbolic calculus recalled in Prop. 7, to verify that all the operators R_1, R_2, R_3, R_4 and R_5 are 0-th order operators. Consequently,

$$\frac{d}{dt}\|w_\nu(t)\|_{L^2}^2 = 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}} T_{\alpha_\nu^2(\gamma^2+|\xi|^2)} u_\nu\big)_{L^2} + Q_2, \tag{34}$$

with $|Q_2| \le C e_\nu(t)$.

Finally, from (32),

$$\frac{d}{dt}\|z_\nu(t)\|_{L^2}^2 \le |2\,\mathrm{Re}\big(u_\nu, \partial_t u_\nu\big)_{L^2}| \le C e_\nu(t). \tag{35}$$

Now we pair the first term in right hand side of (33) with the first term in right hand side of (34). We obtain

$$\begin{aligned}
&|2\,\mathrm{Re}\big(v_\nu, \textstyle\sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j (T_{a_{jk}} \partial_k u_\nu)\big)_{L^2} + 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}} T_{\alpha_\nu^2(\gamma^2+|\xi|^2)} u_\nu\big)_{L^2}| \\
&\le C\, \|v_\nu\|_{L^2}\, \|\zeta_\nu\|_{L^2},
\end{aligned}$$

where

$$\begin{aligned}
\zeta_\nu &= T_{\alpha_\nu^2(\gamma^2+|\xi|^2)} u_\nu + \sum_{j,k} \partial_j (T_{a_{jk}} \partial_k u_\nu) \\
&= T_{\gamma^2 + \sum_{j,k} a_{jk,\nu}\xi_j\xi_k} u_\nu + \sum_{j,k} \partial_j (T_{a_{jk}} \partial_k u_\nu) \\
&= T_{\gamma^2} u_\nu + \sum_{j,k} (T_{a_{jk,\nu}\xi_j\xi_k} u_\nu + T_{\partial_j a_{jk}} \partial_k u_\nu - T_{a_{jk}\xi_j\xi_k} u_\nu).
\end{aligned}$$

We have

$$\|\sum_{j,k} T_{\partial_j a_{jk}} \partial_k u_\nu\|_{L^2} \le C \sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}} \|\nabla u_\nu\|_{L^2} \le C (e_\nu(t))^{\frac{1}{2}},$$

and, from Bernstein's inequalities and (19),

$$\|\sum_{j,k} T_{(a_{jk,\nu}-a_{jk})\xi_j\xi_k} u_\nu\|_{L^2} \le C \sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}}\, 2^{-\nu}\, \|\nabla^2 u_\nu\|_{L^2} \le C (e_\nu(t))^{\frac{1}{2}}.$$

From this we deduce

$$\|\zeta_\nu\|_{L^2} \le C (e_\nu(t))^{\frac{1}{2}}.$$

Summing up, from (33), (34) and (32) we get

$$\frac{d}{dt} e_\nu(t) \le C_1 e_\nu(t) + C_2 (e_\nu(t))^{\frac{1}{2}} \, \|(\tilde{L}u)_\nu\|_{L^2} + |2\,\mathrm{Re}\big(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j ([\Delta_\nu,\ T_{a_{jk}}]\partial_k u)\big)_{L^2}|. \tag{36}$$

4.3 Commutator Estimate

We want to estimate

$$|\sum_{j,k} 2\,\mathrm{Re}\big(v_\nu, T_{\alpha_\nu^{-1/2}} \partial_j ([\Delta_\nu,\ T_{a_{jk}}]\partial_k u)\big)_{L^2}|.$$

We remark that

$$\begin{aligned}
[\Delta_\nu,\ T_{a_{jk}}]w &= \Delta_\nu(S_{\mu-1}a_{jk}\, S_{\mu+2}w) + \Delta_\nu(\sum_{h=\mu}^{+\infty} S_h a_{jk}\, \Delta_{h+3}w) \\
&\quad - S_{\mu-1}a_{jk}\, S_{\mu+2}(\Delta_\nu w) - \sum_{h=\mu}^{+\infty} S_h a_{jk}\, \Delta_{h+3}(\Delta_\nu w) \\
&= \Delta_\nu(S_{\mu-1}a_{jk}\, S_{\mu+2}w) - S_{\mu-1}a_{jk}\, \Delta_\nu(S_{\mu+2}w) \\
&\quad + \sum_{h=\mu}^{+\infty} \Delta_\nu(S_h a_{jk}\, \Delta_{h+3}w) - \sum_{h=\mu}^{+\infty} S_h a_{jk}\, \Delta_\nu(\Delta_{h+3}w) \\
&= [\Delta_\nu,\ S_{\mu-1}a_{jk}]\, S_{\mu+2}w + \sum_{h=\mu}^{+\infty} [\Delta_\nu,\ S_h a_{jk}]\, \Delta_{h+3}w,
\end{aligned}$$

where we recall that μ is a fixed constant (depending on γ, which has been chosen at the beginning of Sect. 4.1). Hence we have

$$\begin{aligned}
&\partial_j([\Delta_\nu T_{a_{jk}}]\partial_k u) \\
&\qquad = \partial_j([\Delta_\nu,\ S_{\mu-1}a_{jk}]\, \partial_k(S_{\mu+2}u)) + \partial_j(\sum_{h=\mu}^{+\infty} [\Delta_\nu,\ S_h a_{jk}]\, \partial_k(\Delta_{h+3}u)).
\end{aligned}$$

Consider first

$$\partial_j([\Delta_\nu,\ S_{\mu-1}a_{jk}]\, \partial_k(S_{\mu+2}u)).$$

The support of the Fourier transform of $[\Delta_\nu,\ S_{\mu-1}a_{jk}]\,\partial_k(S_{\mu+2}u)$ is contained in $\{|\xi|\le 2^{\mu+4}\}$ and $[\Delta_\nu,\ S_{\mu-1}a_{jk}]\,\partial_k(S_{\mu+2}u)$ is identically 0 if $\nu\ge\mu+5$. From Bernstein's inequalities and [4, Th. 35] we deduce that

$$\|\partial_j([\Delta_\nu,\ S_{\mu-1}a_{jk}]\,\partial_k(S_{\mu+2}u))\|_{L^2}\le C\,2^\mu \sup_{j,k}\|a_{jk}\|_{\mathrm{Lip}}\,\|S_{\mu+2}u\|_{L^2}.$$

We have

$$\begin{aligned}\sum_{\nu=0}^{+\infty}2^{-2\nu\theta}|\sum_{j,k}2\,\mathrm{Re}\big(v_\nu,T_{\alpha_\nu^{-1/2}}\partial_j([\Delta_\nu,\ S_{\mu-1}a_{jk}]\,\partial_k(S_{\mu+2}u))\big)_{L^2}|&\\
\le C\,2^\mu\sup_{j,k}\|a_{jk}\|_{\mathrm{Lip}}\sum_{\nu=0}^{\mu+4}2^{-2\nu\theta}\|v_\nu\|_{L^2}(\sum_{h=0}^{\mu+2}\|u_h\|_{L^2})&\\
\le C\,2^{\mu+(\mu+4)\theta}\sup_{j,k}\|a_{jk}\|_{\mathrm{Lip}}\sum_{\nu=0}^{\mu+4}2^{-\nu\theta}\|v_\nu\|_{L^2}\sum_{h=0}^{\mu+4}2^{-h\theta}\|u_h\|_{L^2}&\\
\le C\sup_{j,k}\|a_{jk}\|_{\mathrm{Lip}}\sum_{h=0}^{\mu+4}2^{-2\nu\theta}\,e_\nu(t).&\end{aligned}$$

Consider then

$$\partial_j(\sum_{h=\mu}^{+\infty}[\Delta_\nu,\ S_h a_{jk}]\,\partial_k(\Delta_{h+3}u)).$$

Looking at the support of the Fourier transform, it is possible to see that

$$[\Delta_\nu,\ S_h a_{jk}]\,\partial_k(\Delta_{h+3}u)$$

is identically 0 if $|h+3-\nu|\ge 3$. As a consequence, the sum over h is reduced to at most 5 terms: $\partial_j([\Delta_\nu,\ S_{\nu-5}a_{jk}]\,\partial_k(\Delta_{\nu-2}u)),\ldots,\partial_j([\Delta_\nu,\ S_{\nu-1}a_{jk}]\,\partial_k(\Delta_{\nu+2}u))$. Each of these terms has the support of the Fourier transform contained in the ball $\{|\xi|\le 2^{\nu+4}\}$.

We consider the term $\partial_j([\Delta_\nu,\ S_{\nu-3}a_{jk}]\,\partial_k(\Delta_\nu u))$: for the other terms the estimate will be similar. Again by Bernstein's inequalities and [4, Th. 35] we infer

$$\|\partial_j([\Delta_\nu,\ S_{\nu-3}a_{jk}]\,\partial_k(\Delta_\nu u))\|_{L^2}\le C\,2^\nu\sup_{j,k}\|a_{jk}\|_{\mathrm{Lip}}\,\|\Delta_\nu u\|_{L^2},$$

and then

$$\begin{aligned}|\sum_{j,k}2\,\mathrm{Re}\big(v_\nu,T_{\alpha_\nu^{-1/2}}\partial_j(\sum_{h=\mu}^{+\infty}[\Delta_\nu,\ S_h a_{jk}]\,\partial_k(\Delta_{h+3}u))\big)_{L^2}|&\\
\le C\sup_{j,k}\|a_{jk}\|_{\mathrm{Lip}}\,(e_{\nu-2}(t)+e_{\nu-1}(t)+e_\nu(t)+e_{\nu+1}(t)+e_{\nu+2}(t)).&\end{aligned}$$

Thus we have

$$\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} \Big| \sum_{j,k} 2\,\mathrm{Re}\big(v_\nu,\, T_{\alpha_\nu^{-1/2}} \partial_j \big(\sum_{h=\mu}^{+\infty} [\Delta_\nu,\; S_h a_{jk}]\, \partial_k(\Delta_{h+3} u)\big)\big)_{L^2} \Big| \\ \leq C \sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}} \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t).$$

As a conclusion

$$\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} \Big| \sum_{j,k} 2\,\mathrm{Re}\big(v_\nu,\, T_{\alpha_\nu^{-1/2}} \partial_j ([\Delta_\nu,\; T_{a_{jk}}]\partial_k u)\big)_{L^2} \Big| \leq C_3 \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t), \tag{37}$$

where C_3 depends on γ, θ and $\sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}}$.

4.4 Final Estimate

From (36) and (37) we obtain

$$\begin{aligned} \frac{d}{dt} E_\theta(t) &\leq (C_1 + C_3) \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t) + C_2 \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} (e_\nu(t))^{\frac{1}{2}} \|(\tilde{L}u(t))_\nu\|_{L^2} \\ &\leq (C_1 + C_3) \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t) + C_2 \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} (e_\nu(t))^{\frac{1}{2}} \|(Lu(t))_\nu\|_{L^2} \\ &\quad + C_2 \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} (e_\nu(t))^{\frac{1}{2}} \big\|\big(\sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u)\big)_\nu\big\|_{L^2}. \end{aligned}$$

We have

$$\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} (e_\nu(t))^{\frac{1}{2}} \big\|\big(\sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u)\big)_\nu\big\|_{L^2} \\ \leq \Big(\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t)\Big)^{\frac{1}{2}} \Big(\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} \big\|\big(\sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u)\big)_\nu\big\|_{L^2}^2\Big)^{\frac{1}{2}}.$$

From (15) we deduce

$$\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} \big\|\big(\sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u)\big)_\nu\big\|_{L^2}^2 \leq C \,\Big\| \sum_{j,k} \partial_j((a_{jk} - T_{a_{jk}})\partial_k u)\Big\|_{H^{-\theta}}^2$$

Now, using [10, Prop. 3.5] in the case $\theta \in]0, 1[$ and [14, Th. 5.2.8] in the case $\theta = 0$,

$$\| \sum_{j,k} \partial_j ((a_{jk} - T_{a_{jk}}) \partial_k u) \|^2_{H^{-\theta}} \leq C(\sup_{j,k} \|a_{jk}\|_{\mathrm{Lip}}) \|u(t)\|_{H^{1-\theta}},$$

so that

$$\sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} (e_\nu(t))^{\frac{1}{2}} \| \big(\sum_{j,k} \partial_j ((a_{jk} - T_{a_{jk}}) \partial_k u) \big)_\nu \|_{L^2} \leq C_4 E_\theta(t),$$

and finally

$$\frac{d}{dt} E_\theta(t) \leq C(E_\theta(t) + (E_\theta(t))^{\frac{1}{2}} \|Lu(t)\|_{H^{-\theta}}).$$

The energy estimate (9) easily follows from this last inequality and the Grönwall Lemma.

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Note on the Wigner Distribution and Localization Operators in the Quasi-Banach Setting

Elena Cordero

Abstract Time-frequency analysis have played a crucial role in the development of localization operators in the last 20 years. We present its applications to the study of boundedness and Schatten Class property for such operators. In particular, new sufficient conditions for such operators to belong to the Schatten-von Neumann Class $S_p(L^2(\mathbb{R}^d))$, $0 < p < 1$, are exhibited. As a byproduct, sharp continuity results for the Wigner distribution are also presented.

Keywords Time-frequency analysis · Short-time Fourier transfrom · Wigner distribution · Modulation spaces

1 Introduction

Localization operators have a long-standing tradition among physicists, mathematicians and engineers. A special form of such operators called "Anti-Wick operators" had been used as a quantization procedure by Berezin [5, 29] in 1971. The terminology "Time-frequency localization operators" or simply "localization operators" is due to Daubechies, who wrote the popular papers [11, 12] appeared in 1988. From then onwards so many authors have written contributions on this topic that it is not possible to cite them all. In this note we shall focus on the time-frequency properties of such operators and we will exhibit the results known so far. Much has been done in terms of necessary and sufficient conditions for boundedness of such operators on suitable normed spaces, as well as their belonging to the Schatten-von Neumann Class $S_p(L^2(\mathbb{R}^d))$, $1 < p \leq \infty$. Here we focus on the quasi-Banach setting $0 < p < 1$ and present outcomes in this framework, while reviewing also the known results for the Banach case $p \geq 1$.

First, we introduce the main features of this study.

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The protagonists of time-frequency analysis are the operators of translation and modulation defined by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t), \qquad f \in L^2(\mathbb{R}^d). \tag{1}$$

For a fixed non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz class), the short-time Fourier transform, in short STFT, of $f \in \mathcal{S}'(\mathbb{R}^d)$ (the space of tempered distributions), with respect to the window g, is given by

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^d} f(t)\, \overline{g(t-x)}\, e^{-2\pi i \omega t}\, dt\,. \tag{2}$$

By means of the STFT, the time-frequency localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol a, analysis window function φ_1, and synthesis window function φ_2 can be formally defined as

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\, dx d\omega. \tag{3}$$

In particular, if $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then (3) is a well-defined continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. If $\varphi_1(t) = \varphi_2(t) = e^{-\pi t^2}$, then $A_a = A_a^{\varphi_1,\varphi_2}$ is the classical Anti-Wick operator and the mapping $a \mapsto A_a^{\varphi_1,\varphi_2}$ is understood as a quantization rule, cf. [5, 29] and the recent contribution [14].

In a weak sense, the definition of $A_a^{\varphi_1,\varphi_2}$ in (3) can be rephrased as

$$\langle A_a^{\varphi_1,\varphi_2} f, g\rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g\rangle = \langle a, \overline{V_{\varphi_1} f}\, V_{\varphi_2} g\rangle, \qquad f, g \in \mathcal{S}(\mathbb{R}^d)\,. \tag{4}$$

The definition in (3) has suggested the study of localization operators as a multilinear mapping

$$(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}. \tag{5}$$

In [7, 8, 10, 32, 33, 36] the boundedness of the map in (5) has been widely studied, in dependence on the function spaces of both symbol a and windows φ_1, φ_2. The sharpest Schatten-class results are obtained by choosing modulation space s as spaces for both symbol and windows, as observed in [8] and [10]; in those contributions the focus is limited to the Banach framework. Sharp compactness results for localization operators are contained in [16]. Finally, smoothness and decay of eigenfuctions for localization operators are studied in [4], see also [1–3].

Modulation spaces are (quasi-)Banach spaces that measure the concentration of functions and distributions on the time-frequency plane. Since the STFT is the mean to extract the time-frequency features of a function/distribution, the idea that leads to the definition of modulation space s is the following: *give a (quasi)norm* to the STFT. These spaces will be introduced in the following Sect. 2.2.

Another way to introduce localization operators is as a form of Weyl transform. The latter can be defined by means of another popular time-frequency representation, the cross-Wigner distribution. Namely, given two functions $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$, the *cross-Wigner distribution* $W(f_1, f_1)$ is defined to be

$$W(f_1, f_2)(x, \omega) = \int f_1(x + \frac{t}{2})\overline{f_2(x - \frac{t}{2})}e^{-2\pi i\omega t}\, dt. \tag{6}$$

The quadratic expression $Wf = W(f, f)$ is called the Wigner distribution of f.

Every continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ can be represented as a pseudodifferential operator in the Weyl form L_σ and the connection with the cross-Wigner distribution is provided by

$$\langle L_\sigma f, g\rangle = \langle \sigma, W(g, f)\rangle, \qquad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{7}$$

Localization operators $A_a^{\varphi_1,\varphi_2}$ can be represented as Weyl operators as follows (cf. [6, 7, 33])

$$A_a^{\varphi_1,\varphi_2} = L_{a*W(\varphi_2,\varphi_1)}, \tag{8}$$

so that the Weyl symbol of the localization operator $A_a^{\varphi_1,\varphi_2}$ is given by

$$\sigma = a * W(\varphi_2, \varphi_1)\,. \tag{9}$$

This representation of localization operators in the Weyl form, together with boundedness properties of Weyl operators and sharp continuity properties for the cross-Wigner distribution, yields to Schatten-class results for localization operators. In particular here we present new outcomes in the quasi-Banach setting, while reviewing the known results in the Banach framework, see Theorems 5 and 7 below.

The paper is organized as follows. Section 2 presents the basic definitions and properties of the Schatten-von Neumann Classes $S_p(L^2(\mathbb{R}^d))$, $0 < p \leq \infty$, of the modulation spaces and the time-frequency analysis tools needed to infer our results. Section 3 exhibits the sufficient conditions for localization operators to be in the Schatten-von Neumann classes S_p. To chase this goal, sharp continuity properties for the cross-Wigner distribution are presented. Such result is new in the framework of quasi-Banach modulation spaces and is the main ingredient to prove sufficient Schatten class conditions for localization operators. Section 4 contains necessary Schatten class results for localization operators and ends by showing perspectives and open problems about this topic.

2 Preliminaries on Schatten Classes, Modulation Spaces and Frames

2.1 Schatten-von Neumann Classes

We limit to consider the Hilbert space $L^2(\mathbb{R}^d)$. Let T be a compact operator on $L^2(\mathbb{R}^d)$. Then $T^*T\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is compact, self-adjoint, and non-negative. Hence, we can define the absolute value of T by $|T| = (T^*T)^{\frac{1}{2}}$, acting on $L^2(\mathbb{R}^d)$. Recall that $|T|$ is compact, self-adjoint, and non-negative, hence by the Spectral Theorem we can find an orthonormal basis $(\psi_n)_n$ for $L^2(\mathbb{R}^d)$ consisting of eigenvectors of $|T|$. The corresponding eigenvalues $s_1(T) \geq s_2(T) \geq \cdots \geq s_n(T) \geq \cdots \geq 0$, are called the singular values of T.

If $0 < p < \infty$ and the sequence of singular values is ℓ^p-summable, then T is said to belong to the Schatten-von Neumann class $S_p(L^2(\mathbb{R}^d))$. If $1 \leq p < \infty$, a norm is associated to $S_p(L^2(\mathbb{R}^d))$ by

$$\|T\|_{S_p} := \left(\sum_{n=1}^{\infty} s_n(T)^p\right)^{\frac{1}{p}}. \tag{10}$$

If $1 \leq p < \infty$ then $(S_p(L^2(\mathbb{R}^d)), \|\cdot\|_{S_p})$ is a Banach space whereas, for $0 < p < 1$, $(S_p(L^2(\mathbb{R}^d)), \|\cdot\|_{S_p})$ is a quasi-Banach space since the quantity $\|T\|_{S_p}$ defined in (10) is only a quasinorm.

For completeness, we define $S_\infty(L^2(\mathbb{R}^d))$ to be the space of bounded operators on $L^2(\mathbb{R}^d)$. The Schatten-von Neumann classes are nested, with $S_p \subset S_q$, for details on this topic we refer to [19, 25, 26, 29, 30, 37].

For $2 \leq p < \infty$ and T in $S_p(L^2(\mathbb{R}^d))$, we can express its norm by

$$\|T\|_{S_p}^p = \sup \sum_n \|T\phi_n\|_{L^2}^p, \tag{11}$$

the supremum being over all orthonormal bases $(\phi_n)_n$ of $L^2(\mathbb{R}^d)$. Then, it is a straightforward consequence (see [24, Theorem 12])

$$\left(\sum_n |\langle T\phi_n, \phi_n\rangle|^p\right)^{1/p} \leq \|T\|_{S_p}, \tag{12}$$

for every orthonormal basis $(\phi_n)_n$, $2 \leq p < \infty$. If $T \in S_2(L^2(\mathbb{R}^d))$ then T is called *Hilbert-Schmidt* operator. If $T \in S_1(L^2(\mathbb{R}^d))$ then T is said to be a *trace class* operator and the space S_1 is named the Trace Class.

Remark 1 For $0 < p < 2$, the characterization in (11) does not hold, in general. In fact, a simple example is shown by Bingyang, Khoi and Zhu in the paper [24]. Let us recall it for sake of clarity in the case of the Hilbert space $H = L^2(\mathbb{R}^d)$. Fix an orthonormal basis $(\phi_n)_n$ and consider the function $h \in L^2(\mathbb{R}^d)$ given by

$$h = \sum_{n=1}^{\infty} \frac{\phi_n}{\sqrt{n}\log(n+1)}.$$

Define the rank-one operator on $L^2(\mathbb{R}^d)$ by

$$Tf = \langle f, h \rangle h, \quad f \in L^2(\mathbb{R}^d).$$

We have

$$T\phi_n = \langle \phi_n, h \rangle h = \frac{h}{\sqrt{n}\log(n+1)}, \quad n \geq 1.$$

It follows that

$$\sum_{n=1}^{\infty} \|T\phi_n\|_{L^2}^p = \|h\|_{L^2}^p \sum_{n=1}^{\infty} \frac{1}{[\sqrt{n}\log(n+1)]^p} = \infty$$

for any $0 < p < 2$.

2.2 Modulation Spaces

Weight Functions

In the sequel v will always be a continuous, positive, submultiplicative weight function on $\mathbb{R}^d$, i.e., $v(z_1 + z_2) \leq v(z_1)v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^d$. We say that $m \in \mathscr{M}_v(\mathbb{R}^d)$ if m is a positive, continuous weight function on $\mathbb{R}^d$ *v-moderate*: $m(z_1 + z_2) \leq Cv(z_1)m(z_2)$ for all $z_1, z_2 \in \mathbb{R}^d$. We will mainly work with polynomial weights of the type

$$v_s(z) = \langle z \rangle^s = (1 + |z|^2)^{s/2}, \quad s \in \mathbb{R}, \quad z \in \mathbb{R}^d. \tag{13}$$

Observe that, for $s < 0$, v_s is $v_{|s|}$-moderate.

Given two weight functions m_1, m_2 on $\mathbb{R}^d$, we write

$$(m_1 \otimes m_2)(x, \omega) = m_1(x)m_2(\omega), \quad x, \omega \in \mathbb{R}^d.$$

Modulation Spaces We present the more general definition of such spaces, containing the quasi-Banach setting, introduced first by Y.V. Galperin and S. Samarah in [18].

Definition 1 Fix a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a weight $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ and $0 < p, q \leq \infty$. The modulation space $M^{p,q}_m(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the (quasi)norm

$$\|f\|_{M^{p,q}_m} = \|V_g f\|_{L^{p,q}_m} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_g f(x,\omega)|^p m(x,\omega)^p dx\right)^{\frac{q}{p}} d\omega\right)^{\frac{1}{q}} \tag{14}$$

(obvious changes with $p = \infty$ or $q = \infty$) is finite.

The most known modulation spaces are those $M^{p,q}_m(\mathbb{R}^d)$, with $1 \leq p, q \leq \infty$, introduced by H. Feichtinger in [15]. In that paper their main properties were exhibited; in particular we recall that they are Banach spaces, whose norm does not depend on the window g: different window functions in $\mathcal{S}(\mathbb{R}^d)$ yield equivalent norms. Moreover, the window class $\mathcal{S}(\mathbb{R}^d)$ can be extended to the modulation space $M^{1,1}_v(\mathbb{R}^d)$ (so-called Feichtinger algebra).

For shortness, we write $M^p_m(\mathbb{R}^d)$ in place of $M^{p,p}_m(\mathbb{R}^d)$ and $M^{p,q}(\mathbb{R}^d)$ if $m \equiv 1$.

The modulation spaces $M^{p,q}_m(\mathbb{R}^d)$, $0 < p, q < 1$, where introduced almost 20 years later by Y.V. Galperin and S. Samarah in [18]. In this framework, it appears that the largest natural class of windows universally admissible for all spaces $M^{p,q}_m(\mathbb{R}^d)$, $0 < p, q \leq \infty$ (with weight m having at most polynomial growth) is the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. Many properties related to the quasi-Banach setting are still unexplored.

The focus of this paper is on the quasi Banach setting, which allows to infer new results for localization operators.

In the sequel we shall use inclusion relations for modulation spaces (cf. [18, Theorem 3.4] and [20, Theorem 12.2.2]):

Theorem 1 *Let* $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. *If* $0 < p_1 \leq p_2 \leq \infty$ *and* $0 < q_1 \leq q_2 \leq \infty$ *then* $M^{p_1,q_1}_m(\mathbb{R}^d) \subseteq M^{p_2,q_2}_m(\mathbb{R}^d)$.

Remark 2 In our framework it is important to notice the following inclusion relation for $s > 0$:

$$M^{\infty}_{v_s \otimes 1}(\mathbb{R}^{2d}) \subset M^{p,\infty}(\mathbb{R}^{2d}) \quad \text{if } p > 2d/s. \tag{15}$$

This follows from the recent contribution [22, Theorem 1.5].

Let us recall convolution relations for modulations spaces. They are contained in the contributions [7] and [34] for the Banach framework. The more general case is exhibited in [4].

Proposition 1 *Let $\nu(\omega) > 0$ be an arbitrary weight function on $\mathbb{R}^d$, $0 < p, q, r, t, u, \gamma \leq \infty$, with*

$$\frac{1}{u} + \frac{1}{t} = \frac{1}{\gamma}, \tag{16}$$

and

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \qquad \text{for } 1 \leq r \leq \infty \tag{17}$$

whereas

$$p = q = r, \qquad \text{for } 0 < r < 1. \tag{18}$$

For $m \in \mathscr{M}_v(\mathbb{R}^{2d})$, $m_1(x) = m(x, 0)$ and $m_2(\omega) = m(0, \omega)$ are the restrictions to $\mathbb{R}^d \times \{0\}$ and $\{0\} \times \mathbb{R}^d$, and likewise for v. Then

$$M^{p,u}_{m_1 \otimes \nu}(\mathbb{R}^d) * M^{q,t}_{v_1 \otimes v_2 \nu^{-1}}(\mathbb{R}^d) \subseteq M^{r,\gamma}_m(\mathbb{R}^d) \tag{19}$$

with norm inequality

$$\|f * h\|_{M^{r,\gamma}_m} \lesssim \|f\|_{M^{p,u}_{m_1 \otimes \nu}} \|h\|_{M^{q,t}_{v_1 \otimes v_2 \nu^{-1}}}.$$

2.3 Frame Theory

A sequence of functions $\{b_j : j \in \mathscr{J}\}$ in $L^2(\mathbb{R}^d)$ is a *frame* for the Hilbert space $L^2(\mathbb{R}^d)$ if there exist positive constants $0 < A \leq B < \infty$, such that

$$A\|f\|^2_{L^2} \leq \sum_{j \in \mathscr{J}} |\langle f, b_j \rangle|^2 \leq B\|f\|^2_{L^2}, \quad \forall f \in L^2(\mathbb{R}^d). \tag{20}$$

The constants A and B are called *lower* and *upper* frame bounds, respectively. It is straightforward from (20) (or see, e.g., [23, Pag. 398]) to check the elements of a frame satisfy

$$\|b_j\|_{L^2} \leq \sqrt{B}, \quad \forall j \in \mathscr{J}. \tag{21}$$

Using (21), in [8] we extended the inequality in (12) from orthonormal bases to frames.

Lemma 1 *Let $(b_n)_n$ be a frame for $L^2(\mathbb{R}^d)$, as defined in (20), with upper bound B. If $T \in S_p(L^2(\mathbb{R}^d))$, for $1 \le p \le \infty$, then*

$$\left(\sum_{n=1}^{\infty} |\langle T b_n, b_n\rangle|^p\right)^{1/p} \le B\|T\|_{S_p}. \tag{22}$$

Observe that an orthonormal basis is a special instance of frame with upper bound $B = 1$; hence Lemma 1 provides an alternative proof to the inequality in (12), for every $1 \le p \le \infty$.

In the case $0 < p < 1$, Lemma 1 is false in general. This is a straightforward consequence of the following result [24, Proposition 22]:

Proposition 2 *Suppose $0 < p < 1$ and $(\phi_n)_n$ any orthonormal basis for $L^2(\mathbb{R}^d)$. Then there exists a positive operator $S \in S_p(L^2(\mathbb{R}^d))$ such that $(\langle S\phi_n, \phi_n\rangle)_n \notin \ell^p$.*

Since an orthonormal basis is a frame with frame bounds $A = B = 1$, it follows that the majorization (22) fails for $(\phi_n)_n$ and, consequently, Lemma 1 is false. For $p \ge 1$, a useful consequence of Lemma 1 is as follows (cf. [8, Corollary 2]):

Corollary 1 *Let $(b_n)_n$ be a frame with upper bound B. Let $L \in S_\infty(L^2(\mathbb{R}^d))$ and $T \in S_p(L^2(\mathbb{R}^d))$, with $1 \le p \le \infty$. Then we have*

$$\left(\sum_{n=1}^{\infty} |\langle T b_n, L b_n\rangle|^p\right)^{1/p} \le B\|T\|_{S_p}\|L\|_{S_\infty}. \tag{23}$$

In [13, Proposition 10], see also [27, 28], it is proved that, if $\alpha\beta < 1$ and

$$\varphi := 2^{d/4} e^{-\pi x^2}, \tag{24}$$

then the set of the Gaussian time-frequency shift $(M_{\beta n} T_{\alpha k}\varphi)_{n,k\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^{2d})$ (called Gabor frame). In the sequel we shall also use the Gabor frames on $L^2(\mathbb{R}^{2d})$ given by

$$(M_{\beta n} T_{\alpha k}\Phi)_{k,n\in\mathbb{Z}^{2d}},$$

where Φ is the $2d$-dimensional Gaussian function below

$$\Phi(x,\omega) := 2^{-d} e^{-\pi(x^2+\omega^2)}, \quad (x,\omega) \in \mathbb{R}^{2d}. \tag{25}$$

It is easy to compute (or see, e.g., [20, Lemma 1.5.2]) that

$$V_\varphi\varphi(x,\omega) = 2^{-d/2} e^{-\pi i x\omega} e^{-\frac{\pi}{2}(x^2+\omega^2)}. \tag{26}$$

Definition 2 For $0 < p, q \leq \infty$, $m \in \mathscr{M}_v(\mathbb{Z}^{2d})$, the space $\ell_m^{p,q}(\mathbb{Z}^{2d})$ consists of all sequences $c = (c_{k,n})_{k,n \in \mathbb{Z}^d}$ for which the (quasi-)norm

$$\|c\|_{\ell_m^{p,q}} = \left(\sum_{n \in \mathbb{Z}^d} \left(\sum_{k \in \mathbb{Z}^d} |c_{k,n}|^p m(k,n)^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

(with obvious modification for $p = \infty$ or $q = \infty$) is finite.

For $p = q$, $\ell_m^{p,q}(\mathbb{Z}^{2d}) = \ell_m^p(\mathbb{Z}^{2d})$, the standard spaces of sequences. Namely, in dimension d, for $0 < p \leq \infty$, m a weight function on $\mathbb{Z}^d$, a sequence $c = (c_k)_{k \in \mathbb{Z}^d}$ is in $\ell_m^p(\mathbb{Z}^d)$ if

$$\|c\|_{\ell_m^p} = \left(\sum_{k \in \mathbb{Z}^d} |c_k|^p m(k)^p \right)^{\frac{1}{p}} < \infty.$$

Discrete equivalent modulation spaces norms are produced by means of Gabor frames. The key result is the following characterization for the $M_m^{p,q}$- norm of localization symbols (see [20, Chapter 12] for $1 \leq p, q \leq \infty$, and [18, Theorem 3.7] for $0 < p, q < 1$).

Theorem 2 *Assume $m \in \mathscr{M}_v(\mathbb{R}^{2d})$, $0 < p, q \leq \infty$. Consider the Gabor frame $(M_{\beta n} T_{\alpha k} \Phi)_{k,n \in \mathbb{Z}^{2d}}$ with Gaussian window Φ in (25). Then, for every $a \in M_m^{p,q}(\mathbb{R}^{2d})$,*

$$\|a\|_{M_m^{p,q}(\mathbb{R}^{2d})} \asymp \|(\langle a, M_{\beta n} T_{\alpha k} \Phi \rangle_{n,k \in \mathbb{Z}^{2d}})_{n,k \in \mathbb{Z}^{2d}}\|_{\ell_m^{p,q}(\mathbb{Z}^{4d})}. \tag{27}$$

2.4 Time-Frequency Tools

In the sequel we shall need to compute the STFT of the cross-Wigner distribution, contained below [20, Lemma 14.5.1]:

Lemma 2 *Fix a nonzero $g \in \mathcal{S}(\mathbb{R}^d)$ and let $\Phi = W(g, g) \in \mathcal{S}(\mathbb{R}^{2d})$. Then the STFT of $W(f_1, f_2)$ with respect to the window Φ is given by*

$$V_\Phi(W(f_1, f_2))(z, \zeta) = e^{-2\pi i z_2 \zeta_2} \overline{V_g f_2(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2})} V_g f_1(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2}) . \tag{28}$$

The following properties of the STFT (cf. [8, Lemma 1]) can be used to prove necessary Schatten class conditions for localization operators.

Lemma 3 *If* $z = (z_1, z_2) \in \mathbb{R}^{2d}$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, *then*

$$T_{(z_1,z_2)}(\overline{V_{\varphi_1} f} \cdot V_{\varphi_2} g)(x,\omega) = \overline{V_{\varphi_1}(M_{z_2}T_{z_1} f)}(x,\omega)\, V_{\varphi_2}(M_{z_2}T_{z_1} g)(x,\omega), \tag{29}$$

$$M_{(\zeta_1,\zeta_2)}\big(\overline{V_{\varphi_1} f}\, V_{\varphi_2} g\big)(x,\omega) = \overline{V_{\varphi_1} f(x,\omega)}\, V_{(M_{\zeta_1}T_{-\zeta_2}\varphi_2)}(M_{\zeta_1}T_{-\zeta_2} g)(x,\omega), \tag{30}$$

$$M_\zeta T_z(\overline{V_{\varphi_1} f} V_{\varphi_2} g) = \overline{V_{\varphi_1}(M_{z_1}T_{z_2} f)} V_{(M_{\zeta_1}T_{-\zeta_2}\varphi_2)}(M_{\zeta_1}T_{-\zeta_2}M_{z_1}T_{z_2} g). \tag{31}$$

3 Sufficient Conditions for Schatten Class S_p, $0 < p \leq \infty$

In this Section we present sufficient conditions for Schatten Class properties of localization operators. The Banach case $p \geq 1$ was studied in [7, 8]. The main result (cf. Theorem 5 below) will take care of the full range $0 < p \leq \infty$.

First, we need to recall similar properties for Weyl operators, obtained in several papers, we refer the interested reader to [7, 20, 21, 31, 34].

Theorem 3 *For* $0 < p \leq \infty$, *we have:*

(i) If $0 < p \leq 2$ *and* $\sigma \in M^p(\mathbb{R}^{2d})$, *then* $L_\sigma \in S_p$ *and* $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^p}$.
(ii) If $2 \leq p \leq \infty$ *and* $\sigma \in M^{p,p'}(\mathbb{R}^{2d})$, *then* $L_\sigma \in S_p$ *and* $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^{p,p'}}$.

Proof The proof for $p \geq 1$ can be found in [7, Theorem 3.1], see also references therein. The case $0 < p < 1$ is contained in [35, Theorem 3.4]. □

We now focus on the properties of the cross-Wigner distribution, which enjoys the following property.

Theorem 4 *Assume* $p_i, q_i, p, q \in (0, \infty]$, $i = 1, 2$, $s \in \mathbb{R}$, *such that*

$$p_i, q_i \leq q, \qquad i = 1, 2 \tag{32}$$

and that

$$\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p} + \frac{1}{q}, \qquad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{p} + \frac{1}{q}. \tag{33}$$

Then, if $f_1 \in M^{p_1,q_1}_{v_{|s|}}(\mathbb{R}^d)$ *and* $f_2 \in M^{p_2,q_2}_{v_s}(\mathbb{R}^d)$ *we have* $W(f_1, f_2) \in M^{p,q}_{1\otimes v_s}(\mathbb{R}^{2d})$, *and*

$$\|W(f_1, f_2)\|_{M^{p,q}_{1\otimes v_s}} \lesssim \|f_1\|_{M^{p_1,q_1}_{v_{|s|}}} \|f_2\|_{M^{p_2,q_2}_{v_s}}. \tag{34}$$

Vice versa, assume that there exists a constant $C > 0$ *such that*

$$\|W(f_1, f_2)\|_{M^{p,q}} \le C\|f_1\|_{M^{p_1,q_1}}\|f_2\|_{M^{p_2,q_2}}, \quad \forall f_1, f_2 \in \mathcal{S}(\mathbb{R}^{2d}). \tag{35}$$

Then (32) *and* (33) *must hold.*

Proof *Sufficient Conditions.* The result for the indices $p_i, q_i, p, q \in [1, \infty]$ is proved in [10, Theorem 3.1]. The general case follows easily from that one, since the main tool is provided by the inclusion relations for modulation spaces in (1). We detail its steps for sake of clarity.

First, study the case both $0 < p, q < \infty$. Let $g \in \mathcal{S}(\mathbb{R}^d)$ and set $\Phi = W(g, g) \in \mathcal{S}(\mathbb{R}^{2d})$. If $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, we write $\tilde{\zeta} = (\zeta_2, -\zeta_1)$. Then, from Lemma 2,

$$|V_\Phi(W(f_1, f_2))(z, \zeta)| = |V_g f_2(z + \tfrac{\tilde{\zeta}}{2})|\,|V_g f_1(z - \tfrac{\tilde{\zeta}}{2})|. \tag{36}$$

Hence,

$$\|W(f_1, f_2)\|_{M^{p,q}_{1\otimes v_s}} \asymp \left(\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^{2d}}|V_g f_2(z + \tfrac{\tilde{\zeta}}{2})|^p\,|V_g f_1(z - \tfrac{\tilde{\zeta}}{2})|^p\,dz\right)^{\frac{q}{p}}\langle\zeta\rangle^{sq}\,d\zeta\right)^{1/q}.$$

Making the change of variables $z \mapsto z - \tilde{\zeta}/2$, the integral over z becomes the convolution $(|V_g f_2|^p * |(V_g f_1)^*|^p)(\tilde{\zeta})$, and observing that $(1 \otimes v_s)(z, \zeta) = \langle\zeta\rangle^s = v_s(\zeta) = v_s(\tilde{\zeta})$, we obtain

$$\begin{aligned}\|W(f_1, f_2)\|_{M^{p,q}_{1\otimes v_s}} &\asymp \left(\iint_{\mathbb{R}^{2d}}(|V_g f_2|^p * |(V_g f_1)^*|^p)^{\frac{q}{p}}(\tilde{\zeta})v_s(\tilde{\zeta})^q\,d\zeta\right)^{1/p}\\ &= \|\,|V_g f_2|^p * |(V_g f_1)^*|^p\,\|^{\frac{1}{p}}_{L^{\frac{q}{p}}_{v_{ps}}}.\end{aligned}$$

Hence

$$\|W(f_1, f_2)\|^p_{M^{p,q}_{1\otimes v_s}} \asymp \|\,|V_g f_2|^p * |(V_g f_1)^*|^p\,\|_{L^{\frac{q}{p}}_{v_{ps}}}. \tag{37}$$

Case $0 < p \le q < \infty$.

Step 1 Consider first the case $p \le p_i, q_i$, $i = 1, 2$, satisfying the condition

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p} + \frac{1}{q}, \tag{38}$$

(and hence $p_i, q_i \leq q$, $i = 1, 2$). Since $q/p \geq 1$, we can apply Young's Inequality for mixed-normed spaces [17] and majorize (37) as follows

$$\begin{aligned}\|W(f_1, f_2)\|^p_{M^{p,q}_{1\otimes v_s}} &\lesssim \| \,|V_g f_2|^p\|_{L^{r_2,s_2}_{v_{p|s|}}} \| \,|(V_g f_1)^*|^p\|_{L^{r_1,s_1}_{v_{ps}}} \\ &= \||V_g f_1|^p\|_{L^{r_1,s_1}_{v_{p|s|}}} \| \,|V_g f_2|^p\|_{L^{r_2,s_2}_{v_{ps}}} \\ &= \|V_g f_1\|^p_{L^{pr_1,ps_1}_{v_{|s|}}} \|V_g f_2\|^p_{L^{pr_2,ps_2}_{v_s}},\end{aligned}$$

for every $1 \leq r_1, r_2, s_1, s_2 \leq \infty$ such that

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{s_1} + \frac{1}{s_2} = 1 + \frac{p}{q}. \tag{39}$$

Choosing $r_i = p_i/p \geq 1$, $s_i = q_i/p \geq 1$, $i = 1, 2$, the indices' relation (39) becomes (38) and we obtain

$$\|W(f_1, f_2)\|_{M^{p,q}_{1\otimes v_s}} \lesssim \|V_g f_1\|_{L^{p_1,q_1}_{v_{|s|}}} \|V_g f_2\|_{L^{p_2,q_2}_{v_s}} \asymp \|f_1\|_{M^{p_1,q_1}_{v_{|s|}}} \|f_2\|_{M^{p_2,q_2}_{v_s}}.$$

Now, still assume $p \leq p_i, q_i$, $i = 1, 2$ but

$$\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p} + \frac{1}{q},$$

(hence $p_i, q_i \leq q$, $i = 1, 2$). We set $u_1 = tp_1$, and look for $t \geq 1$ (hence $u_1 \geq p_1$) such that

$$\frac{1}{u_1} + \frac{1}{p_2} = \frac{1}{p} + \frac{1}{q}$$

that gives

$$0 < \frac{1}{t} = \frac{p_1}{p} + \frac{p_1}{q} - \frac{p_1}{p_2} \leq 1$$

because $p_1(1/p + 1/q) - p_1/p_2 \leq p_1(1/p_1 + 1/p_2) - p_1/p_2 = 1$ whereas the lower bound of the previous estimate follows by $1/(tp_1) = 1/p + 1/q - 1/p_2 > 0$ since $p \leq p_2$. Hence the previous part of the proof gives

$$\|W(f_1, f_2)\|_{M^{p,q}_{1\otimes v_s}} \lesssim \|f_1\|_{M^{u_1,q_1}_{v_{|s|}}} \|f_2\|_{M^{p_2,q_2}_{v_s}} \lesssim \|f_1\|_{M^{p_1,q_1}_{v_{|s|}}} \|f_2\|_{M^{p_2,q_2}_{v_s}},$$

where the last inequality follows by inclusion relations for modulations spaces $M^{p_1,q_1}_{v_s}(\mathbb{R}^d) \subseteq M^{u_1,q_1}_{v_s}(\mathbb{R}^d)$ for $p_1 \leq u_1$.

The general case

$$\frac{1}{p_1}+\frac{1}{p_2}\geq\frac{1}{p}+\frac{1}{q},\quad \frac{1}{q_1}+\frac{1}{q_2}\geq\frac{1}{p}+\frac{1}{q},$$

is similar.

Step 2 Assume now that $0 < p_i, q_i \leq q$, $i = 1, 2$, and satisfy relation (33). If at least one out of the indices p_1, p_2 is less than p, assume for instance $p_1 \leq p$, whereas $p \leq q_1, q_2$, then we proceed as follows. We choose $u_1 = p$, $u_2 = q$, and deduce by the results in Step 1 (with $p_1 = u_1$ and $p_2 = u_2$) that

$$\|W(f_1,f_2)\|_{M^{p,q}_{1\otimes v_s}} \lesssim \|f_1\|_{M^{u_1,q_1}_{v_{|s|}}}\|f_2\|_{M^{u_2,q_2}_{v_s}} \lesssim \|f_1\|_{M^{p_1,q_1}_{v_{|s|}}}\|f_2\|_{M^{p_2,q_2}_{v_s}}$$

where the last inequality follows by inclusion relations for modulation spaces, since $p_1 \leq u_1 = p$ and $p_2 \leq u_2 = q$.

Similarly we argue when at least one out of the indices q_1, q_2 is less than p and $p \leq p_1, p_2$ or when at least one out of the indices q_1, q_2 is less than p and at least one out of the indices p_1, p_2 is less than p. The remaining case $p \leq p_i, q_i \leq q$ is treated in Step 1.

Case $0 < p < q = \infty$ The argument are similar to the case $0 < p \leq q < \infty$.

Case $p = q = \infty$ We use (36) and the submultiplicative property of the weight v_s,

$$\begin{aligned}\|W(f_1,f_2)\|_{M^{\infty}_{1\otimes v_s}} &= \sup_{z,\zeta\in\mathbb{R}^{2d}} |V_g f_2(z+\tfrac{\tilde{\zeta}}{2})|\,|V_g f_1(z-\tfrac{\tilde{\zeta}}{2})|v_s(\zeta)\\
&= \sup_{z,\zeta\in\mathbb{R}^{2d}} ||V_g f_2(z)|\,|(V_g f_1)^*(z-\tilde{\zeta})|v_s(\zeta)\\
&= \sup_{z,\zeta\in\mathbb{R}^{2d}} ||V_g f_2(z)|\,|(V_g f_1)^*(z-\tilde{\zeta})|v_s(\tilde{\zeta})\\
&\leq \sup_{z\in\mathbb{R}^{2d}} (\|V_g f_1 v_{|s|}\|_\infty\, |V_g f_2(z) v_s(z)|) = \|V_g f_1 v_{|s|}\|_\infty \|V_g f_2 v_s\|_\infty\\
&\asymp \|f\|_{M^{\infty}_{v_{|s|}}}\|g\|_{M^{\infty}_{v_s}} \leq \|f\|_{M^{p_1,q_1}_{v_{|s|}}}\|f\|_{M^{p_2,q_2}_{v_s}},\end{aligned}$$

for every $0 < p_i, q_i \leq \infty$, $i = 1, 2$.

Case $p > q$ Using the inclusion relations for modulation spaces, we majorize

$$\|W(f_1,f_2)\|_{M^{p,q}_{1\otimes v_s}} \lesssim \|W(f_1,f_2)\|_{M^{q,q}_{1\otimes v_s}} \lesssim \|f_1\|_{M^{p_1,q_1}_{v_{|s|}}}\|f_2\|_{M^{p_2,q_2}_{v_s}}$$

for every $0 < p_i, q_i \leq q$, $i = 1, 2$. Here we have applied the case $p \leq q$ with $p = q$. Notice that in this case condition (35) is trivially satisfied, since from $p_1, q_i \leq q$ we infer $1/p_1 + 1/p_2 \geq 1/q + 1/q$, $1/q_1 + 1/q_2 \geq 1/q + 1/q$. This ends the proof of the sufficient conditions.

Necessary Conditions The proof works exactly the same as that of [10, Theorem 3.5]. In fact, the main point is the use of the $M^{r,s}$-norm of the rescaled Gaussian $\varphi_\lambda(x) = \varphi(\sqrt{\lambda}x)$, with $\varphi(x) = e^{-\pi x^2}$, for which we reckon (see also [9, Lemma 3.2] and [34, Lemma 1.8]):

$$\|\varphi_\lambda\|_{M^{r,s}} \asymp \lambda^{-\frac{d}{2r}}(\lambda+1)^{-\frac{d}{2}(1-\frac{1}{s}-\frac{1}{r})},$$

for every $0 < r, s \leq \infty$. □

Based on the tools developed above, we establish the following Schatten class results for localization operators.

Theorem 5 *For $s \geq 0$, we have the following statements.*

(i) If $0 < p < 1$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ is bounded from $M^{p,\infty}_{1\otimes v_{-s}}(\mathbb{R}^{2d}) \times M^p_{v_s}(\mathbb{R}^d) \times M^p_{v_s}(\mathbb{R}^d)$ into S_p:

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{p,\infty}_{1\otimes v_{-s}}} \|\varphi_1\|_{M^p_{v_s}} \|\varphi_2\|_{M^p_{v_s}} .$$

(ii) If $1 \leq p \leq 2$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ is bounded from $M^{p,\infty}_{1\otimes v_{-s}}(\mathbb{R}^{2d}) \times M^1_{v_s}(\mathbb{R}^d) \times M^p_{v_s}(\mathbb{R}^d)$ into S_p:

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{p,\infty}_{1\otimes v_{-s}}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^p_{v_s}} .$$

(iii) If $2 \leq p \leq \infty$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ is bounded from $M^{p,\infty}_{1\otimes v_{-s}} \times M^1_{v_s} \times M^{p'}_{v_s}$ into S_p:

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{p,\infty}_{1\otimes v_{-s}}} \|\varphi_1\|_{M^1_{v_s}} \|\varphi_2\|_{M^{p'}_{v_s}} .$$

Proof (i) If $\varphi_1 \in M^p_{v_s}(\mathbb{R}^d)$ and $\varphi_2 \in M^p_{v_s}(\mathbb{R}^d)$, then $W(\varphi_2, \varphi_1) \in M^p_{1\otimes v_s}(\mathbb{R}^{2d})$ by (34). Since $a \in M^{p,\infty}_{1\otimes v_{-s}}$, the convolution relation $M^{p,\infty}_{1\otimes v_{-s}}(\mathbb{R}^{2d}) * M^p_{1\otimes v_s}(\mathbb{R}^{2d}) \subseteq M^p(\mathbb{R}^{2d})$ of Proposition 1 implies that the Weyl symbol $\sigma = a * W(\varphi_2, \varphi_1)$ is in $M^p(\mathbb{R}^{2d})$. The result now follows from Theorem 3 (i).

The items (ii) and (ii) are proved similarly, see [7, Theorem 3.1]. □

Corollary 2 *Any localization operators $A_a^{\varphi_1,\varphi_2}$ with symbol a in $M^\infty_{v_s\otimes 1}(\mathbb{R}^{2d})$, $s > 0$, and windows φ_1, φ_2 in $\mathcal{S}(\mathbb{R}^d)$ is a compact operator belonging to the Schatten class $S_p(L^2(\mathbb{R}^d))$, with $p > 2d/s$.*

Proof It immediately follows from the inclusion relations for modulation spaces in (15) and the sufficient conditions in Theorem 5. □

4 Necessary Conditions

The necessary conditions for Schatten class localization operators for the Banach case $p \geq 1$ is contained in the work [8, Theorem 1 (b)], see also [16], who recaptured the results in [8, Theorem 1 (b)] by using different techniques. Before stating the necessary conditions, observe that using the inclusion relations for modulation spaces in Theorem 1, one can rephrase the unweighted sufficient conditions in Theorem 5 as follows.

Theorem 6 *If $1 \leq p \leq \infty$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A_a^{\varphi_1,\varphi_2}$ is bounded from $M^{p,\infty}(\mathbb{R}^{2d}) \times M^1(\mathbb{R}^d) \times M^1(\mathbb{R}^d)$ into $S_p(L^2(\mathbb{R}^d))$, i.e.,*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \leq C\|a\|_{M^{p,\infty}}\|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1}$$

for a suitable constant $C > 0$.

Proof The inequality immediately follows from Theorem 5 and the estimate $\|\varphi_2\|_p \leq \|\varphi_2\|_1$, for any $p > 1$, by the inclusion relation $M^1(\mathbb{R}^d) \subset M^p(\mathbb{R}^d)$. $\square$

The vice versa of the sufficient conditions above is shown hereafter.

Theorem 7 *Consider $1 \leq p \leq \infty$. If $A_a^{\varphi_1,\varphi_2} \in S_p(L^2(\mathbb{R}^d))$ for every pair of windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ with norm estimate*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \leq C\,\|\varphi_1\|_{M^1}\,\|\varphi_2\|_{M^1}, \tag{40}$$

where the constant $C > 0$ depends only on the symbol a, then $a \in M^{p,\infty}(\mathbb{R}^{2d})$.

In what follows we detail the main steps of the proof, in order to underline the tools employed. The key role is played by Corollary 1, together with the characterization of the $M^{p,\infty}(\mathbb{R}^{2d})$-norm of the symbol a via Gabor frames.

Sketch of the Proof of Theorem 7 Consider $0 < \alpha, \beta < 1$, $\Phi(x, \omega) = 2^{-d}e^{-x^2-\omega^2} \in \mathcal{S}(\mathbb{R}^{2d})$ and the Gabor frame $(T_{\alpha k}M_{\beta n}\Phi)_{n,k\in\mathbb{Z}^{2d}}$. We compute the $M^{p,\infty}(\mathbb{R}^{2d})$-norm of the symbol a in $A_a^{\varphi_1,\varphi_2}$ by using the norm characterization in (27)

$$\|a\|_{M^{p,\infty}(\mathbb{R}^{2d})} \asymp \|\langle a, M_{\beta n}T_{\alpha k}\Phi\rangle_{n,k\in\mathbb{Z}^{2d}}\|_{\ell^{p,\infty}(\mathbb{Z}^{4d})}. \tag{41}$$

Using (26) we can write

$$\Phi(x, \omega) = 2^{-d}e^{-\pi(x^2+\omega^2)} = V_\varphi\varphi(x, \omega)\overline{V_\varphi\varphi(x, \omega)}. \tag{42}$$

Now, let $k = (k_1, k_2), n = (n_1, n_2) \in \mathbb{Z}^{2d}$, by (42) and Formula (31), the time-frequency shift of Φ can be expressed by the point-wise product of two STFTs:

$$\begin{aligned} M_{\beta n} T_{\alpha k} \Phi(x, \omega) &= M_{(\beta n_1, \beta n_2)} T_{(\alpha k_1, \alpha k_2)} (V_\varphi \varphi \overline{V_\varphi \varphi})(x, \omega) \\ &= V_{(M_{\beta n_1} T_{-\beta n_2} \varphi)} (M_{\beta n_1} T_{-\beta n_2} M_{\alpha k_2} T_{\alpha k_1} \varphi) \cdot \overline{V_\varphi (M_{\alpha k_2} T_{\alpha k_1} \varphi)}. \end{aligned}$$

Using the weak definition of localization operator given in (4), we can write

$$\langle a, M_{\beta n} T_{\alpha k} \Phi \rangle = \langle A_a^{\varphi, (M_{\beta n_1} T_{-\beta n_2} \varphi)} (M_{\alpha k_2} T_{\alpha k_1} \varphi), M_{\beta n_1} T_{-\beta n_2} M_{\alpha k_2} T_{\alpha k_1} \varphi \rangle. \tag{43}$$

The $M^{p,\infty}$-norm of the symbol a can be recast as

$$\begin{aligned} \|a\|_{M^{p,\infty}} &\asymp \| \langle a, M_{\beta n} T_{\alpha k} \Phi \rangle_{n,k \in \mathbb{Z}^{2d}} \|_{\ell^{p,\infty}(\mathbb{Z}^{4d})} \\ &= \sup_{n \in \mathbb{Z}^{2d}} \left(\sum_{k \in \mathbb{Z}^{2d}} |\langle a, M_{\beta n} T_{\alpha k} \Phi \rangle|^p \right)^{1/p} \\ &= \sup_{(n_1,n_2) \in \mathbb{Z}^{2d}} \left(\sum_{(k_1,k_2) \in \mathbb{Z}^{2d}} |\langle A_a^{\varphi, (M_{\beta n_1} T_{-\beta n_2} \varphi)} (M_{\alpha k_2} T_{\alpha k_1} \varphi), M_{\beta n_1} T_{-\beta n_2} M_{\alpha k_2} T_{\alpha k_1} \varphi \rangle|^p \right)^{1/p} \end{aligned}$$

We apply the assumption (40) to the localization operators $A_a^{\varphi, (M_{\beta n_1} T_{-\beta n_2} \varphi)}$; in fact, for every choice of β, n_1, n_2, the functions $M_{\beta n_1} T_{-\beta n_2} \varphi$ are in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, so that the localization operators satisfy the uniform estimate

$$\| A_a^{\varphi, (M_{\beta n_1} T_{-\beta n_2} \varphi)} \|_{S_p} \leq C \|\varphi\|_{M^1} \| M_{\beta n_1} T_{-\beta n_2} \varphi \|_{M^1} = C \|\varphi\|_{M^1}^2, \tag{44}$$

since the time-frequency shifts are isometry on $M^1(\mathbb{R}^d)$.

Finally, applying Corollary 1 with the Gabor frame $(M_{\alpha k_2} T_{\alpha k_1} \varphi)_{k_1, k_2 \in \mathbb{Z}^d}$ and operators $T = A_a^{\varphi, (M_{\beta n_1} T_{-\beta n_2} \varphi)} \in S_p$ and $L = M_{\beta n_1} T_{-\beta n_2} \in S_\infty$, we can majorize the norm $\|a\|_{M^{p,\infty}}$ as

$$\begin{aligned} &\|a\|_{M^{p,\infty}} \\ &\asymp \sup_{(n_1,n_2) \in \mathbb{Z}^{2d}} \| \langle A_a^{\varphi, (M_{\beta n_1} T_{-\beta n_2} \varphi)} (M_{\alpha k_2} T_{\alpha k_1} \varphi), M_{\beta n_1} T_{-\beta n_2} M_{\alpha k_2} T_{\alpha k_1} \varphi \rangle_{(k_1,k_2) \in \mathbb{Z}^{2d}} \|_{\ell^p(\mathbb{Z}^{2d})} \\ &\lesssim \sup_{(n_1,n_2) \in \mathbb{Z}^{2d}} \| A_a^{\varphi, (M_{\beta n_1} T_{-\beta n_2} \varphi)} \|_{S_p} \\ &\lesssim \sup_{(n_1,n_2) \in \mathbb{Z}^{2d}} \|\varphi\|_{M^1}^2 = \|\varphi\|_{M^1}^2 < \infty, \end{aligned}$$

where in the last inequality we used (44). □

4.1 Conclusion and Perspectives

As it becomes clear from the previous proof, we cannot expect to prove necessary conditions for small p, that is $0 < p < 1$, using similar techniques to the case $p \geq 1$. The main obstruction being the fact that Corollary 1 does not hold for $0 < p < 1$. Observe that the discrete modulation norm via Gabor frames in (41) remains valid also for $0 < p < 1$. In view of the sufficient conditions in Theorem 5, we conjecture that a necessary condition of the type expressed below should hold true.

Theorem 8 *For* $0 < p < 1$, *if* $A_a^{\varphi_1,\varphi_2}$ *is in* $S_p(L^2(\mathbb{R}^d))$ *for every pair of windows* $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ *and there exists a* $C > 0$ *such that*

$$\|A_a^{\varphi_1,\varphi_2}\|_{S_p} \leq C\|\varphi_1\|_{M^p}\|\varphi_2\|_{M^p}, \quad \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d),$$

then $a \in M^{p,\infty}(\mathbb{R}^{2d})$.

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Wavefronts in Traffic Flows and Crowds Dynamics

Andrea Corli and Luisa Malaguti

Abstract In this paper we give an overview of some recent results concerning partial differential equations modeling collective movements, namely, vehicular traffic flows and crowds dynamics. The focus is on traveling-wave solutions to degenerate parabolic equations in one space dimension, even if we briefly discuss models based on different equations. The case of networks is also taken into consideration. The parabolic degeneration opens the possibilities of several different behaviors of the traveling-wave solutions, which are investigated in details.

Keywords Traveling waves · Degenerate diffusion-convection reaction equations · Sharp profiles · Networks · Semi-wavefronts · Crowds dynamics

1 Introduction

The first papers dealing with mathematical models of vehicular traffic flows appeared in the mid 1950s of the last century, and are due to M.J. Lighthill, G.B. Whitham and P.I. Richards [70, 86]. In both papers the model consists of the single equation

$$\rho_t + (\rho v(\rho))_x = 0, \tag{1}$$

which expresses the conservation of the density ρ of the vehicles under the flux $\rho v(\rho)$. The velocity v is a given decreasing function of ρ (at higher densities of cars it corresponds a smaller velocity) and vanishes at some critical threshold

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density $\overline{\rho}$ (no motion is possible if the vehicles are aligned bumper to bumper). At a first sight the model looks as oversimplified: for instance, all vehicles have the same characteristics, they obey deterministically to the same velocity law, the road is assumed straight and homogeneous. Rather surprisingly, however, for suitable choices of v equation (1) catches the main features of the traffic flows. Equation (1) is a *nonlinear hyperbolic conservation law*; as a consequence, its solutions can become discontinuous at finite times even in correspondence to smooth initial data. These discontinuities are called *shock waves* and correspond, roughly speaking, to braking. Accelerations, on the other hand, are modeled by "smooth" *rarefaction waves*, the other elementary ingredient in the description of general solutions to (1). We refer to [54] for a elementary introduction to several features of the traffic flow modeling based on (1) and to [16, 40, 96] for a general information on nonlinear hyperbolic conservation laws.

In spite the fundamental papers of O. Oleinik [81] (see [82] for an English translation) and P.D. Lax [68] laid the foundations for the study of hyperbolic conservation laws, at the beginning mathematicians seemed to discard applications to traffic flows. The baton passed to transportation engineers, who exploited rather quickly this modeling and contributed to extend it to more general situations, checking, at the same time, the solutions produced by (1) against real data; see for instance [80] and several papers in [1].

An open question was how to improve the traffic flow modeling by including one more equation, as it is done for fluid flows. There were several proposals, which were however criticized by Daganzo [41]. In reaction to Daganzo paper, in the 1990s two fundamental papers [6, 104] put independently new life into the subject, opening the way for a boost of models and applications that has not yet come to an end. Nowadays, both the modeling and the mathematical analysis of traffic flows, including that of the close subject of crowds dynamics, have reached a good level of completeness. In particular, *macroscopic* models (such as (1), i.e., based on averaged quantities) may include systems of several equations; they can consider different populations of vehicles, admit lane changing and the possibilities of entries, exits, crossroads, networks of roads. The equations can include nonlocal operators (to take into account the evaluations of drivers about several preceding vehicles), discontinuous terms (modeling, for instance, different characteristics of the road) and stochastic functions (to model the unpredictability of the different ways of driving); they can couple partial with ordinary differential equations (to simulate, for instance, traffic flows with several cars and a few slow vehicles, e.g., buses). We avoid from giving a long list of references and refer instead, for recent results and modeling to the books [39, 48, 49, 88] and to the survey papers [11, 13, 29, 55], where macroscopic models are duly presented. There, one can also find other models, for instance based on systems of ordinary differential equations (which are dubbed *microscopic* models).

The modeling of collective movements is not confined to *hyperbolic* equations. Even in the paper of Lighthill and Whitham [70], the authors proposed to include a second-order term to (1) in order to smear out the shock waves. The corresponding equation,

$$\rho_t + (\rho v(\rho))_x = D\rho_{xx}, \tag{2}$$

where $D > 0$ is a constant diffusion parameter, turns out to be *parabolic*. Unfortunately, Eq. (2) has some drawbacks. First, the propagation speed is infinite, differently from (1). Second, while v can be easily determined by experiments, it is not clear how to determine or mathematically deduce meaningful expressions for D. Third, the Daganzo paradox occurs: the velocity vanishes at the maximal allowable density $\overline{\rho}$, but motion still occurs (both forwards and backwards!) by diffusion. The answers to these issues were provided in several papers. First, the effect of the infinite propagation speed is negligible if the diffusion coefficient is sufficiently small: the total amount of "mass" that is diffused with infinite velocity is negligible. Second, there are now several models providing different forms of the diffusion coefficient [13, 18, 56, 78, 79, 83, 90] which, third, turns out to be *depending on* ρ and, moreover, *degenerate*. An example is

$$D(\rho) = -\rho v'(\rho)\left(\delta + \tau\rho v'(\rho)\right), \tag{3}$$

which was proposed and in [78, 79] for vehicular traffic flows; here, δ is an anticipation distance and τ a reaction time. An analogous expression with $\tau = 0$ was proposed later on in [18] for crowd dynamics. Notice that $D(0) = 0$; slighlty more sophisticated models [13] also prescribe $D(\overline{\rho}) = 0$, according to the principle of "no flux, no diffusion". On the one hand, the degeneracy of D avoids the Daganzo paradox; on the other hand, degenerate parabolic equations have a property of "finite propagation speed" [51, 99], which also contributes to answer to the first issue. A fourth reason for introducing diffusion in (1) is that the density-flow pairs (ρ, q) for Eq. (1) lie on the curve $q = \rho v(\rho)$ in the (ρ, q)-plane. However, experimental data [55, 60] show that this is *not* the case: such pairs usually cover a two-dimensional region. To reproduce this effect, either one considers inviscid second-order models (see for instance [6, 83, 104], and also [47, 91] where a relaxation term is included), or introduces a diffusive term. In the latter case the physical flow is $q = f(\rho) - D(\rho)\rho_x$, see [12, 13, 18, 78], and the density-flow pairs now correctly cover a full two-dimensional region in the (ρ, q)-plane.

Motivated by the previous considerations, the authors and coworkers have recently considered [34–37], in the framework of collective movements, the degenerate parabolic equation

$$\rho_t + f(\rho)_x = (D(\rho)\rho_x)_x + g(\rho), \tag{4}$$

where the general convection term f generalizes the flux $\rho v(\rho)$ and g is introduced to model entries or exists. From the modeling point of view, as we explained above, the diffusivity $D(\rho)$ must satisfy $D(0) = D(\overline{\rho}) = 0$, but we also treat the case when D does not vanish. Usually we assume $D(\rho) \geq 0$ but that are evidences that the case $D(\rho) < 0$ makes sense (it is related to particular road conditions [78] or to overcrowded environments [32, 35]) and can be studied as well. In these recent researches, the focus was on *qualitative* properties of solutions and the analysis was restricted to *traveling-wave solutions*, i.e., solutions $\rho(x,t) = \varphi(x - ct)$ of (4), where c is the propagation speed of the profile φ. Then the analysis of (4) is reduced to that of the ordinary differential equation

$$\big(D(\varphi)\varphi'\big)' + (c\varphi - f(\varphi))' + g(\varphi) = 0. \tag{5}$$

The degeneracy of the parabolic equation (4) implies, in particular, that the profile may be supported by an half-line or even compactly supported, and its behavior at the points where it reaches the values 0 and $\overline{\rho}$ deserves a detailed study. In the analysis of these problems we exploit and generalize the techniques introduced by the second author and coworkers in [71–74], which in turn form a whole thread of mathematical investigations starting from [5], see [52] for general information.

A slightly different model is considered in [22], namely,

$$\rho_t + f(\rho)_x = \left(D(\rho)\Phi(\rho_x)\right)_x , \tag{6}$$

where $\Phi : \mathbb{R} \to (-1, 1)$ is an increasing function which satisfies $\Phi(0) = 0$ and $\Phi(w) \to \pm 1$ as $w \to \pm\infty$. In this model the diffusion *saturates* at spatial discontinuities of the density (i.e., when the gradient become infinite) and several interesting pattern of wavefronts arise; among them, namely, the occurrence of *discontinuous* traveling waves. While the previous techniques can be applied here as well, the very notion of discontinuous solution must be made precise; an account of this theory can be found in [25], see [19] for a comprehensive survey.

The papers outlined in this survey do not take into account all articles dealing with traveling-wave solutions in the modeling of traffic flows or pedestrian dynamics. Several of them comprehend second-order models (i.e., systems of two or even more equations) but are almost always characterized by a constant diffusivities, on the contrary of the focus of this paper. About first-order models, we quote for completeness [87], where a nonlocal flux term is introduced, [93] for an analogous problem involving a road junction, [92] for a "follow-the-leader" ODE model for traffic flow with rough road conditions.

In spite of the fact that we have in mind applications to collective movements, Eq. (4) arises in a variety of physical and biological models, see for instance [52, 59, 77, 99]; our results can be applied as well to those problems. There is no room here to describe the applications of our results to collective movements; for details about this topic, we refer the reader to our aforementioned papers. The same remark holds for proofs.

Here follows the plan of this paper. In Sect. 2 we briefly recall notations, definitions and outline the main technique that is used in the proofs. In Sect. 3 we merge the results of papers [34, 36] into a single framework; the topic there is about source terms g with a single equilibrium point, which give rise to *semi-wavefronts*. The case of *negative* diffusivity [35] is presented in Sect. 4, while Sect. 5 overviews the case of *networks* [37]. The last Sect. 6 deals with a *saturated-diffusion* model [22]. The paper ends with a long list of references, which aims at showing the close interplay of the subject with other different mathematical topics.

2 Notation, Definitions and Main Assumptions

In this section we briefly provide the necessary background to read what follows, in particular concerning traveling-wave solutions. We always assume, unless differently specified, that for some $\overline{\rho} > 0$ we have

$$f \in C^1([0,\overline{\rho}]), \quad g \in C([0,\overline{\rho}]), \quad D \in C^0([0,\overline{\rho}]) \cap C^1((0,\overline{\rho})). \tag{7}$$

We can assume $f(0) = 0$ without any loss of generality. We first give a rigorous definition of traveling-wave solution.

Definition 1 Let $I \subseteq \mathbb{R}$ be an open interval and $\varphi\colon I \to [0,\overline{\rho}]$ a continuous function in I such that φ' is differentiable a.e. and $D(\varphi)\varphi' \in L^1_{\mathrm{loc}}(I)$. For every (x,t) with $x - ct \in I$, we say that $\rho(x,t) = \varphi(x - ct)$ is a *traveling-wave solution* to (4) with wave speed c and wave profile φ if for every $\psi \in C_0^\infty(I)$

$$\int_I \left\{\left(D(\varphi(\xi))\,\varphi'(\xi) - f(\varphi(\xi)) + c\varphi(\xi)\right)\psi'(\xi) - g(\varphi(\xi))\,\psi(\xi)\right\} d\xi = 0. \tag{8}$$

A traveling-wave solution is *global* if $I = \mathbb{R}$; *strict* if $I \neq \mathbb{R}$ and φ is not extendible to $\mathbb{R}$ (i.e., the maximal-existence interval of φ is strictly contained in $\mathbb{R}$); *classical* if φ is differentiable, $D(\varphi)\varphi'$ is absolutely continuous and (5) holds a.e.; *sharp at* $\ell \in [0,\overline{\rho}]$ if $g(\ell) = 0$ and there exists $\xi_0 \in I$ such that $\varphi(\xi_0) = \ell$, with φ classical in $I \setminus \{\xi_0\}$ but not differentiable at ξ_0. A *wavefront solution* is a global traveling-wave solution such that $\varphi(\pm\infty)$ are zeros of g.

When dealing with wavefront solutions, we focus in the following on *monotone* profiles: if $\xi_1 < \xi_2$ then either $\varphi(\xi_1) \leq \varphi(\xi_2)$ or $\varphi(\xi_1) \geq \varphi(\xi_2)$. For simplicity, in the following we use the terminology introduced for solutions to (4) *also for the profiles*. A wavefront profile must satisfy, in particular,

$$\varphi(-\infty) = \ell^-, \qquad \varphi(+\infty) = \ell^+, \tag{9}$$

for some $\ell^- \neq \ell^+ \in [0,\overline{\rho}]$ such that $g(\ell^\pm) = 0$. In case g is missing, every value $\ell^\pm \in [0,\overline{\rho}]$ (with $\ell^- \neq \ell^+$) is admissible.

Traveling waves are clearly not unique: if $\varphi = \varphi(\xi)$ is a profile in an interval I, then also $\tilde{\varphi}(\xi) = \varphi(\xi - \xi_0)$ is a profile in $I - \xi_0$, for every $\xi_0 \in \mathbb{R}$. If the source term g in (4) has only *one* zero, the definition of wavefront solution is adapted in the following way.

Definition 2 Let ρ be a traveling-wave solution of (4), whose wave profile φ is defined in $(\varpi, +\infty)$, $\varpi \in \mathbb{R}$; let $\ell^+ \in [0,\overline{\rho}]$ be such that $g(\ell^+) = 0$. Then, ρ is said a *semi-wavefront solution* of (4) *to* ℓ^+ if φ is monotonic, non-constant and $\varphi(\xi) \to \ell^+$ for $\xi \to +\infty$.

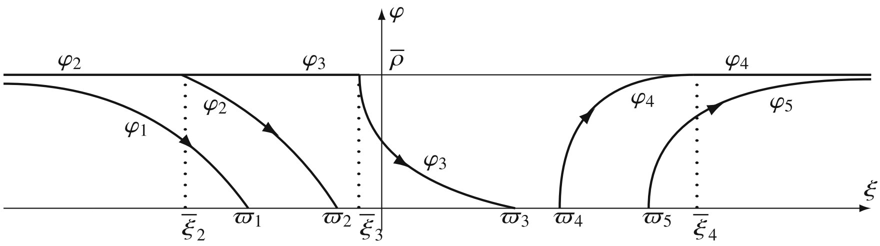

Fig. 1 A strictly decreasing semi-wavefront profile φ_1 from $\overline{\rho}$; a strictly increasing semi-wavefront profile φ_5 to $\overline{\rho}$. Non-strictly decreasing, sharp (at $\overline{\rho}$) semi-wavefront profiles φ_2 and φ_3 from $\overline{\rho}$; a non-strictly increasing, classical semi-wavefront profile φ_4 to $\overline{\rho}$. While φ_4 is smooth at $\overline{\xi}_4$, φ_2 and φ_3 are not smooth at $\overline{\xi}_2$ and $\overline{\xi}_3$, respectively

Analogously, ρ is a *semi-wavefront solution* of (4) *from* ℓ^-, for some $\ell^- \in [0, \overline{\rho}]$, if $g(\ell^-) = 0$, φ is defined $(-\infty, \varpi)$, it is monotonic, non-constant and $\varphi(\xi) \to \ell^-$ as $\xi \to -\infty$.

We refer to Fig. 1 for a representation of some semi-wavefront profiles.

Now, we provide some information on the main tools for solving (5); the first step consists in an order reduction. If $g = 0$, Eq. (5) can be integrated and reduced to the first-order equation

$$D(\varphi)\varphi' + c\varphi - f(\varphi) = C, \tag{10}$$

for some constant C depending on the end states of φ. This equation degenerates where D vanishes.

An order reduction is also possible when g does not change sign in $(0, \overline{\rho})$, see the next Sect. 3. In this case it is possible to prove that every semi-wavefront solution has a wave profile $\varphi(\xi)$ that is strictly monotone where $0 < \varphi(\xi) < \overline{\rho}$, see [34, Proposition 6.1]; hence, it is invertible there, with inverse function $\xi = \xi(\varphi)$, $\varphi \in [0, \overline{\rho})$. A simple computation shows that the function $z(\varphi) := D(\varphi)\varphi'(\xi(\varphi))$, for $\varphi \in (0, \overline{\rho})$, satisfies the singular equation

$$\dot{z}(\varphi) = \dot{f}(\varphi) - c - \frac{D(\varphi)g(\varphi)}{z(\varphi)}, \qquad \varphi \in (0, \overline{\rho}). \tag{11}$$

For clarity, we distinguish derivatives with respect to ξ and ρ (or φ); therefore $\varphi' = d\varphi/d\xi$ while $\dot{z} = dz/d\rho$. Notice the interplay of D and g that appears in a clear way in the right-hand side of (11): roughly speaking, the zeros of either D and g have the same "weight" in the numerator of the fraction in (11). The study of (11) requires an original technique that has been developed in [73] and is based on comparison-type arguments, i.e., on the existence of upper- and lower-solutions. The possible degenerate behavior of D imposes a quite precise construction of these solutions.

3 Semi-Wavefronts

We consider in this section Eq. (4) in the case the diffusivity D and the source term g satisfy

$$D > 0 \text{ in } (0, \overline{\rho}) \quad \text{and} \quad g > 0 \text{ in } [0, \overline{\rho}), \; g(\overline{\rho}) = 0. \tag{12}$$

The assumption (12) on g aims at modeling pedestrian entries only depending on the density; see [7] for different (localized) models for entries and exits. In fact, assume that pedestrians are walking along a long corridor and side access is allowed; if, for instance, g is a decreasing function, then entries are maximum where the density in the corridor is zero and are not possible where the maximal density in reached. We refer to [34] for more information on the modeling. Notice that g has only *one* zero, and this does not make possible wavefronts to exist; only semi-wavefronts may exist. From a mathematical point of view the results presented in this section extend and precise analogous results in [52].

Aiming at the widest generality, we consider in the following Theorem 1 the existence and uniqueness of semi-wavefronts for Eq. (4) by merging the results contained in [34, 36]. As we comment on below, the occurrence of classical or sharp profiles depends on the conditions $D(\overline{\rho}) > 0$ or $D(\overline{\rho}) = 0$.

Theorem 1 *Assume* (7) *and* (12)*. Then, for every* $c \in \mathbb{R}$*, Eq.* (4) *has a strict classical semi-wavefront solution from* $\overline{\rho}$ *and a strict classical semi-wavefront solution to* $\overline{\rho}$*. These solutions are unique up to shifts and their wave profiles are of class* C^2 *in* $(-\infty, \varpi)$ *or* (ϖ, ∞)*, respectively.*

In the previous statement, uniqueness is understood in the class of classical or sharp profiles. The above existence and uniqueness theorem is complemented by several other results; we briefly quote the most important ones.

Behavior of the Profiles at $\varphi = 0$ The behavior of $\varphi'(\xi)$ as $\xi \to \varpi^-$ (see Definition 2) is completely described and depends on the behavior of D at 0. For example, if $D(0) > 0$ then $\varphi'(\xi)$ tends to a strictly negative real number as $\xi \to \varpi^-$; if $D(0) = \dot{D}(0) = 0$, then either $\varphi'(\xi) \to -\infty$ if $c \le c^*$ or $\varphi'(\xi) \to -g(0)/\left(c - f'(0)\right)$ if $c > c^*$. Here, the threshold c^* is a real number depending on the behavior of f, g and D at 0. An explicit expression for c^* is not at disposal, but rather precise estimates can be provided. The case $D(0) = 0 \neq \dot{D}(0)$ is slightly more complicated.

Behavior of the Profiles at $\varphi = \overline{\rho}$ If $D(\overline{\rho}) > 0$ then every profile is classical; assume instead $D(\rho) = 0$. Profiles from $\overline{\rho}$ are sharp if $c < f'(\overline{\rho})$ and classical if $c > f'(\overline{\rho})$; on the contrary, profiles to $\overline{\rho}$ are classical if $c < f'(\overline{\rho})$ and sharp if $c > f'(\overline{\rho})$. If $c = f'(\overline{\rho})$, then profiles are classical if $\dot{D}(\overline{\rho}) < 0$; otherwise they can be either classical or sharp, depending on the order of vanishing at $\overline{\rho}$ of D, g and $c - f'(\rho)$.

Monotony If φ_1 and φ_2 are two profiles from $\overline{\rho}$ with wave speeds $c_1 < c_2$ and $\varpi_1 = \varpi_2 = \varpi$, then $\varphi_2(\xi) < \varphi_1(\xi)$ for every $\xi \in (-\infty, \varpi)$ with $\varphi_2(\xi) < \overline{\rho}$.

Strictly Monotonic Solutions First, assume $D(\overline{\rho}) > 0$. If $g(\rho) \le L(\overline{\rho} - \rho)$ in a left neighborhood of $\overline{\rho}$, then every profile satisfies $\varphi(\xi) < \overline{\rho}$. On the contrary, if $g(\rho) \ge L(\overline{\rho} - \rho)^\alpha$ for some $\alpha \in (0, 1)$ in a left neighborhood of $\overline{\rho}$, then every profile satisfies $\varphi(\xi) \equiv \overline{\rho}$ in $(-\infty, \overline{\xi}]$ (or on $[\overline{\xi}, +\infty)$) for some $\overline{\xi}$.

Second, assume $D(\overline{\rho}) = 0$. Under the same inequalities on g analogous results hold true, requiring however, in the former case, $c > f'(\overline{\rho})$ ($c < f'(\overline{\rho})$) in the case of profiles from (resp., to) $\overline{\rho}$.

Diffusivities with Infinite Slope at 0 Assume either $D > 0$ in $[0, \overline{\rho}]$, $\dot{D}(0) = \pm\infty$ or $D > 0$ in $(0, \overline{\rho}]$, $D(0) = 0$, $\dot{D}(0) = \infty$. Then, under (12) we still have a strict classical semi-wavefront solution from $\overline{\rho}$ for every c. If $D(0) > 0$ then $\varphi'(\xi)$ tends to a strictly negative number when $\xi \to \varpi^-$, while if $D(0) = 0$ then $\varphi'(\xi) \to -\infty$.

Diffusivities with Infinite Slope at $\overline{\rho}$ Assume $D(\overline{\rho}) = 0$ and $\dot{D}(\overline{\rho}) = -\infty$; moreover, assume $(Dg)'(\overline{\rho}) \in (-\infty, 0]$. Then profiles are always classical.

Convergence of Semi-Wavefronts to Wavefronts Consider a source term g_0 satisfying $g(0) = g(\overline{\rho}) = 0$, $g > 0$ in $(0, \overline{\rho})$ and let $(Dg)'(0) < \infty$. In this case, Eq. (5) admits a wavefront φ_0 connecting $\overline{\rho}$ with 0 for every $c \ge c_0^*$, for some $c_0^* \in \mathbb{R}$. Also consider a sequence of source terms g_n satisfying (12), which give rise to semi-wavefront profiles φ_n with the same speed c and satisfying $c \ge c_0^*$. Then $\varphi_n \to \varphi_0$ in $C^1_{\mathrm{loc}}(J)$, where J is the maximal open interval where $0 < \varphi_0 < \overline{\rho}$.

The Case $g < 0$ Instead of (12), assume $g < 0$ in $(0, \overline{\rho}]$ and $g(0) = 0$, while keeping the same assumption on D. These assumptions model the case of exits. Results analogous to those outlined above can be proved.

The Case where g Changes Sign Assume $g > 0$ in $[0, \rho_0)$ and $g < 0$ in $(\rho_0, \overline{\rho}]$, for some ρ_0. By a suitable pasting of the profiles obtained above we can still construct traveling waves under the additional assumption $|g(\rho)| \ge L|\rho_0 - \rho|^\alpha$ for some $\alpha \in (0, 1)$ in a neighborhood of ρ_0.

Applications We refer to [34] for some examples which make more precise the patterns of the profiles as well for an interpretation of the semi-wavefronts for collective movements.

4 The Case of Negative Diffusivity

In the previous Sect. 3 we gave a complete description of the semi-wavefront solutions to Eq. (4), for several different diffusivities and source terms. In those results D could vanish, but staying otherwise positive. In this section we report some recent results obtained in [35] about the case when D changes sign. For simplicity we drop the source term g and consider the equation

$$\rho_t + f(\rho)_x = (D(\rho)\rho_x)_x . \tag{13}$$

Remark that, as far as wavefronts are concerned, the scenery differs a lot from the case where instead g is present: in the current case the end states of a profile φ are completely arbitrary. The *negativity* of D simulates both an unstable and aggregative behavior; it occurs, for instance, in vehicular flows for high car densities and limited sight distance ahead [78]. An analogous modeling can be made in the framework of crowds dynamics, where it simulates panic behaviors in overcrowded environments [32]. We point out that negative diffusivities are also considered in geophysics [44], thermodynamics [62] and biological [58] models.

From an analytic point of view, Eq. (4) becomes a *backward parabolic* equation in the region where D is negative. A general framework to treat backward parabolic equations was originally proposed in [45, 84], for the case $f = 0$; we recall that the problem is not only strongly unstable but also suffers of a loss of uniqueness [57]. In that approach, the solutions to (4) are singled out as the limits for $\varepsilon \to 0$ of solutions of an augmented third-order pseudo-parabolic approximation

$$\rho_t^{\varepsilon} = \left(D(\rho)\rho_x^{\varepsilon}\right)_x + \varepsilon\Psi(\rho^{\varepsilon})_{xxt}.$$

Here above, Ψ satisfies suitable assumptions of sign and growth. This framework has been subsequently developed and extended in several papers, see e.g. [75, 94, 95, 97, 98]. The drawback of this approach is that it is limited, for the moment, to the case $f = 0$; moreover, the third-order approximation has no clear meaning for collective movements. As a consequence, we drop this approach and focus as above just on wavefront solutions.

More precisely, we assume f as in (7) while, for simplicity, $D \in C^1([0, \overline{\rho}])$; the case where $D \in C([0, \overline{\rho}]) \cap C^1((0, \overline{\rho}))$ can be dealt as in Sect. 3. Moreover, we assume that there exists $\alpha \in (0, \overline{\rho})$ such that

$$D > 0 \text{ in } (0, \alpha) \quad \text{and} \quad D < 0 \text{ in } (\alpha, \overline{\rho}). \tag{14}$$

We refer to Fig. 3a. We have $D(\alpha) = 0$ because of the smoothness of D. Clearly, the interesting case for wavefronts is when

$$\ell^- \in [0, \alpha) \quad \text{and} \quad \ell^+ \in (\alpha, \overline{\rho}], \tag{15}$$

since otherwise the results of [52] apply. Notice that for suitable but realistic assumptions on v and on the parameters δ and τ, the diffusivity D provided by (3) behaves as in (14); the same behavior occurs for other models of diffusivities proposed in [13, 56].

Wavefronts for parabolic forward-backward equations as (4) do not suffer of the heavy problems of the general solutions, and were studied by some authors in the case $g \neq 0$ but $f = 0$; we refer to [9, 10, 71] for D changing sign once and

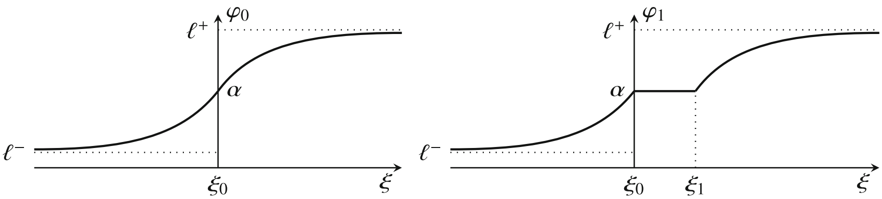

Fig. 2 A profile φ_1 deduced from a profile φ_0 by a stretching of ξ_1 at level α

monostable g, [72] for the bistable case, [46, 66] for D changing sign twice where g is, respectively, monostable and bistable.

The vanishing of D at α implies that any stretching at α of a wavefront profile gives rise to another profile. Refer for instance to Fig. 2, where for simplicity the value α is assumed to be reached at $\xi = 0$: if φ_0 is a profile for $\xi_1 = 0$, then for every $\xi_1 > 0$ the profile φ_1 depicted in Fig. 2 and having a plateau of length ξ_1 is a profile as well.

We can now state the main result of this section.

Theorem 2 *Assume* (14) *and* (15). *Equation* (4) *has a wavefront solution whose profile* φ *satisfies* (9) *if and only if the following three conditions are satisfied:*

$$\frac{f(\alpha) - f(\ell^-)}{\alpha - \ell^-} = \frac{f(\ell^+) - f(\alpha)}{\ell^+ - \alpha} =: c_{\ell^\pm}, \tag{16}$$

$$f > s_{\ell^\pm} \text{ in } (\ell^-, \alpha), \quad f < s_{\ell^\pm} \text{ in } (\alpha, \ell^+), \tag{17}$$

$$\frac{D}{f - s_{\ell^\pm}} \in L^1(I_\alpha), \tag{18}$$

where I_α *is some neighborhood of* α. *We denoted* $s_{\ell^\pm}(\rho) = c_{\ell^\pm}(\rho - \alpha) + f(\alpha)$. *The wave speed of the profile is* $c_{\ell^\pm}$ *and we have* $f'(\alpha) \le c_{\ell^\pm}$. *If* $\xi_0 = \xi_1 = 0$, *then* φ *is unique; in this case,* $\varphi'(\xi) > 0$ *when* $\ell^- < \varphi(\xi) < \ell^+$, $\xi \ne 0$, *while*

$$\lim_{\xi \to 0} \varphi'(\xi) = \begin{cases} \frac{f'(\alpha) - c_{\ell^\pm}}{D'(\alpha)} & \text{if } D'(\alpha) < 0, \\ \infty & \text{if } D'(\alpha) = 0 \text{ and } f'(\alpha) - c_{\ell^\pm} < 0. \end{cases} \tag{19}$$

Notice that by (16) the plot of the function $s_{\ell^\pm}(\rho)$ is the straight line through $\left(\ell^-, f(\ell^-)\right)$ and $\left(\ell^+, f(\ell^+)\right)$; we refer to Fig. 3b for the geometric meaning of conditions (16), (17). We point out that in order that these conditions are satisfied, the function f must change its convexity-concavity; however, the inflection point does not necessarily coincide with α. Notice that non-concave functions f are known [8, 61] to show cluster or oscillatory solutions, which is precisely what we are modeling. We observe that the role of condition (18) is to guarantee that both (non-strictly monotone) profiles from ℓ^- to α and from α to ℓ^+ reach α for a finite value of ξ, see [52, Th. 9.1]. Notice that if $f'(\alpha) < c_{\ell^\pm}$ then (18) is clearly satisfied because

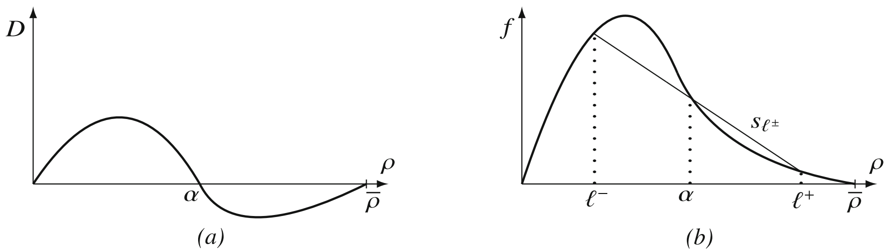

Fig. 3 (**a**): a diffusivity D satisfying assumption (14); (**b**): the flux function f

$D \in C^1$. As a consequence, condition (18) is only needed when $f'(\alpha) = c_{\ell^\pm}$, i.e., when the line $s_{\ell^\pm}$ is tangent to the graph of f at $(\alpha, f(\alpha))$.

As we mentioned above, negative diffusivities are also introduced, e.g., in [44]. In that and in similar cases the region where D is negative is bypassed by inserting in the solution a shock wave, which is uniquely determined by a higher-order regularization (either of pseudo-parabolic type, or of Cahn-Hilliard type), see [103]. On the contrary, our smooth profiles fully enter into the region of negative diffusivity and no artificial wave is added.

As in the case of Theorem 1, the previous result is the starting point for proving several related results that we outline below.

Sharpness of the Profiles Profiles are sharp at α if $D'(\alpha) = 0$ and the subcharacteristic condition $f'(\alpha) < c_{\ell^\pm}$ holds; in the characteristic case $f'(\alpha) = c_{\ell^\pm}$, sharpness can be investigated but depends on suitable technical conditions. Sharpness at 0 and $\overline{\rho}$ can be characterized by adapting the arguments in [36, 37, 52].

The Vanishing-Viscosity Limit Wavefronts are not only important by themselves, but also because they provide smooth approximations to shock waves of the inviscid hyperbolic equation

$$\rho_t + f(\rho)_x = 0. \tag{20}$$

In this way, they contribute to single out unique solutions to (20). More precisely, replace D with εD in (13) to obtain

$$\rho_t + f(\rho)_x = (\varepsilon D(\rho)\rho_x)_x\,, \tag{21}$$

where $\varepsilon > 0$ is a parameter. The issue is whether solutions ρ_ε of (21) converge to a solution ρ_0 of (20) for $\varepsilon \to 0$. The answer is in the affirmative if $D > 0$, see [63] and [40, §6] for a general presentation of the problem; the case $D \geq 0$ is much more tricky and was first considered in [23, 100]. The problem seems open if D changes sign. On the contrary, for wavefronts, we can prove that, in the framework of Theorem 2, the profiles φ_ε joining ℓ^- with ℓ^+ pointwisely converge to the limit expected profile φ_0, which is defined as ℓ^- if $\xi < 0$ and ℓ^+ if $\xi > 0$.

Notice however that the discontinuous solution $\rho_0(x,t) = \varphi_0(x-ct)$ is *not* entropic in the hyperbolic sense [16, Thm. 4.4]; it is so only the analogous solution joining ℓ^- with α, see Fig. 3b. However, even if ρ_0 is not entropic, Theorem 2 shows that it has a viscous profile, where the diffusivity is negative in the nonentropic part of the solution; clearly, such a wave is unstable in the sense of [16, Rem. 4.7], as it was expected. It is interesting to remark that the one-sided sonic case $c_{\ell^\pm} = f'(\ell^+) \neq f'(\ell^-)$ (or $c_{\ell^\pm} = f'(\ell^-) \neq f'(\ell^+)$) was used in [32] (see cases (R1) and (R3)*(a)* there) and gives rise, in the vanishing viscosity limit, to a nonclassical shock [69].

Generalizations of the Conditions on D The case when the signs of D are the opposite of the ones in (14) is easily deduced and provides decreasing profiles. More interesting is the case when D vanishes two times inside $(0,\overline{\rho})$ and satisfies

$$D > 0 \text{ in } (0,\alpha)\cup(\beta,\overline{\rho}) \quad \text{and} \quad D < 0 \text{ in } (\alpha,\beta), \tag{22}$$

for some $0 < \alpha < \beta < \overline{\rho}$, see Fig. 4a. In this case the geometric assumptions (16) and (17) must be modified to require that the straight line through $\left(\ell^-, f(\ell^-)\right)$ and $\left(\ell^+, f(\ell^+)\right)$ also meets both points $(\alpha, f(\alpha))$ and $(\beta, f(\beta))$; moreover, the mutual position of the plot of the function f and of that line must be as in Fig. 4b. As pointed out just above in the similar case (14), it is easy to understand that the limit discontinuous solution ρ_0 to (20), whose profile φ_0 joins ℓ^- on the left to ℓ^+ on the right with a jump propagating with velocity $c_{\ell^\pm}$, is nonentropic because the flux is not convex [16, Remark 4.7]. Once again, the nonentropic part is where the diffusivity is negative. Also shock waves connecting the states ℓ^- and ℓ^+ as in Fig. 4b have been considered in [32] (see case (R3)*(b)* there).

Applications Examples are provided in [35] for different models of the diffusivity and of the velocity. For some of these models, which are based on experimental data, condition (14) really occurs, and the assumptions of Theorem 2 are satisfied, both for models of vehicular traffic flows and pedestrian dynamics.

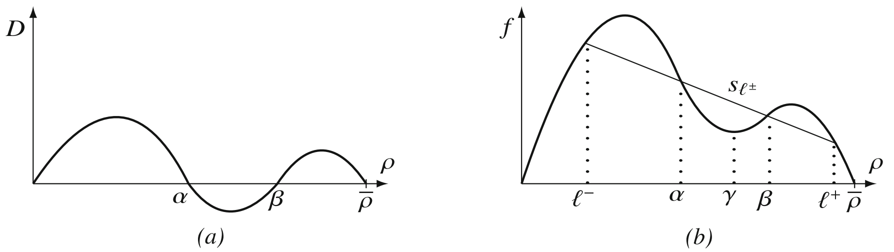

Fig. 4 (**a**): a diffusivity D satisfying (22); (**b**): the flux function f

5 Wavefronts on Networks

The analysis of the previous sections, though rather complete and detailed, involved a *single* road. An interesting issue is how to cope with the (real!) case of *networks of roads*. In recent years, partial differential equations on networks have seen a boost of papers, see for instance [42, 48, 49, 67, 85, 101] as far as *dynamic* equations are concerned, with applications to several different subjects. About *wavefronts in networks of traffic flows*, we review in this section the results in [37], where the network is constituted by a crossroad connecting m incoming roads with n outgoing roads, briefly, a star graph; more general networks can be treated by generalizing this case. Few papers deals with wavefronts in networks; we refer to [101, 102] for the semilinear diffusive case and to [76] for the case of a dispersive equation. We point out that in those papers, as in most modeling of diffusive or dispersive partial differential equations on networks, both the continuity of the unknown functions and the Kirchhoff condition (or variants of it) are imposed at the nodes. These assumptions are natural when dealing with heat or fluid flows, but they are unjustified in the case of traffic modeling, where the density must be allowed to jump at the node while the conservation of the mass must always hold, see [3, 48, 49, 88]. Moreover, such conditions impose rather strong requirements on the existence of the profiles, which often amount to proportionality assumptions on the parameters in play.

We denote $\mathsf{I} = \{1, \ldots, m\}$ and $\mathsf{J} = \{m + 1, \ldots, m + n\}$ the index sets corresponding to incoming and outgoing roads, respectively, and $\mathsf{H} = \mathsf{I} \cup \mathsf{J}$. For $h \in \mathsf{H}$, the traffic in each road is modeled by the scalar diffusive equation

$$\rho_{h,t} + f_h(\rho_h)_x = \big(D_h(\rho_h)\rho_{h,x}\big)_x\,, \tag{23}$$

where ρ_h is the vehicle density. Incoming roads are parametrized by $x \in \mathbb{R}_- = (-\infty, 0]$ while outgoing roads by $x \in \mathbb{R}_+ = [0, \infty)$; the crossroad is located at $x = 0$ for both parameterizations (Fig. 5). We denote the generic road by Ω_h for $h \in \mathsf{H}$; then $\Omega_i := \mathbb{R}_-$ for $i \in \mathsf{I}$ and $\Omega_j := \mathbb{R}_+$ for $j \in \mathsf{J}$. The network is defined as $\mathcal{N} := \prod_{h\in\mathsf{H}} \Omega_h$. We scale the densities in each road so that, according to the previous notation, $\overline{\rho}_h = 1$ for every $h = 1, \ldots, m+n$. We deal with the simple but significative case where f_h and D_h are C^1 functions, the function f_h is positive and

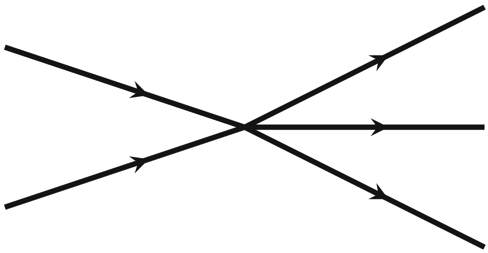

Fig. 5 A star graph. Here $m = 2, n = 3$

strictly concave with $f_h(0) = f_h(1) = 0$; at last, we assume $D_h > 0$ in $(0, 1)$. More general cases can be treated along the lines of the previous sections.

We denote by $F_h(\rho_h, \rho_{h,x}) = f_h(\rho_h) - D_h(\rho_h)\rho_{h,x}$ the parabolic flux along the road h and, motivated by [30, 31], we require that at the crossroad the following coupling among parabolic fluxes

$$F_j\left(\rho_j(t,0^+), \rho_{j,x}(t,0^+)\right) = \sum_{i\in\mathsf{I}} \alpha_{i,j}\, F_i\left(\rho_i(t,0^-), \rho_{i,x}(t,0^-)\right) \quad \text{for a.e. } t \in \mathbb{R},\ \ j \in \mathsf{J}, \tag{24}$$

takes place, for given constant $\alpha_{i,j} \in (0, 1]$ satisfying $\sum_{j\in\mathsf{J}} \alpha_{i,j} = 1$, for $i \in \mathsf{I}$. The coefficients $\alpha_{i,j}$ represent the ratio of vehicles from road i traveling through road j. Summing on j the equations in (24) we see that they imply the conservation of the total flow at the crossroad, which in turn implies the conservation of the mass. Notice that no continuity conditions as $\rho_i(t, 0^-) = \rho_j(t, 0^+)$, for $i \in \mathsf{I}$ and $j \in \mathsf{J}$, is required, on the contrary of what is usually done in the standard modeling of (linear) parabolic flows on networks.

Let ρ_h be a wavefront solution to the Eq. (23) in $\mathbb{R} \times \Omega_h$ with profile $\varphi_h : \mathbb{R} \to [0, 1]$, speed $c_h \in \mathbb{R}$ and end states $\ell_h^\pm$, for $h \in \mathsf{H}$. We say that the vector-valued function $\rho = (\rho_1, \dots, \rho_{m+n})$ is a wavefront solution to the system

$$\rho_{h,t} + f_h(\rho_h)_x = \left(D_h(\rho_h)\rho_{h,x}\right)_x, \qquad h \in \mathsf{H}, \tag{25}$$

in the network $\mathscr{N}$ if (24) holds. The equation for the profile φ_h can be integrated and can be written as

$$D_h\left(\varphi_h(\xi)\right)\varphi_h'(\xi) = g_h\left(\varphi_h(\xi)\right) - g_h(\ell_h^\pm), \tag{26}$$

where $g_h(\rho) := f_h(\rho) - c_h\, \rho$.

We denote $\mathsf{I}_0 = \{i \in \mathsf{I} : c_i = 0\} = \{i \in \mathsf{I} : f_i(\ell_i^-) = f_i(\ell_i^+)\}$, $\mathsf{I}_0^c := \mathsf{I} \setminus \mathsf{I}_0$, and

$$L_{i,j}^\pm = \begin{cases} \ell_i^\pm & \text{if } c_i\, c_j \geq 0, \\ \ell_i^\mp & \text{if } c_i\, c_j < 0. \end{cases} \tag{27}$$

For simplicity we focus on the *nonstationary case*, where $c_j \neq 0$ for every $j \in \mathsf{J}$. In this case, we also define $c_{i,j} = c_i/c_j$, $A_{i,j} = \alpha_{i,j}\, c_{i,j}$, $k_j = \sum_{i\in\mathsf{I}_0^c}(A_{i,j}\, L_{i,j}^\pm) - \ell_j^\pm$.

Here follows the main result of this section, which deals with the *non-stationary* case.

Theorem 3 *Under the above conditions on f_h and D_h, $h \in H$, problem (25)–(24) admits a non-stationary traveling wave if and only if the following condition holds.*

There exist $\ell_1^{\pm}, \ldots, \ell_m^{\pm} \in [0, 1]$ *with* $\ell_i^- < \ell_i^+$, $i \in I$, *such that* $I_0^c \neq \emptyset$ *and for any* $j \in J$:

(i) *there exist* $\ell_j^{\pm} \in [0, 1]$ *satisfying* $f_j(\ell_j^{\pm}) = \sum_{i \in I} \alpha_{i,j} f_i(L_{i,j}^{\pm})$ *and* $f_j(\ell_j^-) \neq f_j(\ell_j^+)$;
(ii) *we have*

$$\frac{g_j\left(\ell_j(c_j\,\xi)\right) - g_j(\ell_j^-)}{D_j\left(\ell_j(c_j\,\xi)\right)} = \sum_{i \in I_0^c} A_{i,j}\,c_{i,j}\,\frac{g_i\left(\varphi_i(c_i\,\xi)\right) - g_i(\ell_i^-)}{D_i\left(\varphi_i(c_i\,\xi)\right)} \quad \text{for a.e. } \xi \in \mathbb{R}, \tag{28}$$

where $\varphi_1, \ldots, \varphi_m$ *are solutions to* (9)–(26) *and*

$$\ell_j(\xi) := \sum_{i \in I_0^c} [A_{i,j}\,\varphi_i\left(c_{i,j}\xi\right)] - k_j \quad \text{for } \xi \in \mathbb{R}.$$

The statement of Theorem 3 is very technical but it reduces the problem of the existence of a wavefront in the network to some *algebraic* and *functional* conditions, which in a sense play the role of the much simpler Kirchhoff conditions of the linear case. Some extensions of the above result can be given, as we explain below.

Stationary and Degenerate Wavefronts A wavefront to Eq. (23) is stationary if $c_h = 0$ and degenerate if either $D_h(0) = 0 = \ell_h^-$ or $D_h(1) = 0 = 1 - \ell_h^+$. In turn, a wavefront ρ to (25) is stationary if each component ρ_h is stationary; completely non-stationary if none of its components is stationary; degenerate if at least one component ρ_h is degenerate; completely degenerate if each of its components is degenerate. Some characterizations of stationary and/or degenerate wavefronts on the network can be given.

Continuity Conditions The above algebraic and functional conditions become much simpler if the continuity condition $\rho_i(t, 0^-) = \rho_j(t, 0^+)$, for $t \in \mathbb{R}$ and $(i, j) \in I \times J$ holds, as well as the characterization of non-stationary wavefronts.

Applications Several examples, in the case $m = 1$ of a single incoming road, are provided in [37], which also make much simpler and explicit the conditions of Theorem 3 for the existence of wavefronts. In particular, they refer to the cases of quadratic and logarithmic fluxes, for constant or linear diffusivities.

6 Wavefronts for Saturated Diffusion Models

A nonlocal model for pedestrian dynamics in $\Omega \subset \mathbb{R}^2$ was proposed in [33, 38] and can be written as

$$\rho_t + \operatorname{div}\left(\rho v(\rho)\left(\nu + \mathcal{I}(\rho)\right)\right) = 0, \tag{29}$$

where $\rho(x_1, x_2, t)$ is the crowd density at point $(x_1, x_2) \in \mathbb{R}^2$, with $0 \le \rho(x_1, x_2, t) \le \overline{\rho}$. The scalar pedestrians' velocity in absence of environmental constraints is denoted by $v = v(\rho)$; the unit vector $\nu = \nu(x_1, x_2) \in \mathbb{R}^2$ is the preferred direction of the pedestrian at (x_1, x_2). The operator $\mathscr{I}(\rho)$ takes into account how a pedestrian deviates from the direction ν by trying to avoid high crowd densities ρ. The operator $\mathscr{I}$ is possibly nonlocal and can involve terms of the form $\nabla\rho * \eta$, where η is a suitable mollifier. In the case η is the Dirac measure, and we choose $\mathscr{I}(\rho) = -\varepsilon\nabla\rho/(\sqrt{1 + \|\nabla\rho\|^2})$, for $\varepsilon > 0$, then we recover the model proposed in [17]:

$$\rho_t + \operatorname{div}(\nu\rho v(\rho)) = \varepsilon \operatorname{div}\Big(\rho v(\rho)\frac{\nabla\rho}{\sqrt{1 + \|\nabla\rho\|^2}}\Big). \tag{30}$$

If moreover ν is constant and $\Omega = \mathbb{R}^2$, then we can look for *plane-wave* solutions, which are solutions of the form $\rho(x_1, x_2, t) = \rho(\mu \cdot \mathbf{x}, t)$, where $\mu \in \mathbb{R}^2$ is a unit vector. In this case ρ must satisfy the equation

$$\rho_t + \mu \cdot \nu\,(\rho v(\rho))_x = \varepsilon\Big(\rho v(\rho)\frac{\rho_x}{\sqrt{1 + |\rho_x|^2}}\Big)_x, \tag{31}$$

for $x \in \mathbb{R}$. This leads to consider equations of the form (6) where, for instance, $\Phi(w) = w/\sqrt{1 + w^2}$. More generally, we assume $D > 0$ in $(0, 1)$, $D(0) = D(1) = 0$ (where we scaled the density to have $\overline{\rho} = 1$ as in the previous section) and

$$\Phi : \mathbb{R} \to (-1, 1), \quad \Phi' > 0, \quad \Phi(0) = 0, \quad \Phi(w) \to \pm 1 \text{ as } w \to \pm\infty. \tag{32}$$

Notice that the assumption $\Phi' > 0$ in (32) implies that (6) is a forward parabolic equation, which degenerates at $\rho = 0$, $\rho = 1$ and if $\rho_x = \pm\infty$, i.e., when the tangent to the graph of $\rho(\cdot, t)$ becomes vertical.

Assumption (32) makes (6) a *flux-saturated* porous media equation. In the case $f = 0$, the existence and uniqueness of solutions to the initial-value problem for these equations with initial data ρ_0 was proved in [14, 43] for D strictly positive and ρ_0 strictly increasing; the case $D(0) = 0$ was considered later by Caselles and co-workers e.g. in [4, 24, 26]. The interesting feature of these equations is that they admit *discontinuous solutions*; this can sound strange because the term $\Phi(u_x)$ has no meaning in $\mathscr{D}'$ for such functions. Indeed, if $u \in BV$, then Du is a Radon measure that can be written as $Du = D^a u + D^j u + D^c u$, where on the right-hand side we have the absolutely continuous part (with respect to the Lebesgue measure), the jump part, and the Cantor part of Du, respectively [2]. In the former papers [14, 43], an *ad hoc* definition of solutions is given; in the latter [4, 24, 26], equivalently, for solutions u with $u(\cdot, t) \in BV$ a.e. it is understood that (6) holds in $\mathscr{D}'$ with $\Phi(\rho_x)$ replaced by $\Phi(D^a\rho)$. Notice that in this case the equation for the profiles becomes

$$\big(D(\varphi)\Phi(D^a\varphi)\big)' + c\varphi' - (f(\varphi))' = 0. \tag{33}$$

In both cases the solutions are obtained as vanishing-viscosity limits, but uniqueness is missing without some further requirements. That is not surprising, since this subject has several common issues with the theory of hyperbolic conservation laws [16, 40, 63, 96]: solutions are required to be *entropic*, see [25]. Additional information and a bibliography on the subject can be found in the survey paper [19].

Under suitable assumptions, a first study of wavefronts for (6) was performed by Rosenau and co-workers [27, 28, 53, 64, 65], see also [89]; however, in those papers, the focus was oriented more toward applications and numerics. We stress that the above assumptions on D do not allow to apply those results here. Later on, discontinuous entropic wavefronts were rigorously analyzed in [19–21, 26], again in the case $f = 0$ but with a source term; we also refer to [15, 50]. However, the case of a parabolic equation with an advection term has never been considered in the framework of Caselles' theory. A general result about wavefronts for Eq. (6) with Φ satisfying (32) is that profiles are smooth if $\left|\ell^+ - \ell^-\right|$ is small and possibly discontinuous otherwise. Such a result was first justified in a special case in [64] and proved in [89] in the case $D = 1$. In [22] we confirm this result in a much more general framework. Now, we give a brief account of [22].

For simplicity, we consider the case of profiles φ having at most *one* singular point ξ_0, i.e., a point where φ can be either continuous, but then non-differentiable, or just discontinuous; in $\mathbb{R} \setminus \{\xi_0\}$ the profiles are classical. Moreover, we deal with *increasing* profiles. Much more general statements can be found in [22]. In every interval where φ is a classical profile, Eq. (33) can be integrated and becomes

$$D(\varphi)\Phi(\varphi') + c\varphi - f(\varphi) = k, \tag{34}$$

for an arbitrary constant k. In turn, in the interval $J = \{\xi : 0 < \varphi(\xi) < 1\}$ Eq. (34) can be written as

$$\varphi' = \Phi^{-1}\left(h(\varphi)\right) \quad \text{for} \quad h(\rho) := \frac{f(\rho) - (c\rho - k)}{D(\rho)}. \tag{35}$$

Since Φ^{-1} is only defined in the interval $(-1, 1)$, it follows that classical solutions φ are valued in the admissible region $\mathscr{A} = \{\rho \in (0, 1) : |h(\rho)| < 1\}$. If ℓ^- and ℓ^+ belong to a *same* interval contained in $\mathscr{A}$ and $\ell^- \neq \ell^+$, then there exists [52] a unique (up to shifts) profile φ joining these two end states; moreover, $k = c\ell^\pm - f(\ell^\pm)$, $h(\ell^\pm) = 0$, and

$$c = \frac{f(\ell^+) - f(\ell^-)}{\ell^+ - \ell^-}. \tag{36}$$

If instead ℓ^- and ℓ^+ belong to two *different* intervals contained in $\mathscr{A}$, then classical solution do not exist.

Solutions in this case can be obtained by inserting a jump in the profile as follows. Let φ be a classical solution to (35) in $(-\infty, \xi_0)$, with $\varphi(-\infty) = \ell^-$, and assume $\lim_{\xi\to\xi_0} \varphi(\xi) = \rho_1$ for $h(\rho_1) = 1$, see Fig. 6(i). By (35) we deduce $\lim_{\xi\to\xi_0} \varphi'(\xi) =$

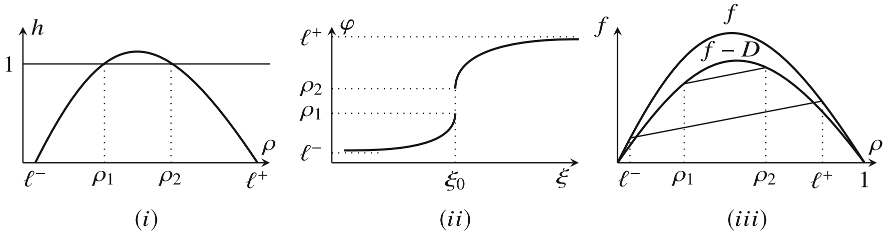

Fig. 6 Formation of singularities in a profile

∞. If h is as in Fig. 6(i), then we can extend φ by letting it jump from ρ_1 to ρ_2, see Fig. 6(ii). By subtracting the expressions $h(\rho_1) = 1$ and $h(\rho_2) = 1$, we deduce

$$c = \frac{f(\rho_2) - f(\rho_1) - (D(\rho_2) - D(\rho_1))}{\rho_2 - \rho_1}. \tag{37}$$

This expression should be compatible with (36): notice that the jump points ρ_1 and ρ_2 must match with the diffusivity D in order to obtain the previous speed c of the profile. If φ can be extended in this way to (ξ_0, ∞) to reach ℓ^+, then we have succeeded in constructing a profile. A geometrical interpretation is given in Fig. 6(iii). The case $\rho_1 = \rho_2$ is easily seen to give rise to a continuous profile whose graph has a vertical tangent with $\varphi'(\xi_0) = \infty$.

Singular points are proved to be only of two kinds (recall that profiles are assumed to be increasing for simplicity):

- We say that $\xi_0 \in \mathscr{C}$ if φ is *continuous* at ξ_0 and the following holds: if $\varphi(\xi_0) \in (0, 1)$, then $\varphi'(\xi_0) = \infty$; if $\varphi(\xi_0) = 0$ or 1, then $\varphi'_+(\xi_0) \neq 0$ or $\varphi'_-(\xi_0) \neq 0$. The first case corresponds to a profile having an interior (i.e., $\varphi(\xi_0) \in (0, 1)$) inflection point at ξ_0 with vertical tangent; in the second case, the profile comes off from 0 or reaches 1 with a non zero (possibly infinite) slope or even the limit of $\varphi'(\xi)$ for $\xi \to \xi_0$ may fail to exist.
- We say that $\xi_0 \in \mathscr{J}$ if φ has a *jump discontinuity* at ξ_0 and the following holds: if $\varphi(\xi_0^\pm) \in (0, 1)$, then $\varphi'(\xi_0) = \infty$; if $\varphi(\xi_0^-) \in \{0, 1\}$, then $\varphi'_+(\xi_0) = \infty$; if $\varphi(\xi_0^+) \in \{0, 1\}$, then $\varphi'_-(\xi_0) = \infty$. The former case states that the plot of φ' must be vertical at both sides of the jump; the latter specifies that if a profile comes off from 0 or reaches 1 (where D degenerates), then the vertical slope condition is only required at one side of the jump point.

As we mentioned above, in general, many singular points may occur in a wavefront. For simplicity, we give below the definition of entropic wavefront in the case that there is at most one of them.

Definition 3 Consider $\ell^-, \ell^+ \in [0, 1]$, $\ell^- < \ell^+$. An increasing function $\varphi\colon \mathbb{R} \to [0, 1]$ is an *entropic wavefront solution* to Eq. (6) with speed c and end states $\ell^\pm$ if φ

satisfies (9), it solves (33) in $\mathscr{D}'(\mathbb{R})$ and there is at most one point ξ_0 as above such that φ is a classical solution to (33) in $\mathbb{R} \setminus \{\xi_0\}$.

It can be proved that φ is a solution to (33) in $\mathscr{D}'(\mathbb{R})$ if and only if

$$c = \frac{\left(f(\varphi(\xi_0^+)) - f(\varphi(\xi_0^-))\right) - \left(D(\varphi(\xi_0^+)) - D(\varphi(\xi_0^-))\right)}{\varphi(\xi_0^+) - \varphi(\xi_0^-)}, \tag{38}$$

and, in turn, if and only if there is a unique constant $\gamma \in \mathbb{R}$ such that φ satisfies in $\mathbb{R} \setminus \{\xi_0\}$ the equation

$$D\left(\varphi(\xi)\right) \Phi\left(\varphi'(\xi)\right) + c\varphi(\xi) - f\left(\varphi(\xi)\right) = \gamma. \tag{39}$$

In [22] we provide a complete description of all the possible pattern of profiles that can arise. For simplicity, we only consider here the case of discontinuous profiles.

Theorem 4 *Under the previous assumptions, fix $\ell^\pm \in [0,1]$ with $\ell^- < \ell^+$ and assume $h > 0$ in (ℓ^-, ℓ^+). Then Eq.* (6) *has an increasing entropic WF φ that satisfies* (9)*, with c given in* (36) *and singular set $\mathscr{J} = \{\xi_0\}$, if there exist $\rho_1, \rho_2 \in [\ell^-, \ell^+]$, with $\varphi(\xi_0^-) = \rho_1 < \rho_2 = \varphi(\xi_0^+)$, such that*

$$h < 1 \ \ in\ (\ell^-, \rho_1) \cup (\rho_2, \ell^+) \quad and \quad h \geq 1 \ \ in\ (\rho_1, \rho_2), \tag{40}$$

and one of the following conditions is satisfied:

(i) $\rho_1, \rho_2 \in (\ell^-, \ell^+)$ *and* $h\left((\ell^-)^+\right) = h\left((\ell^+)^-\right) = 0$*;*
(ii) *either* $\rho_1 = \ell^- = 0$, $\rho_2 < \ell^+$, $h\left((\ell^+)^-\right) = 0$ *or* $\rho_2 = \ell^+ = 1$, $\ell^- < \rho_1$, $h\left((\ell^-)^+\right) = 0$*;*
(iii) $\rho_1 = \ell^- = 0$ *and* $\rho_2 = \ell^+ = 1$.

Conversely, suppose again $\ell^- < \ell^+$ and $h > 0$ in (ℓ^-, ℓ^+). Also assume that for every increasing wavefront φ with $\varphi(\pm\infty) = \ell^\pm$ we have $\mathscr{J} = \{\xi_0\}$ and $\rho_1 := \varphi(\xi_0^-)$, $\rho_2 := \varphi(\xi_0^+)$. Then the above conditions on h hold.

The above result allows to confirm the previous statement about the existence of classical or discontinuous profile according to the size of $|\ell^+ - \ell^-|$. In particular, as it is geometrically clear from Fig. 6, it shows the competition between the hyperbolic regime (where f dominates) and the parabolic regime (where g fully smears out the discontinuities).

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A New Critical Exponent for the Heat and Damped Wave Equations with Nonlinear Memory and Not Integrable Data

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Abstract In this paper, we discuss the influence of assuming L^m regularity of initial data, instead of L^1, on a heat or damped wave equation with nonlinear memory. We find that the interplay between the loss of decay rate due to the presence of the nonlinear memory and to the assumption of initial data in L^m instead of L^1, leads to a new critical exponent for the problem, whose shape is quite different from the one of the critical exponent for L^m theory for the corresponding problem with power nonlinearity $|u|^p$. We prove the optimality of the critical exponent using the test function method.

Keywords Heat equation · Damped wave · Nonlinear memory · Critical exponent

1 Introduction

In this paper, we show how a new critical exponent for global small data solutions arises for a heat equation, or a damped wave equation, if we study the interplay between a nonlinear memory term and the assumption that the initial data are not in L^1. The solutions space change accordingly to the problem considered. The existence result is proved in the space $\mathscr{C}([0,\infty), L^m \cap L^\infty)$, $m \in (1,\infty)$, for the heat model (Sobolev solutions) and in the space $\mathscr{C}([0,\infty), H^1) \cap \mathscr{C}^1([0,\infty), L^2)$ for the wave model (energy solutions). The nonexistence result is proved for solutions in the space $L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^n)$ (weak solutions).

In [1], the authors studied the Cauchy problem for

$$\begin{cases} u_t - \Delta u = F(t,u), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

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where

$$F(t,u) = \int_0^t (t-s)^{-\gamma} \, |u(s,x)|^p \, ds \,, \qquad \gamma \in (0,1)\,, \tag{2}$$

represents a memory term, since it is a fractional Riemann–Liouville integral of a power nonlinearity $|u|^p$.

Assuming initial data in $\mathscr{C}_0 \cap L^1$, the authors observed that the Fujita critical exponent $\bar{p}$ for (1) was not given by scaling arguments, as it happened for the heat equation with power nonlinearity $F = |u|^p$, whose critical exponent is the Fujita exponent $1 + 2/n$ (see [8]). Indeed,

$$\bar{p}_1 = \max\{p_\gamma(n),\ \gamma^{-1}\},$$

where

$$p_\gamma(n) = 1 + \frac{2(2-\gamma)}{(n-2(1-\gamma))_+}\,. \tag{3}$$

The competition between the two exponents $p_\gamma(n)$ and γ^{-1} is related to two different effects of the memory term. On the one hand, the memory term produces a loss of decay rate $t^{1-\gamma}$ with respect to the solution to the corresponding linear Cauchy problem, which modifies the ordinary Fujita exponent into a Fujita exponent $p_\gamma(n)$ not given by scaling. On the other hand, it halts the decay rate of the solution to a maximum vanishing speed given by $t^{-\gamma}$, independently on the decay rate for the corresponding linear Cauchy problem. Indeed, in bounded domains, where the linear heat equation shows an exponential decay in time, instead of a polynomial one, the critical exponent is γ^{-1} (Theorem 1.3 in [1]).

What Is a Critical Exponent

By saying that $\bar{p}$ is a critical exponent, here and in the following, we mean that global solutions exist for initial data, small in a certain norm, for supercritical powers $p > \bar{p}$, whereas global solutions do not exist for subcritical powers $p \in (1, \bar{p})$, even for small initial data in the same norm as above, under a suitable sign assumption for the data. The critical power $p = \bar{p}$ may belong to the existence or nonexistence range of global solutions, according to the problem considered.

Due to the fact that the damped wave equation shows the same decay rate profile of the heat equation as a consequence of the diffusion phenomenon (see [13, 15]), the critical exponent for the damped wave equation with power nonlinearity $F = |u|^p$ is also $1 + 2/n$ (see [14, 16]). Similarly, the critical exponent in presence of a nonlinear memory term for the damped wave is the same for the heat equation.

Assuming small initial data in the energy space and in L^1, global energy solutions to

$$\begin{cases} u_{tt} + u_t - \Delta u = F(t,u), & t \geq 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{4}$$

exist for $p > \bar{p}_1 = \max\{p_\gamma(n), \ \gamma^{-1}\}$, at least in space dimension $n \leq 5$ (see [2]). The difficulty of dealing with higher space dimensions is related to the fact that the damped wave equation inherits the good decay properties of the heat equation (the so-called "diffusion phenomenon"), but the more delicate issues with regularity in L^p spaces, with $p \neq 2$, typical of the undamped wave equation.

However, if one drops the assumption of initial data to be (small) in L^1, it is well-known that the Fujita exponent for heat and damped wave equations with power nonlinearity $|u|^p$ changes. In particular, if the L^1 smallness assumption is replaced by L^m smallness, with $m \in (1,2]$, then the Fujita exponent for (1) and (4) with $F = |u|^p$, is $1+2m/n$ (see [9, 11]). Interestingly, the critical case $p = 1+2m/n$ belongs to the nonexistence range if $m = 1$ (see [17]), and to the existence range if $m \in (1,2]$ (see [10]).

It would then be a natural question to ask what happens if initial data are assumed in L^m for a model with nonlinear memory term as (1) or (4), with F as in (2). A first educated guess which could be formulated is asking if the same phenomenon arising for the problem with power nonlinearity $|u|^p$, also appears: is it true that it is like if the space dimension n is replaced by n/m, so that

$$\bar{p} = \max\{p_\gamma(n/m), \gamma^{-1}\} \ ?$$

The answer is no.

In this paper, we will show that if we take into account of L^m smallness of initial data, then a new critical exponent arises from the competition between the loss of decay rate influenced by the nonlinear memory term, and the loss of decay rate originated from dropping the L^1 assumption for the initial data. This exponent originates from the fact that the two losses do not cumulate (the technical reason is explained later, in Remarks 6 and 7) but their interplay is more complicated.

! The New Critical Exponent

The expected critical exponent is

$$\bar{p} = \max\{p_\gamma(n), \ \gamma^{-1}, \ p_{m,\gamma}(n), \ \tilde{p}_{m,\gamma}(n)\}, \tag{5}$$

where the new two exponents, related to the L^m assumption of the data, are defined by

$$p_{m,\gamma}(n) = 1 + \frac{2m(2-\gamma)}{n}, \tag{6}$$

$$\tilde{p}_{m,\gamma}(n) = 1 + \frac{1-\gamma}{\left(1 - \frac{n}{2}\left(1 - \frac{1}{m}\right)\right)_+} = 1 + \frac{(1-\gamma)m}{(2m - n(m-1))_+}. \tag{7}$$

So far, we are able to prove that the exponent $p_{m,\gamma}(n)$ is really critical, that is, global small data (Sobolev or energy) solutions exist when $p > p_{m,\gamma}(n) = \bar{p}$, and no global weak solutions may exist for suitable sign assumption on the initial data if $1 < p < p_{m,\gamma}(n)$. On the other hand, we are only able to prove that global small data (Sobolev or energy) solutions exist when $p > \tilde{p}_{m,\gamma}(n) = \bar{p}$, but we cannot prove that no global weak solutions may exist if $1 < p < \tilde{p}_{m,\gamma}(n)$.

The expression in (5) may appear complicated, but it is easy to determine what is the maximum in (5), for some given γ, m, n. Indeed, if γ is sufficiently small with respect to m, namely,

$$0 < \gamma \leq 1 - \frac{n}{2}\left(1 - \frac{1}{m}\right), \tag{8}$$

then

$$\bar{p} = \bar{p}_1 = \max\{p_\gamma(n),\ \gamma^{-1}\},$$

in (5). Indeed, condition (8) corresponds to say that $(1+t)^{\frac{n}{2}\left(1-\frac{1}{m}\right)} \leq (1+t)^{1-\gamma}$, i.e., that the loss of decay due to the assumption of L^m smallness of the initial data becomes irrelevant with respect to the loss of decay rate related to the presence of the nonlinear memory term. In this case, one may easily follow the approach in [1] and [2] and prove the global existence of (Sobolev or energy) solutions for $p > \bar{p}_1$, even replacing the L^1 assumption of the data by the L^m assumption.

On the other hand, if

$$0 < 1 - \frac{n}{2}\left(1 - \frac{1}{m}\right) < \gamma, \tag{9}$$

then the situation is opposite and $\bar{p} = \bar{p}_m$, where we define

$$\bar{p}_m = \max\{p_{m,\gamma}(n),\ \tilde{p}_{m,\gamma}(n)\}. \tag{10}$$

As the transition from $p_\gamma(n)$ to γ^{-1} appeared in space dimension $n \geq 3$ at the threshold value $\gamma = (n-2)/n$, which corresponds to $p_\gamma(n) = \gamma^{-1} = n/(n-2)$, the transition from $p_{m,\gamma}(n)$ to $\tilde{p}_{m,\gamma}(n)$ appears in space dimension $n \geq 3$ at the threshold value which corresponds to $p_{m,\gamma}(n) = \tilde{p}_{m,\gamma}(n) = n/(n-2)$, that is,

$$m = \frac{1}{2-\gamma}\,\frac{n}{n-2}.$$

Since for $n \geq 3$ we may write (9) as

$$\frac{n}{n-2(1-\gamma)} < m < \frac{n}{n-2},$$

we may distinguish two cases. If

$$\frac{1}{2-\gamma}\,\frac{n}{n-2} < m < \frac{n}{n-2}, \tag{11}$$

then $\bar{p} = p_{m,\gamma}(n) > \tilde{p}_{m,\gamma}(n)$. If

$$\frac{n}{n-2(1-\gamma)} < m < \frac{1}{2-\gamma}\,\frac{n}{n-2}, \tag{12}$$

then $\bar{p} = \tilde{p}_{m,\gamma}(n) > p_{m,\gamma}(n)$. However, the interval in (12) is empty if, and only if,

$$n \geq \frac{2}{1-\gamma},$$

so that the exponent $\tilde{p}_{m,\gamma}(n)$ only appears for some m when $3 \leq n < 2/(1-\gamma)$.

In the limit case

$$m = \frac{1}{2-\gamma}\,\frac{n}{n-2}, \tag{13}$$

then $\bar{p} = p_{m,\gamma}(n) = \tilde{p}_{m,\gamma}(n) = n/(n-2)$.

We are now ready to state our main results.

Theorem 1 *Let* $n = 1, 2$ *and* $m \in (1, \infty)$ *or* $n \geq 3$ *and* $m \in (1, n/(n-2))$. *Assume that*

$$1 - \frac{n}{2}\left(1 - \frac{1}{m}\right) < \gamma < 1,$$

and that $p \geq \bar{p}_m$*, or* $p > n/(n-2)$ *if* $n \geq 3$ *and* (13) *holds. Then there exists* $\varepsilon > 0$ *such that for any initial data*

$$u_0 \in L^m \cap L^\infty, \quad \text{with } \|u_0\|_{L^m \cap L^\infty} = \|u_0\|_{L^m} + \|u_0\|_{L^\infty} \leq \varepsilon,$$

there is a unique global Sobolev solution $u \in \mathscr{C}([0,\infty), L^m \cap L^\infty)$ *to* (1)*. Moreover, it satisfies the same decay estimates satisfied by the corresponding linear Cauchy problem, that is,* (1) *with* $F = 0$*,*

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{q}\right)} \|u_0\|_{L^m \cap L^\infty}, \tag{14}$$

for any $q \in [m,\infty]$ *if* $n = 1$ *and for any* $q \in [m, n/(n-2))$ *if* $n \geq 2$*. Here and in the following we set* $n/(n-2) = \infty$ *when* $n = 2$*. If* $n \geq 2$ *and* $q = n/(n-2)$*, it satisfies the same decay estimates as above, but with a logarithmic loss of decay, that is,*

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-1+\frac{n}{2}\left(1-\frac{1}{m}\right)} \log(e+t)\, \|u_0\|_{L^m \cap L^\infty}. \tag{15}$$

If $n \geq 3$ *and* $q \in (n/(n-2), \infty]$*, it satisfies the following decay estimate*

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-1+\frac{n}{2}\left(1-\frac{1}{m}\right)} \|u_0\|_{L^m \cap L^\infty}. \tag{16}$$

The constant $C > 0$ *is independent of the initial datum.*

Remark 1 We notice that the critical case $p = \bar{p}_m$ belongs to the existence range when (9) holds, exception given for the special case (13) (see the end of the proof of Theorem 1 for the technical reason). On the other hand, in the case in which (8) holds and $\gamma \geq (n-2)/n$, the critical power $p = p_\gamma(n)$ belongs to the nonexistence range.

Remark 2 We may easily check that if $p \geq \bar{p}_m$, then $m < p$ in Theorem 1. Indeed,

$$p \geq \bar{p}_m \geq p_{m,\gamma}(n) = 1 + \frac{2m(2-\gamma)}{n} > 1 + \frac{2m}{n} > m,$$

where the last inequality is true since it is either trivial if $n = 1, 2$, or equivalent to $m < n/(n-2)$ if $n \geq 3$.

For the sake of brevity, we only state the corresponding result for the damped wave equation in space dimension $n = 1, 2$, but it can be easily extended to space dimension $n = 3, 4, 5$, following as in [2]. Also, we restrict our assumption to $m \in [1, 2]$, to get the existence of energy solutions, in a classical sense, i.e., in $\mathscr{C}([0,\infty), H^1) \cap \mathscr{C}^1([0,\infty), L^2)$. Finally, we notice that $\bar{p}_m = p_{m,\gamma}(n)$ due to the assumption $n = 1, 2$.

Theorem 2 *Let* $n = 1, 2$ *and* $m \in (1, 2]$. *Assume that*

$$1 - \frac{n}{2}\left(1 - \frac{1}{m}\right) < \gamma < 1,$$

and that $p \geq p_{m,\gamma}(n)$. *Then there exists* $\varepsilon > 0$ *such that for any initial data*

$$(u_0, u_1) \in \mathscr{A} = (L^m \cap H^1) \cap (L^m \cap L^2),$$
$$\text{with } \|(u_0, u_1)\|_{\mathscr{A}} = \|u_0\|_{L^m} + \|u_0\|_{H^1} + \|u_1\|_{L^m} + \|u_1\|_{L^2} \leq \varepsilon,$$

there is a unique global energy solution $u \in \mathscr{C}([0, \infty), H^1) \cap \mathscr{C}^1([0, \infty), L^2)$ *to* (4). *Moreover, it satisfies the same decay estimates satisfied by the corresponding linear Cauchy problem, that is,* (4) *with* $F = 0$,

$$\|u(t, \cdot)\|_{L^q} \leq C(1 + t)^{-\frac{n}{2}\left(\frac{1}{m} - \frac{1}{q}\right)} \|(u_0, u_1)\|_{\mathscr{A}}, \tag{17}$$

for any $q \in [2, \infty]$ *if* $n = 1$ *and for any* $q \in [2, \infty)$ *if* $n = 2$. *Also, its derivatives satisfy the following decay estimates*

$$\|u_x(t, \cdot)\|_{L^2} \leq C(1 + t)^{-\frac{1}{4} - \frac{1}{2m}} \|(u_0, u_1)\|_{\mathscr{A}}, \quad \text{if } n = 1, \tag{18}$$

$$\|\nabla u(t, \cdot)\|_{L^2} \leq C(1 + t)^{-\frac{1}{m}} \log(e + t) \|(u_0, u_1)\|_{\mathscr{A}}, \quad \text{if } n = 2, \tag{19}$$

$$\|u_t(t, \cdot)\|_{L^2} \leq C(1 + t)^{-1 + \frac{n}{2}\left(1 - \frac{1}{m}\right)} \|(u_0, u_1)\|_{\mathscr{A}}, \quad \text{if } n = 1, 2. \tag{20}$$

Estimate (18) *is the same decay estimate satisfied by the corresponding linear Cauchy problem.*

Remark 3 We notice that we used that $p > 2$ as a consequence of $p \geq p_{m,\gamma}(n) > 2$ for $n = 1, 2$. This allows us to work only with energy solutions, without the need to employ $L^1 - L^p$ estimates, with $p < 2$, as one may do to extend Theorem 2 to higher space dimension $n = 3, 4, 5$, as done in [2]. We also notice that $H^1 \hookrightarrow L^q$, for any $q \in [2, \infty)$, since the space dimension is $n \leq 2$ in Theorem 2.

So far, we are not able to prove that the critical exponent is $\bar{p}_m$ if $3 \leq n < 2/(1-\gamma)$, in the sense that we are able to prove a nonexistence result of global weak solutions to (1) and (4) only for $p < p_{m,\gamma}(n)$. It remains open to check if the nonexistence of global weak solutions can really be proved for $p \leq \tilde{p}_{m,\gamma}(n)$, when $\bar{p} = \tilde{p}_{m,\gamma}(n)$.

Theorem 3 *Let* $n \geq 1$, $\gamma \in (0, 1)$, $m \in (1, \infty)$. *Assume that*

$$u_0 \in L^1_{\text{loc}}, \quad u_0(x) \geq \varepsilon\, |x|^{-\frac{n}{m}} \log |x|, \text{ for } |x| \gg 1, \tag{21}$$

or, respectively,

$$u_0, u_1 \in L^1_{\mathrm{loc}}, \quad (u_0(x) + u_1(x)) \geq \varepsilon\, |x|^{-\frac{n}{m}} \log|x|, \text{ for } |x| \gg 1, \text{ and } u_0 \geq 0, \tag{22}$$

and that $u \in L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^n)$ *is a global-in-time weak solution to* (1) *or, respectively,* (4). *Then* $p \geq p_{m,\gamma}(n)$.

Remark 4 The sign assumption in (21) and (22) is taken in such a way that it is compatible with $u_0, u_1 \in L^m$ and so Theorem 3 is the counterpart of Theorem 1 and (2) for the L^m assumption of initial data.

Remark 5 Another model for which the critical exponent belongs to the existence range is provided by the fractional diffusive equation

$$\begin{cases} \partial_t^{1+\alpha} u - \Delta u = |u|^p, & t \geq 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

where $\partial_t^{1+\alpha}$, with $\alpha \in (0,1)$ denotes the Caputo fractional derivative. In [7] it is proved that global small data Sobolev solutions exist for

$$p \geq \bar{p} = 1 + \frac{2}{n - 2(1+\alpha)^{-1}}.$$

In this model, the loss of decay rate is proper of the linear Cauchy problem as well, due to the assumption that $u_1 \neq 0$, even if $u_1 \in L^1$, and appears in comparison with the Duhamel operator itself. The loss of decay rate is the motivation for which one has global existence in the critical case, for the same technical reason which also appears for models (1) and (4) in the proofs of Theorems 1 and 2. In [4], the nonexistence counterpart result is provided for the model above and for more general models with fractional derivatives in time.

? Open Problem

It would be interesting to extend the results to the wave equation or to σ-evolution equations, with effective structural damping, namely to the Cauchy problems

$$\begin{cases} u_{tt} + (-\Delta)^\theta u_t + (-\Delta)^\sigma u = F(t,u), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{23}$$

where $\sigma > 0$ and $\theta \in [0, \sigma/2]$. In the case of a power nonlinearity $F = |u|^p$ it has been proved [5] that the critical exponent, assuming small data in L^m, is

$$\bar{p} = 1 + \frac{2m\sigma}{n - 2\theta}.$$

On the other hand, it has been proved [3] that the critical exponent for (23) when $\theta = 1/2$ and $\sigma = 1$, assuming small data in L^1, with nonlinear memory term F as in (2), is

$$\bar{p} = \max\{p_\gamma(n), \gamma^{-1}\}, \quad p_\gamma(n) = 1 + \frac{3-\gamma}{n+\gamma-2}$$

in space dimension $n \geq 2$. The interplay between $\sigma, \theta, \gamma, m$ could lead to some interesting result.

Also, the case in which a nonlinear memory term is related to $|u_t|^p$ is of interest, namely,

$$F = \int_0^t (t-s)^{-\gamma} \, |u_t(s,x)|^p \, ds\,, \qquad \gamma \in (0,1)\,,$$

due to the new phenomenon investigated in [5] about how two different asymptotic profiles appear for structurally damped evolution equations and may influence problems with different power nonlinearities. The easier limit case $\sigma = 2\theta$ has been recently investigated in [6].

2 Proof of Theorem 1

In this section, we prove Theorem 1.

Proof By Duhamel's principle, a function $u \in \mathscr{C}([0,\infty), L^m \cap L^\infty)$ is a global Sobolev solution to (1) if, and only if, it satisfies the equality

$$u(t,\cdot) = E(t,\cdot) * u_0 + \int_0^t \int_0^\tau (\tau - s)^{-\gamma} \, E(t-\tau,\cdot) *_{(x)} |u(s,\cdot)|^p \, ds \, d\tau\,, \tag{24}$$

in $L^m \cap L^\infty$, where E is the fundamental solution to the linear heat equation in, that is,

$$\begin{cases} E_t - \Delta E = 0, & t > 0, \\ E(0,\cdot) = \delta. \end{cases}$$

Explicitly, $E(t,x) = (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{|x|^2}{4t}}$ for any $t > 0$ and $x \in \mathbb{R}^n$.

For any $T > 0$ we define the Banach space

$$X(T) \doteq \mathscr{C}([0,T], L^m \cap L^\infty) \tag{25}$$

with the norm given by

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]} \left\{ \|u(t,\cdot)\|_{L^m} + (1+t)^{-\frac{n}{2m}} \|u(t,\cdot)\|_{L^\infty} \right\}, \tag{26}$$

if $n = 1$, or by

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]} \left\{ \sup_{q\in[m,\bar{q})} (1+t)^{\frac{n}{2}\left(\frac{1}{m}-\frac{1}{q}\right)} \|u(t,\cdot)\|_{L^q} \right.$$
$$\left. + (1+t)^{1-\frac{n}{2}\left(1-\frac{1}{m}\right)} \left((\log(e+t))^{-1} \|u(t,\cdot)\|_{L^{\bar{q}}} + \sup_{q\in(\bar{q},\infty]} \|u(t,\cdot)\|_{L^q} \right) \right\}, \tag{27}$$

where $\bar{q} = n/(n-2)$. Here we recall that $\bar{q} = \infty$ if $n = 2$, due to the notation in Theorem 1.

We introduced the time-dependent weights in the norm in (26) and (27) in such a way that

$$\|u(t,\cdot)\|_{L^q} \le (1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{q}\right)} \|u\|_{X(T)}, \tag{28}$$

for any $q \in [m,\infty]$ if $n = 1$, and for any $q \in [1,\bar{q})$ if $n \ge 2$, for any $t \in [0,T]$. On the other hand,

$$\|u(t,\cdot)\|_{L^q} \le (1+t)^{-1+\frac{n}{2}\left(1-\frac{1}{m}\right)} \log(e+t)\, \|u\|_{X(T)}, \tag{29}$$

if $n \ge 2$, and

$$\|u(t,\cdot)\|_{L^q} \le (1+t)^{-1+\frac{n}{2}\left(1-\frac{1}{m}\right)} \|u\|_{X(T)}, \tag{30}$$

for any $q \in (\bar{q},\infty]$, if $n \ge 3$.

In particular, the solution to the linear Cauchy problem corresponding to (1) (i.e., $E(t,\cdot) * u_0$) is in $X(T)$ for any $T > 0$, due to

$$\|E(t,\cdot) * u_0\|_{L^q} \le C\,(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{q}\right)} \left(\|u_0\|_{L^m} + \|u_0\|_{L^q} \right), \tag{31}$$

for any $q \in [m, p]$, with C independent of t. In particular,

$$\|E *_{(x)} u_0\|_{X(T)} \leq C \left(\|u_0\|_{L^m} + \|u_0\|_{L^\infty}\right). \tag{32}$$

We define the operator G such that, for any $u \in X(T)$,

$$Gu(t,x) \doteq \int_0^t F(\tau, u(\tau,x))\, d\tau = \int_0^t \int_0^\tau (\tau - s)^{-\gamma}\, E(t-\tau, x) *_{(x)} |u(s,x)|^p\, ds\, d\tau\,, \tag{33}$$

then we prove the estimate

$$\|Gu - Gv\|_{X(T)} \leq C\|u - v\|_{X(T)}\left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right), \tag{34}$$

for any $u, v \in X(T)$, where C is independent of $T \in (0, \infty)$. By standard arguments, thanks to (32), estimate (34) leads to the existence of a unique Sobolev solution to (24). Since all of the constants are independent of T we can take $T \to \infty$ and we gain a local and a global existence result simultaneously.

Moreover, thanks to the properties (28) and, respectively, (29) and (30), we also obtain the desired estimate (14) and, respectively, (15) and (16), for the solution to (1).

In order to prove (34), we use the following $(L^1 \cap L^q) - L^q$ estimate for the fundamental solution of the heat equation:

$$\|E(t-\tau,\cdot) * g\|_{L^q} \leq C\,(1 + t - \tau)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)} \left(\|g\|_{L^1} + \|g\|_{L^q}\right), \tag{35}$$

inside the inner integral of (33), for $g(x) = |u(s,x)|^p - |v(s,x)|^p$. By Hölder's inequality, we may estimate

$$\begin{aligned}\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^1} &\leq C\,\|u(s,\cdot) - v(s,\cdot)\|_{L^p}\, \||u(s,\cdot)^{p-1} + |v(s,\cdot)^{p-1}\|_{L^{p'}} \\ &\leq C\,\|u(s,\cdot) - v(s,\cdot)\|_{L^p}\left(\|u(s,\cdot)\|_{L^p}^{p-1} + \|v(s,\cdot)\|_{L^p}^{p-1}\right).\end{aligned}$$

On the other hand, due to the fact that $u, v \in X(T)$, we may then estimate

$$\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^1} \leq C\,(1+s)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{p}\right)p}\, \|u-v\|_{X(T)}\left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right),$$

if $n = 1$ or $p < \bar{q}$, or

$$\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^1} \leq C\,(1+s)^{-p+\frac{n}{2}\left(1-\frac{1}{m}\right)p}\, \|u-v\|_{X(T)}\left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right),$$

if $n \geq 3$ and $p > \bar{q}$. A logarithmic loss $(\log(e+s))^p$ appears above if $p = \bar{q}$.

We proceed similarly for $\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^q}$.

Now we distinguish two cases.

Assume first that $n = 1, 2$, or that (11) holds if $n \geq 3$. This latter corresponds to $p_{m,\gamma}(n) < n/(n-2)$.

Due to $pq \geq p \geq p_{m,\gamma}(n)$, we may estimate

$$\begin{aligned}
&\| |u(s,\cdot)|^p - |v(s,\cdot)|^p \|_{L^1\cap L^q} \\
&\quad \leq C\,(1+s)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{p_{m,\gamma}(n)}\right)p_{m,\gamma}(n)} \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right) \\
&\quad = C\,(1+s)^{\frac{n}{2}\left(1-\frac{1}{m}\right)-2+\gamma} \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right).
\end{aligned}$$

Summarizing, we estimated so far

$$\begin{aligned}
&\|Gu(t,\cdot) - Gv(t,\cdot)\|_{L^q} \leq C\,\|u-v\|_{X(T)}\left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right) \\
&\quad \times \int_0^t (1+t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)} \int_0^\tau (\tau-s)^{-\gamma}\,(1+s)^{\frac{n}{2}\left(1-\frac{1}{m}\right)-2+\gamma}\,ds\,d\tau.
\end{aligned}$$

Due to the fact that the exponent of $(1+t)^{\frac{n}{2}\left(1-\frac{1}{m}\right)-2+\gamma}$ verifies

$$\frac{n}{2}\left(1-\frac{1}{m}\right) - 2 + \gamma > -1,$$

as a consequence of (9), and since $\gamma \in (0,1)$, straightforward computation leads to estimate the inner integral by

$$\int_0^\tau (\tau-s)^{-\gamma}\,(1+s)^{\frac{n}{2}\left(1-\frac{1}{m}\right)-2+\gamma}\,ds \approx (1+\tau)^{-1+\frac{n}{2}\left(1-\frac{1}{m}\right)}. \tag{36}$$

On the other hand,

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}\,(1+\tau)^{-1+\frac{n}{2}\left(1-\frac{1}{m}\right)}\,d\tau \\
&\qquad \approx \begin{cases}
(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{q}\right)} & \text{if } \frac{n}{2}\left(1-\frac{1}{q}\right) < 1, \\
(1+t)^{-1+\frac{n}{2}\left(1-\frac{1}{m}\right)} \log(e+t) & \text{if } \frac{n}{2}\left(1-\frac{1}{q}\right) = 1, \\
(1+t)^{-1+\frac{n}{2}\left(1-\frac{1}{m}\right)} & \text{if } \frac{n}{2}\left(1-\frac{1}{q}\right) > 1.
\end{cases}
\end{aligned}$$

Therefore, we proved (34) in the case that $n = 1, 2$, or that (11) holds if $n \geq 3$.

Assume now that $n \geq 3$ and (12) holds. In this case, due to

$$pq \geq p \geq \tilde{p}(m,\gamma) > n/(n-2),$$

we may estimate

$$\begin{aligned}
&\| |u(s,\cdot)|^p - |v(s,\cdot)|^p \|_{L^1\cap L^q} \\
&\quad \leq C\,(1+s)^{\left(-1+\frac{n}{2}\left(1-\frac{1}{m}\right)\right)\,\tilde{p}_{m,\gamma}(n)}\, \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right) \\
&\quad = C\,(1+s)^{\frac{n}{2}\left(1-\frac{1}{m}\right)-2+\gamma}\, \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right).
\end{aligned}$$

Once again, we obtained (36), so that we proceed as before, proving (34).

It remained open the limit case in which $n \geq 3$ and (13) holds, that is, $\bar{p} = n/(n-2)$. In this case, the situation is more tricky, since from $u \in X(T)$, we only derive the above estimate

$$\begin{aligned}
&\| |u(s,\cdot)|^p - |v(s,\cdot)|^p \|_{L^1\cap L^q} \\
&\qquad \leq C\,(1+s)^{\frac{n}{2}\left(1-\frac{1}{m}\right)-2+\gamma}\, \|u-v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right),
\end{aligned}$$

for $p > \bar{p}$, due to the fact that a logarithmic loss appears when $p = \bar{p}$. For this reason, we have to exclude the case $p = \bar{p} = n/(n-2)$ if (13) holds, in order to obtain (36) and prove (34).

This concludes the proof. □

Remark 6 It is crucial to remark the following difference in the use of the decay estimates for the convolution with the fundamental solution E to the heat equation. On the one hand, in Theorem 1 initial data are only assumed in $L^m \cap L^\infty$, so that the decay rate appearing in (31) for the solution to the corresponding linear problem is "only" $(n/2)(1/q - 1/m)$, in particular it depends on m. On the other hand, the $L^1 - L^q$ estimate (35) is applied to the function $g(s,x) = |u(s,x)|^p - |v(s,x)|^p$. In particular, $g(s,\cdot) \in L^1 \cap L^\infty$, due to $u(s,\cdot), v(s,\cdot) \in L^m \cap L^\infty$, and $p \geq m$ (see Remark 2). The fact that the decay rate $(n/2)(1-1/q)$ in (35) is faster than the decay rate in (31) for any $m > 1$, is the basis for the new effects appearing for the interplay between the L^m data regularity and the presence of a nonlinear memory term.

3 Proof of Theorem 2

In this section, we prove Theorem 2.

Proof Let us denote by E the fundamental solution to the linear damped wave equation in $\mathbb{R}^n$, namely,

$$\begin{cases} E_{tt} + E_t - \Delta E = 0, \quad t > 0, \\ E(0,\cdot) = 0, \\ E_t(0,\cdot) = \delta, \end{cases}$$

and set $\tilde{E} = E + E_t$. Then

$$u^{\,\mathrm{lin}}(t,\cdot) = \tilde{E}(t,\cdot) * u_0 + E(t,\cdot) * u_1$$

is the solution to the linear Cauchy problem for the damped wave equation, that is, (4) with $F = 0$. By Duhamel's principle, a function $u \in \mathscr{C}([0,\infty), H^1) \cap \mathscr{C}^1([0,\infty), L^2)$ is a global energy solution to (4) if, and only if, it satisfies the equality

$$u(t,\cdot) = u^{\,\mathrm{lin}}(t,\cdot) + \int_0^t \int_0^\tau (\tau - s)^{-\gamma}\, E(t-\tau,\cdot) *_{(x)} |u(s,\cdot)|^p \, ds \, d\tau\,, \tag{37}$$

in H^1, and

$$u_t(t,\cdot) = u_t^{\,\mathrm{lin}}(t,\cdot) + \int_0^t \int_0^\tau (t - s)^{-\gamma}\, E_t(t-\tau,\cdot) *_{(x)} |u(s,\cdot)|^p \, ds \, d\tau\,,$$

in L^2.

For any $T > 0$ we define the Banach space

$$X(T) \doteq \mathscr{C}([0,T], H^1) \cap \mathscr{C}^1([0,T], L^2), \tag{38}$$

with the norm given by

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]} \Big\{ (1+t)^{\frac{1}{2m}-\frac{1}{4}} \|u(t,\cdot)\|_{L^2} + (1+t)^{\frac{1}{2m}+\frac{1}{4}} \|u_x(t,\cdot)\|_{L^2} + (1+t)^{\frac{1}{2m}+\frac{1}{2}} \|u_t(t,\cdot)\|_{L^2} \Big\}, \tag{39}$$

if $n = 1$, or by

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]} \Big\{ \sup_{q\in[2,\infty)} (1+t)^{\frac{1}{m}-\frac{1}{q}} \|u(t,\cdot)\|_{L^q} + (1+t)^{\frac{1}{m}} \Big((\log(e+t))^{-1} \|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2} \Big) \Big\} \tag{40}$$

if $n = 2$.

We introduced the time-dependent weights in the norm in (39) and (40) in such a way that

$$\|u(t,\cdot)\|_{L^q} \leq (1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{q}\right)} \|u\|_{X(T)}, \tag{41}$$

for any $q \in [2,\infty]$ if $n = 1$, and for any $q \in [2,\infty)$ if $n = 2$, for any $t \in [0,T]$. On the other hand,

$$\|u_x(t,\cdot)\|_{L^2} \leq (1+t)^{-\frac{1}{4}-\frac{1}{2m}} \|u\|_{X(T)}, \tag{42}$$

if $n = 1$, and

$$\|\nabla u(t,\cdot)\|_{L^2} \leq (1+t)^{-\frac{1}{m}} \log(e+t) \|u\|_{X(T)}, \tag{43}$$

if $n = 2$, whereas

$$\|u_t(t,\cdot)\|_{L^2} \leq (1+t)^{-1+\frac{n}{2}\left(1-\frac{1}{m}\right)} \|u\|_{X(T)}, \tag{44}$$

for $n = 1, 2$.

In particular, the solution u^{lin} is in $X(T)$ for any $T > 0$, due to

$$\|u^{\text{lin}}(t,\cdot)\|_{L^q} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{q}\right)} \|(u_0,u_1)\|_{\mathscr{A}}, \tag{45}$$

for any $q \in [2,\infty]$ if $n = 1$ and for any $q \in [2,\infty)$ if $n = 2$, and

$$\|\nabla u(t,\cdot)\|_{L^2} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}} \|(u_0,u_1)\|_{\mathscr{A}},$$

$$\|u_t(t,\cdot)\|_{L^2} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-1} \|(u_0,u_1)\|_{\mathscr{A}},$$

in space dimension $n = 1, 2$, with C independent of t. In particular,

$$\|u^{\text{lin}}\|_{X(T)} \leq C\|(u_0,u_1)\|_{\mathscr{A}}. \tag{46}$$

We define the operator G for any $u \in X(T)$, as in (33), then the proof of our statement follows as in the proof of Theorem 1, if we prove (34) for any $u, v \in X(T)$, where C is independent of $T \in (0,\infty)$.

In order to prove (34), we use inside the inner integral of (33), for $g(x) = |u(s,x)|^p - |v(s,x)|^p$, the following estimates for the fundamental solution of the damped wave equation:

$$\|E(t-\tau,\cdot) * g\|_{L^q} \leq C(1+t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)} \big(\|g\|_{L^1} + \|g\|_{L^2}\big), \tag{47}$$

$$\|\nabla E(t-\tau,\cdot)*g\|_{L^2} \leq C\,(1+t-\tau)^{-\frac{n}{4}-\frac{1}{2}}\big(\|g\|_{L^1}+\|g\|_{L^2}\big), \tag{48}$$

$$\|E_t(t-\tau,\cdot)*g\|_{L^2} \leq C\,(1+t-\tau)^{-\frac{n}{4}-1}\big(\|g\|_{L^1}+\|g\|_{L^2}\big), \tag{49}$$

where (47) holds for $q \in [2,\infty]$ if $n=1$ and for $q \in [2,\infty)$ if $n=2$ (due to the fact that $H^1(\mathbb{R}^2)$ is not imbedded in $L^\infty(\mathbb{R}^2)$).

As in the proof of Theorem 2, by Hölder's inequality, and due to the fact that $u, v \in X(T)$, for any $p \geq p_{m,\gamma}(n)$, we may estimate

$$\begin{aligned}
&\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{L^1\cap L^2}\\
&\quad\leq C\,(1+s)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{p}\right)p}\,\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big)\\
&\quad\leq C\,(1+s)^{\frac{n}{2}\left(1-\frac{1}{m}\right)-2+\gamma}\,\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big).
\end{aligned}$$

We remark that we used that $p>2$ as a consequence of $p \geq p_{m,\gamma}(n) > 2$.

Summarizing, we estimated so far

$$\begin{aligned}
&\|Gu(t,\cdot)-Gv(t,\cdot)\|_{L^q} \leq C\,\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big)\\
&\quad\times\int_0^t (1+t-\tau)^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}\int_0^\tau (\tau-s)^{-\gamma}\,(1+s)^{\frac{n}{2}\left(1-\frac{1}{m}\right)-2+\gamma}\,ds\,d\tau,
\end{aligned}$$

for $q \in [2,\infty]$ if $n=1$ and for $q \in [2,\infty)$ if $n=2$. Also, we estimated

$$\begin{aligned}
&\|\partial_t^j\nabla^k(Gu(t,\cdot)-Gv(t,\cdot))\|_{L^2} \leq C\,\|u-v\|_{X(T)}\big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\big)\\
&\quad\times\int_0^t (1+t-\tau)^{-\frac{n}{4}-j-\frac{k}{2}}\int_0^\tau (\tau-s)^{-\gamma}\,(1+s)^{\frac{n}{2}\left(1-\frac{1}{m}\right)-2+\gamma}\,ds\,d\tau,
\end{aligned}$$

for $j+k=1$.

As in the proof of Theorem 1, straightforward computation leads to estimate the inner integral by (36). We conclude the proof as in the proof of Theorem 1, noticing that

$$-\frac{n}{2}\left(1-\frac{1}{q}\right) > -1,$$

for any $q \in [2,\infty]$ if $n=1$ and for any $q \in [2,\infty)$ if $n=2$. On the other hand,

$$-\frac{n}{4}-\frac{1}{2}$$

is greater than -1 for $n = 1$, and it equals -1 for $n = 2$, whereas

$$-\frac{n}{4} - 1 < -1,$$

for $n = 1, 2$.

This concludes the proof. □

Remark 7 As in Remark 6, we emphasize the following difference in the use of the decay estimates for the convolution with the fundamental solution E to the damped wave equation. On the one hand, in Theorem 2, initial data are only assumed in $(L^m \cap H^1) \times (L^m \cap L^2)$, so that the decay rate appearing in (45) for the solution to the corresponding linear problem is "only" $(n/2)(1/q - 1/m)$, in particular it depends on m. On the other hand, the $L^1 - L^q$ estimate (47) is applied to the function $g(s, x) = |u(s, x)|^p - |v(s, x)|^p$. In particular, $g(s, \cdot) \in L^1 \cap L^2$, due to $u(s, \cdot), v(s, \cdot) \in H^1 \hookrightarrow L^p \cap L^{2p}$ (see Remark 3). A similar difference also appears for the estimates on the derivatives.

4 Proof of Theorem 3

In order to prove Theorem 3 simultaneously for the heat and for the damped wave equation, we multiply u_{tt} and u_1 in (4) by a parameter $a \in \{0, 1\}$. In this way, for $a = 0$ we recover (1) and for $a = 1$ we recover (4).

Before proving Theorem 3, we shall explain the meaning of weak solutions in $L^p_{\text{loc}}([0, \infty) \times \mathbb{R}^n)$ in the statement. Assume first that u is a smooth solution in $\mathscr{C}^2([0, \infty) \times \mathbb{R}^n)$. If we consider a test function $\varphi \in \mathscr{C}^2_c([0, \infty) \times \mathbb{R}^n)$, multiplying the equation by φ and integrating by parts, we obtain

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} F(t, u)\varphi \, dx dt &= \int_0^\infty \int_{\mathbb{R}^n} (au_{tt} + u_t - \Delta u)\varphi \, dx dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} u \,(a\varphi_{tt} - \varphi_t - \Delta\varphi) \, dx dt \\ &\quad - \int_{\mathbb{R}^n} (au_1 + u_0)\, \varphi(0, x) \, dx \\ &\quad + a \int_{\mathbb{R}^n} u_0 \, \varphi_t(0, x) \, dx. \end{aligned}$$

On the other hand, due to the fact that

$$F(t, u) = \Gamma(1 - \gamma)\, (J^{1-\gamma}_{0|+} |u|^p)(t),$$

where $J^{1-\gamma}_{0|+}$ is the Riemann–Liouville fractional integral of order $1-\gamma$ and starting point 0, we may use the integration by parts rule for fractional integrals [12, (2.2.31)], which gives us

$$\int_0^\infty \int_{\mathbb{R}^n} F(t,u)\varphi \, dx dt = \Gamma(1-\gamma) \int_0^\infty \int_{\mathbb{R}^n} |u|^p \, (J_{\infty|-}\varphi) \, dx dt,$$

where

$$\Gamma(1-\gamma)\,(J_{\infty|-}\varphi)(t) = \int_t^\infty (\tau-t)^{-\gamma} \varphi(\tau,x)\, d\tau.$$

We remark that $J_{\infty|-}\varphi$ is compactly supported, since φ is compactly supported. Indeed, if $\varphi(\tau,x)=0$ for any $\tau \geq T$, then $(J_{\infty|-}\varphi)(t,x)=0$ for any $t \geq T$.

Then we may say that $u \in L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^n)$ is a weak solution to (1) or (4) if, for any test function $\varphi \in \mathscr{C}^2_c([0,\infty)\times\mathbb{R}^n)$, it satisfies the integral equality

$$\begin{aligned}\Gamma(1-\gamma) \int_0^\infty \int_{\mathbb{R}^n} |u|^p (J_{\infty|-}\varphi)\, dx dt &= \int_0^\infty \int_{\mathbb{R}^n} u\,(a\varphi_{tt} - \varphi_t - \Delta\varphi)\, dx dt \\ &\quad - \int_{\mathbb{R}^n} (au_1 + u_0)\,\varphi(0,x)\, dx \\ &\quad + a \int_{\mathbb{R}^n} u_0\, \varphi_t(0,x)\, dx.\end{aligned}$$

It is then clear that smooth, Sobolev and energy solutions are also weak solutions.

To prove Theorem 3, we apply the above integral equality for a suitable test function, and we obtain the necessary condition $p \geq p_{m,\gamma}(n)$ for the global in time existence of weak solutions with suitable sign assumption on the initial data.

Proof For a given $T \geq 1$, we fix

$$\omega = (1-t/T)_+.$$

In particular, $\omega(t)^\beta \in \mathscr{C}^k_c([0,\infty))$, for any $k < \beta$. Let us define $\alpha = 1-\gamma \in (0,1)$. Then it holds (see, for instance, Lemma 4.1 in [3]):

$$D^\alpha_{t|-}\,\omega(t)^\beta = C(\alpha,\beta)\,T^{-\alpha}\omega(t)^{\beta-\alpha}\,, \qquad \text{for any } \beta > \alpha, \tag{50}$$

where

$$C(\alpha,\beta) = \frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha)}\,.$$

We fix $\Psi \in \mathscr{C}_c^\infty$ as a radial test function, such that:

- Ψ is supported in the unit ball B_1;
- $\Psi(x) = 1$, for any $x \in B_{1/2}$;
- $\Psi(x_1) \geq \Psi(x_2)$ if $|x_1| \leq |x_2|$.

For any $R \geq 1$, we denote $\Psi_R(x) \doteq \Psi(x/R)$. Let us fix

$$\beta > (\alpha + 2)p', \quad \text{and} \quad \ell > p', \tag{51}$$

where $p' \doteq p/(p-1)$ is the Hölder conjugate of p, and let

$$\Phi_R(t,x) \doteq \omega(t)^\beta\, \Psi_R(x)^\ell\,, \qquad \varphi \doteq D^\alpha_{t|-}\Phi_R(t,x)\,.$$

Then, φ is supported in $[0,T] \times B_R$, for any $T, R \geq 1$.

Now we use Young inequality to estimate

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^n} |u|\, |a\varphi_{tt} - \varphi_t - \Delta\varphi|\, dxdt\\ &\quad \leq \delta \int_0^\infty \int_{\mathbb{R}^n} |u|^p\, \Phi_R dx\, dt + C_\delta \int_0^T \int_{B_R} |a\varphi_{tt} - \varphi_t - \Delta\varphi|^{p'}\, \Phi_R^{-\frac{1}{p-1}}\, dx\, dt,\end{aligned}$$

for a sufficiently small $\delta > 0$. Thanks to

$$\begin{aligned}\partial_t^2 D^\alpha_{t|-}\omega(t)^\beta &= \frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha)}\, T^{-\alpha}\partial_t^2\omega(t)^{\beta-\alpha}\\ &= \frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha-2)}\, T^{-(\alpha+2)}\omega(t)^{\beta-\alpha-2},\end{aligned}$$

and similarly for $\partial_t D^\alpha_{t|-}\omega(t)^\beta$, due to

$$\text{meas}\,([0,T]) = T\,, \qquad \text{meas}\, = c_n R^n\,, \tag{52}$$

and using (51), we obtain

$$\int_0^T \int_{B_R} |a\varphi_{tt}-\varphi_t-\Delta\varphi|^{p'}\, \Phi_R^{-\frac{1}{p-1}}\, dx\, dt \leq C\left(T^{-(\alpha+1)p'+1}R^n + T^{-\alpha p'+1}\, R^{-2p'+n}\right).$$

We now consider the initial data. We notice that

$$\begin{aligned}\varphi(0,x) &= \Psi_R(x)\, \frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha)}\, T^{-\alpha},\\ \varphi_t(0,x) &= -\Psi_R(x)\, \frac{\Gamma(\beta+1)}{(\beta+2-\alpha)\Gamma(\beta-\alpha-1)}\, T^{-\alpha-1},\end{aligned}$$

Due to assumption (21) or (22), we get

$$\int_{\mathbb{R}^n} (au_1 + u_0)\,\varphi(0,x)\,dx \geq c\varepsilon\, T^{-\alpha}\, R^{n\left(1-\frac{1}{m}\right)}\,\log R,$$

for a sufficiently large $R \gg 1$. Moreover, if we are considering (4), by $u_0(x) \geq 0$, we obtain

$$a\int_{\mathbb{R}^n} u_0\,\varphi_t(0,x)\,dx \leq 0\,.$$

As a consequence, for a sufficiently small $\delta > 0$ and sufficiently large $R \gg 1$, we obtained the estimate

$$0 \leq \int_0^\infty \int_{\mathbb{R}^n} |u|^p\,\Phi_R dx\,dt \leq C\left(T^{-(\alpha+1)p'+1}R^n + T^{-\alpha p'+1}\,R^{-2p'+n}\right) \\ - c\varepsilon\, T^{-\alpha}\, R^{n\left(1-\frac{1}{m}\right)}\,\log R.$$

If we now set $R = \sqrt{T}$, so that

$$T^{-(\alpha+1)p'+1}\,R^n + T^{-\alpha p'+1}\,R^{-2p'+n} = 2T^{-(\alpha+1)p'+1+\frac{n}{2}},$$
$$T^{-\alpha}\,R^{n\left(1-\frac{1}{m}\right)}\,\log R = \frac{1}{2}\,T^{-\alpha+\frac{n}{2}\left(1-\frac{1}{m}\right)}\,\log T,$$

we got a contradiction, as $T \to \infty$, if

$$-(\alpha+1)p' + 1 + \frac{n}{2} < -\alpha + \frac{n}{2}\left(1-\frac{1}{m}\right). \tag{53}$$

We remark that the right-hand side of (53) is positive if, and only if,

$$1-\gamma = \alpha \leq \frac{n}{2}\left(1-\frac{1}{m}\right),$$

consistently with the result obtained for the existence of global (Sobolev or energy) solutions.

Condition (53) reads as

$$(p'-1)(\alpha+1) > \frac{n}{2m}, \quad \text{i.e.} \quad p-1 \geq \frac{2m(\alpha+1)}{n}\,.$$

Replacing $\alpha+1 = 2-\gamma$, we obtain that $p \geq p_{m,\gamma}$ is then a necessary condition for the existence of global weak solutions. This concludes the proof. □

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Blow-Up Results for Semi-Linear Structurally Damped σ-Evolution Equations

Tuan Anh Dao and Michael Reissig

Abstract We would like to prove a blow-up result for Sobolev solutions to the Cauchy problem for semi-linear structurally damped σ-evolution equations, where $\sigma \geq 1$ and $\delta \in [0, \sigma)$ are assumed to be any fractional numbers. To deal with the fractional Laplacian $(-\Delta)^\sigma$ and $(-\Delta)^\delta$ as well-known non-local operators, a modified test function method is applied to prove a blow-up result in the subcritical case and in the critical case as well.

Keywords σ-evolution equations · Structural damping · Critical exponent · Blow-up · Test functions

1 Introduction

The main goal of this paper is to discuss the critical exponent to the following Cauchy problem for semi-linear structurally damped σ-evolution models:

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\delta u_t = |u|^p, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \end{cases} \tag{1}$$

with some $\sigma \geq 1$, $\delta \in [0, \sigma)$ and a given real number $p > 1$. Here, critical exponent $p_{crit} = p_{crit}(n)$ means that for some range of admissible $p > p_{crit}$ there exists

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a global (in time) Sobolev solution for small initial data from a suitable function space. Moreover, one can find suitable small data such that there exists no global (in time) Sobolev solution if $1 < p \leq p_{crit}$. In other words, we have, in general, only local (in time) Sobolev solutions under this assumption for the exponent p.

For the local existence of Sobolev solutions to (1), we address the interested readers to Proposition 9.1 in the paper [2]. The proof of blow-up results in the present paper is based on a contradiction argument by using the test function method. The test function method is not influenced by higher regularity of the data. For this reason, we restrict ourselves to the critical exponent to (1) in the case, where the data are supposed to belong to the energy space. In this paper, we use the following notations.

- For given nonnegative f and g we write $f \lesssim g$ if there exists a constant $C > 0$ such that $f \leq Cg$. We write $f \approx g$ if $g \lesssim f \lesssim g$.
- We denote $\widehat{v} = \widehat{v}(\xi) := \mathfrak{F}_{x\to\xi}\big(v(x)\big)$ as the Fourier transform with respect to the spatial variables of a function $v = v(x)$.
- As usual, H^a with $a \geq 0$ stands for Bessel potential spaces based on L^2.
- We denote by $[b]$ the integer part of $b \in \mathbb{R}$. We put $\langle x\rangle := \sqrt{1+|x|^2}$.
- Moreover, we introduce the following two parameters:

$$\mathrm{k}^- := \min\{\sigma; 2\delta\} \quad \text{and} \quad \mathrm{k}^+ := \max\{\sigma; 2\delta\} \quad \text{if } \delta \in [0,\sigma).$$

In order to state our main result, we recall the global (in time) existence result of small data energy solutions to (1) in the following theorem.

Theorem 1 (Global Existence) *Let $m \in [1,2)$ and $n > m_0 k^-$ with $\frac{1}{m_0} = \frac{1}{m} - \frac{1}{2}$. We assume the conditions*

$$\begin{aligned} &\frac{2}{m} \leq p < \infty && \text{if } n \leq 2k^+,\\ &\frac{2}{m} \leq p \leq \frac{n}{n-2k^+} && \text{if } n \in \Big(2k^+, \frac{4k^+}{2-m}\Big]. \end{aligned}$$

Moreover, we suppose the following condition:

$$p > 1 + \frac{m(k^+ + \sigma)}{n - mk^-}. \tag{2}$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for any small data

$$(u_0, u_1) \in \big(L^m \cap H^{k^+}\big) \times \big(L^m \cap L^2\big)$$

satisfying the assumption $\|u_0\|_{L^m\cap H^{k^+}} + \|u_1\|_{L^m\cap L^2} \leq \varepsilon_0$, we have a uniquely determined global (in time) small data energy solution

$$u \in \mathscr{C}\Big([0,\infty), H^{k^+}\Big) \cap \mathscr{C}^1\Big([0,\infty), L^2\Big)$$

to (1). Moreover, the following estimates hold:

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2(k^+-\delta)}(\frac{1}{m}-\frac{1}{2})+\frac{k^-}{2(k^+-\delta)}}\big(\|u_0\|_{L^m\cap H^{k^+}}+\|u_1\|_{L^m\cap L^2}\big),$$

$$\big\||D|^{k^+}u(t,\cdot)\big\|_{L^2} \lesssim (1+t)^{-\frac{n}{2(k^+-\delta)}(\frac{1}{m}-\frac{1}{2})-\frac{k^+-k^-}{2(k^+-\delta)}}\big(\|u_0\|_{L^m\cap H^{k^+}}+\|u_1\|_{L^m\cap L^2}\big),$$

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2(k^+-\delta)}(\frac{1}{m}-\frac{1}{2})-\frac{\sigma-k^-}{k^+-\delta}}\big(\|u_0\|_{L^m\cap H^{k^+}}+\|u_1\|_{L^m\cap L^2}\big).$$

We are going to prove the following main result.

Theorem 2 (Blow-Up) *Let $\sigma \geq 1$, $\delta \in [0,\sigma)$ and $n > k^-$. We assume that we choose the initial data $u_0 = 0$ and $u_1 \in L^1$ satisfying the following relation:*

$$\int_{\mathbb{R}^n} u_1(x)dx > \epsilon_0, \tag{3}$$

where ϵ_0 is a suitable nonnegative constant. Moreover, we suppose the condition

$$p \in \Big(1, 1+\frac{2\sigma}{n-k^-}\Big]. \tag{4}$$

Then, there is no global (in time) Sobolev solution $u \in \mathscr{C}\big([0,\infty), L^2\big)$ to (1).

Remark 1 We want to underline that the lifespan T_ε of Sobolev solutions to given data $(0, \varepsilon u_1)$ for any small positive constant ε in the subcritical case of Theorem 2 can be estimated as follows:

$$T_\varepsilon \leq C\varepsilon^{-\frac{(2\sigma-k^-)(p-1)}{2\sigma-(n-k^-)(p-1)}} \quad \text{with } C > 0. \tag{5}$$

Nevertheless, catching the sharp lower bound of the lifespan T_ε to verify whether the obtained upper bound in (5) is optimal or not still remains open so far.

Remark 2 If we choose $m = 1$ in Theorem 1, then from Theorem 2 it is clear that the critical exponent $p_{crit} = p_{crit}(n)$ is given by

$$p_{crit}(n) = 1+\frac{2\sigma}{n-2\delta} \quad \text{if } \delta \in \Big[0, \frac{\sigma}{2}\Big] \text{ and } 4\delta < n \leq 4\sigma.$$

However, in the case $\delta \in (\frac{\sigma}{2}, \sigma)$ there appears a gap between the exponents given by $1+\frac{2\delta+\sigma}{n-\sigma}$ from Theorem 1 and $1+\frac{2\sigma}{n-\sigma}$ from Theorem 2 for $2\sigma < n \leq 8\delta$. Related to such a gap in the latter case, quite recently, the authors in [3] have succeeded to indicate the global (in time) existence of small data energy solutions to (1), with $\sigma > 1$, in low space dimensions for any $p > 1+\frac{2\sigma}{n-\sigma}$ by using suitable $L^{r_1} - L^{r_2}$ decay estimates, with $1 \leq r_1 \leq r_2 \leq \infty$, for solutions to the corresponding linear equation, after application of the stationary phase method. For this reason, at least in low space dimensions, we can claim that the critical exponent $p_{crit} = p_{crit}(n)$ in the case $\delta \in (\frac{\sigma}{2}, \sigma)$ with $\sigma > 1$ is

$$p_{crit}(n) = 1 + \frac{2\sigma}{n-\sigma}.$$

2 Preliminaries

In this section, we collect some preliminary knowledge needed in our proofs.

Definition 1 ([8, 10]) Let $s \in (0,1)$. Let X be a suitable set of functions defined on $\mathbb{R}^n$. Then, the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$ is a non-local operator given by

$$(-\Delta)^s: \; v \in X \to (-\Delta)^s v(x) := C_{n,s} \; p.v. \int_{\mathbb{R}^n} \frac{v(x)-v(y)}{|x-y|^{n+2s}} dy$$

as long as the right-hand side exists, where $p.v.$ stands for Cauchy's principal value, $C_{n,s} := \frac{4^s \Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}} \Gamma(-s)}$ is a normalization constant and Γ denotes the Gamma function.

Lemma 1 *Let $q > 0$. Then, the following estimate holds for any multi-index α satisfying $|\alpha| \geq 1$:*

$$\big|\partial_x^\alpha \langle x\rangle^{-q}\big| \lesssim \langle x\rangle^{-q-|\alpha|}.$$

Proof First, we recall the following formula of derivatives of composed functions for $|\alpha| \geq 1$:

$$\partial_x^\alpha h\big(f(x)\big) = \sum_{k=1}^{|\alpha|} h^{(k)}\big(f(x)\big) \left(\sum_{\substack{\gamma_1+\cdots+\gamma_k \leq \alpha \\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\, |\gamma_i|\geq 1}} \big(\partial_x^{\gamma_1} f(x)\big) \cdots \big(\partial_x^{\gamma_k} f(x)\big) \right),$$

where $h = h(z)$ and $h^{(k)}(z) = \frac{d^k h(z)}{dz^k}$. Applying this formula with $h(z) = z^{-\frac{q}{2}}$ and $f(x) = 1 + |x|^2$ we obtain

$$\begin{aligned} \big|\partial_x^\alpha \langle x\rangle^{-q}\big| &\leq \sum_{k=1}^{|\alpha|} (1+|x|^2)^{-\frac{q}{2}-k} \\ &\quad \times \left(\sum_{\substack{\gamma_1+\cdots+\gamma_k \leq \alpha \\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\, |\gamma_i|\geq 1}} \big|\partial_x^{\gamma_1}(1+|x|^2)\big| \cdots \big|\partial_x^{\gamma_k}(1+|x|^2)\big| \right) \end{aligned}$$

$$\le C_1 \sum_{k=1}^{|\alpha|} (1+|x|^2)^{-\frac{q}{2}-k}$$

$$\times \begin{cases} 1 & \text{if } 0 \le |x| \le 1, \\ \left(\displaystyle\sum_{\substack{\gamma_1+\cdots+\gamma_k \le \alpha \\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\ |\gamma_i| \ge 1}} |x|^{2-|\gamma_1|} \cdots |x|^{2-|\gamma_k|} \right) & \text{if } |x| \ge 1, \end{cases}$$

$$\le C_2 \sum_{k=1}^{|\alpha|} (1+|x|^2)^{-\frac{q}{2}-k} \begin{cases} 1 & \text{if} \quad 0 \le |x| \le 1, \\ |x|^{2k-|\alpha|} & \text{if} \quad |x| \ge 1, \end{cases}$$

$$\le \begin{cases} C_2 |\alpha| \langle x \rangle^{-q-2} & \text{if} \quad 0 \le |x| \le 1, \\ C_2 |\alpha| \langle x \rangle^{-q} |x|^{-|\alpha|} & \text{if} \quad |x| \ge 1, \end{cases}$$

where C_1 and C_2 are some suitable constants. This completes the proof. □

Lemma 2 *Let $m \in \mathbb{Z}$, $s \in (0,1)$ and $\gamma := m+s$. If $v \in H^{2\gamma}(\mathbb{R}^n)$, then it holds*

$$(-\Delta)^\gamma v(x) = (-\Delta)^m \big((-\Delta)^s v(x) \big) = (-\Delta)^s \big((-\Delta)^m v(x) \big).$$

One can find the proof of Lemma 2 in Remark 3.2 in [1].

Lemma 3 *Let $m \in \mathbb{Z}$, $s \in (0,1)$ and $\gamma := m+s$. Let $q > 0$. Then, the following estimates hold for all $x \in \mathbb{R}^n$:*

$$\big| (-\Delta)^\gamma \langle x \rangle^{-q} \big| \lesssim \begin{cases} \langle x \rangle^{-q-2\gamma} & \text{if} \quad 0 < q+2m < n, \\ \langle x \rangle^{-n-2s} \log(e+|x|) & \text{if} \quad q+2m = n, \\ \langle x \rangle^{-n-2s} & \text{if} \quad q+2m > n. \end{cases} \tag{6}$$

Proof We follow ideas from the proof of Lemma 1 in [7] devoting to the case $m = 0$ and $s = \frac{1}{2}$, that is, the case $\gamma = \frac{1}{2}$ is generalized to any fractional number $\gamma > 0$. To do this, for any $s \in (0,1)$ we shall divide the proof into two cases: $m = 0$ and $m \ge 1$.
Let us consider the first case $m = 0$. Denoting by $\psi = \psi(x) := \langle x \rangle^{-q}$ we write $(-\Delta)^s \langle x \rangle^{-q} = (-\Delta)^s(\psi)(x)$. According to Definition 1 of fractional Laplacian as a singular integral operator, we have

$$(-\Delta)^s(\psi)(x) := C_{n,\delta}\ p.v. \int_{\mathbb{R}^n} \frac{\psi(x)-\psi(y)}{|x-y|^{n+2s}} dy.$$

A standard change of variables leads to

$$(-\Delta)^s(\psi)(x) = -\frac{C_{n,s}}{2}\ p.v. \int_{\mathbb{R}^n} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy$$

$$= -\frac{C_{n,s}}{2} \lim_{\varepsilon\to 0+} \int_{\varepsilon\le|y|\le 1} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy - \frac{C_{n,s}}{2} \int_{|y|\ge 1} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy.$$

To deal with the first integral, after using a second order Taylor expansion for ψ we arrive at

$$\frac{|\psi(x+y)+\psi(x-y)-2\psi(x)|}{|y|^{n+2s}} \lesssim \frac{\|\partial_x^2\psi\|_{L^\infty}}{|y|^{n+2s-2}}.$$

Thanks to the above estimate and $s \in (0, 1)$, we may remove the principal value of the integral at the origin to conclude

$$(-\Delta)^s(\psi)(x) = -\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy.$$

To prove the desired estimates, we shall divide our considerations into two cases. In the first subcase $\{x : |x| \le 1\}$, we can proceed as follows:

$$\begin{aligned}\big|(-\Delta)^s(\psi)(x)\big| &\lesssim \int_{|y|\le 1} \frac{|\psi(x+y)+\psi(x-y)-2\psi(x)|}{|y|^{n+2s}}dy \\ &\quad + \int_{|y|\ge 1} \frac{|\psi(x+y)+\psi(x-y)-2\psi(x)|}{|y|^{n+2s}}dy \\ &\lesssim \|\partial_x^2\psi\|_{L^\infty} \int_{|y|\le 1} \frac{1}{|y|^{n+2s-2}}dy + \|\psi\|_{L^\infty} \int_{|y|\ge 1} \frac{1}{|y|^{n+2s}}dy.\end{aligned}$$

Due to the boundedness of the above two integrals, it follows immediately

$$\big|(-\Delta)^s(\psi)(x)\big| \lesssim 1 \quad \text{for } |x| \le 1. \tag{7}$$

In order to deal with the second subcase $\{x : |x| \geq 1\}$, we can re-write

$$(-\Delta)^s(\psi)(x) = -\frac{C_{n,s}}{2}\int_{|y|\geq 2|x|}\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy$$
$$-\frac{C_{n,s}}{2}\int_{\frac{1}{2}|x|\leq|y|\leq 2|x|}\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy$$
$$-\frac{C_{n,s}}{2}\int_{|y|\leq\frac{1}{2}|x|}\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy. \quad (8)$$

For the first integral, we notice that the relations $|x+y| \geq |y| - |x| \geq |x|$ and $|x-y| \geq |y| - |x| \geq |x|$ hold for $|y| \geq 2|x|$. Since ψ is a decreasing function, we obtain the following estimate:

$$\Big|\int_{|y|\geq 2|x|}\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy\Big|$$
$$\leq 4|\psi(x)|\int_{|y|\geq 2|x|}\frac{1}{|y|^{n+2s}}dy \lesssim \langle x\rangle^{-q}\int_{|y|\geq 2|x|}\frac{1}{|y|^{1+2s}}d|y|$$
$$\lesssim \langle x\rangle^{-q}|x|^{-2s} \lesssim \langle x\rangle^{-q-2s} \qquad \big(\text{due to } |x|\approx\langle x\rangle \text{ for } |x|\geq 1\big). \quad (9)$$

It is clear that $|y| \approx |x|$ in the second integral domain. Moreover, it follows

$$\Big\{y : \frac{1}{2}|x| \leq |y| \leq 2|x|\Big\} \subset \big\{y : |x+y| \leq 3|x|\big\}, \quad (10)$$
$$\Big\{y : \frac{1}{2}|x| \leq |y| \leq 2|x|\Big\} \subset \big\{y : |x-y| \leq 3|x|\big\}. \quad (11)$$

For this reason, we arrive at

$$\Big|\int_{\frac{1}{2}|x|\leq|y|\leq 2|x|}\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy\Big|$$
$$\lesssim |x|^{-n-2s}\Big(\int_{|x+y|\leq 3|x|}\psi(x+y)dy + \int_{|x-y|\leq 3|x|}\psi(x-y)dy$$
$$+\psi(x)\int_{\frac{1}{2}|x|\leq|y|\leq 2|x|}1dy\Big)$$
$$\lesssim |x|^{-n-2s}\Big(\int_{|x+y|\leq 3|x|}\psi(x+y)dy + \langle x\rangle^{-q}|x|^n\Big), \quad (12)$$

where we used the relation

$$\int_{|x+y|\leq 3|x|}\psi(x+y)dy=\int_{|x-y|\leq 3|x|}\psi(x-y)dy.$$

By the change of variables $r=|x+y|$, we apply the inequality $1+r^2\geq \frac{(1+r)^2}{2}$ to get

$$\begin{aligned}\int_{|x+y|\leq 3|x|}\psi(x+y)dy &\lesssim \int_{r\leq 3|x|}(1+r^2)^{-\frac{q}{2}}\,r^{n-1}dr\lesssim \int_{r\leq 3|x|}(1+r)^{n-q-1}dr\\ &\lesssim \begin{cases}(1+3|x|)^{n-q} & \text{if}\quad 0<q<n,\\ \log(e+3|x|) & \text{if}\quad q=n,\\ 1 & \text{if}\quad q>n.\end{cases}\end{aligned}\tag{13}$$

By $|x|\approx \langle x\rangle$ for $|x|\geq 1$, combining (12) and (13) leads to

$$\begin{aligned}&\Big|\int_{\frac{1}{2}|x|\leq|y|\leq 2|x|}\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy\Big|\\ &\quad\lesssim \begin{cases}\langle x\rangle^{-q-2s} & \text{if}\quad 0<q<n,\\ \langle x\rangle^{-n-2s}\log(e+3|x|) & \text{if}\quad q=n,\\ \langle x\rangle^{-n-2s} & \text{if}\quad q>n.\end{cases}\end{aligned}\tag{14}$$

For the third integral in (8), using again the second order Taylor expansion for ψ we obtain

$$\begin{aligned}&\Big|\int_{|y|\leq\frac{1}{2}|x|}\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy\Big|\\ &\quad\leq \int_{|y|\leq\frac{1}{2}|x|}\frac{|\psi(x+y)+\psi(x-y)-2\psi(x)|}{|y|^{n+2s}}dy\\ &\quad\lesssim \int_{|y|\leq\frac{1}{2}|x|}\max_{\theta\in[0,1]}\big|\partial_x^2\psi(x\pm\theta y)\big|\frac{1}{|y|^{n+2s-2}}dy\\ &\quad\lesssim \int_{|y|\leq\frac{1}{2}|x|}\max_{\theta\in[0,1]}\langle x\pm\theta y\rangle^{-q-2}\frac{1}{|y|^{n+2s-2}}dy\\ &\quad\lesssim \langle x\rangle^{-q-2}\int_{|y|\leq\frac{1}{2}|x|}|y|^{1-2s}d|y|\lesssim \langle x\rangle^{-q-2s}.\end{aligned}\tag{15}$$

Here we used the relation $|x \pm \theta y| \geq |x| - \theta|y| \geq |x| - \frac{1}{2}|x| = \frac{1}{2}|x|$. From (8), (9), (14), and (15) we arrive at the following estimates for $|x| \geq 1$:

$$\big|(-\Delta)^s(\psi)(x)\big| \lesssim \begin{cases} \langle x\rangle^{-q-2s} & \text{if} \quad 0 < q < n, \\ \langle x\rangle^{-n-2s} \log(e + 3|x|) & \text{if} \quad q = n, \\ \langle x\rangle^{-n-2s} & \text{if} \quad q > n. \end{cases} \tag{16}$$

Finally, combining (7) and (16) we may conclude all desired estimates for $m = 0$. Next let us turn to the second case $m \geq 1$. First, a straight-forward calculation gives the following relation:

$$-\Delta\langle x\rangle^{-r} = r\Big((n - r - 2)\langle x\rangle^{-r-2} + (r + 2)\langle x\rangle^{-r-4}\Big) \quad \text{for any } r > 0. \tag{17}$$

By induction argument, carrying out m steps of (17) we obtain the following formula for any $m \geq 1$:

$$\begin{aligned}(-\Delta)^m\langle x\rangle^{-q} = (-1)^m \prod_{j=0}^{m-1}(q + 2j)\Big(&\prod_{j=1}^{m}(-n + q + 2j)\langle x\rangle^{-q-2m} \\ &- C_m^1 \prod_{j=2}^{m}(-n + q + 2j)(q + 2m)\langle x\rangle^{-q-2m-2} \\ &+ C_m^2 \prod_{j=3}^{m}(-n + q + 2j)(q + 2m)(q + 2m + 2)\langle x\rangle^{-q-2m-4} \\ &+ \cdots + (-1)^m \prod_{j=0}^{m-1}(q + 2m + 2j)\langle x\rangle^{-q-4m}\Big).\end{aligned} \tag{18}$$

Then, thanks to Lemma 2, we derive

$$\begin{aligned}(-\Delta)^\gamma\langle x\rangle^{-q} &= (-\Delta)^s\big((-\Delta)^m\langle x\rangle^{-q}\big) \\ &= (-1)^m \prod_{j=0}^{m-1}(q + 2j)\Big(\prod_{j=1}^{m}(-n + q + 2j)\,(-\Delta)^s\langle x\rangle^{-q-2m} \\ &\quad - C_m^1 \prod_{j=2}^{m}(-n + q + 2j)(q + 2m)\,(-\Delta)^s\langle x\rangle^{-q-2m-2}\end{aligned}$$

$$+ C_m^2 \prod_{j=3}^{m}(-n+q+2j)(q+2m)(q+2m+2)\,(-\Delta)^s\langle x\rangle^{-q-2m-4}$$

$$+\cdots+(-1)^m \prod_{j=0}^{m-1}(q+2m+2j)\,(-\Delta)^s\langle x\rangle^{-q-4m}\Big). \tag{19}$$

For this reason, in order to conclude the desired estimates, we only indicate the following estimates for $k = 0, \cdots, m$:

$$\big|(-\Delta)^s\langle x\rangle^{-q-2(m+k)}\big| \lesssim \begin{cases} \langle x\rangle^{-q-2\gamma} & \text{if } 0 < q+2m < n,\\ \langle x\rangle^{-n-2s}\log(e+|x|) & \text{if } q+2m = n,\\ \langle x\rangle^{-n-2s} & \text{if } q+2m > n. \end{cases} \tag{20}$$

Indeed, substituting q by $q+2(m+k)$ with $k = 0, \cdots, m$ and $\gamma = s$ into (6) leads to

$$\big|(-\Delta)^s\langle x\rangle^{-q-2(m+k)}\big| \lesssim \begin{cases} \langle x\rangle^{-q-2\gamma} & \text{if } 0 < q+2(m+k) < n,\\ \langle x\rangle^{-n-2s}\log(e+|x|) & \text{if } q+2(m+k) = n,\\ \langle x\rangle^{-n-2s} & \text{if } q+2(m+k) > n. \end{cases}$$

From these estimates, it follows immediately (20) to conclude (6) for any $m \geq 1$. Summarizing, the proof of Lemma 3 is completed. □

Lemma 4 *Let $s \in (0,1)$. Let ψ be a smooth function satisfying $\partial_x^2\psi \in L^\infty$. For any $R > 0$, let ψ_R be a function defined by*

$$\psi_R(x) := \psi\big(R^{-1}x\big)$$

for all $x \in \mathbb{R}^n$. Then, $(-\Delta)^s(\psi_R)$ satisfies the following scaling properties for all $x \in \mathbb{R}^n$:

$$(-\Delta)^s(\psi_R)(x) = R^{-2s}\big((-\Delta)^s\psi\big)\big(R^{-1}x\big).$$

Proof Thanks to the assumption $\partial_x^2\psi \in L^\infty$, following the proof of Lemma 3 we may remove the principal value of the integral at the origin to conclude

$$\begin{aligned}(-\Delta)^s(\psi_R)(x) &= -\frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\frac{\psi_R(x+y)+\psi_R(x-y)-2\psi_R(x)}{|y|^{n+2s}}dy\\ &= -\frac{C_{n,s}}{2R^{2s}}\int_{\mathbb{R}^n}\frac{\psi\big(R^{-1}x+R^{-1}y\big)+\psi\big(R^{-1}x-R^{-1}y\big)-2\psi\big(R^{-1}x\big)}{|R^{-1}y|^{n+2s}}d(R^{-1}y)\\ &= R^{-2s}\big((-\Delta)^s\psi\big)\big(R^{-1}x\big).\end{aligned}$$

This completes the proof. □

Lemma 5 (One Mapping Property in the Scale of Fractional Spaces $\{H^s\}_{s\in\mathbb{R}}$) *Let $\gamma,\ s\in\mathbb{R}$. Then, the fractional Laplacian*

$$(-\Delta)^\gamma:\ f\to(-\Delta)^\gamma f=\big((-\Delta)^\gamma f\big)(x):=\mathfrak{F}^{-1}\big(|\xi|^{2\gamma}\widehat{f}(\xi)\big)(x)$$

maps isomorphically the space H^s onto $H^{s-2\gamma}$.

This result can be found in Section 2.3.8 in [12].

Lemma 6 *Let $f=f(x)\in H^s$ and $g=g(x)\in H^{-s}$ with $s\in\mathbb{R}$. Then, the following estimate holds:*

$$\Big|\int_{\mathbb{R}^n}f(x)\,g(x)dx\Big|\le\|f\|_{H^s}\,\|g\|_{H^{-s}}.$$

The proof of Lemma 6 can be found in Theorem 16 in [6].

Lemma 7 *Let $s\in\mathbb{R}$. Let $v_1=v_1(x)\in H^s$ and $v_2=v_2(x)\in H^{-s}$. Then, the following relation holds:*

$$\int_{\mathbb{R}^n}v_1(x)\,v_2(x)dx=\int_{\mathbb{R}^n}\widehat{v}_1(\xi)\,\widehat{v}_2(\xi)d\xi.$$

Proof We present the proof from Theorem 16 in [6] to make the paper self-contained. Since the space $\mathcal{S}$ is dense in H^s and H^{-s}, there exist sequences $\{v_{1,k}\}_k$ and $\{v_{2,k}\}_k$ with $v_{1,k}=v_{1,k}(x)\in\mathcal{S}$ and $v_{2,k}=v_{2,k}(x)\in\mathcal{S}$ such that

$$\|v_{1,k}-v_1\|_{H^s}\to 0\quad\text{and}\quad\|v_{2,k}-v_2\|_{H^{-s}}\to 0\quad\text{for }k\to\infty.$$

On the one hand, for $k\to\infty$ we have the relations

$$\widehat{V}_{1,k}(\xi):=(1+|\xi|^2)^{\frac{s}{2}}\,\widehat{v}_{1,k}(\xi)\to\widehat{V}_1(\xi):=(1+|\xi|^2)^{\frac{s}{2}}\,\widehat{v}_1(\xi)\quad\text{in }L^2,$$
$$\widehat{V}_{2,k}(\xi):=(1+|\xi|^2)^{-\frac{s}{2}}\,\widehat{v}_{2,k}(\xi)\to\widehat{V}_2(\xi):=(1+|\xi|^2)^{-\frac{s}{2}}\,\widehat{v}_2(\xi)\quad\text{in }L^2.$$

On the other hand, by Parseval–Plancherel formula we arrive at

$$\begin{aligned}\int_{\mathbb{R}^n}v_{1,k}(x)\,v_{2,k}(x)\,dx&=\big(v_{1,k},\ v_{2,k}\big)_{L^2}=\big(\widehat{v}_{1,k},\ \widehat{v}_{2,k}\big)_{L^2}=\int_{\mathbb{R}^n}\widehat{v}_{1,k}(\xi)\,\widehat{v}_{2,k}(\xi)\,d\xi\\&=\int_{\mathbb{R}^n}(1+|\xi|^2)^{\frac{s}{2}}\,\widehat{v}_{1,k}(\xi)\,(1+|\xi|^2)^{-\frac{s}{2}}\,\widehat{v}_{2,k}(\xi)\,d\xi\\&=\int_{\mathbb{R}^n}\widehat{V}_{1,k}(\xi)\,\widehat{V}_{2,k}(\xi)\,d\xi,\end{aligned}\tag{21}$$

where $(\cdot,\cdot)_{L^2}$ stands for the scalar product in L^2. Moreover, applying Lemma 6 we may estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n}\big(v_{1,k}(x)\,v_{2,k}(x)-v_1(x)\,v_2(x)\big)\,dx\Big|\\
&\quad\leq\Big|\int_{\mathbb{R}^n}\big(v_{1,k}(x)-v_1(x)\big)\,v_{2,k}(x)dx\Big|+\Big|\int_{\mathbb{R}^n}v_1(x)\,\big(v_{2,k}(x)-v_2(x)\big)\,dx\Big|\\
&\quad\leq\|v_{1,k}-v_1\|_{H^s}\,\|v_{2,k}\|_{H^{-s}}+\|v_1\|_{H^s}\,\|v_{2,k}-v_2\|_{H^{-s}}\to 0\quad\text{as } k\to\infty.
\end{aligned}$$

This is equivalent to

$$\int_{\mathbb{R}^n}v_{1,k}(x)\,v_{2,k}(x)\,dx\to\int_{\mathbb{R}^n}v_1(x)\,v_2(x)\,dx\quad\text{as } k\to\infty. \tag{22}$$

In the same way we also derive

$$\int_{\mathbb{R}^n}\widehat{V}_{1,k}(\xi)\,\widehat{V}_{2,k}(\xi)\,d\xi\to\int_{\mathbb{R}^n}\widehat{V}_1(\xi)\,\widehat{V}_2(\xi)\,d\xi\quad\text{as } k\to\infty. \tag{23}$$

Summarizing from (21) to (23) we may conclude

$$\int_{\mathbb{R}^n}v_1(x)\,v_2(x)\,dx=\int_{\mathbb{R}^n}\widehat{V}_1(\xi)\,\widehat{V}_2(\xi)\,d\xi=\int_{\mathbb{R}^n}\widehat{v}_1(\xi)\,\widehat{v}_2(\xi)\,d\xi.$$

Therefore, the proof of Lemma 7 is completed. □

3 Proof of Theorem 2

We divide the proof of Theorem 2 into several cases.

3.1 The Case that Both Parameters σ and δ Are Integers

Proof The proof of this case can be found in the paper [2]. □

3.2 The Case that the Parameter σ Is Integer and the Parameter δ Is Fractional from $(0, 1)$

Proof The first case is devoted to the subcritical case $p < 1 + \frac{2\sigma}{n-\mathrm{k}^-}$. First, we introduce the function $\varphi = \varphi(|x|) := \langle x\rangle^{-n-2\delta}$ and the function $\eta = \eta(t)$ having the following properties:

$$1. \quad \eta \in \mathscr{C}_0^\infty([0,\infty)) \text{ and } \eta(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{1}{2}, \\ \text{decreasing} & \text{for } \frac{1}{2} \le t \le 1, \\ 0 & \text{for } t \ge 1, \end{cases}$$

$$2. \quad \eta^{-\frac{p'}{p}}(t)\big(|\eta'(t)|^{p'} + |\eta''(t)|^{p'}\big) \le C \quad \text{for any } t \in \Big[\frac{1}{2}, 1\Big], \tag{24}$$

where p' is the conjugate of $p > 1$. Let R be a large parameter in $[0, \infty)$. We define the following test function:

$$\varphi_R(t,x) := \eta_R(t)\varphi_R(x),$$

where $\eta_R(t) := \eta\big(R^{-\alpha}t\big)$ and $\varphi_R(x) := \varphi\big(R^{-1}x\big)$ with a fixed parameter $\alpha := 2\sigma - \mathrm{k}^-$. We define the functionals

$$I_R := \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p \varphi_R(t,x)\,dxdt = \int_0^{R^\alpha} \int_{\mathbb{R}^n} |u(t,x)|^p \varphi_R(t,x)\,dxdt$$

and

$$I_{R,t} := \int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} |u(t,x)|^p \varphi_R(t,x)\,dxdt.$$

Let us assume that $u = u(t,x)$ is a global (in time) Sobolev solution from $\mathscr{C}\big([0,\infty), L^2\big)$ to (1). After multiplying the Eq. (1) by $\varphi_R = \varphi_R(t,x)$, we carry out partial integration to derive

$$\begin{aligned} 0 \le I_R = &-\int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\,dx + \int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} u(t,x)\partial_t^2\eta_R(t)\varphi_R(x)\,dxdt \\ &+ \int_0^\infty \int_{\mathbb{R}^n} \eta_R(t)\varphi_R(x)\,(-\Delta)^\sigma u(t,x)\,dxdt \\ &- \int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \partial_t\eta_R(t)\varphi_R(x)\,(-\Delta)^\delta u(t,x)\,dxdt \\ &=: -\int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\,dx + J_1 + J_2 - J_3. \end{aligned} \tag{25}$$

Applying Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$ we may estimate as follows:

$$
\begin{aligned}
|J_1| &\le \int_{\frac{R^\alpha}{2}}^{R^\alpha}\int_{\mathbb{R}^n} |u(t,x)|\,\big|\partial_t^2\eta_R(t)\big|\varphi_R(x)\,dxdt\\
&\lesssim \Big(\int_{\frac{R^\alpha}{2}}^{R^\alpha}\int_{\mathbb{R}^n}\Big|u(t,x)\varphi_R^{\frac{1}{p}}(t,x)\Big|^p\,dxdt\Big)^{\frac{1}{p}}\\
&\qquad\times\Big(\int_{\frac{R^\alpha}{2}}^{R^\alpha}\int_{\mathbb{R}^n}\Big|\varphi_R^{-\frac{1}{p}}(t,x)\partial_t^2\eta_R(t)\varphi_R(x)\Big|^{p'}\,dxdt\Big)^{\frac{1}{p'}}\\
&\lesssim I_{R,t}^{\frac{1}{p}}\Big(\int_{\frac{R^\alpha}{2}}^{R^\alpha}\int_{\mathbb{R}^n}\eta_R^{-\frac{p'}{p}}(t)\big|\partial_t^2\eta_R(t)\big|^{p'}\varphi_R(x)\,dxdt\Big)^{\frac{1}{p'}}.
\end{aligned}
$$

By the change of variables $\tilde{t}:=R^{-\alpha}t$ and $\tilde{x}:=R^{-1}x$, a straight-forward calculation gives

$$
|J_1|\lesssim I_{R,t}^{\frac{1}{p}}\,R^{-2\alpha+\frac{n+\alpha}{p'}}\Big(\int_{\mathbb{R}^n}\langle\tilde{x}\rangle^{-n-2\delta}\,d\tilde{x}\Big)^{\frac{1}{p'}}. \tag{26}
$$

Here we used $\partial_t^2\eta_R(t)=R^{-2\alpha}\eta''(\tilde{t})$ and the assumption (24). Now let us turn to estimate J_2 and J_3. First, by using $\varphi_R\in H^{2\sigma}$ and $u\in\mathscr{C}\big([0,\infty),L^2\big)$ we apply Lemma 7 to conclude the following relations:

$$
\begin{aligned}
\int_{\mathbb{R}^n}\varphi_R(x)\,(-\Delta)^\sigma u(t,x)\,dx &= \int_{\mathbb{R}^n}|\xi|^{2\sigma}\widehat{\varphi}_R(\xi)\,\widehat{u}(t,\xi)\,d\xi\\
&= \int_{\mathbb{R}^n}u(t,x)\,(-\Delta)^\sigma\varphi_R(x)\,dx,\\
\int_{\mathbb{R}^n}\varphi_R(x)\,(-\Delta)^\delta u(t,x)\,dx &= \int_{\mathbb{R}^n}|\xi|^{2\delta}\widehat{\varphi}_R(\xi)\,\widehat{u}(t,\xi)\,d\xi\\
&= \int_{\mathbb{R}^n}u(t,x)\,(-\Delta)^\delta\varphi_R(x)\,dx.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
J_2 &= \int_0^\infty\int_{\mathbb{R}^n}\eta_R(t)\varphi_R(x)\,(-\Delta)^\sigma u(t,x)\,dxdt\\
&= \int_0^\infty\int_{\mathbb{R}^n}\eta_R(t)u(t,x)\,(-\Delta)^\sigma\varphi_R(x)\,dxdt,
\end{aligned}
$$

$$\begin{aligned} J_3 &= \int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \partial_t \eta_R(t) \varphi_R(x)\, (-\Delta)^\delta u(t,x)\, dxdt \\ &= \int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \partial_t \eta_R(t) u(t,x)\, (-\Delta)^\delta \varphi_R(x)\, dxdt. \end{aligned}$$

Applying Hölder's inequality again as we estimated J_1 leads to

$$\begin{aligned} |J_2| &\le I_R^{\frac{1}{p}} \Big(\int_0^{R^\alpha} \int_{\mathbb{R}^n} \eta_R(t) \varphi_R^{-\frac{p'}{p}}(x)\, \big|(-\Delta)^\sigma \varphi_R(x)\big|^{p'}\, dxdt \Big)^{\frac{1}{p'}}, \\ |J_3| &\le I_{R,t}^{\frac{1}{p}} \Big(\int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \eta_R^{-\frac{p'}{p}}(t) \big|\partial_t \eta_R(t)\big|^{p'} \varphi_R^{\frac{-p'}{p}}(x)\, \big|(-\Delta)^\delta \varphi_R(x)\big|^{p'}\, dxdt \Big)^{\frac{1}{p'}}. \end{aligned}$$

In order to control the above two integrals, the key tools rely on results from Lemmas 1, 3 and 4. Namely, at first carrying out the change of variables $\tilde{t} := R^{-\alpha} t$ and $\tilde{x} := R^{-1} x$ we arrive at

$$\begin{aligned} |J_2| &\lesssim I_R^{\frac{1}{p}}\, R^{-2\sigma + \frac{n+\alpha}{p'}} \Big(\int_0^1 \int_{\mathbb{R}^n} \eta(\tilde{t}) \varphi^{-\frac{p'}{p}}(\tilde{x})\, \big|(-\Delta)^\sigma(\varphi)(\tilde{x})\big|^{p'}\, d\tilde{x} d\tilde{t} \Big)^{\frac{1}{p'}} \\ &\lesssim I_R^{\frac{1}{p}}\, R^{-2\sigma + \frac{n+\alpha}{p'}} \Big(\int_{\mathbb{R}^n} \varphi^{-\frac{p'}{p}}(\tilde{x})\, \big|(-\Delta)^\sigma(\varphi)(\tilde{x})\big|^{p'}\, d\tilde{x} \Big)^{\frac{1}{p'}}, \end{aligned}$$

where we note (σ is an integer) that $(-\Delta)^\sigma \varphi_R(x) = R^{-2\sigma} (-\Delta)^\sigma \varphi(\tilde{x})$. Using Lemma 1 implies the following estimate:

$$|J_2| \lesssim I_R^{\frac{1}{p}}\, R^{-2\sigma + \frac{n+\alpha}{p'}} \Big(\int_{\mathbb{R}^n} \langle \tilde{x} \rangle^{-n-2\delta-2\sigma p'}\, d\tilde{x} \Big)^{\frac{1}{p'}}. \tag{27}$$

Next carrying out again the change of variables $\tilde{t} := R^{-\alpha} t$ and $\tilde{x} := R^{-1} x$ and employing Lemma 4 we can proceed J_3 as follows:

$$\begin{aligned} |J_3| &\lesssim I_{R,t}^{\frac{1}{p}}\, R^{-2\delta-\alpha+\frac{n+\alpha}{p'}} \\ &\qquad \times \Big(\int_{\frac{1}{2}}^1 \int_{\mathbb{R}^n} \eta^{-\frac{p'}{p}}(\tilde{t}) \big|\eta'(\tilde{t})\big|^{p'} \varphi^{-\frac{p'}{p}}(\tilde{x})\, \big|(-\Delta)^\delta(\varphi)(\tilde{x})\big|^{p'}\, d\tilde{x} d\tilde{t} \Big)^{\frac{1}{p'}} \\ &\lesssim I_{R,t}^{\frac{1}{p}}\, R^{-2\delta-\alpha+\frac{n+\alpha}{p'}} \Big(\int_{\mathbb{R}^n} \varphi^{-\frac{p'}{p}}(\tilde{x})\, \big|(-\Delta)^\delta(\varphi)(\tilde{x})\big|^{p'}\, d\tilde{x} \Big)^{\frac{1}{p'}}. \end{aligned}$$

Here we used $\partial_t \eta_R(t) = R^{-\alpha} \eta'(\tilde{t})$ and the assumption (24). To deal with the last integral, we apply Lemma 3 with $q = n + 2\delta$ and $\gamma = \delta$, that is, $m = 0$ and $s = \delta$

to get

$$|J_3| \lesssim I_{R,t}^{\frac{1}{p}} R^{-2\delta-\alpha+\frac{n+\alpha}{p'}} \Big(\int_{\mathbb{R}^n} \langle \tilde{x} \rangle^{-n-2\delta}\, d\tilde{x} \Big)^{\frac{1}{p'}}. \tag{28}$$

Because of the assumption (3), there exists a sufficiently large constant $R_0 > 0$ such that it holds

$$\int_{\mathbb{R}^n} u_1(x) \varphi_R(x)\, dx > 0 \tag{29}$$

for all $R > R_0$. Combining the estimates from (25) to (29) we may arrive at

$$0 < \int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\, dx \lesssim I_{R,t}^{\frac{1}{p}} \big(R^{-2\alpha+\frac{n+\alpha}{p'}} + R^{-\alpha-2\delta+\frac{n+\alpha}{p'}} \big)$$
$$+ I_R^{\frac{1}{p}} R^{-2\sigma+\frac{n+\alpha}{p'}} - I_R \lesssim I_R^{\frac{1}{p}} R^{-2\sigma+\frac{n+\alpha}{p'}} - I_R \tag{30}$$

for all $R > R_0$. Moreover, applying the inequality

$$A\, y^{\gamma} - y \leq A^{\frac{1}{1-\gamma}} \quad \text{for any } A > 0,\ y \geq 0 \text{ and } 0 < \gamma < 1$$

leads to

$$0 < \int_{\mathbb{R}^n} u_1(x)\varphi_R(x) dx \lesssim R^{-2\sigma p' + n + \alpha} \tag{31}$$

for all $R > R_0$. It is clear that the assumption (4) is equivalent to $-2\sigma p' + n + \alpha \leq 0$. For this reason, in the subcritical case, that is, $-2\sigma p' + n + \alpha < 0$ letting $R \to \infty$ in (31) we obtain

$$\int_{\mathbb{R}^n} u_1(x)\, dx = 0.$$

This is a contradiction to the assumption (3).
Let us turn to the critical case $p = 1 + \frac{2\sigma}{n-\mathrm{k}^-}$. It follows immediately $-2\sigma + \frac{n+\alpha}{p'} = 0$. Then, repeating some arguments as we did in the subcritical case we may conclude the following estimate:

$$0 < C_0 := \int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\, dx \leq C_1 I_R^{\frac{1}{p}} - I_R,$$

where $C_1 := \Big(\int_{\mathbb{R}^n} \langle \tilde{x} \rangle^{-n-2\delta}\, d\tilde{x} \Big)^{\frac{1}{p'}}$, that is,

$$C_0 + I_R \leq C_1 I_R^{\frac{1}{p}}. \tag{32}$$

From (32) it is obvious that $I_R \leq C_1 I_R^{\frac{1}{p}}$ and $C_0 \leq C_1 I_R^{\frac{1}{p}}$. Hence, we obtain

$$I_R \leq C_1^{p'} \tag{33}$$

and

$$I_R \geq \frac{C_0^p}{C_1^p}, \tag{34}$$

respectively. By substituting (34) into the left-hand side of (32) and calculating straightforwardly, we get

$$I_R \geq \frac{C_0^{p^2}}{C_1^{p+p^2}}.$$

For any integer $j \geq 1$, an iteration argument leads to

$$I_R \geq \frac{C_0^{p^j}}{C_1^{p+p^2+\cdots+p^j}} = \frac{C_0^{p^j}}{C_1^{\frac{p^{j+1}-p}{p-1}}} = C_1^{\frac{p}{p-1}} \Big(\frac{C_0}{C_1^{\frac{p}{p-1}}} \Big)^{p^j}. \tag{35}$$

Now we choose the constant

$$\epsilon_0 = \int_{\mathbb{R}^n} \langle \tilde{x} \rangle^{-n-2\delta}\, d\tilde{x}$$

in the assumption (3). Then, there exists a sufficiently large constant $R_1 > 0$ so that

$$\int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\, dx > \epsilon_0$$

for all $R > R_1$. This is equivalent to

$$C_0 > C_1^{p'} = C_1^{\frac{p}{p-1}}, \quad \text{that is,} \quad \frac{C_0}{C_1^{\frac{p}{p-1}}} > 1.$$

Therefore, letting $j \to \infty$ in (35) we derive $I_R \to \infty$, which is a contradiction to (33). Summarizing, the proof is completed. □

Let us now consider the case of subcritical exponent to explain the estimate for lifespan T_ε of solutions in Remark 1. We assume that $u = u(t, x)$ is a local (in time) Sobolev solution to (1) in $[0, T) \times \mathbb{R}^n$. In order to prove the lifespan estimate, we replace the initial data $(0, u_1)$ by $(0, \varepsilon u_1)$ with a small constant $\varepsilon > 0$, where $u_1 \in L^1$ satisfies the assumption (3). Hence, there exists a sufficiently large constant $R_2 > 0$ so that we have

$$\int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\, dx \geq c > 0$$

for any $R > R_2$. Repeating the steps in the above proofs we arrive at the following estimate:

$$\varepsilon \leq C\, R^{-2\sigma p'+n+\alpha} \leq C\, T^{-\frac{2\sigma p'-n-\alpha}{\alpha}}$$

with $R = T^{\frac{1}{\alpha}}$. Finally, letting $T \to T_\varepsilon^-$ we may conclude (5).

Remark 3 We want to underline that in the special case $\sigma = 1$ and $\delta = \frac{1}{2}$ the authors in [4] have investigated the critical exponent $p_{crit} = p_{crit}(n) = 1 + \frac{2}{n-1}$. If we plug $\sigma = 1$ and $\delta = \frac{1}{2}$ into the statements of Theorem 2, then the obtained results for the critical exponent p_{crit} coincide.

3.3 The Case that the Parameter σ Is Integer and the Parameter δ Is Fractional from $(1, \sigma)$

Proof We follow ideas from the proof of Sect. 3.2. At first, we denote $s_\delta := \delta - [\delta]$. Let us introduce test functions $\eta = \eta(t)$ as in Sect. 3.2 and $\varphi = \varphi(x) := \langle x \rangle^{-n-2s_\delta}$. We can repeat exactly, the estimates for J_1 and J_2 as we did in the proof of Sect. 3.2 to conclude

$$|J_1| \lesssim I_{R,t}^{\frac{1}{p}}\, R^{-2\alpha+\frac{n+\alpha}{p'}}, \tag{36}$$

$$|J_2| \lesssim I_R^{\frac{1}{p}}\, R^{-2\sigma+\frac{n+\alpha}{p'}}. \tag{37}$$

Let us turn to estimate J_3, where δ is any fractional number in $(1, \sigma)$. In the first step, applying Lemma 7 and Hölder's inequality leads to

$$|J_3| \leq I_{R,t}^{\frac{1}{p}} \Big(\int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \eta_R^{-\frac{p'}{p}}(t) \big|\partial_t \eta_R(t)\big|^{p'} \varphi_R^{-\frac{p'}{p}}(x) \, \big|(-\Delta)^\delta \varphi_R(x)\big|^{p'}\, dxdt \Big)^{\frac{1}{p'}}.$$

Now we can re-write $\delta = m_\delta + s_\delta$, where $m_\delta := [\delta] \geq 1$ is integer and s_δ is a fractional number in $(0, 1)$. Employing Lemma 2 we derive

$$(-\Delta)^\delta \varphi_R(x) = (-\Delta)^{s_\delta}\big((-\Delta)^{m_\delta}\varphi_R(x)\big).$$

By the change of variables $\tilde{x} := R^{-1}x$ we also notice that

$$(-\Delta)^{m_\delta}\varphi_R(x) = R^{-2m_\delta}(-\Delta)^{m_\delta}(\varphi)(\tilde{x})$$

since m_δ is an integer. Using the formula (18) we re-write

$$\begin{aligned}(-\Delta)^{m_\delta}\varphi_R(x) = (-1)^{m_\delta} R^{-2m_\delta} \prod_{j=0}^{m_\delta-1}(q+2j)\Big(&\prod_{j=1}^{m_\delta}(-n+q+2j)\langle\tilde{x}\rangle^{-q-2m_\delta} \\ &- C^1_{m_\delta}\prod_{j=2}^{m_\delta}(-n+q+2j)(q+2m_\delta)\langle\tilde{x}\rangle^{-q-2m_\delta-2} \\ &+ C^2_{m_\delta}\prod_{j=3}^{m_\delta}(-n+q+2j)(q+2m_\delta)(q+2m_\delta+2)\langle\tilde{x}\rangle^{-q-2m_\delta-4} \\ &+ \cdots + (-1)^{m_\delta}\prod_{j=0}^{m_\delta-1}(q+2m_\delta+2j)\langle\tilde{x}\rangle^{-q-4m_\delta}\Big),\end{aligned}$$

where $q := n + 2s_\delta$. For simplicity, we introduce the following functions:

$$\varphi_k(x) := \langle x\rangle^{-q-2m_\delta-2k} \quad \text{and} \quad \varphi_{k,R}(x) := \varphi_k(R^{-1}x) = \langle\tilde{x}\rangle^{-q-2m_\delta-2k}$$

with $k = 0, \cdots, m_\delta$. As a result, by Lemma 4 we arrive at

$$\begin{aligned}(-\Delta)^\delta\varphi_R(x) = (-1)^{m_\delta} R^{-2m_\delta}\prod_{j=0}^{m_\delta-1}(q+2j)\Big(&\prod_{j=1}^{m_\delta}(-n+q+2j)\,(-\Delta)^{s_\delta}(\varphi_{0,R})(x) \\ &- C^1_{m_\delta}\prod_{j=2}^{m_\delta}(-n+q+2j)(q+2m_\delta)\,(-\Delta)^{s_\delta}(\varphi_{1,R})(x) \\ &+ C^2_{m_\delta}\prod_{j=3}^{m_\delta}(-n+q+2j)(q+2m_\delta)(q+2m_\delta+2)\,(-\Delta)^{s_\delta}(\varphi_{2,R})(x) \\ &+ \cdots + (-1)^{m_\delta}\prod_{j=0}^{m_\delta-1}(q+2m_\delta+2j)\,(-\Delta)^{s_\delta}(\varphi_{m_\delta,R})(x)\Big)\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{m_\delta} R^{-2m_\delta-2s_\delta} \prod_{j=0}^{m_\delta-1}(q+2j)\Big(\prod_{j=1}^{m_\delta}(-n+q+2j)\,(-\Delta)^{s_\delta}(\varphi_0)(\tilde{x})\\
&\quad - C^1_{m_\delta}\prod_{j=2}^{m_\delta}(-n+q+2j)(q+2m_\delta)\,(-\Delta)^{s_\delta}(\varphi_1)(\tilde{x})\\
&\quad + C^2_{m_\delta}\prod_{j=3}^{m_\delta}(-n+q+2j)(q+2m_\delta)(q+2m_\delta+2)\,(-\Delta)^{s_\delta}(\varphi_2)(\tilde{x})\\
&\quad + \cdots + (-1)^{m_\delta}\prod_{j=0}^{m_\delta-1}(q+2m_\delta+2j)\,(-\Delta)^{s_\delta}(\varphi_{m_\delta})(\tilde{x})\Big)\\
&= R^{-2\delta}(-\Delta)^{\delta}(\varphi)(\tilde{x}).
\end{aligned}
$$

For this reason, performing the change of variables $\tilde{t} := R^{-\alpha}t$ we obtain

$$
\begin{aligned}
|J_3| &\lesssim I_{R,t}^{\frac{1}{p}}\, R^{-2\delta-\alpha+\frac{n+\alpha}{p'}}\\
&\quad\times\Big(\int_{\frac{1}{2}}^{1}\int_{\mathbb{R}^n}\eta^{-\frac{p'}{p}}(\tilde{t})\big|\eta'(\tilde{t})\big|^{p'}\varphi^{-\frac{p'}{p}}(\tilde{x})\,\big|(-\Delta)^{\delta}(\varphi)(\tilde{x})\big|^{p'}\,d\tilde{x}d\tilde{t}\Big)^{\frac{1}{p'}}\\
&\lesssim I_{R,t}^{\frac{1}{p}}\, R^{-2\delta-\alpha+\frac{n+\alpha}{p'}}\Big(\int_{\mathbb{R}^n}\varphi^{-\frac{p'}{p}}(\tilde{x})\,\big|(-\Delta)^{\delta}(\varphi)(\tilde{x})\big|^{p'}\,d\tilde{x}\Big)^{\frac{1}{p'}}.
\end{aligned}
$$

Here we used $\partial_t\eta_R(t) = R^{-\alpha}\eta'(\tilde{t})$ and the assumption (24). After applying Lemma 3 with $q = n + 2s_\delta$ and $\gamma = \delta$, i.e. $m = m_\delta$ and $s = s_\delta$, we may conclude

$$
|J_3| \lesssim I_{R,t}^{\frac{1}{p}}\, R^{-2\delta-\alpha+\frac{n+\alpha}{p'}}\Big(\int_{\mathbb{R}^n}\langle\tilde{x}\rangle^{-n-2s_\delta}\,d\tilde{x}\Big)^{\frac{1}{p'}} \lesssim I_{R,t}^{\frac{1}{p}}\, R^{-2\delta-\alpha+\frac{n+\alpha}{p'}}. \tag{38}
$$

Finally, combining (36)–(38) and repeating arguments as in Sect. 3.2 we may complete the proof of Theorem 2. □

3.4 The Case that the Parameter σ Is Fractional from (1, ∞) and the Parameter δ Is Integer

Proof We follow ideas from the proofs of Sects. 3.2 and 3.3. At first, we denote $s_\sigma := \sigma - [\sigma]$. Let us introduce test functions $\eta = \eta(t)$ as in Sect. 3.2 and $\varphi = \varphi(x) := \langle x\rangle^{-n-2s_\sigma}$. Then, repeating the proof of Sects. 3.2 and 3.3 we may conclude what we wanted to prove. □

3.5 The Case that the Parameter σ Is Fractional from $(1, \infty)$ and the Parameter δ Is Fractional from $(0, 1)$

Proof We follow ideas from the proofs of Sects. 3.2 and 3.4. At first, we denote $s_\sigma := \sigma - [\sigma]$. Next, we put $s^* := \min\{s_\sigma, \delta\}$. It is obvious that s^* is fractional from $(0, 1)$. Let us introduce test functions $\eta = \eta(t)$ as in Sect. 3.2 and $\varphi = \varphi(x) := \langle x\rangle^{-n-2s^*}$. Then, repeating the proof of Sects. 3.2 and 3.4 we may conclude what we wanted to prove. □

3.6 The Case that the Parameter σ Is Fractional from $(1, \infty)$ and the Parameter δ Is Fractional from $(1, \sigma)$

Proof We follow ideas from the proofs of Sects. 3.2 and 3.5. At first, we denote $s_\sigma := \sigma - [\sigma]$ and $s_\delta := \delta - [\delta]$. Next, we put $s^* := \min\{s_\sigma, s_\delta\}$. It is obvious that s^* is fractional from $(0, 1)$. Let us introduce test functions $\eta = \eta(t)$ as in Sect. 3.2 and $\varphi = \varphi(x) := \langle x\rangle^{-n-2s^*}$. Then, repeating the proof of Sects. 3.2 and 3.5 we may conclude what we wanted to prove. □

4 Critical Exponent Versus Critical Nonlinearity

In Remark 2 we explained that for some models (1) we determined the critical exponent $p_{crit} = p_{crit}(n)$ in the scale of power nonlinearities $\{|u|^p\}_{p>1}$. But is this observation sharp? In the paper [5] the authors discussed this issue for the classical damped wave model with power nonlinearity. Here we want to extend this idea to some models of type (1). For this reason, we discuss the following model:

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\delta u_t = |u|^{p_{crit}(n)}\mu(|u|), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \end{cases} \tag{39}$$

where $\sigma \geq 1$, $\delta \in [0, \frac{\sigma}{2}]$ and $p_{crit}(n) = 1 + \frac{2\sigma}{n-2\delta}$ with $n \geq 1$. Here the function $\mu = \mu(|u|)$ is a suitable modulus of continuity.

4.1 Main Results

First we state a global (in time) existence result of small data Sobolev solutions to (39).

Theorem 3 (Global Existence) *Let $\sigma \geq 1$, $\delta \in [0, \frac{\sigma}{2}]$ and $m \in [1, 2)$. Let $0 < \theta \leq \sigma$. We assume the conditions*

$$\begin{cases} 2m_0\delta < n < 2\theta & \text{if} \quad \delta \in [0, \frac{\sigma}{2}), \\ m\sigma < n < 2\theta & \text{if} \quad \delta = \frac{\sigma}{2}. \end{cases} \tag{40}$$

Moreover, we suppose the following assumptions of modulus of continuity:

$$s\mu'(s) \lesssim \mu(s) \tag{41}$$

and

$$\int_0^{C_0} \frac{\mu(s)}{s}\, ds < \infty \tag{42}$$

with a sufficiently small constant $C_0 > 0$. Then, there exists a constant $\varepsilon_0 > 0$ such that for any small data

$$(u_0, u_1) \in \big(L^m \cap H^\theta\big) \times \big(L^m \cap L^2\big)$$

satisfying the assumption $\|u_0\|_{L^m \cap H^\theta} + \|u_1\|_{L^m \cap L^2} \leq \varepsilon_0$, we have a uniquely determined global (in time) small data Sobolev solution

$$u \in \mathscr{C}\big([0, \infty), H^\theta\big)$$

to (39). The following estimates hold:

$$\|u(t, \cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{2})+\frac{\delta}{\sigma-\delta}} \big(\|u_0\|_{L^m \cap H^\theta} + \|u_1\|_{L^m \cap L^2}\big),$$

$$\big\||D|^\theta u(t, \cdot)\big\|_{L^2} \lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{2})-\frac{\theta-2\delta}{2(\sigma-\delta)}} \big(\|u_0\|_{L^m \cap H^\theta} + \|u_1\|_{L^m \cap L^2}\big).$$

Now we state a blow-up result to (39).

Theorem 4 (Blow-Up) *Let $\sigma \geq 1$ and $\delta \in [0, \frac{\sigma}{2}]$ be integer numbers. We assume that we choose the initial data $u_0 = 0$ and $u_1 \in L^1$ satisfying the following relation:*

$$\int_{\mathbb{R}^n} u_1(x)\, dx > 0. \tag{43}$$

Moreover, we suppose the following assumption of modulus of continuity:

$$s^k \mu^{(k)}(s) = o\big(\mu(s)\big) \quad \text{as } s \to +0 \text{ with } k = 1, 2, \tag{44}$$

and

$$\int_0^{C_0} \frac{\mu(s)}{s}\, ds = \infty, \tag{45}$$

where $C_0 > 0$ is a sufficiently small constant. Then, there is no global (in time) Sobolev solution to (39).

In the following we restrict ourselves to prove the blow-up result.

4.2 Proof of Theorem 4

The ideas of the following proof are based on the recent paper [5] of the second author and his collaborators in which the authors focused on their considerations to (39) with $\sigma = 1$ and $\delta = 0$. For simplicity, we use the abbreviations $p_c := p_{crit}(n) = 1 + \frac{2\sigma}{n-2\delta}$ to (39) in the following proof.

Proof of Theorem 4 First, we introduce a test function $\varphi = \varphi(\tau)$ having the following properties:

$$\varphi \in \mathscr{C}_0^\infty([0,\infty)) \text{ and } \varphi(\tau) = \begin{cases} 1 & \text{for } 0 \le \tau \le \frac{1}{2}, \\ \text{decreasing} & \text{for } \frac{1}{2} \le \tau \le 1, \\ 0 & \text{for } \tau \ge 1. \end{cases}$$

Moreover, we also introduce the function $\varphi^* = \varphi^*(\tau)$ satisfying

$$\varphi^*(\tau) = \begin{cases} 0 & \text{for } 0 \le \tau < \frac{1}{2}, \\ \varphi(\tau) & \text{for } \frac{1}{2} \le \tau < \infty. \end{cases}$$

Let R be a large parameter in $[0, \infty)$. We define the following two functions:

$$\varphi_R(t,x) = \Big(\varphi\Big(\frac{|x|^{2(\sigma-\delta)} + t}{R}\Big)\Big)^{n+2(\sigma-\delta)}$$

and

$$\varphi_R^*(t,x) = \Big(\varphi^*\Big(\frac{|x|^{2(\sigma-\delta)} + t}{R}\Big)\Big)^{n+2(\sigma-\delta)}.$$

Then it is clear that

$$\operatorname{supp}\varphi_R \subset Q_R := \big\{(t,x)\ :\ (t,|x|) \in [0,R]\times\big[0,R^{1/(2(\sigma-\delta))}\big]\big\},$$
$$\operatorname{supp}\varphi_R^* \subset Q_R^* := Q_R \setminus \big\{(t,x)\ :\ (t,|x|) \in [0,R/2]\times\big[0,(R/2)^{1/(2(\sigma-\delta))}\big]\big\}.$$

Now we define the functional

$$\begin{aligned} I_R &:= \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^{p_c}\mu\big(|u(t,x)|\big)\varphi_R(t,x)\,dxdt \\ &= \int_{Q_R} |u(t,x)|^{p_c}\mu\big(|u(t,x)|\big)\varphi_R(t,x)\,d(x,t). \end{aligned}$$

Let us assume that $u = u(t,x)$ is a global (in time) Sobolev solution to (39). After multiplying the Eq. (39) by $\varphi_R = \varphi_R(t,x)$, we carry out partial integration to derive

$$\begin{aligned} 0 \le I_R &= -\int_{\mathbb{R}^n} u_1(x)\varphi_R(0,x)\,dx \\ &\quad + \int_{Q_R} u(t,x)\big(\partial_t^2\varphi_R(t,x) + (-\Delta)^\sigma\varphi_R(t,x) - (-\Delta)^\delta\partial_t\varphi_R(t,x)\big)\,d(x,t) \\ &=: -\int_{\mathbb{R}^n} u_1(x)\varphi_R(0,x)\,dx + J_R. \end{aligned}$$

Because of the assumption (43), there exists a sufficiently large constant $R_0 > 0$ such that for all $R > R_0$ it holds

$$\int_{\mathbb{R}^n} u_1(x)\varphi_R(0,x)\,dx > 0.$$

Consequently, we obtain

$$0 \le I_R < J_R \quad \text{for all } R > R_0. \tag{46}$$

In order to estimate J_R, firstly we have

$$\begin{aligned} |\partial_t\varphi_R(t,x)| &\lesssim \Big|\frac{1}{R}\Big(\varphi\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n+2(\sigma-\delta)-1}\varphi'\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big| \\ &\lesssim \frac{1}{R}\Big(\varphi^*\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n+2(\sigma-\delta)-1}. \end{aligned} \tag{47}$$

Further calculations lead to

$$
\begin{aligned}
|\partial_t^2 \varphi_R(t,x)| &\lesssim \Big|\frac{1}{R^2}\Big(\varphi\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n+2(\sigma-\delta)-2}\Big(\varphi'\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^2\Big| \\
&\quad + \Big|\frac{1}{R^2}\Big(\varphi\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n+2(\sigma-\delta)-1}\varphi''\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big| \\
&\lesssim \frac{1}{R^2}\Big(\varphi^*\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n+2(\sigma-\delta)-2}. \qquad (48)
\end{aligned}
$$

To control $(-\Delta)^\sigma \varphi_R(t,x)$, we shall apply Lemma 8 as a main tool. Indeed, we divide our consideration into three sub-steps as follows:

Step 1: Applying Lemma 8 with $h(z) = \frac{z^{\sigma-\delta}+t}{R}$ and $z = f(x) = |x|^2$ we derive the following estimate for $|\alpha| \geq 1$:

$$
\begin{aligned}
&\Big|\partial_x^\alpha\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big| \\
&\leq \sum_{k=1}^{|\alpha|}\frac{|x|^{2(\sigma-\delta)-2k}}{R}\left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=|\alpha| \\ |\gamma_i|\geq 1}}\big|\partial_x^{\gamma_1}(|x|^2)\big|\cdots\big|\partial_x^{\gamma_k}(|x|^2)\big|\right) \\
&\leq \sum_{k=1}^{|\alpha|}\frac{|x|^{2(\sigma-\delta)-2k}}{R}\left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=|\alpha| \\ 1\leq|\gamma_i|\leq 2}}\big|\partial_x^{\gamma_1}(|x|^2)\big|\cdots\big|\partial_x^{\gamma_k}(|x|^2)\big|\right) \\
&\lesssim \sum_{k=1}^{|\alpha|}\frac{|x|^{2(\sigma-\delta)-2k}}{R}\left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=|\alpha| \\ 1\leq|\gamma_i|\leq 2}}|x|^{2-|\gamma_1|}\cdots|x|^{2-|\gamma_k|}\right) \\
&\lesssim \sum_{k=1}^{|\alpha|}\frac{|x|^{2(\sigma-\delta)-2k}}{R}|x|^{2k-|\alpha|} \lesssim \frac{|x|^{2(\sigma-\delta)-|\alpha|}}{R}.
\end{aligned}
$$

This estimate holds for $|\alpha| \leq 2(\sigma-\delta)$. But we may conclude that it holds for all $|\alpha| \geq 1$, too and small $|x|$. More precisely, the singularity appearing in the case $|\alpha| > 2(\sigma-\delta)$ does not really bring any difficulty in the further treatment.

Step 2: Applying Lemma 8 with $h(z) = \varphi(z)$ and $z = f(x) = \frac{|x|^{2(\sigma-\delta)}+t}{R}$ we get for all $|\alpha| \geq 1$ the following estimate:

$$\begin{aligned}
&\Big|\partial_x^\alpha \varphi\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big| \\
&\quad \leq \sum_{k=1}^{|\alpha|} \Big|\varphi^{(k)}\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big| \\
&\qquad \times \left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=|\alpha| \\ 1\leq|\gamma_i|\leq 2(\sigma-\delta)}} \Big|\partial_x^{\gamma_1}\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big| \cdots \Big|\partial_x^{\gamma_k}\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big|\right) \\
&\leq \sum_{k=1}^{|\alpha|} \Big|\varphi^{(k)}\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big| \\
&\quad \times \left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=|\alpha| \\ 1\leq|\gamma_i|\leq 2(\sigma-\delta)}} \frac{|x|^{2(\sigma-\delta)-|\gamma_1|}}{R} \cdots \frac{|x|^{2(\sigma-\delta)-|\gamma_k|}}{R}\right) \\
&\lesssim \sum_{k=1}^{|\alpha|} \Big(\frac{|x|^{2(\sigma-\delta)}}{R}\Big)^k |x|^{-|\alpha|} \lesssim \frac{|x|^{2(\sigma-\delta)-|\alpha|}}{R} \quad \big(\text{since } \ |x|^{2(\sigma-\delta)} \leq R \text{ in } Q_R^*\big).
\end{aligned}$$

Step 3: Applying Lemma 8 with $h(z) = z^{n+2(\sigma-\delta)}$ and $z = f(x) = \varphi\big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\big)$ we obtain

$$\begin{aligned}
&\big|(-\Delta)^\sigma \varphi_R(t,x)\big| \lesssim \sum_{|\alpha|=2\sigma} \big|\partial_x^\alpha \varphi_R(t,x)\big| \qquad (49)\\
&\quad \lesssim \sum_{k=1}^{2\sigma} \Big(\varphi\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n+2(\sigma-\delta)-k} \\
&\qquad \times \left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=2\sigma \\ |\gamma_i|\geq 1}} \Big|\partial_x^{\gamma_1}\varphi\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big| \cdots \Big|\partial_x^{\gamma_k}\varphi\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big|\right)
\end{aligned}$$

$$\lesssim \sum_{k=1}^{2\sigma} \Big(\varphi^*\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n+2(\sigma-\delta)-k}$$
$$\times \sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=2\sigma \\ |\gamma_i|\geq 1}} \frac{|x|^{2(\sigma-\delta)-|\gamma_1|}}{R} \cdots \frac{|x|^{2(\sigma-\delta)-|\gamma_k|}}{R}$$

$$\lesssim \sum_{k=1}^{2\sigma} \Big(\varphi^*\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n+2(\sigma-\delta)-k} \frac{|x|^{2k(\sigma-\delta)-2\sigma}}{R^k}$$
$$\lesssim \frac{1}{R^{\frac{\sigma}{\sigma-\delta}}}\Big(\varphi^*\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n-2\delta} \quad \big(\text{since } |x|^{2(\sigma-\delta)} \approx R \text{ in } Q_R^*\big). \tag{50}$$

It is clear that if $\delta = 0$, then $\big|(-\Delta)^\delta \partial_t \varphi_R(t,x)\big|$ was estimated in (47). For the case $\delta \in (0, \frac{\sigma}{2}]$, we can proceed in an analogous way as we controlled $\big|(-\Delta)^\sigma \varphi_R(t,x)\big|$ to derive

$$\big|(-\Delta)^\delta \partial_t \varphi_R(t,x)\big| \lesssim \frac{1}{R^{\frac{\sigma}{\sigma-\delta}}}\Big(\varphi^*\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n+2(\sigma-2\delta)-1}. \tag{51}$$

From (47) to (51), we arrive at the following estimate:

$$\big|\partial_t^2 \varphi_R(t,x) + (-\Delta)^\sigma \varphi_R(t,x) - (-\Delta)^\delta \partial_t \varphi_R(t,x)\big|$$
$$\lesssim \frac{1}{R^{\frac{\sigma}{\sigma-\delta}}}\Big(\varphi^*\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n-2\delta} = \frac{1}{R^{\frac{\sigma}{\sigma-\delta}}}\big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}.$$

Hence, we may conclude

$$J_R = |J_R| \lesssim \frac{1}{R^{\frac{\sigma}{\sigma-\delta}}} \int_{Q_R} |u(t,x)| \big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}\, d(x,t). \tag{52}$$

Now we focus our attention to estimate the above integral. To do this, we introduce the function $\Psi(s) = s^{p_c}\mu(s)$. Then, we derive

$$\Psi\Big(|u(t,x)| \big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}\Big)$$
$$= |u(t,x)|^{p_c} \big(\varphi_R^*(t,x)\big)^{\frac{p_c(n-2\delta)}{n+2(\sigma-\delta)}} \mu\Big(|u(t,x)| \big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}\Big)$$
$$\leq |u(t,x)|^{p_c} \varphi_R^*(t,x) \mu\big(|u(t,x)|\big) = \Psi\big(|u(t,x)|\big)\, \varphi_R^*(t,x). \tag{53}$$

Here we used the increasing property of the function $\mu = \mu(s)$ and the relation

$$0 \leq \big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}} \leq 1.$$

Due to the assumption (44), we may verify that Ψ is a convex function on a small interval $(0, c_0]$ by the following relation:

$$\Psi''(s) = s^{p_c-2}\Big(p_c(p_c-1)\mu(s) + 2p_c\, s\mu'(s) + s^2\mu''(s)\Big) \geq 0.$$

Moreover, we can choose a convex continuation of Ψ outside this interval to guarantee that Ψ is convex on $[0, \infty)$. Applying Proposition 1 with $h(s) = \Psi(s)$, $f(t,x) = |u(t,x)|\big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}$ and $\gamma \equiv 1$ gives the following estimate:

$$\Psi\Big(\frac{\int_{Q_R^*} |u(t,x)|\big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}\, d(x,t)}{\int_{Q_R^*} 1\, d(x,t)}\Big) \leq \frac{\int_{Q_R^*} \Psi\Big(|u(t,x)|\big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}\Big)\, d(x,t)}{\int_{Q_R^*} 1\, d(x,t)}.$$

We may compute

$$\int_{Q_R^*} 1\, d(x,t) \approx R^{1+\frac{n}{2(\sigma-\delta)}}.$$

Hence, we get

$$\begin{aligned}\Psi\Big(\frac{\int_{Q_R^*} |u(t,x)|\big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}\, d(x,t)}{R^{1+\frac{n}{2(\sigma-\delta)}}}\Big) &\leq \frac{\int_{Q_R^*} \Psi\Big(|u(t,x)|\big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}\Big)\, d(x,t)}{R^{1+\frac{n}{2(\sigma-\delta)}}} \\ &\leq \frac{\int_{Q_R} \Psi\Big(|u(t,x)|\big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}\Big)\, d(x,t)}{R^{1+\frac{n}{2(\sigma-\delta)}}}. \end{aligned} \tag{54}$$

Combining the estimates (53) and (54) we may arrive at

$$\Psi\Big(\frac{\int_{Q_R^*} |u(t,x)|\big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}\,d(x,t)}{R^{1+\frac{n}{2(\sigma-\delta)}}}\Big) \leq \frac{\int_{Q_R} \Psi\big(|u(t,x)|\big)\,\varphi_R^*(t,x)\,d(x,t)}{R^{1+\frac{n}{2(\sigma-\delta)}}}. \tag{55}$$

Since $\mu = \mu(s)$ is a strictly increasing function, it immediately follows that $\Psi = \Psi(s)$ is also a strictly increasing function on $[0,\infty)$. For this reason, from (55) we deduce

$$\begin{aligned}\int_{Q_R} &|u(t,x)|\big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}\,d(x,t)\\ &= \int_{Q_R^*} |u(t,x)|\big(\varphi_R^*(t,x)\big)^{\frac{n-2\delta}{n+2(\sigma-\delta)}}\,d(x,t)\\ &\leq R^{1+\frac{n}{2(\sigma-\delta)}}\,\Psi^{-1}\Big(\frac{\int_{Q_R} \Psi\big(|u(t,x)|\big)\,\varphi_R^*(t,x)\,d(x,t)}{R^{1+\frac{n}{2(\sigma-\delta)}}}\Big).\end{aligned} \tag{56}$$

From (46), (52) and (56) we may conclude

$$I_R \lesssim R^{\frac{n-2\delta}{2(\sigma-\delta)}}\,\Psi^{-1}\Big(\frac{\int_{Q_R} \Psi\big(|u(t,x)|\big)\,\varphi_R^*(t,x)\,d(x,t)}{R^{1+\frac{n}{2(\sigma-\delta)}}}\Big) \tag{57}$$

for all $R > R_0$. Next we introduce the following two functions:

$$g(r) = \int_{Q_R} \Psi\big(|u(t,x)|\big)\,\varphi_r^*(t,x)\,d(x,t) \quad \text{with } r \in (0,\infty)$$

and

$$G(R) = \int_0^R g(r) r^{-1}\,dr.$$

Then, we re-write

$$\begin{aligned}G(R) &= \int_0^R \Big(\int_{Q_R} \Psi\big(|u(t,x)|\big)\,\varphi_r^*(t,x)\,d(x,t)\Big) r^{-1}\,dr\\ &= \int_{Q_R} \Psi\big(|u(t,x)|\big)\Big(\int_0^R \Big(\varphi^*\Big(\frac{|x|^{2(\sigma-\delta)}+t}{r}\Big)\Big)^{n+2(\sigma-\delta)} r^{-1}\,dr\Big)\,d(x,t).\end{aligned}$$

Carrying out change of variables $\tilde{r} = \frac{|x|^{2(\sigma-\delta)}+t}{r}$ we derive

$$
\begin{aligned}
G(R) &= \int_{Q_R} \Psi\big(|u(t,x)|\big)\Big(\int_{\frac{|x|^{2(\sigma-\delta)}+t}{R}}^{\infty} \big(\varphi^*(\tilde{r})\big)^{n+2(\sigma-\delta)}\,\tilde{r}^{-1}\,d\tilde{r}\Big)\,d(x,t)\\
&\le \int_{Q_R} \Psi\big(|u(t,x)|\big)\Big(\int_{1/2}^{1} \big(\varphi^*(\tilde{r})\big)^{n+2(\sigma-\delta)}\,\tilde{r}^{-1}\,d\tilde{r}\Big)\,d(x,t)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \big(\text{since } \operatorname{supp}\varphi^* \subset [1/2,1]\big)\\
&\le \int_{Q_R} \Psi\big(|u(t,x)|\big)\Big(\int_{1/2}^{1} \big(\varphi(\tilde{r})\big)^{n+2(\sigma-\delta)}\,\tilde{r}^{-1}\,d\tilde{r}\Big)\,d(x,t)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \big(\text{since } \varphi^* \equiv \varphi \text{ in } [1/2,1]\big)\\
&\le \int_{Q_R} \Psi\big(|u(t,x)|\big)\Big(\varphi\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n+2(\sigma-\delta)}\Big(\int_{1/2}^{1} \tilde{r}^{-1}\,d\tilde{r}\Big)\,d(x,t)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \big(\text{since } \varphi \text{ is decreasing}\big)\\
&\le \log(1+e)\int_{Q_R} \Psi\big(|u(t,x)|\big)\Big(\varphi\Big(\frac{|x|^{2(\sigma-\delta)}+t}{R}\Big)\Big)^{n+2(\sigma-\delta)}\,d(x,t)\\
&= \log(1+e) I_R. \qquad (58)
\end{aligned}
$$

Moreover, it holds the following relation:

$$
G'(R)R = g(R) = \int_{Q_R} \Psi\big(|u(t,x)|\big)\,\varphi_R^*(t,x)\,d(x,t). \qquad (59)
$$

From (57) to (59) we get

$$
\frac{G(R)}{\log(1+e)} \le I_R \le C_1 R^{\frac{n-2\delta}{2(\sigma-\delta)}}\,\Psi^{-1}\Big(\frac{G'(R)}{R^{\frac{n}{2(\sigma-\delta)}}}\Big)
$$

for all $R > R_0$ and with a suitable positive constant C_1. This implies

$$
\Psi\Big(\frac{G(R)}{C_2 R^{\frac{n-2\delta}{2(\sigma-\delta)}}}\Big) \le \frac{G'(R)}{R^{\frac{n}{2(\sigma-\delta)}}}
$$

for all $R > R_0$ and $C_2 := C_1 \log(1+e) > 0$. By the definition of the function Ψ, the above inequality is equivalent to

$$
\Big(\frac{G(R)}{C_2 R^{\frac{n-2\delta}{2(\sigma-\delta)}}}\Big)^{p_c}\mu\Big(\frac{G(R)}{C_2 R^{\frac{n}{2(\sigma-\delta)}}}\Big) \le \frac{G'(R)}{R^{\frac{n}{2(\sigma-\delta)}}}
$$

for all $R > R_0$. Therefore, we have

$$\frac{1}{C_3 R}\mu\Big(\frac{G(R)}{C_2 R^{\frac{n-2\delta}{2(\sigma-\delta)}}}\Big) \leq \frac{G'(R)}{\big(G(R)\big)^{p_c}}$$

for all $R > R_0$ and $C_3 := C_2^{p_c} > 0$. Because $G = G(R)$ is an increasing function, for all $R > R_0$ it holds the following inequality:

$$\frac{1}{C_3 R}\mu\Big(\frac{G(R_0)}{C_2 R^{\frac{n-2\delta}{2(\sigma-\delta)}}}\Big) \leq \frac{G'(R)}{\big(G(R)\big)^{p_c}}.$$

After denoting $\tilde{s} := R$ and integrating two sides over $[R_0, R]$ we arrive at

$$\begin{aligned}\frac{1}{C_3}\int_{R_0}^{R}\frac{1}{\tilde{s}}\,\mu\Big(\frac{1}{C_4 \tilde{s}^{\frac{n-2\delta}{2(\sigma-\delta)}}}\Big)\,d\tilde{s} &\leq \int_{R_0}^{R}\frac{G'(\tilde{s})}{\big(G(\tilde{s})\big)^{p_c}}\,d\tilde{s}\\ &= \frac{n-2\delta}{2\sigma}\Big(\frac{1}{\big(G(R_0)\big)^{\frac{2\sigma}{n-2\delta}}} - \frac{1}{\big(G(R)\big)^{\frac{2\sigma}{n-2\delta}}}\Big)\\ &\leq \frac{n-2\delta}{2\sigma\big(G(R_0)\big)^{\frac{2\sigma}{n-2\delta}}},\end{aligned}$$

where $C_4 := \frac{C_2}{G(R_0)} > 0$. Letting $R \to \infty$ leads to

$$\frac{1}{C_3}\int_{R_0}^{\infty}\frac{1}{\tilde{s}}\,\mu\Big(\frac{1}{C_4 \tilde{s}^{\frac{n-2\delta}{2(\sigma-\delta)}}}\Big)\,d\tilde{s} \leq \frac{n-2\delta}{2\sigma\big(G(R_0)\big)^{\frac{2\sigma}{n-2\delta}}}.$$

Finally, using change of variables $s = C_4 \tilde{s}^{\frac{n-2\delta}{2(\sigma-\delta)}}$ we may conclude

$$C\int_{C_0}^{\infty}\frac{\mu\big(\frac{1}{s}\big)}{s}\,ds \leq \frac{n-2\delta}{2\sigma\big(G(R_0)\big)^{\frac{2\sigma}{n-2\delta}}},$$

where $C := \frac{2\sigma}{C_3(n-2\delta)} > 0$ and $C_0 := C_4 R_0^{\frac{n-2\delta}{2(\sigma-\delta)}} > 0$ is a sufficiently large constant. This is a contradiction to the assumption (45). Summarizing, the proof of Theorem 4 is completed. □

Remark 4 From the condition (42) in Theorem 3 and the condition (45) in Theorem 4, we recognize that determining the critical exponent $p_{crit} = 1 + \frac{2\sigma}{n-2\delta}$ in the scale of power nonlinearities $\{|u|^p\}_{p>1}$ is really sharp to (39) in the case $\delta \in [0, \frac{\sigma}{2}]$, i.e. for "parabolic like models". However, up to now this observation remains an open problem for "σ-evolution like models" in the remaining case

$\delta \in (\frac{\sigma}{2}, \sigma]$, the so-called "hyperbolic like models" or "wave like models" in the case $\sigma = 1$.

Appendix

Proposition 1 (A Generalized Jensen's Inequality) *Let $\gamma = \gamma(x)$ be a defined and nonnegative function almost everywhere on Ω, provided that γ is positive in a set of positive measure. Then, for each convex function h on $\mathbb{R}$ the following inequality holds:*

$$h\left(\frac{\int_\Omega f(x)\gamma(x)\,dx}{\int_\Omega \gamma(x)\,dx}\right) \leq \frac{\int_\Omega h\big(f(x)\big)\gamma(x)\,dx}{\int_\Omega \gamma(x)\,dx},$$

where f is any nonnegative function satisfying all the above integrals are meaningful.

The proof of this result can be found in [5, 9].

Lemma 8 (Useful Lemma) *The following formula of derivative of composed function holds for any multi-index α:*

$$\partial_\xi^\alpha h\big(f(\xi)\big) = \sum_{k=1}^{|\alpha|} h^{(k)}\big(f(\xi)\big)\left(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leq\alpha \\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\ |\gamma_i|\geq 1}} \big(\partial_\xi^{\gamma_1} f(\xi)\big)\cdots\big(\partial_\xi^{\gamma_k} f(\xi)\big)\right),$$

where $h = h(z)$ and $h^{(k)}(z) = \frac{d^k h(z)}{d\,z^k}$.

The result can be found in [11] at page 202.

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Critical Exponent for a Class of Semilinear Damped Wave Equations with Decaying in Time Propagation Speed

Marcelo Rempel Ebert and Jorge Marques

Dedicated to Massimo Cicognani and Michael Reissig (The main tools used in this paper was introduced by M. Reissig to the first author on the occasion of his first visit to Freiberg in 2010) to their 60th Birthday.

Abstract We consider the Cauchy problem on $\mathbf{R}_0^+ \times \mathbf{R}^n$ for the semilinear damped wave equation

$$u_{tt}(t,x) - a^2(t)\Delta u(t,x) + b(t)u_t(t,x) = |u(t,x)|^p$$

with decreasing in time coefficients, the propagation speed $a(t) = (1+t)^{-\ell}$, $\ell \in (0,1)$, the scale-invariant dissipation $b(t) = \beta(1+t)^{-1}$, $\beta > 0$, and a power nonlinearity of order $p > 1$. The solution u^0 of the corresponding linear Cauchy problem will be represented in the explicit form using Fourier multipliers operators with multipliers expressed in terms of special functions. Our main goal is to prove a global in time existence result when initial data belongs to the space $H^m(\mathbf{R}^n) \times H^{m-1}(\mathbf{R}^n)$, $m \geq 1$. We are focused in finding the critical exponent $p_c(n,\ell)$ such that if $1 < p < p_c(n,\ell)$ there exist small data for which u blow-up in finite time. We also prove that if $p \geq p_c(n,\ell)$ the global solution has the same long time behavior as u^0. In order to estimate u we use Duhamel's principle to represent u and then we apply $L^2 - L^2$ estimates of u^0.

Keywords Scale-invariant damped waves · Global existence · Critical exponent

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1 Introduction

In this paper, we obtain the global existence of small data solutions to the Cauchy problem for the semilinear damped wave equation with decreasing in time propagation speed

$$\begin{cases} u_{tt} - (1+t)^{-2\ell}\Delta u + \frac{\beta}{1+t}u_t = f(u), & t \geq 0,\ x \in \mathbf{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbf{R}^n, \end{cases} \tag{1}$$

with $f(u) = |u|^p$, $p > 1$, $\ell \in (0,1)$, $\beta > 0$.

If u is a solution to (1), then

$$\lambda^h u(\lambda(1+t), \lambda^{1-\ell}x)\,, \qquad \text{with} \qquad h \doteq \frac{2}{p-1}\,,$$

is a solution to the equation in (1) for any $\lambda > 0$, with initial data $\lambda^h u_0(\lambda^{1-\ell}x)$. We have

$$\lambda^h \|u_0(\lambda^{1-\ell}\cdot)\|_{L^r} = \lambda^{h-\frac{n(1-\ell)}{r}}\|u_0\|_{L^r}\,, \qquad r \in [1,2],$$

so that the L^r norm is invariant if, and only if, $h - \frac{n(1-\ell)}{r} = 0$. If we assume small data with additional regularity L^r, then one is tempted to say that, at least for $\beta \geq \beta_\sharp(n,\ell)$, with $\beta_\sharp(n,\ell) > 0$ sufficiently large, the critical exponent $p_c(r,n,\ell)$ to (1) is

$$p_c(r,n,\ell) \doteq 1 + \frac{2r}{n(1-\ell)}.$$

If $\ell = 0$ and $\beta \geq \frac{5}{3}$ for $n = 1$, $\beta \geq 3$ for $n = 2$, or $\beta \geq n+2$ for $n \geq 3$ by assuming data in the energy space with additional regularity $L^1(R^n)$, a global (in time) existence result for (1) was proved in [3] for $p > p_F(n) \doteq 1 + \frac{2}{n}$, the well known Fujita index [13]. The exponent p_F is critical for this model, that is, for $p \leq p_F$ and suitable, arbitrarily small data, there exists no global weak solution [5]. As conjectured by [7] and [8], if β becomes smaller with respect to the space dimension, the critical exponent increases to $\max\{p_S(n+\beta), 1+\frac{2}{n}\}$, where p_S is the Strauss exponent for the semilinear undamped wave equation [16, 22]. In [18] the authors shed some light on this problem and gave the explicit value for the threshold $\beta_\star = \frac{n^2+n+2}{n+2}$, namely, for $\beta \in [0,\beta_\star)$ we have "wave-like" models, whereas for $\beta \geq \beta_\star$ we have "heat-like" models. Moreover, for $\beta \in (0,\beta_\star)$ and $1 < p \leq p_S(n+\beta)$ they proved a blow-up result and gave the upper bound for the lifespan of solutions to (1). It is worth noticing that if $\beta \in [0,\beta_\star)$, then $p_F(n) < p_S(n+\beta)$ and, $p_F(n) = p_S(n+\beta_\star)$.

As far as we know, it is still an open problem to prove global existence of small data solutions for $p > p_F(n)$ in the cases $\frac{4}{3} < \beta < \frac{5}{3}$ for $n = 1$, $2 < \beta < 3$ for $n = 2$, or $\beta_\star < \beta < n + 2$ for $n \geq 3$.

If we remove the assumption that the initial data are in $L^1(R^n)$ and we only assume that they are in the energy space, then the critical exponent to (1) is modified into $1 + \frac{4}{n}$ and one may lower the thresholds required for β (see Theorem 4 in [3]). For the classical damped wave equation, this phenomenon has been investigated in [20].

In [1], the authors proposed a classification of non-effective and effective dissipation, respectively, for the damped wave equation

$$u_{tt}(t,x) - a^2(t)\Delta u(t,x) + b(t)u_t(t,x) = 0$$

with increasing speed of propagation. The authors derived sharp estimates for solutions to the Cauchy problem and, in the case of effective dissipation, i.e.,

$$b(t)\frac{A(t)}{a(t)} \to \infty, \ as\ t \to \infty, \qquad A(t) = 1 + \int_0^t a(\tau)\,d\tau,$$

derived global existence (in time) results for the semilinear Cauchy problem with power nonlinearities [2] (see [6] for the case $a(t) \equiv 1$). A similar classification was introduced in [9] in the case $a \in L^1$. A natural generalization for the model (1) is to consider a positive and decreasing speed of propagation $a(t)$, with $a \notin L^1$. But in this paper we restrict ourselves to the case that a is a polynomial function, since it includes interesting models by itself, for instance, if $\ell = \frac{2}{3}$ in (1), the considered model coincides with the non-singular wave equation in the Einstein de Sitter space-time [14, 15].

In this paper we derive higher order energy estimates for solutions to the linear Cauchy problem associate to (1). Then, as in Theorem 6 in [3] (see Remark 5 below), assuming small data in the energy space and for β sufficiently large, in Theorem 2 we prove a global existence result. In Theorem 1 we prove the non-existence part, so it is verified that the critical exponent for the global existence of small data energy solutions to (1) is given by

$$p_c(n,\ell) \doteq 1 + \frac{4}{n(1-\ell)}. \tag{2}$$

Our main goal in this paper is to complete the gaps that appear on ℓ and on the space dimension n in Theorem 6 in [3]. By using additional $H^m(\mathbf{R}^n)$ regularity, with $m > \frac{n}{2}$, in Theorem 3 we are able to deal with higher space dimensions and enlarge the admissible range for $\ell \in (0, 1)$. In particular, we are able to deal with the non-singular wave equation in the Einstein de Sitter space-time for space dimension $n = 3$.

2 Main Results

The next result explains that for $1 < p < p_c(n, \ell)$ Sobolev solutions to (1) in general can not exist globally in time even if the data are supposed to be very small.

Theorem 1 *Let* $f(u) = |u|^p$, $p > 1$, $\ell \in (0, 1)$ *and* $\beta > \ell$ *in (1). Moreover, assume that* $u_0, u_1 \in L^1_{loc}(\mathbf{R}^n)$ *verifies*

$$u_0(x) + u_1(x) \geq \varepsilon(1 + |x|)^{-\frac{n}{2}}(\log(e + |x|)^{-1} \tag{3}$$

for some $\varepsilon \in (0, 1)$. *Then there exists no global (in time) weak solution to (1) for any*

$$p \in \left(1, 1 + \frac{4}{n(1-\ell)}\right).$$

Remark 1 Condition (3) implies that $u_0 + u_1 \notin L^{2-\delta}(\mathbf{R}^n)$ for all $\delta \in (0, 1]$.

Remark 2 By replacing (3) by

$$u_0(x) + u_1(x) \geq \varepsilon(1 + |x|)^{-\frac{n}{r}}(\log(e + |x|)^{-1}, \qquad r \in (1, 2]$$

for some $\varepsilon \in (0, 1)$ and, following the proof of Theorem 1, one may conclude that there exists no global (in time) weak solution to (1) for any

$$p \in \left(1, 1 + \frac{2r}{n(1-\ell)}\right).$$

Remark 3 Theorem 1 is optimal only for $\beta \geq \beta^\sharp$, where $\beta^\sharp$ is expected to be $\beta^\sharp = \ell + \frac{4n(1-\ell)}{n(1-\ell)+4}$. For $\beta \in [0, \beta^\sharp)$, one may try to follow [18] and prove a non-existence result of global (in time) weak solutions for $1 < p < p_0$, with $1 + \frac{4}{n(1-\ell)} < p_0$.

Theorem 2 *Let* $\ell \in (0, 1)$, $\beta > 1$ *for* $n = 1$ *and* $\beta \geq \ell + \frac{4n(1-\ell)}{n(1-\ell)+4}$ *for* $2 \leq n < \frac{4}{1+\ell}$. *If*[1]

$$1 + \frac{4}{n(1-\ell)} < p \leq 1 + \frac{2}{[n-2]_+}$$

then there exists $\epsilon > 0$ *such that for any initial data*

$$(u_0, u_1) \in \mathscr{D} = H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n), \qquad ||(u_0, u_1)||_{\mathscr{D}} \leq \epsilon,$$

[1] By $[x]_+$ we denote the positive part of $x \in \mathbf{R}$, i.e. $[x]_+ = \max\{x, 0\}$.

there exists a unique weak solution $u \in C([0,\infty), H^1(\mathbf{R}^n) \cap C^1([0,\infty), L^2(\mathbf{R}^n))$ *to (1). Moreover, the solution satisfies the following estimates*[2]

$$||u(t,\cdot)||_{\dot{H}^k} \lesssim (1+t)^{-\min\{k(1-\ell),\frac{\beta-\ell}{2}\}}||(u_0,u_1)||_{\mathscr{D}} \tag{4}$$

for any $k \in [0,1]$ *and*

$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\min\{1,\frac{\ell+\beta}{2}\}}||(u_0,u_1)||_{\mathscr{D}}. \tag{5}$$

Remark 4 The lower bound for β can be written as

$$\ell + \frac{4n(1-\ell)}{n(1-\ell)+4} = \ell + \frac{4}{p_c(n,\ell)} = 2 - \ell - \frac{2(1-\ell)[4-(1+\ell)n]}{n(1-\ell)+4}.$$

The condition $n < \frac{4}{1+\ell}$ in Theorem 2 implies $p_c(n,\ell) < \frac{n}{[n-2]_+}$ and $\ell + \frac{4}{p_c(n,\ell)} < 2-\ell$. Moreover, the decay in (4) and (5) changes according to $\beta < 2-\ell$ or $\beta \geq 2-\ell$.

Remark 5 By applying the change of variable

$$v(\tau,x) = u(t,x), \qquad 1+\tau = \frac{(1+t)^{1-\ell}}{1-\ell},$$

the Cauchy problem (1) takes the form

$$\begin{cases} v_{\tau\tau} - \Delta v + \frac{\beta-\ell}{(1-\ell)(1+\tau)}v_\tau = g(v), & \tau \geq \frac{\ell}{1-\ell},\ x \in \mathbf{R}^n, \\ v(\frac{\ell}{1-\ell},x) = u_0(x), & x \in \mathbf{R}^n, \\ v_\tau(\frac{\ell}{1-\ell},x) = u_1(x), & x \in \mathbf{R}^n. \end{cases}$$

with $g(v) = [(1-\ell)(1+\tau)]^{\frac{2\ell}{1-\ell}}|v|^p$. In this way, Theorem 2 is essentially included in Theorem 6 of [3]. But we included it in this paper in order that this result can be compared with the next theorem.

In the next result, the novelty is to use higher regularity $H^m(\mathbf{R}^n)$, $m > \frac{n}{2}$, in order to relax the conditions on the parameters ℓ and n in Theorem 6 of [3], in particular, the condition $n < \frac{4}{1+\ell}$ in Theorem 2. In this way we can also take $\frac{1}{3} \leq \ell < 1$ for space dimension $n = 3$, in particular, if $\ell = \frac{2}{3}$ we may derive a global existence result for the non-singular wave equation in the Einstein de Sitter model [15].

[2]Let $f, g : \Omega \subset \mathbf{R}^n \to \mathbf{R}$ be two functions. From now one we use the notation $f \lesssim g$ if there exists a constant $C > 0$ such that $f(y) \leq Cg(y)$ for all $y \in \Omega$.

Theorem 3 *Let $n \geq 3$, $\ell \in (0,1)$ and $\beta \geq \ell + 2m(1-\ell)$ with m such that $\frac{n}{2} < m \leq p_c(n,\ell)$. If $p > p_c(n,\ell)$, then there exists $\epsilon > 0$ such that for any initial data*

$$(u_0, u_1) \in \mathscr{D} = H^m(\mathbf{R}^n) \times H^{m-1}(\mathbf{R}^n), \qquad ||(u_0,u_1)||_{\mathscr{D}} \leq \epsilon,$$

there exists a unique weak solution $u \in C([0,\infty), H^m(\mathbf{R}^n)) \cap C^1([0,\infty), H^{m-1}(\mathbf{R}^n))$ to (1). Moreover, the solution satisfies the following estimates

$$\|u(t,\cdot)\|_{\dot{H}^k} \lesssim (1+t)^{-k(1-\ell)}\, ||(u_0,u_1)||_{\mathscr{D}}, \qquad k = 0, m, \tag{6}$$

$$\|u_t(t,\cdot)\|_{\dot{H}^k} \lesssim (1+t)^{k(\ell-1)-1}\, ||(u_0,u_1)||_{\mathscr{D}}, \qquad k = 0, m-1. \tag{7}$$

Example 1 Let $\ell = \frac{2}{3}$ in (1). For sufficiently large β and $p > 1 + \frac{12}{n}$, the conclusion of Theorem 2 holds for $n = 1, 2$, whereas Theorem 3 holds for $n = 3, 4, 5$.

3 Representation of Solutions to Parameter Dependent Cauchy Problems

Let $s \geq 0$ be a parameter. We need to solve a family of parameter dependent linear ($f(u) = 0$) Cauchy problems corresponding to (1):

$$\begin{cases} u_{tt}(t,x) - (1+t)^{-2\ell}\Delta u(t,x) + \frac{\beta}{1+t} u_t(t,x) = 0, \ t > s, \\ u(s,x) = g_1(s,x), \\ u_t(s,x) = g_2(s,x). \end{cases} \tag{8}$$

We begin by applying Fourier transform to the solution of (8), then we denote the partial Fourier transform of $u : \mathbf{R_0^+} \times \mathbf{R_0^+} \times \mathbf{R^n} \to \mathbf{C}$ with respect to x and its inverse, respectively, by

$$\mathscr{F}[u](t,s,\xi) = \hat{u}(t,s,\xi) = (2\pi)^{-n/2} \int_{\mathbf{R^n}} e^{-ix\xi} u(t,s,x)\, dx \tag{9}$$

and

$$\mathscr{F}^{-1}[\hat{u}](t,s,x) = u(t,s,x) = (2\pi)^{-n/2} \int_{\mathbf{R^n}} e^{ix\xi} \hat{u}(t,s,\xi)\, d\xi\,. \tag{10}$$

Following Ebert and Reissig [9], we make the change of variables $\tau = \frac{(1+t)^{1-\ell}}{1-\ell}|\xi|$ and $v(\tau, s) = \hat{u}(t, s, \xi)$. If $u(t, s, x)$ is the solution of (8) then $v(\tau, s)$ satisfies

$$\begin{cases} v''(\tau) + \frac{\beta-\ell}{(1-\ell)\tau} v'(\tau) + v(\tau) = 0 \\ v\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) = \hat{g_1}(s, \xi) \\ v'\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) = \frac{\hat{g_2}(s,\xi)}{|\xi|}. \end{cases} \tag{11}$$

Moreover, if we are looking for a solution in the product form $v(\tau, s) = \tau^{\rho} w(\tau, s)$, then $w(\tau, s)$ is a solution of the Bessel's differential equation of order $\pm\rho$:

$$\tau^2 w''(\tau) + \tau w'(\tau) + (\tau^2 - \rho^2) w(\tau) = 0, \tag{12}$$

where $\rho = \frac{1-\beta}{2(1-\ell)}$. We will use the set of Hankel functions, $\{H_{\rho}^{+}(\tau), H_{\rho}^{-}(\tau)\}$ to write the general solution of the ODE (12). First, according to Wirth's paper [23] we introduce an auxiliary function

$$\psi_{j,\gamma,\delta}(t, s, \xi) = |\xi|^j \begin{vmatrix} H_{\gamma}^{-}\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & H_{\gamma+\delta}^{-}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \\ H_{\gamma}^{+}\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & H_{\gamma+\delta}^{+}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \end{vmatrix}, \tag{13}$$

where j, γ, δ, s are real parameters. Since $H_{\gamma}^{\pm} = J_{\gamma} \pm i Y_{\gamma}$, we can rewrite it in the form

$$\psi_{j,\gamma,\delta}(t, s, \xi) = 2i|\xi|^j \begin{vmatrix} J_{\gamma}\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & J_{\gamma+\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \\ Y_{\gamma}\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & Y_{\gamma+\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \end{vmatrix} \tag{14}$$

if $\gamma, \gamma + \delta \in \mathbf{Z}$, or

$$\psi_{j,\gamma,\delta}(t, s, \xi) = 2i \csc(\gamma\pi)|\xi|^j \begin{vmatrix} J_{-\gamma}\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & J_{-\gamma-\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \\ (-1)^{\delta} J_{\gamma}\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & J_{\gamma+\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \end{vmatrix} \tag{15}$$

if $\gamma, \gamma + \delta \notin \mathbf{Z}$, where J_{γ}, Y_{γ} denote the Bessel functions of the first and second kind, respectively. We then determine the Fourier multipliers and the first order partial derivatives with respect to t to represent $\hat{u}$ and $\hat{u}_t$ in an explicit form.

Lemma 1 *Let $u = u(t, s, x)$ be the solution of (8). Then the partial Fourier transform of u with respect to x, $\hat{u}$, is represented by*

$$\hat{u}(t, s, \xi) = m_0(t, s, \xi)\hat{g_1}(s, \xi) + m_1(t, s, \xi)\hat{g_2}(s, \xi) \tag{16}$$

with Fourier multipliers and the first order partial derivatives with respect to t given by

$$\partial_t^j m_k = \frac{(-1)^k \pi i}{4(1-\ell)}(1+s)^{1+(\beta-1)/2}(1+t)^{(1-\beta)/2-j\ell}\psi_{1+j-k,\rho+k-1,1-j-k}, \tag{17}$$

where $\rho = \frac{1-\beta}{2(1-\ell)}$, $k, j = 0, 1$.

Proof Let $v_1(\tau) = \tau^\rho H_\rho^+(\tau)$, $v_2(\tau) = \tau^\rho H_\rho^-(\tau)$. Then the solution $v(\tau, s) = \hat{u}(t, s, \xi)$ is written as

$$v(\tau, s) = c_1(s, \xi)v_1(\tau) + c_2(s, \xi)v_2(\tau). \tag{18}$$

We solve the system

$$\begin{bmatrix} c_1(s,\xi) \\ c_2(s,\xi) \end{bmatrix} = \frac{1}{W(v_1(\tau_s), v_2(\tau_s))} \times \begin{bmatrix} v_2'(\tau_s) & -v_2(\tau_s) \\ -v_1'(\tau_s) & v_1(\tau_s) \end{bmatrix} \times \begin{bmatrix} v(\tau_s, s, \xi) \\ v'(\tau_s, s, \xi) \end{bmatrix}$$

with the initial data (11) on $\tau_s = \frac{(1+s)^{1-\ell}|\xi|}{1-\ell}$. Plugging $v_1'(\tau_s) = \tau_s^\rho H_{\rho-1}^+(\tau_s)$, $v_2'(\tau_s) = \tau_s^\rho H_{\rho-1}^-(\tau_s)$ into the system we get

$$\begin{bmatrix} c_1(s,\xi) \\ c_2(s,\xi) \end{bmatrix} = \frac{\tau_s^\rho}{W(v_1(\tau_s), v_2(\tau_s))} \times \begin{bmatrix} H_{\rho-1}^-(\tau_s) & -H_\rho^-(\tau_s) \\ -H_{\rho-1}^+(\tau_s) & H_\rho^+(\tau_s) \end{bmatrix} \times \begin{bmatrix} v(\tau_s, s, \xi) \\ v'(\tau_s, s, \xi) \end{bmatrix}.$$

Since the Wronskian $W(v_1(\tau_s), v_2(\tau_s)) = -\frac{4i}{\pi}\tau_s^{2\rho-1}(1+s)^{-\ell}|\xi|$ we simplify

$$\frac{\tau_s^\rho}{W(v_1(\tau_s), v_2(\tau_s))} = \frac{i\pi}{4}(1-\ell)^{\rho-1}(1+s)^{1-(1-\ell)\rho}|\xi|^{1-\rho}.$$

Multiplying by the factor τ^ρ we have

$$\tau^\rho \begin{bmatrix} c_1(s,\xi) \\ c_2(s,\xi) \end{bmatrix} = h(t, s, \xi) \times \begin{bmatrix} H_{\rho-1}^-(\tau_s) & -H_\rho^-(\tau_s) \\ -H_{\rho-1}^+(\tau_s) & H_\rho^+(\tau_s) \end{bmatrix} \times \begin{bmatrix} v(\tau_s, s, \xi) \\ v'(\tau_s, s, \xi) \end{bmatrix}$$

where

$$h(t, s, \xi) = \frac{i\pi}{4(1-\ell)}(1+s)^{1-(1-\ell)\rho}(1+t)^{(1-\ell)\rho}|\xi|.$$

Multiplying the first component by $H_\rho^+(\tau)$ and the second one by $H_\rho^-(\tau)$ and adding its components we obtain (17) for $j = 0$. To evaluate the first order partial derivatives with respect to t we use the chain rule, $\partial_t \tau = (1+t)^{-\ell}|\xi|$ and

$$\rho H_\rho^\pm(\tau) + \tau \partial_\tau (H_\rho^\pm(\tau)) = \tau H_{\rho-1}^\pm(\tau).$$

□

By Duhamel's principle, the solution of (1) is represented by

$$u(t,x) = u^0(t,0,x) + \int_0^t u^1(t,s,x)\,ds \tag{19}$$

where $u^0 = u^0(t,0,x)$ is the solution of the linear Cauchy problem

$$\begin{cases} u_{tt}(t,x) - (1+t)^{-2\ell}\Delta u(t,x) + \frac{\beta}{1+t} u_t(t,x) = 0, \ \ t > 0, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{20}$$

and $u^1 = u^1(t,s,x)$ is the solution of (8) for $g_1(s,x) = 0$ and $g_2(s,x) = |u(s,x)|^p$. We can give explicit representation formulas for these solutions using the convolution $*_{(x)}$[3] in the variable x.

Corollary 1 *Let $K_0(t,0,x)$ and $K_1(t,0,x)$ be fundamental solutions of (20), that is, the distributional solution with data*[4] $(u_0,u_1) = (\delta_0, 0)$ *and* $(u_0,u_1) = (0,\delta_0)$ *respectively. Then*

$$u^0(t,0,x) = K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x)\,, \tag{21}$$

where

$$K_k(t,0,x) = \mathscr{F}^{-1}[m_k](t,0,x),\ k = 0,1.$$

Corollary 2 *Let $K_1(t,s,x)$ be the fundamental solution of the linear Cauchy problem (8), that is, the distributional solution with data* $(u_0,u_1) = (0,\delta_0)$. *Then*

$$u^1(t,s,x) = K_1(t,s,x) *_{(x)} |u(s,x)|^p\,, \tag{22}$$

[3]Let $f, g : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}$ be two regular functions. We use the notation $f *_{(x)} g$ to indicate the convolution with respect to the space variable of the functions f and g, i.e.,

$$(f *_{(x)} g)(t,x) = \int_{\mathbf{R}^n} f(t,y)\cdot g(t,x-y)\,dy.$$

[4]Here δ_0 denotes the Dirac measure at point $x = 0$.

where

$$K_1(t,s,x) = -\frac{\pi i}{4(1-\ell)}(1+s)^{1+(\beta-1)/2}(1+t)^{(1-\beta)/2}\mathscr{F}^{-1}[\psi_{0,\rho,0}](t,s,x).$$

4 $L^2 - L^2$ Estimates

In Sect. 4 assume $\beta \neq 1$. In order to obtain an estimate of (16) we have to distinguish between large and small values for τ . We divide the extended phase space $\mathbf{R}_0^+ \times \mathbf{R}_0^+ \times \mathbf{R}^+$ into three zones. Given a fixed $N > 0$, we define the zone of high frequencies

$$Z_1 = \{(t,s,|\xi|) : |\xi| > N(1-\ell)(1+s)^{\ell-1}\};$$

and the zones of low frequencies

$$Z_2 = \{(t,s,|\xi|) : N(1-\ell)(1+t)^{\ell-1} \le |\xi| \le N(1-\ell)(1+s)^{\ell-1}\};$$

$$Z_3 = \{(t,s,|\xi) : |\xi| < N(1-\ell)(1+t)^{\ell-1}\},$$

splited by the boundary $\{(t,|\xi|) : (1+t)^{1-\ell}|\xi| = N(1-\ell)\}$.

By Plancherel's theorem $L^2 - L^2$ estimates of the solution correspond to L^∞ estimates of the corresponding Fourier multipliers.

Lemma 2 *Let* $\ell \in (0,1)$*,* $\gamma \neq 0$*,* $|\delta| \le j$ *and* $k \ge 0$*. It holds*

$$|\xi|^k|\psi_{j,\gamma,\delta}(t,s,\xi)| \lesssim \tag{23}$$
$$\begin{cases} (1+s)^{(\ell-1)/2}(1+t)^{(\ell-1)/2}|\xi|^{k+j-1} & \text{if } (t,\xi) \in Z_1 \\ (1+s)^{(1-\ell)(1/2-k-j)}(1+t)^{(\ell-1)/2} & \text{if } |\gamma| \le k+j-1/2 \text{ and } (t,\xi) \in Z_2 \\ (1+s)^{(\ell-1)|\gamma|}(1+t)^{(1-\ell)(|\gamma|-k-j)} & \text{if } |\gamma| > k+j-1/2 \text{ and } (t,\xi) \in Z_2 \\ (1+s)^{(\ell-1)|\gamma|}(1+t)^{(1-\ell)(|\gamma|-k-j)} & \text{if } (t,\xi) \in Z_3. \end{cases}$$

for all $s \ge 0$ *and* $t \ge s$.

Proof For any $N \in (0,1)$, the following properties hold:

$$|H_\gamma^\pm(\tau)| \lesssim \tau^{-\frac{1}{2}},\ \tau \in [N,\infty); \tag{24}$$

$$|H_\gamma^\pm(\tau)| \lesssim \tau^{-|\gamma|},\ \tau \in (0,N),\ \gamma \neq 0; \tag{25}$$

$$|J_\gamma(\tau)| \lesssim \tau^\gamma,\ \tau \in (0,N); \tag{26}$$

$$|Y_\gamma(\tau)| \lesssim \tau^{-\gamma},\ \tau \in (0,N),\ \gamma \neq 0. \tag{27}$$

To conclude the estimates in zones Z_1 and Z_2 we may use the representation (13), estimates (24) and (25), whereas in the zone Z_3 we use (14)–(15) and (26)–(27). □

In the next propositions we will get decay estimates for the $\dot{H}^k$ norm of solutions, in which the regularity of the data is influenced by the zone Z_1.

Proposition 1 *Let* $\ell \in (0,1)$, $0 < \beta \neq 1$ *and* $k \geq 0$. *If* $u_0 \in H^k(\mathbf{R^n})$ *then* $w(t,x) = K_0(t,0,x) *_{(x)} u_0(x)$ *satisfies the following estimates:*

$$||w(t,\cdot)||_{\dot{H}^k} \lesssim ||u_0||_{H^k} \times$$
$$\begin{cases} (1+t)^{(k+2)\ell-\beta-k-1} & \text{if } \beta \leq (2k+3)\ell - 2(k+1) \\ (1+t)^{(\ell-\beta)/2} & \text{if } (2k+3)\ell - 2(k+1) < \beta < (1-2k)\ell + 2k \\ (1+t)^{k\ell-k} & \text{if } \beta \geq (1-2k)\ell + 2k. \end{cases}$$

Proof In the zone Z_2, by Lemma 2 we obtain

$$|\xi|^k |\psi_{1,\rho-1,1}(t,0,\xi)| \lesssim$$
$$\begin{cases} (1+t)^{(2(k+2)\ell-\beta-2k-3)/2} & \text{if } \beta \leq (2k+3)\ell - 2(k+1) \\ (1+t)^{(\ell-1)/2} & \text{if } (2k+3)\ell - 2(k+1) < \beta < (1-2k)\ell + 2k \\ (1+t)^{(2k\ell+\beta-2k-1)/2} & \text{if } \beta \geq (1-2k)\ell + 2k, \end{cases}$$

then from (17) it follows that

$$|\xi|^k |m_0(t,0,\xi)| \lesssim$$
$$\begin{cases} (1+t)^{(k+2)\ell-\beta-k-1} & \text{if } \beta \leq (2k+3)\ell - 2(k+1) \\ (1+t)^{(\ell-\beta)/2} & \text{if } (2k+3)\ell - 2(k+1) < \beta < (1-2k)\ell + 2k \\ (1+t)^{k\ell-k} & \text{if } \beta \geq (1-2k)\ell + 2k. \end{cases}$$

In Z_3, we get the same estimate than in Z_2 since

$$|\xi|^k |m_0(t,0,\xi)| \lesssim \begin{cases} (1+t)^{(k+2)\ell-\beta-k-1} & \text{if } \beta \leq 2\ell - 1 \\ (1+t)^{k\ell-k} & \text{if } \beta > 2\ell - 1 \end{cases}$$

and in Z_1 we obtain

$$|\xi|^k |m_0(t,0,\xi)| \lesssim (1+t)^{(\ell-\beta)/2} |\xi|^k.$$

Thanks to $u_0 \in H^k(\mathbf{R^n})$, we have that $|\xi|^k m_0(t,0,\cdot)\hat{u_0} \in L^2(\mathbf{R^n})$ for all $t \geq 0$, thus the proof is concluded. □

Proposition 2 *Let $\ell \in (0,1)$, $0 < \beta \neq 1$ and $k \geq 0$. If $u_0 \in H^{k+1}(\mathbf{R^n})$ then the partial derivative with respect to time of $w(t,x) = K_0(t,0,x) *_{(x)} u_0(x)$ satisfies the estimates:*

$$||w_t(t,\cdot)||_{\dot{H}^k} \lesssim ||u_0||_{H^{k+1}} \times$$
$$\begin{cases} (1+t)^{(k+2)\ell-\beta-k-2} & \text{if } \beta \leq (2k+5)\ell - 2(k+2) \\ (1+t)^{-(\beta+\ell)/2} & \text{if } (2k+5)\ell - 2(k+2) < \beta < -(2k+1)\ell + 2(k+1) \\ (1+t)^{k(\ell-1)-1} & \text{if } \beta \geq -(2k+1)\ell + 2(k+1). \end{cases}$$

Proof In the zone Z_2, by Lemma 2 we obtain

$$|\xi|^k |\psi_{2,\rho-1,0}(t,0,\xi)| \lesssim$$
$$\begin{cases} (1+t)^{(2(k+3)\ell-\beta-2k-5)/2} & \text{if } \beta \leq (2k+5)\ell - 2(k+2) \\ (1+t)^{(\ell-1)/2} & \text{if } (2k+5)\ell - 2(k+2) < \beta < -(2k+1)\ell + 2(k+1) \\ (1+t)^{(2(k+1)\ell+\beta-2k-3)/2} & \text{if } \beta \geq -(2k+1)\ell + 2(k+1), \end{cases}$$

then from (17) it follows that

$$|\xi|^k |\partial_t m_0(t,0,\xi)| \lesssim$$
$$\begin{cases} (1+t)^{((k+2)\ell-\beta-k-2} & \text{if } \beta \leq (2k+5)\ell - 2(k+2) \\ (1+t)^{-(\beta+\ell)/2} & \text{if } (2k+5)\ell - 2(k+2) < \beta < -(2k+1)\ell + 2(k+1) \\ (1+t)^{k\ell-k-1} & \text{if } \beta \geq -(2k+1)\ell + 2(k+1). \end{cases}$$

In Z_3, we get the same estimate than in Z_2 since

$$|\xi|^k |\partial_t m_0(t,0,\xi)| \lesssim \begin{cases} (1+t)^{(k+2)\ell-\beta-k-2} & \text{if } \beta \leq 2\ell - 1 \\ (1+t)^{k\ell-k-1} & \text{if } \beta > 2\ell - 1 \end{cases}$$

and in Z_1 we obtain

$$|\xi|^k |\partial_t m_0(t,0,\xi)| \lesssim (1+t)^{-(\beta+\ell)/2} |\xi|^{k+1}.$$

Thanks to $u_0 \in H^{k+1}(\mathbf{R^n})$, we have that $|\xi|^k \partial_t m_0(t,0,\cdot)\hat{u_0} \in L^2(\mathbf{R^n})$ for all $t \geq 0$, thus the proof is concluded. □

Proposition 3 *Let $\ell \in (0,1)$, $0 < \beta \neq 1$ and $k \geq 0$. If $g_1 \doteq 0$ and $g_2(s,\cdot) \in H^{[k-1]_+}(\mathbf{R^n})$ then the solution u of the problem (8) satisfies estimates:*

$$||u(t,s,\cdot)||_{\dot{H}^k} \lesssim ||g_2(s,\cdot)||_{L^2} \times \begin{cases} (1+s)^{\beta}(1+t)^{k\ell-\beta-k+1} & \text{if } \beta < 1 \\ (1+s)(1+t)^{k\ell-k} & \text{if } \beta > 1 \end{cases}$$

for $k \in [0, 1/2]$ *and*

$$||u(t,s,\cdot)||_{\dot{H}^k} \lesssim ||g_2(s,\cdot)||_{L^2} \times$$
$$\begin{cases} (1+s)^{\beta}(1+t)^{k\ell-\beta-k+1} & \textit{if } \beta \le (2k-1)\ell + 2(1-k) \\ (1+s)^{\beta}(1+t)^{(\ell-\beta)/2} & \textit{if } (2k-1)\ell + 2(1-k) < \beta < 1 \\ (1+s)(1+t)^{(\ell-\beta)/2} & \textit{if } 1 < \beta < (1-2k)\ell + 2k \\ (1+s)(1+t)^{k\ell-k} & \textit{if } \beta \ge (1-2k)\ell + 2k \end{cases}$$

for $k \in (1/2, 1]$ *and*

$$||u(t,s,\cdot)||_{\dot{H}^k} \lesssim \Big(||g_2(s,\cdot)||_{L^2} + (1+s)^{(1-\ell)(k-1)}||g_2(s,\cdot)||_{\dot{H}^{k-1}}\Big) \times$$
$$\begin{cases} (1+s)^{\beta}(1+t)^{k\ell-\beta-k+1} & \textit{if } \beta \le (2k-1)\ell + 2(1-k) \\ (1+s)^{\beta}(1+t)^{(\ell-\beta)/2} & \textit{if } (2k-1)\ell + 2(1-k) < \beta < 1 \\ (1+s)(1+t)^{(\ell-\beta)/2} & \textit{if } 1 < \beta < (1-2k)\ell + 2k \\ (1+s)(1+t)^{k\ell-k} & \textit{if } \beta \ge (1-2k)\ell + 2k \end{cases}$$

for $k > 1$.

Proof In the following estimates we use Lemma 2 and (17). In the zone Z_2, we obtain

$$|\xi|^k|\psi_{0,\rho,0}(t,s,\xi)| \lesssim \begin{cases} (1+s)^{(\beta-1)/2}(1+t)^{(1-\beta)/2+k(\ell-1)} & \text{if } \beta < 1 \\ (1+s)^{(1-\beta)/2}(1+t)^{(\beta-1)/2+k(\ell-1)} & \text{if } \beta > 1 \end{cases}$$

for $k \in [0, 1/2]$ and

$$|\xi|^k|\psi_{0,\rho,0}(t,s,\xi)| \lesssim$$
$$\begin{cases} (1+s)^{(\beta-1)/2}(1+t)^{(2k\ell-\beta-2k+1)/2} & \text{if } \beta \le (2k-1)\ell + 2(1-k) \\ (1+s)^{(\beta-1)/2}(1+t)^{(\ell-1)/2} & \text{if } (2k-1)\ell + 2(1-k) < \beta < 1 \\ (1+s)^{(1-\beta)/2}(1+t)^{(\ell-1)/2} & \text{if } 1 < \beta < (1-2k)\ell + 2k \\ (1+s)^{(1-\beta)/2}(1+t)^{(2k\ell+\beta-2k-1)/2} & \text{if } \beta \ge (1-2k)\ell + 2k \end{cases}$$

for $k > 1/2$. Then

$$|\xi|^k|m_1(t,s,\xi)| \lesssim \begin{cases} (1+s)^{\beta}(1+t)^{k\ell-\beta-k+1} & \text{if } \beta < 1 \\ (1+s)(1+t)^{k\ell-k} & \text{if } \beta > 1 \end{cases}$$

for $k \in [0, 1/2]$ and

$$|\xi|^k |m_1(t,s,\xi)| \lesssim \begin{cases} (1+s)^\beta (1+t)^{k\ell-\beta-k+1} & \text{if } \beta \le (2k-1)\ell + 2(1-k) \\ (1+s)^\beta (1+t)^{(\ell-\beta)/2} & \text{if } (2k-1)\ell + 2(1-k) < \beta < 1 \\ (1+s)(1+t)^{(\ell-\beta)/2} & \text{if } 1 < \beta < (1-2k)\ell + 2k \\ (1+s)(1+t)^{k\ell-k} & \text{if } \beta \ge (1-2k)\ell + 2k \end{cases}$$

for $k > 1/2$. In the zone Z_3 we have the same estimates for all k since

$$|\xi|^k |m_1(t,s,\xi)| \lesssim \begin{cases} (1+s)^\beta (1+t)^{k\ell-\beta-k+1} & \text{if } \beta < 1 \\ (1+s)(1+t)^{k\ell-k} & \text{if } \beta > 1. \end{cases}$$

In the zone Z_1 we may estimate

$$|\xi|^k |m_1(t,s,\xi)| \lesssim |\xi|^{k-1} (1+s)^{(\beta+\ell)/2} (1+t)^{(\ell-\beta)/2}.$$

Now we have to analyze two cases. If $k \le 1$ we can estimate

$$|\xi|^{k-1} \lesssim (1+s)^{(1-\ell)(1-k)}$$

so that

$$|\xi|^k |m_1(t,s,\xi)| \lesssim (1+s)^{(\beta+\ell)/2+(1-\ell)(1-k)} (1+t)^{(\ell-\beta)/2}$$

and we obtain the same estimates that as in Z_2. However, if $k > 1$ it holds

$$|\xi|^k |m_1(t,s,\xi)| \lesssim (1+s)^{(1-\ell)(k-1)} |\xi|^{k-1} \times \begin{cases} (1+s)^\beta (1+t)^{k\ell-\beta-k+1} & \text{if } \beta \le (2k-1)\ell + 2(1-k) \\ (1+s)^\beta (1+t)^{(\ell-\beta)/2} & \text{if } (2k-1)\ell + 2(1-k) < \beta < 1 \\ (1+s)(1+t)^{(\ell-\beta)/2} & \text{if } 1 < \beta < (1-2k)\ell + 2k \\ (1+s)(1+t)^{k\ell-k} & \text{if } \beta \ge (1-2k)\ell + 2k. \end{cases}$$

Thanks to $g_2(s,\cdot) \in H^{[k-1]_+}(\mathbf{R^n})$, we have that $|\xi|^k m_1(t,s,\cdot)\hat{g_2}(s,\cdot) \in L^2(\mathbf{R^n})$ for all $s \ge 0$ and $t \ge s$, thus the proof is concluded. □

Proposition 4 *Let $\ell \in (0,1)$ and $0 < \beta \ne 1$ and $k \ge 0$. If $g_1 \doteq 0$ and $g_2(s,\cdot) \in H^k(\mathbf{R^n})$ then the partial derivative with respect to time of the solution u of the* *problem (8) satisfies the following estimates:*

$$\|u_t(t,s,\cdot)\|_{\dot H^k} \lesssim \begin{cases} (1+s)^{\beta}(1+t)^{k(\ell-1)-\beta}\left(\|g_2(s,\cdot)\|_{L^2}+(1+s)^{(1-\ell)k}\|g_2(s,\cdot)\|_{\dot H^k}\right) \\ \textit{if } \beta \le (1+2k)\ell-2k\,, \\ (1+s)^{\beta}(1+t)^{-(\beta+\ell)/2}\left(\|g_2(s,\cdot)\|_{L^2}+(1+s)^{(\ell-\beta)/2}\|g_2(s,\cdot)\|_{\dot H^k}\right) \\ \textit{if } (1+2k)\ell-2k<\beta\le\ell\,, \\ (1+s)^{\beta}(1+t)^{-(\beta+\ell)/2}\|g_2(s,\cdot)\|_{L^2\cap\dot H^k} \\ \textit{if } \ell<\beta<1\,, \\ (1+s)(1+t)^{-(\beta+\ell)/2}\|g_2(s,\cdot)\|_{L^2\cap\dot H^k} \\ \textit{if } 1<\beta\le 2-\ell\,, \\ (1+s)(1+t)^{-(\beta+\ell)/2}\left(\|g_2(s,\cdot)\|_{L^2}+(1+s)^{(\beta+\ell-2)/2}\|g_2(s,\cdot)\|_{\dot H^k}\right) \\ \textit{if } 2-\ell<\beta<-(2k+1)\ell+2(k+1)\,, \\ (1+s)(1+t)^{k(\ell-1)-1}\left(\|g_2(s,\cdot)\|_{L^2}+(1+s)^{(1-\ell)k}\|g_2(s,\cdot)\|_{\dot H^k}\right) \\ \textit{if } \beta\ge-(2k+1)\ell+2(k+1)\,. \end{cases}$$

Remark 6 If $k=0$ these estimates are reduced to

$$\|u_t(t,s,\cdot)\|_{L^2} \lesssim \|g_2(s,\cdot)\|_{L^2}\times\begin{cases} (1+s)^{\beta}(1+t)^{-\beta} & \text{if } \beta\le\ell \\ (1+s)^{\beta}(1+t)^{-(\beta+\ell)/2} & \text{if } \ell<\beta<1 \\ (1+s)(1+t)^{-(\beta+\ell)/2} & \text{if } 1<\beta<2-\ell \\ (1+s)(1+t)^{-1} & \text{if } \beta\ge 2-\ell. \end{cases}$$

Proof In the following estimates we use Lemma 2 and (17). First, in the zone Z_2,

$$|\xi|^k|\psi_{1,\rho,-1}(t,s,\xi)| \lesssim \begin{cases} (1+s)^{(\beta-1)/2}(1+t)^{(2(k+1)\ell-\beta-2k-1)/2} \\ \text{if } \beta\le(2k+1)\ell-2k\,, \\ (1+s)^{(\ell-1)(k+1/2)}(1+t)^{(\ell-1)/2} \\ \text{if } (2k+1)\ell-2k<\beta<-(2k+1)\ell+2(k+1)\,, \\ (1+s)^{(1-\beta)/2}(1+t)^{(2(k+1)\ell+\beta-2k-3)/2} \\ \text{if } \beta\ge-(2k+1)\ell+2(k+1)\,, \end{cases}$$

and

$$|\xi|^k|\partial_t m_1(t,s,\xi)| \lesssim \begin{cases} (1+s)^{\beta}(1+t)^{k\ell-\beta-k} \\ \text{if } \beta\le(2k+1)\ell-2k\,, \\ (1+s)^{((2k+1)\ell+\beta-2k)/2}(1+t)^{-(\beta+\ell)/2} \\ \text{if } (2k+1)\ell-2k<\beta<-(2k+1)\ell+2(k+1)\,, \\ (1+s)(1+t)^{k\ell-k-1} \\ \text{if } \beta\ge-(2k+1)\ell+2(k+1)\,. \end{cases}$$

Then, in the zone Z_3,

$$|\xi|^k|\partial_t m_1(t,s,\xi)| \lesssim \begin{cases} (1+s)^\beta(1+t)^{k\ell-\beta-k} & \text{if } \beta < 1 \\ (1+s)(1+t)^{k\ell-k-1} & \text{if } \beta > 1. \end{cases}$$

Hence, in $Z_2 \cup Z_3$,

$$|\xi|^k|\partial_t m_1(t,s,\xi)| \lesssim$$
$$\begin{cases} (1+s)^\beta(1+t)^{k\ell-\beta-k} & \text{if } \beta \le (2k+1)\ell - 2k \\ (1+s)^\beta(1+t)^{-(\beta+\ell)/2} & \text{if } (2k+1)\ell - 2k < \beta < 1 \\ (1+s)(1+t)^{-(\beta+\ell)/2} & \text{if } 1 < \beta < -(2k+1)\ell + 2(k+1) \\ (1+s)(1+t)^{k\ell-k-1} & \text{if } \beta \ge -(2k+1)\ell + 2(k+1). \end{cases}$$

Finally, in the zone Z_1,

$$|\xi|^k|\partial_t m_1(t,s,\xi)| \lesssim (1+s)^{(\beta+\ell)/2}(1+t)^{-(\beta+\ell)/2}|\xi|^k.$$

Thanks to $g_2(s,\cdot) \in H^k(\mathbf{R^n})$, we have that $|\xi|^k\partial_t m_1(t,s,\cdot)\hat{g_2}(s,\cdot) \in L^2(\mathbf{R^n})$ for all $s \ge 0$ and $t \ge s$, thus the proof is concluded. □

5 Non-existence via Test Function Method

The following proof is a modified version of Theorem 3 in [4]:

Proof (Theorem 1) By applying the change of variable

$$v(\tau,x) = u(t,x), \qquad 1+\tau = \frac{(1+t)^{1-\ell}}{1-\ell},$$

the Cauchy problem (1) takes the form

$$\begin{cases} v_{\tau\tau} - \Delta v + \frac{\mu}{(1+\tau)}v_\tau = f(\tau)|v|^p, & \tau \ge \tau_0,\ x \in \mathbf{R^n} \\ v(\tau_0,x) = u_0(x), & x \in \mathbf{R^n}, \\ v_\tau(\tau_0,x) = u_1(x), & x \in \mathbf{R^n}, \end{cases} \tag{28}$$

with $f(\tau) = [(1-\ell)(1+\tau)]^{\frac{2\ell}{1-\ell}}$, $\mu = \frac{\beta-\ell}{1-\ell}$ and $\tau_0 = \frac{\ell}{1-\ell}$. In this way, it is sufficient to prove a non-existence of global (in time) result for (28).

We fix a nonnegative, decreasing test function $\varphi \in \mathscr{C}_c^\infty([0,\infty))$ with $\varphi = 1$ in $[0, 1/2]$ and $\text{supp}(\varphi) \subset [0,1]$, and a nonnegative, radial test function $\psi \in \mathscr{C}_c^\infty(\mathbf{R^n})$, such that $\psi = 1$ in the ball $B_{1/2}$, and $\text{supp}(\psi) \subset B_1$.

We also assume $\psi(x) \leq \psi(y)$ when $|x| \geq |y|$. Here B_r denotes the ball of radius r, centered at the origin. We may assume that

$$\varphi^{-\frac{p'}{p}} \left(|\varphi''|^{p'} + \left| \partial_\tau \left(\frac{\mu}{1+\tau} \varphi_R \right) \right|^{p'} \right), \qquad \psi^{-\frac{p'}{p}} |\Delta \psi|^{p'}, \qquad \text{are bounded,} \tag{29}$$

where $p' = p/(p-1)$. Then, for $R \geq 1$ and $R^{-1}\tau_0 \leq \frac{1}{2}$, we define:

$$\varphi_R(\tau) = \varphi(R^{-1}\tau), \quad \psi_R(x) = \psi(R^{-1}x). \tag{30}$$

Let us assume that v is the global (in time) weak solution to (28). Let $R > 0$, and also assume that $R \leq T$, if u is a local solution in $[0, T] \times \mathbf{R^n}$. Integrating by parts, and recalling that $\varphi_R(\tau) = 1$ for $\tau \in [0, \frac{1}{2}]$, we obtain

$$\begin{aligned} I_R = & \int_{\tau_0}^{\infty} \int_{\mathbf{R^n}} v \left(\varphi_R'' \psi_R + \varphi_R \Delta \psi_R - \partial_\tau \left(\frac{\mu}{1+\tau} \varphi_R \right) \psi_R \right) dx d\tau \\ & - \int_{\mathbf{R^n}} \psi_R(x) \left(\frac{\mu}{1+\tau_0} u_0(x) + u_1(x) \right) dx, \end{aligned}$$

where

$$I_R = \int_{\tau_0}^{\infty} \int_{\mathbf{R^n}} f(\tau) |v|^p \varphi_R \psi_R \, dx d\tau.$$

We may now apply Young inequality to estimate

$$\int_{\tau_0}^{\infty} \int_{\mathbf{R^n}} |v| \left(|\varphi_R''| \, \psi_R + \varphi_R \, |\Delta \psi_R| + \psi_R \left| \partial_\tau \left(\frac{\mu}{1+\tau} \varphi_R \right) \right| \right) dx d\tau \leq \frac{1}{p} I_R +$$
$$\frac{1}{p'} \int_{\tau_0}^{\infty} \int_{\mathbf{R^n}} (\varphi_R \psi_R f)^{-\frac{p'}{p}} \left(|\varphi_R''| \, \psi_R + \varphi_R \, |\Delta \psi_R| + \psi_R \left| \partial_\tau \left(\frac{\mu}{1+\tau} \varphi_R \right) \right| \right)^{p'} dx d\tau.$$

Due to

$$\varphi_R^{(k)}(\tau) = R^{-k} (\varphi^{(k)})(R^{-1}\tau), \qquad \Delta \psi_R(x) = R^{-2} (\Delta \psi)(R^{-1}x),$$

recalling (29), we may estimate

$$\int_{\tau_0}^{\infty} \int_{\mathbf{R^n}} (\varphi_R \psi_R f(\tau))^{-\frac{p'}{p}} |\varphi_R'' \psi_R|^{p'} \, dx d\tau \leq C \, R^{-\frac{2\ell}{1-\ell} \frac{p'}{p} - 2p' + n + 1},$$
$$\int_{\tau_0}^{\infty} \int_{\mathbf{R^n}} (\varphi_R \psi_R f(\tau))^{-\frac{p'}{p}} |\varphi_R \Delta \psi_R|^{p'} \, dx d\tau \leq C \, R^{-\frac{2\ell}{1-\ell} \frac{p'}{p} - 2p' + n + 1},$$

$$\int_{\tau_0}^{\infty}\int_{\mathbf{R^n}} (\varphi_R \psi_R f(\tau))^{-\frac{p'}{p}} \left|\partial_\tau\left(\frac{\mu}{1+\tau}\varphi_R\right)\right|^{p'} dx d\tau \leq C\, R^{-\frac{2\ell}{1-\ell}\frac{p'}{p}-2p'+n+1}.$$

Summarizing, we proved that

$$\frac{1}{p'} I_R \leq C\, R^{-\frac{2\ell}{1-\ell}\frac{p'}{p}-2p'+n+1} - \int_{\mathbf{R^n}} \psi_R(x)\left(\frac{\mu}{1+\tau_0}u_0(x)+u_1(x)\right) dx.$$

Recalling that $R^{-1}\tau_0 \leq \frac{1}{2}$ and assumption (3), there exists $c > 0$ such that

$$\begin{aligned}\int_{\mathbf{R^n}} \psi_R(x)\left(\frac{\mu u_0(x)}{1+\tau_0}+u_1(x)\right) dx &\geq \frac{\varepsilon}{R}\int_{\mathbf{R^n}} (1+|x|)^{-\frac{n}{2}}(\log(e+|x|)^{-1}\,\psi_R(x)dx \\ &\geq c\varepsilon\, R^{n-\frac{n}{2}-1}(\log(e+R))^{-1}.\end{aligned}$$

As a consequence:

$$\begin{aligned} I_R &\leq C\, R^{-\frac{2\ell}{1-\ell}\frac{p'}{p}-2p'+n+1} - c\varepsilon\, R^{n-\frac{n}{2}-1}(\log(e+R))^{-1} \\ &= R^n\left(C\, R^{-\frac{2\ell}{1-\ell}\frac{p'}{p}-2p'+1} - c\varepsilon\, R^{-\frac{n}{2}-1}(\log(e+R))^{-1}\right).\end{aligned}$$

Assume, by contradiction, that the solution v is global.

In the subcritical case $p < 1+\frac{4}{n(1-\ell)}$, it follows that

$$\frac{2\ell}{1-\ell}\frac{p'}{p}+2p'-1 > 1+\frac{n}{2}$$

and $I_R < 0$, for any sufficiently large R, and this contradicts the fact that $I_R \geq 0$. Therefore, v cannot be a global (in time) solution and this concludes the proof. □

6 Global Existence Results

By Duhamel's principle, a function $u \in X$, where X is a suitable space, is a solution to (1) if, and only if, it satisfies the equality

$$u(t,x) = u^0(t,x) + \int_0^t K_1(t,s,x) *_{(x)} f(u(s,x))\, ds\,, \qquad \text{in } X, \tag{31}$$

with $f(u(s,x)) = |u|^p$, u^0 and $K_1(t,s,x) *_{(x)} f$ are the solutions to the linear Cauchy problem (20) and (8) with $g_1 \equiv 0$, respectively. The proof of our global existence results is based on the following scheme: We define an appropriate data function space

$$\mathscr{D} \doteq H^{s+1}(\mathbf{R^n}) \times H^s(\mathbf{R^n}),$$

and an evolution space for solutions

$$X(T) \doteq C([0,T], H^{s+1}(\mathbf{R^n}) \cap C^1([0,T], H^s(\mathbf{R^n})),$$

with $s = 0$ in Theorem 2 and $s = m$ in Theorem 3, equipped with a norm related to the estimates of solutions to the linear Cauchy problem (20) such that

$$\|u^0\|_X \le C\,\|(u_0, u_1)\|_{\mathscr{D}}.$$

For any $u \in X$, we define the operator P by

$$P : u \in X(T) \to Pu(t,x) := u^0(t,x) + Fu(t,x),$$

with

$$Fu(t,x) \doteq \int_0^t K_1(t,s,x) *_{(x)} f(u(s,x))\,ds\,,$$

then we prove the estimates

$$\|Pu\|_X \le C\,\|(u_0,u_1)\|_{\mathscr{D}} + C_1(t)\|u\|_X^p\,,$$

$$\|Pu - Pv\|_X \le C_2(t)\|u-v\|_X\big(\|u\|_X^{p-1} + \|v\|_X^{p-1}\big)\,.$$

The estimates for the image Pu allow us to apply Banach's fixed point theorem. In this way we get simultaneously a unique solution to $Pu = u$ locally in time for large data and globally in time for small data. To prove the local (in time) existence we use that $C_1(t), C_2(t)$ tend to zero as t goes to zero, whereas to prove the global (in time) existence we use $C_1(t) \le C$ and $C_2(t) \le C$ for all $t \ge 0$.

Proof *(Theorem 2)* We follow closely [3]. We define the space

$$X(T) \doteq C([0,T), H^1) \cap C^1([0,T], L^2),$$

equipped with the norm

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]} \Big\{\|u(t,\cdot)\|_{L^2} + (1+t)^{1-\ell}\|\nabla u(t,\cdot)\|_{L^2} + (1+t)\|u_t(t,\cdot)\|_{L^2}\Big\} \tag{32}$$

if $\beta \geq 2-\ell$, or

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]} \Big\{ \|u(t,\cdot)\|_{L^2} + (1+t)^{\frac{\beta-\ell}{2}} \Big(\|\nabla u(t,\cdot)\|_{L^2} + \|u(t,\cdot)\|_{\dot{H}^{\frac{\beta-\ell}{2(1-\ell)}}} \Big) + \\ + (1+t)^{\frac{\beta+\ell}{2}} \|u_t(t,\cdot)\|_{L^2} \Big\},$$

if $\beta \in (1, 2-\ell)$. In the following we only verify how to prove the global (in time) existence in time. Thanks to Propositions 1–4, $u^0 \in X(T)$ and it satisfies

$$\|u^0\|_X \leq C\, \|(u_0, u_1)\|_{\mathscr{D}}.$$

It remains to show the estimates

$$\|Fu\|_X \leq C\|u\|_X^p\,, \tag{33}$$

$$\|Fu - Fv\|_X \leq C\|u-v\|_X \big(\|u\|_X^{p-1} + \|v\|_X^{p-1}\big)\,. \tag{34}$$

Let us prove (33). Applying Proposition 3 we have for $k \in [0,1]$ the estimates

$$\|Fu(t,\cdot)\|_{\dot{H}^k} \lesssim \int_0^t (1+s)(1+t)^{-\min\{k(1-\ell), \frac{\beta-\ell}{2}\}} \||u(s,\cdot)|^p\|_{L^2} ds.$$

If $\beta \geq 2-\ell$, taking into account of the norm in (32), by using Gagliardo-Nirenberg inequality we may estimate

$$\||u(s,\cdot)|^p\|_{L^2} = \|u(s,\cdot)\|_{L^{2p}}^p \lesssim \|u(s,\cdot)\|_{L^2}^{(1-\theta)p} \|\nabla u(s,\cdot)\|_{L^2}^{\theta p} \lesssim (1+s)^{(\ell-1)\theta p} \|u\|_{X(T)}^p,$$

with $\theta p = \frac{n(p-1)}{2}$ and $p \leq \frac{n}{[n-2]_+}$.

If $\beta \in (1, 2-\ell)$, we use the fractional Sobolev embedding to estimate

$$\||u(s,\cdot)|^p\|_{L^2} \lesssim ||u(s,\cdot)||_{\dot{H}^{\bar{k}}}^p, \qquad \bar{k} = \frac{n}{2}\left(1 - \frac{1}{p}\right). \tag{35}$$

In space dimension $n=1$, thanks to $u \in X(T)$ and $\bar{k} < \frac{1}{2} < \frac{\beta-\ell}{2(1-\ell)}$, by Gagliardo-Nirenberg inequality we conclude that

$$||u(s,\cdot)||_{\dot{H}^{\bar{k}}} \lesssim \|u(s,\cdot)\|_{L^2}^{(1-\theta)} \|u(s,\cdot)\|_{\dot{H}^{\frac{\beta-\ell}{2(1-\ell)}}}^{\theta} \lesssim (1+s)^{(\ell-1)\bar{k}} \|u\|_{X(T)},$$

with $\bar{k} = \theta \frac{\beta-\ell}{2(1-\ell)}$.

Now, let us consider space dimension $n \geq 2$. If $\beta \geq \ell + 2\bar{k}(1-\ell)$, using again Gagliardo-Nirenberg inequality with $\theta \frac{\beta-\ell}{2(1-\ell)} = \bar{k}$ we conclude that

$$||u(s,\cdot)||_{\dot{H}^{\bar{k}}} \lesssim \|u(s,\cdot)\|_{L^2}^{(1-\theta)} \|u(s,\cdot)\|^{\theta}_{\dot{H}^{\frac{\beta-\ell}{2(1-\ell)}}} \lesssim (1+s)^{(\ell-1)\bar{k}} \|u\|_{X(T)}.$$

Therefore, for $\beta \geq \ell + 2\bar{k}(1-\ell)$ we have

$$\begin{aligned}\|Fu(t,\cdot)\|_{\dot{H}^k} &\lesssim (1+t)^{-\min\{k(1-\ell),\frac{\beta-\ell}{2}\}} \int_0^t (1+s)^{1+(\ell-1)\frac{n(p-1)}{2}} ds \|u\|^p_{X(T)} \\ &\lesssim (1+t)^{-\min\{k(1-\ell),\frac{\beta-\ell}{2}\}} \|u\|^p_{X(T)},\end{aligned}$$

for $1 + \frac{4}{n(1-\ell)} < p \leq 1 + \frac{\beta-\ell}{n(1-\ell)-\beta+\ell}$ and for all $k \in [0,1]$. Now, let us consider the case

$$\beta \in \left[\ell + \frac{4n(1-\ell)}{n(1-\ell)+4}, \ell + 2\bar{k}(1-\ell)\right],$$

which is not empty for $p > p_c(n,\ell)$. The condition $\beta < \ell + 2\bar{k}(1-\ell)$ is equivalent to $p > 1 + \frac{\beta-\ell}{n(1-\ell)-\beta+\ell}$. Hence, using that $u \in X(T)$ and Gagliardo-Nirenberg inequality (now in different basis) it follows for $p \leq \frac{n}{[n-2]_+}$ that

$$||u(s,\cdot)||_{\dot{H}^{\bar{k}}} \lesssim \|\nabla u(s,\cdot)\|^{\theta}_{L^2} \|u(s,\cdot)\|^{1-\theta}_{\dot{H}^{\frac{\beta-\ell}{2(1-\ell)}}} \lesssim (1+s)^{\frac{(\ell-\beta)}{2}} \|u\|_{X(T)}.$$

Again we conclude

$$\begin{aligned}\|Fu(t,\cdot)\|_{\dot{H}^k} &\lesssim (1+t)^{-\min\{k(1-\ell),\frac{\beta-\ell}{2}\}} \int_0^t (1+s)^{1+\frac{(\ell-\beta)p}{2}} ds \|u\|^p_{X(T)} \\ &\lesssim (1+t)^{-\min\{k(1-\ell),\frac{\beta-\ell}{2}\}} \|u\|^p_{X(T)},\end{aligned}$$

for $p > 1 + \frac{\beta-\ell}{n(1-\ell)-\beta+\ell} \geq \frac{4}{\beta-\ell}$ thanks to $\beta \geq \ell + \frac{4n(1-\ell)}{n(1-\ell)+4}$.

Similarly we conclude

$$\|\partial_t Fu(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\min\{1,\frac{\ell+\beta}{2}\}} \|u\|^p_{X(T)},$$

for $p > p_c(n,\ell)$.

The considerations to prove (34) are the following: due to the Mean Value Theorem we have

$$||u|^p - |v|^p| \leq C_0 |u-v| \big(|u|^{p-1} + |v|^{p-1}\big),$$

and using Hölder's inequality we get

$$\| |u|^p - |v|^p \|_{L^2} \le C_0 \|u - v\|_{L^{2p}} \big(\|u\|_{L^{2p}}^{p-1} + \|v\|_{L^{2p}}^{p-1} \big).$$

Now we proceed as in the proof of (33) to derive (34) and the proof of Theorem 2 is concluded. □

Remark 7 By using an idea from [19], one may include $p = 1 + \frac{4}{n(1-\ell)}$ in the statements of Theorem 2. We only sketch the idea for large $\beta > 0$. By using the change of variable in Remark 5, it is sufficient to discuss the global existence of small data solutions to the Cauchy problem

$$\begin{cases} v_{\tau\tau} - \Delta v + \frac{\beta-\ell}{(1-\ell)(1+\tau)} v_\tau = g(v), & \tau \ge \frac{\ell}{1-\ell},\ x \in \mathbf{R}^n, \\ v(\frac{\ell}{1-\ell}, x) = u_0(x), & x \in \mathbf{R}^n, \\ v_\tau(\frac{\ell}{1-\ell}, x) = u_1(x), & x \in \mathbf{R}^n, \end{cases} \tag{36}$$

with $g(v) = [(1-\ell)(1+\tau)]^{\frac{2\ell}{1-\ell}} |v|^p$. Now, for $\beta > 0$ sufficiently large, we may use the following estimates for solutions v^0 to the linear problem associate to (36):

For initial data in energy space $H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ we have (see Theorem 4 in [3])

$$\|v^0(\tau, \cdot)\|_{\dot{H}^k} \lesssim (1+\tau)^{-k} \|(u_0, u_1)\|_{H^1 \times L^2}, \quad k \in [0, 1],$$

whereas for a parameter dependent initial data $(v^0(s, x), v^0_\tau(s, x)) = (0, u_1(x))$ we have (see Remark 3 in [3]):

$$\|v^0(\tau, \cdot)\|_{\dot{H}^k} \lesssim (1+\tau)^{-\frac{n}{2}-k} (1+s) \Big(\|u_1\|_{L^1} + (1+s)^{\frac{n}{2}} \|u_1\|_{L^2} \Big) \quad k \in [0, 1].$$

Hence, following as in the proof of Theorem 2 we may estimate

$$\|Fv(\tau, \cdot)\|_{\dot{H}^k} \lesssim (1+\tau)^{-\frac{n}{2}-k} \int_0^t (1+s) \Big(\|g(v(s, \cdot)\|_{L^1} + (1+s)^{\frac{n}{2}} \|g(v(s, \cdot))\|_{L^2} \Big) ds.$$

Assuming the a priori estimate for solutions to (36)

$$\|v(\tau, \cdot)\|_{\dot{H}^k} \lesssim (1+\tau)^{-k} \|v\|_{X(T)}, \quad k \in [0, 1],$$

by Gagliardo-Nirenberg inequality we may estimate

$$\begin{aligned} \|g(v(s, \cdot))\|_{L^1} = (1+s)^{\frac{2\ell}{1-\ell}} \|v(s, \cdot)\|_{L^p}^p &\lesssim (1+s)^{\frac{2\ell}{1-\ell}} \|v(s, \cdot)\|_{L^2}^{(1-\theta)p} \|\nabla v(s, \cdot)\|_{L^2}^{\theta p} \\ &\lesssim (1+s)^{\frac{2\ell}{1-\ell} - p\theta} \|v\|_{X(T)}^p, \end{aligned}$$

with $\theta p = \frac{n(p-2)}{2}$, and

$$\|g(v(s,\cdot))\|_{L^2} = (1+s)^{\frac{2\ell}{1-\ell}}\|v(s,\cdot)\|_{L^{2p}}^p \lesssim (1+s)^{\frac{2\ell}{1-\ell}}\|v(s,\cdot)\|_{L^2}^{(1-\theta)p}\|\nabla v(s,\cdot)\|_{L^2}^{\theta p}$$
$$\lesssim (1+s)^{\frac{2\ell}{1-\ell}-p\theta}\|v\|_{X(T)}^p,$$

with $\theta p = \frac{n(p-1)}{2}$ and $p \le \frac{n}{[n-2]_+}$. Therefore, we conclude that

$$\|Fv(\tau,\cdot)\|_{\dot H^k} \lesssim (1+\tau)^{-\frac{n}{2}-k}\int_{\frac{\ell}{1-\ell}}^{\tau}(1+s)^{1+\frac{2\ell}{1-\ell}-\frac{n(p-2)}{2}}ds\|v\|_{X(T)}^p \lesssim (1+\tau)^{-k}\|v\|_{X(T)}^p,$$

thanks to

$$2+\frac{2\ell}{1-\ell}-\frac{n(p-2)}{2}-\frac{n}{2}\le 0,$$

for $p \ge 1+\frac{4}{n(1-\ell)}$.

Proof ***(Theorem 3)*** We define the space

$$X(T) \doteq C([0,T), H^m(\mathbf{R^n})) \cap C^1([0,T], H^{m-1}(\mathbf{R^n})),$$

equipped with the norm

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]}\Big\{\|u(t,\cdot)\|_{L^2} + (1+t)^{m(1-\ell)}\|u(t,\cdot)\|_{\dot H^m} + (1+t)\|u_t(t,\cdot)\|_{L^2}$$
$$+(1+t)^{m(1-\ell)+\ell}\|u_t(t,\cdot)\|_{\dot H^{m-1}}\Big\}.$$

Thanks to Propositions 1–4 and $\beta \ge \ell + 2m(1-\ell) > 2-\ell$ for $n \ge 3$, $u^0 \in X(T)$ and it satisfies

$$\|u^0\|_X \le C\,\|(u_0,u_1)\|_{\mathscr{D}}.$$

Applying Gagliardo-Nirenberg inequality with $H^m(\mathbf{R^n})$ regularity and $m > \frac{n}{2}$, we may estimate

$$\||u(s,\cdot)|^p\|_{L^2} = \|u(s,\cdot)\|_{L^{2p}}^p \lesssim \|u(s,\cdot)\|_{L^2}^{(1-\theta)p}\|(-\Delta)^{\frac{m}{2}}u(s,\cdot)\|_{L^2}^{\theta p}$$
$$\lesssim (1+s)^{m(\ell-1)\theta p}\|u\|_{X(T)}^p$$

with $\theta p = \frac{n(p-1)}{2m}$. Hence, for $i+j \le 1$ we have

$$\|\nabla^j\partial_t^i Fu(t,\cdot)\|_{L^2} \lesssim \int_0^t (1+t)^{j(\ell-1)-i}(1+s)\||u(s,\cdot)|^p\|_{L^2}ds$$

$$\lesssim \|u\|_{X(T)}^p \int_0^t (1+t)^{j(\ell-1)-i}(1+s)^{1+\frac{n}{2}(p-1)(\ell-1)}ds$$
$$\lesssim (1+t)^{j(\ell-1)-i}\|u\|_{X(T)}^p,$$

thanks to $p > 1 + \frac{4}{n(1-\ell)}$.

In order to estimate $\|(-\Delta)^{\frac{m}{2}} Fu(t,\cdot)\|_{L^2}$ and $\|(-\Delta)^{\frac{m-1}{2}} \partial_t Fu(t,\cdot)\|_{L^2}$, we may use that $H^m(\mathbf{R^n})$, with $m > \frac{n}{2}$, is imbedded into $L^\infty(\mathbf{R^n})$. Indeed, thanks to Corollary 3, for $p > \max\{1, m-1\}$ we may estimate

$$\||u(s,\cdot)|^p\|_{\dot{H}^{m-1}} \le C\|u(s,\cdot)\|_{\dot{H}^{m-1}}\|u(s,\cdot)\|_{L^\infty}^{p-1}.$$

Now, for $u \in X(T)$ we have

$$\|u(s,\cdot)\|_{\dot{H}^{m-1}} \lesssim (1+s)^{(m-1)(\ell-1)}\|u\|_{X(T)},$$

and applying Proposition 5 from the Appendix, we may estimate

$$\|u(s,\cdot)\|_{L^\infty} \lesssim \|u(s,\cdot)\|_{L^2}^{(1-\theta)}\|(-\Delta)^{\frac{m}{2}}u(s,\cdot)\|_{L^2}^{\theta} \lesssim (1+s)^{m(\ell-1)\theta}\|u\|_{X(T)},$$

with $\theta = \frac{n}{2m}$. Hence

$$\||u(s,\cdot)|^p\|_{\dot{H}^{m-1}} \lesssim (1+s)^{\frac{n}{2}(\ell-1)(p-1)+(m-1)(\ell-1)}\|u\|_{X(T)}^p,$$

and

$$\|Fu(t,\cdot)\|_{\dot{H}^m}$$
$$\lesssim \int_0^t (1+s)(1+t)^{m(\ell-1)}\left\{|||u(s,\cdot)|^p||_{L^2} + (1+s)^{(1-\ell)(m-1)}|||u(s,\cdot)|^p||_{\dot{H}^{m-1}}\right\}ds$$
$$\lesssim (1+t)^{m(\ell-1)}\|u\|_{X(T)}^p \int_0^t (1+s)^{1+\frac{n}{2}(p-1)(\ell-1)}ds \lesssim (1+t)^{m(\ell-1)}\|u\|_{X(T)}^p,$$

for $p > 1 + \frac{4}{n(1-\ell)}$. Moreover,

$$\|\partial_t Fu(t,\cdot)\|_{\dot{H}^{m-1}}$$
$$\lesssim \int_0^t (1+s)(1+t)^{m(\ell-1)-\ell}\left\{|||u(s,\cdot)|^p||_{L^2} + (1+s)^{(1-\ell)(m-1)}|||u(s,\cdot)|^p||_{\dot{H}^{m-1}}\right\}ds$$
$$\lesssim (1+t)^{m(\ell-1)-\ell}\|u\|_{X(T)}^p \int_0^t (1+s)^{1+\frac{n}{2}(p-1)(\ell-1)}ds \lesssim (1+t)^{m(\ell-1)-\ell}\|u\|_{X(T)}^p.$$

We now prove the Lipschitz condition (34).

To estimate $\|\nabla^j \partial_t^i (Fu - Fv)(t,\cdot)\|_{L^2}$, $i + j = 0, 1$, we follow as in the proof of Theorem 2, but to estimate $\|(Fu - Fv)(t,\cdot)\|_{\dot{H}^m}$ and $\|\partial_t (Fu - Fv)(t,\cdot)\|_{\dot{H}^{m-1}}$ we use

$$Pu - Pv = Fu - Fv = \int_0^t K_1(t,s,x) *_{(x)} \left(|u(s,x)|^p - |v(s,x)|^p\right) ds$$

$$= p \int_0^t K_1(t,s,x) *_{(x)} \Big(\int_0^1 |v + \tau(u-v)|^{p-2}(v + \tau(u-v))d\tau\Big)(s,x)(u-v)(s,x)\, ds.$$

Indeed, applying Minkowski's integral inequality and Proposition 3 with $k = m$ gives

$$\|(Fu - Fv)(t,\cdot)\|_{\dot{H}^m} \lesssim \int_0^t \left\|K_1(t,s,\cdot) *_{(x)} \left(|u(s,\cdot)|^p - |v(s,\cdot)|^p\right)\right\|_{\dot{H}^{m-1}} ds$$

$$\lesssim \int_0^t (1+t)^{m(\ell-1)}(1+s) \Big\|\Big(\int_0^1 |v + \tau(u-v)|^{p-2}(v + \tau(u-v))d\tau\Big)(u-v)(s,\cdot)\Big\|_{L^2} ds$$

$$+ \int_0^t (1+t)^{m(\ell-1)}(1+s)^{1+(1-\ell)(m-1)} \times$$

$$\Big\|\Big(\int_0^1 |v + \tau(u-v)|^{p-2}(v + \tau(u-v))d\tau\Big)(u-v)(s,\cdot)\Big\|_{\dot{H}^{m-1}} ds.$$

We immediately get

$$\Big\||v + \tau(u-v)|^{p-2}(v + \tau(u-v))(u-v)(s,\cdot)\Big\|_{L^2} \lesssim \|v + \tau(u-v)\|_{L^\infty}^{p-1} \|u - v\|_{L^2}$$

$$\lesssim (1+s)^{\frac{n}{2}(\ell-1)(p-1)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right) \|u - v\|_{X(T)}.$$

The application of the fractional Leibniz rule from Proposition 7 from the Appendix yields

$$\Big\||v + \tau(u-v)|^{p-2}(v + \tau(u-v))(u-v)(s,\cdot)\Big\|_{\dot{H}^{m-1}}$$

$$\lesssim \|v + \tau(u-v)\|_{L^{r_1(p-1)}}^{p-1} \|u - v\|_{\dot{H}^{m-1,r_2}} +$$

$$\|(u-v)(s,\cdot)\|_{L^{r_4}} \Big\||v + \tau(u-v)|^{p-2}(v + \tau(u-v))(s,\cdot)\Big\|_{\dot{H}^{m-1,r_3}},$$

under the conditions

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4} = \frac{1}{2}.$$

Now let us estimate all the terms appearing in the above integrals. By using Proposition 5 from the Appendix we arrive at the estimate

$$\begin{aligned}
&\big\| |v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,\cdot)\big\|_{L^{r_1}} = \|v+\tau(u-v)\|^{p-1}_{L^{r_1(p-1)}}\\
&\lesssim \|v+\tau(u-v)\|^{(1-\theta_1)(p-1)}_{L^2}\big\|(-\triangle)^{\frac{m}{2}}(v+\tau(u-v))\big\|^{\theta_1(p-1)}_{L^2}\\
&\lesssim (1+s)^{m(\ell-1)\theta_1(p-1)}\Big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\Big),
\end{aligned}$$

with

$$0 \le \theta_1 = \frac{n}{m}\Big(\frac{1}{2}-\frac{1}{r_1(p-1)}\Big) < 1.$$

By using Proposition 5 from the Appendix we get for the second term

$$\|(-\triangle)^{\frac{m-1}{2}}(u-v)\|_{L^{r_2}} \lesssim \|u-v\|^{1-\theta_2}_{L^2}\big\|(-\triangle)^{\frac{m}{2}}(u-v)\big\|^{\theta_2}_{L^2} \lesssim (1+s)^{m(\ell-1)\theta_2}\|u-v\|_{X(T)},$$

under the condition

$$\frac{m-1}{m} \le \theta_2 = \frac{n}{m}\Big(\frac{1}{2}-\frac{1}{r_2}+\frac{m-1}{n}\Big) \le 1, \quad \text{that is,} \quad 2 \le r_2 \le \frac{2n}{n-2}.$$

We choose $r_2 = \frac{2n}{n-2} > 2$, i.e., $\theta_2 = 1$. Hence $r_1 = n$ and

$$\begin{aligned}
&\|(v+\tau(u-v))(s,\cdot)\|^{p-1}_{L^{r_1(p-1)}}\|(u-v)(s,\cdot)\|_{\dot H^{m-1,r_2}}\\
&\lesssim (1+s)^{(m-1)(\ell-1)+\frac{n}{2}(\ell-1)(p-1)}\Big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\Big)\|u-v\|_{X(T)}.
\end{aligned}$$

In the same way we estimate

$$\|(u-v)(s,\cdot)\|_{L^{r_4}} \lesssim \|u-v\|^{1-\theta_4}_{L^2}\big\|(-\triangle)^{\frac{m}{2}}(u-v)\big\|^{\theta_4}_{L^2} \lesssim (1+s)^{m(\ell-1)\theta_4}\|u-v\|_{X(T)},$$

with

$$0 \le \theta_4 = \frac{n}{m}\Big(\frac{1}{2}-\frac{1}{r_4}\Big) < 1, \quad \text{that is,} \quad r_4 \ge 2.$$

For $p > \max\{1, m\}$, by Proposition 6 from the Appendix

$$\begin{aligned}
&\Big\| |v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,\cdot)\Big\|_{\dot H^{m-1,r_3}}\\
&\lesssim (v+\tau(u-v))(s,\cdot)\Big\|_{\dot H^{m-1,r_3}}\|(v+\tau(u-v))(s,\cdot)\|^{p-2}_{L^\infty}.
\end{aligned}$$

As we did before

$$\|(v+\tau(u-v))(s,\cdot)\|_{L^\infty}^{p-2} \lesssim (1+s)^{\frac{n}{2}(\ell-1)(p-2)}\big(\|u\|_{X(T)}^{p-2}+\|v\|_{X(T)}^{p-2}\big)$$

and with $r_3 = r_2 = \frac{2n}{n-2}$ we get

$$\Big\|(v+\tau(u-v))(s,\cdot)\Big\|_{\dot H^{m-1,r_3}} \lesssim (1+s)^{m(\ell-1)}\big(\|u\|_{X(T)}+\|v\|_{X(T)}\big).$$

Hence, $r_4 = n$ and

$$\begin{aligned}
&\|(u-v)(s,\cdot)\|_{L^{r_4}}\Big\||v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,\cdot)\Big\|_{\dot H^{m-1,r_3}} \\
&\lesssim (1+s)^{(m-1)(\ell-1)+\frac{n}{2}(\ell-1)(p-1)}\Big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\Big)\,\|u-v\|_{X(T)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(Fu-Fv)(t,\cdot)\|_{\dot H^m} \\
&\lesssim (1+t)^{m(\ell-1)}\int_0^t (1+s)^{1+\frac{n}{2}(\ell-1)(p-1)}\,ds\Big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\Big)\,\|u-v\|_{X(T)} \\
&\lesssim (1+t)^{m(\ell-1)}\Big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\Big)\,\|u-v\|_{X(T)},
\end{aligned}$$

for $p > 1+\frac{4}{n(1-\ell)}$. Similarly we conclude

$$\|\partial_t(Fu-Fv)(t,\cdot)\|_{\dot H^{m-1}} \lesssim (1+t)^{m(\ell-1)-\ell}\Big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\Big)\,\|u-v\|_{X(T)}.$$

Summarizing all the estimates implies

$$\|Pu-Pv\|_{X(t)} \lesssim \|u-v\|_{X(T)}\Big(\|u\|_{X(T)}^{p-1}+\|v\|_{X(T)}^{p-1}\Big)$$

for any $u, v \in X(T)$. □

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Appendix

In the Appendix we list some notations used through the paper and results of Harmonic Analysis which are important tools for proving results on the global (in time) existence of small data energy solutions for semi-linear models with power non-linearities. Through this paper, we use the following.

For $s \geq 0$, we denote by $|D|^s f = \mathscr{F}^{-1}(|\xi|^s \hat{f})$ and $\langle D\rangle^s f = \mathscr{F}^{-1}(\langle\xi\rangle^s \hat{f})$, with $\langle\xi\rangle^s = (1+|\xi|^2)^{\frac{s}{2}}$.

For any $q \in [1,\infty]$, we denote by $L^q(\mathbf{R}^n)$ the usual Lebesgue space over $\mathbf{R}^n$. Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then

$$H^{s,p}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \|\langle D\rangle^s u\|_{L^p(\mathbb{R}^n)} = \|u\|_{H^s_p(\mathbb{R}^n)} < \infty\},$$

$$\dot{H}^{s,p}(\mathbb{R}^n) = \{u \in \mathscr{Z}'(\mathbb{R}^n) : \||D|^s u\|_{L^p(\mathbb{R}^n)} = \|u\|_{\dot{H}^s_p(\mathbb{R}^n)} < \infty\}$$

are called Bessel and Riesz potential spaces, respectively. If $p = 2$, then we use the notations $H^s(\mathbb{R}^n)$ and $\dot{H}^s(\mathbb{R}^n)$, respectively. In the definition of the Riesz potential spaces we use the space of distributions $\mathscr{Z}'(\mathbb{R}^n)$. This space of distributions can be identified with the factor space $\mathcal{S}'/\mathscr{P}$, where $\mathcal{S}'$ denotes the dual of Schwartz space and $\mathscr{P}$ denotes the set of all polynomials.

We recall that $H^{s,q}(\mathbf{R}^n) = W^{s,q}(\mathbf{R}^n)$, the usual Sobolev space, for any $q \in (1,\infty)$ and $s \in \mathbf{N}$.

The following inequality can be found in [12], Part 1, Theorem 9.3.

Proposition 5 (Fractional Gagliardo-Nirenberg Inequality) *Let $1 < p, p_0, p_1 < \infty$, $\sigma > 0$ and $s \in [0,\sigma)$. Then it holds the following fractional Gagliardo-Nirenberg inequality for all $u \in L^{p_0}(\mathbb{R}^n) \cap \dot{H}^{\sigma,p_1}(\mathbb{R}^n)$:*

$$\|u\|_{\dot{H}^{s,p}} \lesssim \|u\|_{L^{p_0}}^{1-\theta} \|u\|_{\dot{H}^{\sigma,p_1}}^{\theta},$$

where $\theta = \theta_{s,\sigma}(p,p_0,p_1) = \frac{\frac{1}{p_0}-\frac{1}{p}+\frac{s}{n}}{\frac{1}{p_0}-\frac{1}{p_1}+\frac{\sigma}{n}}$ *and* $\frac{s}{\sigma} \leq \theta \leq 1$.

We present here a result for fractional powers [21].

Proposition 6 *Let $p > 1$, $f(u) = |u|^p$ or $f(u) = |u|^{p-1}u$ and $u \in H^{s,m}$, where $s \in \big(\frac{n}{m}, p\big)$, $1 < m < \infty$. Then the following estimate holds*:

$$\|f(u)\|_{H^{s,m}} \leq C\|u\|_{H^{s,m}} \|u\|_{L^\infty}^{p-1}.$$

In [11] the following corollary was derived:

Corollary 3 *Let $f(u) = |u|^p$ or $f(u) = |u|^{p-1}u$, with $p > \max\{1,s\}$ and $u \in H^{s,m} \cap L^\infty$, $1 < m < \infty$. Then the following estimate holds*:

$$\|f(u)\|_{\dot{H}^{s,m}} \leq C\|u\|_{\dot{H}^{s,m}}\|u\|_{L^\infty}^{p-1}.$$

The next result combines in some sense some familiar results as Leibniz rule for the product of two functions and Hölder's inequality for derivatives of fractional order (Theorem 7.6.1 in [17]).

Proposition 7 *Let us assume* $s > 0$ *and* $1 \leq r \leq \infty$, $1 < p_1, p_2, q_1, q_2 < \infty$ *satisfying the relation*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then the following fractional Leibniz rules hold:

$$\|\,|D|^s(u\,v)\|_{L^r} \lesssim \|\,|D|^s u\|_{L^{p_1}}\|v\|_{L^{p_2}} + \|u\|_{L^{q_1}}\|\,|D|^s v\|_{L^{q_2}}$$

for any $u \in \dot{H}^{s,p_1}(\mathbf{R}^n) \cap L^{q_1}(\mathbf{R}^n)$ *and* $v \in \dot{H}^{s,q_2}(\mathbf{R}^n) \cap L^{p_2}(\mathbf{R}^n)$,

$$\|\langle D\rangle^s(u\,v)\|_{L^r} \lesssim \|\langle D\rangle^s u\|_{L^{p_1}}\|v\|_{L^{p_2}} + \|u\|_{L^{q_1}}\|\langle D\rangle^s v\|_{L^{q_2}}$$

for any $u \in H^{s,p_1}(\mathbf{R}^n) \cap L^{q_1}(\mathbf{R}^n)$ *and* $v \in H^{s,q_2}(\mathbf{R}^n) \cap L^{p_2}(\mathbf{R}^n)$.

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Local Solvability of Some Partial Differential Operators with Non-smooth Coefficients

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Abstract In this paper we will analyze the local solvability property of some second order linear degenerate partial differential operators with non-smooth coefficients. We will start by considering some operators with $C^{\alpha,1}$ coefficients, with $\alpha = 0, 1$, having a kind of affine structure. Next, we will study operators with a more general structure having $C^{0,1}$ or L^∞ coefficients. In both cases the local solvability will be analyzed at multiple characteristic points where the principal symbol may possibly change sign.

Keywords Local solvability · A priori estimates · Degenerate second order operators · Non-smooth coefficients

1 Introduction

In this paper we shall consider the local solvability problem for two classes of second order linear partial differential operators with multiple characteristics, denoted by P_1 and P_2 respectively, given by

$$P_1(x, D) = \sum_{j=1}^{N} X_j(x, D)^* g|g| X_j(x, D) + i X_0(x, D) + a_0, \tag{1}$$

$$P_2(x, D) = \sum_{j=1}^{N} X_j(x, D)^* |h| X_j(x, D) + i X_0(x, D) + a_0, \tag{2}$$

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where $X_0(x, D), \dots, X_N(x, D)$ $(D_j = -i\partial_j)$, are first order PDOs (partial differential operators) with no lower order terms defined on an open set $\Omega \subset \mathbb{R}^n$ and having real or complex coefficients (i.e. iX_j are complex or real vector fields), g is an affine real function such that $S := g^{-1}(0) \neq \emptyset$, $dg|_S \neq 0$, $h \in C^1(\Omega)$ such that $S := h^{-1}(0) \neq \emptyset$, $dh|_S \neq 0$, and a_0 is a continuous function on Ω with complex values.

In particular

- In P_1 we assume $X_1(x, D), \dots, X_N(x, D)$ to have *constant* real or complex coefficients and $X_0(x, D)$ to have *affine* real coefficients;
- In P_2 we assume $X_0(x, D), \dots, X_N(x, D)$ to have smooth complex coefficients and $X_0(x, D)$ to have smooth *real* coefficients.

The local solvability problem both for P_1 and P_2 is considered around the points of the set S where the operators are degenerate and where the principal symbol may possibly change sign. In fact the operators of the form P_1 have a real principal symbol with the aforementioned property, while, because of their structure, the operators of the form P_2 do not show this behaviour (however, being degenerate, their local solvability is not guaranteed). The reason why we consider the solvability problem specifically at these points is due to the fact that the changing sign property of the principal symbol affects the local solvability of the operator and, in general, it adds degeneracy. This is indeed true even for principal type pseudo-differential operators whose local solvability was completely characterized after the resolution of the Nirenber-Treves conjecture. It was Nils Dencker (see [3]) who proved that condition (Ψ) is necessary and sufficient for the local solvability of principal type pseudo-differential operators (with C^∞ coefficients), and, recall, (Ψ) is a condition on the sign of the principal symbol (see [12] and [16]). In particular, condition (Ψ) is satisfied if the imaginary part $\mathrm{Im}\, p$ of the principal symbol p of a pseudo-differential operator $P = Op(p)$ does not change sign from minus to plus when one moves in the positive direction of a bicharacteristic of $\mathrm{Re}\, p$, therefore it is essentially a sign condition on the principal symbol.

For general multiple characteristics PDOs necessary and sufficient conditions for the local solvability to hold are not available. Condition (Ψ) above, as already observed, is a sign condition on the principal symbol and it is not well suited to capture the more complicated geometric structure which characterizes the multiple characteristic setting where the lower order part of the symbol has to be taken into account. However, even though (in general) (Ψ) does not apply to the multiple characteristics case, there are results concerning necessary and/or sufficient conditions for classes of pseudo-differential operators with multiple characteristics. Here we want to mention results about operators with multiple characteristics having a structure similar to that of the classes P_1 and P_2.

Operators of the form $P_1 P_2 + Q$ (where P_1, P_2, Q are first order operators) with double characteristics are studied in a paper by Helffer [10] (where he actually studies the hypoellipticity problem, which is also related to the local solvability problem) and by Treves [26] (in which he studies the solvability of operators of the form $X_1(x, D)X_2(x, D)+iY(x, D)+a_0$, where iX_1, iX_2, iY are *real* vector fields).

Kohn considered in [14] operators of the form $\sum_{j=1}^{N} X_j^* X_j$, where the vector fields iX_j are *complex* and satisfy a suitable Hörmander condition of rank 2. Mendoza and Uhlmann in [18] (see also [17]), studied necessary and sufficient conditions for the local solvability of operators with a principal symbol of the form $\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j$, where $a_{i,j}(x)$ is symmetric and nondegenerate on the characteristic set. About operators on Lie groups, Müller, Ricci and Peloso studied operators which are sums of squares of left-invariant vector fields (see [21, 22, 25]). Moreover, results about the semi-global solvability of operators with transversal multiple symplectic characteristics are given by Parenti and Parmeggiani in [23] (see also [24]) .

The local solvability problem for degenerate operators with non-smooth coefficients has not been deeply analyzed, possibly because of the lack of a complete pseudo-differential approach. In particular, the Fefferman–Phong inequality, which is a key tool to get the a priori estimate from which the local solvability follows, is not (in general) available in the non-smooth coefficients setting, therefore we shall derive the needed estimates by exploiting the structure of the operators.

The classes P_1 and P_2 above, fully analyzed in [4] (see also [5]), are inspired by previous classes analyzed by Colombini et al. in [1] and by Federico and Parmeggiani in [7, 8] (see also [24] for a survey), and which are in turn an elaboration of the Kannai operator (see [14]).

Let us remark, shortly, that the Kannai operator is a very interesting example to look at in order to understand the problem in the multiple characteristic setting, and, in particular, to see the dependence of the local solvability property not only on the principal symbol of the operator but also on the lower order part, namely, on the subprincipal symbol. In fact, the hypoellipticity of the Kannai operator on $\mathbb{R}^n$ is not enough to guarantee the local solvability at the points around which the principal symbol changes sign. However, as shown in [1, 7, 8], the behaviour of the subprincipal symbol dictates, somehow, the local solvability at the multiple characteristics points.

Due to the previous considerations also in the cases we consider here the lower order part of the symbol plays a crucial role in obtaining the local solvability property. Let us also stress that in the present paper we focus on giving sufficient conditions for the local solvability of the classes P_1 and P_2.

We conclude this introduction by giving the plan of the paper. In Sect. 1 we recall the definition of locally solvable partial differential operator and give a natural definition of *local solvability in the non-smooth sense* for operators with non-smooth coefficients. Moreover, we shall also recall the estimate needed to prove the local solvability and that we will use to get the results in the subsequent sections. In Sect. 3 we analyze the problem for the class P_1, state the hypotheses in this case and give two solvability theorems, one in the real coefficients case and one in the complex coefficients case. We will also sketch the proof and explain the differences in the two cases listed before. In Sect. 4 we study the class P_2, state the local solvability result and show some examples of operators in the class. Finally Sect. 5 contains some final remarks and open problems.

2 Local Solvability and a Priori Estimates

Since the classes P_1 and P_2 considered here are composed by non-smooth coefficients PDOs, we will need to give a suitable definition of local solvability applicable to this case. We shall first recall the definition of local solvability for PDOs with smooth coefficients, and, afterwards, we will modify the definition according to our context.

Definition 1 (Local Solvability) Let P be an mth-order partial differential operator with smooth coefficients on an open set $\Omega \subset \mathbb{R}^n$. We say that *$P$ is locally solvable at $x_0 \in \Omega$* if there exists a neighborhood $V \subset \Omega$ of x_0 such that for all $v \in C^\infty(\Omega)$ there is $u \in \mathscr{D}'(\Omega)$ satisfying $Pu = v$ in V.

Definition 2 (H^s to $H^{s'}$ Local Solvability) Let P be an mth-order partial differential operator with smooth coefficients on an open set $\Omega \subset \mathbb{R}^n$. Given $s, s' \in \mathbb{R}$ and $x_0 \in \Omega$ we say that *P is H^s to $H^{s'}$ locally solvable near x_0* if there is a compact $K \subset \Omega$ with $x_0 \in \mathring{K}$ (the interior of K) such that for all $v \in H^s_{\mathrm{loc}}(\Omega)$ there exists $u \in H^{s'}_{\mathrm{loc}}(\Omega)$ with $Pu = v$ in $\mathring{K}$. We will call the number $s - s'$ *the gain of smoothness* (near x_0) of the solution. We will say that *P is H^s to $H^{s'}$ locally solvable near $V \subset \Omega$* if P is H^s to $H^{s'}$ locally solvable near x_0 for all $x_0 \in V$. When one has H^s to $H^{s'}$ local solvability for all $s \in \mathbb{R}$ where $s' = s + m - r$, then one calls r *the loss of derivatives*.

Definition 3 (Local Solvability in the Non-smooth Sense) Let P, defined on $\Omega \subset \mathbb{R}^n$, be an mth-order partial differential operator such that both P and its adjoint P^* have (at least) L^∞ coefficients. We say that P is L^2 to L^2 locally solvable in the non-smooth sense at $x_0 \in \Omega$ if there exists a compact set $K \subset \Omega$, with $x_0 \in U = \mathring{K}$ (where $\mathring{K}$ denotes the interior of K), such that for all $f \in L^2_{\mathrm{loc}}(\Omega)$ there exists $u \in L^2_{\mathrm{loc}}(\Omega)$ such that

$$(u, P^*\varphi) = (f, \varphi), \quad \forall \varphi \in C_0^\infty(K),$$

where $(\cdot, \cdot)$ denotes the L^2-scalar product.

Remark 1 Notice that, in order to have local solvability in the non-smooth sense for an operator P, we ask both P and its adjoint P^* to have L^∞ coefficients (L^∞_{loc} if $\Omega = \mathbb{R}^n$). This is because, as for operators in the class P_2 we may have L^∞ coefficients, we need to guarantee that P_2^* is such that, given $u \in L^2_{\mathrm{loc}}(\Omega)$, the L^2-scalar product identity

$$(u, P_2^*\varphi) = (f, \varphi), \quad \forall \varphi \in C_0^\infty(K), \quad K \Subset \Omega$$

is well defined. Moreover we remark that operators in the class P_1 always have the required property.

The technique used to prove local solvability results relies deeply on the validity of some a priori estimates and on the application of the Hahn–Banach theorem. In fact, in order to prove the L^2 to L^2 local solvability result for a smooth operator P at a point $x_0 \in \Omega$, and, similarly, to prove the L^2 to L^2 local solvability in the non-smooth case, it is enough to prove the following *solvability estimate*.

Solvability Estimate
We say that a partial differential operator P satisfies the solvability estimate at $x_0 \in \Omega$ if there exists a compact set $K \subset \Omega$ containing x_0 in its interior and a positive constant C_K such that

$$\|P^*\varphi\| \geq C_K\|\varphi\|, \quad \forall \varphi \in C_0^\infty(K),$$

where $\|\cdot\|$ denotes the L^2-norm.

The solvability estimate has to be satisfied by the adjoint operator P^*. If this is true, given $f \in L^2_{\mathrm{loc}}(\Omega)$ fixed, we get that the anti-linear form

$$\ell : E := P^*(C_0^\infty(K)) \to \mathbb{C}, \quad \ell(P^*\varphi) = (f, \varphi) := \int_{\mathbb{R}^n} f\overline{\varphi}dx,$$

is bounded on the subspace $E \subset L^2_{\mathrm{comp}}(\Omega)$ and can be extended to a bounded anti-linear form ℓ' on $L^2_{\mathrm{comp}}(\Omega) = (L^2_{\mathrm{loc}}(\Omega))^*$ such that $\ell'|_E = \ell$ by means of the Hahn–Banach theorem. Finally this will give the existence of $u \in L^2_{\mathrm{loc}}(\Omega)$ such that

$$(f, \varphi) =: \ell'(P^*\varphi) = (u, P^*\varphi) = (Pu, \varphi), \quad \forall \varphi \in C_0^\infty(K),$$

that is, P is L^2 to L^2 locally solvable at x_0.

3 Local Solvability of the Class P_1

This section is devoted to the analysis of the local solvability property of operators in the class P_1 which are of the form (1).

Recall that in this case all the vector fields $iX_j(x, D) = X_j(D)$, $j \neq 0$, have constant *real* or *complex* coefficients, $iX_0(x, D)$ is a vector field with *affine real* coefficients, and g is an *affine real* function such that $S := g^{-1}(0) \neq \emptyset$, $dg|_S \neq 0$.

We are interested in the local solvability of P_1 at the points of the set S where the operator is degenerate and has a principal symbol that changes sign in the neighborhood of each point of S.

Notice that operators of the form P_1 may have $C^{0,1}$ or $C^{1,1}$ coefficients ($C^{0,1}_{\mathrm{loc}}$ or $C^{1,1}_{\mathrm{loc}}$ if $\Omega = \mathbb{R}^n$) depending on the tangency or transversality of the vector fields iX_j, $j = 1, \ldots, N$, to the set S.

We state now the hypotheses we consider on the class P_1 and that will be assumed both in the real and in the complex case.

(H1) $iX_0 g > 0$ on S;
(H2) $\forall j = 1, \dots, N$, $\forall K \subset \Omega$, $\exists C > 0$ such that

$$|\{X_j, X_0\}(\xi)|^2 \leq C \sum_{k=1}^{N} |X_k(\xi)|^2, \quad \forall \xi \in \mathbb{R}^n,$$

where $X_j(x, \xi)$ denotes the (principal) symbol of $X_j(x, D)$ and $\{\cdot, \cdot\}$ stands for the Poisson bracket.

Terminology We will call *real case* the case in which all the vector fields iX_j are real; conversely we will call *complex case* the case in which each iX_j with $j \neq 0$ is complex.

Remark 2 Condition (H1) is a transversality condition which implies the nondegeneracy of X_0 at S. Note that the (principal) symbol of iX_0 represents the subprincipal symbol of the operator P_1.

Let us remark that when dealing with multiple characteristics PDOs the analysis of the principal symbol only is not enough to get solvability results, therefore the lower order part of the symbol can not be neglected.

As we shall see below, condition (H2) is a technical condition that permits to control the leading part of the operator by means of the first order nondegenerate part given by iX_0.

3.1 Local Solvability in the Real Case

We are now ready to state the solvability result for the class P_1 in (1) in the real case.

Theorem 1 *Let P_1 be an operator of the form* (1) *satisfying (H1) and (H2). Then P_1 is L^2 to L^2 locally solvable in the non-smooth sense at each point of S.*

Sketch of the Proof We give here the sketch of the proof by listing the main steps (see [4] for the details). Recall that the goal is to obtain the *solvability estimate* from which the result follows.

First Step The first step is to reduce the problem, after an affine change of variables, to the study of

$$P_1 = \sum_{j=1}^{N} X_j^* x_1 |x_1| X_j + i X_0 + a_0,$$

where the iX_j are new suitable vector fields. We remark here that P_1 is invariant under affine changes of variables, therefore the new operator still satisfies hypotheses $(H1)$ and $(H2)$.

Second Step The second step is to prove an intermediate estimate (intermediate because it is not the solvability estimate yet), that we shall call *main estimate*, given in the following proposition.

Proposition 1 (The Main Estimate) *Let* $S = \{x \in \mathbb{R}^n; x_1 = 0\}$. *Then for all* $x_0 \in S$ *there exist a compact set* K_0 *containing* x_0 *in its interior and three positive constants* $C = C(K_0)$, $c = c(K_0)$ *and* $\varepsilon_0 = \varepsilon_0(K_0)$, *with* $\varepsilon_0 \to 0$ *as* $K_0 \to \{x_0\}$, *such that for all compact sets* $K \subset K_0$

$$\|P_1^* u\|^2 \geq \frac{1}{4}\|X_0 u\|^2 + c(P_0 u, u) - C\|u\|^2, \quad \forall u \in C_0^\infty(K),$$

where

$$P_0(x, D) = \sum_{j=1}^{N} (X_j^* |x_1| X_j - \varepsilon_0^2 [X_j, X_0]^* |x_1| [X_j, X_0]),$$

and where $[\cdot, \cdot]$ *denotes the commutator bracket.*

Third Step Here the point is to pass from the main estimate to the *solvability estimate*. In the smooth coefficients case one can pass from the (suitable) main estimate to the solvability estimate by applying the Fefferman–Phong inequality on P_0 (this is possible by virtue of the form of P_0) and a Poincaré inequality on X_0. In the non-smooth coefficients case we replace the use of the Fefferman–Phong inequality by the use of the following two lemmas.

Lemma 1 *If condition (H2) holds, then, for each index* $j \in \{1, \cdots, N\}$, *we have*

$$i[X_j, X_0](D) = \sum_{k=1}^{N} c_k X_k(D), \quad c_k \in \mathbb{R}.$$

This lemma is very important in order to control the term (P_0u, u) in the main estimate. In fact we can then write

$$(P_0(x, D)u, u) = \sum_{j=1}^{N} \Big((X_j^*|x_1|X_j - \varepsilon_0^2[X_j, X_0]^*|x_1|[X_j, X_0])u, u \Big)$$

$$= \sum_{j=1}^{N} \Big(\||x_1|^{1/2}X_ju\|^2 - \varepsilon_0^2\||x_1|^{1/2}[X_j, X_0]u\|^2 \Big),$$

and get the required control of the term (P_0u, u) stated in Lemma 2 below by using Lemma 1.

Lemma 2 *Consider $x_0 \in S$ and K_0 as in the proposition above (Main estimate). Then, suitably shrinking K_0 to a compact set containing x_0 in its interior, and that we still denote by K_0, we have that for all $K \subset K_0$, with $x_0 \in \mathring{K}$, we have*

$$(P_0\varphi, \varphi) \geq 0, \quad \forall \varphi \in C_0^\infty(K).$$

Finally, by using Lemma 2 in the *main estimate* and absorbing the L^2-error by means of a Poincaré inequality on X_0 (recall, X_0 is nondegenerate at S), we get that P_1 satisfies the solvability estimate at $x_0 \in S$, and, in particular, we have the L^2 to L^2 local solvability of P_1 at x_0. □

3.2 Local Solvability in the Complex Case

We now focus on the local solvability of P_1 at S is the complex case. Since we deal with complex coefficients we need to assume the following additional condition:

(H3) $X_jg = 0$ on $S = g^{-1}\{0\}$, for every $j \neq 0$.

The reason for this tangency condition will be clearer in the sketch of the proof, which, essentially, will differ from the one in the real case in the third step.

Theorem 2 *Let P_1 be an operator of the form (1) satisfying (H1), (H2) and (H3). Then P_1 is L^2 to L^2 locally solvable in the non-smooth sense at each point of S.*

Sketch of the Proof The proof follows exactly the same lines of the proof of the real case. We repeat both the first and the second step of the proof of Theorem 1 since the *main estimate* still holds in this case. What differs from the real case is the third step, that is, the way to pass from the *main estimate* to the *solvability estimate.* In the proof of Theorem 1 we used Lemma 1 to get Lemma 2 and, as a consequence, the solvability estimate. Unfortunately Lemma 1 does not hold in the complex case, and this is the reason why we need to require the additional condition (H3) in this setting. We then replace Lemma 1 with the following lemma.

Lemma 3 *Consider* $x_0 \in S$ *and* K_0 $(x_0 \in \mathring{K}_0)$ *as in Proposition 1 (Main estimate). Then we can shrink* K_0 *to a compact set that we keep denoting by* K_0 *, with* $x_0 \in \mathring{K}_0$*, so that*

$$\varepsilon_0^2 \sum_{j=1}^{N} |\{X_j, X_0\}(\xi)|^2 \leq \sum_{j=1}^{N} |X_j(\xi)|^2, \quad \forall \xi \in \mathbb{R}^n. \tag{3}$$

Notice that $[X_j, X_0]$ has constant coefficients, hence (3) applies for all $\xi \in \mathbb{R}^n$. Moreover, due to condition $(H3)$, after application of the first step (where we reduce the problem to the case $g(x) = x_1$) we have that

$$[X_j, X_0](D_{x_1}, D_{x'}) = [X_j, X_0](D_{x'}) \quad \text{and} \quad X_j(D_{x_1}, D_{x'}) = X_j(D_{x_1}, 0) + X_j(0, D_{x'}).$$

Then, as a consequence of Lemma 3, we get

$$\varepsilon_0^2 \sum_{j=1}^{N} |\{X_j, X_0\}(\xi')|^2 \leq \sum_{j=1}^{N} |X_j(0, \xi')|^2, \quad \forall \xi' \in \mathbb{R}^{n-1},$$

where $\xi = (\xi_1, \xi') \in \mathbb{R}_{\xi_1} \times \mathbb{R}^{n-1}_{\xi'}$. By application of Plancharel theorem in the variable $x' \in \mathbb{R}^{n-1}$ $(x = (x_1, x'))$ on the first term of $(P_0 u, u)$, and, afterwards, by using Lemma 3, we get the control of $\sum_{j=1}^{N} \| \, |x_1| [X_j, X_0] u \|_0^2$ by $\sum_{j=1}^{N} \| |x_1| X_j u \|_0^2$, which, in particular, implies Lemma 2 (i.e. $(P_0 \varphi, \varphi) \geq 0$) (see [4] for details).

Finally we apply a Poincaré inequality on the term $\|X_0 \varphi\|^2$ in the *main estimate* to obtain, by shrinking the compact K_0 if necessary, the solvability estimate and thus the result. □

Focus on Condition (H3)

In the complex case we assumed the additional tangency condition (H3). Is it possible to remove this tangency condition? The answer if positive if we assume the following condition $(H2')$ in place of $(H2)$:

$(H2')$ $\exists C > 0$ such that $|\{X_j, X_0\}(\zeta)|^2 \leq C \sum_{j=1}^{N} |X_j(\zeta)|^2, \quad \forall \zeta \in \mathbb{C}^n.$

Remark 3 Condition $(H2')$ is stronger than $(H2)$. Moreover the following relations are satisfied

- $(H2') \Longrightarrow (H2)$
- $(H2) \not\Longrightarrow (H2')$
- $(H2) + (H3) \not\Longrightarrow (H2')$.

These relations show that $(H2')$ is in general too strong to require.

We show the second and the third property in the previous remark through some counterexamples.

Example 1 ($(H2) \not\Longleftarrow (H2')$)

This is an example of an operator of the form (2) with complex coefficients which satisfies conditions $(H1)$ and $(H2)$ but not condition $(H3)$. We use this example to show that $(H2) \not\Longleftarrow (H2')$. Let $N = 1$, $n = 2$ and

$$P = X_1(x, D)x_1|x_1|X_1(x, D) + iX_0(x, D),$$

$$X_1(x, D) = (1 + i)D_1 + (2 + i)D_2, \;\; X_0(x, D) = (3x_1 + 1)D_1 + (6x_1 - x_2)D_2.$$

If $(H2')$ were true then for all $\zeta \in \mathbb{C}^2$ such that $X_1(\zeta) = 0$ we would also have $\{X_0, X_1\}(\zeta) = 0$.

Since there exists $\zeta_0 = (13 + i, 8 + 2i)$ such that $X_1(\zeta_0) = 0$ and $\{X_0, X_1\}(\zeta) = 20 + 12i$, we get that $(H2')$ does not hold. Note also that, since neither $(H2')$ nor $(H3)$ are satisfied, then Theorem 2 do not apply to this operator and we can not conclude the local solvability by using our result.

Example 2 ($(H2) + (H3) \not\Longleftarrow (H2')$)

Now we provide an example of operator satisfying $(H1)$, $(H2)$ and $(H3)$ and we use it to show that $(H2) + (H3) \not\Longleftarrow (H2')$. Let $N = 1$, $n = 3$ and

$$P = X_1(x, D)x_1|x_1|X_1(x, D) + iX_0(x, D),$$

$$X_1(x, D) = (2 + i)D_2 + D_3, \;\; X_0(x, D) = D_1 + x_2 D_2.$$

Once again there exists $\zeta_0 = (0, 2 + i, 0)$ such that $X_1(\zeta_0) = 0$ and $\{X_0, X_1\}(\zeta_0) = 1 + 3i$, hence $(H2')$ does not hold. However the hypotheses of Theorem 2 are satisfied and the operator is locally solvable at each point of $S := \{x_1 = 0\}$

A Special Case

There is a special case in which P_1 has complex coefficients (recall, the first order part iX_0 is always real) and $(H3)$ is not needed, namely, when $N = 1$ and X_1 is *essentially real*, that is, $\mathrm{Re}\, X_1(D)$ and $\mathrm{Im}\, X_1(D)$ are linearly dependent. In this specific case we have that $(H2) \Longrightarrow (H2')$ and $(H3)$ is not needed.

4 Local Solvability for the Class P_2

In this section we discuss the local solvability of the class P_2, that is,

$$P_2(x, D) = \sum_{j=1}^{N} X_j(x, D)^* |h| X_j(x, D) + i X_0(x, D) + a_0,$$

defined over $\Omega \subset \mathbb{R}^n$, where $X_0(x, D), \ldots, X_N(x, D)$ have smooth (not necessarily constant) complex coefficients and $X_0(x, D)$ has smooth (not necessarily affine) *real* coefficients. The function h is not assumed to be affine real but to be $C^1(\Omega; \mathbb{R}^n)$. Moreover we consider h such that $S := h^{-1}\{0\} \neq \emptyset$ and $dh|_S \neq 0$.

We remark here that in [4] all the fields X_j, for all $j = 0, \ldots, N$, are assumed to have smooth real coefficients. However, the proof of the solvability result works exactly the same if we consider the vector field $X_j(x, D)$, for all $j \neq 0$, having smooth complex coefficients, therefore here we consider the problem in this more general setting directly.

Operators of the form P_2 do not have a changing sign principal symbol around the points of the set S. However, they are still degenerate around these points, therefore the local solvability is not guaranteed. Moreover the degree of degeneracy depends on the interplay between the degeneracy of the vector fields in the second order part of the operator and the vanishing of the function h. In fact, as we shall see below, we do not impose a nondegeneracy condition on the iX_j with $j \neq 0$.

Notice that the coefficients of P_2 may be $C^{0,1}(\Omega)$ (for instance if h is an affine function) if all the vector fields iX_j, $j \neq 0$, are tangent to S, and they are $L^\infty(\Omega)$ ($L^\infty_{\text{loc}}(\Omega)$ if $\Omega = \mathbb{R}^n$) otherwise.

Once again, because of the reasons already mentioned, we study the local solvability of the class at the points of the set S.

We assume in the present case only the following condition:

$(H1)$ $iX_0 h \neq 0$ on S.

We are now ready to give the statement of the result for the class P_2

Theorem 3 *Let P_2 be such that $(H1)$ is satisfied. Then P_2 is L^2 to L^2 locally solvable in the non-smooth sense at each point of S.*

Sketch of the Proof The goal is, once more, to establish the solvability estimate for P_2.

In this case, in contrast to the case previously analyzed, a direct estimate of $\|P_2^* \varphi\|$ does not work, therefore we proceed with a Carleman estimate. We can summaryze the proof in three steps.

First Step First, given a point $x_0 \in S$ and $\lambda \in \mathbb{R}$ (that will be chosen in the next step), we prove that there exists a compact set $K_0 \subset \Omega$ (containing x_0 in its interior) such that the quantity $2\mathsf{Re}(P_2^* \varphi, e^{2\lambda f} \varphi)$ can be estimated from below as follows (see [4]):

$$2\mathsf{Re}(P_2^*\varphi, e^{2\lambda h}\varphi) \geq (1-\delta|\lambda|)\sum_{j=1}^{N} \||h|^{1/2}X_j\varphi\|_{L^\infty(K_0)}+$$

$$|\lambda|(c_0 - \frac{1}{\delta}\||h|^{1/2}\|_{L^\infty(K_0)}\sum_{j=1}^{N}\||h|^{1/2}X_j\varphi\|_{L^\infty(K_0)} - \frac{\|d_{X_0}\varphi\|_{L^\infty(K_0)} + \|a_0\|_{L^\infty(K_0)}}{\lambda})\|e^{\lambda h}\varphi\|^2,$$

for all $\varphi \in C_0^\infty(K_0)$, where the constant c_0 is a positive constant that comes from condition $(H1)$ on X_0.

Second Step We then choose $\lambda := \lambda_0$ in such a way that

$$c_0 - \frac{\|d_{X_0}\varphi\|_{L^\infty(K_0)} + \|a_0\|_{L^\infty(K_0)}}{\lambda_0} \geq \frac{c_0}{2},$$

and fix $\delta = \frac{1}{2|\lambda_0|}$. We then shrink K_0 around x_0 to a compact that we keep denoting by K_0, in such a way that

$$\frac{c_0}{2} - 2\||\lambda_0||h|^{1/2}\|_{L^\infty(K_0)}\sum_{j=1}^{N}\||h|^{1/2}X_j\varphi\|_{L^\infty(K_0)} \geq c_0/4.$$

This is possible because, recall, $h(x_0) = 0$.

Third Step We finally get, for all $\varphi \in C_0^\infty(K_0)$, the inequality

$$2\mathsf{Re}(P_2^*\varphi, e^{2\lambda h}\varphi) \geq |\lambda_0|\frac{c_0}{4}\|e^{\lambda_0 h}\varphi\|^2,$$

that, after application of the Cauchy-Schwartz inequality on the right hand side, gives

$$e^{2\lambda_0\|h\|_{L^\infty(K_0)}}\|P_2^*\varphi\|\|\varphi\| \geq |\lambda_0|\frac{c_0}{4}e^{-2\lambda_0\|h\|_{L^\infty(K_0)}}\|u\|^2, \quad \forall\varphi \in C_0^\infty(K_0),$$

from which the solvability estimate follows. □

We conclude this section by giving some examples of operators in the class P_2.

Example 1
Let $n \geq 2$, $g \in C^\infty(\mathbb{R}^n;\mathbb{R})$ such that $g(x) \neq 0$ for all $x \in S := \{x_1 = 0\}$, and $a_0 \in C^\infty(\mathbb{R}^n)$. Then

$$P(x,D) = \sum_{j=1}^{n}((x_1x_j)^pD_{x_j})^*|x_1|((x_1x_j)^pD_{x_j}) + ig(x)D_{x_1} + a_0$$

is L^2 to L^2 locally solvable in the non-smooth sense in S.

Example 2
Let $f_j \in C^\infty(\mathbb{R}^n; \mathbb{C})$, for $j = 1, \dots, n$, $h \in C^1(\mathbb{R}^n; \mathbb{R})$ such that $\partial_k h(x) \neq 0$ (for some $k \in \{1, \dots, n\}$) for all $x \in S := h^{-1}\{0\}$, and $g \in C^\infty(\mathbb{R}^n; \mathbb{R})$ such that $g|_S \neq 0$. Then

$$P(x, D) = \sum_{j=1}^{n} (f_j D_j)^* |h| (f_j D_j) + igD_k$$

is L^2 to L^2 locally solvable in the non-smooth sense in S.

5 Final Remarks and Open Problems

We want to conclude this paper with some final remarks about the classes considered above.

The local solvability results for the two classes P_1 and P_2 are given at the points of degeneracy where the function appearing in the second order part vanishes. If we consider the classes outside the set S we get operators of the form considered in [8] where a local solvability result outside of S is given by using Carleman estimates. Therefore solvability results out of S are available for these operators.

We want to stress that in the non-smooth coefficients case most probably no better results than L^2 to L^2 can be proved. Moreover, the regularity of the coefficients can not be weakened, for instance to $C^{0,\alpha}$ with $\alpha < 1$, since, otherwise, the adjoint operator would not have L^∞ regularity and the definition of local solvability in the non-smooth sense does not work.

Note also that the two classes above contain evolution operators with non-smooth coefficients (where iX_0 can be considered as the evolution direction). It would be interesting to consider the Cauchy problem for operators with this specific form and get conditions for the local well-posedness to hold.

In the cases analyzed here, the vector field iX_0 is always assumed to have real coefficients. What about the case when iX_0 is complex (Schrödinger type operators)? For classes with smooth coefficients of the same form the problem has been analyzed in [6] and [8] where some local solvability results are given. An other open question is: what about the local well-posedness of the related Cauchy problem in this setting? We want to mention that the local well-posedness for degenerate Schrödinger operators has not been intensively studied. We have a significant result due to Cicognani and Reissig in [2] where the local well-posedness of the linear Cauchy problem for degenerate Schrödinger operators with degenerate time-dependent coefficients is considered. Schrödinger operators of the same form considered in [2] have also been analyzed in [9] where it is shown that some weighted smoothing estimates are satisfied by the solutions both of the linear and of the nonlinear problem.

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On Exceptional Times for Pointwise Convergence of Integral Kernels in Feynman–Trotter Path Integrals

Hans G. Feichtinger, Fabio Nicola, and S. Ivan Trapasso

Abstract In the first part of the paper we provide a survey of recent results concerning the problem of pointwise convergence of integral kernels in Feynman path integrals, obtained by means of time-frequency analysis techniques. We then focus on exceptional times, where the previous results do not hold, and we show that weaker forms of convergence still occur. In conclusion we offer some clues about possible physical interpretation of exceptional times.

Keywords Feynman-Trotter formula · Path integral · Modulation spaces · Short-time Fourier transform

1 Introduction

Integration over infinite-dimensional spaces of paths plays a relevant role in modern quantum physics. This machinery first appeared in a 1948 paper [21] by Richard Feynman, shortly followed by Feynman [22] where path integrals paved the way to the celebrated Feynman diagrams, hence to a completely new way to investigate field theories.

Let us briefly recall the most important features of the functional integral formulation of (non-relativistic) quantum mechanics. The interested reader may consult the textbook [23] for a comprehensive introduction to the subject from a physical perspective. Recall that the state of a particle in $\mathbb{R}^d$ at time $t \in \mathbb{R}$ is represented by the wave function $\psi(t,x)$, $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, such that $\psi(t,\cdot) \in$

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$L^2(\mathbb{R}^d)$. The time evolution of a state $\varphi(x)$ at $t = 0$ is regulated by the Cauchy problem for the Schrödinger equation:

$$\begin{cases} i\hbar\partial_t\psi = (H_0 + V(x))\psi \\ \psi(0, x) = \varphi(x), \end{cases} \tag{1}$$

where $0 < \hbar \leq 1$ is a parameter (representing the Planck constant), $H_0 = -\hbar^2\triangle/2$ is the free particle Hamiltonian and V is a real-valued potential; we set $m = 1$ for the mass of the particle. The map $U(t, s) : \psi(s, \cdot) \mapsto \psi(t, \cdot)$, $t, s \in \mathbb{R}$, is a unitary operator on $L^2(\mathbb{R}^d)$ and is called *propagator* or *evolution operator*; we set $U(t)$ for $U(t, 0)$. Since $U(t)$ is a linear operator we may formally represent it as an integral operator with distribution kernel u_t, namely

$$\psi(t, x) = \int_{\mathbb{R}^d} u_t(x, y)\varphi(y)dy.$$

The kernel u_t (actually known as propagator in physics) is interpreted as the transition amplitude from the position y at time 0 to the position x at time t. In his papers Feynman essentially provided a recipe for how to compute this kernel, involving all the possible *interfering alternative paths* from y to x that could be followed by the particle. In particular, each path would contribute to the total probability amplitude with a phase factor proportional to the *action functional* corresponding to the path:

$$S[\gamma] = S(t, 0, x, y) = \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau))d\tau,$$

where L is the Lagrangian of the corresponding classical system. In a nutshell, a formal representation of the kernel is

$$u_t(x, y) = \int e^{\frac{i}{\hbar}S[\gamma]}\mathscr{D}\gamma, \tag{2}$$

underpinning some integration procedure over the infinite-dimensional space of paths satisfying the conditions above. Notice that (a still formal) application of the stationary phase principle shows that the semiclassical limit $\hbar \to 0$ selects the classical trajectory, in according with the principle of stationary action of classical mechanics.

1.1 The Mathematics of Path Integrals

In spite of the popularity and the successful predictions of path integrals, it is not clear what the meaning of (2) could be from a mathematical point of view. This is in fact an open subfield of functional analysis and there have been several attempts to provide a rigorous and satisfactory theory of path integrals with the support of techniques ranging from infinite-dimensional analysis to operator theory, but also from stochastics to geometry. We cannot hope to frame here more than seventy years of literature; we suggest the monographs [2, 27, 37, 39] as points of departure as well as the article [1] for a broad overview. We remark that only in recent times techniques from time-frequency analysis have been fruitfully used in the study of mathematical path integrals, see for instance [41, 43, 44]; see also [50] for an expository paper on the topic.

Among the several frameworks mentioned above we focus here on the so-called *sequential approach*, introduced by Nelson in [40]. The reasons behind this choice are manifold; first, it is probably the mathematical scheme which best meets Feynman's original insight and some of its features are nowadays part of the custom in physics literature, cf. [32, 38]. Moreover, the perturbative nature of this approach is very well suited to certain function spaces and operators related to time-frequency analysis, as will be elucidated later.

Nelson's approach relies on two issues. Recall that the evolution operator for the Schrödinger equation with $V = 0$, namely $U_0(t) = e^{-\frac{i}{\hbar}tH_0}$, $H_0 = -\hbar^2\triangle/2$, is a Fourier multiplier; an explicit representation can be derived after standard computation (cf. [45, Sec. IX.7]):

$$e^{-\frac{i}{\hbar}tH_0}\varphi(x) = \frac{1}{(2\pi i t\hbar)^{d/2}}\int_{\mathbb{R}^d} \exp\left(\frac{i}{\hbar}\frac{|x-y|^2}{2t}\right)\varphi(y)dy, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{3}$$

The second ingredient is a well-known tool from the theory of operator semigroups. Under suitable conditions on the domain of H_0 and on the potential V[1], the *Trotter product formula* holds for the semigroup generated by $H = H_0 + V$:

$$e^{-\frac{i}{\hbar}t(H_0+V)} = \lim_{n\to\infty}\left(e^{-\frac{i}{\hbar}\frac{t}{n}H_0}e^{-\frac{i}{\hbar}\frac{t}{n}V}\right)^n,$$

where the limit is intended in the strong topology of operators in $L^2(\mathbb{R}^d)$. Combining these two results gives that the complete propagator $e^{-\frac{i}{\hbar}tH}$ can be expressed as limit of integral operators (cf. [45, Thm. X.66]):

$$e^{-\frac{i}{\hbar}t(H_0+V)}\varphi(x) = \lim_{n\to\infty}\left(2\pi\hbar i\frac{t}{n}\right)^{-\frac{nd}{2}}\int_{\mathbb{R}^{nd}} e^{\frac{i}{\hbar}S_n(t;x_0,\ldots,x_{n-1},x)}\varphi(x_0)\,dx_0\ldots dx_{n-1}, \tag{4}$$

[1]For instance one may consider a potential V such that $H_0 + V$ is essentially self-adjoint on $D(H_0) \cap D(V)$, cf. [46, Sec. VIII.8].

where we set

$$S_n\left(t; x_0, \ldots, x_{n-1}, x\right) = \sum_{k=1}^{n} \frac{t}{n} \left[\frac{1}{2} \left(\frac{|x_k - x_{k-1}|}{t/n} \right)^2 - V\left(x_k\right) \right], \quad x_n = x.$$

The role of the phase $S_n\left(t; x_0, \ldots, x_n\right)$ may be clarified by the following argument: given the points $x_0, \ldots, x_{n-1}, x \in \mathbb{R}^d$, let $\overline{\gamma}$ be the polygonal path through the vertices $x_k = \overline{\gamma}\left(kt/n\right)$, $k = 0, \ldots, n$, $x_n = x$, parametrized as

$$\overline{\gamma}\left(\tau\right) = x_k + \frac{x_{k+1} - x_k}{t/n} \left(\tau - k\frac{t}{n} \right), \qquad \tau \in \left[k\frac{t}{n}, (k+1)\frac{t}{n} \right], \qquad k = 0, \ldots, n-1. \tag{5}$$

Hence $\overline{\gamma}$ prescribes a classical motion with constant velocity along each segment. The action functional for such path is given by

$$S\left[\overline{\gamma}\right] = \sum_{k=1}^{n} \frac{1}{2} \frac{t}{n} \left(\frac{|x_k - x_{k-1}|}{t/n} \right)^2 - \int_0^t V(\overline{\gamma}(\tau)) d\tau.$$

According to Feynman's heuristics, the relation in (4) should be interpreted as the definition of an integral over all polygonal paths while $S_n\left(x_0, \ldots, x_n, t\right)$ is a Riemann-like, finite-dimensional approximation of the action functional evaluated on them. The regime $n \to \infty$ is then intuitively clear: the set of polygonal paths becomes the set of all paths and we recover (2).

1.2 Convergence at the Level of Integral Kernels

The sequential approach discussed above seems to suggest that path integral can be made mathematically rigorous at the level of operators rather than integral kernels. This remark is reinforced by the achievements of different mathematical theories of path integrals relying on the standard operator-theoretic approach to quantum mechanics. Consider for instance the so-called *time slicing approximation approach* introduced by Fujiwara in celebrated papers like [25, 26]—see also the monograph [27] for a systematic exposition; broadly speaking, the philosophy underlying these works is to design sequences of finite-dimensional approximation operators on $L^2(\mathbb{R}^d)$ (in particular, oscillatory integral operators) and then prove convergence to the exact propagator $U(t)$ in some operator topology on L^2.

Actually, there are good reasons for not being completely satisfied with this state of affairs. The lesson of Feynman's original formulation strongly motivates a focus shift from operators to their kernels, in particular to the problem of *pointwise* convergence of the integral kernels in (4) to the kernel u_t of the propagator. This may appear as an unaffordable problem in general since non-regular or even purely distribution kernels may show up, thus the problem of convergence can be hard or even pointless. A strong clue pointing in this direction comes from the already mentioned papers by Fujiwara, where convergence in a finer topology at the level of integral kernels is proved for sufficiently small time intervals and smooth potentials with at most quadratic growth.

We describe below the recent results obtained by two of the authors in [43], where techniques of time-frequency analysis are fruitfully used to prove pointwise convergence of integral kernels in the framework provided by the sequential approach. In contrast with the aforementioned results by Fujiwara we consider bounded potentials (the minimal regularity assumption is continuity) and we obtain the desired convergence for the kernels in suitable topologies which imply pointwise convergence. Our results are global in time, namely they hold for any fixed $t \in \mathbb{R}\setminus\widetilde{E}$, where $\widetilde{E}$ is a set of *exceptional times*. We describe below the most important features of this set from both the mathematical and physical points of view and provide explicit examples. For the moment we confine ourselves to remark that exceptional times are to be expected: recall that the involved kernels are in general tempered distributions in $\mathcal{S}'(\mathbb{R}^d)$ in view of the Schwartz kernel theorem and the problem of pointwise convergence is well-posed only when the kernels are actually functions. One may still wonder whether there is convergence at exceptional times in some weaker distribution sense. We are able to prove global-in-time convergence in this fashion, again supported by the framework of time-frequency analysis techniques and function spaces. In order to precisely state and prove the claimed results we devote the next section to collect some preparatory material.

2 Preliminaries

2.1 Notation

We set $x^2 = x \cdot x$, for $x \in \mathbb{R}^d$, where $x \cdot y$ is the scalar product on $\mathbb{R}^d$. The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. The brackets $\langle f, g\rangle$ denote the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx$ on $L^2(\mathbb{R}^d)$, but also other related dualities described below.

The conjugate exponent p' of $p \in [1,\infty]$ is defined by $1/p + 1/p' = 1$. The symbol $\lesssim$ means that the underlying inequality holds up to a positive constant factor $C > 0$. For any $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$ we set $\langle x\rangle^s := (1 + |x|^2)^{s/2}$. We choose the following normalization for the Fourier transform:

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx, \qquad \xi \in \mathbb{R}^d.$$

We define the translation and modulation operators: for any $x, \xi \in \mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$,

$$(T_x f)(y) := f(y-x), \qquad \left(M_\xi f\right)(y) := e^{2\pi i \xi \cdot y} f(y).$$

These operators can be extended by duality on tempered distributions. The composition $\pi(x,\xi) = M_\xi T_x$ constitutes a so-called *time-frequency shift*.

Given a linear space of distributions $X \subset \mathcal{S}'(\mathbb{R}^d)$, we set

$$X_{\text{comp}} := \{u \in X \ : \ \text{supp}(u) \text{ is a compact subset of } \mathbb{R}^d\},$$

$$X_{\text{loc}} := \{u \in \mathcal{S}'(\mathbb{R}^d) \ : \ \varphi u \in X \ \forall \varphi \in C_c^\infty(\mathbb{R}^d)\}.$$

In the rest of the paper we set $\hbar = 1$ for convenience, since we are not concerned with semiclassical aspects.

2.2 Modulation Spaces

The short-time Fourier transform (STFT) of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ is defined by

$$V_g f(x,\xi) := \langle f, \pi(x,\xi) g\rangle = \mathcal{F}(f \cdot T_x g)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i y \cdot \xi} f(y)\, \overline{g(y-x)}\, dy. \tag{6}$$

The monograph [28] contains a comprehensive treatment of the mathematical properties of this time-frequency representation, especially those mentioned below. We stress that the STFT is deeply connected with other well-known phase-space transforms, in particular the Wigner distribution

$$W(f,g)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i y \cdot \xi} f\left(x + \frac{y}{2}\right) \overline{g\left(x - \frac{y}{2}\right)}\, dy. \tag{7}$$

Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the *modulation space* $M_s^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L_s^{p,q}(\mathbb{R}^{2d})$ (mixed weighted Lebesgue space), that is:

$$\|f\|_{M_s^{p,q}} = \|V_g f\|_{L_s^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p \, dx \right)^{q/p} \langle \xi \rangle^{qs} d\xi \right)^{1/q} < \infty,$$

with trivial modification if p or q is ∞. If $p = q$, we write M^p instead of $M^{p,p}$, while for the unweighted case ($s = 0$) we set $M_0^{p,q} \equiv M^{p,q}$.

It can be proved that $M^{p,q}_s(\mathbb{R}^d)$ is a Banach space whose definition does not depend on the choice of the window g. We mention that many common function spaces are intimately related with modulation spaces: for instance,

(i) $M^2(\mathbb{R}^d)$ coincides with the Hilbert space $L^2(\mathbb{R}^d)$;
(ii) $M^2_s(\mathbb{R}^d)$ coincides with the usual L^2-based Sobolev space $H^s(\mathbb{R}^d)$;
(iii) the following continuous embeddings with Lebesgue spaces hold:

$$M^{p,q}_r(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,q}_s(\mathbb{R}^d), \qquad r > d/q' \text{ and } s < -d/q.$$

In particular,

$$M^{p,1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,\infty}(\mathbb{R}^d).$$

For these and other embeddings we address the reader to [14–16, 28].

We wish to focus on distinguished members of the family of modulation spaces. The Banach–Gelfand triple $(M^1(\mathbb{R}^d), L^2(\mathbb{R}^d), M^\infty(\mathbb{R}^d))$ proved to be a very fruitful generalization of the standard triple $(\mathcal{S}(\mathbb{R}^d), L^2(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ for the purposes of time-frequency analysis, see [5, 17, 35] for further details. The space $M^1(\mathbb{R}^d)$ is also known as the *Feichtinger algebra* [14] and it does enjoy a large number of particularly nice properties. We stress that $\mathcal{S}(\mathbb{R}^d) \subset M^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ is the completion of $M^1(\mathbb{R}^d)$ with respect to $\|\cdot\|_{L^2}$ norm. Moreover $(M^1(\mathbb{R}^d))' = M^\infty(\mathbb{R}^d)$ under the duality

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^{2d}} V_g f(z) \overline{V_g \varphi}(z) dz, \quad f \in M^1(\mathbb{R}^d), \ \varphi \in M^\infty(\mathbb{R}^d),$$

for any $g \in \mathcal{S}(\mathbb{R}^d) \backslash \{0\}$, without loss of generality with $\|g\|_2 = 1$. Finally, $M^1(\mathbb{R}^d)$ is isometrically invariant under Fourier transform and arbitrary time-frequency shifts, and the embedding $M^1(\mathbb{R}^d) \hookrightarrow M^{p,q}(\mathbb{R}^d)$ hold for all $1 \le p, q \le \infty$. An additional benefit of this extended framework is that one may derive a streamlined and self-consistent presentation of the mathematical foundations of signal analysis with a limited amount of technicalities, cf. [20].

The role of (M^1, L^2, M^∞) as a Gelfand triple is further reinforced by the *Feichtinger kernel theorem* [4, 13, 18, 19].

Theorem 1

(i) Every distribution $k \in M^\infty(\mathbb{R}^{2d})$ defines a bounded linear operator $T : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ according to

$$\langle Tf, g \rangle = \langle k, g \otimes \overline{f} \rangle, \quad \forall f, g \in M^1(\mathbb{R}^d),$$

with $\|T\|_{M^1 \to M^\infty} \lesssim \|k\|_{M^\infty}$.

(ii) Any linear bounded operator $T : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ arises in this way for a unique kernel $k \in M^\infty(\mathbb{R}^{2d})$; moreover $\|k\|_{M^\infty} \lesssim \|T\|_{M^1 \to M^\infty}$.

Another interesting modulation space is $M^{\infty,1}(\mathbb{R}^d)$, also known as the *Sjöstrand class* since it was highlighted in [49] as an exotic symbol class still yielding bounded pseudodifferential operators on $L^2(\mathbb{R}^d)$ (see the next section for further details, also [29, 30]). In order to specify the regularity of functions in this space recall the definition of the Fourier–Lebesgue space: for $s \in \mathbb{R}$ we set

$$f \in \mathscr{F}L^1_s(\mathbb{R}^d) \quad \Leftrightarrow \quad \|f\|_{\mathscr{F}L^1_s} = \int_{\mathbb{R}^d} |\mathscr{F}f(\xi)|\, \langle\xi\rangle^s d\xi < \infty.$$

Proposition 1 ([28] and [44, Prop. 3.4])

1. $M^{\infty,1}(\mathbb{R}^d) \subset (\mathscr{F}L^1)_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subset C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.
2. $(M^{\infty,1})_{\mathrm{loc}}(\mathbb{R}^d) = (\mathscr{F}L^1)_{\mathrm{loc}}(\mathbb{R}^d) = (\mathscr{F}\mathscr{M})_{\mathrm{loc}}(\mathbb{R}^d)$, *where* $\mathscr{F}\mathscr{M}(\mathbb{R}^d)$ *is the space of Fourier transforms of (finite) complex measures on* $\mathbb{R}^d$.
3. $\mathscr{F}\mathscr{M}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d)$.

The equality $(\mathscr{F}L^1)_{\mathrm{loc}}(\mathbb{R}^d) = (\mathscr{F}\mathscr{M})_{\mathrm{loc}}(\mathbb{R}^d)$ is an immediate consequence of the fact that $L^1(\mathbb{R}^d)$ is an ideal in the convolution algebra $\mathscr{M}(\mathbb{R}^d)$.

Moreover, $M^{\infty,1}(\mathbb{R}^d)$ is a Banach algebra under pointwise product. In fact, precise conditions are known on p, q and s in order for $M^{p,q}_s$ to be a Banach algebra with respect to pointwise multiplication.

Proposition 2 ([47, Thm. 3.5 and Cor. 2.10]) *Let* $1 \le p, q \le \infty$ *and* $s \in \mathbb{R}$. *The following facts are equivalent.*

(i) $M^{p,q}_s(\mathbb{R}^d)$ *is a Banach algebra for pointwise multiplication*[2] .
(ii) $M^{p,q}_s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$.
(iii) Either $s = 0$ *and* $q = 1$ *or* $s > d/q'$.

We deduce that also the modulation spaces $M^\infty_s(\mathbb{R}^d)$ with $s > d$ are Banach algebras for pointwise multiplication. In particular we have $M^\infty_s(\mathbb{R}^d) \hookrightarrow M^{\infty,1}(\mathbb{R}^d)$ for $s > d$ and the following characterization holds:

$$C^\infty_b(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \,:\, \left|\partial^\alpha f\right| \le C_\alpha \quad \forall \alpha \in \mathbb{N}^d \right\} = \bigcap_{s \ge 0} M^\infty_s(\mathbb{R}^d); \tag{8}$$

see [31, Lemma 6.1] for further details.

[2] To be precise, we provide conditions under which the embedding $M^{p,q}_s \cdot M^{p,q}_s \hookrightarrow M^{p,q}_s$ is continuous; this means that the algebra property holds up to a constant. It is a rather standard result that that there exists an equivalent norm for which the previous estimate holds with $C = 1$ (cf. [48, Thm. 10.2]). This setting will be tacitly assumed whenever concerned with Banach algebras from now on.

2.3 Weyl Operators

The success of time-frequency analysis in the theory of pseudodifferential operators mainly relies on the following equality:

$$\langle \sigma^{\mathrm{w}} f, g\rangle = \langle \sigma, W(g, f)\rangle, \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d), \tag{9}$$

where $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is the *symbol* of the *Weyl operator* $\sigma^{\mathrm{w}} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, which can be formally represented as

$$\sigma^{\mathrm{w}} f(x) := \int_{\mathbb{R}^{2d}} e^{2\pi i (x-y)\cdot\xi} \sigma\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi,$$

while $W(g, f)$ is the Wigner transform defined in (7). The main benefit of a time-frequency approach to Weyl operators is that very general symbol classes may be taken into account, in particular modulation spaces—recall that classical symbol classes are usually defined by means of decay/smoothness conditions, such as the Hördmander classes $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ [34]. Moreover, most of the properties of σ^{w} are intimately connected to those of the Wigner transform, the latter being very well established nowadays [11, 28].

The composition of Weyl transforms induces a bilinear form on symbols, the so-called *twisted product*: this means that the composition of two operators $\sigma^{\mathrm{w}} \circ \rho^{\mathrm{w}}$ is in fact a Weyl operator with special symbol denoted by $\sigma \# \rho$. Explicit formulas for $\sigma \# \rho$ are known (cf. [51]) but we are more interested in the algebra structure induced on symbol spaces. It is indeed a peculiar feature of $M^{\infty,1}(\mathbb{R}^{2d})$, as well as of $M^{\infty}_s(\mathbb{R}^{2d})$ with $s > 2d$, to enjoy a double Banach algebra structure:

- a commutative one with respect to the pointwise multiplication as a consequence of Proposition 2;
- a non-commutative one with respect to the twisted product of symbols [31, 49]; for instance, $\sigma, \rho \in M^{\infty,1}(\mathbb{R}^{2d}) \Longrightarrow \sigma \# \rho \in M^{\infty,1}(\mathbb{R}^{2d})$.

Furthermore, it turns out that the latter algebraic structure can be related to a characterizing sparse behaviour satisfied by pseudodifferential operators with symbols in those spaces, the so-called *almost diagonalization property* with respect to time-frequency shifts; it can be proved that $\sigma \in M^{\infty}_s(\mathbb{R}^{2d})$ if and only if, for some (hence any) $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$,

$$|\langle \sigma^w \pi(z) g, \pi(w) g\rangle| \le C\langle w - z\rangle^{-s}, \quad z, w \in \mathbb{R}^{2d}.$$

In a similar fashion, $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ if and only if there exists $H \in L^1(\mathbb{R}^{2d})$ such that

$$|\langle \sigma^w \pi(z) g, \pi(w) g\rangle| \le H(w - z), \quad z, w \in \mathbb{R}^{2d}.$$

The reader may consult [6, 8–10, 29, 31] for further details on this topic.

3 Pointwise Convergence of Integral Kernels

The main results in [44] require us to consider a slightly generalized version of the free Hamiltonian operator H_0 in (1). Let a be a quadratic homogeneous polynomial on $\mathbb{R}^{2d}$, namely

$$a(x,\xi) = \frac{1}{2}x \cdot Ax + \xi \cdot Bx + \frac{1}{2}\xi \cdot C\xi,$$

for some symmetric matrices $A, C \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times d}$. The solution of (1) with $H_0 = a^{\mathrm{w}}$ (the Weyl transform of a) and $V = 0$ is given by

$$\psi(t,x) = e^{-itH_0}\varphi(x) = \mu(\mathscr{A}_t)\varphi(x),$$

where $\mu(\mathscr{A}_t)$ is a *metaplectic operator*—see [11, Sec. 15.1.3] and also [3, 24] for a complete derivation of this classic result. A precise characterization of metaplectic operators would lead us too far, hence we just outline their main features. First, recall that the phase-space flow governed by the Hamilton equations

$$\dot{z} = J\nabla_z a(z) = \mathbb{A}z, \quad \mathbb{A} = \begin{pmatrix} B & C \\ -A & -B^\top \end{pmatrix},$$

defines a mapping

$$\mathbb{R} \ni t \mapsto \mathscr{A}_t = e^{(t)\mathbb{A}} = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \in \mathrm{Sp}(d,\mathbb{R}). \tag{10}$$

In sloppy terms, any symplectic matrix $S \in \mathrm{Sp}(d,\mathbb{R})$ is associated with a unitary bounded operator $\mu(S)$ on $L^2(\mathbb{R}^d)$ which satisfies the intertwining property

$$\mu(S)^{-1}\sigma^{\mathrm{w}}\mu(S) = (\sigma \circ S)^{\mathrm{w}}, \quad \sigma \in \mathcal{S}'(\mathbb{R}^{2d}).$$

In particular, the classical flow $\mathscr{A}_t$ is associated (up to a complex phase factor) with a family of unitary operators on $L^2(\mathbb{R}^d)$ (for details see [28], Thm. 9.4.2) An explicit formula for $\mu(\mathscr{A}_t)$ may be provided in some special cases: for all $t \in \mathbb{R}$ such that $\mathscr{A}_t$ is a *free symplectic matrix*, namely such that the upper-right block B_t is invertible, the corresponding metaplectic operator may be represented as a *quadratic Fourier transform* [11, Sec. 7.2.2], namely

$$\mu(\mathscr{A}_t)\varphi(x) = c_t|\det B_t|^{-1/2}\int_{\mathbb{R}^d} e^{2\pi i\Phi_t}(x,\xi)\varphi(y)dy, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{11}$$

for suitable $c_t \in \mathbb{C}$, $|c_t| = 1$, where

$$\Phi_t(x, y) = \frac{1}{2}x \cdot D_t B_t^{-1} x - y \cdot B_t^{-1} x + \frac{1}{2} y \cdot B_t^{-1} A_t y, \qquad x, y \in \mathbb{R}^d. \tag{12}$$

This representation of $\mu(\mathscr{A}_t)$ is a main ingredient of our results, hence we stress that it does hold for any $t \in \mathbb{R} \setminus \widetilde{E}$, where we define the set of *exceptional times* as

$$\widetilde{E} = \{t \in \mathbb{R} : \det B_t = 0\}. \tag{13}$$

Some of the properties of this set can be immediately deduced from the fact that it is indeed the zero set of an analytic function: apart from the case $\widetilde{E} = \mathbb{R}$ (which trivially happens when $H = 0$), $\widetilde{E}$ is a discrete (hence at most countable) subset of $\mathbb{R}$ which always includes $t = 0$—in particular $\widetilde{E} = \{0\}$ in the case of the free Schrödinger equation ($V = 0$).

We now apply a version of Trotter formula from the theory of operator semigroups. It is known that $H_0 = a^{\mathrm{w}}$ is a self-adjoint operator on the maximal domain (see [33])

$$D(H_0) = \{\psi \in L^2(\mathbb{R}^d) : H_0\psi \in L^2(\mathbb{R}^d)\}.$$

For our purposes it is enough to assume that V is a bounded perturbation of H_0, namely $V \in \mathscr{B}(L^2(\mathbb{R}^d))$; notice that $V \in L^\infty(\mathbb{R}^d)$ is then a suitable choice, even for possibly complex-valued potentials.

Then, we have (cf. for instance [12, Cor. 2.7 and Ex. 2.9])

$$e^{-it(H_0+V)} = \lim_{n\to\infty} E_n(t), \qquad E_n(t) = \left(e^{-i\frac{t}{n}H_0} e^{-i\frac{t}{n}V}\right)^n, \tag{14}$$

where the convergence is intended in the strong operator topology in $L^2(\mathbb{R}^d)$. Let us denote by $e_{n,t}(x, y)$ the distribution kernel of $E_n(t)$ and by $u_t(x, y)$ that of $U(t) = e^{-it(H_0+V)}$.

We assume $V \in L^\infty(\mathbb{R}^d)$, and we tune its regularity as follows. In view of the discussion on modulation spaces in the previous section, we have available a scale of decreasing regularity spaces.

1. The best option for our purposes is given by $C_b^\infty(\mathbb{R}^d)$, the space of smooth bounded functions with bounded derivatives of any order.
2. At an intermediate level we have the (scale of) modulation spaces $M_s^\infty(\mathbb{R}^d)$, $s > 2d$, which contain bounded continuous functions becoming less regular as $s \searrow 2d$—the parameter s can be thought of as a measure of (fractional) differentiability.
3. We finally have a maximal space $M^{\infty,1}(\mathbb{R}^d)$, where the partial regularity of the previous level is completely lost. It is still a space of bounded continuous functions which locally enjoy the mild regularity of the Fourier transform of a L^1 function.

Let us first state our main result at the intermediate regularity encoded by $M^{\infty}_s(\mathbb{R}^d)$.

Theorem 2 *Let $H_0 = a^{\mathrm{w}}$ as discussed above and $V \in M^{\infty}_s(\mathbb{R}^d)$, with $s > 2d$. Let $\mathscr{A}_t$ denote the classical flow associated with H_0 as in (10). For any $t \in \mathbb{R} \setminus \widetilde{E}$:*

1. *the distributions $e^{-2\pi i \Phi_t} e_{n,t}$, $n \geq 1$, and $e^{-2\pi i \Phi_t} u_t$ belong to a bounded subset of $M^{\infty}_s(\mathbb{R}^{2d})$;*
2. *$e_{n,t} \to u_t$ in $(\mathscr{F}L^1_r)_{\mathrm{loc}}(\mathbb{R}^{2d})$ for any $0 < r < s-2d$, hence uniformly on compact subsets.*

The first claim ensures the kernel convergence problem is well posed under the given assumptions, since the kernels are indeed bounded continuous functions, while the second one characterizes the regularity at which convergence occurs—which clearly implies pointwise convergence.

We expect to improve the convergence result in the smooth context in view of the characterization given in (8).

Corollary 1 *Let $H_0 = a^{\mathrm{w}}$ as discussed above and $V \in C^{\infty}_b(\mathbb{R}^d)$. Let $\mathscr{A}_t$ denote the classical flow associated with H_0 as in (10). For any $t \in \mathbb{R} \setminus \widetilde{E}$:*

1. *the distributions $e^{-2\pi i \Phi_t} e_{n,t}$, $n \geq 1$, and $e^{-2\pi i \Phi_t} u_t$ belong to a bounded subset of $C^{\infty}_b(\mathbb{R}^{2d})$;*
2. *$e_{n,t} \to u_t$ in $C^{\infty}(\mathbb{R}^{2d})$, hence uniformly on compact subsets together with any derivatives.*

This result should be compared with the results by Fujiwara in [26], where convergence at the level of kernels in C^{∞}_b-sense for short times was proved. In spite of different assumptions and approximation schemes, we stress that our result is global in time.

We conclude with a convergence result in the same spirit, for potentials in the Sjöstrand class.

Theorem 3 *Let $H_0 = a^{\mathrm{w}}$ as discussed above and $V \in M^{\infty,1}(\mathbb{R}^d)$. Let $\mathscr{A}_t$ denote the classical flow associated with H_0 as in (10). For any $t \in \mathbb{R} \setminus \widetilde{E}$:*

1. *the distributions $e^{-2\pi i \Phi_t} e_{n,t}$, $n \geq 1$, and $e^{-2\pi i \Phi_t} u_t$ belong to a bounded subset of $M^{\infty,1}(\mathbb{R}^{2d})$;*
2. *$e_{n,t} \to u_t$ in $(\mathscr{F}L^1)_{\mathrm{loc}}(\mathbb{R}^{2d})$, hence uniformly on compact subsets.*

We stress that a typical potential setting in the papers by Albeverio and coauthors is “harmonic oscillator plus a bounded perturbation”, the latter in the form of the Fourier transform of a (finite) complex measure on $\mathbb{R}^d$—cf. [2] and the references therein. While those results rely on completely different techniques (in particular, infinite-dimensional oscillatory integral operators), in view of the embedding $\mathscr{F}\mathscr{M}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d)$ proved in [43, Prop. 3.4] we are able to cover this class of potentials too.

In addition to the regularity properties mentioned insofar, our choice of modulation space is particularly well suited to the problem in view of the rich algebraic structure discussed in Sect. 2. The key of the proofs is that for $t \in \mathbb{R} \setminus \widetilde{E}$ the approximate operator $E_n(t)$ can be expressed in integral form and a manageable form of the kernel $e_{n,t}$ can be derived. In particular, with the help of some technical lemmas we are able to write

$$\begin{aligned} E_n\,(t)\,\varphi(x) &= a^{\mathrm{w}}_{n,t}\,\mu\,(\mathscr{A}_t)\,\varphi(x) \\ &= c(t)\,|\det B_t|^{-1/2}\int_{\mathbb{R}^d} e^{2\pi i \Phi_t(x,y)}\widetilde{a_{n,t}}\,(x,y)\,\varphi\,(y)\,dy, \end{aligned} \tag{15}$$

where Φ_t is as in (12) and $\{a_{n,t}\}, \{\widetilde{a_{n,t}}\} \subset M^{\infty}_s(\mathbb{R}^{2d})$ are bounded sequences of symbols for fixed $t \in \mathbb{R} \setminus \widetilde{E}$.

4 Results on Integral Kernels at Exceptional Times

The occurrence of a set of exceptional times in Theorems 2 and 3 comes not as a surprise from a mathematical point of view: it may happen indeed that the integral kernel of the evolution operator degenerates into a distribution. A standard example of this phenomenon is provided by the harmonic oscillator, namely

$$i\partial_t\psi = -\frac{1}{4\pi}\triangle\psi + \pi|x|^2\psi.$$

The integral kernel of the corresponding evolution operator is known as the *Mehler kernel* and can be explicitly characterized [11, 36]: for $k \in \mathbb{Z}$,

$$u_t(x,y) = \begin{cases} c(k)|\sin t|^{-d/2}\exp\left(\pi i\frac{x^2+y^2}{\tan t} - 2\pi i\frac{x\cdot y}{\sin t}\right) & (\pi k < t < \pi(k+1)) \\ c'(k)\delta((-1)^k x - y) & (t = k\pi) \end{cases}, \tag{16}$$

for suitable phase factors $c(k), c'(k) \in \mathbb{C}$. This shows the expected degenerate behaviour at integer multiples of π, which is consistent with the fact that the associated classical flow $\mathscr{A}_t$ is given by

$$\mathscr{A}_t = \begin{pmatrix} (\cos t)I & (\sin t)I \\ -(\sin t)I & (\cos t)I \end{pmatrix},$$

where $I \in \mathbb{R}^{d\times d}$ is the identity matrix. Hence we retrieve $\widetilde{E} = \{t \in \mathbb{R} : \sin t = 0\} = \{k\pi : k \in \mathbb{Z}\}$.

We may wonder whether convergence of integral kernels still occurs in some distributional sense, hopefully better than the broadest one (that is $\mathcal{S}'(\mathbb{R}^{2d})$). In view of the discussion in Sect. 2 on the triple (M^1, L^2, M^∞), a suitable setting may be provided by M^∞. We have indeed a general result for the kernels of strongly convergent sequences of operators in L^2.

Theorem 4 *Let* $\{A_n\} \subset \mathscr{B}(L^2(\mathbb{R}^d))$, $n \in \mathbb{N}$, *be a sequence of bounded linear operators on* $L^2(\mathbb{R}^d)$ *with associated distribution kernels* $\{a_n\} \subset \mathcal{S}'(\mathbb{R}^{2d})$, *and* $A \in \mathscr{B}(L^2(\mathbb{R}^d))$ *with distribution kernel* $a \in \mathcal{S}'(\mathbb{R}^{2d})$. *Assume that* $A_n \to A$ *in the strong operator topology. Then:*

1. $a_n, a \in M^\infty(\mathbb{R}^{2d})$, $n \in \mathbb{N}$;
2. $a_n \to a$ *in the weak-* topology on* $M^\infty(\mathbb{R}^{2d})$.

In particular we have $a_n \to a$ *in* $\mathscr{F}L^\infty_{\text{loc}}(\mathbb{R}^{2d})$, *the latter space endowed with the topology* $\sigma((\mathscr{F}L^\infty)_{\text{loc}}(\mathbb{R}^{2d}), (\mathscr{F}L^1)_{\text{comp}}(\mathbb{R}^{2d}))$.

Proof We have that $\{A_n\}$ is a bounded sequence in $\mathscr{B}(L^2(\mathbb{R}^d))$ as a consequence of the uniform boundedness principle, hence also in $\mathscr{B}(M^1(\mathbb{R}^d), M^\infty(\mathbb{R}^d))$. The Feichtinger kernel theorem (Theorem 1) yields that the kernels a_n belong to a bounded subset of $M^\infty(\mathbb{R}^{2d})$. Similarly, $A \in \mathscr{B}(L^2(\mathbb{R}^d)) \Rightarrow a \in M^\infty(\mathbb{R}^{2d})$. For the second part of the claim we remark that $A_n \to A$ in the strong operator topology implies that $a_n \to a$ in $\mathcal{S}'(\mathbb{R}^{2d})$. Therefore, for any fixed non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ we have $V_g a_n \to V_g a$ pointwise in $\mathbb{R}^{2d}$. Moreover, we have the estimate $|V_g a_n(x,\xi)| \le C$, for some constant $C > 0$ independent of n by the first part of the proof. Hence, for any $\varphi \in M^1(\mathbb{R}^d)$ we have

$$\begin{aligned}\langle a_n, \varphi\rangle &= \int_{\mathbb{R}^{2d}} V_g a_n(x,\xi)\overline{V_g\varphi(x,\xi)}dxd\xi \\ &\to \int_{\mathbb{R}^{2d}} V_g a(x,\xi)\overline{V_g\varphi(x,\xi)}dxd\xi = \langle a, \varphi\rangle,\end{aligned}$$

by the dominated convergence theorem. □

It would be interesting to prove the boundedness of a_n in $M^\infty(\mathbb{R}^{2d})$ in Theorem 4 without using the uniform boundedness principle, although it could be not immediate.

A straightforward application of this result allows us to prove global-in-time convergence of integral kernels, although in a weaker sense than before.

Corollary 2 *Assume* $V \in L^\infty(\mathbb{R}^d)$. *Let* $e_{n,t} \in \mathcal{S}'(\mathbb{R}^{2d})$ *be the distribution kernel of the Feynman-Trotter parametrix* $E_n(t)$ *in* (14) *and* $u_t \in \mathcal{S}'(\mathbb{R}^{2d})$ *be the kernel of the Schrödinger evolution operator* $U(t)$ *associated with the Cauchy problem* (1). *For any* $n \in \mathbb{N}$ *and* $t \in \mathbb{R}$ *we have* $e_{n,t}, u \in M^\infty(\mathbb{R}^{2d})$. *Moreover,* $e_{n,t} \to u_t$ *in the weak-* topology on* $M^\infty(\mathbb{R}^{2d})$ *for any fixed* $t \in \mathbb{R}$.

For more regular potentials we expect that the conclusion of Corollary 2 can be improved. Let us first provide a version of the Trotter formula for potentials in $M^{\infty,1}(\mathbb{R}^d)$, with strong convergence on $M^1(\mathbb{R}^d)$.

Theorem 5 *Assume $V \in M^{\infty,1}(\mathbb{R}^d)$. Let $\{E_n(t)\}$ be the sequence of Feynman-Trotter parametrices defined in (14) and $U(t)$ be the Schrödinger evolution operator $U(t)$ associated with the Cauchy problem (1). For any fixed $t \in \mathbb{R}$ we have*

$$\lim_{n\to\infty} E_n(t) = U(t), \qquad \lim_{n\to\infty} E_n(t)^* = U(t)^*$$

in the strong topology of operators acting on $M^1(\mathbb{R}^d)$.

Proof We prove that $E_n(t) \to U(t)$ strongly in $\mathscr{B}(M^1(\mathbb{R}^d))$; the claim concerning adjoint operators follows by similar arguments since $U(t)^* = U(-t)$ and $E_n(t)^* = \left(e^{i\frac{t}{n}V}e^{i\frac{t}{n}H_0}\right)^n$.

As already observed, we know that the operator H_0 with domain $D(H_0) = \{\varphi \in L^2(\mathbb{R}^d) : H_0\varphi \in L^2(\mathbb{R}^d)\}$ is self-adjoint [33]. Let $U_0(t) = e^{-itH_0}$ be the corresponding strongly continuous unitary group on $L^2(\mathbb{R}^d)$. The well-posedness of the Schrödinger equation $i\partial_t\psi = H_0\psi$ in $M^1(\mathbb{R}^d)$ (see e.g. [7]) implies that the restriction of $U_0(t)$ to $M^1(\mathbb{R}^d)$ defines a strongly continuous group on $M^1(\mathbb{R}^d)$, its generator being the restriction of H_0 to the subspace $\{\varphi \in M^1(\mathbb{R}^d) : H_0\varphi \in M^1(\mathbb{R}^d)\}$, as a consequence of known results on subspace semigroups, cf. [12, Chapter 2, Sec. 2.3]. Since the pointwise multiplication by $V \in M^{\infty,1}(\mathbb{R}^d)$ defines a bounded operator on $M^1(\mathbb{R}^d)$, the desired result follows from the classical Trotter formula ([12, Cor. 2.7 and Ex. 2.9]). □

We provide an equivalent formulation of the previous result for the corresponding integral kernels, which is indeed a partial counterpart of the pointwise convergence results of Sect. 3.

Theorem 6 *Under the same assumptions of Theorem 3, for all $t \in \mathbb{R}$ and $\varphi \in M^1(\mathbb{R}^d)$, the functions*

$$\langle e_{n,t}(x,\cdot),\varphi\rangle, \quad \langle e_{n,t}(\cdot,y),\varphi\rangle, \quad \langle u_t(x,\cdot),\varphi\rangle, \quad \langle u_t(\cdot,y),\varphi\rangle$$

belong to $M^1(\mathbb{R}^d)$.

Moreover

$$\langle e_{n,t}(x,\cdot),\varphi\rangle \to \langle u_t(x,\cdot),\varphi\rangle, \quad \langle e_{n,t}(\cdot,y),\varphi\rangle \to \langle u_t(\cdot,y),\varphi\rangle$$

in $M^1(\mathbb{R}^d)$, hence in $L^p(\mathbb{R}^d)$ for every $1 \le p \le \infty$.

The last conclusion follows from the continuous embedding $M^1(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, for every $1 \le p \le \infty$.

Remark 1 We expect other improvements of Theorem 4 to hold in the case where $A_n = E_n(t)$, $A = U(t)$. In particular, convergence result for the corresponding integral kernels could be investigated in the context of mixed modulation spaces and generalized kernel theorems in the spirit of [4]. We will not engage in such formulation here in order to avoid quite technical discussions.

5 Physics at Exceptional Times

In spite of the attempts to shed light on the nature of exceptional times and the partial results in the previous section, a physical interpretation of exceptional times is still not clear at the moment. This non-trivial question also appears in the form of an enigmatic exercise in the textbook [23, Problem 3-1] by Feynman and Hibbs. While dimensional analysis and heuristic arguments may provide some hints, a precise answer still seems to be missing.

We give our contribution to this discussion with a short argument which elucidates the nature of exceptional times in terms of measurable quantities. Recall that $B(u, r)$ denotes the ball with center $u \in \mathbb{R}^d$ and radius $r > 0$ in $\mathbb{R}^d$. Following the custom in physics we adopt below the Bra-ket notation, and we identify states with their wave functions in the position representation.

Fix $x_0, y_0 \in \mathbb{R}^d$ and $a, b > 0$, and consider the normalised wave-packets

$$|A\rangle = \frac{1}{\sqrt{|B(y_0, a)|}} \mathbb{1}_{B(y_0,a)}, \quad |B\rangle = \frac{1}{\sqrt{|B(x_0, b)|}} \mathbb{1}_{B(x_0,b)}.$$

The corresponding transition amplitude from the state $|A\rangle$ to $|B\rangle$ under the Hamiltonian $H = H_0 + V$ as in Theorem 3, namely

$$I = I(t, x_0, y_0, a, b) = \langle B|U(t)|A\rangle, \quad t \in \mathbb{R},$$

trivially satisfies the estimate

$$|I(t, x_0, y_0, a, b)| \le 1, \quad \forall t \in \mathbb{R},\ x_0, y_0 \in \mathbb{R}^d,\ a, b > 0.$$

This bound cannot be improved at exceptional times: consider for instance the case where $t = 0$, $x_0 = y_0$ and $a = b$, which yields $I = 1$. Nevertheless, we have the following result.

Proposition 3 *Under the same assumptions of Theorem 3, for all $t \in \mathbb{R} \setminus \widetilde{E}$ and $x_0, y_0 \in \mathbb{R}^d$ we have*

$$\lim_{a,b\to 0} \frac{I(t, x_0, y_0, a, b)}{(ab)^{d/2}} = C\overline{u_t(x_0, y_0)},$$

where $C = C(d) = |B(0, 1)|$.

Proof An explicit computation yields

$$\frac{I(t, x_0, y_0, a, b)}{C(ab)^{d/2}} = \frac{1}{C^2(ab)^d} \int_{B(x_0,b)} \int_{B(y_0,a)} \overline{u_t(x, y)} dy dx,$$

and the conclusion follows by the continuity of $u_t(x, y)$ in $\mathbb{R}^{2d}$, because $u_t \in \big(\mathscr{F}L^1\big)_{\mathrm{loc}}(\mathbb{R}^{2d})$ for $t \in \mathbb{R} \setminus \widetilde{E}$ by Theorem 3. □

This result shows that while $|I| \leq 1$ in general, for a non-exceptional time $t \in \mathbb{R} \setminus \widetilde{E}$ we have that $|I| \sim (ab)^{d/2}$ as $a, b \to 0$. In particular $|I| \to 0$ as $a, b \to 0$ except (possibly) for exceptional times.

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Decay Estimates for a Klein–Gordon Model with Time-Periodic Coefficients

Giovanni Girardi and Jens Wirth

Abstract In this paper we consider a Klein–Gordon model with time-dependent periodic coefficients. The aim is to investigate how the presence of the mass term influences energy estimates with respect to the case of vanishing mass, already treated by J. Wirth (Hiroshima Math J 38:397–410, 2008). The approach is based on a diagonalisation argument for high frequencies and a contradiction argument for bounded frequencies.

Keywords Wave equation · Damped Klein-Gordon models · Periodic coefficients · Long time decay estimates

1 Introduction

In [18] the second author considered the linear Cauchy problem for a damped wave equation with time-periodic dissipation term $b(t)$,

$$\begin{cases} u_{tt} - \Delta u + 2b(t)u_t = 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

and proved that the solution to (1) satisfies the well-known Matsumura-type estimate obtained for constant dissipation by A. Matsumura in [8], that is

$$\|\partial_t^k \nabla^j u(t,\cdot)\|_{L^2} \le C(1+t)^{-j-\frac{k}{2}}(\|u_0\|_{H^{j+k}} + \|u_1\|_{H^{j+k-1}}), \tag{2}$$

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for $j,\ k = 0,\ 1$ and C a positive constant independent on the initial data. In this paper we generalise these results and consider the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + 2b(t)u_t + m^2(t)u = 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \end{cases} \tag{3}$$

with positive time-periodic dissipation $b(t)$ and mass $m(t)$. We study how the presence of a periodic mass term influences the decay estimates for the solution to (3).

Let us first explain, why such a problem is interesting and how it relates to known results from the literature. There exist many papers in which decay estimates for the solution to wave models of the form (3) are investigated under different assumptions on the coefficients $b(t)$ and $m(t)$. The survey articles [12] and [19] provide for an overview of results; moreover, we refer to the works of M. Reissig and K. Yagdjian [14], of F. Hirosawa and M. Reissig [7], of M. Reissig and J. Smith [13], as well as the papers of the second author [15], [16]. In the latter two papers a classification of dissipation terms as *non-effective* or *effective* is introduced, which distinguishes the dissipation terms according to their strength and influence on the large-time behaviour of solutions. In all these results a control on the amount of oscillations present in the coefficients is essential.

To understand this and the meaning of this classification we consider the Cauchy problem (1) with the coefficient b assumed to be a bounded, non-negative, sufficiently smooth function satisfying a condition of the form

$$|\partial_t^k b(t)| \le C_k \frac{b(t)}{(1+t)^k} \qquad \text{for } k = 1,\ 2. \tag{4}$$

Then, we distinguish between two cases. First, if

$$\limsup_{t\to\infty} t b(t) < 1. \tag{5}$$

we say that b is *non-effective*, in the sense that the solution behaves in an asymptotic sense like a free wave multiplied by a decay factor, that is there exists a solution $v = v(t,x)$ to the wave equation $v_{tt} - \Delta v = 0$ such that

$$\begin{pmatrix} \nabla u(t,x) \\ u_t(t,x) \end{pmatrix} \sim \frac{1}{\lambda(t)} \begin{pmatrix} \nabla v(t,x) \\ v_t(t,x) \end{pmatrix}, \qquad t \to \infty,$$

the asymptotic equivalence understood in an appropriate L^p-sense and with $\lambda = \lambda(t)$ given as

$$\lambda(t) = \exp\Big(\frac{1}{2}\int_0^t b(\tau)\,d\tau\Big).$$

The initial data to the free wave $v = v(t, x)$ are uniquely determined by the solution $u = u(t, x)$ and thus by the initial data u_0 and u_1. Thus, a modified form of scattering is valid. On the other hand, if

$$\lim_{t\to\infty} tb(t) = \infty$$

holds true we say that the dissipation b is *effective*; in this case solutions to damped wave equation are asymptotically related to solutions $w = w(t, x)$ of the parabolic heat equation $w_t = \Delta w$, i.e.

$$u(t, x) \sim w(t, x)$$

holds true again in an appropriate L^p-sense. This can be made precise in the form of the so-called *diffusion phenomenon* for damped waves; see [17] for the time-dependent dissipation case or the papers of Nishihara [10] and Narazaki [9] for the case of constant dissipation.

Wave models with mass and dissipation of the form (3) were considered by the second author and Nunes in [11]. This paper provides in particular $L^p - L^q$ decay estimates in the non-effective case. In [4] the first author considered with M. D'Abbico and M. Reissig the Cauchy problem (3) in the case in which the damping term is effective and dominates the mass term, i.e. $m(t) = o(b(t))$ as $t \to \infty$, again under control assumptions on the oscillations of the coefficients. In that paper it is shown that under a simple condition on the interaction between $b(t)$ and $m(t)$, one can prove that the solutions to (3) satisifies the estimate

$$\|u(t,\cdot)\|_{L^2} \leq C\,\gamma(t)\,\|(u_0, u_1)\|_{H^1\times L^2}, \tag{6}$$

where we define

$$\gamma(t) = \exp\left(-\int_0^t \frac{m^2(\tau)}{b(\tau)}\,d\tau\right). \tag{7}$$

Thus, the decreasing function $\gamma = \gamma(t)$ in (7) represents the influence on the estimates of the mass term with respect to the damping term. In particular, estimate (6) shows that the presence of the mass term produces an additional decay which becomes faster as the mass term becomes more influent. In fact, in [5] the first author proved an exponential decay in the case of dominant mass, that is

$$\|u(t,\cdot)\|_{L^2} \leq C\,\exp\left(-\delta\int_0^t b(\tau)\,d\tau\right)\,\|(u_0, u_1)\|_{H^1\times L^2}, \tag{8}$$

provided that $\liminf_{t\to\infty} m(t)/b(t) > 1/4$. This latter estimate is almost the same as for the solution to the Cauchy problem to a damped Klein–Gordon model with constant coefficients $b(t) \equiv 1$ and $m(t) \equiv 1$, that is

$$\begin{cases} u_{tt} - \Delta u + u_t + u = 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \end{cases} \tag{9}$$

All these cited papers have in common that they use assumptions on derivatives of the coefficients as in (4) to avoid a bad influence of oscillations. That oscillations may have deteriorating influences was shown for example by K. Yagdjian in [20] for a wave equation with time-periodic speed of propagation. In this case (many) solutions have exponentially growing energy. Controlling oscillations is done by requiring estimates for derivatives of the coefficients.

It is clear that for dissipative wave equations oscillations in the positive dissipation term can not lead to solutions with increasing energy. Therefore, it is interesting to ask whether conditions on derivatives of the coefficient are indeed necessary for proving large-time decay estimates for solutions of (1). A first step to look into that was done in [18], where the author proved that the solution to (1) satisifies estimate (2) without any condition on the oscillations of $b = b(t)$ provided that b is periodic. This led to the conjecture that estimate (2) can be obtained with a general dissipation term $b = b(t)$, with $tb(t) \to \infty$, without further assumptions on derivatives. However, it is still an open problem how to prove such a result.

In the present paper we also avoid assumptions on the derivatives of the coefficients $b(t)$ and $m(t)$ assuming only that they are positive, periodic and of bounded variation. We are going to prove an exponential decay by using the same technique used as in [18] combined with a perturbation argument for the mass term. We remark that the presence of the mass term simplifies the study of the estimates at small frequencies; in fact, in this zone it is not necessary to use tools of Floquet theory as in the case of vanishing mass: we use only a contradiction argument together with some results of spectral theory of matrices.

The study of decay estimates for the solution to the linear problem (3) has an important application in the study of global (in time) existence results for the corresponding nonlinear problem

$$\begin{cases} u_{tt} - \Delta u + 2b(t)u_t + m^2(t)u = h(t,u), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \end{cases}$$

with nonlinearity $h(t,u) = (1 + \int_0^t 1/b(\tau)\,d\tau)^{\omega}|u(t,\cdot)|^p$ for a $\omega \in [-1,\infty)$. Such applications can be found for example in [2, 3] in the purely dissipative case and in [4–6] for equations including mass terms.

The paper is organized as follows: In Sect. 2 we give the basic assumptions on the Cauchy problem and we state our main results that are Theorems 1 and 2; in Sect. 3 we make considerations and discuss properties of the fundamental solution to (3) and the associated monodromy operator. In Sect. 4 we treat the case of constant mass for small frequencies and we prove a fundamental lemma useful for the proof of the main theorems. Finally in Sect. 5 the main theorems are proved.

2 Main Results

In this paper we suppose that the coefficient $b = b(t)$ is a non-negative and continuous periodic function of bounded variation, i.e., we assume that its weak derivative is essentially bounded, $b' \in L^\infty$. We further suppose that the coefficient $m = m(t)$ is measurable and periodic with the same period. We denote the period of both coefficients by T. The first result concerns constant mass terms and provides an exponential decay result.

Theorem 1 *Suppose $m \equiv m_0 \in \mathbb{R}$ is constant. There exists $\delta > 0$ such that the solution $u = u(t, x)$ to the Cauchy problem (3) satisfies*

$$\|u(t,\cdot)\|_{L^2} \le Ce^{-\delta t}(\|u_0\|_{L^2} + \|u_1\|_{H^{-1}}),$$
$$\|\nabla u(t,\cdot)\|_{L^2} \le Ce^{-\delta t}(\|u_0\|_{H^1} + \|u_1\|_{L^2}),$$
$$\|u_t(t,\cdot)\|_{L^2} \le Ce^{-\delta t}(\|u_0\|_{H^1} + \|u_1\|_{L^2}),$$

where δ and C are positive constants depending on the coefficient b and on m_0.

If the mass term is non-constant, the exponential decay is obtained under a smallness condition for the deviation of the mass-term from a constant.

Theorem 2 *Let $m_0 \in \mathbb{R}$ and $m_1 = m_1(t)$ a measurable T-periodic function such that $\sup_{t\ge 0} |m_1(t)| = 1$. Then, there exists ϵ sufficiently small such that the solution to*

$$\begin{cases} u_{tt} - \Delta u + 2b(t)u_t + m_\epsilon^2(t)u = 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \end{cases} \tag{10}$$

with $m_\epsilon^2(t) = m_0^2 + \epsilon m_1(t)$ satisfies

$$\|u(t,\cdot)\|_{L^2} \le Ce^{-\sigma t}(\|u_0\|_{L^2} + \|u_1\|_{H^{-1}}),$$
$$\|\nabla u(t,\cdot)\|_{L^2} \le Ce^{-\sigma t}(\|u_0\|_{H^1} + \|u_1\|_{L^2}),$$
$$\|u_t(t,\cdot)\|_{L^2} \le Ce^{-\sigma t}(\|u_0\|_{H^1} + \|u_1\|_{L^2}),$$

where σ and C are positive constant depending on m_0, m_1, b and ϵ.

Remark 1 It is still an open problem to understand which is the largest value that ϵ can assume in order to guarantee an exponential decay of the energy. A possible estimate of ϵ is given in the proof of Theorem 2: from estimate (36) it is clear that the value of ϵ depends on how large we choose N, such that the line $|\xi| = N$ divides the phase space in small and large frequencies. In particular, the value of N depends only on the dissipation and does not depend on the mass term.

3 Representation of Solution

In a first step we derive properties of the representation of solutions for the Cauchy problem

$$u_{tt} - \Delta u + 2b(t)u_t + m^2(t)u = 0, \qquad u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \tag{11}$$

with $b = b(t) \geq 0$ and $m = m(t) \geq 0$ both periodic of period T. We denote the mean value of $b(t)$ as

$$\beta = \frac{1}{T}\int_0^T b(t)\,dt.$$

A partial Fourier transform with respect to the spatial variables reduces the problem to an ordinary differential equation

$$\hat{u}_{tt} + |\xi|^2\hat{u} + 2b(t)\hat{u}_t + m^2(t)\hat{u} = 0, \tag{12}$$

parameterised by $|\xi| \in \mathbb{R}$. To reformulate this as first order system, we introduce the symbol $\langle\xi\rangle_{m(t)} := \sqrt{|\xi|^2 + m^2(t)}$ and we define the new variable $V = (\langle\xi\rangle_{m(t)}\hat{u}, D_t\hat{u})^T$. Then we obtain the system $D_t V = A(t,\xi)V$ with

$$A(t,\xi) = \begin{pmatrix} 0 & \langle\xi\rangle_{m(t)} \\ \langle\xi\rangle_{m(t)} & 2ib(t) \end{pmatrix}, \tag{13}$$

using the Fourier derivative $D_t = -i\partial_t$. We want to study the fundamental solution $\mathscr{E} = \mathscr{E}(t,s,\xi)$ to (13), that is the matrix-valued solution to the Cauchy problem

$$D_t\mathscr{E}(t,s,\xi) = A(t,\xi)\mathscr{E}(t,s,\xi), \qquad \mathscr{E}(s,s,\xi) = I. \tag{14}$$

In particular, we consider the family of monodromy matrices $\mathscr{M}(t,\xi) = \mathscr{E}(t+T,t,\xi)$. The fundamental solution to (14) can be represented by the Peano–Baker series

$$\mathscr{E}(t,s,\xi) = I + \sum_{\ell=1}^{\infty} i^k \int_s^t A(t_1,\xi)\int_s^{t_1} A(t_2,\xi)\cdots\int_s^{t_{\ell-1}} A(t_\ell,\xi)\,dt_\ell\cdots dt_1. \tag{15}$$

The T-periodicity of coefficients implies periodicity of the matrix $A(t,\xi)$ and hence the T-translation invariance of the fundamental solution, i.e. $\mathscr{E}(t+T,s+T,\xi) = \mathscr{E}(t,s,\xi)$. Thus, the the monodromy matrix $\mathscr{M}(t,\xi)$ is T-periodic. Moreover, since $\mathscr{E}(t,s,\xi)\mathscr{E}(s,t,\xi) = I$ it follows that $\mathscr{E}(t,s,\xi)$ satisfies $D_s\mathscr{E}(t,s,\xi) = -\mathscr{E}(t,s,\xi)A(s,\xi)$, and, therefore, $\mathscr{M}(t,\xi)$ satisfies the equation

$$D_t\mathscr{M}(t,\xi) = [A(t,\xi),\mathscr{M}(t,\xi)], \qquad \mathscr{M}(T,\xi) = \mathscr{M}(0,\xi).$$

In what follows we will distinguish between small and large frequencies and provide estimates for $\mathscr{M}$.

3.1 Large Frequencies

For large frequencies we want to prove that the monodromy matrix is uniformly contractive, i.e.

$$\|\mathscr{M}(t,\xi)\| < 1 \tag{16}$$

holds true uniformly in $t \in [0,T]$ and $|\xi| \geq N$ for a constant N chosen large enough. The choice of N does not depend on the coefficient $m = m(t)$. In order to prove (16) we apply two steps of diagonalization. We consider the unitary matrices

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad M^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and define the new variable $V^{(0)} = M^{-1}V$, which satisfies

$$D_tV^{(0)} = (D(t,\xi) + R(t,\xi))V^{(0)}$$

with

$$D(t,\xi) = \begin{pmatrix} \langle\xi\rangle_{m(t)} & 0 \\ 0 & -\langle\xi\rangle_{m(t)} \end{pmatrix}, \qquad R(t,\xi) = ib(t)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Next, we define $D_1 = D + \operatorname{diag} R$ and $R_1 = R - \operatorname{diag} R$ and construct a matrix $N_1 = N_1(t,\xi)$ with

$$D_tN_1 = [D_1,N_1] + R_1, \tag{17}$$

and $N_1(0,\xi) = I$. Thus, the requirement for N_1 is equivalent to the operator identity

$$(D_t - D_1 - R_1)N_1 - N_1(D_t - D_1) = D_tN_1 - [D_1,N_1] - R_1N_1 = R_1(I - N_1).$$

Hence by denoting $R_2 = -N_1^{-1} R_1 (I - N_1)$ we obtain

$$(D_t - D_1 - R_1) N_1 = N_1 (D_t - D_1 - R_2)$$

and as a consequence, provided that N_1 is invertible, we obtain that the new unknown $V^{(1)} = N_1^{-1} V^{(0)}$ satisfies the transformed equation

$$D_t V^{(1)} = (D_1 + R_2) V^{(1)}$$

with improved remainder allowing us later on to prove (16).

Since $N_1 = N_1(t, \xi)$ satisfies equation (17) and D_1 is diagonal, we find $D_t \operatorname{diag} N_1 = 0$. Thus, we can use a matrix N_1 of the form

$$N_1 = \begin{pmatrix} 1 & n^- \\ n^+ & 1 \end{pmatrix},$$

with

$$D_t n^\pm(t, \xi) = \mp \langle \xi \rangle_{m(t)} n^\pm(t, \xi) + i b(t).$$

The initial conditions $n^\pm(0, \xi) = 0$ giving $N_1(0, \xi) = I$ imply

$$n^\pm(t, \xi) = \int_0^t e^{\mp i \int_s^t \langle \xi \rangle_{m(r)}\, dr} b(s)\, ds.$$

Integrating by parts, we obtain

$$|n^\pm(t, \xi)| = \left| \left[\frac{\mp i}{\langle \xi \rangle_{m(s)}} e^{\mp i \int_s^t \langle \xi \rangle_{m(r)}\, dr} b(s) \right]_0^t - \int_0^t \frac{\mp i}{\langle \xi \rangle_{m(s)}} e^{\mp i \int_s^t \langle \xi \rangle_{m(r)}\, dr} b'(s) ds \right|$$

and using that $b = b(t)$ is of bounded variation we find a constant $C > 0$ such that

$$|n^\pm(t, \xi)| \le C(1 + t)|\xi|^{-1}. \tag{18}$$

Thus we get that $n^\pm(t, \xi) \to 0$ when $\xi \to \infty$, uniformly in $[0, 2T]$. Then we can conclude that $N_1(t, \xi) \to I$ and therefore $N^{-1}(t, \xi) \to I$ uniformly in $t \in [0, 2T]$ as $|\xi| \to \infty$. Hence $\|R_2(t, \xi)\| \to 0$ as $|\xi| \to \infty$ uniformly in $t \in [0, 2T]$. Thus the supremum on the left hand side in the following formula tends to 1 as $N \to \infty$ and we fix N such that

$$\sup_{|\xi| \ge N} \sup_{t \in [0, T]} \|N_1(t + T, \xi)\| e^{\int_t^{t+T} \|R_2(s, \xi)\| ds} \|N_1^{-1}(t, \xi)\| \le e^{\beta T/2} \tag{19}$$

holds true. Note that this choice of N can be made indepent of the coefficient $m = m(t)$, in fact $R_1 = R_1(t,\xi)$ does not depend on $m(t)$, and $N_1(t,\xi)$, $N_1^{-1}(t,\xi)$ tend both to I uniformly with respect to $m(t)$.

In order to prove the desired estimate (16) we go back to the original problem. We define $\lambda(t) := \exp(\int_0^t b(\tau)d\tau)$. Then, for each $|\xi| > N$ the fundamental solution $\mathscr{E}(t,s,\xi)$ to $D_t V = A(t,\xi)V$ with A defined in (13) is given by

$$\mathscr{E}(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)} M N_1(t,\xi)\tilde{\mathscr{E}}_0(t,s,\xi)Q(t,s,\xi)N_1^{-1}(t,\xi)M^{-1}, \tag{20}$$

for all $t \in [0,T]$, where

$$\tilde{\mathscr{E}}_0(t,s,\xi) = \begin{pmatrix} e^{i\int_s^t \langle\xi\rangle_{m(\tau)}d\tau} & 0 \\ 0 & e^{-i\int_s^t \langle\xi\rangle_{m(\tau)}d\tau} \end{pmatrix}$$

and $Q = Q(t,s,\xi)$ is the solution to the Cauchy problem

$$D_t Q(t,s,\xi) = \tilde{\mathscr{E}}_0(s,t,\xi)R_2(t,\xi)\tilde{\mathscr{E}}_0(t,s,\xi)Q(t,s,\xi), \qquad Q(s,s,\xi) = I.$$

Let $\mathscr{R}_2(t,s,\xi) = \tilde{\mathscr{E}}_0(s,t,\xi)R_2(t,\xi)\tilde{\mathscr{E}}_0(t,s,\xi)$. Then by using the Peano-Beaker formula again we can represent $Q(t,s,\xi)$ as

$$Q(t,s,\xi) = I + \sum_{\ell=1}^{\infty} i^k \int_s^t \mathscr{R}_2(t_1,s,\xi)\int_s^{t_1}\mathscr{R}_2(t_2,t_1,\xi)\cdots\int_s^{t_{k-1}}\mathscr{R}_2(t_k,t_{k-1},\xi)dt_k\ldots dt_1.$$

Since $\|\mathscr{R}_2(t,s,\xi)\| = \|R_2(t,\xi)\|$ we conclude

$$\|Q(t,s,\xi)\| \le \exp\Big(\int_s^t \|R_2(\tau,\xi)\|d\tau\Big). \tag{21}$$

By (20) we can represent the monodromy matrix $\mathscr{M}(t,\xi) = \mathscr{E}(t,s,\xi)$ as

$$\mathscr{M}(t,\xi) = \frac{\lambda(t)}{\lambda(t+T)} M N_1(t+T,\xi)\tilde{\mathscr{E}}_0(t+T,t,\xi)Q(t+T,t,\xi)N_1^{-1}(t+T,\xi)M^{-1}.$$

Since $\lambda(t)/\lambda(t+T) = e^{-\beta T}$ the desired result $\|\mathscr{M}(t,\xi)\| \le e^{-\beta T/2} < 1$ for each $t \in [0,T]$ and each $|\xi| \ge N$ follows by (19) and (21). Hence we obtain

Lemma 1 *There exists a constant N depending only on T, $\|b'\|_\infty$ and $\|b\|_\infty$ such that the monodromy matrix $\mathscr{M}(t,\xi)$ satisfies*

$$\|\mathscr{M}(t,\xi)\| \le e^{-\beta T/2}$$

uniformly on $t \in \mathbb{R}$ and $|\xi| \ge N$ and independent of the mass term $m(t)$.

Remark 2 In the case of constant dissipation $b \equiv 1$ it is possible to give an explicit value of admissible N. In fact, in such case we find explicitely

$$n^{\pm}(t,\xi) = \frac{\mp i}{\langle\xi\rangle_{m(t)}} - \frac{\mp i}{\langle\xi\rangle_{m(0)}} e^{\mp i \int_0^t \langle\xi\rangle_{m(r)}\,dr}.$$

Moreover, it holds

$$\det N_1(t,\xi) = 1 - n^+(t,\xi)n^-(t,\xi),$$

and we can estimate

$$\begin{aligned} n^+(t,\xi)n^-(t,\xi) &= \left(\frac{1}{\langle\xi\rangle_{m(t)}^2} - \frac{2}{\langle\xi\rangle_{m(t)}\langle\xi\rangle_{m(0)}}\cos\left(\int_0^t \langle\xi\rangle_{m(r)}\,dr\right) + \frac{1}{\langle\xi\rangle_{m(0)}^2}\right) \\ &\le \left(\frac{\langle\xi\rangle_{m(t)} + \langle\xi\rangle_{m(0)}}{\langle\xi\rangle_{m(t)}\langle\xi\rangle_{m(0)}}\right)^2 \le \frac{2}{N}. \end{aligned}$$

In the last estimate we used that the function $(x+y)/(xy)$ is decreasing with respect to x and y and it holds $\langle\xi\rangle_{m(t)}, \langle\xi\rangle_{m(0)} \ge N$. Toghether with estimate (18) this allows to conclude that

$$\|R_2(t,\xi)\| \lesssim \frac{1}{N-2}.$$

Thus, estimate (19) is satisfied, for instance, if we take $N > 4$.

4 Small Frequencies: Constant Mass

In this section we want to prove that there exists $k \in \mathbb{N}$ such that

$$\|\mathscr{M}^k(t,\xi)\| < 1 \tag{22}$$

uniformly in $|\xi| \le N$ and $t \in [0,T]$ provided that the mass term is constant. Thus, in this section, we restrict our study to the Cauchy problem

$$v_{tt} - \Delta v + 2b(t)v_t + m_0^2 v = 0 \qquad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x). \tag{23}$$

In particular, we denote by $\mathscr{E}_0(t,s,\xi)$ the fundamental solution associated to the system $D_t V = A_0(t,\xi)V$ with

$$A_0(t,\xi) = \begin{pmatrix} 0 & \langle\xi\rangle_{m_0} \\ \langle\xi\rangle_{m_0} & 2ib(t) \end{pmatrix}, \qquad \langle\xi\rangle_{m_0} = \sqrt{|\xi|^2 + m_0^2}. \tag{24}$$

Let $\mathscr{M}_0(t,\xi)=\mathscr{E}_0(t+T,0,\xi)$ be the corresponding family of monodromy matrices. In order to get our aim we will prove at first that the spectrum spec $\mathscr{M}_0(t,\xi)$ is contained in the open ball $\{\eta\in\mathbb{C}||\eta|<1\}$.

Since it holds

$$\mathscr{M}_0(t,\xi)\mathscr{E}_0(t,0,\xi)=\mathscr{E}_0(t+T,0,\xi)=\mathscr{E}_0(t+T,T,\xi)\mathscr{E}_0(T,0,\xi)=\mathscr{E}_0(t,0,\xi)\mathscr{M}_0(0,\xi),$$

we conclude that for each $t\in[0,T]$ the monodromy matrix $\mathscr{M}_0(t,\xi)$ is similar to $\mathscr{M}_0(0,\xi)$ and, hence, has the same spectrum. Moreover, as both $b(t)$ and $m(t)$ are real; the Eq. (12) has real solutions and it follows that $\mathscr{M}_0(t,\xi)$ is similar to a real-valued matrix. Furthermore, by Liouville Theorem we know that

$$\det\mathscr{M}_0(0,\xi)=e^{i\int_0^T\operatorname{tr}A_0(\tau,\xi)d\tau}=e^{-2\beta T}. \tag{25}$$

Hence, for each $\xi\in\mathbb{R}^n$ the eigenvalues $\eta_1(\xi)$, $\eta_2(\xi)$ of $\mathscr{M}_0(0,\xi)$ are either real, in the form $\eta_2(\xi)=\eta_1^{-1}(\xi)e^{-2\beta T}$, or complex conjugate with $|\eta_1(\xi)|=|\eta_2(\xi)|=e^{-\beta T}$. In the latter case it is clear that spec $\mathscr{M}_0(0,\xi)\subset\{\xi\in\mathbb{R}^n||\xi|=\exp(-\beta T)\}$. In the case in which the eigenvalues are real we need to prove that for each $\xi\in\mathbb{R}^n$ both $\eta_1(\xi)$ and $\eta_2(\xi)$ have modulus less then 1. We will prove this by using a contradiction argument.

Suppose that there exists $\bar\xi\in\mathbb{R}^n$ such that the monodromy matrix $\mathscr{M}_0(0,\bar\xi)$ has an eigenvalue of modulus 1, i.e, $\eta_1(\bar\xi)=\pm1$ and so $\eta_2(\bar\xi)=\pm e^{-2\beta T}$. Let $\mathbf{c}=(c_1,c_2)$ be an eigenvector corresponding to $\eta_1(\bar\xi)$. Then, we can find a domain $\Omega_R=\{x\in\mathbb{R}^n||x|\le R\}$ (with R depending on m_0) and a function $\Phi=\Phi(x)$ defined on Ω_R such that $-|\bar\xi|^2-m_0^2$ is an eigenvalue for the Dirichlet Laplacian with normal eigenfunction $\Phi=\Phi(x)$, i.e.

$$-\Delta\Phi(x)=(|\bar\xi|^2+m_0^2)\Phi(x),\qquad \Phi(x)=0 \text{ on } \partial\Omega_R. \tag{26}$$

Let us consider $v=v(t,x)$ the solution to the Cauchy problem, with Dirichlet boundary condition on Ω_R

$$\begin{cases} v_{tt}-\Delta v+2b(t)v=0,\\ v(0,x)=c_1\langle\bar\xi\rangle_{m_0}^{-1}\Phi(x),\quad v_t(0,x)=ic_2\Phi(x),\\ v(t,\cdot)\equiv0\quad \text{on } \partial\Omega_R \text{ for each } t\ge0.\end{cases} \tag{27}$$

In particular, we look for a solution in the form

$$v(t,x)=f(t)\Phi(x),$$

and we show that $f=f(t)$ is T-periodic (or $2T$-periodic). Since, $\Phi=\Phi(x)$ satisfies the Dirichlet problem (26), the partial differential equation $v_{tt}-\Delta v+2b(t)v=0$ turns into the ordinary differential equation $v_{tt}+|\bar\xi|^2v+2b(t)v_t+m_0^2v=$

0, with x regarded as a parameter. In particular, $f = f(t)$ satisfies the ordinary differential equation

$$f''(t) + 2b(t)f'(t) + (|\bar{\xi}|^2 + m_0^2)f(t) = 0. \tag{28}$$

Moreover, the corresponding solution $v(t,x) = f(t)\Phi(x)$, satisfies the Cauchy problem

$$D_t \begin{pmatrix} \langle\bar{\xi}\rangle_{m_0} v(t,x) \\ D_t v(t,x) \end{pmatrix} = \begin{pmatrix} 0 & \langle\bar{\xi}\rangle_{m_0} \\ \langle\bar{\xi}\rangle_{m_0} & 2ib(t) \end{pmatrix} \begin{pmatrix} \langle\bar{\xi}\rangle_{m_0} v(t,x) \\ D_t v(t,x) \end{pmatrix}$$
$$\begin{pmatrix} \langle\bar{\xi}\rangle_{m_0} v(t,x) \\ D_t v(t,x) \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Phi(x).$$

This system can be solved by using the fundamental solution $\mathscr{E}_0(t,0,\bar{\xi})$; in particular, we have that

$$\begin{pmatrix} \langle\bar{\xi}\rangle_{m_0} v(t,x) \\ D_t v(t,x) \end{pmatrix}\Big|_{t=T} = \mathscr{M}_0(0,\bar{\xi}) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Phi(x) = \pm \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Phi(x).$$

We conclude that $f = f(t)$ is T-periodic (or $2T$-periodic) and $f(0) = c_1\langle\bar{\xi}\rangle_{m_0}^{-1}$. This gives a contradiction: if we denote the energy of this solution as

$$E(u,t) = 1/2\|v_t(t,\cdot)\|^2_{L^2(\Omega_R)} + 1/2\|\nabla v\|^2_{L^2(\Omega_R)},$$

we obtain

$$\frac{d}{dt}E(v,t) = -b(t)\|v_t\|^2_{L^2(\Omega_R)} = -b(t)|f'(t)|^2.$$

But, by integrating the previous equation we obtain that

$$-\int_0^T b(t)|f'(t)|^2\,dt = 0,$$

that is not possible since $f = f(t)$ can not be constant, by Eq. (28) as $(|\bar{\xi}|^2 + m_0^2) > 0$ for each $\bar{\xi} \in \mathbb{R}^n$. Thus, $\pm 1 \notin \operatorname{spec} \mathscr{M}_0(t,\xi)$ for each $\xi \in \mathbb{R}^n$, and therefore the spectral radius $\rho(\mathscr{M}_0(t,\xi)) < 1$ for all $\xi \in \mathbb{R}^n$.

By the spectral radius formula, we know that

$$\lim_{k\to\infty} \|\mathscr{M}_0^k(t,\xi)\|^{\frac{1}{k}} = \rho(\mathscr{M}_0(t,\xi)) < 1.$$

Thus, we conclude that for each $t \in [0, T]$ and $\xi \in \mathbb{R}^n$ there exists $k = k(t, \xi) \in \mathbb{N}$ such that

$$\|\mathscr{M}_0^k(t, \xi)\| < 1. \tag{29}$$

We want to show that we can find a number k such that the condition (29) holds uniformly with respect to $t \in [0, T]$ and $|\xi| \in [0, N]$.

Let us define for each $k \in \mathbb{N}$ the set $\mathscr{U}_k = \{(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^n \mid \|\mathscr{M}_0^k(t, \xi)\| < 1\}$. It is open due to the continuity of the monodromy matrix $M_0^k(t, \xi)$; moreover, it holds $\mathscr{U}_k \subset \mathscr{U}_\ell$, for $k \leq \ell$. Then, by (29) we have that the compact set $\mathscr{C} = \{(t, \xi) | 0 \leq t \leq T, |\xi| \leq N\}$ is contained in $\bigcup_k \mathscr{U}_k$. By compactness we find $k \in \mathbb{N}$ such that $\mathscr{C} \subset \mathscr{U}_k$. This concludes the proof of estimate (22). By continuity of $\mathscr{M}_0^k(t, \xi)$ in both variables, the estimate is uniform. Hence we obtain

Lemma 2 *For constant mass term m_0 and fixed $N > 0$ there exists a number k such that the monodromy matrix for the problem with constant mass satisfies*

$$\sup_{|\xi| \leq N} \sup_{t \in [0, T]} \|\mathscr{M}_0^k(t, \xi)\| < 1.$$

5 Proof of the Main Theorems

5.1 Proof of Theorem 1

In order to prove Theorem 1 we distinguish between small and large frequencies.

Let $|\xi| \geq N$. Then the monodromy matrix $\mathscr{M}(t, \xi)$ is estimated in Lemma 1. Let $t \geq 0$, $t = \ell T + s$, with $\ell \in \mathbb{N}$ and $s \in [0, T]$. Then, we obtain

$$\|\mathscr{E}(t, 0, \xi)\| = \|\mathscr{M}^\ell(s, \xi)\mathscr{E}(s, 0, \xi)\| \leq e^{-\ell\beta T/2}\|\mathscr{E}(s, 0, \xi)\|.$$

Moreover, since $b(t) > 0$ we know that $\|\mathscr{E}(s, 0, \xi)\| \leq 1$ and therefore we find

$$\|\mathscr{E}(t, 0, \xi)\| \leq e^{-\delta_0(t-T)},$$

by defining $\delta_0 := \beta/2 > 0$. We remark that this estimate for large frequencies is valid for arbitrary periodic mass terms.

For the remainder of the proof assume that $m^2(t) \equiv m_0^2$ constant and $|\xi| \leq N$. By Lemma 2 there exists $k \in \mathbb{N}$ depending only on m_0 such that the matrix $\mathscr{M}_0^k(t, \xi)$ is a contraction uniform in t and ξ. Let $t = \ell k T + s \geq 0$ for some $\ell \in \mathbb{N}$ and $s \in [0, kT]$. Then, we obtain the exponential decay

$$\|\mathscr{E}_0(t, 0, \xi)\| = \|\mathscr{M}_0^{k\ell}(s, \xi)\mathscr{E}_0(s, 0, \xi)\| \leq e^{-\delta_1(t-kT)}, \tag{30}$$

where we set $\delta_1 := (kT)^{-1}\log(c_1(N)^{-1}) > 0$ and

$$c_1(N) := \sup_{|\xi|\le N}\sup_{t\in[0,T]} \|\mathscr{M}_0^k(t,\xi)\| < 1. \tag{31}$$

Going back to the original problem (3), we find

$$\begin{pmatrix} \langle\xi\rangle_{m_0}\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix} = \mathscr{E}(t,0,\xi)\begin{pmatrix}\langle\xi\rangle_{m_0}\hat{u}_0(\xi) \\ \hat{u}_1(\xi)\end{pmatrix}.$$

Thus, we find that

$$\|u(t,\cdot)\|_{L^2} \le \sup_{\xi\in\mathbb{R}^n}\|\mathscr{E}_0(t,0,\xi)\|(\|u_0\|_{L^2}+\|u\|_{H^{-1}}),$$

$$\|\nabla u(t,\cdot)\|_{L^2} \le \sup_{\xi\in\mathbb{R}^n}\|\mathscr{E}_0(t,0,\xi)\|(\|u_0\|_{H^1}+\|u_1\|_{L^2}),$$

$$\|u_t(t,\cdot)\|_{L^2} \le \sup_{\xi\in\mathbb{R}^n}\|\mathscr{E}_0(t,0,\xi)\|(\|u_0\|_{H^1}+\|u_1\|_{L^2}).$$

The proof of Theorem 1 with $C = e^{\delta_1 kT}$ follows immediately by estimate (30).

5.2 *Proof of Theorem 2*

Let $u = u(t,x)$ the solution to (10) where $m_\epsilon^2(t) = m_0^2 + \epsilon m_1(t)$, whit $m_1(t)$ periodic of period T and m_0 a sufficiently large constant such that $m_0^2 + \epsilon m_1(t) > 0$. The corresponding system is

$$D_t V_\epsilon = A_\epsilon(t,\xi)V_\epsilon = \begin{pmatrix} 0 & \langle\xi\rangle_{m_\epsilon(t)} \\ \langle\xi\rangle_{m_\epsilon(t)} & 2ib(t)\end{pmatrix} V_\epsilon, \tag{32}$$

where $V_\epsilon = (\langle\xi\rangle_{m_\epsilon(t)}\hat{u^\epsilon}, D_t\hat{u^\epsilon})$. In order to obtain our result we need to estimate $\|\mathscr{E}_\epsilon(t,0,\xi)\|$, where we denoted by $\mathscr{E}_\epsilon$ the fundamental solution to the system (32). In particular, $\mathscr{E}_0$ solves $D_t V_0 = A_0(t,\xi)V_0$ where

$$D_t V_0 = A_0(t,\xi)V_0 = \begin{pmatrix} 0 & \langle\xi\rangle_{m_0} \\ \langle\xi\rangle_{m_0} & 2ib(t)\end{pmatrix} V_0. \tag{33}$$

We again distinguish between small and large frequencies. If $|\xi| \ge N$, as in the case of constant mass we conclude

$$\|\mathscr{E}_\epsilon(t,0,\xi)\| \le e^{-\delta_0(t-T)},$$

where we recall $\delta_0 = \beta/2 > 0$ by making use of Lemma 1.

If $|\xi| \leq N$, there exists $k \in \mathbb{N}$ given by Lemma 2 such that the matrix $\mathscr{M}_0^k(t,\xi)$ is a contraction uniformly in $t \in [0,T]$ and $|\xi| \in [0,N]$. We write $t = \ell kT + s \geq 0$ for some $\ell \in \mathbb{N}$ and $s \in [0,kT]$; then, we have

$$\mathscr{E}_\epsilon(t,0,\xi) = \mathscr{M}_\epsilon^{k\ell}(s,\xi)\mathscr{E}_\epsilon(s,0,\xi); \tag{34}$$

we can treat the fundamental solution as a perturbation of constant case

$$\begin{aligned}\|\mathscr{E}_\epsilon(t,s,\xi)\| &\leq \|\mathscr{E}_\epsilon(t,s,\xi) - \mathscr{E}_0(t,s,\xi)\| + \|\mathscr{E}_0(t,s,\xi)\| \\ &\leq \|\mathscr{E}_\epsilon(t,s,\xi) - \mathscr{E}_0(t,s,\xi)\| + e^{-\delta(t-s-kT)},\end{aligned}$$

where we recall $\delta_1 = (kT)^{-1}\log(c_1(N)^{-1}) > 0$ and $c_1(N)$ as in (31). In order to estimate the difference $\|\mathscr{E}_\epsilon(t,s,\xi) - \mathscr{E}_0(t,s,\xi)\|$ we use that for each $\epsilon \geq 0$ the fundamental solution $\mathscr{E}_\epsilon$ satisfies the integral equation

$$\mathscr{E}_\epsilon(t,s,\xi) = I + \int_s^t A_\epsilon(\tau,\xi)\mathscr{E}_\epsilon(\tau,s,\xi)\,ds,$$

such that

$$\begin{aligned}\mathscr{E}_\epsilon(t,s,\xi) - \mathscr{E}_0(t,s,\xi) &= \int_s^t A_\epsilon(\tau,\xi)(\mathscr{E}_\epsilon(\tau,s,\xi) - \mathscr{E}_0(\tau,s,\xi))\,ds \\ &\quad + \int_s^t (A_\epsilon(\tau,\xi) - A_0(\tau,\xi))\mathscr{E}_0(\tau,s,\xi)\,ds.\end{aligned}$$

By using the Gronwall inequality we get

$$\|\mathscr{E}_\epsilon(t,s,\xi) - \mathscr{E}_0(t,s,\xi)\| \leq \int_s^t \|\mathscr{E}_0(\tau,s,\xi)\|\,\|A_\epsilon(\tau,\xi) - A_0(\tau,\xi)\|\,ds \cdot e^{\int_s^t \|A_\epsilon(\tau,\xi)\|\,d\tau};$$

here, for any $\tau > 0$ and $\xi \in \mathbb{R}^n$, since we are assuming $\sup\limits_{t\geq 0}|m_1(t)| = 1$ we can estimate

$$\|A_\epsilon(\tau,\xi) - A_0(\tau,\xi)\| \leq \frac{\epsilon}{\langle\xi\rangle_{m_0}}, \qquad \|A_0(\tau,\xi)\| \leq \langle\xi\rangle_{m_0} + 2b(\tau),$$

and so

$$\|A_\epsilon(\tau,\xi)\| \leq C_\epsilon(\xi) + \langle\xi\rangle_{m_0} + 2b(\tau), \qquad C_\epsilon(\xi) = \frac{\epsilon}{\langle\xi\rangle_{m_0}}.$$

Thus, recalling that $\mathscr{M}_\epsilon^k(s,\xi) = \mathscr{E}_\epsilon(s+kT,s,\xi)$, we find

$$\begin{aligned}
\|\mathscr{M}_\epsilon^k(s,\xi) - \mathscr{M}_0^k(s,\xi)\| &\le C_\epsilon(\xi)e^{C_\epsilon(\xi)kT}e^{(\langle\xi\rangle_{m_0}+2\beta)kT}\int_s^{s+kT}\|\mathscr{E}_0(\tau,s,\xi)\|\,d\tau \\
&\le C_\epsilon(\xi)e^{C_\epsilon(\xi)kT}e^{(\langle\xi\rangle_{m_0}+2\beta)kT}\int_s^{s+kT}e^{-\delta_1(\tau-s-kT)}\,d\tau \\
&\le C_\epsilon(\xi)e^{C_\epsilon(\xi)kT}e^{(\langle\xi\rangle_{m_0}+2\beta)kT}\int_s^{s+kT}e^{-\delta_1(\tau-s-kT)}\,d\tau \\
&\le \frac{C_\epsilon(\xi)}{\delta_1}e^{C_\epsilon(\xi)kT}e^{(\langle\xi\rangle_{m_0}+2\beta)kT}(e^{\delta_1 kT}-1).
\end{aligned}$$

Therefore, recalling that $\exp(\delta_1 kT) = c_1(N)^{-1}$, we can conclude

$$\begin{aligned}
\sup_{|\xi|\le N}\sup_{s\in[0,T]}\|\mathscr{M}_\epsilon^k(s,\xi)\| \le{}& \sup_{|\xi|\le N}\sup_{s\in[0,T]}\|\mathscr{M}_0^k(s,\xi)\| \\
&+\sup_{|\xi|\le N}\left\{\frac{C_\epsilon(\xi)}{\delta_1}e^{C_\epsilon(\xi)kT}e^{(\langle\xi\rangle_{m_0}+2\beta)kT}(c_1(N)^{-1}-1)\right\} \\
={}& c_1(N)+\sup_{|\xi|\le N}\left\{\frac{C_\epsilon(\xi)}{\delta_1}e^{C_\epsilon(\xi)kT}e^{(\langle\xi\rangle_{m_0}+2\beta)kT}(c_1(N)^{-1}-1)\right\}.
\end{aligned}$$

By (34) we get the desired result

$$\sup_{|\xi|\le N}\sup_{s\in[0,T]}\|\mathscr{M}_\epsilon^k(s,\xi)\| < 1,$$

by choosing ϵ sufficiently small such that

$$\frac{C_\epsilon(\xi)}{\delta_1}e^{C_\epsilon(\xi)kT}e^{(\langle\xi\rangle_{m_0}+2\beta)kT}(c_1(N)^{-1}-1) < 1-c_1(N). \tag{35}$$

Let us introduce $W = W(x)$ the Lambert W-function defined in the set $\mathbb{R}_+ := \{x \in \mathbb{R} : x \ge 0\}$ such that for any $x \in \mathbb{R}_+$ it holds $x = W(x)e^{W(x)}$. The function W is increasing (see [1] for more details); thus, recalling the definition of δ_1, we find that estimate (35) is equivalent to ask

$$\epsilon \le \frac{\langle\xi\rangle_{m_0}}{kT}W\Big(c_1(N)\log(c_1(N)^{-1})e^{-(\langle\xi\rangle_{m_0}+\beta)kT}\Big),$$

for any $\xi \in [0,N]$, that is

$$\epsilon \le \frac{m_0}{kT}W\Big(c_1(N)\log(c_1(N)^{-1})e^{-(\langle N\rangle_{m_0}+\beta)kT}\Big). \tag{36}$$

Remark 3 In the case of constant dissipation $b \equiv 1$ we know an admissible value of N (see Remark 2); however, even in this case, with this technique it is not possible to give an explicit value of ε; in fact, both k and $c_1(N)$ are not determined.

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Conditional Stability of Semigroups and Periodic Solutions to Evolution Equations

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Abstract We prove the existence and uniqueness of periodic solutions to linear and semilinear evolution equations. Our method is based on the analysis of the conditional stability of the semigroups generated by the corresponding linear equations and connection with the choice of the initial data from which emanates the periodic solution. We also give applications to damped wave equations and damped Timoshenko beam systems.

Keywords Semigroups · Conditional φ-stability · Periodic solutions · Damped wave equations · Damped Timoshenko beam systems

1 Introduction

Consider the semilinear evolution equation of the form

$$u'(t) - Au(t) = g(u)(t) \tag{1}$$

where A is a generator of a strongly continuous semigroup $(\mathscr{T}(t))_{t\geq 0}$, and the given Nemyskii's operator g maps T-periodic functions to T-periodic functions. The research for existence and uniqueness of a T-periodic solution to (1) is one of important research directions related to asymptotic behavior of solutions to evolution equations. There some approaches used for that research suitable for large classes of differential equations, such as Massera methodology [8, 15], Tikhonov's fixed-point principle [10] or the Lyapunov functionals [14] (which can be applied to some specific equations), and the most well-known approaches for establishing

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the existence of a periodic solution are the ultimate boundedness of solutions and the compactness of Poincaré map realized through some compact embeddings (see [1, 6, 10, 11, 13, 14] and the references therein). However, in several applications, e.g., to partial differential equations in unbounded domains or to equations that have unbounded solutions, such compact embeddings are no longer valid, and the existence of bounded solutions is difficult to obtain since one has to carefully choose an appropriate initial vector (or data) to guarantee the boundedness of the solution emanating from that vector. One way to overcome such difficulties is to use the so-called Massera-type theorem, that is roughly speaking that if a differential equation has a bounded solution then it has a periodic one. However, to apply the Massera's principle ones have to use somehow the compactness at least at the level of weak-* topology (e.g., Banach-Alaoglu theorem). Actually, we have invoked this Massera's methodology combining with interpolation spaces to prove the existence of periodic solutions to Navier-Stokes equations around a rotating obstacle in [9] and to general fluid flow problems in [4]. In those works we have used the interpolation functors in combination with ergodic method (see [9]) or with topological arguments (see [4]). Note that there is an approach described in [7] allowing $2\pi ki/T$ being in the spectrum of A for some $k \in \mathbb{Z}$ under the requirement that such $2\pi ki/T$ are semisimple eingenvalues of A.

In the present paper, we propose another approach toward the existence and uniqueness of the periodic solution to the abstract evolution equation (1). Namely, we use the boundedness and conditional φ-stability of the corresponding semigroups (see Definition 1 below) to construct a Cauchy sequence which converges to the initial vector from which emanates a periodic solution. This approach seems more direct and simpler than the approaches used in [4, 9] since we do not use the interpolation functors. The other advantage of our approach here is lying in the fact that we do not use any compactness arguments. Consequently, we can prove the existence and uniqueness of general linear inhomogeneous evolution equations in a direct and more elementary manner. Our main result is contained in Theorems 1 and 2. Then, in Sect. 3, we apply the abstract results to hyperbolic semigroups and damped wave equations.

2 Periodic Solutions to Evolution Equations

Let us first consider the following linear evolution equation on a Banach space Y

$$\begin{cases} u' - Au = f(t) \\ \quad u(0) = u_0 \in Y. \end{cases} \tag{2}$$

where A generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on Y, and f belongs to $C_b(\mathbb{R}_+, Y) := \{h : \mathbb{R}_+ \to Y \mid h \text{ is continuous and } \sup_{t\geqslant 0} \|h(t)\|_Y < \infty\}$ endowed with norm $\|h\|_{C_b(\mathbb{R}_+,Y)} := \sup_{t\geqslant 0} \|h(t)\|_Y$.

By a *mild solution* of (2) we mean a function $u : \mathbb{R}_+ \to Y$ satisfying the integral equation

$$u(t) = \mathscr{T}(t)u_0 + \int_0^t \mathscr{T}(t-s)f(s)ds. \tag{3}$$

Also, we assume the following standard assumption on the semigroup.

Definition 1 Let $\varphi : (0, \infty) \to \mathbb{R}$ be a continuous function satisfying $\lim_{t\to\infty} \varphi(t) = 0$. The semigroup $(\mathscr{T}(t))_{t\geq 0}$ is called *conditionally* φ*-stable* if

$$\|\mathscr{T}(t)x\|_Y \leqslant \varphi(t)\|x\|_Y \text{ for all } x \in Y \text{ such that } \sup_{t\geq 0} \|\mathscr{T}(t)x\|_Y < \infty. \tag{4}$$

We then come to our first result for linear equation stated in the following theorem.

Theorem 1 *Let* $(\mathscr{T}(t))_{t\geq 0}$ *be a conditionally* φ*-stable semigroup as in Definition 1. Let* $f \in C_b(\mathbb{R}_+, Y)$ *and suppose that there exists* $x_0 \in Y$ *such that the mild solution* $u(t) = \mathscr{T}(t)x_0 + \int_0^t \mathscr{T}(t-s)f(s)ds,\ t \geq 0$*, belongs to* $C_b(\mathbb{R}_+, Y)$ *and satisfies* $\|u\|_{C_b(\mathbb{R}_+,Y)} \leqslant M\|f\|_{C_b(\mathbb{R}_+,Y)}$*. Then, if* f *is* T*-periodic in time, there exists a unique* T*-periodic mild solution* $\hat{u}$ *of* (2) *with*

$$\|\hat{u}\|_{C_b(\mathbb{R}_+,Y)} \leqslant \tilde{M}\|f\|_{C_b(\mathbb{R}_+,Y)} \textit{ for } \tilde{M} := (M+T) \sup_{0\leqslant t\leqslant T} \|\mathscr{T}(t)\|. \tag{5}$$

Proof By the hypothesis of the theorem, we have that the mild solution u of (2) with $u(0) = x_0$ (i.e., $u(t) = \mathscr{T}(t)x_0 + \int_0^t \mathscr{T}(t-s)f(s)ds, t \geq 0$) belongs to $C_b(\mathbb{R}_+, Y)$.

We next prove that $\{u(nT)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Indeed, putting $w(t) = u(t + (m-n)T)$ for arbitrary fixed natural numbers $m > n \in \mathbb{N}$, using the periodicity of f we now prove that w can be rewritten as

$$w(t) = \mathscr{T}(t)u((m-n)T) + \int_0^t \mathscr{T}(t-s)f(s)ds \text{ for all } t \geq 0. \tag{6}$$

Indeed,

$$\begin{aligned}
w(t) &= u(t+(m-n)T) \\
&= \mathscr{T}(t+(m-n)T)u(0) + \int_0^{t+(m-n)T} \mathscr{T}(t+(m-n)T-s)f(s)ds \\
&= \mathscr{T}(t)\mathscr{T}((m-n)T)u(0) + \int_0^{(m-n)T} \mathscr{T}(t)\mathscr{T}((m-n)T-s)f(s)ds + \\
&\quad + \int_{(m-n)T}^{t+(m-n)T} \mathscr{T}(t+(m-n)T-s)f(s)ds
\end{aligned}$$

$$= \mathscr{T}(t)\left(\mathscr{T}((m-n)T)u(0) + \int_0^{(m-n)T} \mathscr{T}((m-n)T - s)f(s)ds\right) +$$

$$+ \int_{(m-n)T}^{t+(m-n)T} \mathscr{T}(t + (m-n)T - s)f(s)ds$$

$$= \mathscr{T}(t)u((m-n)T) + \int_0^t \mathscr{T}(t-s)f(s)ds.$$

Therefore, (6) follows.

The boundedness of u and therefore of w implies that the function $u(t) - w(t) = \mathscr{T}(t)(u(0) - w(0))$, $t \in \mathbb{R}_+$, is bounded, i.e., $\sup_{t\geq 0} \|\mathscr{T}(t)(u(0) - w(0))\|_Y < \infty$. Hence, the relation in (4) yields

$$\|u(t) - w(t)\|_Y = \|\mathscr{T}(t)(u(0) - w(0))\|_Y \leqslant \varphi(t)\|u(0) - w(0)\|_Y \leqslant C\varphi(t), \quad t > 0$$

for $C := 2\|u\|_{C_b(\mathbb{R}_+,Y)}$ independent of m, n.

Taking $t := nT$ on the above inequality we obtain

$$\|u(nT) - u(mT)\|_Y \leqslant C\varphi(nT)$$

for all $m > n \in \mathbb{N}$. From the fact $\lim_{t\to\infty} \varphi(t) = 0$ it follows that $\{u(nT)\}_{n\in\mathbb{N}}$ is Cauchy sequence in Y. Since Y is a Banach space, the sequence $\{u(nT)\}_{n\in\mathbb{N}}$ is convergent in Y, and we put

$$u^* := \lim_{n\to\infty} u(nT) \in Y.$$

Taking now u^* as initial value, we then prove that the mild solution $\hat{u}(t) = \mathscr{T}(t)u^* + \int_0^t \mathscr{T}(t-s)f(s)ds$ is T-periodic. To do this, we put $v(t) := \mathscr{T}(t + nT)x_0 + \int_0^{t+nT} \mathscr{T}(t + nT - s)f(s)ds$ for every fixed $n \in \mathbb{N}$ and all $t \geq 0$, i.e., $v(t) = u(t + nT)$ for

$$u(t) = \mathscr{T}(t)x_0 + \int_0^t \mathscr{T}(t-s)f(s)ds \tag{7}$$

as in previous step.

Again, by the periodicity of f we obtain that v satisfies

$$v(t) = \mathscr{T}(t)u(nT) + \int_0^t \mathscr{T}(t-s)f(s)ds$$

for u being defined as in (7).

We then have

$$\|\hat{u}(T) - v(T)\|_Y = \|\mathcal{T}(T)(\hat{u}(0) - v(0))\|_Y \leqslant \|\mathcal{T}(T)\|\|\hat{u}(0) - v(0)\|_Y.$$

This means

$$\|\hat{u}(T) - u((n+1)T)\|_Y \leqslant \|\mathcal{T}(T)\|\|u^* - u(nT)\|_Y.$$

Letting now $n \to \infty$ and using the fact that $\lim_{n\to\infty} u(nT) = u^* = \hat{u}(0)$ in Y (see above) we obtain

$$\hat{u}(T) = \hat{u}(0).$$

Therefore, $\hat{u}(t)$ is $T-$periodic. The inequality (5) follows from the facts that $\|u^*\|_Y \leqslant \|u\|_{C_b}$ and $\|\hat{u}\|_{C_b} = \sup_{0\leqslant t\leqslant T} \|\hat{u}(t)\|_Y$ thanks to the periodicity of $\hat{u}$.

The uniqueness of the T-periodic solution follows from (4). Namely, if u and v are two T-periodic solutions of Eq. (3) with initial values u_0 and v_0, respectively, then $u(t) - v(t) = \mathcal{T}(t)(u_0 - v_0)$, and from the fact that $u(t) - v(t)$ is bounded it follows from (4) that $\|u(t) - v(t)\|_Y = \|\mathcal{T}(t)(u_0 - v_0)\|_Y \leqslant \varphi(t)\|u_0 - v_0\|_Y$.

Therefore, $\lim_{t\to\infty} \|u(t) - v(t)\|_Y = 0$. This, together with periodicity and continuity of u and v, follows that $u(t) = v(t)$ for all $t \in \mathbb{R}_+$. □

We now consider the following semi-linear evolution equation

$$\begin{cases} u'(t) = Au(t) + g(u)(t) \\ u(0) = u_0 \in Y, \end{cases} \tag{8}$$

where the operators A satisfy the above hypotheses for linear equations, and the Nemytskii's operator $g : C_b(\mathbb{R}_+, Y) \to C_b(\mathbb{R}_+, Y)$ satisfies:

(1) $\|g(0)\|_{C_b(\mathbb{R}_+,Y)} \leqslant \gamma$ where γ is a non-negative constant,

(2)g maps T-periodic functions to T-periodic functions,

(3) there exist positive constants ρ and L such that

$$\|g(v_1) - g(v_2)\|_{C_b(\mathbb{R}_+,Y)} \leqslant L\|v_1 - v_2\|_{C_b(\mathbb{R}_+,Y)} \text{ for all } v_1, v_2 \in C_b(\mathbb{R}_+, Y) \text{ with } \|v_1\|_{C_b(\mathbb{R}_+,Y)}, \|v_2\|_{C_b(\mathbb{R}_+,Y)} \leqslant \rho. \tag{9}$$

Furthermore, by the *mild solution* to (8) we mean the function u satisfying the following equation

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-s)g(u)(\tau)d\tau \text{ for all } t \geq 0. \tag{10}$$

We then come to our next result on the existence and uniqueness of the periodic mild solution to Eq. (8).

Theorem 2 *Let the hypotheses of Theorem 1 be satisfied, and let g satisfy the conditions in (9). Then, if L and γ are small enough, Eq. (8) has one and only one mild T-periodic solution $\hat{u}$ on a small ball of $C_b(\mathbb{R}_+, Y)$.*

Proof Consider the following ball $\mathscr{B}_\rho^T$ defined by

$$\mathscr{B}_\rho^T := \{v \in C_b(\mathbb{R}_+, Y) : v \text{ is } T\text{-periodic and } \|v\|_{C_b(\mathbb{R}_+,Y)} \leqslant \rho\}. \tag{11}$$

We then define the following transformation Φ given as follows: Consider the equation

$$u'(t) = Au(t) + g(v)(t). \tag{12}$$

Then, for $v \in \mathscr{B}_\rho^T$ we set

$$\Phi(v) = u \text{ where } u \in C_b(\mathbb{R}_+, Y) \text{ is the unique } T\text{-periodic mild solution to Equation (12).} \tag{13}$$

We will prove that if L and γ are sufficiently small, then the transformation Φ acts from $\mathscr{B}_\rho^T$ into itself and is a contraction. To do this, taking any $v \in \mathscr{B}_\rho^T$, by the properties of g given in (9) we have that

$$\|g(v)\|_{C_b(\mathbb{R}_+,Y)} \leqslant \|g(v) - g(0)\|_{C_b(\mathbb{R}_+,Y)} + \|g(0)\|_{C_b(\mathbb{R}_+,Y)} \leqslant K\rho + \gamma. \tag{14}$$

Applying Theorem 1 for the right-hand side $g(v)(t)$ instead of $f(t)$ and using inequality (5) we obtain that for $v \in \mathscr{B}_\rho^T$ there exists a unique T-periodic mild solution u to (12) satisfying

$$\|u\|_{C_b(\mathbb{R}_+,Y)} \leqslant \tilde{M}\|g(v)\|_{C_b(\mathbb{R}_+,Y)} \leqslant \tilde{M}(L\rho + \gamma). \tag{15}$$

Therefore, if L and γ are small enough, the map Φ acts from $\mathscr{B}_\rho^T$ into itself. Then, by Formula (3) with $g(v)$ instead of f, we have the following representation of Φ

$$\Phi(v)(t) = e^{tA}u(0) + \int_0^t e^{(t-\tau)A} g(v)(\tau)d\tau \text{ for } \Phi(v) = u. \tag{16}$$

Furthermore, for $v_1, v_2 \in \mathscr{B}_\rho^T$ by the representation (16) we obtain that the function $u := \Phi(v_1) - \Phi(v_2)$ is the unique T-periodic mild solution to the equation

$$u'(t) = Au(t) + g(v_1)(t) - g(v_2)(t).$$

Thus, again by Theorem 1 we arrive at

$$\begin{aligned}\|\Phi(v_1) - \Phi(v_2)\|_{C_b(\mathbb{R}_+,Y)} &\leqslant \tilde{M}\|g(v_1) - g(v_2)\|_{C_b(\mathbb{R}_+,Y)} \\ &\leqslant 2\tilde{M}L\|v_1 - v_2\|_{C_b(\mathbb{R}_+,Y)}. \end{aligned} \quad (17)$$

We hence obtain that if L, and γ are sufficiently small, then $\Phi : \mathscr{B}_\rho^T \to \mathscr{B}_\rho^T$ is a contraction. Therefore, for these values of L and γ, there exists a unique fixed point $\hat{u}$ of Φ, and by definition of Φ, this function $\hat{u}$ is the unique T-periodic mild solution to Eq. (8). □

3 The Case of Hyperpolic Semigroups and Damped Wave Equations

In this section we apply our abstract results in the previous section to the case that the semigroup $(\mathscr{T}(t))_{t\geq 0}$ is hyperbolic (or admits an exponential dichotomy). Precisely, in that case we will prove that $(\mathscr{T}(t))_{t\geq 0}$ is conditionally φ-stable for $\varphi(t) = Me^{-\nu t}$, $t \geq 0$, which is an exponential decaying function (here $\nu > 0$).

3.1 General Framework for Hyperbolic Semigroups

We start by recalling the notion of hyperbolic semigroups in the following definition taken from [3].

Definition 2 A strongly continuous semigroup $(\mathscr{T}(t))_{t\geq 0}$ on a Banach space Y is *called hyperbolic* (or *admitting an exponential dichotomy*) if and only if there exists a (linear, bounded) projection P on Y and constants $M, \nu > 0$ such that each $\mathscr{T}(t)$ commutes with P, satisfies $T(t)\ker P = \ker P$, and

$$\begin{aligned}&\|\mathscr{T}(t)x\| \leqslant Me^{-\nu t}\|x\| \text{ for all } t \geq 0 \text{ and } x \in \mathrm{Im}P := PY,\\ &\|\mathscr{T}(t)x\| \geqslant \frac{e^{\nu t}}{M}\|x\| \text{ for all } t \geq 0 \text{ and } x \in \mathrm{Ker}P := (I-P)Y.\end{aligned} \quad (18)$$

In this case, the projection P is called the *dichotomy projection* for the hyperbolic semigroup $(\mathscr{T}(t))_{t\geq 0}$, whereas M, ν are called *dichotomy constants*.

Especially, the semigroup $(\mathscr{T}(t))_{t\geq 0}$ is called *exponentially stable* if it is hyperbolic with the dichotomy projection $P = Id$, the identity operator on Y.

It is obvious from above definition that if $(\mathscr{T}(t))_{t\geq 0}$ is hyperbolic then the restriction $\mathscr{T}(t)\mid_{\mathrm{Ker}P}$ of $\mathscr{T}(t)$ to $\mathrm{Ker}P$ is an isomorphism $\mathscr{T}(t)\mid_{\mathrm{Ker}P}$: $\mathrm{Ker}P \to \mathrm{Ker}P$. We denote its inverse by $\mathscr{T}(-t) := (\mathscr{T}(t)\mid_{\mathrm{Ker}P})^{-1}$ for $t > 0$. That is to say, the restriction of the semigroup $(\mathscr{T}(t))_{t\geq 0}$ to Ker$P$ can be extended to a group $(\mathscr{T}(t))_{t\in\mathbb{R}}$ on the Banach space Ker$P$. Moreover, the space PY can be characterized by $PY = \{x \in Y : \sup_{t\geq 0} \|\mathscr{T}(t)x\| < \infty\}$. Also, we have the following important remark for latter use.

Remark 1 If $(\mathscr{T}(t))_{t\geq 0}$ is hyperbolic then it is obvious that $(\mathscr{T}(t))_{t\geq 0}$ is conditional φ-stable with the function $\varphi(t) = Me^{-\nu t}$ for all $t \geq 0$.

We will prove the existence and uniqueness of T-periodic mild solutions to linear equation (2) and to semilinear equation (8), respectively. To do so, we first have to prove that the Eq. (2) has at least a bounded mild solution so that we can apply Theorem 1 to obtain the existence of T-periodic mild solutions. To this purpose we now present some preliminaries for latter use.

If $(\mathscr{T}(t))_{t\geq 0}$ is hyperbolic with dichotomy projection P and constants $N, \nu > 0$, then the *Green's function* is defined as follows:

$$\mathscr{G}(t) := \begin{cases} P\mathscr{T}(t) & \text{for } t \geqslant 0, \\ -\mathscr{T}(t)(I-P) & \text{for } t < 0. \end{cases} \tag{19}$$

Here note that for $t < 0$ we have $\mathscr{T}(t) := (\mathscr{T}(-t)\mid_{\ker P})^{-1}$ which is defined on $\ker P = (I - P)Y$.

Also, $\mathscr{G}(t)$ satisfies the estimate

$$\|\mathscr{G}(t)\| \leqslant (1 + \|P\|)Me^{-\nu|t|} \text{ for } t \in \mathbb{R}. \tag{20}$$

The following lemma gives the form of bounded solutions of Eqs. (3) and (10).

Lemma 1 *Let the semigroup $(\mathscr{T}(t))_{t\geq 0}$ be hyperbolic with the dichotomy projection P and constants $M, \nu > 0$. Let $f \in C_b(\mathbb{R}_+, Y)$ and let $g : C_b(\mathbb{R}_+, Y) \to C_b(\mathbb{R}_+, Y)$ satisfy conditions in (9). Then, the following assertions hold true.*

(a) Let $v \in C_b(\mathbb{R}_+, Y)$ be the solution to Eq. (3) (i.e., the mild solution to (2)). Then, v can be rewritten in the form

$$v(t) = \mathscr{T}(t)\xi_0 + \int_0^\infty \mathscr{G}(t-\tau)f(\tau)d\tau \text{ for some } \xi_0 \in PY, \tag{21}$$

where $\mathscr{G}(t)$ is the Green's function determined as in (19).

(b) Let $u \in C_b(\mathbb{R}_+, Y)$ be a solution to Eq. (10) such that $\sup_{t\geq 0} \|u(t)\|_Y \leqslant \rho$ for a fixed $\rho > 0$. Then, for $t \geq 0$ this function u can be rewritten in the form

$$u(t) = \mathscr{T}(t)v_0 + \int_0^\infty \mathscr{G}(t-\tau)g(u)(\tau)d\tau \text{ for some } v_0 \in PY, \tag{22}$$

for $\mathscr{G}$ as in Item (a).

Proof

(a) Denote by $\|\cdot\|$ the norm in Y, and by $\|\cdot\|_{C_b}$ the norm in $C_b(\mathbb{R}_+, Y)$. Put $y(t) := \int_0^\infty \mathscr{G}(t-\tau)f(\tau)d\tau$ for $t \geq 0$. Since $f \in C_b(\mathbb{R}_+, Y)$, using estimate (20) we obtain that

$$\|y(t)\| \leqslant (1+\|P\|)M\|f\|_{C_b}\int_0^\infty e^{-\nu|t-\tau|}d\tau \leqslant \frac{2(1+\|P\|)M\|f\|_{C_b}}{\nu} \text{ for all } t \geq 0.$$

Moreover, it is straightforward to see that $y(\cdot)$ satisfies the equation

$$y(t) = \mathscr{T}(t)y(0) + \int_0^t \mathscr{T}(t-\tau)f(\tau)d\tau \text{ for } t \geq 0.$$

Since $v(t)$ is a solution of the Eq. (3) we obtain that $v(t) - y(t) = \mathscr{T}(t)(v(0) - y(0))$ for $t \geq 0$. Put now $\xi_0 = v(0) - y(0)$. The boundedness of $v(\cdot)$ and $y(\cdot)$ on $[0, \infty)$ implies that $\xi_0 \in PY$. Finally, since $v(t) = \mathscr{T}(t)\xi_0 + y(t)$ for $t \geq 0$, the equality (21) follows.

(b) Similarly as in Item (a) we put $y(t) := \int_0^\infty \mathscr{G}(t-\tau)g(u)(\tau)d\tau$ for $t \geq 0$. Since g satisfies the conditions in (9) and using estimate (20) we obtain that

$$\begin{aligned}\|y(t)\| &\leqslant (1+\|P\|)M\int_0^\infty e^{-\nu|t-\tau|}(\|g(u)(\tau) - g(0)(\tau)\| + \|g(0)(\tau)\|)d\tau \\ &\leqslant (1+\|P\|)M(L\rho+\gamma)\int_0^\infty e^{-\nu|t-\tau|}d\tau \\ &\leqslant \frac{2(1+\|P\|)M(L\rho+\gamma)}{\nu} \text{ for } t \geq 0.\end{aligned}$$

Also, it is straightforward to see that $y(\cdot)$ satisfies the equation

$$y(t) = \mathscr{T}(t)y(0) + \int_0^t \mathscr{T}(t-\tau)g(u)(\tau)d\tau \text{ for } t \geq 0.$$

Since $u(t)$ is a solution of the Eq. (10) we obtain that $u(t) - y(t) = \mathscr{T}(t)(u(0) - y(0))$ for $t \geq 0$. Put now $v_0 = u(0) - y(0)$. The boundedness of $u(\cdot)$ and $y(\cdot)$ on $\mathbb{R}_+$ implies that $v_0 \in PY$. Finally, from equality $u(t) = \mathscr{T}(t)v_0 + y(t)$ for $t \geq 0$ the equality (22) follows. □

Remark 2 Straightforward computations show that the converses of statements (a) and (b) are also true, i.e., a solution of Eq. (21) satisfies Eq. (3) for $t \geq 0$, and that of Eq. (22) satisfies Eq. (10) for $t \geq 0$.

We next will show the existence of bounded solutions to (3) and (10) (i.e., bounded mild solutions to (2) and (8), respectively) and hence that of periodic solutions in the following theorem.

Theorem 3 *Consider equations* (3) *and* (10). *Let semigroup* $(\mathcal{T}(t))_{t\geq s\geq 0}$ *be hyperbolic with the dichotomy projection* P *and constants* M, ν. *Let further that* $f \in C_b(\mathbb{R}_+, Y)$ *be* T*-periodic and that* $g : C_b(\mathbb{R}_+, Y) \to C_b(\mathbb{R}_+, Y)$ *satisfy the conditions in* (9) *with given constants* ρ, L, *and* γ. *Then, the following assertions hold true.*

(a) Equation (3) *has a unique* T*-periodic solution.*
(b) For sufficiently small L, γ *Eq.* (10) *has a unique* T*-periodic solutions.*

Proof

(a) For a given $f \in C_b(\mathbb{R}_+, Y)$ taking $\xi_0 = 0 \in PY$ in (21) we have that Eq. (3) has a bounded solution

$$u(t) = \int_0^\infty \mathcal{G}(t-\tau)f(\tau)d\tau. \tag{23}$$

and this solution can be estimated using inequality (20) by

$$\|u\|_{C_b} \leqslant \frac{2M(\|P\|+1)}{\nu}\|f\|_{C_b}. \tag{24}$$

From Remark 1 we obtain that $(\mathcal{T}(t))_{t\geq 0}$ is conditionally φ-stable with $\varphi(t) = Me^{-\nu t}, t \geq 0$. Then, applying Theorem 1 we obtain that for T-periodic function $f \in C_b(\mathbb{R}_+, Y)$ there exists a T-periodic solution $\hat{u}$ of (3) (i.e., a T-periodic mild solution of (2)) satisfying

$$\|\hat{u}\|_{C_b} \leqslant \tilde{M}\|f\|_{C_b} \tag{25}$$

where $\tilde{M} := \left(\frac{2M(\|P\|+1)}{\nu} + T\right) \sup\limits_{0\leqslant t\leqslant T} \|\mathcal{T}(t)\|$.

The uniqueness of the T-periodic solution follows from the fact that for two continuous and T-periodic (hence bounded on $\mathbb{R}_+$) solutions $\hat{u}$ and $\hat{v}$ we obtain by using the form for bounded solutions (21) that $\|\hat{u}(t) - \hat{v}(t)\| = \|\mathcal{T}(t)(u_0 - v_0)\| \leqslant Me^{-\nu t}\|u_0 - v_0\| \to 0$ as $t \to \infty$ since $u_0, v_0 \in PX$. This, together with the periodicity, implies $\hat{u}(t) = \hat{v}(t)$ for all $t \geq 0$, finishing the proof of Assertion (a).

(b) By assertion (a), for each T-periodic input function f, the linear problem (3) has a unique T-periodic solution $\hat{u}$ satisfying inequality (25). Therefore, the assertion (b) then follows from Theorem 2.

□

We now prove the conditional stability of periodic solutions to (10). To do this, we first denote by $B_r(x)$ (by $\mathcal{B}_r(v)$) the ball in Y (in $C_b(\mathbb{R}_+, Y)$, respectively) centered at x (at v) with radius r.

Theorem 4 *Let the assumptions of the Theorem 3 hold. Suppose that* $\hat{u}$ *is the* T*-periodic solution of* (10) *obtained in assertion (b) of Theorem 3. Let* $\mathcal{B}_\rho(0)$ *be the* *ball containing $\hat{u}$ as in assertion (b) of Theorem 3. Suppose further that there exists a positive constant L_1 such that $\|g(v_1) - g(v_2)\|_{C_b} \leqslant L_1\|v_1 - v_2\|_{C_b}$ for all $v_1, v_2 \in \mathscr{B}_{2\rho}(0)$. Then, if L_1 is small enough, there corresponds to each $v_0 \in B_{\frac{\rho}{2M}}(P\hat{u}(0)) \cap PX$ one and only one solution $u(t)$ of the Eq. (10) on $\mathbb{R}_+$ satisfying the conditions $Pu = v_0$ and $u \in \mathscr{B}_\rho(\hat{u})$. Moreover, the following estimate is valid for $u(t)$ and $\hat{u}(t)$:*

$$\|u(t) - \hat{u}(t)\| \leqslant Ce^{-\mu t}\|Pu(0) - P\hat{u}(0)\| \text{ for } t \geq 0, \tag{26}$$

for some positive constants C and μ independent of u and $\hat{u}$.

Proof For $v_0 \in B_{\frac{\rho}{2M}}(P\hat{u}(0)) \cap PY$ we will prove that the transformation F defined by

$$(Fw)(t) = \mathscr{T}(t)v_0 + \int_0^\infty \mathscr{G}(t-\tau)(g(w)(\tau))d\tau \text{ for } t \geq 0$$

maps from $\mathscr{B}_\rho(\hat{u})$ into $\mathscr{B}_\rho(\hat{u})$ and is a contraction.

In fact, for $w(\cdot) \in \mathscr{B}_\rho(\hat{u})$ we have that

$$\|w\|_{C_b} \leqslant \|w - \hat{u}\|_{C_b} + \|\hat{u}\|_{C_b} \leqslant 2\rho \tag{27}$$

and $\|g(w) - g(\hat{u})\|_{C_b} \leqslant L_1\|w - \hat{u}\|_{C_b} \leqslant L_1\rho$. Therefore, putting

$$y(t) := (Fw)(t) = \mathscr{T}(t)v_0 + \int_0^\infty \mathscr{G}(t-\tau)(g(w)(\tau))d\tau \text{ for } t \geq 0$$

we obtain

$$\begin{aligned}\|y(t) - \hat{u}(t)\| &\leqslant Me^{-\nu t}\|v_0 - P(0)\hat{u}(0)\| + \\ &\quad + (1 + \|P\|)M\int_0^\infty e^{-\nu|t-\tau|}d\tau\|g(w) - g(\hat{u})\|_{C_b} \\ &\leqslant M\|v_0 - P\hat{u}(0)\| + \frac{2(1+\|P\|)ML_1\rho}{\nu}\end{aligned}$$

for all $t \geq 0$. Therefore,

$$\|Fw - \hat{u}\|_{C_b} \leqslant M\|v_0 - P\hat{u}(0)\| + \frac{2(1+\|P\|)ML_1\rho}{\nu}.$$

Using now the fact that $\|v_0 - P\hat{u}(0)\| \leqslant \frac{\rho}{2M}$ we obtain that if L_1 is small enough, then the transformation F acts form $\mathscr{B}_\rho(\hat{u})$ into $\mathscr{B}_\rho(\hat{u})$.

Now, for $x, z \in \mathscr{B}_\rho(\hat{u})$ (similarly as in (27) we have $\|x\|_{C_b}, \|z\|_{C_b} \leqslant 2\rho$) we estimate

$$\begin{aligned}\|(Fx)(t) - (Fz)(t)\| &\leqslant \int_0^\infty \|\mathscr{G}(t-\tau)\|\|(g(x)(\tau) - g(z)(\tau))\|d\tau\\ &\leqslant (1+\|P\|)M\int_0^\infty e^{-\nu|t-\tau|}d\tau\|g(x)-g(z)\|_{C_b} \text{ for all } t \geq 0.\end{aligned}$$

Therefore,

$$\|Fx - Fz\|_{C_b} \leqslant \frac{2(1+\|P\|)ML_1}{\nu}\|x(\cdot) - z(\cdot)\|_{C_b}.$$

Using now the fact that $\frac{2(1+\|P\|)ML_1}{\nu} < 1$ we obtain that $F : \mathscr{B}_\rho(\hat{u}) \to \mathscr{B}_\rho(\hat{u})$ is a contraction. Thus, there exists a unique $u \in \mathscr{B}_\rho(\hat{u})$ such that $Fu = u$. By definition of F we have that u is the unique solution in $\mathscr{B}_\rho(\hat{u})$ of the Eq. (22) for $t \geq 0$. By Lemma 1 and Remark 2 we have that u is the unique solution in $\mathscr{B}_\rho(\hat{u})$ of the Eq. (10).

Finally, we prove the estimate (26). To do this, since both $\hat{u}$ and u are bounded on $\mathbb{R}_+$, we can use the formula (22) to write

$$u(t) - \hat{u}(t) = \mathscr{T}(t)(Pu(0) - P\hat{u}(0)) + \int_0^\infty \mathscr{G}(t-\tau)(g(u)(\tau) - g(\hat{u})(\tau))d\tau.$$

Therefore,

$$\begin{aligned}\|u(t) - \hat{u}(t)\| \leqslant\; & Me^{-\nu t}\|Pu(0) - P\hat{u}(0)\| +\\ & + (1+\|P\|)M\int_0^\infty e^{-\nu|t-\tau|}\|g(u)(\tau) - g(\hat{u})(\tau)\|d\tau\\ \leqslant\; & Me^{-\nu t}\|Pu(0) - P\hat{u}(0)\| +\\ & + (1+\|P\|)ML_1\int_0^\infty e^{-\nu|t-\tau|}\|u(\tau) - \hat{u}(\tau)\|d\tau.\end{aligned}$$

Applying now a Gronwall-type inequality [2, Corollary III.2.3] we obtain for $\beta := (1+\|P\|)ML_1 < \frac{\nu}{2}$ that

$$\|u(t) - \hat{u}(t)\| \leqslant Ce^{-\mu t}\|Pu(0) - P\hat{u}(0)\| \text{ for } \mu := \sqrt{\nu^2 - 2\nu\beta},\ C := \frac{2M\nu}{\nu + \sqrt{\nu^2 - 2\nu\beta}}.$$

The proof is complete. □

Remark 3 The assertion of the above theorem shows us the *conditional stability* of the periodic solution $\hat{u}$ in the sense that for any other solution u such that $Pu(0) \in$

$B_{\frac{\rho}{2M}}(P\hat{u}(0)) \cap PY$ and u being in a small ball $\mathscr{B}_\rho(\hat{u})$ we have $\|u(t) - \hat{u}(t)\| \to 0$ exponentially as $t \to \infty$ (see inequality (26)).

For an exponentially stable semigroup $(\mathscr{T}(t))_{t\geq 0}$ (see the last part of Definition 2) we have the following corollary which is a direct consequence of Theorem 4.

Corollary 1 *Let the assumptions of the Theorem 3 hold, and let $\hat{u}$ be the periodic solution of (10) obtained in assertion (b) of Theorem 3. Suppose further that the semigroup $(\mathscr{T}(t))_{t\geq 0}$ is exponentially stable. Then, the periodic solution $\hat{u}$ is exponentially stable in the sense that for any other solution $u \in C_b(\mathbb{R}_+, Y)$ of (10) such that $\|u(0) - \hat{u}(0)\|$ is small enough we have*

$$\|u(t) - \hat{u}(t)\| \leqslant Ce^{-\mu t}\|u(0) - \hat{u}(0)\| \text{ for all } t \geq 0 \tag{28}$$

for some positive constants C and μ independent of u and $\hat{u}$.

Proof We just apply Theorem 4 for $P = Id$ to obtain the assertion of the theorem. □

3.2 Applications to Semilinear Damped Wave Equations

In this subsection we consider applications of results obtained in the previous subsection to damped wave equations. To that purpose, suppose $\mathscr{A}$ is a selfadjoint, positive definite operator with compact resolvent in a Hilbert space H and $r : D(\mathscr{A}^{\frac{1}{2}}) \to H$ is of class C^1 with $r(0) = 0,\ r'(0) = 0$. We consider the following abstract damped wave equation:

$$\begin{cases} \ddot{u} + \alpha\dot{u} + \mathscr{A}u + \omega u = r(u) + f(t),\ t > 0 \\ u(0) = u_0;\ \dot{u}(0) = u_1;\ u_0, u_1 \in H, \end{cases} \tag{29}$$

where $\alpha > 0$, $\omega \in \mathbb{R}$ are constant and $f \in C_b(\mathbb{R}_+, H)$ is the external force.

To transform this equation to the first order problem we set $v = \dot{u}$ and handle with the variable $U = \begin{pmatrix} u \\ v \end{pmatrix}$ which belongs to the space $X = D(\mathscr{A}^{\frac{1}{2}}) \times H$. Then we obtain the following equations

$$\begin{cases} \partial_t U = AU + g(U),\ t > 0 \\ U(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X; \end{cases} \tag{30}$$

where the matrix A is defined as $A = \begin{pmatrix} 0 & I \\ -\mathscr{A} - \omega & -\alpha \end{pmatrix}$ with the domain $D(\mathscr{A}) \times H$, and $g(U) = \begin{pmatrix} 0 \\ r(u) + f(t) \end{pmatrix}$. It was proved in [5, p. 4724] that the operator A generates a hyperbolic C_0−semigroup $(e^{tA})_{t\geq 0}$ if $-\omega \notin \sigma(\mathscr{A})$. Moreover, since the operator r is of C^1 and $r(0) = r'(0) = 0$, it follows that r is locally Lipschitz with a small Lipschitz constant in a small neighborhood of 0. Therefore, the operator g satisfies condition in (9) with $Y = X$, $g(0) = f$ and with the Lipschitz constant being small if the radius ρ is small. Thus, applying Theorems 3 and 4 we obtain the following results for the damped wave equation (29).

Theorem 5 *Let $\mathscr{A}$ be a selfadjoint, positive definite operator with compact resolvent in a Hilbert space H, $\alpha > 0$, and $\omega \in \mathbb{R}$ such that $-\omega \notin \sigma(\mathscr{A})$. Suppose $r : D(\mathscr{A}^{\frac{1}{2}}) \to H$ is of class C^1 with $r(0) = r'(0) = 0$. Let $f \in C_b(\mathbb{R}_+, H)$ be T-periodic. Then, if $\|f\|_{C_b(\mathbb{R}_+,H)}$ is small enough, the Eq. (29) has a unique T-periodic mild solution $\hat{u}$ in a small neighborhood of 0. Moreover, this solution $\hat{u}$ is conditional stable in the sense of Theorem 4.*

We next present two examples of the above results.

Example 1 Consider the damped wave equation with nonlinear forcing

$$\partial_t^2 u + \alpha \partial_t u - \Delta u = h(u) + f(t);\ t \in \mathbb{R}_+,\ x \in \Omega, \tag{31}$$

where Ω is a bounded domain with C^2-boundary in $\mathbb{R}^n$, $n = 1, 2, 3$, with homogeneous Dirichlet or Neumann boundary conditions, and $\alpha > 0$ is a constant. The nonlinear term h is given such that h is of class C^1 with $h(0) = 0$. Then, putting $\omega = -h'(0) - 1$, the Eq. (31) is equivalent to

$$\partial_t^2 u + \alpha \partial_t u + (I - \Delta)u + \omega u = h(u) - h'(0)u + f(t);\ t \in \mathbb{R}_+,\ x \in \Omega.$$

The above equation can be rewritten in the form

$$\partial_t^2 u + \alpha \partial_t u + \mathscr{A} u + \omega u = r(u) + f(t);\ t \in \mathbb{R}_+$$

for $\mathscr{A} = I - \Delta$ and $r(u) := h(u) - h'(0)u$. Then with the choice $H := L_2(\Omega)$, it is well-known that $\mathscr{A} = I - \Delta$ with the domain $D(\mathscr{A}) = H_0^1(\Omega) \cap H^2(\Omega)$ is selfadjoint, positive definite, and has compact resolvent. Moreover, by Sobolev embeddings, it easy to to see that the corresponding operator r is a C^1 map from $D(\mathscr{A}^{\frac{1}{2}}) \subset H^1(\Omega)$ to H. Therefore, Theorems 5 and 4 apply, provided that $-\omega \notin \sigma(\mathscr{A})$, i.e., $-h'(0) \notin \sigma(\Delta)$, which shows that for f being T-periodic and sufficiently small, the damped wave equation (31) has a unique T-periodic mild solution $\hat{u}$ in a small ball of $C_b(\mathbb{R}_+, H)$, and this periodic solution is conditionally stable.

Example 2 Consider the damped Timoshenko beam system with nonlinear load

$$\begin{aligned} &\partial_t^2 w + \alpha\partial_t w - \kappa\partial_x(\varphi + \partial_x w) = \partial_w\Psi(w,\varphi) + f(t);\ t \in \mathbb{R}_+,\ x \in [0,l],\\ &\partial_t^2\varphi + \alpha\partial_t\varphi + \kappa(\varphi + \partial_x w) - \epsilon\partial_x^2\varphi = \partial_\varphi\Psi(w,\varphi);\ t \in \mathbb{R}_+,\ x \in [0,l], \end{aligned} \tag{32}$$

with the boundary conditions

$$w(t,0) = \varphi(t,0) = 0 \text{ (clamped end)},\ \partial_x w(t,l) + \varphi(t,l) = \partial_x\varphi(t,l) = 0 \text{ (free end)}.$$

For the details of modeling and physical derivations of damped Timoshenko beam we refer the reader to [12, Sect. 9]. Here, the constants α, κ, ϵ are positive, and $\Psi : \mathbb{R}^2 \to \mathbb{R}$ is of class C^2 with $\nabla\Psi(0) = 0$. Then, we choose $H = L_2(0,l)^2$ and

$$\mathscr{A} = \begin{pmatrix} -\kappa\partial_x^2 & -\kappa\partial_x \\ \kappa\partial_x & \kappa I - \epsilon\partial_x^2 \end{pmatrix} - \nabla^2\Psi(0) - \omega,$$

equipped with the boundary conditions, and $r(u) = \nabla\Psi(u) - \nabla^2\Psi(0)u$, where $u = (w,\varphi)^T$. Then the assumptions of Theorems 5 and 4 are fulfilled, provided that $-\omega \geq 0$ is sufficiently large and $-\omega \notin \sigma(\mathscr{A})$. Therefore, Theorems 5 and 4 can now be applied to obtained that for a T-periodic function $f \in C_b(\mathbb{R}_+, H)$ with $\|f\|_{C_b}$ being sufficiently small, the systems of the damped Timoshenko beam has a unique T-periodic mild solution $\hat{u}$ in a small ball of $C_b(\mathbb{R}_+, H)$, and $\hat{u}$ is conditionally stable.

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Anomalous Solutions to Nonlinear Hyperbolic Equations

Michael Oberguggenberger

Abstract The behavior of sufficiently regular solutions to semilinear hyperbolic equations has attracted a great deal of attention in the past decades, concerning local/global existence, finite time blow-up, critical exponents, and propagation of singularities. Solutions of lower regularity may exhibit unexpected (anomalous) propagation of singularities. The purpose of this paper is to present various striking examples that seemingly have not been addressed in the literature so far. The key issue is the interpretation of the nonlinear operations.

Keywords Semilinear wave equations · Anomalous solutions · Propagation of singularities · Multiplication of distributions

1 Introduction

This paper serves to display various unusual, or *anomalous* solutions to semilinear wave equations

$$\frac{1}{c^2}\partial_t^2 u - \Delta u = f(x,t,u), \quad u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x) \tag{1}$$

in space dimension $n \geq 1$, and to advection-reaction equations

$$\frac{1}{c}\partial_t u + \partial_x u = f(x,t,u), \quad u(x,0) = u_0(x) \tag{2}$$

in one space dimension as prototypical hyperbolic partial differential equations. For nonlinearities of the form $f(x,t,u) = \pm|u|^p$ or $\pm|u|^{p-1}u$, the main research direction in the past decades has been to find bounds on the exponent p and

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the regularity of the initial data, asking about the existence of global solutions with small or large initial data, local solutions, self-similar solutions, blow-up in finite time or stability of blow-up. The reader is referred to the discussion in the monograph [13], the survey article from the 1990s [36], a collection of currently known critical exponents [24] and some of the papers discussing the development of the field [16, 37]. Relevant literature on self-similar solutions and stationary solutions as building blocks will be quoted at the appropriate place in Sect. 4.

In order not to introduce additional singularities, the nonlinear function f will be assumed to be smooth here (actually of the form $f(x, t, u) = g(x)u^p$ with integer $p \geq 2$).

In the 1980s and 1990s, a central question has been propagation of singularities, which started with the discovery of Jeffrey Rauch and Michael Reed [29, 30] that in semilinear hyperbolic equations and systems, singularities do not only propagate out from initial singularities along characteristics or bicharacteristics as in the linear case, but may be created at later times by the interaction of previous singularity bearing (bi-)characteristics. For example, an initial singularity at the origin in problem (1) may lead to singularities in the solution that fill up the solid light cone [2]. For a survey of the vast literature up to around 1990 we refer to the monograph [3]. Rauch and Reed coined the term *anomalous singularities* for this phenomenon.

The results on anomalous singularities required sufficient overall regularity of the solution, for example H^s_{loc}-regularity with $s > (n+1)/2$, and the mechanism for creating the anomalous singularities was still based on characteristics, bicharacteristics and their interaction.

The anomalous solutions presented in this paper are distinguished by (a) lower regularity than in the previous literature and (b) propagation along non-characteristic curves. The majority of examples is based on non-regular solutions to the corresponding stationary elliptic equation. Derivatives are always understood in the sense of distributions. In an attempt to categorize the solutions, four types will be singled out:

Type I: products defined by Hörmander's wave front set criterion;
Type II: products and powers evaluated by Nemytskii operators;
Type III: limits of weak asymptotic solutions;
Type IV: sequential solutions, especially very weak solutions in the sense of Ruzhansky.

It is worth noting that all constructed solutions come with a certain assertion of uniqueness.

The plan of the paper is as follows. Section 2 serves to recall results on anomalous propagation of singularities for sufficiently regular solutions, for reasons of comparison. Section 3 addresses Type I solutions, introducing the employed multiplication of distributions and discussing the question of regularization. Section 4 will exhibit seemingly harmless solutions lying in an L^p-space on which the nonlinear operations are defined and continuous (Type II). In Sect. 5 it will be shown that the solutions from Sect. 4 arise as limits of nets of asymptotic solutions (satisfying the equations up to an error term converging weakly to zero, Type III). In

Sect. 6 nets of smooth functions $(u_\varepsilon)_{\varepsilon>0}$ will be constructed that solve the equations at each fixed $\varepsilon > 0$, but need not necessarily converge as $\varepsilon \to 0$ (Type IV). Nevertheless, their regularity properties can be characterized by suitable estimates on their growth in terms of negative powers of ε as $\varepsilon \to 0$. The appendix serves to recall some notions required to define the products arising in Type I solutions.

The author has been aware of the existence of these anomalous solutions since the early 1980s, but due to a lack of explanation, hesitated to publish them so far. It is hoped that this publication will arouse interest in these types of solutions among the community. Many more examples of similar nature are known, collected by the author and in joint work with Hideo Deguchi [10].

What concerns notation, H^s denotes the usual Sobolev space based on L^2; $\mathcal{C}^k$ denotes the space of k-times differentiable functions, $\mathcal{C}_b^k$ the subspace of functions with bounded derivatives up to order k. The notation for spaces of test functions and distributions follows [35]. The Fourier transform is defined as $\mathcal{F}\varphi(\xi) = \int \mathrm{e}^{-2\pi \mathrm{i} x\xi}\varphi(x)\,\mathrm{d}x$.

2 Propagation of Singularities for Regular Solutions

This section serves to recall results from the 1980s on propagation of singularities for solutions to semilinear hyperbolic systems. These results hold for sufficiently regular solutions (L^∞_{loc} in one space dimension, H^s_{loc} for $s > (n+1)/2$ in space dimension n). We do not strive for full generality—the quoted results will be contrasted with the much less regular solutions to be constructed in the following sections.

We start with $(m \times m)$-systems of first order hyperbolic equations in one space dimension, considering the initial value problem

$$\begin{aligned} (\partial_t + \Lambda\partial_x)u(x,t) &= f(x,t,u(x,t)),\ (x,t) \in R \\ u(x,0) &= u_0(x),\ x \in R_0 \end{aligned} \tag{3}$$

where $R_0 \subset \mathbb{R}$ is an interval and $R \subset \mathbb{R} \times [0,\infty)$ is its domain of determinacy. Here $u = (u_1,\dots,u_m)$, $\Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_m)$ with real and constant entries λ_i, and $f = (f_1,\dots,f_m)$ is smooth. Let $x_1,\dots,x_k \in R_0$ and denote by S_0 the union of characteristic lines emanating from $x_1,\dots,x_k$. Following [30], construct the forward characteristic lines starting at the intersection points of S_0 and call this set S_1. Let S_2 be the set of forward characteristic lines starting from the intersection points of S_1. Continue recursively to construct a sequence of sets S_j. Let S be the closure of $\bigcup_{j=0}^\infty S_j$ intersected with R.

Proposition 1 *Let $u \in (L^\infty(R))^m$ satisfy (3) in the sense of distributions and take on the initial data $u_0 \in (L^\infty(R_0))^m$. Suppose that u_0 is $\mathcal{C}^\infty$ with each derivative uniformly bounded on the complement of the finitely many points $x_1,\dots,x_k$. Then*

u is $\mathcal{C}^\infty$ on $R \setminus S$ and all derivatives of u have continuous extensions from each connected component of $R \setminus S$ to its closure.

Proof This is Theorem 1 from [30]. □

Remark 1

(a) If the function f is linear, then the solution u is in $\mathcal{C}^\infty$ on $R \setminus S_0$—singularities can only lie on characteristic curves tracing back to the singularities of the initial data. In the nonlinear case, the solution is not $\mathcal{C}^\infty$ on $S \setminus S_0$, in general. The singularities belonging to $S \setminus S_0$ in the nonlinear case have been termed *anomalous singularities* by the authors.
(b) In the scalar case and in the case of (2×2)-systems (thus $m = 1$ or $m = 2$), $S = S_0$, so no anomalous singularities arise.

Next we recall a result of [28] on propagation of singularities for semilinear wave equations. Consider the initial value problem

$$\begin{aligned}(\partial_t^2 - \Delta)v(x,t) &= f(v(x,t)),\ (x,t) \in \mathbb{R}^n \times \mathbb{R},\\ v(x,0) &= v_0(x),\ \partial_t v(x,0) = v_1(x),\ x \in \mathbb{R}^n,\end{aligned} \tag{4}$$

where f is a polynomial with $f(0) = 0$, Δ denotes the n-dimensional Laplace operator, and $u_0 \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$, $u_1 \in H^{s-1}_{\mathrm{loc}}(\mathbb{R}^n)$ with $s > (n+1)/2$. Note that $H^s_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$ is an algebra in this case, even contained in the space of continuous functions, so $f(u)$ is classically defined.

Proposition 2 *Let $s > (n+1)/2$ and $v \in H^s_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$ satisfy (4) in the sense of distributions. Suppose that v_0 and v_1 belong to $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$. Then v is $\mathcal{C}^\infty$ on $\{|x| > |t|\}$, and it belongs to $H^{s+1+\sigma}_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R})$ on $\{|x| < |t|\}$ for all $\sigma < s - (n+1)/2$.*

Proof This follows from Theorem 3.1, together with Theorem 1.1 of [28]. □

Remark 2 In space dimension $n = 1$, the solution v is actually $\mathcal{C}^\infty$ in $\{|x| < |t|\}$, as follows from the Corollary to Theorem 2 in [29] as well as the earlier paper [31].

It is known that the solution is not necessarily better than $H^{s+1+\sigma}$ in $\{|x| < |t|\}$ in space dimension $n \geq 2$. For a survey of the state of the art around 1990, see [3].

3 Type I Solutions: Multiplication of Distributions

In this section, we address weak solutions to nonlinear equations where the involved products or powers exist in the sense of Hörmander's wave front set criterion [18]. The examples will be based on the one-dimensional distribution

$$u_0(x) = \frac{1}{x + \mathrm{i}0} = \lim_{\varepsilon \to 0} \frac{1}{x + \mathrm{i}\varepsilon} = \mathrm{vp}\frac{1}{x} - \mathrm{i}\pi\delta(x) \tag{5}$$

also denoted by $\delta_+(x)$ in the physics literature. Here $\mathrm{vp}\frac{1}{x}$ denotes the principal value distribution $\mathrm{vp}\frac{1}{x} = \partial_x \log|x|$ and $\delta(x)$ is the Dirac measure. The Fourier transform of $u_0(x)$ and its auto-convolution are

$$(\mathcal{F}u_0)(\xi) = -2\pi \mathrm{i}\, H(\xi) \quad \text{and} \quad (\mathcal{F}u_0 * \mathcal{F}u_0)(\xi) = -4\pi^2 \xi H(\xi)$$

where H denotes the Heaviside function. In particular, the wavefront set of u_0 is $\{(0, \xi) : \xi > 0\}$, thus u_0^2 exists according to Hörmander's criterion. Actually, it can simply be computed as Fourier product (see Appendix),

$$u_0^2 = \mathcal{F}^{-1}(\mathcal{F}u_0 * \mathcal{F}u_0),$$

as well as all its powers. It holds that

$$u_0^2(x) = \Big(\frac{1}{x + \mathrm{i}0}\Big)^2 = -\Big(\frac{1}{x + \mathrm{i}0}\Big)' = \mathrm{Pf}\frac{1}{x^2} + \mathrm{i}\pi\delta'(x) = -u_0'(x) \tag{6}$$

where $\mathrm{Pf}\frac{1}{x^2}$ is the Hadamard finite part distribution, and

$$2u_0^3(x) = 2\Big(\frac{1}{x + \mathrm{i}0}\Big)^3 = \Big(\frac{1}{x + \mathrm{i}0}\Big)'' = u_0''(x). \tag{7}$$

3.1 A Nonlinear Advection-Reaction Equation

Proposition 3 *The distribution $u(x, t) \equiv u_0(x)$ given by (5) is a weak solution to the initial value problem*

$$\frac{1}{c}\partial_t u + \partial_x u + u^2 = 0, \quad u(x, 0) = u_0(x) \tag{8}$$

for whatever $c \in \mathbb{R}, c \neq 0$, where the square is understood in the sense of Hörmander's product.

Proof It is clear from (6) that $\partial_x u + u^2 = 0$ and that $\partial_t u = 0$. □

Clearly, the mechanism producing this result is that the stationary solution satisfies the nonlinear differential relation $u_0' = -u_0^2$. Further reasons why a genuine distribution can satisfy such a relation will be discussed below. At first we wish to point out that the solution given in Proposition 3 exhibits anomalous propagation of singularities. Indeed,

$$\operatorname{sing\,supp} u = \{(x, t) : x = 0, t \geq 0\}$$

while the expected singular support from Proposition 1 or Remark 1(b) should be $\{(x,t) : x = ct, t \geq 0\}$. To be sure, u_0 does not belong to L^∞ as required in Proposition 1.

Remark 3 It should be noted that anomalous propagation of singularities is not confined to stationary solutions. The following example, due to Deguchi [10], shows that any anomalous propagation speed is possible. Indeed,

$$u(x,t) = \frac{1}{ax + bct + \mathrm{i}0} \tag{9}$$

with $a + b = 1$ solves Eq. (8) with initial data $u_0(x) = 1/(ax + \mathrm{i}0)$, noting that the Fourier product respects affine transformations of the independent variables. The singular support is

$$\operatorname{sing\,supp} u = \{(x,t) : ax = bct, t \geq 0\},$$

which is a non-characteristic line if $a \neq b$.

Remark 4 One possible explanation why the mentioned nonlinear differential relation, as well as similar relations for the higher derivatives, hold for the specific distribution (5) can be obtained by studying its representation as a boundary value of an analytic function. Indeed, every distribution $v \in \mathcal{D}'(\mathbb{R})$ can be represented as the boundary value of a function $\widehat{v}(z)$, analytic in $\mathbb{C} \setminus \operatorname{supp}(v)$, in the sense

$$v(x) = \lim_{\varepsilon \to 0} (\widehat{v}(x + \mathrm{i}\varepsilon) - \widehat{v}(x - \mathrm{i}\varepsilon)) \tag{10}$$

in $\mathcal{D}'(\mathbb{R})$, see e.g. [38]. If v is a distribution of compact support, $\widehat{v}(z)$ is given by the Fantappiè indicatrix

$$\widehat{v}(z) = \frac{1}{2\pi\mathrm{i}}\Big\langle v(x), \frac{1}{x - z}\Big\rangle$$

and in general by a partition of unity procedure. Further, $|\widehat{v}(z)|$ grows at most like a negative power of $|\operatorname{Im} z|$ as $\operatorname{Im} z \to 0$, locally uniformly in $\operatorname{Re} z$. The representation $\widehat{v}(z)$ is unique up to a function analytic on $\mathbb{C}$. Further, every function $\widehat{v}(z)$, analytic in $\mathbb{C} \setminus \mathbb{R}$ and satisfying the growth condition has a distributional boundary value in the sense of (10).

If the support of $\widehat{v}(z)$ is contained in $\{\operatorname{Im} z > 0\}$, the representation is unique. Thus the space of distributions $\mathcal{H}_+(\mathbb{R})$ whose Fantappiè parametrix has support in the upper complex half plane is isomorphic to the space of analytic functions in the upper complex half plane satisfying the mentioned growth condition. However, the latter space is a differential algebra, the differential-algebraic structure of which can be transported to $\mathcal{H}_+(\mathbb{R})$, rendering it a differential algebra [38]. (Similar constructions have also been elaborated in [19].)

This is exactly the case with $u_0(x)$ given by (5) for which

$$\widehat{u}_0(z) = \begin{cases} \frac{1}{z}, & \operatorname{Im} z > 0, \\ 0, & \operatorname{Im} z < 0. \end{cases}$$

In the algebra of analytic functions in the upper half plane, the functional relation

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\Big(\frac{1}{z}\Big) = (-1)^k k!\Big(\frac{1}{z}\Big)^{k+1}, \quad z \neq 0$$

is valid. In this way, formulas (6) and (7) are explained. The differential-algebraic relations persist in the boundary values.

Analytic Regularization

It will be instructive to study the behavior of approximate solutions when the initial data are regularized. The first obvious possibility is to consider the analytic regularization defining the distribution $u_0(x) = 1/(x + \mathrm{i}0)$. We wish to solve the regularized problem

$$\frac{1}{c}\partial_t u_\varepsilon + \partial_x u_\varepsilon + u_\varepsilon^2 = 0, \quad u_\varepsilon(x, 0) = u_{0\varepsilon}(x) = \frac{1}{x + \mathrm{i}\varepsilon}. \tag{11}$$

Solving (11) by the method of characteristics results in the unique classical solution

$$u_\varepsilon(x, t) = \frac{u_{0\varepsilon}(x - ct)}{1 + ctu_{0\varepsilon}(x - ct)} = \frac{\frac{1}{x-ct+\mathrm{i}\varepsilon}}{1 + ct\frac{1}{x-ct+\mathrm{i}\varepsilon}} = \frac{1}{x + \mathrm{i}\varepsilon}.$$

Thus, by simple arithmetic, $u_\varepsilon(x, t) \equiv u_{0\varepsilon}(x)$ and so the solution given in Proposition 3 coincides with the weak limit of approximate solutions when the initial data are replaced by their analytic regularization.

Regularization by Convolution with a Mollifier

The purpose of this subsection is to show that the convergence of the approximate solution is a peculiarity of the analytic regularization and does not hold if the initial data are regularized by convolution with a standard Friedrichs mollifier $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon)$ with $\varphi \in \mathcal{D}(\mathbb{R})$, $\int \varphi(x)\,\mathrm{d}x = 1$. For the sake of the argument, we take $\varphi \geq 0$ symmetric, $\operatorname{supp}\varphi \subset (-1, 1)$. Thus let

$$U_{0\varepsilon}(x) = (u_0 * \varphi_\varepsilon)(x)$$

and let $U_\varepsilon(x,t)$ be the corresponding classical solution to (11) with initial condition $U_\varepsilon(x,0) = U_{0\varepsilon}(x)$. By the method of characteristics,

$$U_\varepsilon(x,t) = \frac{(\mathrm{vp}\frac{1}{\mathrm{x}} * \varphi_\varepsilon)(x-ct) - \mathrm{i}\pi\varphi_\varepsilon(x-ct)}{1 + ct\big((\mathrm{vp}\frac{1}{\mathrm{x}} * \varphi_\varepsilon)(x-ct) - \mathrm{i}\pi\varphi_\varepsilon(x-ct)\big)}.$$

In particular,

$$U_\varepsilon(ct-\varepsilon,t) = \frac{(\mathrm{vp}\frac{1}{\mathrm{x}} * \varphi_\varepsilon)(-\varepsilon)}{1 + ct(\mathrm{vp}\frac{1}{\mathrm{x}} * \varphi_\varepsilon)(-\varepsilon)}. \tag{12}$$

We show that the solution $U_\varepsilon(x,t)$ blows up at latest at

$$t_\varepsilon = \frac{-1/c}{(\mathrm{vp}\frac{1}{\mathrm{x}} * \varphi_\varepsilon)(-\varepsilon)} = \frac{1/c}{(\mathrm{vp}\frac{1}{\mathrm{x}} * \varphi_\varepsilon)(\varepsilon)}$$

and that this number is of order ε as $\varepsilon \to 0$. Thus there is no global solution, when Friedrichs regularization is used.

Indeed, starting from the defining formula

$$(\mathrm{vp}\frac{1}{\mathrm{x}} * \varphi_\varepsilon)(x) = \lim_{\eta\to 0}\int_{|x-y|\geq\eta} \frac{\varphi_\varepsilon(y)}{x-y}\,\mathrm{d}y,$$

some simple manipulations using the support properties of φ lead to

$$(\mathrm{vp}\frac{1}{\mathrm{x}} * \varphi_\varepsilon)(-\varepsilon) = \lim_{\eta\to 0}\int_{-1+\eta/\varepsilon}^{\infty} \frac{\varphi(y)}{-\varepsilon(1+y)}\,\mathrm{d}y = -\frac{1}{\varepsilon}\int_{\mathrm{supp}\,\varphi} \frac{\varphi(y)}{1+y}\,\mathrm{d}y = -\frac{1}{\varepsilon}C_\varphi$$

where C_φ is a positive constant. This shows that the denominator in (12) is indeed zero at $t_\varepsilon = \varepsilon/cC_\varphi$, while the numerator is nonzero.

Separation in Real and Imaginary Part

One might argue that the complex valued initial value problem (11) is actually a real valued, nonstrictly hyperbolic system. This is indeed the case; the real and imaginary part of the analytically regularized solution are

$$u_\varepsilon(x,t) = \frac{1}{x+\mathrm{i}\varepsilon} = v_\varepsilon(x,t) + \mathrm{i}\,w_\varepsilon(x,t) = \frac{x}{x^2+\varepsilon^2} - \mathrm{i}\frac{\varepsilon}{x^2+\varepsilon^2}.$$

The hyperbolic system for the real and imaginary part is

$$\begin{aligned}\partial_t v_\varepsilon + \partial_x v_\varepsilon &= -v_\varepsilon^2 + w_\varepsilon^2,\\ \partial_t w_\varepsilon + \partial_x w_\varepsilon &= -2v_\varepsilon w_\varepsilon.\end{aligned}$$

Here $v_\varepsilon(x,t) \to \mathrm{vp}\frac{1}{x}$ and $w_\varepsilon(x,t) \to -\pi\delta(x)$ as $\varepsilon \to 0$. However, it is well-known (and rather immediate) that v_ε^2 and w_ε^2 do not converge in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \to 0$. Thus the individual terms in the first line make no sense in the limit. (By purely arithmetic manipulations involving $1/(x+\mathrm{i}\varepsilon)$ and $1/(x-\mathrm{i}\varepsilon)$ and their limits, the limit in the right-hand side of the second line is seen to exist and to equal $-\pi\delta'(x)$.)

3.2 A Nonlinear Wave Equation

In the same vein, the distribution $u_0(x)$ can serve to produce a solution to a semilinear wave equation in one space dimension.

Proposition 4 *The distribution $u(x,t) = u_0(x)$ given by (5) is a weak solution to the initial value problem*

$$\frac{1}{c^2}\partial_t^2 u - \partial_x^2 u + 2u^3 = 0, \quad u(x,0) = u_0(x),\ \partial_t u(x,0) = 0 \tag{13}$$

for whatever $c > 0$, where the cubic term is understood in the sense of Hörmander's product.

Proof It is clear from (7) that $-\partial_x^2 u + 2u^3 = 0$ and that $\partial_t u = 0$. □

In the real-valued case, the wave equation (13) has a so-called defocusing nonlinearity. For initial data (u_0, u_1) in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, it would have a unique global finite energy solution [36], belonging to $\mathcal{C}([0,\infty) : H^1(\mathbb{R})) \cap \mathcal{C}^1([0,\infty) : L^2(\mathbb{R}))$. By local existence theory, it could also be extended to small negative times, and hence would belong to L^∞_{loc} in an open neighborhood of the half plane. As in Remark 2, the Corollary to Theorem 2 in [29] would imply that a singularity in the initial data at $x = 0$ can only spread along the characteristic lines $x = \pm ct$. Clearly, the solution given in Proposition 13 neither has the required regularity properties nor does it show the expected singularity propagation.

Remark 5

(a) The distribution $u_0(x)$ is homogeneous of degree -1. Thus $u(x,t) = u_0(x)$ is a self-similar solution to (13), satisfying $\mu u(\mu x, \mu t) = u(x,t)$ for all $\mu > 0$.
(b) The function $u(x,t)$ from Eq. (9) may serve as an example of a non-stationary solution to a nonlinear wave equation which exhibits anomalous propagation of singularities. Indeed, when $a^2 - b^2 = 1$, it solves Eq. (13) with initial data

$u(x,0) = 1/(ax + \mathrm{i}0)$, $\partial_t u(x,0) = 0$. The initial singularity propagates along the line $\{(x,t) : ax + bct = 0, t \geq 0\}$, which is non-characteristic if $a \neq \pm b$.

4 Type II Solutions: Nemytskii Operators

This section addresses weak solutions, whereby the nonlinear terms are defined by Nemytskii operators. We recall the *pseudofunctions* R_λ, meromorphic functions of $\lambda \in \mathbb{C}$ with values in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ [11, Chapter 17]. For $\mathrm{Re}\,\lambda > -n$ they are given by

$$\langle R_\lambda, \varphi \rangle = \int |x|^\lambda \varphi(x)\,\mathrm{d}x$$

and can be analytically continued to $\mathbb{C} \setminus \{-n - 2k : k \in \mathbb{N}\}$. Outside the poles, they satisfy

$$\Delta R_\lambda = \lambda(\lambda + n - 2) R_{\lambda - 2}.$$

In particular, when $\lambda > 2 - n$ and $p = 1 - 2/\lambda$, R_λ belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $(R_\lambda)^p = R_{\lambda p}$ and it satisfies the elliptic equation

$$\Delta R_\lambda = \lambda(\lambda + n - 2)(R_\lambda)^p,$$

where the derivatives are understood in the weak sense and the pth power as the evaluation of the Nemytskii operator $L^p_{\mathrm{loc}}(\mathbb{R}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

We note that for $\lambda \in \mathbb{R} \setminus \{-n - 2k : k \in \mathbb{N}\}$, R_λ is homogeneous of degree λ, and $R_\lambda \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$, if $\lambda > (2-n)/2$.

As examples to be discussed further, we only consider two cases in which p is a positive integer. In the context of propagation of singularities, fractional powers are not interesting for our purpose, because they represent non-smooth nonlinearities. We use the solutions R_λ as examples of peculiar rotationally symmetric stationary solutions to nonlinear wave equations.

Example 1 Let $n = 3$ and $\lambda = -1/2$ (then $\lambda(\lambda + n - 2) = -1/4$). Let $u_0(x) = |x|^{-1/2}$. Then $u_0 \in L^5_{\mathrm{loc}}(\mathbb{R}^3)$, and $u(x,t) \equiv u_0(x)$ satisfies the nonlinear wave equation

$$\frac{1}{c^2}\partial_t^2 u - \Delta u - \frac{1}{4}u^5 = 0, \quad u(x,0) = u_0(x),\ \partial_t u(x,0) = 0 \tag{14}$$

for whatever $c > 0$.

Example 2 Let $n = 4$ and $\lambda = -1$ (then $\lambda(\lambda + n - 2) = -1$). Let $u_0(x) = |x|^{-1}$. Then $u_0 \in L^3_{\text{loc}}(\mathbb{R}^4)$, and $u(x,t) \equiv u_0(x)$ satisfies the nonlinear wave equation

$$\frac{1}{c^2}\partial_t^2 u - \Delta u - u^3 = 0, \quad u(x,0) = u_0(x),\ \partial_t u(x,0) = 0 \tag{15}$$

for whatever $c > 0$.

In all these cases, derivatives are understood in the weak sense and the powers of u exist as locally integrable functions, actually as evaluations of the continuous map $u \to u^p$ from $L^p_{\text{loc}} \to L^1_{\text{loc}}$. Note that the nonlinear operation is taken outside the space of distributions, and the result is embedded afterwards.

Remark 6

(a) As u_0 is nonnegative, we might replace u^5 by $|u|^5$ or $|u|^4 u$. In any case, we are dealing with so-called focusing nonlinearities.
(b) Recall that $u(x,t)$ is a self-similar solution to the nonlinear wave equation

$$\frac{1}{c^2}\partial_t^2 u - \Delta u \pm |u|^p = 0, \tag{16}$$

if $u(x,t) = \mu^\alpha u(\mu t, \mu x)$ for all $\mu > 0$, where necessarily $\alpha = 2/(p-1)$. On the other hand, $u_0 = R_\lambda$ is homogeneous of degree λ, that is, $u_0(sx) = s^\lambda u_0(x)$ for $s > 0$. It also satisfies Eq. (16) when $\lambda - 2 = \lambda p$, i.e., $\lambda = -2/(p-1)$. Thus the special solutions exhibited here are self-similar solutions to the nonlinear wave equation. However, they do not fall into the classes of functions considered e.g. in [4, 20, 26, 27, 32]. It should be noted that solutions to nonlinear elliptic equations have also been used in the literature. They can serve for constructing solutions of finite life span, but also for proving the existence of (time-dependent) self-similar solutions [7, 12, 21, 22].

5 Type III: Weak Asymptotic Solutions

A net of smooth functions $(u_\varepsilon)_{\varepsilon>0}$ is a called a *weak asymptotic solution* [8] to a nonlinear partial differential equation, such as Eq. (16), if it has a limit in the space of distributions and if it satisfies the equation up to an error term which tends to zero weakly as $\varepsilon \to 0$.

The basic example derives again from a nonlinear elliptic equation. Indeed, in $\mathbb{R}^n$, we start from the relation

$$\Delta(|x|^2 + \varepsilon^2)^q = \big((2qn + 4q(q-1))|x|^2 + 2qn\varepsilon^2\big)(|x|^2 + \varepsilon^2)^{q-2}.$$

We will simply work out two special cases that correspond to the ones in Examples 1 and 2.

Example 3 Let $n = 3$ and $q = -1/4$. By simple arithmetic,

$$\big(2qn + 4q(q-1)\big)|x|^2 + 2qn\varepsilon^2 = -\frac{1}{4}(|x|^2 - \varepsilon^2) - \frac{5}{4}\varepsilon^2$$

and so

$$\Delta(|x|^2 + \varepsilon^2)^{-1/4} = -\frac{1}{4}(|x|^2 + \varepsilon^2)^{-5/4} - \frac{5}{4}\varepsilon^2(|x|^2 + \varepsilon^2)^{-9/4}.$$

Thus

$$u_\varepsilon(x,t) = (|x|^2 + \varepsilon^2)^{-1/4}$$

satisfies the nonlinear wave equation

$$\frac{1}{c^2}\partial_t^2 u_\varepsilon - \Delta u_\varepsilon - \frac{1}{4}u_\varepsilon^5 - \frac{5}{4}\varepsilon^2 u_\varepsilon^9 = 0 \tag{17}$$

for whatever $c > 0$. An easy calculation shows that $\varepsilon^2 u_\varepsilon^9$ converges to zero in $\mathcal{D}'(\mathbb{R}^3)$ as $\varepsilon \to 0$. Thus u_ε is a weak asymptotic solution to the nonlinear wave equation (14) with initial data converging to $u_0(x) = |x|^{-1/2}$. As in Example 1 we set $u(x,t) = u_0(x)$. By the continuity assertions for Type II solutions,

$$u_\varepsilon \to u, \quad u_\varepsilon^5 \to u^5 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^3) \quad \text{as } \varepsilon \to 0,$$

thus each term in Eq. (17) converges to the corresponding term in Eq. (14). Further, u_ε is a smooth approximation to u; as $\varepsilon \to 0$, a singularity emerges at $x = 0$.

It is of interest to note that the solution to the regularized Eq. (17) is unique. This emphasizes again the anomaly in the propagation of singularities in the initial value problem (14).

Lemma 1 *Let $n = 1$, $n = 2$ or $n = 3$. Assume that $u_0 \in \mathcal{C}_b^1(\mathbb{R}^n)$, $u_1 \in \mathcal{C}_b^0(\mathbb{R}^n)$ and let f be smooth. Given any $T > 0$, the initial value problem*

$$\frac{1}{c^2}\partial_t^2 u - \Delta u = f(u), \quad u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x) \tag{18}$$

has at most one weak solution in $\mathcal{C}_b^0(\mathbb{R}^n \times [0,T])$.

Proof Let $S(t)$ be the fundamental solution of the Cauchy problem, that is, $S(t)$ is the inverse Fourier transform of $\sin(c|\xi|t)/|\xi|$. In space dimensions $n = 1, 2, 3$, $S(t)$ is a finite measure of total mass ct. The solution is given by

$$u(.,t) = \frac{d}{dt}S(t) * u_0 + S(t) * u_1 + \int_0^t S(t-s) * f(u(.,s))\,\mathrm{d}s.$$

By Young's inequality, the L^∞-estimate

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^n)} \le C(t)\|u_0, \nabla u_0, u_1\|_{L^\infty(\mathbb{R}^n)} + \int_0^t (ct - cs)\|u(.,s)\|_{L^\infty(\mathbb{R}^n)}\,\mathrm{d}s$$

holds, where C(t) is a constant depending linearly on t. Applying this estimate to the difference $u - v$ of two solutions with the same initial data, writing $f(u) - f(v) = (u - v)g(u, v)$ with g smooth and applying Gronwall's inequality shows that $u = v$. □

Example 4 Let $n = 4$ and $q = -1/2$ and let

$$u_\varepsilon(x,t) = (|x|^2 + \varepsilon^2)^{-1/2}.$$

By the same arguments as in Example 3 one sees that u_ε satisfies the nonlinear wave equation

$$\frac{1}{c^2}\partial_t^2 u_\varepsilon - \Delta u_\varepsilon - u_\varepsilon^3 - 3\varepsilon^2 u_\varepsilon^5 = 0 \tag{19}$$

for whatever $c > 0$. Again, one shows that $\varepsilon^2 u_\varepsilon^5$ converges to zero in $\mathcal{D}'(\mathbb{R}^4)$ as $\varepsilon \to 0$, and u_ε is a weak asymptotic solution to the nonlinear wave equation (15) with initial data converging to $u_0(x) = |x|^{-1}$. With $u(x,t) \equiv u_0(x)$, one has again

$$u_\varepsilon \to u, \quad u_\varepsilon^3 \to u^3 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^4) \quad \text{as } \varepsilon \to 0,$$

thus each term in Eq. (19) converges to the corresponding term in Eq. (15). The same behavior as in Example 3 is observed.

Due to the continuity of the Nemytskii operators, the weak asymptotic solutions constructed here are consistent with the solutions presented in Sect. 4.

6 Type IV: Sequential Solutions

In this section, we address solutions defined by nets of smooth functions which do not necessarily converge. To introduce the concept, let Ω be an open subset of $\mathbb{R}^n$ and let P be a possibly nonlinear partial differential operator which is a smooth function of its arguments, $Pu = P(x, u, \partial u, \ldots)$. Let $(u_\varepsilon)_{\varepsilon>0}$ be a net of functions belonging to $\mathcal{C}^\infty(\Omega)$. If $Pu_\varepsilon = 0$ for all sufficiently small $\varepsilon > 0$, then the net $(u_\varepsilon)_{\varepsilon>0}$ is called a *sequential solution* of the equation $Pu = 0$, following e.g. [33]. The net $(u_\varepsilon)_{\varepsilon>0}$ may or may not converge. Even if $(u_\varepsilon)_{\varepsilon>0}$ converges, individual terms in $P(x, u, \partial u, \ldots)$ may or may not converge. However, if $(u_\varepsilon)_{\varepsilon>0}$ converges to a distribution u, together with all individual terms in $P(x, u, \partial u, \ldots)$, then u can be called a *proper weak solution* to $Pu = 0$ [23].

Restricting the class of sequential solutions to *moderate nets* allows one to establish a regularity theory for sequential solutions, even if they diverge. A net of smooth functions $(u_\varepsilon)_{\varepsilon>0}$ on Ω is called *moderate*, if for all compact subsets K of Ω and all multi-indices $\alpha \in \mathbb{N}_0^n$ there exists $b \geq 0$ such that

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-b}) \quad \text{as } \varepsilon \to 0.$$

The net of smooth functions $(u_\varepsilon)_{\varepsilon>0}$ on Ω is called *negligible*, if for all compact subsets K of Ω, all multi-idices $\alpha \in \mathbb{N}_0^n$ and all $a \geq 0$,

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{a}) \quad \text{as } \varepsilon \to 0.$$

Following [15, 34], a moderate net satisfying $Pu_\varepsilon = 0$ for all sufficiently small $\varepsilon > 0$ is called a *very weak solution* to the equation $Pu = 0$. If $(u_\varepsilon)_{\varepsilon>0}$ is moderate and $Pu_\varepsilon = n_\varepsilon$ where $(n_\varepsilon)_{\varepsilon>0}$ is a negligible net, then $(u_\varepsilon)_{\varepsilon>0}$ is a *Colombeau solution* to the equation $Pu = 0$. (As a matter of fact, its equivalence class in the Colombeau algebra $\mathcal{G}(\Omega)$ is a solution in the differential-algebraic sense [6, 17, 25].)

Finally, a net $(u_\varepsilon)_{\varepsilon>0}$ is said to possess the $\mathcal{G}^\infty$-property, if for all compact subsets K of Ω there is $b \geq 0$ such that for all multi-indices $\alpha \in \mathbb{N}_0^n$,

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-b}) \quad \text{as } \varepsilon \to 0.$$

(Note the change in quantifiers: the local order of growth is the same for all derivatives.) The significance of this notion is that it generalizes $\mathcal{C}^\infty$-smoothness from distributions to moderate nets. In fact, if $w \in \mathcal{E}'(\Omega)$ is a compactly supported distribution and φ_ε is a mollifier ($\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ with φ smooth, rapidly decaying and $\int \varphi(x)\,\mathrm{d}x = 1$), then

- $w_\varepsilon = w * \varphi_\varepsilon | \Omega$ defines a moderate net;
- $(w_\varepsilon)_{\varepsilon>0}$ has the $\mathcal{G}^\infty$-property if and only if $w \in \mathcal{C}^\infty(\Omega)$.

The $\mathcal{G}^\infty$-singular support of a moderate net $(u_\varepsilon)_{\varepsilon>0}$ is defined as the complement of the largest open subset $\omega \subset \Omega$ such that $(u_\varepsilon|\omega)_{\varepsilon>0}$ has the $\mathcal{G}^\infty$-property on ω. The same notions can be introduced for nets of smooth functions defined on the closure of an open subset of $\mathbb{R}^n$, thereby enabling the study of initial value problems or boundary value problems.

Replacing $\mathcal{C}^\infty$ by $\mathcal{G}^\infty$, classical regularity theory and propagation of singularities for linear partial differential equations can be literally transferred to the setting of moderate nets in the case of linear equations (with possibly non-smooth coefficients). Here are some specific results in this direction: $\mathcal{G}^\infty$-singularities in the linear wave equation propagate along the light cone in any space dimension, [25]. For wave equations in one space dimension with piecewise constant coefficient, propagation of $\mathcal{G}^\infty$-singularities occurs along characteristic lines emanating from the initial point singularity, with reflection/diffraction at the points of discontinuity of the coefficient, [9]. The $\mathcal{G}^\infty$-wave front set of the kernels of Fourier integral operators can be computed analogously to the classical case, and $\mathcal{G}^\infty$-singularities in solutions to first order hyperbolic equations propagate along the Hamiltonian flow [14].

6.1 Moderate Sequential Solutions to an Advection-Reaction Equation

We are going to construct moderate sequential solutions to the advection-reaction equation in one space dimension

$$\frac{1}{c}\partial_t u + \partial_x u + \frac{2}{p}\, x\, u^{p+1} = 0, \quad u(x,0) = u_0(x) \tag{20}$$

where—for simplicity—p is a positive integer. We first note that for continuous initial data, there is at most one solution.

Lemma 2 *Assume that $u_0 \in \mathcal{C}_b^0(\mathbb{R})$, $c \neq 0$ and let f be smooth. Given any $T > 0$, the initial value problem*

$$\frac{1}{c}\partial_t u + \partial_x u = f(x,t,u), \quad u(x,0) = u_0(x) \tag{21}$$

has at most one weak solution in $\mathcal{C}_b^0(\mathbb{R}^n \times [0,T])$.

Proof Indeed, if u is a solution, it solves the integral equation

$$u(x,t) = u_0(x-ct) + \int_0^t f(x-ct+cs, s, u(x-ct+cs,s))\,\mathrm{d}s.$$

Uniqueness follows by the same argument as in the proof of Lemma 1. □

It is immediately checked that, for each $\varepsilon > 0$, the smooth function

$$u_\varepsilon(x,t) \equiv u_{0\varepsilon}(x) = (x^2+\varepsilon^2)^{-1/p} \tag{22}$$

is a solution to the initial value problem

$$\frac{1}{c}\partial_t u_\varepsilon + \partial_x u_\varepsilon + \frac{2}{p}\, x\, u_\varepsilon^{p+1} = 0, \quad u_\varepsilon(x,0) = (x^2+\varepsilon^2)^{-1/p}. \tag{23}$$

According to Lemma 2, the solution is unique. It is clear that the net $(u_\varepsilon)_{\varepsilon>0}$ is moderate, hence it defines a moderate sequential solution to (20).

Lemma 3 *The net $(u_{0\varepsilon})_{\varepsilon>0}$ converges for $p \geq 3$ and diverges for $p = 1, 2$. In particular, $(u_{0\varepsilon}^{p+1})_{\varepsilon>0}$ diverges for every $p > 0$.*

Proof For $p \geq 3$, $u_0(x) = |x|^{-2/p}$ belongs to the space of locally integrable functions, and $u_{0\varepsilon}(x) = (x^2 + \varepsilon^2)^{-1/p}$ converges to it in that space.

Let $p = 2$ and take a test function $\varphi \geq 0$ such that $\varphi(x) = 1$ on $[-1, 1]$. Then

$$\langle u_{0\varepsilon}, \varphi \rangle = \int_{-\infty}^{\infty} \frac{\varphi(x)}{\sqrt{x^2+\varepsilon^2}}\,\mathrm{d}x \geq \int_{-1}^{1} \frac{1}{\sqrt{x^2+\varepsilon^2}}\,\mathrm{d}x = \int_{-1/\varepsilon}^{1/\varepsilon} \frac{1}{\sqrt{y^2+1}}\,\mathrm{d}y \to \infty$$

as $\varepsilon \to 0$. A similar argument shows that $(x^2 + \varepsilon^2)^{-q}$ diverges for $q > 1/2$. Thus $u_{0\varepsilon}(x) = (x^2 + \varepsilon^2)^{-1/p}$ diverges when $p < 2$ as well, in particular, for $p = 1$. Further, $u_{0\varepsilon}^{p+1}(x) = (x^2 + \varepsilon^2)^{-1-1/p}$ diverges for every $p > 0$. □

This shows that even in the convergent case $p \geq 2$, the limit $u = \lim_{\varepsilon\to 0} u_\varepsilon$ is not a proper solution of Eq. (20).

The Special Case $p = 2$

Let us have a more detailed look at the (divergent) case $p = 2$. Then the function

$$u_\varepsilon(x, t) = (x^2 + \varepsilon^2)^{-1/2}, \tag{24}$$

at fixed $\varepsilon > 0$, is a solution to the advection-reaction equation

$$\frac{1}{c}\partial_t u_\varepsilon + \partial_x u_\varepsilon + x\, u_\varepsilon^3 = 0, \quad u_\varepsilon(x, 0) = (x^2 + \varepsilon^2)^{-1/2}. \tag{25}$$

According to Lemma 2, this solution is unique. We may study its $\mathcal{G}^\infty$-regularity properties.

Proposition 5 *The $\mathcal{G}^\infty$-singular support of $(u_\varepsilon)_{\varepsilon>0}$ is $\{(0, t) : t \geq 0\}$.*

Proof Let $\chi(x) = (x^2 + 1)^{-1/2}$. Then $(u_\varepsilon(x, t) = (x^2 + \varepsilon^2)^{-1/2} = \chi_\varepsilon(x) = \varepsilon^{-1}\chi(x/\varepsilon)$. It is straightforward to show that the kth derivative of χ is of the form

$$\chi^{(k)}(x) = P_k(x)(x^2 + 1)^{-k-1/2}$$

where P_k is a polynomial of degree k. Therefore,

$$\chi_\varepsilon^{(k)}(x) = \varepsilon^{-k-1} P_k\Big(\frac{x}{\varepsilon}\Big)\Big(\frac{x^2}{\varepsilon^2} + 1\Big)^{-k-1/2} = \varepsilon^k P_k\Big(\frac{x}{\varepsilon}\Big)(x^2 + \varepsilon^2)^{-k-1/2}.$$

When $|x| \geq x_0 > 0$, the latter expression is bounded independently of $\varepsilon > 0$. Thus $(u_\varepsilon)_{\varepsilon>0}$ has the $\mathcal{G}^\infty$-property in the region $\{(x, t) : |x| > 0, t \geq 0\}$.

On the other hand, $\chi(x)$ is the derivative of arsinh x, whose Taylor expansion shows that $\chi^{(k)}(x) \neq 0$ when k is an even integer. Thus

$$\chi_\varepsilon^{(k)}(0) = \varepsilon^{-k-1}\chi^{(k)}(0)$$

does not have the $\mathcal{G}^\infty$-property: the line $x = 0$ is contained in the $\mathcal{G}^\infty$-singular support. □

This shows that the moderate sequential solution to (25) exhibits anomalous propagation of singularities. The initial $\mathcal{G}^\infty$-singularity at $x = 0$ is not propagated along the line $x = ct$ as in the linear case, but rather remains at $x = 0$ for all times.

Remark 7 Actually, the classical initial value problem $\frac{1}{c}\partial_t v + \partial_x v + x\,v^3 = 0$, $v(x, 0) = v_0(x)$ can be solved explicitly. Transformation to characteristic coordinates $s = t$, $y = x - ct$ leads to an ordinary differential equation and to the solution

$$v(x,t) = \frac{v_0(x-ct)}{\sqrt{\left(x^2-(x-ct)^2\right)v_0^2(x-ct)+1}}.$$

Inserting $v_0(x) = (x^2+\varepsilon^2)^{-1/2}$ it turns out that by simple arithmetic, $v(x,t) = (x^2+\varepsilon^2)^{-1/2}$, supporting the fact that $u_\varepsilon(x,t)$ as given above by (24) is indeed the solution. The same phenomenon also happens for $p \neq 2$ in (22) and (23).

6.2 Moderate Sequential Solutions to a Nonlinear Wave Equation

Taking a further x-derivative, it is seen that $u_\varepsilon(x,t)$ given by (24) also solves the one-dimensional nonlinear wave equation

$$\frac{1}{c^2}\partial_t^2 u_\varepsilon - \partial_x^2 u_\varepsilon + u_\varepsilon^3 + 3x^2 u_\varepsilon^5 = 0, \quad u_\varepsilon(x,0) = (x^2+\varepsilon^2)^{-1/2},\ \partial_t u_\varepsilon(x,0) = 0$$

for every $c > 0$. In this case, standard energy estimates can be used to show that the solution is unique.

Lemma 4 *Given $v_0 \in H^1(\mathbb{R})$, $v_1 \in L^2(\mathbb{R})$ of finite energy (defined by (27) below), the equation*

$$\frac{1}{c^2}\partial_t^2 v - \partial_x^2 v + v^3 + 3x^2 v^5 = 0, \quad v(x,0) = v_0(x),\ \partial_t v(x,0) = v_1(x) \tag{26}$$

has a unique solution $v \in \mathcal{C}([0,\infty) : H^1(\mathbb{R})) \cap \mathcal{C}^1([0,\infty) : L^2(\mathbb{R}))$ of finite energy, where $c > 0$.

Proof It is quite obvious that the energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty}\Big(|\partial_t v|^2 + c^2|\partial_x v|^2 + |v|^4 + 3x^2|v|^6\Big)\,\mathrm{d}x \tag{27}$$

is conserved. The proof follows standard arguments (see e.g. [36]). □

At fixed $\varepsilon > 0$, $u_\varepsilon(x,0) = (x^2+\varepsilon^2)^{-1/2}$ belongs to $H^1(\mathbb{R})$ and, together with $\partial_t u_\varepsilon(x,0) = 0$, forms initial data of finite energy. Thus the stationary solution $u_\varepsilon(x,t) = u_\varepsilon(x,0)$ is the unique solution in this sense. The net $(u_\varepsilon)_{\varepsilon>0}$ provides a moderate sequential solution to the nonlinear wave equation (26). Its $\mathcal{G}^\infty$-singular support $\{(x,t), x = 0, t \geq 0\}$ has been computed in Proposition 5. Again, this differs from the linear case [14] and the nonlinear, classical case (Propositions 1, 2), according to which the singular support should be $\{(x,t), |x| = ct, t \geq 0\}$.

Anomalous propagation of singularities persists for sequential solutions.

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Appendix: On Multiplication of Distributions

Let $S, T \in \mathcal{S}'(\mathbb{R}^n)$. The $\mathcal{S}'$-convolution of S and T is said to exist, if

$$(\varphi * \check{S})T \in \mathcal{D}'_{L^1}(\mathbb{R}^n), \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where $\check{S}(x) = S(-x)$. In this case, the convolution is defined by $\langle S * T, \varphi\rangle = \langle(\varphi * \check{S})T, 1\rangle$, and $S * T$ belongs to $\mathcal{S}'(\mathbb{R}^n)$.

Let $u, v \in \mathcal{S}'(\mathbb{R}^n)$. If the $\mathcal{S}'$-convolution of $\mathcal{F}u$ and $\mathcal{F}v$ exists, one may define the *Fourier product*

$$u \cdot v = \mathcal{F}^{-1}(\mathcal{F}u * \mathcal{F}v).$$

The definition can be localized [1] as follows. Assume that for every $x \in \mathbb{R}^n$ there is a neighborhood Ω_x and $\chi_x \in \mathcal{D}(\mathbb{R}^n)$, $\chi_x \equiv 1$ on Ω_x, such that the $\mathcal{S}'$-convolution of $\mathcal{F}(\chi_x u)$ and $\mathcal{F}(\chi_x v)$ exists. Locally near x, the product $u \cdot v$ is defined to be $\mathcal{F}^{-1}(\mathcal{F}(\chi_x u) * \mathcal{F}(\chi_x v))$. Globally, it is defined by a partition of unity argument.

A special case arises when the distributions satisfy Hörmander's wave front set criterion [18], requiring that for every $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, $(x,\xi) \in \mathrm{WF}(u)$ implies $(x,-\xi) \notin \mathrm{WF}(v)$.

In space dimension $n = 1$, a very convenient case arises when $\operatorname{supp} \mathcal{F}u$ and $\operatorname{supp} \mathcal{F}v$ are contained in $[0, \infty)$. (In particular, Hörmander's criterion is fulfilled.) The basic example used in Sect. 3 is

$$u_0(x) = \frac{1}{x + \mathrm{i}0} = \lim_{\varepsilon \to 0} \frac{1}{x + \mathrm{i}\varepsilon} = \mathrm{vp}\frac{1}{x} - \mathrm{i}\pi\delta(x)$$

whose Fourier transform is $(\mathcal{F}u_0)(\xi) = -2\pi \mathrm{i} H(\xi)$. The auto-convolution results in $(\mathcal{F}u_0 * \mathcal{F}u_0)(\xi) = -4\pi^2 \xi H(\xi)$. Thus $u_0^2 = \mathcal{F}^{-1}(\mathcal{F}u_0 * \mathcal{F}u_0)$ exists as Fourier product, and the formula shows that $u_0^2(x) = -u_0'(x)$. The remaining formulas used in Sect. 3 follow in the same way.

A more general definition of the product of distributions on $\mathbb{R}^n$ can be obtained by regularization and passage to the limit. The *model product* of u and v is defined as

$$[u \cdot v] = \lim_{\varepsilon \to 0} (u * \varphi_\varepsilon)(v * \varphi_\varepsilon)$$

provided the limit exists for all mollifiers φ_ε of the form $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ with $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\int \varphi(x)\,\mathrm{d}x = 1$, and is independent of the chosen mollifier. If the Fourier product exists, so does the model product.

In the one-dimensional case ($n = 1$), a yet more general definition is obtained by using the representation by boundary values of analytic functions, which was discussed in Sect. 3. Given $u \in \mathcal{D}'(\mathbb{R})$, let

$$\widetilde{u}_\varepsilon(x) = \widehat{u}(x + \mathrm{i}\varepsilon) - \widehat{u}(x - \mathrm{i}\varepsilon),$$

with the right-hand side as in (10). It was seen in Sect. 3 that $u(x) = \lim_{\varepsilon \to 0} \widetilde{u}_\varepsilon(x)$. If $u \in \mathcal{D}'_{L^1}(\mathbb{R})$, $\widetilde{u}_\varepsilon$ is obtained by convolving u with the special mollifier $\psi_\varepsilon(x) = \varepsilon/(\pi(x^2 + \varepsilon^2))$. The *Tillmann product* [38] of two distributions u, v is defined by

$$u \circ v = \lim_{\varepsilon \to 0} \widetilde{u}_\varepsilon \cdot \widetilde{v}_\varepsilon$$

provided the limit exists. The definition does not work in higher space dimensions; there, harmonic regularization should be used [5]. In any case, the powers in (6) and (7) can also be understood in the sense of the Tillmann product.

Hörmander's criterion implies the existence of the Fourier product, which implies the existence of the model product and in turn also the existence of the Tillmann product. None of the implications can be reversed.

The other products used in this paper enter at different levels. For example, the most basic product of a smooth function with a distributions enters below Hörmander's criterion. The product in $H^s_{\mathrm{loc}}(\mathbb{R}^n)$ when this space is an algebra ($s > n/2$) enters as a subcase of the Fourier product, but is independent of Hörmander's criterion. The Nemytskii operators in the form of a continuous map $L^p_{\mathrm{loc}} \times L^q_{\mathrm{loc}} \to L^1_{\mathrm{loc}}$, $1/p + 1/q = 1$, enter at the level of the model product, but are independent of the Fourier product criterion. For more details on these circle of ideas, see [25].

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An Introduction to the Gabor Wave Front Set

Luigi Rodino and S. Ivan Trapasso

Abstract In this expository note we present an introduction to the Gabor wave front set. As is often the case, this tool in microlocal analysis has been introduced and reinvented in different forms which turn out to be equivalent or intimately related. We provide a short review of the history of this notion and then focus on some recent variations inspired by function spaces in time-frequency analysis. Old and new results are presented, together with a number of concrete examples and applications to the problem of propagation of singularities.

Keywords Wave front set · Modulation spaces · Microlocal analysis · Propagation of singularities · Schrödinger equation

1 Introduction

A central notion in microlocal analysis of partial differential equations is the wave front set [29]. In somewhat rough terms, the wave front set of a distribution u is the collection of all the points of the phase space (x_0, ξ_0), $\xi_0 \neq 0$, where the lack of regularity of u at x_0 is detected on the spectral side by a characteristic behaviour in the direction ξ_0. Giving a rigorous meaning to this heuristic model provides a fine scale of technical tools for the microlocal study of singularities of pseudodifferential operators and their propagation. It should be stressed that wave front sets play a major role in the mathematical theory of quantum fields [21, 49]. We cannot frame here the long tradition of studies on the wave front set and its applications; a

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complete historical and technical account may be found in the monograph [29] by Hörmander, who first introduced wave front sets in [26].

In recent times the notion of wave front set has benefited from the perspective of time-frequency analysis [33, 44, 45, 50]. The spirit of Gabor analysis may be condensed in the idea of simultaneous analysis of distributions with respect to both time and frequency variables; several techniques and function spaces were introduced in the last decades to carry out this program [25]. The affinities with the notion of wave front set, where the regularity is measured by a simultaneous analysis of points and directions, are evident.

The purpose of this introductory paper is to present some of the contributions in this respect, in particular we focus on the Gabor wave front set [50]. To be precise, the idea of a *global* wave front set showed up many times under several different guises; a historical account on the issue with many pointers to the literature is given in Sect. 3, while in Sect. 2 we collected some preliminary material from microlocal and time-frequency analysis.

In Sect. 4 we provide a more technical description of the Gabor wave front set. In particular, we highlight the most important results of the papers [11, 28] and [50], together with a number of detailed examples. New results for the wave front set in the context of modulation space regularity are derived in Sect. 4.3. We conclude with a brief review of applications to propagation of singularities.

Most of the technical proofs are omitted to keep the presentation at an introductory level. We hope that this overview may be useful as a point of departure for the interested reader, as well as a practical summation of the most relevant results on the topic.

2 Preliminaries

2.1 Notation

We set $x^2 = x \cdot x$, for $x \in \mathbb{R}^n$, where $x \cdot y = xy$ is the scalar product on $\mathbb{R}^n$. The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^n)$, the space of temperate distributions by $\mathcal{S}'(\mathbb{R}^n)$. The brackets $\langle f, g\rangle$ denote the extension to $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ of the inner product $\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx$ on $L^2(\mathbb{R}^n)$.

The conjugate exponent p' of $p \in [1,\infty]$ is defined by $1/p + 1/p' = 1$. The symbol $\lesssim$ means that the underlying inequality holds up to a positive constant factor $C > 0$. For any $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$ we set $\langle x\rangle^s := (1 + |x|^2)^{s/2}$. We choose the following normalization for the Fourier transform:

$$\hat{f}(\xi) = \mathcal{F} f(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\xi} f(x)dx, \qquad \xi \in \mathbb{R}^n.$$

We define the translation and modulation operators: for any $x, \xi \in \mathbb{R}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$,

$$(T_x f)(y) := f(y-x), \qquad \left(M_\xi f\right)(y) := e^{2\pi i \xi y} f(y).$$

These operators can be extended by duality on temperate distributions. The composition $\pi(x,\xi) = M_\xi T_x$ constitutes a *time-frequency shift*.

Recall that $\Gamma \subset \mathbb{R}^n$ is a *conic subset* of $\mathbb{R}^n$ if it is invariant under multiplication by positive real numbers, namely $x \in \Gamma \Rightarrow \lambda x \in \Gamma$ for any $\lambda > 0$.

The *symplectic group* $\mathrm{Sp}(n, \mathbb{R})$ consists of all $2n \times 2n$ invertible matrices $S \in \mathrm{GL}\,(2n, \mathbb{R})$ such that

$$S^\top J S = S J S^\top = J, \quad J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

where J is the canonical symplectic matrix and O and I are the $n \times n$ zero and identity matrices respectively.

In the rest of the paper we identify the cotangent set $T^*\mathbb{R}^n$ of $\mathbb{R}^n$ with $\mathbb{R}^{2n}$ to lighten the notation.

2.2 Modulation Spaces

The short-time Fourier transform (STFT) of a temperate distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ with respect to the window function $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ is defined by

$$V_\varphi u(x,\xi) := \mathcal{F}(u \cdot T_x \varphi)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i y \xi} u(y)\, \overline{\varphi(y-x)}\, dy. \tag{1}$$

The reader may want to consult the monograph [25] for a comprehensive treatment of the mathematical properties of this time-frequency representation, in particular those mentioned below. We highlight that the STFT is intimately connected with other well-known phase-space transforms, in particular the Wigner distribution

$$W(u,\varphi)(x,\xi) = \int_{\mathbb{R}^d} e^{-2\pi i y \xi} u\left(x + \frac{y}{2}\right) \overline{\varphi\left(x - \frac{y}{2}\right)}\, dy. \tag{2}$$

As far as the regularity is concerned, the STFT of a possibly wild distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ is a well-behaved function; in particular, we have that $V_\varphi u \in C(\mathbb{R}^{2n})$ and there exist constants $C > 0$ and $N \ge 0$ such that $|V_\varphi u(z)| \le C \langle z \rangle^N$ for all $z \in \mathbb{R}^{2n}$. Furthermore, $V_\varphi u \in \mathcal{S}(\mathbb{R}^{2n}) \Leftrightarrow u \in \mathcal{S}(\mathbb{R}^n)$. It turns out that the STFT is one-to-one in $\mathcal{S}'(\mathbb{R}^n)$, as a result of the following *inversion formula*: for $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ such that $\langle \varphi, \psi \rangle \neq 0$ we have

$$u = \frac{1}{\langle \varphi, \psi \rangle} \int_{\mathbb{R}^{2n}} V_\varphi u(z) \pi(z) \psi dz, \tag{3}$$

to be interpreted in the distribution sense - namely, the right-hand side is a temperate distribution whose action on $\phi \in \mathcal{S}(\mathbb{R}^n)$ coincides with $\langle u, \phi \rangle$. Notice in particular that if we choose $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ with $\|\varphi\|_{L^2} = 1$ and set $\psi = \varphi$ we have

$$|V_\varphi u(w)| = \left| \int_{\mathbb{R}^{2n}} V_\varphi u(z) V_\varphi \varphi(w - z) dz \right|, \quad w \in \mathbb{R}^{2n}. \tag{4}$$

This argument generalizes to the following pointwise inequality ("change-of-window lemma") which will be used below.

Lemma 1 ([25, Lem. 11.3.3]) *Let $\varphi_1, \varphi_2, \phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\langle \phi, \varphi_1 \rangle \neq 0$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Therefore*

$$|V_{\varphi_2} u(x, \xi)| \le \frac{1}{|\langle \phi, \varphi_1 \rangle|} (|V_{\varphi_1} u| * |V_{\varphi_2} \phi|)(x, \xi), \quad \forall (x, \xi) \in \mathbb{R}^{2n}.$$

When speaking of weight functions below we refer to some positive function $m \in L^\infty_{\text{loc}}(\mathbb{R}^{2n})$ such that $m(z + \zeta) \lesssim m(z) \langle \zeta \rangle^r$ for some $r \ge 0$ and any $z, \zeta \in \mathbb{R}^{2n}$ - that is, m is $\langle \cdot \rangle^r$-moderate.

Given a non-zero window $\varphi \in \mathcal{S}(\mathbb{R}^n)$, a weight function m on $\mathbb{R}^{2n}$ and $1 \le p, q \le \infty$, the *modulation space* $M^{p,q}_m(\mathbb{R}^n)$ consists of all the temperate distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $V_\varphi u \in L^{p,q}_m(\mathbb{R}^{2n})$ (mixed weighted Lebesgue space), that is:

$$\|u\|_{M^{p,q}_m} = \|V_\varphi u\|_{L^{p,q}_s} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\varphi u(x, \xi)|^p m(x, \xi)^p \, dx \right)^{q/p} d\xi \right)^{1/q} < \infty,$$

with trivial modification if p or q is ∞. If $p = q$, we write M^p instead of $M^{p,p}$, while for the unweighted case ($m = 1$) we write $M^{p,q}$.

It can be proved that $M^{p,q}_m(\mathbb{R}^n)$ is a Banach space whose definition does not depend on the choice of the window φ (in the sense that different windows yield equivalent norms). The standard weight used in the rest of the paper is $m(z) = v_s(z) = \langle z \rangle^s$ for some $s \in \mathbb{R}$. We mention that many common function spaces are intimately related with modulation spaces: for instance, $M^2(\mathbb{R}^n)$ coincides with the Hilbert space $L^2(\mathbb{R}^n)$, while if $m(x, \xi) = \langle \xi \rangle^s$ for $s \ge 0$ we have that $M^2_m(\mathbb{R}^n)$ coincides with the usual L^2-based Sobolev space $H^s(\mathbb{R}^n)$. Furthermore, the following characterizations hold for any $1 \le p, q \le \infty$:

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{s \ge 0} M^{p,q}_{v_s}(\mathbb{R}^n), \quad \mathcal{S}'(\mathbb{R}^n) = \bigcup_{s \ge 0} M^{p,q}_{v_{-s}}(\mathbb{R}^n). \tag{5}$$

Another perspective on modulation spaces is provided by inspecting the definition of the STFT $V_\varphi u$: it may be thought of as a continuous expansion of the function u with respect to the uncountable system $\{\pi(z)\varphi : z = (x, \xi) \in \mathbb{R}^{2n}\}$. Notice that $\pi(z)\varphi$ is a wave packet highly concentrated near z in phase space. For short, we have $V_\varphi u(x, \xi) = \langle u, \pi(x, \xi)\varphi\rangle$ in the sense of the (extension to the duality $\mathcal{S}' - \mathcal{S}$ of the) inner product on L^2. This perspective is further reinforced by the role of *frame theory* and discrete time-frequency representations. Given a non-zero window function $\varphi \in L^2(\mathbb{R}^n)$ and a subset $\Lambda \subset \mathbb{R}^{2n}$, we say that the collection of the time-frequency shifts of φ along Λ is a *Gabor system*, namely

$$\mathcal{G}(\varphi, \Lambda) = \{\pi(z)\varphi : z \in \Lambda\}.$$

For instance one may consider separable lattices such as

$$\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z} = \{(\alpha k, \beta n) : k, n \in \mathbb{Z}\},$$

for lattice parameters $\alpha, \beta > 0$; we write $\mathcal{G}(g, \alpha, \beta)$ for the corresponding Gabor system. Recall that a *frame* for a Hilbert space $\mathcal{H}$ is a sequence $\{x_j\}_{j\in J} \subset \mathcal{H}$ such that there exist constants $A, B > 0$ (frame bounds) such that

$$A \|x\|^2_{\mathcal{H}} \le \sum_{j\in J} |\langle x, x_j\rangle|^2 \le B \|x\|^2_{\mathcal{H}} ,$$

$$\forall x \in \mathcal{H}.$$

Roughly speaking, the paradigm of frame theory consists in decomposing a vector x along the frame, then studying the action of operators on such elementary pieces and finally reconstructing the image vector. The entire process is encoded by the *frame operator*

$$S : \mathcal{H} \ni x \mapsto \sum_{j\in J} \langle x, x_j\rangle x_j \in \mathcal{H}.$$

If a Gabor system $\mathcal{G}(\varphi, \Lambda)$ is a frame for $L^2(\mathbb{R}^n)$ it is called *Gabor frame*. Notice that the Gabor frame operator reads

$$Sf = \sum_{z\in\Lambda} V_g f(z)\pi(z)g,$$

and is a positive, bounded invertible operator on $L^2(\mathbb{R}^n)$. A remarkable result of frame theory is that a function can be reconstructed from its Gabor coefficients by means of the following discrete analogue of (3):

$$u = \sum_{z\in\Lambda} V_\varphi u(z)\pi(z)\widetilde{\varphi}, \tag{6}$$

where $\widetilde{\varphi} = S^{-1}\varphi$ is the *canonical dual window* and the sum is unconditionally convergent in L^2. Notice that $\varphi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \widetilde{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ if $\mathcal{G}(\varphi, \Lambda)$ is a Gabor frame [32].

Moreover, the reconstruction formula (6) extends to $u \in M_m^{p,q}(\mathbb{R}^n)$ for all $1 \leq p, q \leq \infty$ and weight function m on $\mathbb{R}^{2n}$, with unconditional convergence in the modulation space norm if $1 \leq p, q < \infty$ (weak unconditional otherwise). In addition, an equivalent discrete norm for $M_m^{p,q}(\mathbb{R}^n)$ is given by

$$\|u\|_{M_s^{p,q}} = \left(\sum_{n \in \mathbb{Z}^n} \left(\sum_{k \in \mathbb{Z}^n} |V_\varphi u(\alpha k, \beta n) m(\alpha k, \beta n)|^p \right)^{q/p} \right)^{1/q}.$$

2.3 Pseudodifferential Operators

In the spirit of time-frequency analysis we define Weyl operators starting from the relation

$$\langle \sigma^{\mathrm{w}} f, g \rangle = \langle \sigma, W(g, f) \rangle, \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d), \tag{7}$$

where $W(g, f)$ is the Wigner transform defined in (2) and $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is the *symbol* of the *Weyl operator* $\sigma^{\mathrm{w}} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, which can be formally represented as

$$\sigma^{\mathrm{w}} f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i (x-y)\xi} \sigma\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi.$$

Other quantization rules may be covered in a similar fashion. In particular, for $\tau \in [0, 1]$ we define

$$\langle \mathrm{Op}_\tau(\sigma) f, g \rangle = \langle \sigma, W_\tau(g, f) \rangle, \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d), \tag{8}$$

where the Wigner distribution is generalized as

$$W_\tau(f, g)(x, \xi) = \int_{\mathbb{R}^d} e^{-2\pi i y \xi} f(x + \tau y) \overline{g(x - (1-\tau) y)}\, dy.$$

We refer to the papers [2, 8, 14, 15] for results in this general framework. Notice that we recapture the Weyl quantization for $\tau = 1/2$, while the case $\tau = 0$ corresponds to the Kohn-Nirenberg quantization. In the rest of the paper we will focus on Weyl operators, but most of the stated results can be transferred to other kind of pseudodifferential operators in view of the identity

$$\mathrm{Op}_{\tau_1}(a) = \mathrm{Op}_{\tau_2}(T_{\tau_1,\tau_2}a), \quad T_{\tau_1,\tau_2}a = e^{2\pi i(\tau_1-\tau_2)D_x D_\xi}a, \quad a \in \mathcal{S}'(\mathbb{R}^{2n}). \tag{9}$$

Nevertheless, there is a distinctive property characterizing the Weyl calculus among other quantization rules, which is known as *symplectic covariance*. Recall indeed that $S \in \mathrm{Sp}(n,\mathbb{R})$ can be associated with a unitary operator $\mu(S)$ on $L^2(\mathbb{R}^n)$, called *metaplectic operator*, which satisfies the intertwining property

$$\mu(S)^{-1}\sigma^{\mathrm{w}}\mu(S) = (\sigma \circ S)^{\mathrm{w}}, \quad \sigma \in \mathcal{S}'(\mathbb{R}^{2n}).$$

In fact, the map $\mu : S \mapsto \mu(S)$ defines a metaplectic operator only up to a constant complex factor of modulus one. We will not focus on technical details concerning the metaplectic representation; the reader may consult [20, 25, 54] for a precise account on the issue.

A major advantage of the time-frequency analysis approach to pseudodifferential operators is that general symbol classes may be considered, in particular modulation spaces. Recall the definition of the classical Hördmander classes [30].

Definition 1 Let $m \in \mathbb{R}$. The symbol class $S^m_{0,0}$ is the subspace of smooth functions $a \in C^\infty(\mathbb{R}^{2n})$ such that

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}} \langle\xi\rangle^{-m}|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| < \infty, \quad \forall\alpha,\beta \in \mathbb{N}^n.$$

It is a Fréchet space with the obvious seminorms.

For $a \in S^m_{0,0}$ we have that a^{w} is continuous on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$; moreover the map $T_{0,1/2}$ is an automorphism of $S^m_{0,0}$. Composition of Weyl operators with symbols in $S^m_{0,0}$ classes is well behaved: if $a \in S^m_{0,0}$ and $b \in S^n_{0,0}$, then $a^{\mathrm{w}} \circ b^{\mathrm{w}}$ is again a Weyl operator with symbol $a\#b \in S^{m+n}_{0,0}$ - the latter is known as the *Weyl (or twisted) product* of a and b. While explicit formulas are known for $a\#b$ in general, we stress that the calculus associated with symbols in $S^m_{0,0}$ is highly non-trivial due to the lack of asymptotic expansions for Weyl product of symbols.

A somewhat better behaviour is showed by *Shubin symbol classes* [52], defined as follows.

Definition 2 Let $m \in \mathbb{R}$. The symbol class G^m is the subspace of smooth functions $a \in C^\infty(\mathbb{R}^{2n})$ such that

$$\sup_{z\in\mathbb{R}^{2n}} \langle z\rangle^{-m+|\alpha|}|\partial_z^\alpha a(z)| < \infty, \quad \forall\alpha \in \mathbb{N}^{2n}.$$

It is a Fréchet space with the obvious seminorms.

We confine ourselves to recall that $\bigcap_{m\in\mathbb{R}} G^m = \mathcal{S}(\mathbb{R}^{2n})$ and the Weyl product is a bilinear continuous map $\# : G^m \times G^n \to G^{m+n}$. We also set $G^\infty = \bigcup_{m\in\mathbb{R}} G^m$.

3 A Short History of the Gabor Wave Front Set

By analogy with the classical Huygens' construction of a propagating wave, Hörmander ([27], 1971) called *wave front set* of a distribution u the subset $WF(u)$ of $\mathbb{R}^n_x \times (\mathbb{R}^n_\xi \setminus \{0\})$ defined by examining the behaviour at infinity of the Fourier transform $\hat{u}$. Namely, the point (x_0, ξ_0), $\xi_0 \neq 0$, does *not* belong to $WF(u)$ if there exist a function $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\varphi(x_0) \neq 0$, and a conic neighbourhood $\Gamma_{\xi_0} \subset \mathbb{R}^n$ of ξ_0 such that

$$|\widehat{\varphi u}(\xi)| \leq C_N \langle\xi\rangle^{-N} \quad \forall \xi \in \Gamma_{\xi_0},\ N \in \mathbb{N}, \tag{10}$$

for a suitable constant $C_N > 0$. Here and below we assume $u \in \mathcal{S}'(\mathbb{R}^n)$, though the preceding estimate applies obviously to $u \in \mathcal{D}'(\mathbb{R}^n)$ or $u \in \mathcal{D}'(\Omega)$ with Ω open subset of $\mathbb{R}^n$ such that $x_0 \in \Omega$ and $\mathrm{supp}(\varphi) \subset \Omega$.

An alternative definition can be given in terms of classical pseudodifferential operators with polyhomogeneous symbol with respect to the ξ variables:

$$p(x, \xi) = p_m(x, \xi) + \dots, \tag{11}$$

where p_m satisfies $p_m(x, \lambda\xi) = \lambda^m p(x, \xi)$ for $\lambda > 0$ and $\xi \neq 0$. Precisely, $(x_0, \xi_0) \notin WF(u)$ if and only if there exists $p(x, \xi)$ with $p_m(x_0, \xi_0) \neq 0$ such that $p(x, D)u \in C^\infty(U_{x_0})$ for some neighbourhood U_{x_0} of x_0. The statement does not depend on the quantization rule that we adopt to define $p(x, D)$.

Afterwards, several variables of the definition of $WF(u)$ appeared. Our attention is focused on the *global wave front set* of Hörmander ([28], 1989), which we denote here by $WF_G(u)$. To define $WF_G(u)$ for $u \in \mathcal{S}'(\mathbb{R}^n)$ we may imitate the preceding argument in terms of pseudodifferential operators, by taking now polyhomogeneous symbols in the $z = (x, \xi)$ variable, as in Shubin [52]:

$$p(z) = p_m(z) + \dots, \tag{12}$$

with $p_m(\lambda z) = \lambda^m p_m(z)$ for $\lambda > 0$, and similarly for lower order terms. Then, $z_0 = (x_0, \xi_0) \notin WF_G(u)$, $z_0 \notin 0$, if there exists $p(z)$ with $p_m(z_0) \neq 0$ such that $p(x, D)u \in \mathcal{S}(\mathbb{R}^n)$. Willing to give a direct definition, we may replace the Fourier transform with the integral transformation

$$Tu(x, \xi) = \int_{\mathbb{R}^n} e^{-2\pi i t\xi} e^{-|t-x|^2/2} u(t)dt. \tag{13}$$

We have that $z_0 = (x_0, \xi_0) \notin WF_G(u)$ if and only if there exists a conic neighbourhood Γ_{z_0} of z_0 in $\mathbb{R}^{2n}$ such that

$$|Tu(z)| \leq C_N \langle z \rangle^{-N}, \quad \forall z \in \Gamma_{z_0},\ N \in \mathbb{N}. \tag{14}$$

In the next sections we shall review the main properties of $WF_G(u)$ and present some variants of the definition. We continue here by listing some papers of the last 30 years, where $WF_G(u)$ was reinvented, without reference to the original contribution by Hörmander [28].

Let us first mention Nakamura ([39], 2005), who introduced the so-called *homogeneous wave front set* to study propagation of singularities for the Schrödinger equation through methods typically used in semiclassical analysis. Schulz and Wahlberg ([51], 2017) proved recently that the homogeneous wave front set coincides with $WF_G(u)$. In turn, Ito ([31], 2006) clarified the connection of the homogeneous wave front set with the *quadratic scattering wave front set* of Wunsch ([55], 1999), see also [38].

To complete this survey, we may mention the related definition of the *scattering wave front set* of Melrose [36], Melrose and Zworski [37], coinciding with the *SG wave front set* of Cordes [16] and Coriasco and Maniccia [17].

Roughly speaking, the scattering/SG wave front set consists of three components: $WF(u)$, $WF(\hat{u})$ and a third component similar to $WF_G(u)$ where analysis is limited to rays through $z_0 = (x_0, \xi_0)$, with $x_0 \in \mathbb{S}^{n-1}_x$ and $\xi_0 \in \mathbb{S}^{n-1}_\xi$. The enormous developments of the corresponding SG-microlocal analysis are somewhat outside our present perspective, see for instance [18] for references.

A new approach to $WF_G(u)$ was proposed by Rodino and Wahlberg ([50], 2014) where the original contribution by Hörmander [28] was finally recognized and a further equivalent definition was given in terms of time-frequency analysis. Namely, the integral transform in (13) coincides with the Bargmann-Gabor transform of u, that is a short-time Fourier transform with Gaussian window, see [24] and the textbook [25]. It is then natural to replace Tu with the discrete Gabor frame representation of u, possibly with more general windows, and impose in the cone Γ_{z_0} a rapid decay of the Gabor coefficients, see the next section for the details. In [50] the authors gave the name *Gabor wave front set* to the associated wave front set and introduced the notation $WF_G(u)$, where the subscript G stands both for global and Gabor.

In the last 5 years, this new approach and the new name were adopted by a number of authors working in the area of time-frequency analysis. Let us try to give a short account. As already evident from the original work of Hörmander [28], the main application concerns the propagation of microlocal singularities for the Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = H(x,D)u(t,x) \\ u(0,x) = u_0(x) \end{cases}. \tag{15}$$

A basic example is the quantum harmonic oscillator, corresponding to the Hamiltonian $H(x,\xi) = |x|^2 + |\xi|^2$. In fact, starting from the Gabor-Fourier integral representation of the Schrödinger propagator in [9, 10] one can deduce in a natural way propagation in terms of $WF_G(u)$, see [7, 11–13]. The analysis is extended to the case of non-self-adjoint Hamiltonians in [6, 40, 47, 48, 53] and semilinear equations in [41]. In all these papers the definition of $WF_G(u)$ is modified by replacing the $\mathscr{S}$-decay in (14) with other regularity conditions in order to best fit with the features of the Hamiltonian. In particular, in [7, 11, 13] the authors reconsider $WF_G(u)$ in the framework of weighted modulation spaces M^p introduced by Feichtinger, see [22] and [25]. In this connection we address to the next sections, where we present an alternative definition in terms of Gabor frames.

In [41], to study the non-linear properties of $WF_G(u)$, attention is addressed to $M^2 = L^2$ regularity with weight $\langle z\rangle^s$, $z = (x,\xi) \in \mathbb{R}^{2n}$, corresponding to the spaces Q^s of Shubin [52]. In [51] the authors consider the same variant of $WF_G(u)$, under the action of localization operators. In [53] the *polynomial Gabor wave front set* is defined assuming (14) satisfied for a fixed value of N.

In [4, 6, 12] the $\mathscr{S}$-decay is replaced by *analytic* and *Gelfand-Shilov* decay. To be precise, $z_0 = (x_0, \xi_0)$ does not belong to such wave front sets if there exists a conic neighbourhood Γ_{z_0} of z_0 in $\mathbb{R}^{2n}$ such that

$$|Tu(z)| \leq Ce^{-\epsilon\langle z\rangle^r}, \quad z \in \Gamma_{z_0}, \tag{16}$$

for some fixed $r > 0$ and positive constants C and ϵ. The case $r = 1$ corresponds to the analytic Gabor wave front set. In [3] the definition is generalized to ultradifferentiable classes by assuming

$$|Tu(z)| \leq Ce^{-\omega(z)}, \quad z \in \Gamma_{z_0}, \tag{17}$$

for a given weight function $\omega(z)$.

Observe that in [13] and [12] the notion of $WF_G(u)$ is generalized to that of Gabor Ψ-filter, respectively in the analytic and modulation space setting. This allows one to get rid of the homogeneity assumption on the Hamiltonian.

The research related to the Gabor wave front set, or other wave front sets from the point of view of time-frequency analysis, is very intensive at present and it is impossible to give complete references. Let us limit ourselves to further mention [5, 19, 35, 42, 43, 46].

4 Gabor Wave Front Set: Theory and Practice

In this section we focus on the Gabor wave front set WF_G introduced in the preceding historical account.

4.1 The Global Wave Front Set of Hörmander

We briefly review the main properties of the global wave front set $WF(u)$ introduced by Hörmander in [28]. We need to introduce some preparatory notions.

Definition 3 The conic support of $a \in \mathscr{S}'(\mathbb{R}^{2n})$ is the set conesupp(a) of all $z \in \mathbb{R}^{2n} \setminus \{0\}$ such that any open conic neighbourhood Γ_z of z in $\mathbb{R}^{2n} \setminus \{0\}$ satisfies

$$\overline{\mathrm{supp}(a) \cap \Gamma_z} \quad \text{is not compact in } \mathbb{R}^{2n}.$$

Definition 4 Let $a \in G^m$ for some $m \in \mathbb{R}$. We say that a point $z_0 \in \mathbb{R}^{2n} \setminus \{0\}$ is *non-characteristic* for a if there exist positive constants $A, \epsilon > 0$ and an open conic set $\Gamma \subset \mathbb{R}^{2n} \setminus \{0\}$ such that

$$|a(z)| \geq \epsilon \langle z \rangle^m, \quad z \in \Gamma, \ |z| \geq A.$$

We define char(a) as the subset of $\mathbb{R}^{2n} \setminus \{0\}$ containing all the non-characteristic points for a.

Notice that

$$\mathrm{conesupp}(a) \cup \mathrm{char}(a) = \mathbb{R}^{2n} \setminus \{0\}, \quad a \in G^m.$$

We are now ready to define the global wave front set.

Definition 5 Let $u \in \mathscr{S}'(\mathbb{R}^n)$. We say that a point $z_0 \in \mathbb{R}^{2n} \setminus \{0\}$ does not belong to $WF(u)$ if there exist $m \in \mathbb{R}$ and $a \in G^m$ such that $a^{\mathrm{w}} u \in \mathscr{S}(\mathbb{R}^n)$ and $z_0 \notin$ char(a).

We collect below some properties satisfied by $WF(u)$, following [50].

Proposition 1 *Let* $u \in \mathscr{S}'(\mathbb{R}^n)$.

(i) $WF(u)$ *is a closed conic subset of* $\mathbb{R}^{2n} \setminus \{0\}$.
(ii) $WF(u)$ *is symplectically invariant:*

$$z_0 \in WF(u) \Rightarrow Sz_0 \in WF(\mu(S)u), \quad S \in \mathrm{Sp}(n, \mathbb{R}).$$

(iii) For $a \in G^m$ *the following inclusions hold:*

$$WF(a^{\mathrm{w}} u) \subseteq WF(u) \cap \mathrm{conesupp}(a) \subseteq WF(u) \subseteq WF(a^{\mathrm{w}} u) \cup \mathrm{char}(a).$$

In particular, if char$(a) = \emptyset$ *then* $WF(a^{\mathrm{w}} u) = WF(u)$.
(iv) If $a \in G^m$ *and* conesupp$(a) \cap WF(u) = \emptyset$ *then* $a^{\mathrm{w}} u \in \mathscr{S}(\mathbb{R}^n)$.
(v) $WF(u) = \emptyset$ *if and only if* $u \in \mathscr{S}(\mathbb{R}^n)$.

4.2 The Gabor Wave Front Set at Schwartz Regularity

Let us give a concise review of the Gabor wave front set in the context of Schwartz regularity, following [50]. First we introduce a continuous version of the Gabor wave front set characterized by rapid decay of the phase space representation of a distribution.

Definition 6 Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. We say that $z_0 \in \mathbb{R}^{2n} \setminus \{0\}$ does not belong to the set $WF'(u)$ if there exists an open conic neighbourhood Γ_{z_0} of z_0 in $\mathbb{R}^{2n} \setminus \{0\}$ such that

$$\sup_{z\in\Gamma_{z_0}} \langle z\rangle^N |V_\varphi u(z)| < \infty \quad \forall N \in \mathbb{N}. \tag{18}$$

It is a direct consequence of the definition that $WF'(u)$ is a closed conic subset of $\mathbb{R}^{2n} \setminus \{0\}$. The definition of $WF'(u)$ is well-posed in the sense that the Schwartz decay of $V_\varphi u$ in a conic neighbourhood does not depend on the window function φ, as detailed below.

Proposition 2 ([50, Cor. 3.3]) *Let $u \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$ and $z_0 \in \mathbb{R}^{2n}\setminus\{0\}$. Assume that there exists an open conic neighbourhood Γ_{z_0} of z_0 in $\mathbb{R}^{2n} \setminus \{0\}$ such that condition (18) holds. For any open conic neighbourhood Γ'_{z_0} of z_0 in $\mathbb{R}^{2n} \setminus \{0\}$ such that $\overline{\Gamma'_{z_0} \cap \mathbb{S}^{2n-1}} \subseteq \Gamma_{z_0}$ and any $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ we have*

$$\sup_{z\in\Gamma'_{z_0}} \langle z\rangle^N |V_\psi u(z)| < \infty \quad \forall N \in \mathbb{N}.$$

In the spirit of time-frequency analysis it is interesting to study the discrete variant of $WF'(u)$ obtained by replacing the full phase-space cone Γ_{z_0} in (18) with its restriction to suitable lattice points. This remark leads to the definition of the Gabor wave front set $WF_G(u)$.

Definition 7 Let $u \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and a separable lattice $\Lambda = \alpha\mathbb{Z}^n \times \beta\mathbb{Z}^n$ where $\alpha, \beta > 0$ are such that $\mathcal{G}(\varphi, \Lambda)$ is a Gabor frame. We say that $z_0 \in \mathbb{R}^{2n} \setminus \{0\}$ does not belong to the Gabor wave front set $WF_G(u)$ if there exists an open conic neighbourhood Γ_{z_0} of z_0 in $\mathbb{R}^{2n} \setminus \{0\}$ such that

$$\sup_{\lambda\in\Lambda\cap\Gamma_{z_0}} \langle \lambda\rangle^N |V_\varphi u(z)| < \infty \quad \forall N \in \mathbb{N}. \tag{19}$$

While it is clear that $WF_G(u) \subseteq WF'(u)$, it is a remarkable result that the other inclusion holds too, cf. [50, Thm. 3.5], that is

$$\boxed{WF_G(u) = WF'(u), \quad u \in \mathcal{S}'(\mathbb{R}^n).} \tag{20}$$

This characterization also shows that the definition of $WF_G(u)$ is independent of the choice of the Gabor frame $\mathscr{G}(\varphi, \Lambda)$ used in (19). Moreover, it can be proved that all these results still hold for more general lattices $\Lambda = \mathscr{A}\mathbb{Z}^{2n}$, where $\mathscr{A} \in \mathrm{GL}\,(2n, \mathbb{R})$. In the rest of the paper we will discard the notation $WF'(u)$ and we compute $WF_G(u)$ according to (18) whenever convenient.

Another important achievement in [50] is the proof of the fact that the Gabor wave front set coincides with Hörmander's global wave front set. We prefer not to include a discussion of this issue in order to keep the presentation at an introductory level. We just mention that a key ingredient in the proof is a precise characterization of the Gabor wave front set of Weyl operators with symbols in $S^m_{0,0}$ classes.

Proposition 3 *Let $m \in \mathbb{R}$. For $a \in S^m_{0,0}$ we have*

$$WF_G(a^{\mathrm{w}}u) \subseteq \mathrm{conesupp}(a), \quad u \in \mathscr{S}'(\mathbb{R}^n).$$

In particular, for $m = 0$ we have

$$WF_G(a^{\mathrm{w}}u) \subseteq WF_G(u) \cap \mathrm{conesupp}(a), \quad u \in \mathscr{S}'(\mathbb{R}^n).$$

We determine below the Gabor wave front set of some special distributions in order to get a taste of this notion and also to prepare material for applications to Schrödinger equations.

Example 1 Fix $z_0 = (x_0, \xi_0) \in \mathbb{R}^{2n}$. The Gabor wave front set is invariant under time-frequency shifts, namely

$$WF_G(\pi(z_0)u) = WF_G(u), \quad u \in \mathscr{S}'(\mathbb{R}^n).$$

This is indeed a consequence of the invertibility of time-frequency shifts and Proposition 3, since

$$\pi(z_0) = \sigma^{\mathrm{w}}, \quad \sigma(x, \xi) = e^{\pi i x_0 \xi_0} e^{2\pi i (x\xi_0 - \xi x_0)} \in S^0_{0,0}.$$

Example 2 (Dirac Delta) Consider the Dirac distribution centered at $x_0 \in \mathbb{R}^n$, namely $\delta_{x_0} \in \mathscr{S}'(\mathbb{R}^n)$. In view of the previous example we can assume $x_0 = 0$ without loss of generality, namely $WF_G(\delta_{x_0}) = WF_G(\delta_0)$ for all $x_0 \in \mathbb{R}^n$. Let us compute the STFT of δ_0: for a fixed window $\varphi \in \mathscr{S}(\mathbb{R}^n) \setminus \{0\}$,

$$V_\varphi \delta_0(x, \xi) = \langle \delta_0, M_\xi T_x \varphi \rangle = \overline{\varphi(-x)}.$$

This implies that $|V_\varphi \delta_0(0, \lambda\xi)| = |\varphi(0)|$ for all $\lambda > 0$ and $\xi \in \mathbb{R}^n$. If we further assume $\varphi(0) \neq 0$ we see that $\{0\} \times (\mathbb{R}^n \setminus \{0\}) \subseteq WF_G(\delta_0)$. To conclude, let $C > 0$

and consider the conic subset $\Gamma = \{(x,\xi) \in \mathbb{R}^{2n} \setminus \{0\} : |\xi| < C|x|\}$. Let $z_0 = (x_0,\xi_0) \in \Gamma$; then

$$\sup_{z\in\Gamma} \langle z\rangle^N |V_\varphi \delta_0(z)| \lesssim \sup_{x\in\mathbb{R}^n} \langle x\rangle^N |\varphi(-x)| < \infty,$$

hence $z_0 \notin WF_G(\delta_0)$. This argument allows us to conclude that

$$\boxed{WF_G(\delta_{x_0}) = WF_G(\delta_0) = \{0\} \times (\mathbb{R}^n \setminus \{0\}).}$$

We remark that in the case of δ_{x_0} the Gabor wave front set is less informative than the classical Hörmander wave front set [29], which reads $WF_H(\delta_{x_0}) = \{x_0\} \times (\mathbb{R}^n \setminus \{0\})$ and coincides with the SG wave front set $WF_{\mathcal{S}}$ by Coriasco and Maniccia [17].

Example 3 (Pure Frequency) Fix $\xi_0 \in \mathbb{R}^n$ and consider the distribution $u(t) = e^{2\pi i t\xi_0}$. In order to determine its Gabor wave front set we apply again the invariance property under phase-space shifts, namely

$$WF_G(u) = WF_G(M_{\xi_0} 1) = WF_G(1).$$

For a fixed window $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ we have

$$V_\varphi 1(x,\xi) = \langle 1, M_\xi T_x \varphi\rangle = \langle \delta_0, T_\xi M_{-x} \hat{\varphi}\rangle = e^{-2\pi i x\xi} \hat{\varphi}(-\xi),$$

hence $|V_\varphi 1(\lambda x, 0)| = |\hat{\varphi}(0)|$ for any $\lambda > 0$ and $x \in \mathbb{R}^n$. It is not restrictive to assume $\hat{vp}(0) \neq 0$, thus we conclude $(\mathbb{R}^n \setminus \{0\}) \times \{0\} \subseteq WF_G(1)$. The same arguments used in Example 2 yield

$$\boxed{WF_G\left(e^{2\pi i \xi_0 \cdot}\right) = WF_G(1) = (\mathbb{R}^n \setminus \{0\}) \times \{0\}.}$$

To compare with other wave front sets, notice that the classical wave front set is not able to detect any singularity since $u \in C^\infty(\mathbb{R}^n)$, hence $WF_H(u) = \emptyset$. However, the SG wave front set is again more precise, yielding $WF_{\mathcal{S}}(u) = (\mathbb{R}^n \setminus \{0\}) \times \{\xi_0\}$.

Example 4 (Fresnel Chirp) Fix $c \in \mathbb{R}\setminus\{0\}$ and consider the linear chirp (also known as Fresnel function) $u(t) = e^{\pi i c t^2}$. Straightforward computations for the STFT of u with Gaussian window $\varphi(t) = e^{-\pi t^2}$ (cf. for instance [1]) provide

$$|V_\varphi u(x,\xi)| = (1+c^2)^{-n/4} e^{-\pi|\xi - cx|^2/(1+c^2)}. \tag{21}$$

We deduce that the STFT rapidly decays in any open cone in $\mathbb{R}^{2n} \setminus \{0\}$ which does not include the hyperplane $\xi = cx$. Arguing as in the previous example we conclude that

$$\boxed{WF_G\left(e^{\pi i c|\cdot|^2}\right) = \{(x, cx) \,:\, x \in \mathbb{R}^n \setminus \{0\}\}.}$$

We stress that the Gabor wave front set is superior in detecting singularities than other notions in this case, which is characterized by varying frequency. Notice indeed that $WF_H(u) = \emptyset$, while $WF_{\mathcal{S}}(u) = (\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})$.

4.3 Modulation Space Setting

In Sect. 2.2 we introduced modulation spaces by conditioning the (weighted and mixed) Lebesgue regularity of the phase-space representation (STFT) of their members. This notion suggests a natural generalization of the Gabor wave front set WF_G by relaxing the Schwartz decay in (18) as follows, cf. [11].

Definition 8 Let $1 \le p \le \infty$, $s \ge 0$, $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. We say that $z_0 \in \mathbb{R}^{2n} \setminus \{0\}$ does not belong to $WF_G^{p,s}(u)$ if there exists an open conic neighbourhood Γ_{z_0} of z_0 in $\mathbb{R}^{2n} \setminus \{0\}$ such that $V_\varphi u \in L^p_{v_s}(\Gamma_{z_0})$, that is

$$\int_{\Gamma_{z_0}} |V_\varphi u(z)|^p \langle z\rangle^{sp} dz < \infty, \tag{22}$$

with obvious modification in the case where $p = \infty$.

It is clear from the definition that $WF_G^{p,s}(u)$ is a closed conic subset of $\mathbb{R}^{2n} \setminus \{0\}$.

We remark that other kinds of microlocal analysis at modulation space regularity may be taken into account. In this respect we mention the wave front set $WF_{M^{p,q}_m}(u)$ introduced in [44, 45] and defined as follows. First define for $f \in \mathcal{S}'(\mathbb{R}^n)$ the set $\Sigma(f)$ as the complement in $\mathbb{R}^n \setminus \{0\}$ of the subset which contains all $\bar{\xi} \in \mathbb{R}^n \setminus \{0\}$ such that

$$\left(\int_{\Gamma_{\bar{\xi}}} \left(\int_{\mathbb{R}^n} \left|V_\varphi f(x,\xi)\right|^p m(x,\xi)^p dx\right)^{q/p} d\xi\right)^{1/q} < \infty,$$

for some conic neighbourhood $\Gamma_{\bar{\xi}}$ of $\bar{\xi}$ in $\mathbb{R}^n \setminus \{0\}$. Hence, for $1 \le p, q \le \infty$, a weight function m on $\mathbb{R}^{2n}$ and $u \in \mathcal{D}'(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ open, $WF_{M^{p,q}_m}(u)$ consists of elements $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \setminus \{0\}$ such that $\xi_0 \in \Sigma(\phi u)$ for any $\phi \in C_c^\infty(\Omega)$ with $\phi(x_0) \neq 0$. It is a remarkable result that modulation spaces are microlocally equivalent to Fourier-Lebesgue spaces, in the sense of [45, Thm. 6.1]. We also refer to [33] for a discrete version of this analysis.

We prove below the independence of the window φ in the definition of $WF_G^{p,s}$, cf. [11] for more general results.

Proposition 4 *Let* $1 \le p \le \infty$, $s \ge 0$, $u \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ *and* $z_0 \in \mathbb{R}^{2n} \setminus \{0\}$. *Assume that there exists an open conic neighbourhood* Γ_{z_0} *of* z_0 *in* $\mathbb{R}^{2n} \setminus \{0\}$ *such that condition* (22) *holds. For any open conic neighbourhood* Γ'_{z_0} *of* z_0 *in* $\mathbb{R}^{2n} \setminus \{0\}$ *such that* $\overline{\Gamma'_{z_0} \cap \mathbb{S}^{2n-1}} \subseteq \Gamma_{z_0}$ *and any* $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ *we have*

$$\int_{\Gamma'_{z_0}} |V_\psi u(z)|^p \langle z\rangle^{sp} dz < \infty. \tag{23}$$

Proof Let us first recall the change-of-window estimate in Lemma 1, namely

$$|V_\psi u(z)| \lesssim (|V_\varphi u| * |V_\psi \varphi|)(z), \quad z \in \mathbb{R}^{2n}.$$

Since $V_\psi \varphi \in \mathcal{S}(\mathbb{R}^{2n})$ for $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$, for any $N \ge 0$ we have

$$|V_\psi u(z)| \lesssim \int_{\mathbb{R}^{2n}} \langle z-w\rangle^{-N} |V_\varphi u(w)| dw.$$

Therefore, to prove the desired estimate (23) it is enough to show that, for a suitable choice of $n \ge 0$ we have

$$\left\| \int_{\mathbb{R}^{2n}} F(\cdot, w) dw \right\|_{L^p(\Gamma'_{z_0})} < \infty,$$

where we set $F(z,w) = F_n(z,w) = \langle z\rangle^s \langle z-w\rangle^{-N} |V_\varphi u(w)|$.

We conveniently split the domain of integration in $\int_{\mathbb{R}^{2n}} F(\cdot, w)dw$ in two parts, namely Γ_{z_0} and $\mathbb{R}^{2n} \setminus \Gamma_{z_0}$. Let us first consider $\mathbb{R}^{2n} \setminus \Gamma_{z_0}$ and notice that

$$\langle z-w\rangle \gtrsim \max\{\langle z\rangle, \langle w\rangle\}, \quad z \in \Gamma'_{z_0},\ w \in \mathbb{R}^{2n} \setminus \Gamma_{z_0}. \tag{24}$$

Furthermore, in view of the characterization of $\mathcal{S}'(\mathbb{R}^n)$ in (5) we deduce that $u \in M^p_{v_{-r}}(\mathbb{R}^n)$ for some $r \ge 0$. Therefore, for $z \in \Gamma'_{z_0}$ we may write

$$\begin{aligned}\int_{\mathbb{R}^{2n}\setminus\Gamma_{z_0}} F(z,w)dw &\le \int_{\mathbb{R}^{2n}\setminus\Gamma_{z_0}} \langle z\rangle^s \langle w\rangle^r \langle z-w\rangle^{-N} \frac{|V_\varphi u(w)|}{\langle w\rangle^r} dw \\ &\lesssim \left(\langle\cdot\rangle^{r+s-N} * \frac{|V_\varphi u(\cdot)|}{\langle\cdot\rangle^r} \right)(z).\end{aligned}$$

It is then enough to assume $N > r + s + 2n$ to conclude

$$\left\| \int_{\mathbb{R}^{2n} \setminus \Gamma_{z_0}} F(\cdot, w) dw \right\|_{L^p(\Gamma'_{z_0})} \lesssim \left\| \langle \cdot \rangle^{r+s-N} \right\|_{L^1(\mathbb{R}^{2n})} \|u\|_{M^p_{v_{-r}}(\mathbb{R}^n)} < \infty.$$

For the remaining part we have

$$\begin{aligned} \int_{\Gamma_{z_0}} F(z, w) dw &\leq \int_{\Gamma_{z_0}} \langle z \rangle^s \langle w \rangle^{-s} \langle z - w \rangle^{-s} \langle z - w \rangle^{s-N} |V_\varphi u(w)| \langle w \rangle^s dw \\ &\lesssim \int_{\Gamma_{z_0}} \langle z - w \rangle^{s-N} |V_\varphi u(w)| \langle w \rangle^s dw \\ &\lesssim \left(\langle \cdot \rangle^{s-N} * \left(1_{\Gamma_{z_0}}(\cdot) |V_\varphi u(\cdot)| \langle \cdot \rangle^s \right) \right)(z), \end{aligned}$$

where $1_{\Gamma_{z_0}}$ is the characteristic function of the set Γ_{z_0}. Assumption (22) finally yields

$$\left\| \int_{\Gamma_{z_0}} F(\cdot, w) dw \right\|_{L^p(\Gamma'_{z_0})} \lesssim \left\| \langle \cdot \rangle^{s-N} \right\|_{L^1(\mathbb{R}^{2n})} \left\| V_\varphi u \right\|_{L^p_{v_s}(\Gamma_{z_0})} < \infty.$$

□

In complete analogy with the Gabor wave front set WF_G introduced in Definition 19 we consider a discrete version of $WF_G^{p,s}$.

Definition 9 Let $1 \leq p \leq \infty$, $s \geq 0$, $\varphi \in \mathscr{S}(\mathbb{R}^n) \setminus \{0\}$ and $u \in \mathscr{S}'(\mathbb{R}^n)$. Consider a separable lattice $\Lambda = \alpha\mathbb{Z}^n \times \beta\mathbb{Z}^n$ where $\alpha, \beta > 0$ are such that $\mathscr{G}(\varphi, \Lambda)$ is a Gabor frame. We say that $z_0 \in \mathbb{R}^{2n} \setminus \{0\}$ does not belong to $\widetilde{WF_G^{p,s}}(u)$ if there exists an open conic neighbourhood Γ_{z_0} of z_0 in $\mathbb{R}^{2n} \setminus \{0\}$ such that $V_\varphi u \in L^p_{v_s}(\Gamma_{z_0})$, that is

$$\sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} |V_\varphi u(\lambda)|^p \langle \lambda \rangle^{sp} < \infty, \tag{25}$$

with obvious modification in the case where $p = \infty$.

We show that the discrete and continuous modulation Gabor wave front set coincide. Therefore, modulation space regularity in a conic neighbourhood of a phase space direction is a condition as strong as modulation space regularity restricted to the points of the same cone which belong to a suitable lattice.

Theorem 1 *Let* $1 \leq p \leq \infty$, $s \geq 0$ *and* $u \in \mathscr{S}'(\mathbb{R}^n)$. *Then* $WF_G^{p,s}(u) = \widetilde{WF_G^{p,s}}(u)$.

Proof We give the proof only in the case where $p < \infty$, since the case $p = \infty$ requires trivial modification. We first prove that $z_0 \notin \widetilde{WF_G^{p,s}}(u) WF_G^{p,s}(u)$, namely that (9) implies (8). In view of the reconstruction formula (6) we write $u = u_1 + u_2$, where

$$u_1 = \sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} V_\varphi u(\lambda) \pi(\lambda) \widetilde{\varphi}, \qquad u_2 = \sum_{\lambda \in \Lambda \setminus \Gamma_{z_0}} V_\varphi u(\lambda) \pi(\lambda) \widetilde{\varphi},$$

where $\widetilde{\varphi} = S^{-1}\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ is the canonical dual window. It is therefore enough to show that $V_\varphi u_1, V_\varphi u_2 \in L^p_{v_s}(\Gamma_{z_0})$. Let us start with $V_\varphi u_1$.

$$\begin{aligned}
\|V_\varphi u_1\|^p_{L^p_{v_s}(\Gamma_{z_0})} &= \int_{\Gamma_{z_0}} \left|V_\varphi u_1\right|^p (z) \langle z \rangle^{ps} dz \\
&\leq \int_{\Gamma_{z_0}} \sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} \left(\left|V_\varphi u(\lambda)\right| \left|V_{\widetilde{\varphi}} \varphi(z - \lambda)\right| \langle z \rangle^s \right)^p dz.
\end{aligned}$$

We use the subadditivity of the weight, namely the identity $\langle z \rangle^s \leq \langle z - \lambda \rangle^{-s} \langle \lambda \rangle^s$ to get

$$\|V_\varphi u_1\|^p_{L^p_{v_s}(\Gamma_{z_0})} \leq \int_{\Gamma_{z_0}} \sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} \left(\left|V_\varphi u(\lambda)\right| \langle \lambda \rangle^s \left|V_{\widetilde{\varphi}} \varphi(z - \lambda)\right| \langle z - \lambda \rangle^{-s} \right)^p dz.$$

Let us set $f(\lambda) = \left|V_\varphi u(\lambda)\right| \langle \lambda \rangle^s$ and $g(z - \lambda) = \left|V_{\widetilde{\varphi}} \varphi(z - \lambda)\right| \langle z - \lambda \rangle^{-s}$ for the sake of clarity. Notice that $g(z - \lambda) \lesssim \langle z - \lambda \rangle^{-N-s}$ for arbitrary $N \geq 0$. Hence, by Hölder's inequality we have

$$\begin{aligned}
\|V_\varphi u_1\|^p_{L^p_{v_s}(\Gamma_{z_0})} &\leq \int_{\Gamma_{z_0}} \sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} \left(f(\lambda) g(z - \lambda) \right)^p dz \\
&= \int_{\Gamma_{z_0}} \sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} \left(f(\lambda) g(z - \lambda)^{1/p} g(z - \lambda)^{1-1/p} \right)^p dz \\
&\leq \int_{\Gamma_{z_0}} \left(\sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} f(\lambda)^p g(z - \lambda) \right) \left(\sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} g(z - \lambda) \right)^{p/p'} dz \\
&\leq C \int_{\Gamma_{z_0}} \sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} f(\lambda)^p g(z - \lambda) dz,
\end{aligned}$$

where

$$C = \sup_{z \in \mathbb{R}^{2n}} \|g(z - \cdot)\|_{\ell^1}^{p/p'} < \infty.$$

We conclude by Minkowski inequality:

$$\begin{aligned}\left\| V_\varphi u_1 \right\|_{L^p_{v_s}(\Gamma_{z_0})}^p &\leq C \int_{\Gamma_{z_0}} \sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} f(\lambda)^p g(z - \lambda) dz \\ &\leq C \sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} f(\lambda)^p \int_{\Gamma_{z_0}} g(z - \lambda) dz \\ &\leq C' \sum_{\lambda \in \Lambda \cap \Gamma_{z_0}} f(\lambda)^p < \infty,\end{aligned}$$

where we set

$$C' = C \int_{\Gamma_{z_0}} g(z - \lambda) dz < \infty,$$

and used the assumption (9) in the last step.

It remains to prove that $V_\varphi u_2 \in L^p_{v_s}(\Gamma_{z_0})$, namely

$$\begin{aligned}\left\| V_\varphi u_2 \right\|_{L^p_{v_s}(\Gamma_{z_0})}^p &= \int_{\Gamma_{z_0}} \left| V_\varphi u_2 \right|^p (z) \langle z \rangle^{ps} dz \\ &\leq \int_{\Gamma_{z_0}} \sum_{\lambda \in \Lambda \setminus \Gamma_{z_0}} \left(\left| V_\varphi u(\lambda) \right| \left| V_{\widetilde{\varphi}} \varphi(z - \lambda) \right| \langle z \rangle^s \right)^p dz.\end{aligned}$$

Recall from Sect. 2.2 that the STFT has at most polynomial growth, that is $|V_\varphi u(\lambda)| \lesssim \langle \lambda \rangle^r$ for some $r \geq 0$. Moreover, since $V_{\widetilde{\varphi}} \varphi \in \mathcal{S}(\mathbb{R}^{2n})$ we have $|V_{\widetilde{\varphi}} \varphi(z - \lambda)| \lesssim \langle z - \lambda \rangle^{-N}$ for any $N \geq 0$. As a consequence of (24) we have

$$\begin{aligned}\left\| V_\varphi u_2 \right\|_{L^p_{v_s}(\Gamma_{z_0})}^p &\leq \int_{\Gamma_{z_0}} \sum_{\lambda \in \Lambda \setminus \Gamma_{z_0}} \left(\left| V_\varphi u(\lambda) \right| \left| V_{\widetilde{\varphi}} \varphi(z - \lambda) \right| \langle z \rangle^s \right)^p dz \\ &\lesssim \int_{\Gamma_{z_0}} \sum_{\lambda \in \Lambda \setminus \Gamma_{z_0}} \left(\langle \lambda \rangle^r \langle z - \lambda \rangle^{-N} \langle z \rangle^s \right)^p dz\end{aligned}$$

$$\lesssim \int_{\Gamma_{z_0}} \sum_{\lambda\in\Lambda\setminus\Gamma_{z_0}} \left(\langle\lambda\rangle^{r-N/2}\langle z\rangle^{s-N/2}\right)^p dz$$

$$\left(\int_{\Gamma_{z_0}} \langle z\rangle^{p(s-N/2)}\right)\left(\sum_{\lambda\in\Lambda\setminus\Gamma_{z_0}} \langle\lambda\rangle^{p(r-N/2)}\right) < \infty,$$

where the conclusion follows after choosing N large enough.

We need to prove now that $z_0 \notin WF_G^{p,s}(u) \Rightarrow z_0 \notin \widetilde{WF_G^{p,s}}(u)$, that is (8) implies (9). We essentially argue as before after inverting the role of discrete and continuous norms and reconstruction formulae. To be concrete we prove that $V_\varphi u \in \ell^p_{v_s}(\Lambda\cap\Gamma_{z_0})$. In view of the inversion formula for the STFT in (3) we set $u = u_1'+u_2'$, where

$$u_1' = \int_{\Gamma_{z_0}} V_\varphi u(z)\pi(z)\varphi dz, \quad u_2' = \int_{\mathbb{R}^{2n}\setminus\Gamma_{z_0}} V_\varphi u(z)\pi(z)\varphi dz.$$

It is enough to prove that $V_\varphi u_1', V_\varphi u_2' \in \ell^p_{v_s}(\Lambda\cap\Gamma_{z_0})$. Let us first prove the claim for $V_\varphi u_1'$, having in mind (4). We have

$$\begin{aligned}\left\|V_\varphi u_1'\right\|_{\ell^p_{v_s}(\Lambda\cap\Gamma_{z_0})} &= \sum_{\lambda\in\Lambda\cap\Gamma_{z_0}} \left|V_\varphi u_1(\lambda)\right|^p \langle\lambda\rangle^{sp} \\ &\lesssim \sum_{\lambda\in\Lambda\cap\Gamma_{z_0}} \left(\int_{\Gamma_{z_0}} \left|V_\varphi u(z)\right|\left|V_\varphi\varphi(\lambda-z)\right| dz\right)^p \\ &\lesssim \sum_{\lambda\in\Lambda\cap\Gamma_{z_0}} \left(\int_{\Gamma_{z_0}} \left|V_\varphi u(z)\right|\langle z\rangle^s \left|V_\varphi\varphi(\lambda-z)\right|\langle\lambda-z\rangle^{-s} dz\right)^p.\end{aligned}$$

We set $f(z) = \left|V_\varphi u(z)\right|\langle z\rangle^s$ and $h(\lambda-z) = \left|V_\varphi\varphi(\lambda-z)\right|\langle\lambda-z\rangle^{-s}$ in order to lighten the notation. Therefore, by applying again Hölder's inequality we get

$$\begin{aligned}\left\|V_\varphi u_1'\right\|_{\ell^p_{v_s}(\Lambda\cap\Gamma_{z_0})} &\lesssim \sum_{\lambda\in\Lambda\cap\Gamma_{z_0}} \left(\int_{\Gamma_{z_0}} f(z)h(\lambda-z)dz\right)^p \\ &\le \sum_{\lambda\in\Lambda\cap\Gamma_{z_0}} \left(\int_{\Gamma_{z_0}} f(z)^p h(\lambda-z)dz\right)\left(\int_{\Gamma_{z_0}} h(z-\lambda)dz\right)^{p/p'} \\ &\le \|h\|_{L^1}^{p/p'} \sum_{\lambda\in\Lambda\cap\Gamma_{z_0}} \int_{\Gamma_{z_0}} f(z)^p h(\lambda-z)dz \\ &\le C\int_{\Gamma_{z_0}} f(z)^p dz < \infty,\end{aligned}$$

where we used the assumption (8) in the last step and we set

$$C = \|h\|_{L^1}^{p/p'} \sup_{z\in\mathbb{R}^n} \sum_{\lambda\in\Lambda\cap\Gamma_{z_0}} h(z-\lambda) < \infty.$$

The proof of $V_\varphi u_2' \in \ell^p_{v_s}(\Lambda \cap \Gamma_{z_0})$ follows the same pattern of the proof of $V_\varphi u_2 \in L^p_{v_s}(\Gamma_{z_0})$ above, hence is left to the interested reader. □

Remark 1 As a consequence of the previous identification and Proposition 4 we have that $\widetilde{WF_G^{p,s}}(u)$ does not depend on the Gabor frame $\mathscr{G}(\varphi, \Lambda)$ used in (9). Moreover, it is clear from the definition that $u \in M^p_{v_s}(\mathbb{R}^n)$ if and only if $WF_G^{p,s}(u) = \emptyset$, in view of the compactness of the sphere $\mathbb{S}^{2n-1}$.

The modulation space Gabor wave front set is very well suited to the study of Weyl operators with low regular symbols, as detailed in the following result.

Proposition 5 ([11, Prop. 5.3]) *Let* $1 \le p \le \infty$, $a \in M^\infty_{1\otimes v_\gamma}(\mathbb{R}^{2n})$ *with* $\gamma > 2n$ *and* $0 < 2s < \gamma - 2n$. *For any* $u \in M^p_{-s}(\mathbb{R}^n)$ *we have*

$$WF_G^{p,s}(a^{\mathrm{w}}u) \subset WF_G^{p,s}(u).$$

This should be compared with Proposition 2, having in mind that $\bigcap_{\gamma\ge 0} M^\infty_{1\otimes v_\gamma}(\mathbb{R}^{2n}) = S^0_{0,0}$.

4.4 Propagation of Singularities

We conclude this survey with some easy examples of application of the Gabor wave front set to propagation of microlocal singularities for Schrödinger equations. We refer to [11, 41] for a broader treatment of the topic, see also the other references cited in the historical account above.

Let us fix the setting of our investigation. We consider the Cauchy problem for the Schrödinger equation, namely

$$\begin{cases} i\partial_t u(t,x) = Hu(t,x) \\ u(0,x) = u_0(x) \end{cases}, \tag{26}$$

where $H = a^{\mathrm{w}}$ is the Weyl quantization of a real-valued quadratic polynomial in $\mathbb{R}^{2n}$, namely

$$a(x,\xi) = \frac{1}{2}xAx + \xi Bx + \frac{1}{2}\xi C\xi, \tag{27}$$

for some symmetric matrices $A, C \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$. The phase-space analysis of the Schrödinger propagator $U(t) : u_0(x) \mapsto u(t,x)$ is intimately related to the corresponding Hamiltonian system, that is

$$\dot{z} = J\nabla_z a(z) = \mathbb{A}z, \quad \mathbb{A} = \begin{pmatrix} B & C \\ -A & -B^\top \end{pmatrix}.$$

The classical phase-space flow $\mathscr{A}_t = e^{t\mathbb{A}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectic diffeomorphism and the following result on the propagation of singularities holds in our setting.

Theorem 2 *Consider the Cauchy problem* (26) *with the assumption specified above. We have that* $U(t) \in \mathscr{B}(M^p_{v_r}(\mathbb{R}^n))$ *for all* $t \in \mathbb{R}$, $1 \le p \le \infty$ *and* $r \in \mathbb{R}$. *If* $u_0 \in \mathscr{S}'(\mathbb{R}^n)$ *then*

$$WF_G(U(t)u_0) = \mathscr{A}_t(WF_G(u_0)), \quad t \in \mathbb{R}.$$

If in particular $u_0 \in M^p_{v_{-s}}(\mathbb{R}^n)$ *for some* $1 \le p \le \infty$ *and* $s \ge 0$ *then*

$$WF_G^{p,s}(U(t)u_0) = \mathscr{A}_t(WF_G^{p,s}(u_0)), \quad t \in \mathbb{R}.$$

More refined results for general Hamiltonians and potential perturbations can be found in [11]. We stress that this is one of the rare case where propagation of singularities for Schrödinger operators with non-smooth potentials is taken into account.

Example 5 (The Free Particle) Let us first consider the free case, namely $H = -\triangle/2$ - which corresponds to $a(x,\xi) = \pi\xi^2$. It is well known that the solution of (26) can then be expressed as

$$u(t,x) = (K_t * u_0)(x), \quad K_t(x) = \frac{1}{(2\pi i t)^{n/2}} e^{ix^2/(2t)}.$$

An easy computation reveals that the corresponding Hamiltonian flow is given by

$$\mathscr{A}_t(x,\xi) = (x + 2\pi t\xi, \xi), \quad (x,\xi) \in \mathbb{R}^{2n}.$$

Let us consider the initial datum $u_0 = \delta_0$, so that $U(t)u_0(x) = K_t(x)$. Therefore, using the results in Example 2 we get

$$WF_G(U(t)u_0) = \mathscr{A}_t(WF_G(\delta_0)) = \{(x,\xi) \in \mathbb{R}^{2n} : x = 2\pi t\xi, \xi \neq 0\}.$$

Notice that a pure frequency initial state, namely $u_0(x) = e^{2\pi i x\xi_0}$ for $\xi_0 \in \mathbb{R}^n$, evolves as $u(t,x) = e^{-2\pi^2 it\xi_0^2} e^{2\pi i x\xi_0}$, hence the wave front set is stationary:

$$WF_G(U(t)u_0) = WF_G(u_0) = \{(x,0) \in \mathbb{R}^{2n}, x \neq 0\}.$$

Example 6 (The Harmonic Oscillator) Consider now the Hamiltonian

$$H = -\frac{1}{4\pi}\triangle + \pi x^2,$$

that is the Weyl quantization of the symbol $a(x,\xi)$ as in (27) with $A = C = (2\pi)I$, $B = 0$, where I is the $n \times n$ identity matrix—see [23, Sec. 4.3] and [7, Sec. 4] for a detailed derivation. The classical flow can be explicitly computed:

$$\mathscr{A}_t = \begin{pmatrix} (\cos t)I & (\sin t)I \\ -(\sin t)I & (\cos t)I \end{pmatrix}, \quad t \in \mathbb{R}.$$

Therefore, by taking into account the initial datum $u_0 = 1$ and Example 3 above we have for any $t \in \mathbb{R}$

$$WF_G(U(t)u_0) = \mathscr{A}_t(WF_G(1)) = \{(x,\xi) = ((\cos t)y, -(\sin t)y) \in \mathbb{R}^{2n},\ y \neq 0\}.$$

Let us examine the behaviour of the wave front set in the interval $t \in [0, \pi/2]$ for the sake of concreteness. For $t = 0$ we have $WF_G(u_0) = (\mathbb{R}^n \setminus \{0\}) \times \{0\}$, while for $t = \pi/2$ we have $WF(U(\pi/2)u_0) = \{0\} \times (\mathbb{R}^n \setminus \{0\})$. We see that for $t \in (0, \pi/2)$ the singularities are propagated by clockwise rotation in phase space. Let us stress the connection with the structure of the propagator, whose distribution kernel is given by the *Mehler formula* [20, 34]: for $k \in \mathbb{Z}$,

$$K_t(x,y) = \begin{cases} c(k)|\sin t|^{-n/2} \exp\left(\pi i \frac{x^2+y^2}{\tan t} - 2\pi i \frac{xy}{\sin t}\right) & (\pi k < t < \pi(k+1)) \\ c'(k)\delta((-1)^k x - y) & (t = k\pi) \end{cases}, \tag{28}$$

for suitable phase factors $c(k), c'(k) \in \mathbb{C}$.

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On the Regularity of Characteristic Functions

Winfried Sickel

Abstract In this survey we shall deal with the regularity of characteristic functions $\mathcal{X}_E$ of subsets E of $\mathbb{R}^d$ in the framework of Besov spaces. We will describe a number of necessary and sufficient conditions to guarantee membership in a Besov space of given smoothness s and with integrability p. Several examples are discussed in detail.

Keywords Characteristic functions · Indicator functions · Regularity · Besov spaces · Snowflake domain · Twindragon · Modified Nikodym domains

1 Introduction

Let E be a nontrivial measurable subset of $\mathbb{R}^d$ such that $0 < |E| < \infty$. By $|E|$ we denote the Lebesgue measure of E and by $\mathcal{X}_E$ the associated characteristic function. For $1 \le p \le \infty$ and $s \ge 0$ we have

$$\mathcal{X}_E \in L_p(\mathbb{R}^d) \quad \text{for all } E\,, \qquad \mathcal{X}_E \not\in W^1_p(\mathbb{R}^d) \quad \text{for all } E\,,$$

and

$$\mathcal{X}_E \not\in C^s(\mathbb{R}^d) \quad \text{for all } E\,.$$

Neither the Lebesgue spaces $L_p(\mathbb{R}^d)$ nor the first order Sobolev spaces $W^1_p(\mathbb{R}^d)$ nor the Hölder spaces $C^s(\mathbb{R}^d)$ allow to distinguish the regularity of those characteristic functions. Intuitively it is clear that these functions have different regularity depending on the quality of the boundary (whatever this means at this moment). To make this clear we have to deal with notions of fractional smoothness $s \in (0, 1)$ related to spaces with $p < \infty$. There are several possibilities. Not only for simplicity

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we have decided here for Besov spaces $B^s_{p,q}(\mathbb{R}^d)$, see Sect. 2 for a definition. Alternatively we could have chosen Bessel potential spaces $H^s_p(\mathbb{R}^d)$ or even more general Lizorkin–Triebel spaces $F^s_{p,q}(\mathbb{R}^d)$. Parts of the results obtained below carry over from Besov spaces to the spaces $F^s_{p,q}(\mathbb{R}^d)$, but there will be also exceptions, mainly in limiting situations. We will not go into details here.

In this survey we will discuss various notions describing the regularity of the boundary ∂E and to compare this with the regularity of $\mathcal{X}_E$ in Besov spaces. Mostly they will stem from fractal geometry, but not exclusively. For convenience of the reader we have collected some basic facts from fractal geometry in the Appendix at the end of this paper. The paper is written in a way that it is readable also for non-experts in function spaces. The author had spend some time to look for proofs as simple as possible. Only in a few cases we did not include the known but more complicated proofs. This makes the paper essentially self-contained. A certain number of examples is treated in detail.

The motivation of the author to deal with this topic originated from the theory of pointwise multipliers for Besov spaces. Here a function $f : \mathbb{R}^d \to \mathbb{R}$ is called a pointwise multiplier for $B^s_{p,q}(\mathbb{R}^d)$ if $f \cdot g$ belongs to $B^s_{p,q}(\mathbb{R}^d)$ for all $g \in B^s_{p,q}(\mathbb{R}^d)$. The question, whether the characteristic function of the half space $E := \mathbb{R}^d_+$ is a pointwise multiplier for Besov and Bessel potential spaces, has attracted a lot of attention since the early sixties. Later Gulisashvili [30, 31], see also Maz'ya and Shaposhnikova [46, 5.5.2], had found necessary and sufficient conditions on a set $E \subset \mathbb{R}^d$ such that $\mathcal{X}_E$ is a pointwise multiplier in specific situations. For a function f to be a pointwise multiplier for $B^s_{p,q}(\mathbb{R}^d)$ it is necessary that f belongs at least locally to $B^s_{p,q}(\mathbb{R}^d)$ itself. Hence, the regularity of the characteristic function $\mathcal{X}_E$ is part of the pointwise multiplier problem for Besov spaces. In my opinion it is interesting enough to be considered as an independent problem.

There will be a continuation of this survey dealing with characteristic functions as pointwise multipliers for Besov spaces.

The paper is organized as follows. Section 2 is devoted to the function spaces under consideration. In Sect. 3 we will discuss the maximal smoothness of characteristic functions related to the case $s = 1/p$. Section 4 contains results on less regular characteristic functions, i.e., we consider $0 < s < 1/p$.

1.1 Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the integers and $\mathbb{R}$ the real numbers. The letter $d \in \mathbb{N}$ is always reserved for the underlying dimension in $\mathbb{R}^d$. As usual, a domain in $\mathbb{R}^d$ is an open, non-trivial and simply connected set. For a subset E of $\mathbb{R}^d$ we denote it's complement by F and the set of inner points of F by $\mathring{F}$. Furthermore, we put

$$\partial E = \partial F := \{x \in \mathbb{R}^d : \ \mathrm{dist}\,(x, E) = \mathrm{dist}\,(x, F) = 0\} .$$

Several times we will work with dyadic cubes. Here by a dyadic cube we mean a cube of type

$$Q_{j,k} := \{x \in \mathbb{R}^d : \, 2^{-j} k_\ell \le x_\ell < 2^{-j}(k_\ell + 1), \; \ell = 1, \dots, d\}, \qquad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^d .$$

With $\lambda\, Q$ we denote the cube having the same centre as Q itself, the sides of $\lambda\, Q$ and Q are parallel and the side-length of $\lambda\, Q$ is the side-length of Q multiplied with $\lambda > 0$. A ball with center in x and radius r will be denoted by $B(x, r)$.

If X and Y are two normed spaces, the symbol $X \hookrightarrow Y$ indicates that the identity operator is continuous. For two sequences $(a_n)_n$ and $(b_n)_n$ of nonnegative real numbers we will write $a_n \lesssim b_n$ if there exists a constant $c > 0$ such that $a_n \le c\, b_n$ for all n. We use $a_n \asymp b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$.

2 Besov Spaces

Nowadays Besov spaces represent a standard version of regularity used in various branches of mathematics. One of their advantages consists in the possibility to describe them in quite different ways. For our purpose the most appropriate one is the characterization by differences.

Definition 1 Let $1 \le p, q \le \infty$.

(i) Let $0 < s < 1$. Then $B^s_{p,q}(\mathbb{R}^d)$ is the collection of all real-valued functions $f \in L_p(\mathbb{R}^d)$ such that

$$\| f \,|B^s_{p,q}(\mathbb{R}^d)\| := \| f \,|L_p(\mathbb{R}^d)\| + \bigg(\int_{|h|<1} |h|^{-sq} \Big(\int_{\mathbb{R}^d} |f(x+h) - f(x)|^p dx \Big)^{q/p} \frac{dh}{|h|^d} \bigg)^{1/q} < \infty$$

(usual modification if p and/or q are equal to infinity).

(ii) Let $1 \le s < 2$. Then $B^s_{p,q}(\mathbb{R}^d)$ is the collection of all real-valued functions $f \in L_p(\mathbb{R}^d)$ such that

$$\| f \,|B^s_{p,q}(\mathbb{R}^d)\| := \| f \,|L_p(\mathbb{R}^d)\| + \bigg(\int_{|h|<1} |h|^{-sq} \Big(\int_{\mathbb{R}^d} |f(x+2h) - 2f(x+h) + f(x)|^p dx \Big)^{q/p} \frac{dh}{|h|^d} \bigg)^{1/q} < \infty$$

(usual modification if p and/or q are equal to infinity).

Remark 1

(i) Besov spaces can be defined for all $s \in \mathbb{R}$ and all $p, q \in (0, \infty]$ (partly by using simply higher order differences). But for us the above definition will be sufficient.
(ii) Besov spaces are Banach spaces. They can be characterized also in terms of the modulus of smoothness, in a Fourier analytic way, by atoms, molecules and wavelets etc.. Standard references are the monographs by Besov, Il'yin, Nikol'skij [6, 7], Nikol'skij [48], Peetre [51] and Triebel [61, 62, 66].

Normally most important are the parameters p and s. The parameter q may be considered as a fine-index which only comes into play in limiting situations. There will be two cases, namely $q = \infty$ and $q = p$, which will be more important for us then the other. In case $q = \infty$ the norm reads as

$$\| f \,|B^s_{p,\infty}(\mathbb{R}^d)\| := \| f \,|L_p(\mathbb{R}^d)\| + \sup_{|h|<1} |h|^{-s} \left(\int_{\mathbb{R}^d} |f(x+h) - f(x)|^p dx \right)^{1/p}$$

if $0 < s < 1$ and

$$\| f \,|B^s_{p,\infty}(\mathbb{R}^d)\| := \| f \,|L_p(\mathbb{R}^d)\| + \sup_{|h|<1} |h|^{-s} \left(\int_{\mathbb{R}^d} |f(x+2h) - 2f(x+h) + f(x)|^p dx \right)^{1/p}$$

if $1 \le s < 2$. In case $q = p$ we first observe that we can replace $\int_{|h|<1} \ldots$ by $\int_{\mathbb{R}^d} \ldots$ (since the additional term $(\int_{|h|\ge 1} \ldots)^{1/p}$ is dominated by a constant $C(s, p, d)$ (independent of f) times $\| f \,|L_p(\mathbb{R}^d)\|$). A change of variables finally results in the following equivalent norms for $B^s_{p,p}(\mathbb{R}^d)$:

$$\| f \,|B^s_{p,p}(\mathbb{R}^d)\|^* := \| f \,|L_p(\mathbb{R}^d)\| + \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x-y|^{sp+d}} dx\, dy \right)^{1/p}$$

if $0 < s < 1$ and

$$\| f \,|B^s_{p,p}(\mathbb{R}^d)\|^* := \| f \,|L_p(\mathbb{R}^d)\| + \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(2y-x) - 2f(y) + f(x)|^p}{|x-y|^{sp+d}} dx\, dy \right)^{1/p}$$

if $1 \le s < 2$. If $\mathbb{R}^d$ is replaced by a smooth bounded domain Ω these norms are often called Gagliardo norms. Many times we shall employ so-called elementary embeddings. They express the monotonicity of the Besov spaces with respect to s and q. Here we mean the following

$$B^{s_0}_{p,q_0}(\mathbb{R}^d) \hookrightarrow B^{s_0}_{p,\infty}(\mathbb{R}^d) \hookrightarrow B^{s_1}_{p,1}(\mathbb{R}^d) \hookrightarrow B^{s_1}_{p,q_1}(\mathbb{R}^d)\,, \qquad (1)$$

where q_0, q_1 are arbitrary in $[1, \infty]$ and $0 < s_1 < s_0$.

Remark 2

(i) To restrict the values of h by $|h| < 1$ is always artificial. If a is an arbitrary positive real number, then the restriction $|h| < a$ leads to an equivalent norm.
(ii) Officially Besov spaces have been introduced by Besov in his Phd thesis published in the papers [3] and [4] in 1959/1961. However, Nikol'skij had already introduced the classes $B^s_{p,\infty}(\mathbb{R}^d)$ in 1951 and Gagliardo had considered $B^s_{p,p}(\mathbb{R}^d)$ in 1956 (in connection with trace problems for $W^1_p(\Omega)$).

3 The Limiting Case $s = 1/p$

As we shall see below, the smoothness s of a characteristic function $\mathcal{X}_E$ of a measurable set $E \subset \mathbb{R}^d$, $0 < |E| < \infty$, will be dominated in any case by $1/p$. With this problem we will deal first. Afterwards we will characterize those sets E such that $\mathcal{X}_E$ has maximal regularity.

3.1 Necessary Conditions

Let us start with a very simple example. We choose $d = 1$ and consider the characteristic function $\mathcal{X}$ of the interval (0, 1). For $1 \le p < \infty$ and $0 < h < 1$ we have

$$\int_{-\infty}^{\infty} |\mathcal{X}(x+h) - \mathcal{X}(x)|^p dx = \int_{-h}^{0} 1\, dx + \int_{1-h}^{1} 1\, dx = 2h\,.$$

The same argument applies for $-1 < h < 0$. Hence

$$\| \mathcal{X}(\cdot + h) - \mathcal{X}(\cdot)\, | L_p(\mathbb{R})\| = |2h|^{1/p}\,, \qquad |h| < 1\,. \qquad (2)$$

For $1 < p < \infty$ this immediately implies $\mathcal{X} \in B^s_{p,\infty}(\mathbb{R})$ if $0 < s \le 1/p$ and $\mathcal{X} \notin B^s_{p,\infty}(\mathbb{R})$ if $1/p < s < 1$. Since Besov spaces are monotone in s, see (1), we conclude $\mathcal{X} \notin B^s_{p,\infty}(\mathbb{R})$ for all $s > 1/p$.

Now we apply the same method to the case of a more general set E. Recall, $F := \mathbb{R}^d \setminus E$. For $h \in \mathbb{R}^d$ we define

$$E(h) := \{x \in E : \ x + h \notin E\}\,;$$
$$F(h) := \{x \in F : \ x + h \notin F\}\,.$$

It follows

$$\| \mathcal{X}_E(\cdot + h) - \mathcal{X}_E(\cdot)\,|L_p(\mathbb{R}^d)\|^p = \int_{E(h)} 1\,dx + \int_{F(h)} 1\,dx$$
$$= |E(h)| + |F(h)|\,. \qquad (3)$$

Hence, we have a first result.

Lemma 1 *Let* $1 \le p < \infty$ *and* $0 < s < 1$. *Then* $\mathcal{X}_E$ *belongs to* $B^s_{p,\infty}(\mathbb{R}^d)$ *if and only if*

$$\sup_{|h|<1} |h|^{-s}\,(|E(h)| + |F(h)|)^{1/p} < \infty\,. \qquad (4)$$

There is an easy but interesting consequence of Lemma 1. Let $1 < p < \infty$. Observe that $\mathcal{X}_E \in B^s_{1,\infty}(\mathbb{R}^d)$ implies $\mathcal{X}_E \in B^{s/p}_{p,\infty}(\mathbb{R}^d)$ and vice versa.

Figure 1 below shows shifted versions of the supports of characteristic functions of a circle and of a rectangle, respectively. The shaded regions are just $E(h) \cup F(h)$ in these cases. De facto it is "seen" that $|E(h)| + |F(h)| \asymp |h|,\ |h| < 1$.

As a consequence we obtain a second result.

Lemma 2 *Let* $d \ge 2$. *Let* $1 \le p < \infty$ *and* $s > 0$. *Then the characteristic function* $\mathcal{X}_E$ *of either a ball or a cuboid, i.e., the cartesian product of d segments, belongs to* $B^s_{p,\infty}(\mathbb{R}^d)$ *if and only if* $s \le 1/p$.

Proof Only $p = 1$ requires an additional comment. Obviously

$$\int_{\mathbb{R}^d} |\mathcal{X}_E(x + 2h) - 2\mathcal{X}_E(x + h) + \mathcal{X}_E(x)|dx$$
$$\le \int_{\mathbb{R}^d} |\mathcal{X}_E(x + 2h) - \mathcal{X}_E(x + h)|dx + \int_{\mathbb{R}^d} |\mathcal{X}_E(x + h) - \mathcal{X}_E(x)|dx$$
$$= 2(|E(h)| + |F(h)|)\,.$$

This explains sufficiency. Necessity follows from Theorem 1 below. □

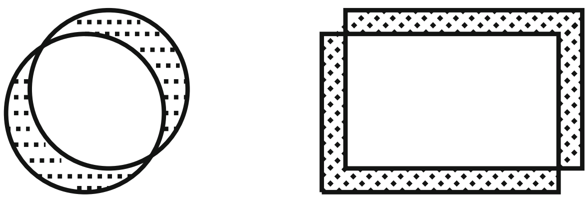


Fig. 1 $E(h) \cup F(h)$ for circle and rectangle

It will be the main aim of this subsection to show that $s = 1/p$ is a barrier for the smoothness of characteristic functions $\mathcal{X}_E$ in general. Our point of departure is a generalization of a theorem of Titchmarsh, due to Gulisashvili [30].

Proposition 1 *If for some ball B, $B \subset \mathbb{R}^d$, and $f \in L_1^{\ell oc}(\mathbb{R}^d)$ we have*

$$\lim_{|h|\to 0} \frac{1}{|h|} \int_B |f(x+h) - f(x)|\, dx = 0$$

then $f \equiv$ const almost everywhere on B.

Now we turn to an application of this Proposition 1. Let $E \subset \mathbb{R}^d$, $0 < |E| < \infty$. Then the function $g(x) := |\mathcal{X}(x+h) - \mathcal{X}(x)|$, $x \in \mathbb{R}^d$, only takes values in $\{0, 1\}$. This implies

$$\int_B |\mathcal{X}_E(x+h) - \mathcal{X}_E(x)|\, dx = \int_B |\mathcal{X}_E(x+h) - \mathcal{X}(x)|^p\, dx$$

for all h and all $1 \le p < \infty$. Next we need to recall an equivalent characterization of Besov spaces in terms of modulus of smoothness. We put

$$\omega_p(f,t) := \sup_{|h|<t} \Big(\int_{\mathbb{R}^d} |f(x+h) - f(x)|^p\, dx \Big)^{1/p}, \quad f \in L_p^{\ell oc}(\mathbb{R}^d)\, .$$

Let $1 \le q < \infty$, $1 \le p < \infty$ and $0 < s < 1$. Then there exist positive constants A, B such that

$$\begin{aligned} A\, \| f\, |B^s_{p,q}(\mathbb{R}^d)\| &\le \| f\, |L_p(\mathbb{R}^d)\| + \Big(\sum_{j=0}^{\infty} (2^{js} \omega_p(f, 2^{-js}))^q \Big)^{1/q} \\ &\le B\, \| f\, |B^s_{p,q}(\mathbb{R}^d)\| \end{aligned}$$

holds for all $f \in B^s_{p,q}(\mathbb{R}^d)$, we refer, e.g., to [61, 2.5.12]. A simple monotonicity argument yields that

$$\lim_{h\to 0} \frac{1}{|h|} \Big(\int_{\mathbb{R}^d} |f(x+h) - f(x)|^p\, dx \Big)^{1/p} = 0$$

for any $f \in B^s_{p,q}(\mathbb{R}^d)$. If $f = \mathcal{X}_E$ then the assumption $\mathcal{X}_E \in B^{1/p}_{p,q}(\mathbb{R}^d)$ $(1 < p < \infty,\ 1 \le q < \infty)$ and Proposition 1 yield that $\mathcal{X}_E$ is constant on any ball B. But this is in contradiction with $E \subset \mathbb{R}^d$, $0 < |E| < \infty$.

Theorem 1 *Let $1 \le p < \infty$ and $1 \le q < \infty$. Then there exists no subset $E \subset \mathbb{R}^d$, $0 < |E| < \infty$, such that $\mathcal{X}_E \in B^{1/p}_{p,q}(\mathbb{R}^d)$.*

Proof The case $1 < p < \infty$, $1 \le q < \infty$ has been treated above. It remains to consider $p = 1$. Let us assume $\mathcal{X}_E \in B^1_{1,q}(\mathbb{R}^d)$. Since the function

$$g_h(x) := |\mathcal{X}(x+2h) - 2\mathcal{X}(x+h) + \mathcal{X}(x)|\,, \quad x \in \mathbb{R}^d\,,$$

can only take values from the set $\{0, 1, 2\}$, we obtain

$$\| \, g_h \, | L_1(\mathbb{R}^d)\| \le \| \, g_h \, | L_p(\mathbb{R}^d)\|^p \le 2^{p-1} \, \| \, g_h \, | L_1(\mathbb{R}^d)\| \, .$$

Let $1 \le r < \infty$. It follows

$$\int_{|h|\le 1} \Big(|h|^{-1} \, \| \, g_h \, | L_1(\mathbb{R}^d)\| \Big)^{r/p} \frac{dh}{|h|^d} \asymp \int_{|h|\le 1} \Big(|h|^{-1/p} \, \| \, g_h \, | L_p(\mathbb{R}^d)\| \Big)^{r} \frac{dh}{|h|^d} \, .$$

Since E has finite measure, this implies $\mathcal{X}_E \in B^1_{1,q}(\mathbb{R}^d)$ if and only if $\mathcal{X}_E \in B^{1/p}_{p,pq}(\mathbb{R}^d)$. For $1 < p < \infty$ and $q < \infty$ we may apply our arguments from above. This yields the claim for $p = 1$. □

Hence we conclude that the maximal regularity of a characteristic function in the framework of Besov spaces is given by the class $B^{1/p}_{p,\infty}(\mathbb{R}^d)$ for some p. Only in case $d = 1$ the spaces $B^{1/p}_{p,\infty}(\mathbb{R}^d)$, $1 \le p \le \infty$, are comparable. Then we have

$$B^1_{1,\infty}(\mathbb{R}) \hookrightarrow B^{1/p_0}_{p_0,\infty}(\mathbb{R}) \hookrightarrow B^{1/p_1}_{p_1,\infty}(\mathbb{R}) \hookrightarrow B^0_{\infty,\infty}(\mathbb{R})\,, \quad 1 \le p_0 < p_1 \le \infty\,.$$

The characteristic function $\mathcal{X}$ of the interval $(0, 1)$ not only belongs to $B^1_{1,\infty}(\mathbb{R})$, it belongs to $BV(\mathbb{R})$, the space of functions of bounded variation (which represents a strictly smaller class). This will play a role in the next subsection.

3.2 *Characteristic Functions with Maximal Regularity*

Here we follow Gulisashvili [30]. Therefore we consider functions of bounded variation which are integrable on $\mathbb{R}^d$.

Recall, a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$ is of bounded variation if its first order partial derivatives (in the distributional sense) are bounded Borel measures. The space $BV \cap L_1(\mathbb{R}^d)$ will be endowed with the norm

$$\| \, f \, | BV \cap L_1(\mathbb{R}^d)\| := \sum_{j=1}^d \Big| \frac{\partial f}{\partial x_j} \Big| + \| \, f \, | L_1(\mathbb{R}^d)\| \, .$$

where $|\frac{\partial f}{\partial x_j}|$ denotes the total variation of the measure. The symbol $\mathcal{H}^s$ refers to the s-dimensional Hausdorff measure, see the Appendix for details. Then the perimeter of a set E is the quantity

$$\mathrm{per}\, E := \liminf_{j\to\infty} \mathscr{H}^{d-1}(\partial M_j)\,,$$

where the limit is taken with respect to all sequences $(M_j)_j$ of sets with a smooth boundary (or polyhedra) such that

$$\lim_{j\to\infty} \| \mathcal{X}_E - \mathcal{X}_{M_j} |L_1(\mathbb{R}^d)\| = 0\,.$$

A basic fact in the theory of the BV spaces is the Kronrod–Federer–Fleming–Rishel formula

$$\| f |BV(\mathbb{R}^d)\| = \int_{-\infty}^{\infty} \mathrm{per}\,(\{x \in \mathbb{R}^d :\ f(x) > t\})\, dt\,,$$

see, e.g., Fleming, Rishel [26] and Burago, Zalgaller [11]. In particular, it follows

$$\mathcal{X}_E \in BV(\mathbb{R}^d) \qquad \text{if and only if} \qquad \mathrm{per}\, E < \infty\,. \tag{5}$$

Next we recall the definition of the space $Lip\,(1,1)(\mathbb{R}^d)$. A function $f:\ \mathbb{R}^d \to \mathbb{R}$ belongs to $Lip\,(1,1)(\mathbb{R}^d)$ if $f \in L_1(\mathbb{R}^d)$ and $\sup_{t>0} t^{-1}\,\omega_1(f,t) < \infty$. The norm is given by

$$\| f |Lip\,(1,1)(\mathbb{R}^d)\| := \| f |L_1(\mathbb{R}^d)\| + \sup_{t>0} t^{-1}\,\omega_1(f,t)\,.$$

Hardy and Littlewood proved that $BV \cap L_1(\mathbb{R})$ coincides with $Lip\,(1,1)(\mathbb{R})$. The generalization to the case $d > 1$ has been proved by Gulisashvili [30].

Proposition 2 *It holds $BV \cap L_1(\mathbb{R}^d) = Lip\,(1,1)(\mathbb{R}^d)$ as sets. There exist positive constants A, B such that*

$$A\ \sup_{t>0} t^{-1}\,\omega_1(f,t) \le \| f |BV(\mathbb{R}^d)\| \le B\ \sup_{t>0} t^{-1}\,\omega_1(f,t)$$

holds for all $f \in L_1(\mathbb{R}^d)$.

Summarizing we get the following.

Lemma 3 *Let $E \subset \mathbb{R}^d$ be a measurable set satisfying $0 < |E| < \infty$.*

(i) Let per $E < \infty$*. Then $\mathcal{X}_E$ belongs to $B^{1/p}_{p,\infty}(\mathbb{R}^d)$ for all $p,\ 1 \le p < \infty$.*
(ii) Let $\mathcal{X}_E \in B^{1/p_0}_{p_0,\infty}(\mathbb{R}^d)$ for some $p_0,\ 1 \le p_0 < \infty$. Then per $E < \infty$*, $\mathcal{X}_E \in BV(\mathbb{R}^d)$ and $\mathcal{X}_E \in B^{1/p}_{p,\infty}(\mathbb{R}^d)$ for all $p,\ 1 \le p < \infty$, follows.*

Proof

Step 1. Proof of (i). Thanks to (5) and per $E < \infty$ we know that $\mathcal{X}_E \in BV \cap L_1(\mathbb{R}^d)$. Since this space coincides with $Lip(1,1)(\mathbb{R}^d)$, see Proposition 2, we conclude

$$\sup_{|h|<1} |h|^{-1} \int_{\mathbb{R}^d} \Big|\Big(\mathcal{X}_E(x+2h) - \mathcal{X}_E(x+h)\Big) - \Big(\mathcal{X}_E(x+h) - \mathcal{X}_E(x)\Big)\Big|\, dx$$

$$\le 2 \sup_{|h|<1} |h|^{-1} \int_{\mathbb{R}^d} |\mathcal{X}_E(x+h) - \mathcal{X}_E(x)|\, dx < \infty\,,$$

i.e., $\mathcal{X}_E \in B^1_{1,\infty}(\mathbb{R}^d)$. We put $\tilde{g}_h(x) := \mathcal{X}_E(x+h) - \mathcal{X}_E(x),\ x \in \mathbb{R}^d$. Observe that $|\tilde{g}(x)| \in \{0,1\}$ for all x. Hence, for all $p \in (1,\infty)$ we get

$$|h|^{-1}\, \|\, \tilde{g}_h\, |L_1(\mathbb{R}^d)\| \asymp |h|^{-1}\, \|\, \tilde{g}_h\, |L_p(\mathbb{R}^d)\|^p \tag{6}$$

with hidden constants independent of h. This yields that $\mathcal{X}_E$ belongs to $B^{1/p}_{p,\infty}(\mathbb{R}^d)$ for all these p.

Step 2. Proof of (ii). Once again we use (6). Since $\mathcal{X}_E \in L_1(\mathbb{R}^d)$ is guaranteed by $|E| < \infty$ we conclude that $\mathcal{X}_E \in B^{1/p_0}_{p_0,\infty}(\mathbb{R}^d)$ implies that $\mathcal{X}_E \in B^{1/p}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1,\infty)$. We get a bit more. We also obtain that

$$\sup_{|h|<1} |h|^{-1} \int_{\mathbb{R}^d} |\mathcal{X}_E(x+h) - \mathcal{X}_E(x)|\, dx < \infty.$$

Now we employ (3) and find

$$\begin{aligned}\sup_{0<t<1} t^{-1} \sup_{|h|<t} \int_{\mathbb{R}^d} |\mathcal{X}_E(x+h) - \mathcal{X}_E(x)|\, dx & \\ = \sup_{0<t<1} t^{-1} \sup_{|h|<t} (E(h) + F(h)) & \\ \le \sup_{|h|<1} |h|^{-1}\, (E(h) + F(h)) =: I < \infty\,. &\end{aligned}$$

Because of the trivial estimate

$$\sup_{t\ge 1} t^{-1} \sup_{|h|<t} \int_{\mathbb{R}^d} |\mathcal{X}_E(x+h) - \mathcal{X}_E(x)|\, dx \le I + 2\,|E| < \infty$$

we conclude $\mathcal{X}_E \in Lip(1,1) \cap L_1(\mathbb{R}^d)$ and therefore $\mathcal{X}_E \in BV \cap L_1(\mathbb{R}^d)$, see Proposition 2. Finally, formula (5) yields the claim.

□

The second main result in this subsection we get as an immediate consequence.

Theorem 2 *Let* $E \subset \mathbb{R}^d$ *and* $0 < |E| < \infty$. *Then the following assertions are equivalent:*

(i) $\mathrm{per}\, E < \infty$;
(ii) $\sup_{|h|<1} |h|^{-1} (|E(h)| + |F(h)|) < \infty$;
(iii) $\mathcal{X}_E \in BV(\mathbb{R}^d)$;
(iv) $\mathcal{X}_E \in Lip(1,1)(\mathbb{R}^d)$;
(v) $\mathcal{X}_E \in B^{1/p_0}_{p_0,\infty}(\mathbb{R}^d)$ *for some* $p_0,\ 1 \le p_0 < \infty$.
(vi) $\mathcal{X}_E \in B^{1/p}_{p,\infty}(\mathbb{R}^d)$ *for all* $p,\ 1 \le p < \infty$.

Proof Part (i) implies (iii) by using (5). Proposition 2 yields the implication (iii) $\to$ (iv). From Lemma 3 we derive (iv) $\to$ (v) and (v) $\to$ (vi). Lemma 1 shows (vi) $\to$ (ii) and at the same time (ii) $\to$ (v) ($p_0 = 1$). Finally, Lemma 3 helps to close the circle since (v) $\to$ (i). □

3.3 Examples

Characteristic functions of balls and of rectangles (cuboids) we have already considered. Now we turn to more complicated domains. As usual, a domain is an open connected set in $\mathbb{R}^d$. First we apply a well-known fact in the theory of Besov spaces. The classes $B^s_{p,q} \cap L_\infty(\mathbb{R}^d),\ s > 0,\ 1 \le p,q \le \infty$, are algebras under pointwise multiplication, i.e., there exists a positive constant c such that

$$\| f \cdot g \,|B^s_{p,q}\| \le c \Big(\| f \,|B^s_{p,q}(\mathbb{R}^d)\| \,\| g \,|L_\infty(\mathbb{R}^d)\| + \| g \,|B^s_{p,q}(\mathbb{R}^d)\| + \| f \,|L_\infty(\mathbb{R}^d)\|\Big)$$

holds for all $f, g \in B^s_{p,q} \cap L_\infty(\mathbb{R}^d)$. We refer to Peetre [50] and [52, 4.6]. In addition we shall use that Besov spaces are invariant under rotations, translations and reflections. The combination of these two facts leads to a large number of further examples sharing the same smoothness properties as the characteristic function of a cube. For example, multiplying the characteristic function of a cube with an rotated, shifted and properly enlarged version of it we get that the characteristic function of a triangle has maximal regularity as well. Hence, any domain which allows a finite triangulation, has an associated characteristic function with maximal regularity. All these examples are covered by the classes of characteristic functions which we will consider below. The most important but probably not the most interesting examples are given by characteristic functions of elementary Lipschitz domains. Concerning these domains we shall make use of the following definition, picked up from Burenkov [12, 4.3]. In this definition we shall apply the notation $x = (x', x_d),\ x' = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1},\ x_d \in \mathbb{R}$.

Definition 2 Let $d \geq 2$. An open bounded set E is called elementary Lipschitz domain if there exist a function φ and numbers $0 < D_1 \leq D_2 < \infty$, $a_1, \dots a_d$, $b_1, \dots, b_{d-1}, L$ such that

(i) $\mathrm{diam}\,(E) \leq D_2$;
(ii) $E = \{x \in \mathbb{R}^d : \; a_d < x_d < \varphi(x'), \; x' \in W\}$;
(iii) $W := \{x' \in \mathbb{R}^{d-1} : \; a_i < x_i < b_i, \; i = 1, \dots, d-1\}$;
(iv) $a_d + D_1 \leq \varphi(x'), \; x' \in W$;
(v) $|\varphi(x') - \varphi(y')| \leq L\,|x' - y'|, \quad x', \; y' \in W$.

For elementary Lipschitz domains it is easy to prove that the associated characteristic function has maximal regularity.

Lemma 4 *Let E be an elementary Lipschitz domain. Then $\mathscr{X}_E \in BV \cap B^{1/p}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1, \infty)$.*

Proof We will apply Theorem 2(ii).

Step 1. For positive δ we define

$$\partial E^\delta := \{x \in \mathbb{R}^d : \; \mathrm{dist}\,(x, \partial E) \leq \delta\}\,. \tag{7}$$

Usually ∂E^δ is called the δ-neighbourhood of ∂E. Observe, in our particular case we have

$$\partial E = \partial W^* \cup \{(x', x_d) : \; x' \in \partial W\,, \; a_d \leq x_d \leq \varphi(x')\} \cup \{(x', \varphi(x')) : \; x' \in W\}\,,$$

where $\partial W^* := \{(x', a_d) : \; x' \in W\}$. In what follows we concentrate on the last part since the remaining part of the boundary is either regular or can be treated similarly as the last part. Suppose $0 < \delta < D_1/L$, where L denotes the Lipschitz constant of φ. Let

$$G := \{(x', \varphi(x')) : \; x' \in W\}.$$

We claim that

$$\partial G^\delta \subset \Omega := \Big\{(x', x_d) : \; x' \in W\,, \; \varphi(x') - (L+1)\delta < x_d < \varphi(x') + (L+1)\delta\Big\}\,.$$

Let $x \in G^\delta$ and suppose $\mathrm{dist}\,(x, G) = \rho \leq \delta$. Hence, there is a point $y \in G$ such that $|x - y| = \rho$. Clearly, $y = (y', \varphi(y'))$. This yields

$$|x_d - \varphi(x')| \leq |x_d - \varphi(y')| + |\varphi(y') - \varphi(x')| \leq |x_d - \varphi(y')| + L|x' - y'|\,. \tag{8}$$

Since $|x_d - \varphi(y')| \leq \mathrm{dist}\,(x, G) = \rho$ and $|x' - y'| \leq \rho$, we find

$$|x_d - \varphi(x')| \leq (L+1)\delta$$

and therefore $x \in \Omega$.

Step 2. There is an obvious relation between the δ-neighbourhood of ∂E and $E(h) \cup F(h)$. We have

$$E(h) \cup F(h) \subset \partial E^{\delta}, \quad |h| = \delta. \tag{9}$$

Applying the result of Step 1 we find

$$|E(h)| + |F(h)| \le |\partial E^{\delta}| \le |\Omega| = 2\,(L+1)\,\delta\,|W|, \quad \delta = |h|.$$

By Theorem 2 the claim follows.

□

As already mentioned above, Besov spaces are invariant under rotations, translations and reflections. This has an immediate consequence.

Corollary 1 *Let E be a domain which can be written as the union of the closures of a finite number of pairwise disjoint domains $E_1, \ldots, E_N$ such that any of the E_j, $j = 1, \ldots, N$, is the image of an elementary Lipschitz domain under a finite number of rotations, translations and reflections. Then $\mathcal{X}_E \in BV \cap B^{1/p}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1, \infty)$.*

Proof Lemma 4 yields

$$\mathcal{X}_{E_j} \in BV \cap B^{1/p}_{p,\infty}(\mathbb{R}^d)$$

for all $p \in [1, \infty)$ and all j. Since $|\partial E_j| = 0$, $j = 1, \ldots, N$, see Lemma 5 below, we have

$$\mathcal{X}_E = \sum_{j=1}^{N} \mathcal{X}_{E_j}\,.$$

Therefore, Corollary 1 is a consequence of Lemma 4. □

Figure 2 shows a domain with a polygonal boundary of finite length, covered by Corollary 1. Now we turn to examples in $\mathbb{R}^3$. In Fig. 3, we have a

polyhedral cone and in Fig. 4 we see an Icosahedron. Both are elementary Lipschitz domains. But Lipschitz regularity of the boundary is not necessary for maximal regularity of the associated characteristic function. Here are a few examples. First we take the domain $A \subset \mathbb{R}^2$ with boundary ∂A given by the Astroid. The determining functional equation of this curve is given by

$$x^{2/3} + y^{2/3} = 1\,, \qquad x, y \in \mathbb{R}\,. \tag{10}$$

Afterwards we consider the rotation of this curve around the y-axis resulting in the domain $A_{\text{rot}} \subset \mathbb{R}^3$.

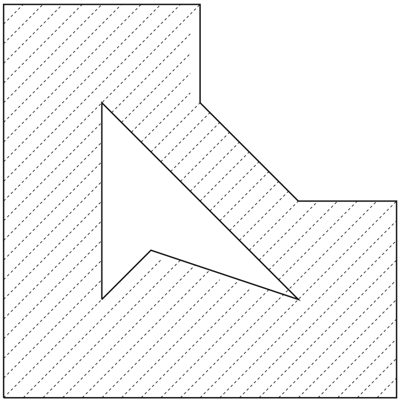


Fig. 2 A domain with a polygonal boundary in the plane

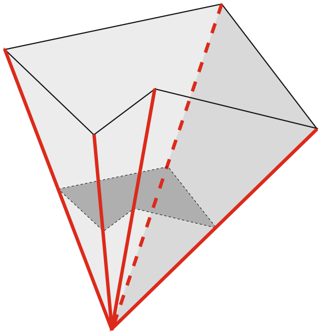


Fig. 3 A polyhedral cone

Fig. 4 The icosahedron

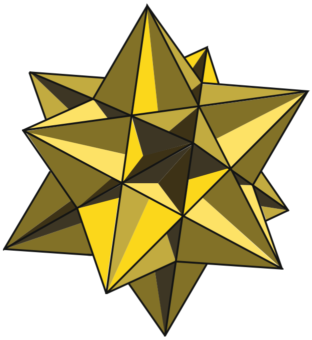


Figure 5 shows a vertical cut through A_{rot}, which gives us the domain A itself. Obviously the boundary ∂A has Hölder regularity $\alpha = 2/3$, see (10), and is therefore not Lipschitz (in four isolated points). Concerning the δ-neigbourhood it is easy to show that there exists a positive constant c such that

$$|A^\delta| \le c\,|h|\,, \qquad |h| < 1\,.$$

Fig. 5 The Astroid

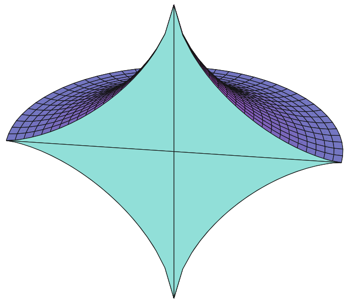
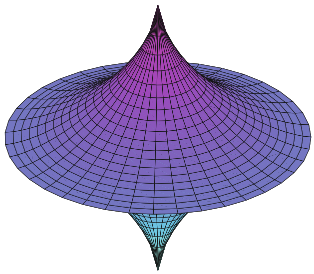


Fig. 6 The rotated Astroid



Hence, Theorem 2 yields $\mathscr{X}_A \in BV \cap B^{1/p}_{p,\infty}(\mathbb{R}^2)$ for all $p \in [1, \infty)$. Figure 6 shows the set A_{rot} itself. Obviously the boundary $\partial A_{\mathrm{rot}}$ is not Lipschitz in north and south pole and on the equator. However, we can argue as in case of A itself, i.e., there exists a positive constant C such that

$$|A^{\delta}_{\mathrm{rot}}| \le c\,|h|\,, \qquad |h| < 1\,.$$

Hence, Theorem 2 yields $\mathscr{X}_{A_{\mathrm{rot}}} \in BV \cap B^{1/p}_{p,\infty}(\mathbb{R}^3)$ for all $p \in [1, \infty)$.

The next example is even simpler, see Fig. 7. Let $\varepsilon \in (0, 1)$. We define

$$E_\varepsilon := \{(x, y) \in \mathbb{R}^2 : \ -1 < x < 1,\ \ |x|^\varepsilon < y < 1\}\,.$$

The domain E_ε has a boundary with Hölder regularity $\alpha = \varepsilon$. So the Hölder regularity can be arbitrarily small. However, the same argument as above can be applied. For any ε there exists a positive constant c_ε such that

$$|E^{\delta}_{\varepsilon}| \le c_\varepsilon\,|h|\,, \qquad |h| < 1\,.$$

Hence, Theorem 2 yields that the characteristic function of the domain E_ε belongs to $BV \cap B^{1/p}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1, \infty)$.

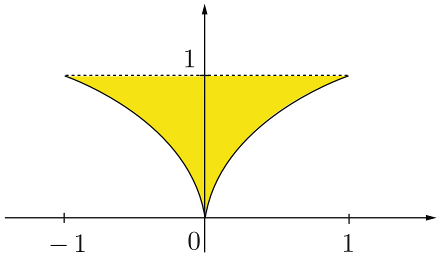


Fig. 7 A typical non-Lipschitz domain

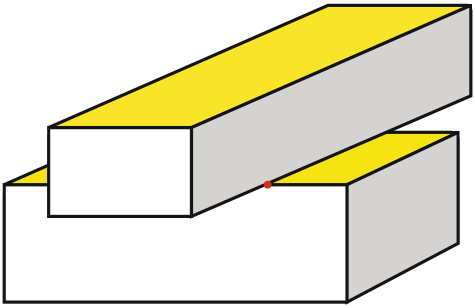


Fig. 8 A polyhedral domain which is not Lipschitz

Now we turn to $d = 3$ again. There is a famous example of a polyhedral domain in $\mathbb{R}^3$ which is not a Lipschitz domain, see Fig. 8. A convenient reference is given by Dobrowolski [20], see page 103.

The red dot indicates one of the critical points of the boundary when one tries to describe the neigbourhood as an elementary Lipschitz domain.

But in our situation it is simpler. We may apply Corollary 1. By the obvious splitting of the domain into the two subdomains, each of them given by one cuboid, it is immediate that the associated characteristic function has maximal regularity, see Lemma 2. This is the reason why we avoided the notion of a Lipschitz domain in Corollary 1. The class of domains used in this corollary covers the class of the Lipschitz domains, but is more general.

4 Less Regular Characteristic Functions

Now we turn to characteristic functions of sets with a more wild boundary. First we will investigate some necessary conditions.

4.1 Necessary Conditions

Let us start with some basics.

Lemma 5 *Let E be a bounded domain. If $\mathcal{X}_E \in B^s_{p,q}(\mathbb{R}^d)$ for some $s > 0$, $1 \le p < \infty$ and $1 \le q \le \infty$, then $|\partial E| = 0$ follows.*

Proof We employ the Withney-type decomposition of E into dyadic cubes, cf. Stein [58, VI.1] for details. By dyadic cubes we mean cubes of the type

$$Q_{j,k} := \{x \in \mathbb{R}^d : \ 2^{-j}k_\ell \le x_\ell < 2^{-j}(k_\ell + 1),\ \ell = 1, \ldots, d\},$$

$j \in \mathbb{Z},\ k \in \mathbb{Z}^d$. Hence

$$E = \bigcup_{Q \in \mathcal{F}} Q$$

where $Q = Q_{j,k}$ for some nonnegative integer j and $k \in \mathbb{Z}^d$, $\mathcal{F}$ denotes a subset of the set of all dyadic cubes and the cubes Q are pairwise disjoint. To each point $x \in \partial E$ we can associate a sequence of points $(x^j)_j \subset E$ approaching x. Each of the points x^j belongs to one of the dyadic cubes $Q \in \mathcal{F}$ and these cubes have the property

$$\operatorname{diam} Q \le \operatorname{dist}(Q, \partial E) \le 4 \operatorname{diam} Q .$$

Consequently, for any $\varepsilon > 0$ and each $x \in \partial E$ there exist $x^j \in E$ and a cube $Q(x^j) \in \mathcal{F}$, $x \in Q(x^j)$ such that $\operatorname{diam} Q(x^j) < \varepsilon$. Since Besov spaces are monotonically ordered with respect to s and q, see (1), we may concentrate on the classes $B^s_{p,p}(\mathbb{R}^d)$ for some small positive $s < 1$. It follows

$$\begin{aligned}
\Big(\| \mathcal{X}_E \,|B^s_{p,p}(\mathbb{R}^d)\|^*\Big)^p &\ge \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\mathcal{X}_E(x) - \mathcal{X}_E(y)|^p}{|x-y|^{d+sp}}\, dy\, dx \\
&\ge \int_{\partial E} \int_{Q(x^j)} \frac{1}{|x-y|^{d+sp}}\, dy\, dx \\
&\ge \int_{\partial E} \int_{Q(x^j)} (\operatorname{diam} Q(x^j))^{-(d+sp)}\, dy\, dx \\
&\ge C\, |\partial E|\, (\operatorname{diam} Q(x^j))^{-sp} \\
&\ge C\, |\partial E|\, \varepsilon^{-sp},
\end{aligned}$$

where C does not depend on ε. This proves the claim. □

Now we will continue with a more serious result due to Jaffard and Meyer [35]. To establish further necessary conditions we need to have additional information on the set E, in particular we need to know how thick the sets E and $F \setminus \partial E$ are in a neighbourhood of the boundary. We define

$$\partial E_+ = \Big\{ x \in \partial E \ : \ \exists \mu > 0 \quad \text{such that } \forall \varepsilon,\ 0 < \varepsilon \le 1,\ \exists A_\varepsilon\,,\, B_\varepsilon \text{ satisfying} \\ A_\varepsilon \subset B(x,\varepsilon) \cap E,\ B_\varepsilon \subset B(x,\varepsilon) \cap F, \ \text{ and } \ |A_\varepsilon| \cdot |B_\varepsilon| \ge \mu \varepsilon^{2d} \Big\}. \tag{11}$$

Let A be a subset of $\mathbb{R}^d$. By $\dim_P(A)$ we denote the packing dimension, cf. the Appendix.

Theorem 3 ([35, Thm. 2.2]) *Let E be a nontrivial subset of $\mathbb{R}^d$. Suppose $\mathcal{X}_E$ belongs to $B^s_{p,p}(\mathbb{R}^d)$ for some $s > 0$ and $1 \le p < \infty$. Then $\dim_P(\partial E_+) \le d - sp$.*

Remark 3

(i) Jaffard and Meyer [35] worked with a slightly modified definition for the set ∂E_+. They replaced $|A_\varepsilon|, |B_\varepsilon| \ge \mu\, \varepsilon^{2d}$ by the more restrictive conditions $|A_\varepsilon| \ge \mu\, \varepsilon^d$ and $|B_\varepsilon| \ge \mu\, \varepsilon^d$. But this change has no relevance for the proof. Since we shall not apply Theorem 3 below we skip the proof.
(ii) It seems that the method of proof does not apply to the Besov spaces with $p \neq q$ (but it extends to Lizorkin–Triebel spaces).

Of course, of interest are those domains E satisfying $\partial E = \partial E_+$. We discuss some examples.

(a) **John domains.** We say that a bounded domain E is a John domain provided there is a constant $C \ge 1$ and a distinguished point $x_0 \in E$, so that each point $x \in E$ can be joint to x_0 (inside E) by a rectifiable curve $\gamma : [0, \ell] \to E$, $\gamma(0) = x$, $\gamma(\ell) = x_0$, parameterized by arc-length (ℓ may depend on x), and such that the distance to the boundary satisfies

$$\mathrm{dist}\,(\gamma(t), \partial E) > C^{-1}\, t\, .$$

We refer to Martio, Sarvas [43] or Hajlasz, Koskela [32]. Relatives of John domains are investigated by Besov, we refer to Definition 6 below and [5], [7]. A direct consequence of the definition of John domains is the observation that for all $x \in \partial E$ there exists a $\mu > 0$ such that for all $\varepsilon \in (0,1)$ there exists a ball A_ε satisfying $A_\varepsilon \subset B(x,\varepsilon) \cap E$ and $|A_\varepsilon| \ge \mu\, \varepsilon^n$.

Now, select a cube Q such that $E \subset Q$ and $\mathrm{dist}\,(\partial E, \partial Q) > 1$. For a given set A we denote by $\mathring{A}$ the set of all inner points of A. Define $G := F \cap Q$. If E and $\mathring{G}$ are John domains then we conclude that $\partial E = \partial E_+$.

(b) (ε, δ) **domains**. Let $0 < \varepsilon < \infty$ and $0 < \delta \le \infty$. Then a domain E is called an (ε,δ) domain whenever $x, y \in E$ and $|x - y| < \delta$, there is a rectifiable arc $\gamma \subset E$ joining x to y and satisfying

$$\ell(\gamma) \le \frac{1}{\varepsilon}\, |x - y|$$

($\ell(\gamma)$ denotes the length of the arc γ) and

$$\mathrm{dist}\,(z, \partial E) \ge \varepsilon \frac{|x-z|\,|y-z|}{|x-y|} \qquad \text{for all} \quad z \in \gamma\, .$$

It is known that for an (ε,δ) domain it holds $|\partial E| = 0$. One of the key properties of (ε,δ) domains is the following. Denote by W_1 the collection of all dyadic cubes which form the Whitney decomposition of E. By W_2 we denote the collection of all dyadic cubes which form the Whitney decomposition of $\mathring{F}$. Then, for each cube $Q \in W_2$ with sidelength $\ell(Q) \le \varepsilon\,\delta/(16d)$ there exists a cube $Q^* \in W_1$ such that

$$1 \le \frac{\ell(Q^*)}{\ell(Q)} \le 4 \qquad \text{and} \qquad \operatorname{dist}(Q, Q^*) \le C\,\ell(Q)$$

where $C = C(d)$ but independent of Q and E. For all these properties we refer to Jones [36]. Hence, for E being an (ε, δ) domain we have $\partial E = \partial E_+$.

(c) **Regular domains**. A domain E is called regular if it satisfies the measure density condition: there exists a constant $c > 0$ such that for all $x \in E$ and all $r \in (0, 1]$

$$|B(x, r) \cap E| \ge c\, r^d\,.$$

If E and $\mathring{F}$ are regular then $\partial E = \partial E_+$ follows.

(d) **Extension and embedding domains**. We say that a bounded domain $\Omega \subset \mathbb{R}^d$ is a $B^s_{p,p}$-extension domain if every function $u \in B^s_{p,p}(\Omega)$ can be extended to a function $\tilde{u} \in B^s_{p,p}(\mathbb{R}^d)$, the mapping $u \mapsto \tilde{u}$ is continuous and there exists a constant $C = C(d, p, s, \Omega)$ such that

$$\|\, \tilde{u}\, |B^s_{p,p}(\mathbb{R}^d)\| \le C\, \|\, u\, |B^s_{p,p}(\Omega)\|\,.$$

Here we use the following definition for $B^s_{p,p}(\Omega),\ 0 < s < 1,\ 1 \le p \le \infty$. A function $u \in L_p(\Omega)$ belongs to $B^s_{p,p}(\Omega)$ if

$$\|\, f\, |B^s_{p,p}(\Omega)\|^* := \|\, f\, |L_p(\Omega)\| + \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{sp+d}} dx\, dy \right)^{1/p} < \infty\,. \tag{12}$$

Often these spaces are denoted by $W^s_p(\Omega)$ and called Sobolev spaces of fractional order s on Ω. In a remarkable paper Zhou [68] proved the following. Let $d \ge 2$ and Ω a domain in $\mathbb{R}^d$. Then the following assertions are equivalent:

- Ω is a regular domain;
- Ω is a $B^s_{p,p}$-extension domain for all $s \in (0, 1)$ and all $p \in [1, \infty)$;
- Ω is a $B^s_{p,p}$-extension domain for some $s \in (0, 1)$ and some $p \in [1, \infty)$.

In addition Zhou was able to prove that a similar characterization takes place when the existence of a continuous extension operator is replaced by the validity and continuity of the standard Sobolev embeddings into Lebsgue spaces/Hölder spaces. We refer to [68] for more details.

4.2 Sufficient Conditions: Approximation by Piecewise Constant Functions

Now we turn to sufficient conditions. There are several ways to attack this problem. All methods are related to specific characterizations of Besov spaces. The first one is given by the characterization in terms of best approximation by piecewise constant functions.

Let us turn to Lemma 1 again. There we have already a sharp result. However, to make it more easy to deal with, we may use a further easy observation already employed in the proof of Lemma 4. Recall, the δ-neighbourhood ∂E^δ of ∂E has been defined in (7). We have

$$E(h) \cup F(h) \subset \partial E^\delta , \quad |h| = \delta ,$$

see (9). As a consequence, if $|h|^{-s} \, |\partial E^{|h|}|$ stays bounded in a neighborhood of 0 the function $\mathcal{X}_E$ belongs to $B^{s/p}_{p,\infty}(\mathbb{R}^d)$. For later reference we fix this. Concerning the definition of upper Minkowski content and upper Minkowski dimension (box counting dimension) we refer to the Appendix below.

Lemma 6 *Let* $E \subset \mathbb{R}^d$ *such that* $0 < |E| < \infty$. *Let* $1 \le p < \infty,\ 0 < s \le 1$ *and* $0 < a \le 1$.

(i) *If*

$$\sup_{0<\delta<a} \ \delta^{-s} \, |\partial E^\delta| < \infty ,$$

then $\mathcal{X}_E \in B^{s/p}_{p,\infty}(\mathbb{R}^d)$.

(ii) *If the* $d - s$*-dimensional upper Minkowski content of* ∂E*, denoted by* $\mathcal{M}^{*(d-s)}(\partial E)$*, is finite, then* $\mathcal{X}_E \in B^{s/p}_{p,\infty}(\mathbb{R}^d)$.

(iii) *If the upper Minkowski dimension* $\overline{\dim}_M \partial E = t$*, then* $\mathcal{X}_E \in B^{s'}_{p,\infty}(\mathbb{R}^d)$ *for all* $s' < \frac{d-t}{p}$.

Proof Part (i) follows directly from Lemma 1. In view of the definition of the Minkowski content part (ii) is just a reformulation of (i). Finally (iii) is a consequence of (A.1). □

Remark 4 We recall a result from Falconer [22, Prop. 9.6]. Let $\mathcal{S}$ be an m-tuple of contractions on a closed subset D of $\mathbb{R}^d$ such that

$$|S_i(x) - S_i(y)| \le r_i \, |x - y| , \qquad x, y \in D ,$$

where $r_i \in (0,1)$ for all $i = 1, \ldots, m$. Then the invariant set K satisfies $\dim_H K \le s$ and $\dim_M K \le s$, where s is the unique number for which

$$\sum_{j=1}^{N} r_i^s = 1\,.$$

Let E be a bounded domain with boundary $\partial E = K$. Hence, Lemma 6 yields $\mathscr{X}_E \in B^{s'}_{p,\infty}(\mathbb{R}^d)$ for all $s' < \frac{d-s}{p}$ and all $p \in [1,\infty)$.

But we can do a little bit better. For $f \in L_p(\mathbb{R}^d)$ we define

$$E_j(f)_p := \inf\Big\{ \| f - g \,|L_p(\mathbb{R}^d)\| : \quad g \in L_p(\mathbb{R}^d) \quad \text{and} \\ g \ \text{ is constant on the dyadic cubes } \ Q_{j,k}\,,\ k \in \mathbb{Z}^d \Big\}\,,\ j \in \mathbb{N}_0\,.$$

The number $E_j(f)_p$ expresses the minimal error in approximating f with first order splines (piecewise constant functions) with respect to the dyadic cubes $Q_{j,k},\ k \in \mathbb{Z}^d$. By assumption any approximant has the form

$$g = \sum_{k\in\mathbb{Z}^d} \alpha_{j,k}\, \mathscr{X}_{j,k}\,. \tag{13}$$

Here the $\alpha_{j,k}$ are appropriate real numbers and $\mathscr{X}_{j,k}$ denotes the characteristic function of the dyadic cube $Q_{j,k}$. There is a well-known characterization of Besov spaces in terms of these numbers $E_j(f)_p$. Let $1 \le p < \infty$, $1 \le q \le \infty$ and $0 < s < 1/p$. Then $f \in B^s_{p,q}(\mathbb{R}^d)$ if and only if $f \in L_p(\mathbb{R}^d)$ and

$$\Big(\sum_{j=0}^{\infty} \big[2^{js}\, E_j(f)_p \big]^q \Big)^{1/q} < \infty\,, \tag{14}$$

cf., e.g., Oswald [49]. Let E be a bounded domain in $\mathbb{R}^d$. Choosing the approximant g in (13) such that $\alpha_{j,k} = 1$ as long as $Q_{j,k} \subset E$ and $\alpha_{j,k} = 0$ otherwise, then it follows

$$\Big\| \mathscr{X}_E - \sum_{k\in\mathbb{Z}^d} \alpha_{j,k}\, \mathscr{X}_{j,k} \Big| L_p(\mathbb{R}^d) \Big\|^p = \sum_{k:\, |Q_{j,k}\cap\partial E|>0} \int_{Q_{j,k}} |\chi_E(x)|^p dx \\ \le \Big| \{x \in E : \ \mathrm{dist}\,(x, \partial E) \le \sqrt{d}\, 2^{-j}\} \Big|\,. \tag{15}$$

For a subset E of $\mathbb{R}^d$ and $\delta > 0$ we put

$$\partial E_+^{\delta} := \{x \in E : \ \mathrm{dist}\,(x, \partial E) \le \delta\}\,, \tag{16}$$

i.e., we concentrate on that part of the neigbourhood of the boundary which is part of E.

Theorem 4 *Let E be a bounded domain in $\mathbb{R}^d$. Let $1 \le p < \infty,\ 1 \le q \le \infty$ and $0 < s < 1/p$. Suppose*

$$\int_0^1 \delta^{-sq}\, |\partial E_+^\delta|^{q/p}\, \frac{d\delta}{\delta} < \infty \qquad \text{if} \quad q < \infty$$

$$\text{and} \qquad \sup_{0<\delta<1} \delta^{-s}\, |\partial E_+^\delta|^{1/p} < \infty \qquad \text{if} \quad q = \infty\, .$$

Then $\mathcal{X}_E \in B^s_{p,q}(\mathbb{R}^d)$ holds.

Proof The condition $|E| < \infty$ implies $\mathcal{X}_E \in L_p$. Let $q < \infty$. As a consequence of (15) and obvious monotonicity arguments we have

$$\begin{aligned}
\sum_{j=0}^\infty 2^{jsq}\, E_j(\mathcal{X}_E)_p^q &\le \sum_{j=0}^\infty 2^{jsq}\, |\partial E_+^{\sqrt{d}2^{-j}}|^{q/p} \\
&\le d^{s/2} \sum_{j=0}^\infty \int_{\sqrt{d}2^{-j-1}}^{\sqrt{d}2^{-j}} \delta^{-sq} \left|\partial E_+^{2\sqrt{d}\delta}\right|^{q/p} \frac{d\delta}{\delta} \\
&\le d^{s/2}\, (2\sqrt{d})^{sq} \int_0^{2d} t^{-sq}\, |\partial E_+^t|^{q/p} \frac{dt}{t}\, .
\end{aligned}$$

Since

$$\int_1^{2d} t^{-sq}\, |\partial E_+^t|^{q/p} \frac{dt}{t} \le C(s,q,d)\, |E|^{q/p},$$

the claim follows from (14). In case $q = \infty$ the needed modifications are obvious. □

Remark 5 As mentioned above, in case $q = \infty$ our sufficient condition is close to the property that $\mathcal{M}^{*(d-sp)}(\partial E) < \infty$. The usefulness of the (upper) Minkowski content in connection with the regularity of characteristic functions has been pointed out at several places, e.g. Strichartz [59] (but traced there to Madych), Jaffard and Meyer [35, Prop.2.1], Runst, S. [52, 2.3.1] and Sickel [57].

There is a further improvement possible. In our context it is quite easy to find the best approximation of $\mathcal{X}_E$. For $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$ we define

$$\alpha_{j,k} := \begin{cases} 1 & \text{if} \quad Q_{j,k} \subset E; \\ 1 & \text{if} \quad |Q_{j,k} \cap E| \ge |Q_{j,k}|/2; \\ 0 & \text{otherwise.} \end{cases}$$

It follows

$$\Big\| \mathcal{X}_E - \sum_{k\in\mathbb{Z}^d} \alpha_{j,k}\, \mathcal{X}_{j,k} \Big| L_p(\mathbb{R}^d)\Big\|$$
$$= \Big(\sum_{k:\, 0<|Q_{j,k}\cap E|<2^{-jd-1}} |Q_{j,k}\cap E| + \sum_{k:\, |Q_{j,k}\cap E|\geq 2^{-jd-1}} |Q_{j,k}\cap F|\Big)^{1/p}$$
$$= E_j(f)_p\,.$$

If we change the definition of the $\alpha_{j,k}$ for one cube $Q_{j,k}$, then it is easy to see that the error increases. This explains the last identity. Now we obtain an analog of Lemma 1.

Lemma 7 *Let E be a bounded nontrivial domain in $\mathbb{R}^d$. Let $1 \leq p < \infty,\ 1 \leq q \leq \infty$ and $0 < s < 1/p$. Then $\mathcal{X}_E$ belongs to $B^s_{p,q}(\mathbb{R}^d)$ if and only if*

$$\Big(\sum_{j=0}^{\infty} 2^{jsq}\Big[\sum_{k\in\mathbb{Z}^d} \min(|Q_{j,k}\cap E|, |Q_{j,k}\cap F|)\Big]^{q/p}\Big)^{1/q} < \infty$$

(standard modification for $q = \infty$).

Both, Lemmas 1 and 7 seem to have the disadvantage that they are not of great help with respect to the understanding of concrete examples.

4.3 Examples: I

First we continue our study of elementary domains.

Definition 3 Let $d \geq 2$. We define an elementary domain with Hölder continuous boundary of order $\alpha \in (0, 1]$ by replacing (v) in Definition 2 by

$$|\varphi(x') - \varphi(y')| \leq L\, |x' - y'|^\alpha\,, \quad x',\ y' \in W\,.$$

Lemma 8 *Let $d \geq 2$. Let $\alpha \in (0, 1)$. Let E be an elementary domain with Hölder continuous boundary of order α. Then $\mathcal{X}_E \in B^{\alpha/p}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1, \infty)$.*

Proof The proof is almost the same as in case of Lemma 4. We indicate the needed modifications only. By applying the same notation as there we have to change the estimate (8). This yields in case $\rho \leq \delta < 1$

$$|x_d - \varphi(x')| \leq |x_d - \varphi(y')| + L|x' - y'|^\alpha \leq \rho^\alpha (L + 1)\,.$$

Hence $\partial G^\delta \subset \Omega$, where

$$\Omega := \{(x', x_d) \in \mathbb{R}^d : x' \in W , \ \varphi(x') - (L+1)\delta^\alpha < x_d < \varphi(x') + (L+1)\delta^\alpha\} .$$

The remaining part of the boundary is regular, i.e., for the sets $E(h)$ and $F(h)$ we conclude

$$|E(h)| + |F(h)| \le |\partial E^\delta| \le C\,\delta^\alpha , \quad \delta = |h|$$

with a constant $C > 0$ independent on δ. Now the claim follows from Lemma 1. □

4.4 On the Dimension of Graphs of Functions and Consequences

There is a certain number of contributions in the literature where the problem of the Hausdorff or Minkowski dimension of a graph of a function is studied, we refer, e.g., to Carvalho and Caetano [16], Deliu and Jawerth [19], Falconer [22, Cor. 11.2], Hunt [33], Kamont and Wolnik [38], Kaplan et al. [39] and Triebel [63, Thm. 16.2].

In view of Lemma 6 any bound of the Minkowski dimension of the graph results in an estimate for the smoothness of the characteristic function of the associated domain. The most prominent example is the family of Weierstrass functions. Here we will have a short look onto the simplified version

$$f_\lambda(t) := \sum_{k=1}^\infty \lambda^{-\beta} \sin(\lambda^k t) , \qquad t \in \mathbb{R} , \quad 0 < \beta < 1 , \quad \lambda > 1 .$$

For more general Weierstrass functions we refer to Kaplan et al. [39] and Hunt [33]. Since f_λ represents a lacunary Fourier series, the regularity in periodic Besov spaces $B^s_{\infty,\infty}(\mathbb{T})$ is well understood. For the case $\lambda = 2$ one may consult [53, Chapt. 3], for the general case $\lambda \neq 2$ one has to apply in addition some arguments from Triebel [60, 2.2.1], replacing the dyadic resolution of unity by more general resolutions of unity (depending on λ). It follows $f_\lambda \in B^\beta_{\infty,\infty}(\mathbb{T})$ and this is just the periodic subspace of $C^\beta(\mathbb{R}) = B^\beta_{\infty,\infty}(\mathbb{R})$, see also [53, Chapt. 3], since $0 < \beta < 1$. Define $a_\lambda := \min_{t\in\mathbb{R}} f_\lambda(t)$ and

$$\Omega_\lambda := \{(x, y) : 0 < x < 2\pi , \ a_\lambda - \frac{1}{2} < y < f_\lambda(x)\} .$$

The Fig. 9 shows the graph of the function f_2 on $[0, 2\pi]$, i.e., below of the graph we see Ω_2.

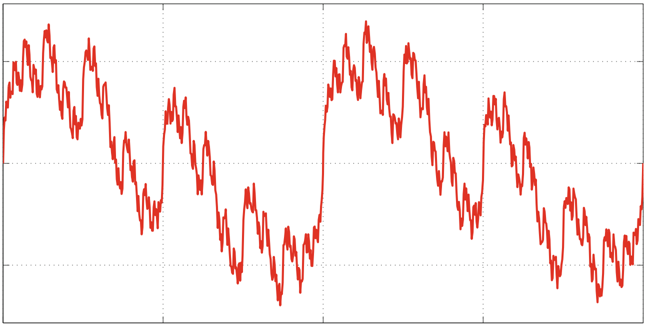


Fig. 9 A special Weierstrass function

Then

$$\mathcal{X}_{\Omega_\lambda} \in B^{\beta/p}_{p,\infty}(\mathbb{R}^d) \qquad \text{for all} \quad p \in [1,\infty)$$

follows. We refer also to Falconer [22, Ex. 11.3].

Let us mention that Triebel [63, proof of Thm. 16.2], [65] has constructed another example of a Hölder continuous function of order $\alpha \in (0,1)$ such that the characteristic function $\mathcal{X}_\Omega$ of the associated domain Ω satisfies

$$\mathcal{X}_\Omega \in B^{\alpha/p}_{p,\infty}(\mathbb{R}^d) \qquad \text{for all} \quad p \in [1,\infty)$$

and

$$\mathcal{X}_\Omega \notin B^{s}_{p,\infty}(\mathbb{R}^d) \qquad \text{for all} \quad s > \frac{\alpha}{p}\,.$$

We make a short summary. Hölder continuity of the boundary of order $\alpha \in (0,1]$ is a sufficient condition for regularity of order α/p but by no means necessary. Triebel's example shows that for the class C^α itself the result is unimprovable. However, also our examples from Figs. 5, 6, and 7 show, that Hölder regularity and Lipschitz regularity are not well adapted to our problem of determining the smoothness of $\mathcal{X}_E$.

There is one more general class of domains we would like to investigate.

4.5 Domains with a Boundary Being an h-Set

We follow Bricchi [8, 9], but see also [63–65] and [54].

Definition 4 Let $h : (0,1] \to (0,\infty)$ be a positive non-decreasing function such that there exists a positive constant c with

$$\frac{h(2^{-j-k})}{h(2^{-j})} \ge c\, 2^{-kd} \qquad \text{for all} \quad j,k \in \mathbb{N}_0\,. \tag{17}$$

Let Γ be a non-empty compact set in $\mathbb{R}^d$. Then Γ is called an h-set if there exists a finite Radon measure μ in $\mathbb{R}^d$ satisfying

$$\operatorname{supp}\mu = \Gamma \qquad \text{and} \qquad \mu(B(y,r)) \asymp h(r)\,, \quad y \in \Gamma\,, \quad 0 < r < 1\,. \tag{18}$$

Observe that for any such function h there exists at least one such set Γ (for an explicit construction we refer to [8]). We recall a few more properties of h-sets. Again our references are [8, 9].

Lemma 9 *Let Γ be an h-set. Then the following assertions are true.*

(i) *All h-measures related to Γ are equivalent to the generalized Hausdorff measure $\mathcal{H}^h$ restricted to Γ (see the Appendix below for a definition).*
(ii) *The related Radon measure μ is a doubling measure, i.e., there exists a constant $c > 0$ such that*

$$\mu(B(y,2r)) \le c\,\mu(B(y,r)) \qquad \textit{for all} \quad y \in \Gamma \quad \textit{and all} \quad 0 < r < 1\,.$$

(iii) *For any $t \in (0,1]$ and any $y \in \Gamma$ one has*

$$\dim_H \Gamma \cap B(y,t) = \liminf_{r\to 0} \frac{\log h(r)}{\log r} \qquad \textit{and} \qquad \dim_P \Gamma \cap B(y,t) = \limsup_{r\to 0} \frac{\log h(r)}{\log r}$$

There is a list of examples in [9]. All these functions are defined on a small intervall $(0,a)$, $0 < a < 1$, and then suitably prolonged on the whole $(0,1]$.

- $h_1(r) = r^\delta \quad 0 \le \delta \le d$;
- $h_2(r) = r^\delta\, |\log r|^b\,, \quad 0 < \delta < d,\ b \in \mathbb{R}$;
- $h_3(r) = |\log r|^b\,, \quad b < 0$;
- $h_4(r) = r^d\, |\log r|^b\,, \quad b > 0$;
- $h_5(r) = r^\delta \exp(b\,|\log r|^\kappa)\,, \quad 0 < \delta < d,\ b \in \mathbb{R}, 0 < \kappa < 1$;
- $h_6(r) = r^\delta\, S(r)$, where S is a slowly varying function.

Here a slowly varying function $S : (0,1] \to \mathbb{R}$ is a positive measurable function such that $\lim_{r\to 0} S(\lambda r)/S(r) = 1$ for all $\lambda \in (0,1]$.

The most important special case is the first one. The compact sets Γ related to h_1 are called δ-sets (in most of the cases the letter d is used instead of δ, but d has already a different meaning). δ-sets are discussed at various places, sometimes they are also called *regular* or *Ahlfors regular* sets, see, e.g., Bechtel and Egert [2], Frazer [27], Jonsson and Wallin [37], Schneider and Vybíral [54] or Triebel [63, 65, 66].

The main step to understand domains E with ∂E being an h-set is made with the following lemma, see Bricchi [8].

Lemma 10 *Let E be a bounded domain in $\mathbb{R}^d$ with boundary ∂E being an h-set. Then there exists a constant $c > 0$ such that*

$$|\partial E^r_+| \le c \, \frac{r^d}{h(r)}\,, \qquad 0 < r < 1\,.$$

Proof The proof is based on the fact that the finite Radon measure controls the thickness of ∂E^r for r sufficiently small.

The starting point is the Whitney decomposition of E, see [58]. More exactly, let $\mathscr{F}$ denote the collection of all dyadic cubes representing the Whitney decomposition of E , i.e.,

$$E = \bigcup_{j=0}^{\infty} \bigcup_{\ell=0}^{M_j} Q_{j,\ell(j)}\,, \tag{19}$$

all the cubes $Q_{j,\ell(j)}$ are pairwise disjoint and

$$\sqrt{d}\, 2^{-j} \le \mathrm{dist}\,(Q_{j,\ell(j)}, \partial E) \le \sqrt{d}\, 2^{-j+2}\,.$$

We shall need an estimate of the numbers M_j. Let

$$E_j := \{x \in E:\ \sqrt{d}\, 2^{-j-1} \le \mathrm{dist}\,(x, \partial E) \le 4\sqrt{d}\, 2^{-j+1}\}\,, \qquad j \in \mathbb{N}\,.$$

By $\mathscr{F}_j$ we denote the collection of all $\ell \in \mathbb{Z}^d$ such that the dyadic cube $Q_{j,\ell} \in \mathscr{F}$ is contained in E_j. Then, if $k \in \mathscr{F}_j$, the cube $3\sqrt{d}\, Q_{j,k}$ intersects Γ. Furthermore, there exists a point $y^k \in \Gamma$ such that the cube P_k, side-length $\sqrt{d}2^{-j}$, sides parallel to the axes and with center in y^k, is contained in $3\sqrt{d}\, Q_{j,k}$. Let us denote the centre of $Q_{j,k}$ by x^k. Then, by definition, x^k is the centre of $3\sqrt{d}\, Q_{j,k}$ as well and $|x^k - x^\ell| \ge 2^{-j}$, $k \neq \ell$. Hence, every $y \in \Gamma$ is contained in at most $C = C(d)$ (independent of j) cubes $3\sqrt{d}\, Q_{j,k}$ with $k \in \mathscr{F}_j$. Let μ be the associated finite Radon measure on Γ. By assumption on μ it follows

$$\infty > C\,\mu(\Gamma) \ge \sum_{k\in\mathscr{F}_j} \mu\Big(3\sqrt{d}\, Q_{j,k} \cap \Gamma\Big) \ge \sum_{k\in\mathscr{F}_j} \mu(P_k \cap \Gamma) \asymp |\mathscr{F}_j|\, h(2^{-j})\,.$$

Here $|\mathscr{F}_j|$ denotes the cardinality of $\mathscr{F}_j$. Hence

$$\sup_{j=0,1,\ldots} M_j\, h(2^{-j}) \le C\,\mu(\Gamma)\,. \tag{20}$$

This inequality is the key step in the proof. The inequality (20) can be turned immediately into an estimate of the Lebesgue measure of the sets ∂E^r_+. For a moment we put $r := \sqrt{d}2^{-j-1}$, $j \in \mathbb{N}$. Then

$$\partial E^r_+ \subset \bigcup_{\ell \in \mathscr{F}_j} 3\sqrt{d}\, Q_{j,\ell}$$

and therefore

$$|\partial E^r_+| \le \frac{C}{h(2^{-j})} \Big(3\sqrt{d}\, 2^{-j}\Big)^d \le c\, \frac{r^d}{h(r)}\,,$$

where c is a positive constant independent of j. In the last step we used the monotonicity of h and the doubling property, see Lemma 9. □

In view of Theorem 4 the Lemma 10 implies the following.

Corollary 2 *Let E be a bounded domain in $\mathbb{R}^d$ with boundary ∂E being an h-set.*

(i) *Let $1 \le p < \infty$ and $0 < s < 1/p$. Then the characteristic function $\mathscr{X}_E$ belongs to $B^s_{p,\infty}(\mathbb{R}^d)$ if*

$$\sup_{0<r<1} \frac{r^{d-sp}}{h(r)} < \infty\,.$$

(ii) *Let $1 \le p, q < \infty$ and $0 < s < 1/p$. Then the characteristic function $\mathscr{X}_E$ belongs to $B^s_{p,q}(\mathbb{R}^d)$ if*

$$\int_0^1 r^{(\frac{d}{p}-s-\frac{1}{q})q}\, h(r)^{-\frac{q}{p}}\, dr < \infty\,.$$

As an immediate consequence we get the following.

Corollary 3 *Let E be a bounded domain in $\mathbb{R}^d$ with boundary ∂E being an δ-set for some $d-1 < \delta < d$. Let $1 \le p < \infty$. Then we have $\mathscr{X}_E \in B^{\frac{d-\delta}{p}}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1,\infty)$.*

Remark 6

(i) Corollary 3 originates from Triebel [65, Thm. 3, Rem. 9] and Schneider, Vybíral [54]. The proofs in [65] and [54] are partly different. They are based on the characterization of Besov spaces by atoms.
(ii) Also Triebel [65] and Schneider, Vybíral [54] have dealt with h-sets. However, for more general sets than δ-sets the sufficient condition

$$I_{s,p,q} := \sup_{j\in\mathbb{N}_0} \bigg(\sum_{k=0}^{\infty} 2^{ksq}\Big(\frac{h(2^{-j})}{h(2^{-j-k})}\, 2^{-kd}\Big)^{q/p}\bigg)^{1/q} < \infty$$

for $\mathcal{X}_E$, to belong to $B^s_{p,q}(\mathbb{R}^d)$, given in the quoted papers, is in general stronger than that one from Corollary 2. It is not difficult to see that

$$\Big(\int_0^1 r^{(\frac{d}{p}-s-\frac{1}{q})q}\, h(r)^{-\frac{q}{p}}\, dr\Big)^{1/q} \le c_h\, I_{s,p,q}$$

always holds with some constant c_h, depending on h. As an example for the non-equivalence of these quantities may serve $h_2(r) := r^\delta\, |\log r|^b,\ 0 < r < 1$. Let E denote a bounded domain with boundary being an h-set with respect to h_2. In case $d - 1 < \delta < d$ and $b > 0$ Corollary 2 yields $\mathcal{X}_E \in B^{\frac{d-\delta}{p}}_{p,q}(\mathbb{R}^d)$ if $b > p/q$. But $I_{s,p,q} = \infty$, $s = \frac{d-\delta}{p}$ for all $q < \infty$. However, let us mention that Triebel, Schneider and Vybíral showed that $I_{s,p,q} < \infty$ implies $\mathcal{X}_E \in B^{\frac{d-\delta}{p}}_{p,q,selfs}(\mathbb{R}^d)$, a smaller space than the corresponding Besov space. The classes $B^s_{p,q,selfs}(\mathbb{R}^d)$ are of some relevance in connection with pointwise multipliers of Besov spaces.

Particular examples of δ-sets are self-similar sets, see the Appendix.

Corollary 4 *Let K be a bounded domain in $\mathbb{R}^d$ with boundary ∂K being a self-similar set satisfying the assumptions in Proposition 5 with $s = \delta$, see the Appendix. Let $1 \le p < \infty$. Then we have $\mathcal{X}_K \in B^{\frac{d-\delta}{p}}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1,\infty)$.*

Now we turn to the next concrete example.

4.6 The Twindragon

The twindragon is a space filling curve with a fractal boundary, see Fig. 10. More information, also about relatives (heighway dragon, Levy dragon) of this curve, may be found in Wikipedia, see https://en.wikipedia.org/wiki/Dragon-curve.
Let $T \subset \mathbb{R}^2$ denote the set which is filled by this curve. It is known that ∂T is a self-similar set, which satisfies the assumptions of Proposition 5 in the Appendix below. It holds that $\dim_H \partial T = \dim_M \partial T = \delta$, where δ is the unique solution of

$$\Big(\frac{1}{\sqrt{2}}\Big)^\delta + 2\Big(\frac{1}{2\sqrt{2}}\Big)^\delta = 1\,,$$

given by

$$\delta := \log_2\left(\frac{1 + \sqrt[3]{73 - 6\sqrt{87}} + \sqrt[3]{73 + 6\sqrt{87}}}{3}\right) \sim 1.5236\,,$$

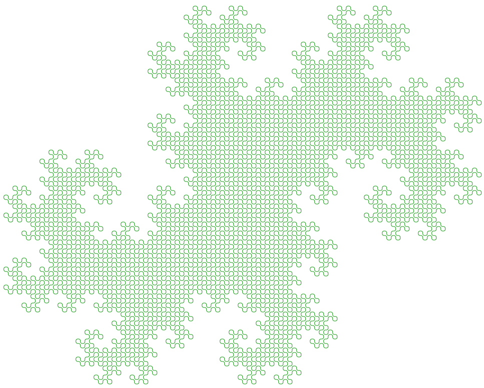


Fig. 10 The twindragon

see Mandelbrot [42, p. 78]. Most important for us is the existence of a finite Radon measure on ∂T, which turns ∂T into a δ-set. Here we may take the Hausdorff measure $\mathscr{H}^\delta$ restricted to ∂T. Hence, as a consequence of Corollary 3 we conclude

$$\mathcal{X}_T \in B^{\frac{2-\delta}{p}}_{p,\infty}(\mathbb{R}^2) \qquad \text{for all} \quad p \in [1,\infty).$$

Let us mention that we do not know whether this number δ is optimal. In particular, we do not know whether Theorem 3 is applicable. If that would be the case, we could conclude that this number δ is best possible.

There are further interesting properties of $\mathcal{X}_T$, in particular of interest in the theory of wavelets. It can be used as a scaling function, we refer to Gröchenig, Madych [29] and Wojtaszczyk [67, 5.3]. It is not difficult to see that the associated wavelets have the same regularity as $\mathcal{X}_T$ has.

4.7 Some Sufficient Conditions: Quasiballs

An essential step forward has been done by Faraco and Rogers [25]. These authors worked with quasiballs.

A homeomorphism $f : \mathbb{R}^d \to \mathbb{R}^d$ is called *K-quasiconformal* if there is a constant $K < \infty$ such that for all $x \in \mathbb{R}^d$

$$K(x) := \limsup_{\varepsilon\to 0} \frac{\max_{a:\,|x-a|=\varepsilon} |f(x)-f(a)|}{\min_{b:\,|x-b|=\varepsilon} |f(x)-f(b)|} \le K\,.$$

A K-quasiball is the image of the unit ball under a K-quasiconformal mapping. For $d=2$ also the name quasicircle is commonly used.

Theorem 5 ([25, Thm. 1.3]) *Let* $1 \le p < \infty$, $0 < s < 1$ *and let* $E \subset \mathbb{R}^d$ *be a* K*-quasiball. Then*

$$\| \mathcal{X}_E \,|B^s_{p,p}(\mathbb{R}^d)\| \asymp \Big(|E| + \int_0^{\delta^*} \delta^{-ps}\, |\partial E^\delta|\, \frac{d\delta}{\delta}\Big)^{1/p},$$

where $\delta^* := \inf\{\delta :\ E \subset \partial E^\delta\}$.

The proof is not short enough to be included into this survey. The more interesting part in Theorem 5 is the estimate of $\| \mathcal{X}_E \,|B^s_{p,p}(\mathbb{R}^d)\|$ from below, because this part is missing in Theorem 4. In general there is some gap between the sufficient conditions in Theorem 4 and the necessary condition in Theorem 3. However, in case of certain domains with a fractal boundary they almost touch. For later use we formulate a simple consequence, already known to [25].

Corollary 5 *Let* $1 \le p < \infty$, $0 < s < 1$ *and let* $E \subset \mathbb{R}^d$ *be a* K*-quasiball. If we assume* $\mathcal{X}_E \in B^s_{p,p}(\mathbb{R}^d)$, *then*

$$\lim_{\delta \to 0} \delta^{-s}\, |\partial E^\delta|^{1/p} = 0$$

follows.

Proof The mapping $\delta \mapsto |\partial E^\delta|$ is monotone in δ. Hence

$$\int_0^{\delta^*} \delta^{-ps}\, |\partial E^\delta|\, \frac{d\delta}{\delta} \asymp \sum_{k=k_0}^{\infty} 2^{kps}\, |\partial E^\delta|\,,$$

where k_0 has to be chosen in dependence of δ^*. This yields the claim. □

Remark 7 A reformulation of Corollary 5 (just by definition) reads as follows. Under the given restrictions we obtain $\mathcal{M}^{*d-s}(\partial E) = 0$.

The most beautiful example we discuss next.

4.8 The Snowflake Domain

The standard construction of the von Koch curve is as follows, see Fig. 11. We start with an equilateral triangle. Then we subdivide each side into three equal parts and remove the middle one. This middle part is replaced by an equilateral triangle again.

Sidelength is now 1/3 of the original one. This procedure is iterated. After a few further iterations one obtains Fig. 12 which might be seen as a reasonable approximation of the von Koch curve. The domain Ω with the *von Koch curve* as its boundary is called the *snowflake domain*.

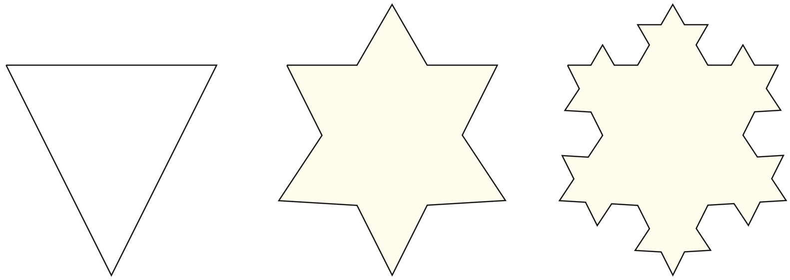


Fig. 11 The first three steps of the construction of the von Koch curve

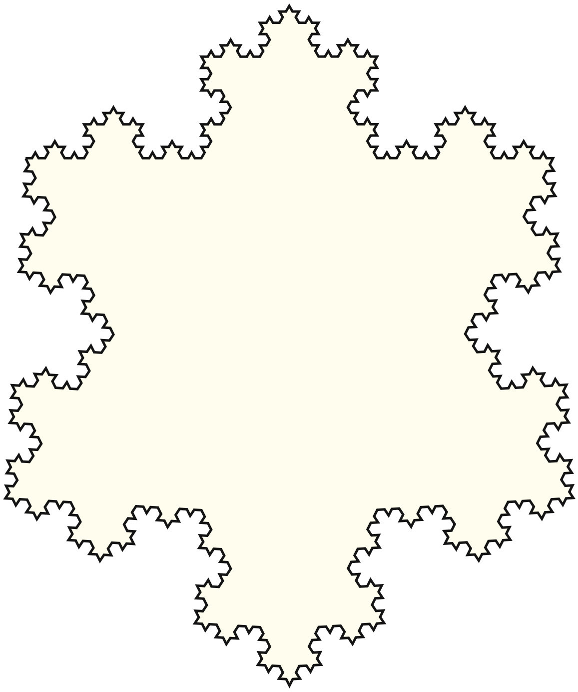


Fig. 12 The Snowflake domain

We collect a few facts about its properties.

(i) Ω is a (ε, ∞) domain, see [36];
(ii) Ω is a John domain, see [10];
(iii) Ω is a quasiball, see [47, 1.2];
(iv) Ω is a selfsimilar set, which fulfils the conditions in Proposition 5 in the Appendix, see [22, 9.2] and [44, p. 67];

(v) $\partial\Omega_+ = \partial\Omega$, see (i);
(vi) $\dim_H \partial\Omega = \dim_M \partial\Omega = \log 4/\log 3$, see, e.g., Falconer [22, Ex.9.5]);
(vii) $\partial\Omega$ is a δ-set with $\delta = \log 4/\log 3$, see Proposition 5 in the Appendix.
(viii) $0 < \mathscr{H}^{\log 4/\log 3}(\partial\Omega) \le \mathscr{M}_*^{\log 4/\log 3}(\partial\Omega) \le \mathscr{M}^{*\,\log 4/\log 3}(\partial\Omega)$,
see Proposition 5 in the Appendix and (A.2).

As a combination of Theorem 5, Corollary 5 and property (viii) we obtain now the following.

Corollary 6 ([25, Cor. 1.4]) *Let $1 \le p < \infty$. The characteristic function $\mathscr{X}_\Omega$ of the snowflake domain belongs to $B^s_{p,p}(\mathbb{R}^2)$ if and only if $s < \frac{1}{p}\left(2 - \frac{\log 4}{\log 3}\right)$.*

This result has a counterpart for $q = \infty$.

Theorem 6 *Let $1 \le p < \infty$. The characteristic function $\mathscr{X}_\Omega$ of the snowflake domain Ω belongs to $B^s_{p,\infty}(\mathbb{R}^2)$ if and only if $s \le \big(2 - \log 4/\log 3\big)/p$.*

Proof Sufficiency follows from Proposition 5, see the Appendix, and Corollary 3. If we assume that $\mathscr{X}_\Omega \in B^t_{p,\infty}(\mathbb{R}^2)$ for some $t > \frac{2-s}{p}$ then by the elementary embeddings of the Besov spaces in (1) it follows $\mathscr{X}_\Omega \in B^{(2-s)/p}_{p,p}(\mathbb{R}^2)$. But this contradicts Corollary 6. □

The author conjectures that, for fixed $p \in [1,\infty)$, the smallest Besov space containing $\mathscr{X}_\Omega$ is given by $B^s_{p,\infty}(\mathbb{R}^2)$ with $s := \frac{2-\log 4/\log 3}{p}$.

4.9 The Rotating Snowflake

The Fig. 13 below is obtained by first shifting an approximation of the snowflake domain Ω in the (x, y)-plane to the right such that it will be located to the right of $x = 1$. Afterwards this shifted domain is rotated around the y-axes. In the limit the outcome in $\mathbb{R}^3$ is denoted by Ω_{rot}. What we have in mind is a spiked car tyre.

Lemma 11 *Let $1 \le p < \infty$. The characteristic function $\mathscr{X}_{\Omega_{\mathrm{rot}}}$ of the rotating snowflake domain belongs to $B^s_{p,\infty}(\mathbb{R}^3)$ if $s \le \frac{1}{p}\left(2 - \frac{\log 4}{\log 3}\right)$.*

Proof Lemma 10 yields

$$|\Omega^r| \le c\, r^{2-s}\,, \qquad r \in (0,1)\,, \qquad s = \frac{\log 4}{\log 3}\,.$$

Hence, because of

$$|\Omega^r_{\mathrm{rot}}| \asymp |\Omega^r|\,, \qquad 0 < r < 1\,,$$

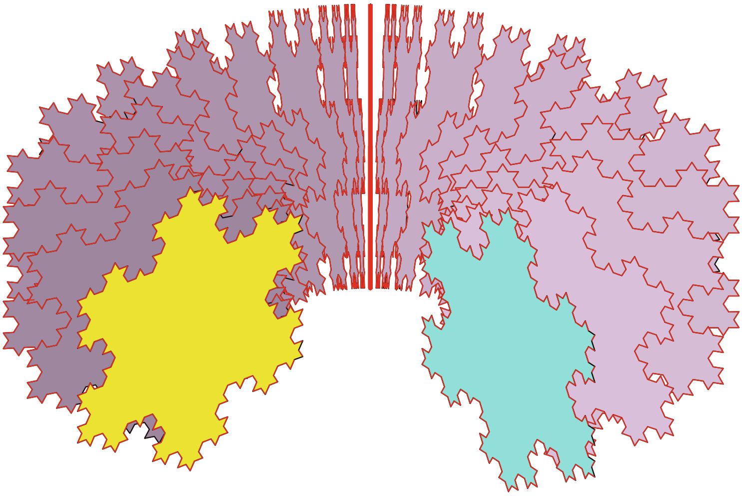


Fig. 13 The rotated Snow flake domain

we get the same inequality for $|\Omega^r_{\mathrm{rot}}|$. Lemma 6(i) can be used to complete the argument. □

4.10 Some Sufficient Conditions: The Aikawa Dimension

This time we shall work with a sufficient condition related to the Aikawa dimension of the boundary of a domain. In [1] Aikawa introduced the following definition of a fractal dimension (for simplicity we concentrate on the situation in $\mathbb{R}^d$ and the Lebesgue measure).

Definition 5 Let A be a subset of $\mathbb{R}^d$ and let $G(A)$ be the set of those t > 0 for which there exists a constant c_t such that

$$\int_{B(x,r)} \mathrm{dist}\,(y,A)^{t-d}\, dy \le c_t\, r^{t-d} \qquad \text{for all} \quad x \in A \quad \text{and all} \quad r \in (0, \mathrm{diam}\,(A))\,.$$

Then the Aikawa dimension of A is defined to be $\dim_{\mathcal{AJ}} A = \inf G(A)$.

Our point of departure is Lemma 1. Let $p = 1,\ 0 < s < 1$ and $|h| < a < 1$. First, observe that

$$E(h) = E^a(h) = \{x \in E:\ \mathrm{dist}\,(x, \partial E) < a\,,\ x + h \not\in E\}$$

and similarly for $F(h) = F^a(h)$. Furthermore

$$|h|^{-s} \int_{E^a(h)} dx \le \int_{E^a} \operatorname{dist}(x, \partial E)^{-s}\, dx\,.$$

This is almost all what is needed to prove the following supplement to Lemma 1.

Theorem 7 *Let $1 \le p < \infty$ and $0 < s < 1$. Let E be a bounded domain.*

(i) *If*

$$\sup_{x \in \partial E} \int_{B(x,1)} \operatorname{dist}(y, \partial E)^{-s}\, dy < \infty\,,$$

then $\mathcal{X}_E$ belongs to $B^{s/p}_{p,\infty}(\mathbb{R}^d)$.

(ii) *If $\dim_{\mathcal{A}\mathcal{J}} \partial E = t$, then $\mathcal{X}_E$ belongs to $B^{s'}_{p,\infty}(\mathbb{R}^d)$ for all $s' < \frac{d-t}{p}$.*

Proof The sets E^a and F^a can be covered by finitely many balls $B(x^k, 1)$, $x^k \in \partial E$, since E is bounded. Hence

$$\int_{E^a} \operatorname{dist}(x, \partial E)^{-s}\, dx + \int_{F^a} \operatorname{dist}(x, \partial E)^{-s}\, dx$$

is finite if

$$\int_{B(x^k,1)} \operatorname{dist}(x, \partial E)^{-s}\, dx < \infty$$

for all k. This proves (i). On the other hand part (ii) is an obvious consequence of (i). □

Remark 8

(i) For deciding about membership of $\mathcal{X}_E$ in a Besov space we do not need the full power of the Aikawa dimension since we only work with balls of radius 1. This will be different when we switch to the question whether $\mathcal{X}_E$ is a pointwise multiplier for a Besov space. For more details we refer to Frazier and Jawerth [28], Bechtel and Egert [2] and [56, 57].

(ii) It is interesting to notice that on $\mathbb{R}^d$ the probably more popular Assouad dimension $\dim_A$ and the Aikawa dimension coincide. We refer to Lehrbäck and Tuominen [41] and Fraser [27] for more details.

(iii) On $\mathbb{R}^d$ we have the following chain of inequalities

$$\dim_H \partial E \le \underline{\dim}_M \partial E \le \overline{\dim}_M \partial E \le \dim_A \partial E = \dim_{\mathcal{A}\mathcal{J}} \partial E\,.$$

Let E be a bounded domain with the boundary being a δ-set for some $d-1<\delta<d$. Then $\dim_A \partial E = \dim_M \partial E = \dim_H \partial E = \delta$. We refer to Frazer [27], see also [2].

Mainly Besov [5], but see also [7, 2.8], has worked with domains satisfying a flexible horn condition.

Definition 6 The domain Ω satisfies a flexible horn condition if there exist $\delta_0 > 0$ and $T > 0$ such that for any $x \in \Omega$ there exist an arc

$$\gamma(t,x) := (\gamma_1(t,x), \ldots, \gamma_d(t,x)), \qquad 0 \le t \le T,$$

with the following properties.

(i) For all $i \in \{1, \ldots, d\}$ the functions $\gamma_i(t,x)$ are absolutely continuous with respect to t and $|\gamma_i(u,x)| \le 1$ for almost all $u \in [0,T]$.
(ii) $\gamma(0,x) = 0$ and $x + \bigcup_{0 \le t \le T} \Big(\gamma(t,x) + t\delta_0[-1,1]^d\Big) \subset \Omega$.

This is quite close to the definition of a John domain.

Lemma 12 *Let* $1 \le p < \infty$.

(i) *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain which satisfies a flexible horn condition with parameters* δ_0 *and* T. *Then there exists a positive number* $s \le 1$ *such that* $\mathcal{X}_\Omega$ *belongs to* $B^{s/p}_{p,\infty}(\mathbb{R}^d)$.
(ii) *Let* $\Omega \subset \mathbb{R}^d$ *be a John domain. Then there exists a positive number* $s \le 1$ *such that* $\mathcal{X}_\Omega \in B^{s/p}_{p,\infty}(\mathbb{R}^d)$.

Proof In both cases it is known that the Aikawa dimension of the boundary $\partial\Omega$ is positive. In case (i) this is proved in Besov [5]. For John domains we refer to Hajlasz and Koskela [32]. □

4.11 The Distance Zeta Function of a Set

Let A be a bounded subset of $\mathbb{R}^d$. In the recent book [40] Lapidus, Radunović and Žubrinić studied the function

$$\zeta_A(s) := \int_{A^\delta} \mathrm{dist}\,(x,A)^{s-d}\,dx, \qquad s \in \mathbb{C},$$

where A^δ denotes the δ-neigbourhood of A. The chosen fixed $\delta > 0$ is of no importance in their context. They call ζ_A the *distance zeta function* of A. For us of interest are Lemmas 2.1.3 and 2.1.6 in [40]. They read as follows.

Proposition 3 *Let A be an arbitrary subset of* $\mathbb{R}^d$ *and let* δ *be an arbitrary positive number.*

(i) *If* $\sigma > d - \overline{\dim}_M A$, *then* $\int_{A^\delta} \operatorname{dist}(x, A)^{-\sigma}\, dx = +\infty$.
(ii) *If* $-\infty < \sigma < d - \overline{\dim}_M A$, *then* $\int_{A^\delta} \operatorname{dist}(x, A)^{-\sigma}\, dx < \infty$.

Consequently, if $0 < s < d - \overline{\dim}_M \partial E$, then in view of Theorem 7(i) we obtain $\mathcal{X}_\Omega \in B^{s/p}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1, \infty)$, which is just a different proof of Lemma 6(iii). Part (i) of Proposition 3 illustrates that on this way we can not improve our conclusion.

4.12 Some Further Examples

When looking at the two examples of the twindragon and the snowflake domain one could conjecture that the following formula holds:

$$\mathcal{X}_E \in B^s_{p,\infty}(\mathbb{R}^d) \qquad \text{and} \qquad s = \frac{1}{p}\Big(d - \dim_M \partial E\Big) = \frac{1}{p}\Big(d - \dim_H \partial E\Big).$$

In what follows we shall investigate a two-parameter family $E_{\alpha,\gamma}$ of domains in the plane, see Fig. 14, with a quite different behaviour. These domains are related to the shark-domain on the cover of the monograph of Maz'ya [45] (and on the cover of its Russian edition). In a certain sense the domains under consideration are also limit cases of the classical Nikodym domains, cf. [45, 1.1.4].

Let $\gamma \ge \alpha > 1$. Then we define

$$\beta_j := \sum_{\ell=1}^{j} \ell^{-\alpha}, \qquad \beta := \sum_{\ell=1}^{\infty} \ell^{-\alpha} \quad \text{and} \quad \delta_j := \frac{1}{4(2j+2)^\gamma}, \qquad j \in \mathbb{N}.$$

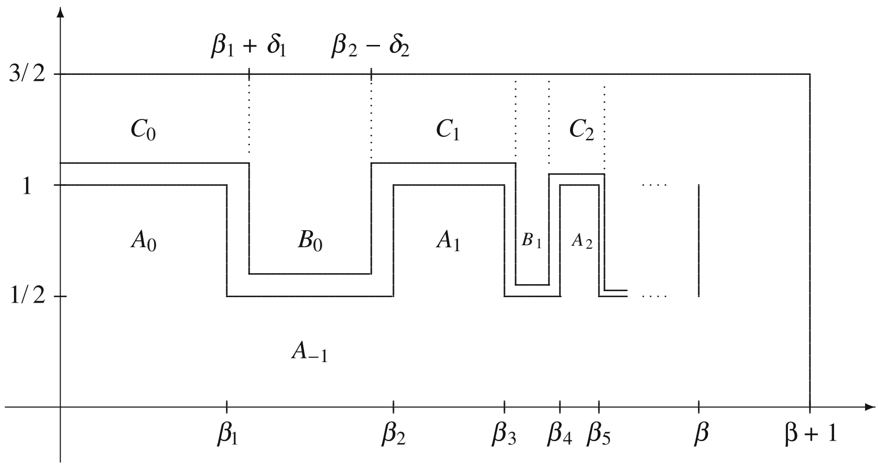


Fig. 14 A (modified) Nikody domain

Further we put

$$A_{-1} := \Big\{(x,y):\ 0 < x < \beta\,, \quad 0 < y < \frac{1}{2}\Big\}\,,$$

$$A_0 := \Big\{(x,y):\ 0 < x < 1\,, \quad \frac{1}{2} \le y < 1\Big\}\,,$$

$$A_j := \Big\{(x,y):\ \beta_{2j} < x < \beta_{2j+1}\,, \quad \frac{1}{2} \le y < 1\Big\}\,, \quad j = 1, 2, \ldots\,,$$

$$B_j := \Big\{(x,y): \beta_{2j+1} + \delta_j < x < \beta_{2j+2} - \delta_j,\ \frac{1}{2} + \delta_j < y < \frac{3}{2}\Big\},\ j = 0, 1, \ldots,$$

$$C_0 := \Big\{(x,y):\ 0 < x < \beta_1 + \delta_0\,, \quad \frac{1}{2} + \delta_0 < y < \frac{3}{2}\Big\}\,,$$

$$C_j := \Big\{(x,y): \beta_{2j} - \delta_{j-1} \le x < \beta_{2j+1} + \delta_j,\ \frac{1}{2} + \delta_j < y < \frac{3}{2}\Big\},\ j = 1, 2, \ldots,$$

$$D := \Big\{(x,y):\ \beta < x < \beta + 1\,, \quad 0 < y < \frac{3}{2}\Big\} \cup \Big\{(\beta, y):\ 0 < y < \frac{1}{2} \text{ or } 1 < y < \frac{3}{2}\Big\}\,,$$

and

$$E_{\alpha,\gamma} := \left(\bigcup_{j=-1}^{\infty} A_j\right) \cup \left(\bigcup_{j=0}^{\infty} B_j\right) \cup \left(\bigcup_{j=0}^{\infty} C_j\right) \cup D\,.$$

What we have in mind are two combs where the teeth come closer and closer together. Just by looking at the neigbourhood of the line $\{(\beta, y):\ 1/2 < y < 1\}$ it is clear that $E_{\alpha,\gamma}$ is neither an (ε, δ)-domain nor an John domain nor a domain satisfying a flexible horn condition in the sense of Besov. They do not belong to the regular domains as well.

Proposition 4 *Let $1 \le p < \infty$ and $\gamma \ge \alpha > 1$. Then the sets $E_{\alpha,\gamma}$ have the following properties.*

(i) $\dim_M(\partial E_{\alpha,\gamma}) = 1 + 1/\alpha$.
(ii) $\dim_H(\partial E_{\alpha,\gamma}) = \dim_P(\partial E_{\alpha,\gamma}) = 1$.
(iii) $\chi_{E_{\alpha,\gamma}} \in B^s_{p,\infty}(\mathbb{R}^2)$ *if and only if* $s\,p \le (1 - 1/\gamma)$.
(iv) Let $1 \le q < \infty$. *Then* $\chi_{E_{\alpha,\gamma}} \in B^s_{p,q}(\mathbb{R}^2)$ *if and only if* $s\,p < (1 - 1/\gamma)$.

The rather technical proofs can be found in [56]. Let $\gamma > \alpha$. Obviously we have

$$\frac{1}{p}\Big(d - \dim_M \partial E_{\alpha,\gamma}\Big) = \frac{1}{p}\Big(1 - \frac{1}{\alpha}\Big) < \frac{1}{p}\Big(1 - \frac{1}{\gamma}\Big) < \frac{1}{p} = \frac{1}{p}\Big(d - \dim_H \partial E_{\alpha,\gamma}\Big)\,.$$

Clearly, in case of these domains neither the Hausdorff dimension nor the Minkowski dimension characterize the smoothness s of the characteristic function. Furthermore, from our knowledge on this family $E_{\alpha,\gamma}$ we can derive the following conclusions.

- Let $s \in (0,1)$, $p \in [1,\infty)$ and $q \in [1,\infty]$ be fixed. Then there exists a set $E \subset \mathbb{R}^2$, $0 < |E| < \infty$, such that $\mathcal{X}_E \notin B^s_{p,q}(\mathbb{R}^2)$.
- Let $s \in (0,1)$ be fixed. Then for any $s' \in (s,1)$ there exists a set $E \subset \mathbb{R}^2$, $0 < |E| < \infty$, such that the Minkowski dimension of ∂E equals $2-s$ and

$$\mathcal{X}_E \in B^{s'/p}_{p,q}(\mathbb{R}^2) \qquad \text{for all} \quad p \in [1,\infty) \quad \text{and} \quad q \in [1,\infty].$$

  Hence, our sufficient conditions given in Lemma 6 and Theorem 4 are not sharp in general.
- Let $p \in [1,\infty)$ and $q \in [1,\infty]$ be fixed. Then for any $s \in (0,1]$ there exists a set $E \subset \mathbb{R}^2$, $0 < |E| < \infty$, such that the Hausdorff and the packing dimension of ∂E equals 1 and

$$\mathcal{X}_E \notin B^{s/p}_{p,q}(\mathbb{R}^2)\,.$$

Summarizing one observes that in general the Hausdorff dimension and the packing dimension of ∂E are too small to characterize the smoothness of $\mathcal{X}_E$. On the other hand the Minkowski dimension of ∂E is oversized for a characterization of the smoothness of $\mathcal{X}_E$ in many cases.

4.13 The Mandelbrot Set

We finish this subsection with one well-known extreme example, the famous Mandelbrot set, see Fig. 15. This set, denoted by D, satisfies $\dim_H D = 2$ and $\dim_H \partial D = 2$, see Shishikura [55]. Obviously this implies $\dim_M \partial D = 2$. Hence, in view of Lemma 6, we do not expect any positive smoothness of $\mathcal{X}_D$.

References with respect to the Mandelbrot set are, e.g., [42] and [22, 14.2].

Concerning the smoothness of $\mathcal{X}_D$ there is at least a chance that it belongs to some Besov spaces $B^{0,b}_{p,\infty}(\mathbb{R}^2)$ of logarithmic smoothness $b > 0$, characterized by the norm

$$\|f|B^{0,b}_{p,\infty}(\mathbb{R}^d)\| := \|f|L_p(\mathbb{R}^d)\| + \sup_{|h|<1/2} (-\log|h|)^b \bigg(\int_{\mathbb{R}^d} |f(x+h)-f(x)|^p dx\bigg)^{1/p}.$$

Recently, those function spaces have showed up in various publications, see, e.g., [13–15, 17, 18].

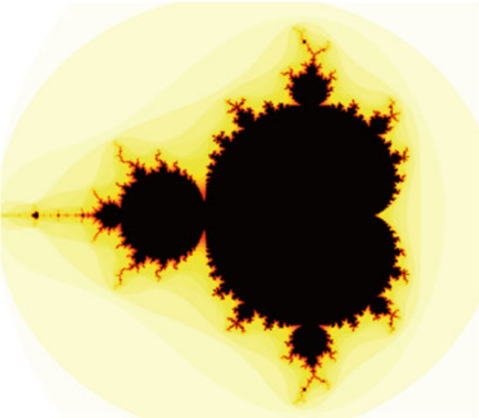


Fig. 15 The Mandelbrot set

4.14 A Final Comment

The three methods, to obtain sufficient conditions for the regularity of $\mathscr{X}_E$, discussed in this section, seem to be more adapted to situations where

$$\lim_{t \to 0} \dim_H \Big(\partial E \cap B(y,t) \Big)$$

exists and does not depend on $y \in \partial E$, compare with Lemma 9(iii). If this quantity depends on y as in case of the domains $E_{\alpha,\gamma}$, then we need more sophisticated criteria.

Appendix

We recall some basic notions from fractal geometry. Our main sources are the monographs of Falconer [21, 22] and Mattila [44].

Fractal Dimensions

Here we recall Hausdorff, Minkowski and packing dimension as well as the Minkowski content.

Hausdorff Dimension

Let A be a subset of $\mathbb{R}^d$. A countable (or finite) collection of sets U_i with diameter $\operatorname{diam} U_i$ is called a δ-cover of A if

$$A \subset \bigcup_{i=1}^{\infty} U_i \qquad \text{and} \qquad 0 < \operatorname{diam} U_i \le \delta$$

for all i. Let s be a nonnegative real number. For any $\delta > 0$ we put

$$\mathscr{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s \ : \ (U_i)_{i=1}^{\infty} \text{ is a } \delta\text{-cover of } A \right\}.$$

We shall write

$$\mathscr{H}^s(A) := \lim_{\delta \to 0} \mathscr{H}^s_\delta(A).$$

This limit exists in $[0, \infty]$ for any subset of $\mathbb{R}^d$. $\mathscr{H}^s(A)$ is called the s-dimensional Hausdorff measure of A. If $s = d \in \mathbb{N}$ we have

$$\mathscr{H}^d(A) = \frac{2^d\,|A|}{|B(0,1)|}$$

where $|A|$ and $|B(0,1)|$ refer to the d-dimensional Lebesgue measure of these sets.

Definition 7 The Hausdorff dimension of a set $A \subset \mathbb{R}^d$ is given by

$$\dim_H A := \sup\{s \ : \ \mathscr{H}^s(A) > 0\} = \inf\{t \ : \ \mathscr{H}^t(A) < \infty\}.$$

We also need a generalization due to Bricchi [8, 9]. Let $h : (0,1] \to (0,\infty)$ be a positive non-decreasing function such that there exists a positive constant c with

$$\frac{h(2^{-j-k})}{h(2^{-j})} \ge c\, 2^{-kd} \qquad \text{for all} \quad j, k \in \mathbb{N}_0.$$

Then, for a set $A \subset \mathbb{R}^d$, we put $h(A) := h(\operatorname{diam} A)$ if $A \neq \emptyset$ and $h(\emptyset) := 0$. The set function

$$\mathscr{H}^h(A) := \lim_{\delta \to 0} \left(\inf \left\{ \sum_{i=1}^{\infty} h(U_i) \ : \ (U_i)_{i=1}^{\infty} \text{ is a } \delta\text{-cover of } A \right\} \right)$$

is called the Hausdorff measure corresponding to h.

Minkowski Dimensions

Let A be a non-empty bounded subset of $\mathbb{R}^d$. For $0 < \varepsilon < \infty$, let

$$N(A,\varepsilon) := \min\left\{k : \ A \subset \bigcup_{i=1}^{k} B(x_i,\varepsilon) \text{ for some } x_i \in \mathbb{R}^d\right\}.$$

$N(A,\varepsilon)$ is sometimes called covering number.

Definition 8 The upper and lower Minkowski dimension of a set $E \subset \mathbb{R}^d$ are defined by

$$\overline{\dim}_M A := \inf\{s : \ \limsup_{\varepsilon\downarrow 0} N(A,\varepsilon)\,\varepsilon^s = 0\}$$

and

$$\underline{\dim}_M A := \inf\{s : \ \liminf_{\varepsilon\downarrow 0} N(A,\varepsilon)\,\varepsilon^s = 0\}\,.$$

In case $\overline{\dim}_M A = \underline{\dim}_M A$ we call this number the Minkowski dimension of A.

It follows

$$\dim_H A \le \underline{\dim}_M A \le \overline{\dim}_M A \le d\,,$$

see Mattila [44, pp. 78]. Let us mention that the Minkowski dimension is sometimes also called box counting dimension.

Minkowski Content

Recall, for a given set $A \subset \mathbb{R}^d$ the family of δ-neighbourhoods A^δ, $\delta > 0$, are defined as

$$A^\delta := \{x \in \mathbb{R}^d : \ \mathrm{dist}\,(x,A) \le \delta\}\,.$$

Definition 9 The s-dimensional upper Minkowski content of A is defined by

$$\mathscr{M}^{*s}(A) := \limsup_{\delta\downarrow 0}\,(2\delta)^{s-d}\,|A^\delta|$$

and the s-dimensional lower Minkowski content of A by

$$\mathscr{M}^{s}_{*}(A) := \liminf_{\delta\downarrow 0}\,(2\delta)^{s-d}\,|A^\delta|\,.$$

The Minkowski content and the Minkowski dimension are related as follows

$$\overline{\dim}_M A = \inf\{s : \mathscr{M}^{*s}(A) = 0\} = \sup\{s : \mathscr{M}^{*s}(A) > 0\}, \qquad \text{(A.1)}$$
$$\underline{\dim}_M A = \inf\{s : \mathscr{M}^{s}_{*}(A) = 0\} = \sup\{s : \mathscr{M}^{s}_{*}(A) > 0\},$$

A useful relation between Minkowski content and Hausdorff measure is given by

$$2^{-s-d}\,|B(0,1)|\,\mathscr{H}^s(A) \le \mathscr{M}^s_*(A), \qquad \text{(A.2)}$$

see, e.g., Mattila [44, pp. 79].

Packing Dimension

We define upper and lower packing dimension as follows

$$\overline{\dim}_P A = \inf\left\{\sup_i \overline{\dim}_M A_i : \quad A = \bigcup_{i=1}^{\infty} A_i\,,\; A_i \quad \text{is bounded}\right\},$$
$$\underline{\dim}_P A = \inf\left\{\sup_i \underline{\dim}_M A_i : \quad A = \bigcup_{i=1}^{\infty} A_i\,,\; A_i \quad \text{is bounded}\right\},$$

where A is an arbitrary subset of $\mathbb{R}^d$. If both numbers coincide, they are called packing dimension of A.

Self-Similar and Sub-self-similar Sets

A mapping $S : \mathbb{R}^d \to \mathbb{R}^d$ is called a similarity with ratio r if

$$|S(x) - S(y)| = r\,|x - y|, \qquad x, y \in \mathbb{R}^d.$$

If $0 < r < 1$ we say that S is contracting. Suppose $\mathscr{S} := (S_1, \dots S_N),\; N \ge 2$, is a finite sequence of similarities with contraction ratios $r_1, \dots, r_N \in (0,1)$. Then there exists a unique non-empty compact set K such that

$$K = \bigcup_{j=1}^{N} S_j(K).$$

This set K will be called self-similar. A non-empty compact set $K \subset \mathbb{R}^d$ is called sub-self-similar for $\mathcal{S}$ if

$$K \subset \bigcup_{j=1}^{N} S_j(K)\,,$$

see [23]. Furthermore, $\mathcal{S}$ satisfies the open set condition if there exists a bounded non-empty open set O such that

$$\bigcup_{j=1}^{N} S_j(O) \subset O \qquad \text{and} \qquad (S_i(O) \cap S_j(O)) = \emptyset \quad \text{if} \quad i \neq j\,.$$

We shall need the following two results, see Hutchinson [34] and Falconer [22, Thm. 9.3].

Proposition 5 ([22, Thm. 9.3]) *If $\mathcal{S}$ satisfies the open set condition, then the invariant set K is self-similar and $0 < \mathcal{H}^s(K) < \infty$, whence $s = \dim_H K$, where s is the unique number for which*

$$\sum_{j=1}^{N} r_i^s = 1\,. \tag{A.3}$$

Moreover, there are positive and finite numbers a and b such that

$$a\, r^s \le \mathcal{H}^s(K \cap B(x,r)) \le b\, r^s \qquad \textit{for} \quad x \in K,\ \ 0 < r < 1\,.$$

In addition $\dim_H K = \dim_M K$.

There is a partial generalization to sub-self-similar sets which covers boundaries of self-similar sets as well, see [23].

Proposition 6 ([24, Cor. 3.4], [23, Thm. 3.5]) *Let $\mathcal{S}$ satisfy the open set condition and let the non-empty compact set K be sub-self-similar for $\mathcal{S}$. Define s as the unique solution of (A.3). Then $0 < \mathcal{H}^s(K)$ and $s = \dim_H K = \dim_M K$.*

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Small Data Wave Maps in Cyclic Spacetime

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Dedicated to Michael Reissig on his 60th birthday.

Abstract We study the initial value problem for the wave maps defined on the cyclic spacetime with the target Riemannian manifold that is responsive (see definition of the self coherence structure) to the parametric resonance phenomena. In particular, for arbitrary small and smooth initial data we construct blowing up solutions of the wave map if the metric of the base manifold is periodic in time.

Keywords Wave maps · Cyclic spacetime · Parametric resonance · Global existence

1 Introduction

In this note we study a wave map

$$\phi : (L, g_{\mu\nu}) \longrightarrow (M, h_{ab}),$$

where L is an $n+1$-dimensional Lorentzian manifold and the target M is a m-dimensional Riemannian manifold. The map ϕ is a *wave map* if it is a stationary point for the Lagrangian functional

$$\mathscr{L}[\phi] = \int_L \frac{1}{2} g^{\mu\nu}(x) h_{ab}(\phi) \nabla_\mu \phi^a \nabla_\nu \phi^b \, d\mu_g .$$

The Lagrangian is written in local coordinates on the target, for which the notation $\phi^a = \phi^a(x^\mu)$ is used. We denote by $d\mu_g$ the measure with respect to the metric $g^{\mu\nu}$

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on the spacetime. Here the convention to write $g^{\mu\nu}(x) = (g_{\mu\nu}(x))^{-1}$ and $h^{ab}(\phi) = (h_{ab}(\phi))^{-1}$ for the inverse of two metric tensors is used. These tensors are used also in raising indexes. A stationary point for the Lagrangian functional implies the following system of equations

$$\Box u^b - \Gamma^b_{cd}(u) g^{\mu\nu}(x) \nabla_\mu u^c \nabla_\nu u^d = 0\,,$$

where $\Box$ is the d'Alembert (or wave) operator

$$\Box := -\nabla_\mu \nabla^\mu$$

and Γ^b_{cd} are the Christoffel symbols on the target manifold (M, h) defined as

$$\Gamma^i_{j,k}(u) := \frac{1}{2} \sum_{l=1}^{m} h^{il} \left(\frac{\partial}{\partial u^j} h_{kl} + \frac{\partial}{\partial u^k} h_{jl} - \frac{\partial}{\partial u^l} h_{kj} \right).$$

For the Minkowski spacetime $\mathbb{R}^{1+n}$ to a Riemannian manifold M, the wave map satisfies the system of equations

$$\Box u^i + \sum_{j,k=1}^{m} \Gamma^i_{j,k}(u) \left(\dot{u}^j \dot{u}^k - \nabla u^j \cdot \nabla u^k \right) = 0\,, \quad i = 1, \ldots m, \tag{1}$$

where $\Box = \partial^2/\partial t^2 - \Delta$ and Δ is the Laplacian in L. Here $\dot{u}$ denotes the partial derivative with respect to time, and ∇ denotes the gradient in x.

For Eq. (1) consider the Cauchy problem with the initial conditions

$$u^i(0, x) = u^i_0(x), \quad u^i_t(0, x) = u^i_1(x)\,, \quad i = 1, \ldots m, \quad x \in \mathbb{R}^n\,. \tag{2}$$

It is known (see, e.g., Theorem 6.4.11 [7]) the following *local existence result*: if $\Gamma^i_{j,k}(u)$ are C^∞ functions and $u^i_0(x) \in H^{s+1}(\mathbb{R}^n)$ and $u^i_1(x) \in H^s(\mathbb{R}^n)$ for some integer $s > (n+2)/2$ then the problem (1)–(2) has for some $T > 0$ a solution $u \in C^2([0, T] \times \mathbb{R}^n)$.

For the wave map from the Minkowski spacetime $\mathbb{R}^{1+n}$, $n \geq 4$, to a Riemanian manifold M the global in time existence of the small data solution can be derived from Theorem 6.5.2 [7]. Klainerman and Machedon [8] proved that the Cauchy problem for (1) is locally in time well-posed in the Sobolev space $H^s(\mathbb{R}^{1+n})$ for any $s > n/2$ if $\Gamma^i_{j,k}(u)$ are analytic and $n = 3$. Klainerman and Selberg [9] extended this result to $n \geq 2$.

Sideris [19] considered wave maps (1) on the Minkowski spacetime, where $\Gamma^i_{j,k}(u)$ are smooth functions on $\mathbb{R}^m$ with the property

$$\Gamma^i_{j,k}(u^1, 0, \cdots, 0) = 0 \quad \text{for all} \quad u^1 \in \mathbb{R},\ 1 \leq i, j, k \leq m\,. \tag{3}$$

Since the nonlinearities in (1) are cubic, small amplitude solutions are known to exist (see, e.g., [7]). In [19] the component u^1 need not to be small.

Georgiev and Schirmer in [4] generalized the spacetime estimates obtained by Klainerman and Machedon to wave equations on manifolds with nonconstant metric. They applied these estimates to the question of global existence of low-regularity solution for small data of nonlinear wave equations on Minkowski space $\mathbb{R}^{1+3}$ satisfying the null condition. The null forms are expressions of the form $g^{\mu\nu}\nabla_\mu u \nabla_\nu v$ or $\nabla_\mu u \nabla_\nu v - \nabla_\nu u \nabla_\mu v$, where u, v are the functions on L. These estimates were then applied on the Einstein cylinder (after Penrose compactification) to prove that if $(u(0), u_t(0)) \in H^{2,1}(\mathbb{R}^3) \times H^{1,2}(\mathbb{R}^3)$ is sufficiently small, then a semilinear wave equations $\left(\partial_t^2 - \Delta\right) u = F(u, \nabla u, u_t)$ with F satisfying the null condition has a global solution.

In connection with low dimension n we recall conjecture of Klainerman that states: *Let* $(\mathbb{H}^2, h)$ *be the standard hyperbolic plane. Then classical wave maps originating on* $\mathbb{R}^{2+1}$ *exist for arbitrary smooth initial data.*

The answer to the Klainerman's conjecture as well as the scattering result for the wave map are given by Krieger and Schlag in [10, 11]. In particular, it is proved in [11] that if M is a hyperbolic Riemann surface, and the initial data $(u(0), \partial_t u(0)) : S_0 \longrightarrow M \times TM$ are smooth and $u(0) = const,\ \partial_t u(0) = 0$ outside of some compact set, then the wave map evolution u of these data as a map $\mathbb{R}^{2+1} \longrightarrow M$ exists globally as a smooth function.

In [14] the stability of the last result under perturbation of the metric g in L, that is, in the perturbed Minkowski spacetime, is investigated. More precisely, Nishitani and Yagdjian [14] considered the case of the Riemannian manifold (M, h), which belongs to one-parameter family of manifolds containing the Euclidean half-space and the Poincaré upper half-plane model $(\mathbb{H}^2, h)$. In fact, that family consists of the Riemannian manifolds, which are the half-plane $\{(u^1, u^2) \in \mathbb{R}^2 \mid u^2 > 0\}$ equipped with the metric $h_{ij} du^i du^j = \dfrac{1}{(u^2)^l}\left((du^1)^2 + (du^2)^2\right)$, where the parameter l is a real number. For $l = 0$ the metric is Euclidean, while for $l = 2$ it is the metric of the standard hyperbolic plane. Those are the only two manifolds of this family which have constant curvature. In [14] is proved that the only stationary solutions of equation (1) are the constant solutions and that the global in time solvability can be destroyed by parametric resonance phenomena. (For the scalar quasilinear wave equation it was proved in [22].) For the parametric resonance phenomena in the scalar wave map-type hyperbolic equations see [23] and references therein. Then, according to [20] (see also references therein) the parametric resonance phenomena in the linear scalar wave equations can be localized in the space.

Nakanishi and Ohta [13] studied the Cauchy problem for the nonlinear wave equation

$$\begin{cases} \Box u + f(u)\left(\dot{u}^2 - |\nabla u|^2\right) = 0\,, & (t, x) \in \mathbb{R}^{1+n}\,, \\ u(0, x) = u_0(x), \quad \dot{u}(0, x) = u_1(x)\,, & x \in \mathbb{R}^n\,, \end{cases} \tag{4}$$

where $u = u(t,x)$ is a scalar real-valued unknown function, f is a real-valued smooth function. In [13] the following condition

$$\int_0^\infty \exp\left(\int_0^s f(r)dr\right) ds = \infty \quad \text{and} \quad \int_{-\infty}^0 \exp\left(\int_0^s f(r)dr\right) ds = \infty \tag{5}$$

is suggested to be necessary and sufficient condition (Theorem 2.1 [13]) for the existence of a global classical solution $u \in C^\infty(\mathbb{R}^{1+n})$ for the problem (4) for any $u_0, u_1 \in C^\infty(\mathbb{R}^n)$. Note here, that the initial data u_0, u_1 are not assumed to be small. The equation of (4) is a model and special case for wave maps.

In the case of nonflat base manifold L, the wave maps are less investigated although they are of considerable interest in the general relativity context. The Cauchy problem for the wave maps in the perturbed Minkowski spacetime is considered in [1] and [14] (cyclic universe). More precisely, assume that $V = S \times \mathbb{R}$, with S an n-dimensional orientable smooth manifold, and let g be a Robertson-Walker metric $g = -dt^2 + a^2(t)\sigma$, with the scale function $a = a(t)$, where $\sigma = \sigma_{ij}\, dx^i\, dx^j$ is a given, smooth, time independent metric on S, with non-zero injectivity radius.

Let $(S \times \mathbb{R}, g)$ be a Robertson-Walker expanding universe with the metric $g = -dt^2 + a^2(t)\sigma$, while (S, σ) is a smooth Riemannian manifold of dimension $n \le 3$ with non-zero injectivity radius and $a = a(t)$ a positive increasing function of t such that $1/a(t)$ is integrable on $[t_0, \infty)$. Hence a domain of influence is permanently restricted (see, also, [23, Sec.8]). Let (M, h) be a proper Riemannian manifold regularly embedded in $\mathbb{R}^N$ such that Riem(h) is uniformly bounded. Then according to Choquet-Bruhat [1] there exists a global wave map from $(S \times [t_0, \infty), g)$ into (M, h) taking Cauchy data φ, ψ with $D\varphi$ and ψ in H^1 if the integral of $1/a(t)$ on $[t_0, \infty)$ is less than some corresponding number $M(a, b)$. The number $M(a, b)$ depends on the initial data. Thus, (see Corollary on page 45 [1]) under hypothesis of the theorem, for any finite value of the integral of $1/a(t)$ on $[t_0, \infty)$ there is an open set U of initial data in $H^1 \times H^1$ such that if $(D\varphi, \psi) \in U$, then there exists a global wave map taking the Cauchy data (φ, ψ). In particular, this is true for the curved spacetime of the de Sitter model of universe with the scale function $a(t) = \exp(\Lambda t),\ \Lambda > 0$.

D'Ancona and Zhang [2] derived the global existence of equivariant wave maps from the so-called admissible manifolds to general targets for the small initial data of critical regularity. Both base and target manifolds are assumed rotationally symmetric manifolds with global metrics

$$L \ : \ dr^2 + g(r)^2 d\omega^2_{\mathbb{S}^{n-1}}\,, \qquad M \ : \ d\phi^2 + h(\phi)^2 d\phi^2_{\mathbb{S}^{\ell-1}}\,,$$

where $d\omega^2_{\mathbb{S}^{n-1}}$ and $d\phi^2_{\mathbb{S}^{\ell-1}}$ are the standard metrics on the unit sphere. The solution has a form $u = (\phi, \chi)$ in coordinates on M, the radial component $\phi = \phi(t,r)$ depends only on time t and r, the radial coordinate on L, while the angular component $\chi = \chi(\omega)$ depends only on the angular coordinate ω on L. Thus,

$\chi : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{\ell-1}$ is a harmonic polynomial map of degree k, whose energy density is $k(k+n-2)$ for some integer $k \geq 1$, while ϕ satisfies the *$\bar{\ell}$-equivariant wave map equation*

$$\phi_{tt} - \phi_{rr} - (n-1)\frac{h'(r)}{h(r)}\phi_r + \frac{\bar{\ell}}{h(r)^2} g(\phi)g'(\phi) = 0\,, \tag{6}$$

where $\bar{\ell} = k(k+n-2)$. For (6) the authors consider the Cauchy problem with initial data

$$\phi(0,r) = \phi_0(r), \quad \phi_t(0,r) = \phi_1(r)\,.$$

When $g(r) = r$ the problem for (6) reduces to the equation originally studied in [17, 18]. It is proved in [2] that on the admissible manifolds the wave flow satisfies smoothing and Strichartz estimates. The metric h of the base manifold is assumed to have a limit $h^{\frac{1-n}{2}}(h^{\frac{n-1}{2}})''$ as $r \to \infty$. The existence of small equivariant wave maps on admissible manifolds is proved in the critical space $H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}$, and, moreover, the solution enjoys additional L^pL^q integrability properties determined by the Strichartz estimates.

In the present paper we consider the wave map from the perturbed Minkowski spacetime, with the periodic in time perturbation, into Riemannian manifold that is responsive (see self coherence structure below) to the parametric resonance generated by the metric h. The result of the present note requires some assumption on the ordinary differential equation related to the parametric resonance generated by the periodic metric in L. Consider the ordinary differential equation

$$y_{tt}(t) + \Big(\lambda b^2(t) - q(t)\Big)\, y(t) = 0 \tag{7}$$

with the periodic positive smooth non-constant function $b = b(t)$ and parameter $\lambda \in \mathbb{R}$. Let

$$q(t) = \frac{n}{4}\left(\frac{n}{4} - 1\right)\left(\frac{\dot{b}(t)}{b(t)}\right)^2 - \frac{n}{2}\frac{\ddot{b}(t)}{b(t)}\,.$$

Assumption ISIN ([14]) *There exists the nonempty open instability interval $\Lambda \subset (0,\infty)$ for Eq. (7).*

We consider a wave map such that in the global chart of M it can be written as a system of equations

$$u^i_{tt} - n\frac{\dot{b}(t)}{b(t)}u^i_t - b^2(t)\Delta u^i + \sum_{j,k}\Gamma^i_{j,k}(u^1,\ldots,u^m)\left(u_t{}^j u_t{}^k - b^2(t)\nabla u^j \cdot \nabla u^k\right) = 0, \tag{8}$$

$i = 1, \ldots, m$, where $b = b(t)$ is a smooth positive periodic function. We are concerned with the small data global in time solution to the Cauchy problem for Eq. (8). Our main result shows that the global solvability is not a stable property under small perturbations of the wave map if the Riemannian manifold *M possesses a distinguished geodesic* (or *intrinsic self coherence structure*) in the sense of the following definition.

Definition 1 ([24]) Riemannian or Lorentzian manifold *M possesses a distinguished geodesic* (or *intrinsic self coherence structure*) if in some chart the straight half-line $\mathbb{L}_+ = \{(a_1 t, \ldots, a_m t) \,|\, t \in (0, \infty)\}$ is covered by the geodesics.

The intrinsic self coherence structure can be characterized explicitly in the terms of Christoffel symbols $\Gamma^i_{j,k}$ as follows.

Lemma 1 ([24]) *If in some chart of the Riemannian manifold M the segment I of the straight line $\mathbb{L} = \{(a_1 t, \ldots, a_m t) \,|\, t \in \mathbb{R}\}$ is covered by a smooth non-constant geodesic, then there is a function $f(t)$ such that*

$$\sum_{j,k=1}^{m} \Gamma^i_{j,k}(a_1 t, \ldots, a_m t) a_j a_k = a_i f(t) \;\; \textit{for all} \;\; t \in (a, b) \subseteq \mathbb{R} \;\; \text{and} \quad i = 1, \ldots, m. \tag{9}$$

Conversely, if in some chart there exists a continuously differentiable function $f = f(t)$ such that (9) holds for all points of the segment $I \subseteq \mathbb{L}$, then there is a geodesic covering the segment I.

The main result of this paper is given by the following theorem.

Theorem 1 *Let $b = b(t)$ be a periodic, non-constant, smooth, and positive function defined on $\mathbb{R}$, satisfying condition ISIN. Assume that the Riemannian manifold M possesses intrinsic self coherence structure and for the function $f(t)$, $t \in \mathbb{R}$, the Nakanishi-Ohta condition (5) does not hold, that is,*

$$\int_0^\infty \exp\left(\int_0^s f(r)dr\right) ds < \infty \quad \text{or} \quad \int_{-\infty}^0 \exp\left(\int_0^s f(r)dr\right) ds < \infty\,. \tag{10}$$

Then for every n, s, and for every positive δ there are initial data $u^i_0, u^i_1 \in C_0^\infty(\mathbb{R}^n)$, $i = 1, \ldots, m$, such that

$$\sum_{i=1}^{m} \|u^i_0\|_{(s+1)} + \|u^i_1\|_{(s)} \le \delta\,, \tag{11}$$

but the solution $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^n)$ to the problem with the prescribed data

$$u^i(0, x) = u^i_0(x), \quad u^i_t(0, x) = u^i_1(x)\,, \quad i = 1, \ldots, m, \quad x \in \mathbb{R}^n\,, \tag{12}$$

for the wave map (8) does not exist.

Remark 1 Assume that $(u(s), 0, \ldots, 0)$ is geodesic and for the function

$$f(t) = \Gamma^1_{1,1}(t, 0, 0, \ldots, 0), \quad t \in \mathbb{R}_+,$$

the Nakanishi-Ohta condition (5) is not fulfilled. Then the statement of the theorem holds. That is true also for any other coordinate axis.

Remark 2 If (3) is fulfilled, then the system (8) obeys intrinsic self coherence structure and the Nakanishi-Ohta condition (5) is fulfilled. According to [19] the large data global solution exists for wave map without periodic perturbation ($b(t) \equiv 0$). The small amplitude solutions are known to exist (see, e.g., [7]). According to Theorem 1 (see also [23]) the periodic perturbation $b(t)$ destroys global in time solvability even for the arbitrarily small data.

Following arguments of the proof Theorem 2.1 [13] one can verify the assertion of the next remark for the case of flat manifold although we do not know if there is small data global existence for the case of non-flat M.

Remark 3 The Cauchy problem for the system

$$u^i_{tt} - n\frac{\dot{b}(t)}{b(t)}u^i_t - b^2(t)\Delta u^i + f^i(u^i)\left(\left({u_t}^i\right)^2 - b^2(t)|\nabla u^i|^2\right) = 0, \quad i = 1, \ldots, m\,,$$

with conditions (12) has a global solution $(u^1(x,t), \ldots, u^m(x,t)) \in C^\infty$ for every $(u^1_\ell(x), \ldots, u^m_\ell(x)) \in C^\infty(\mathbb{R}^n) \times \ldots \times C^\infty(\mathbb{R}^n)$, $\ell = 0, 1$, if and only if the condition

$$\int_0^\infty \exp\left(\int_0^s f^i(r)dr\right) ds < \infty \text{ or } \int_{-\infty}^0 \exp\left(\int_0^s f^i(r)dr\right) ds < \infty\,, \;\; i = 1, \ldots, m\,.$$

is fulfilled.

The proof of the next theorem is given in Sect. 3.

Theorem 2 *Let $b = b(t)$ be a defined on $\mathbb{R}$, periodic, smooth, and positive function. Assume that the Riemannian manifold M possesses intrinsic self coherence structure and the Cauchy problem for (8) has a global solution* $(u^1(x,t), \ldots, u^m(x,t)) \in C^2(\mathbb{R}_+ \times \mathbb{R}^n)$ *for every initial data* $(u^1_i(x), \ldots, u^m_i(x)) \in C^\infty(\mathbb{R}^n) \times \ldots \times C^\infty(\mathbb{R}^n)$, $i = 0, 1$. *Then the Nakanishi-Ohta condition (5) is fulfilled.*

Note that the initial data u_0, u_1 are not assumed small. Existence of the distinguished geodesics allows also to extend result of [13] from the wave map type equations to the wave map with the non-oscillating coefficients for some non-small initial data. That will be proved in a forthcoming paper.

The present paper is organized as follows. In Sect. 2 we illustrate Theorem 1 by several examples. Then, in Sect. 3, we lower the system of equations to the single scalar equation. In Sect. 4 we describe some elements of Floquet-Lyapunov theory with its application to the parametric resonance in the ordinary differential equations. In Sects. 5 and 6 we complete the proofs of Theorems 1 and 2, respectively. The final Sect. 7 is devoted to the proof of Lemma 1.

2 Illustration of Theorem 1 by Examples

In the spacetime with the metric tensor

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix}, \quad |g| = a^{2n}(t),$$

the covariant D'Alembert operator is defined as follows:

$$\square_g u = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ik} \frac{\partial}{\partial x^k} u \right) = \frac{\partial^2}{\partial t^2} u + n \frac{\dot{a}(t)}{a(t)} \frac{\partial}{\partial t} u - \frac{1}{a^2(t)} \Delta u \,.$$

If we denote $b(t) = 1/a(t)$, then

$$\square_g u = \frac{\partial^2}{\partial t^2} u - n \frac{b'(t)}{b(t)} \frac{\partial}{\partial t} u - b(t)^2 \Delta u \,.$$

The corresponding wave map equation is (8). Cyclic spacetime with the periodic smooth positive scale factor $a = a(t)$ is one of the models of the cosmology (see [15, Ch. 9]).

Example 1 Consider the system (8) with $m = 2$:

$$\begin{cases} \left(\partial_t^2 - n \frac{\dot{b}(t)}{b(t)} \partial_t - b^2(t) \Delta \right) u^1 \\ \qquad + \sum_{j,k=1}^2 \Gamma^1_{j,k}(u^1, u^2) \left(\dot{u}^j \dot{u}^k - b^2(t) \nabla u^j \cdot \nabla u^k \right) = 0, \\ \left(\partial_t^2 - n \frac{\dot{b}(t)}{b(t)} \partial_t - b^2(t) \Delta \right) u^2 \\ \qquad + \sum_{j,k=1}^2 \Gamma^2_{j,k}(u^1, u^2) \left(\dot{u}^j \dot{u}^k - b^2(t) \nabla u^j \cdot \nabla u^k \right) = 0. \end{cases} \tag{13}$$

We define in M the diagonal metric tensor $h_{ik}(u^1, u^2) := h(u^1, u^2)\delta_{ik}$. Then, the Christoffel symbols are:

$$\Gamma^i_{j,k} = \frac{1}{2h(u^1, u^2)} \left(\frac{\partial}{\partial u^j} h(u^1, u^2) \delta_{ki} + \frac{\partial}{\partial u^k} h(u^1, u^2) \delta_{ji} - \frac{\partial}{\partial u^i} h(u^1, u^2) \delta_{kj} \right),$$

where $i, j, k = 1, 2$. Hence,

$$\Gamma^1_{1,1} = -\Gamma^1_{2,2} = \Gamma^2_{2,1} = \Gamma^2_{1,2} = \frac{1}{2h(u^1, u^2)}\left(\frac{\partial}{\partial u^1}h(u^1, u^2)\right),$$
$$\Gamma^1_{2,1} = \Gamma^1_{1,2} = \Gamma^2_{2,2} = -\Gamma^2_{1,1} = \frac{1}{2h(u^1, u^2)}\left(\frac{\partial}{\partial u^2}h(u^1, u^2)\right).$$

The Gaussian curvature of the surface with such metric is

$$K = -\frac{1}{h(u^1, u^2)}\Delta \ln h(u^1, u^2)\,.$$

The wave map Eq. (8) reads

$$\begin{cases} \left(\partial_t^2 - n\frac{\dot b(t)}{b(t)}\partial_t - b^2(t)\Delta\right)u^1 \\ \qquad +\frac{1}{2h(u^1, u^2)}\left(\frac{\partial}{\partial u^1}h(u^1, u^2)\right)(\dot u^1\dot u^1 - b^2(t)\nabla u^1\cdot\nabla u^1) \\ \qquad +\frac{1}{h(u^1, u^2)}\left(\frac{\partial}{\partial u^2}h(u^1, u^2)\right)(\dot u^1\dot u^2 - b^2(t)\nabla u^1\cdot\nabla u^2) \\ \qquad -\frac{1}{2h(u^1, u^2)}\left(\frac{\partial}{\partial u^1}h(u^1, u^2)\right)(\dot u^2\dot u^2 - b^2(t)\nabla u^2\cdot\nabla u^2) = 0, \\ \left(\partial_t^2 - n\frac{\dot b(t)}{b(t)}\partial_t - b^2(t)\Delta\right)u^2 \\ \qquad -\frac{1}{2h(u^1, u^2)}\left(\frac{\partial}{\partial u^2}h(u^1, u^2)\right)(\dot u^1\dot u^1 - b^2(t)\nabla u^1\cdot\nabla u^1) \\ \qquad +\frac{1}{h(u^1, u^2)}\left(\frac{\partial}{\partial u^1}h(u^1, u^2)\right)(\dot u^1\dot u^2 - b^2(t)\nabla u^1\cdot\nabla u^2) \\ \qquad +\frac{1}{2h(u^1, u^2)}\left(\frac{\partial}{\partial u^2}h(u^1, u^2)\right)(\dot u^2\dot u^2 - b^2(t)\nabla u^2\cdot\nabla u^2) = 0\,. \end{cases}$$

If $b(t) = const > 0$, the small amplitude solutions of (13) exist globally. Now we focus on the case with a half-diagonal $\mathbb{L}_+ = \{(t, \dots, t)\,|\, t \in (0, \infty)\} \subset \mathbb{D}$. We note that

$$\sum_{j,k=1}^{2}\Gamma^1_{jk}(u^1, u^2) = \frac{1}{h}\left(\frac{\partial}{\partial u^2}h\right), \qquad \sum_{j,k=1}^{2}\Gamma^2_{jk}(u^1, u^2) = \frac{1}{h}\left(\frac{\partial}{\partial u^1}h\right).$$

Assume that

$$\frac{\partial h}{\partial u^k}(u^1, u^2) = \frac{\partial h}{\partial u^l}(u^1, u^2) \quad \text{if} \quad u^1 = u^2 \quad \text{for} \quad k, l = 1, 2.$$

Then, due to the last assumption on $h = h(u^1, u^2)$ we set $a_1 = a_2 = 1$ and obtain the function f appearing in (9)

$$f(\xi) := \sum_{j,k=1}^{2} \Gamma^1_{jk}(\xi,\xi) = \sum_{j,k=1}^{2} \Gamma^2_{jk}(\xi,\xi) \quad \text{if} \quad \xi \in \mathbb{R}_+ .$$

To determine the geodesics, let (U, φ) be a parametrization of the manifold M and let $\alpha : I \to M$ be a curve parametrized by arc length, whose trace is contained in $\varphi(U)$. Write

$$\alpha(s) = \varphi(u(s), v(s)) ,$$

where $u = u(s)$ and $v = v(s)$ are real-valued functions of s. Then α is a geodesic if

$$\begin{cases} \ddot{u}(s) + \dfrac{1}{2h(u,v)}\left(\dfrac{\partial}{\partial u}h(u,v)\right)(\dot{u}(s))^2 \\ \qquad + \dfrac{1}{h(u,v)}\left(\dfrac{\partial}{\partial v}h(u,v)\right)\dot{u}(s)\dot{v}(s) - \dfrac{1}{2h(u,v)}\left(\dfrac{\partial}{\partial u}h(u,v)\right)(\dot{v}(s))^2 = 0 , \\ \ddot{v}(s) - \dfrac{1}{2h(u,v)}\left(\dfrac{\partial}{\partial v}h(u,v)\right)(\dot{u}(s))^2 \\ \qquad + \dfrac{1}{h(u,v)}\left(\dfrac{\partial}{\partial u}h(u,v)\right)\dot{u}(s)\dot{v}(s) + \dfrac{1}{2h(u,v)}\left(\dfrac{\partial}{\partial v}h(u,v)\right)(\dot{v}(s))^2 = 0 . \end{cases}$$

We claim that there exists a geodesic curve that lies in the diagonal $\mathbb{D}$. Indeed, set $u(s) = v(s)$. Then, the equation of geodesic and the unit speed equation read

$$\ddot{u}(s) + \frac{1}{h(u(s),u(s))}\left(\frac{\partial}{\partial u}h(u(s),u(s))\right)(\dot{u}(s))^2 = 0 ,$$
$$1 = h(u(s),u(s))2(\dot{u}(s))^2 .$$

From the second equation the solution $u = u(s)$ can be given implicitly by

$$\int_0^{u(s)} \sqrt{h(r,r)}\, dr = \frac{1}{\sqrt{2}} s + C . \tag{14}$$

Setting $h(u^1, u^2) = (1 + u_1^2 + u_2^2)^\alpha$, we check condition (5):

$$\int_0^{\pm\infty} \exp\left(\int_0^s f(r)dr\right) ds = \int_0^{\pm\infty} (1 + 2s^2)^\alpha ds = \int_0^{\pm\infty} h(s,s)ds = \pm\infty .$$

Hence, condition (5) is equivalent to the inequality $\alpha > -\frac{1}{2}$. For the case of $h(u, v) = (1 + u^2 + v^2)^\alpha$ Eq. (14) for the geodesics leads to the function $u = u(s)$ that is defined implicitly by

$$uF\left(\frac{1}{2}, -\frac{\alpha}{2}; \frac{3}{2}; -2u^2\right) = \frac{1}{\sqrt{2}}s + C\,. \tag{15}$$

If $\alpha = -1$, then condition (5) is violated and (15) simplifies to $u(s) = C_1e^s + C_2e^{-s}$, that implies

$$u(s) = v(s) = C_1e^s + C_2e^{-s}\,.$$

The non-constant geodesic that belongs to the diagonal $\mathbb{D}$ and starts at the origin is given by

$$u(s) = v(s) = \frac{1}{\sqrt{2}}\sinh(s)\,.$$

For the case of $h(u^1, u^2) = (1 + u_1^2 + u_2^2)^{-1}$ on the diagonal $\mathbb{D}$ the Christoffel symbols are

$$\Gamma^1_{1,1} = -\Gamma^1_{2,2} = \Gamma^2_{2,1} = \Gamma^2_{1,2} = \Gamma^1_{2,1} = \Gamma^1_{1,2} = \Gamma^2_{2,2} = -\Gamma^2_{1,1} = -\frac{1}{\sqrt{2}}\tanh(s)\mathrm{sech}(s).$$

The Gaussian curvature of the surface with the metric $h(u^1, u^2) = (1 + u_1^2 + u_2^2)^\alpha$ is

$$K = -\frac{1}{h(u^1, u^2)}\Delta \ln h(u^1, u^2) = -4\alpha(1 + u^2 + v^2)^{-\alpha-2}\,.$$

It is also a scalar curvature. It is constant iff $\alpha = -2$.

Example 2 Define the metric $h(u, v) = (1+v)^{-\ell}$, $\ell \geq 0$ on $M = \{(u, v) \in \mathbb{R} \mid v > -1\}$, then the Christoffel symbols are

$$\Gamma^1_{2,1} = \Gamma^1_{1,2} = \Gamma^2_{2,2} = -\Gamma^2_{1,1} = -\frac{\ell}{2(1+v)}$$

while the equations for the geodesics are

$$\begin{cases} \ddot{u}(s) - \dfrac{\ell}{(1+v)}\dot{u}(s)\dot{v}(s) = 0\,, \\ \ddot{v}(s) + \dfrac{\ell}{2(1+v)}\left(\dot{u}(s)\right)^2 - \dfrac{\ell}{2(1+v)}\left(\dot{v}(s)\right)^2 = 0\,. \end{cases}$$

If $\ell = 2$ this system has a solution $u(s) = u(0),\ v(s) = Ce^s - 1$, that is, a vertical half-line in the positive half-plane. The geodesic starting at the origin is $u(s) = 0,\ v(s) = e^s - 1$. Then,

$$f(t) = -\frac{\ell}{2(1+t)}, \quad \int_0^\infty \exp\left(\int_0^s f(r)dr\right) ds = \int_0^\infty (1+s)^{-\frac{\ell}{2}} ds < \infty$$

implies $\ell > 2$. For the case of $\ell \in [0, 2)$ the nonexistence of the global solution for arbitrary small data is proved in [14]. The global existence of arbitrary small data solutions for the case of $\ell = 2$ and non-constant periodic $b = b(t)$ remains an open problem.

Example 3 Assume now that $h(u^1, u^2) = (1 + u_1^2 + u_2^4)^\alpha = (1 + u^2 + v^4)^\alpha$. Then, the Christoffel symbols are

$$\Gamma^1_{1,1} = -\Gamma^1_{2,2} = \Gamma^2_{2,1} = \Gamma^2_{1,2} = \frac{\alpha u}{u^2 + v^4 + 1},$$
$$\Gamma^1_{2,1} = \Gamma^1_{1,2} = \Gamma^2_{2,2} = -\Gamma^2_{1,1} = \frac{2\alpha v^3}{u^2 + v^4 + 1},$$

and the equations for the geodesics are

$$\begin{cases} \ddot{u}(s) + \dfrac{\alpha u}{(1+u^2+v^4)} (\dot{u}(s))^2 \\ \qquad + \dfrac{4v^3\alpha}{(1+u^2+v^4)} \dot{u}(s)\dot{v}(s) - \dfrac{\alpha u}{(1+u^2+v^4)} (\dot{v}(s))^2 = 0, \\ \ddot{v}(s) - \dfrac{2v^3\alpha}{(1+u^2+v^4)} (\dot{u}(s))^2 \\ \qquad + \dfrac{2\alpha u}{(1+u^2+v^4)} \dot{u}(s)\dot{v}(s) + \dfrac{2v^3\alpha}{(1+u^2+v^4)} (\dot{v}(s))^2 = 0. \end{cases}$$

The curve $v(s) = 0$ is geodesic if

$$\ddot{u}(s) + \frac{\alpha u(s)}{(1+u^2(s))} (\dot{u}(s))^2 = 0, \quad 1 = h(u(s), u(s))(\dot{u}(s))^2,$$

that is,

$$\ddot{u}(s) + \frac{\alpha u(s)}{(1+u^2(s))} (\dot{u}(s))^2 = 0, \quad 1 = (1+u^2(s))^\alpha (\dot{u}(s))^2.$$

With the function $f(t) = \alpha t/(1+t^2)$ we observe

$$\int_0^\infty \exp\left(\int_0^s f(r)dr\right) ds = \int_0^\infty \exp\left(\int_0^s \frac{\alpha r}{1+r^2} dr\right) ds = \int_0^\infty (1+s^2)^{\alpha/2} ds < \infty.$$

The condition (5) implies $\alpha > -1$.

The line $u(s) = 0$ is also a geodesic and with the function $f(t) = 2\alpha t^3/(1+t^4)$, together with condition (5),

$$\int_0^{\infty} \exp\left(\int_0^s f(r)dr\right) ds = \int_0^{\infty} \exp\left(\int_0^s \frac{2\alpha r^3}{1+r^4} dr\right) ds = \int_0^{\infty} (1+s^4)^{\alpha/2} ds < \infty$$

we obtain $\alpha > -1/2$. Thus, the choice of the geodesic line is essential. The Gaussian curvature of the surface with the metric $h(u^1, u^2) = (1 + u_1^2 + u_2^4)^{\alpha} = (1 + u^2 + v^4)^{\alpha}$ is

$$K = -2\alpha \left(u^2 \left(6v^2 - 1\right) - 2v^6 + v^4 + 6v^2 + 1\right) \left(u^2 + v^4 + 1\right)^{-\alpha-2} .$$

It is also a scalar curvature.

The next example shows that small perturbation of the diagonal metric tensor does not eliminate the blow up phenomenon.

Example 4 Let $M = \mathbb{R}^m$ be provided with the metric defined by the metric tensor $h_{ik}(u) = h(u)(\delta_{ik} + H_{ik}(u))$, where $u = (u^1, \dots, u^m)$ and $h = h(u)$ is a smooth positive function. We denote by M such Riemannian manifold. Assume that $H(u)$ is a smooth matrix function with the matrix norm $\|H(u)\| < 1$ and that on the diagonal $\mathbb{D}$ of M

$$\frac{\partial}{\partial u^k} H(u) = 0, \quad H(u) = 0 \quad \text{if} \quad u \in \mathbb{D}, \quad \forall k = 1, 2, \dots, m,$$
$$\frac{\partial}{\partial u^k} h(u^1, \dots, u^m) = \frac{\partial}{\partial u^l} h(u^1, \dots, u^m) \quad \text{if} \quad u \in \mathbb{D}, \quad \forall k, l = 1, 2, \dots, m.$$

The Christoffel symbols for the metric $h_{ik}(u)$ on the diagonal $\mathbb{D}$ are

$$\Gamma^i_{jk}(u) = \frac{1}{2} \frac{1}{h(u)} \left(\frac{\partial}{\partial u^j} h(u) \delta_{ki} + \frac{\partial}{\partial u^k} h(u) \delta_{ji} - \frac{\partial}{\partial u^i} h(u) \delta_{jk} \right),$$

and

$$\sum_{j,k=1}^{m} \Gamma^i_{jk}(u) = \frac{1}{2} m \frac{1}{h(u)} \frac{\partial}{\partial u^1} h(u), \quad i = 1, \dots, m, \quad u \in \mathbb{D} .$$

The diagonal $\mathbb{D}$ is a geodesic. Indeed, we set the initial conditions

$$u^1(0) = \dots = u^m(0) = 0 , \qquad \frac{du^1}{ds}(0) = \dots = \frac{du^m}{ds}(0) = (mh(1, \dots, 1))^{-1/2} ,$$

and consider the function $\tilde{u} = \tilde{u}(s)$ that solves the Cauchy problem

$$\frac{d^2\tilde{u}}{ds^2} + \frac{1}{2}m\frac{1}{h(u)}\frac{\partial}{\partial u^1}h(u)\left(\frac{d\tilde{u}}{ds}\right)^2 = 0\,, \quad \tilde{u}(0) = 0, \quad \frac{d\tilde{u}}{ds}(0) = (mh(1,\dots,1))^{-1/2}\,.$$

Then the function $u(s) = (\tilde{u}(s),\dots,\tilde{u}(s))$ is a geodesics that lies in $\mathbb{D}$. Therefore, if we define

$$f(u) := \frac{m}{2h(u)}\frac{\partial}{\partial u^1}h(u), \quad u \in \mathbb{D},$$

then, with $a_1 = \ldots = a_m = 1$, condition (9) is fulfilled:

$$\sum_{j,k=1}^{m}\Gamma^1_{jk}(u) = \sum_{j,k=1}^{m}\Gamma^2_{jk}(u) = \ldots = \sum_{j,k=1}^{m}\Gamma^m_{jk}(u) = f(u), \quad u \in \mathbb{D}.$$

In order to verify condition (10) we specify $h(u) = (1 + u_1^2 + \ldots + u_m^2)^\alpha$, then

$$f(u) := \frac{m\alpha u}{1 + mu^2}, \quad u \in \mathbb{R},$$

$$\int_0^\infty \exp\left(\int_0^s f(r)dr\right)ds = \int_0^\infty \exp\left(\int_0^s \frac{m\alpha r}{1 + mr^2}dr\right)ds = \int_0^\infty (1 + ms^2)^{\alpha/2}ds < \infty\,.$$

Condition (10) implies $\alpha < -1$.

Example 5 Let $b(t) = \sqrt{1 + \varepsilon\sin(t)}$, where $\varepsilon \in (0,1)$, be a periodic, non-constant, smooth, and positive function defined on $\mathbb{R}$. Assuming $m = 2$, it follows

$$\begin{cases} \left(\partial_t^2 - n\dfrac{\varepsilon\cos(t)}{2(1+\varepsilon\sin(t))}\partial_t - (1+\varepsilon\sin(t))\Delta\right)u + |\dot{v}|^2 - (1+\varepsilon\sin(t))|\nabla v|^2 = 0, \\ \left(\partial_t^2 - n\dfrac{\varepsilon\cos(t)}{2(1+\varepsilon\sin(t))}\partial_t - (1+\varepsilon\sin(t))\Delta\right)v + |\dot{u}|^2 - (1+\varepsilon\sin(t))|\nabla u|^2 = 0. \end{cases}$$

Then, for every n, s, and for every positive δ, there are data $u_0, v_0, u_1, v_1 \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|u_0\|_{(s+1)} + \|u_1\|_{(s)} + \|v_0\|_{(s+1)} + \|v_1\|_{(s)} \le \delta$$

but the solution $u, v \in C^2(\mathbb{R}_+ \times \mathbb{R}^n)$ to the problem with data

$$u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \;\; v(0,x) = v_0(x), \;\; v_t(0,x) = v_1(x)\,, \;\; x \in \mathbb{R}^n$$

does not exist. If $\varepsilon = 0$, then a small data solution exists globally. The Riemannian curvature of this spacetime with $n = 3$ is

$$-\frac{3\varepsilon(\varepsilon\cos(2t) + 3\varepsilon + 2\sin(t))}{2(\varepsilon\sin(t) + 1)^2},$$

which is sign changing in time.

3 Lowering to the Scalar Equation

The main idea is to use a composition of the solution of the wave equation in L with the distinguished geodesic of the target manifold M. This composition is a wave map. For the properly chosen geodesic such wave map blows up for the large time (see also [14]). Consider the system of equations

$$u^i_{tt} - n\frac{\dot{b}(t)}{b(t)}u^i_t - b^2(t)\Delta u^i + \sum_{j,k}\Gamma^i_{j,k}(u^1,\ldots,u^m)\left(u^j_t u^k_t - b^2(t)\nabla u^j\cdot\nabla u^k\right) = 0,$$

$i = 1,\ldots m$, where $\Gamma^i_{j,k}(u)$, $b(t)$ are C^∞ functions satisfying condition (9). The choice of the initial data

$$u^i(0,x) = a_i u_0(x), \quad u^i_t(0,x) = a_i u_1(x), \qquad i = 1,\ldots m, \quad x\in\mathbb{R}^n,$$

for the system of equations and the intrinsic self coherent structure of the manifold force a unique local solution to be on the track of the distinguished geodesic. This allows the lowering of the wave map system to the scalar equation. Indeed, if we consider the Cauchy problem for the auxiliary scalar equation

$$\begin{cases} u_{tt} - n\dfrac{\dot{b}(t)}{b(t)}u_t - b^2(t)\Delta u + f(u)\left(u_t^2 - b^2(t)\nabla u\cdot\nabla u\right) = 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x\in\mathbb{R}^n, \end{cases} \tag{16}$$

then according to the uniqueness of the solution we have

$$u^1(t,x) = a^1u(t,x),\ u^2(t,x) = a^2u(t,x),\ \ldots,\ u^m(t,x) = a^m u(t,x)$$

for all $x\in\mathbb{R}^n$, $t\geq 0$. Thus we can restrict ourselves to the Cauchy problem (16) for the auxiliary scalar equation, where $f(u)$, $b(t)$ are C^∞ functions and $f(u)$ is from condition (9). For this Cauchy problem we find arbitrarily small smooth initial data and prove that the solution blows up in finite time. This implies that the solution to the problem (8) and (12) blows up in finite time, and competes the proof of Theorem 1.

Consider the equation of (16). By the Hopf-Cole-Nakanishi-Ohta transformation

$$v = G(u) := \int_0^u \exp\left(\int_0^s f(r)dr\right) ds \,, \tag{17}$$

Eq. (16) is transformed into the linear wave equation

$$v_{tt} - n\frac{\dot{b}(t)}{b(t)}v_t - b^2(t)\Delta v = 0\,. \tag{18}$$

Since $G \in C^2(\mathbb{R})$ and $G' > 0$, there exists the inverse of G:

$$H := G^{-1} \in C^2(a, b), \tag{19}$$

where we denote

$$a := \lim_{u\to-\infty} G(u)\,, \qquad b := \lim_{u\to\infty} G(u)\,. \tag{20}$$

Next we apply the partial Liouville transformation that eliminates the first derivative v_t in (18). More precisely, we set

$$v = b^{\frac{n}{2}}(t)w\,, \quad b(t) = 1/a(t),$$

then

$$\begin{aligned}
&v_{tt} - n\frac{\dot{b}(t)}{b(t)}v_t - b^2(t)\Delta v \\
&= b^{\frac{n}{2}}(t)\left[w_{tt} - b^2(t)\Delta w + \left\{\frac{n}{2}\left(1-\frac{n}{2}\right)\left(\frac{d}{dt}\frac{1}{b(t)}\right)^2 b^2(t) - \frac{n}{2}\left(\frac{d^2}{dt^2}\frac{1}{b(t)}\right)b(t)\right\} w\right].
\end{aligned}$$

Thus, we have to study the following linear hyperbolic equation

$$w_{tt} - b^2(t)\Delta w + \left(\frac{n}{2}\left(1-\frac{n}{2}\right)\left(\frac{d}{dt}\frac{1}{b(t)}\right)^2 b^2(t) - \frac{n}{2}\left(\frac{d^2}{dt^2}\frac{1}{b(t)}\right)b(t)\right) w = 0$$

with the 1-periodic positive smooth function $b = b(t)$.

4 Floquet-Lyapunov Theory: Parametric Resonance in ODE

We are going to apply the Floquet-Lyapunov theory for ordinary differential equations with the periodic coefficients. Consider the ordinary differential equation:

$$W_{tt} + \left(\lambda b^2(t) + \frac{n}{2}\left(1-\frac{n}{2}\right)\left(\frac{d}{dt}\frac{1}{b(t)}\right)^2 b^2(t) - \frac{n}{2}\left(\frac{d^2}{dt^2}\frac{1}{b(t)}\right)b(t)\right)w = 0$$

with the periodic positive smooth non-constant function $b = b(t)$ and parameter $\lambda \in \mathbb{R}$.

It is more convenient to rewrite this equation by means of the new positive periodic function

$$\alpha(t) := b^2(t),$$

then

$$W_{tt} + \left\{\lambda\alpha(t) - \frac{n}{4}\left[\frac{3}{2}\left(\frac{\dot\alpha(t)}{\alpha(t)}\right)^2 - \frac{\ddot\alpha(t)}{\alpha(t)}\right] - \frac{n}{8}\left(\frac{n}{2}-1\right)\left(\frac{\dot\alpha(t)}{\alpha(t)}\right)^2\right\}W = 0\,.$$

Consider now the equation

$$y_{tt}(t) + (\lambda\alpha(t) - q(t))\,y(t) = 0 \tag{21}$$

with the periodic coefficients $\alpha(t) = b^2(t)$ and

$$q(t) = \frac{n}{4}\left[\frac{3}{2}\left(\frac{\dot\alpha(t)}{\alpha(t)}\right)^2 - \frac{\ddot\alpha(t)}{\alpha(t)}\right] - \frac{n}{8}\left(\frac{n}{2}-1\right)\left(\frac{\dot\alpha(t)}{\alpha(t)}\right)^2 .$$

The first part of the last expression is the so-called Schwarz derivative for the antiderivative of $\alpha(t)$. For equation (21) the spectrum of the eigenvalue problem with the boundary condition

$$y(0) = y(1) = 0$$

is discrete. Equation (21) can be written also as a system of differential equations for the vector-valued function $x(t) = {}^t(w_t, w)$:

$$\frac{d}{dt}x(t) = A(t)x(t)\,, \qquad \text{where} \quad A(t) := \begin{pmatrix} 0 & -\lambda\alpha(t) + q(t) \\ 1 & 0 \end{pmatrix}.$$

Let the matrix-valued function $X_\lambda(t,t_0)$, depending on λ, be a solution of the Cauchy problem

$$\frac{d}{dt}X = A(t)X\,, \qquad X(t_0,t_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22}$$

Thus, $X_\lambda(t,t_0)$ gives a fundamental solution to Eq. (21). In what follows we often omit the subindex λ of $X_\lambda(t,t_0)$. The Liouville formula

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} S(\tau)d\tau\right),$$

where $W(t) := \det X(t,t_0)$, $S(t) := \sum_{k=1}^{2} A_{kk}(t)$ with $S(t) \equiv 0$ guarantees the existence of the inverse matrix $X_\lambda(t,t_0)^{-1}$. For the matrix $X(1,0)$ we will use a notation

$$X_\lambda(1,0) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

This matrix is called a *monodromy matrix* and its eigenvalues are called *multipliers* of system (22). Thus, the monodromy matrix is the value at $t = 1$ of the fundamental matrix $X(t,0)$ defined by the initial condition $X(0,0) = I$, and the multipliers are the roots of the equation

$$\det\left[X(1,0) - \mu I\right] = 0\,.$$

Due to Theorem 2.3.1 [3] there exist the open instability intervals. The Assumption ISIN states that there exists the nonempty open instability interval $\Lambda \subset (0,\infty)$ for Eq. (21).

One can find in [3, 12] the detailed description of functions $\alpha = \alpha(t)$ and $q = q(t)$ satisfying this condition. For instance, in Theorem 4.4.1 [3] one can find asymptotic formula, which allows to estimate the length of the instability intervals of the equation obtained from (21) by Liouville transformation. Then, according to the next lemma one can find in the instability interval Λ a number λ such that a non-diagonal element of the monodromy matrix does not vanish. Moreover, this property is stable under small perturbations of λ.

Lemma 2 ([23]) *Let $b(t)$ be a non-constant, positive, smooth function defined on $\mathbb{R}$, which is 1-periodic. Then, there exists an open subset $\Lambda^0 \subset \Lambda$ such that $b_{21} \neq 0$ for all $\lambda \in \Lambda^0$.*

Next we use the periodicity of $b = b(t)$ and the eigenvalues $\mu_0 > 1$, $\mu_0^{-1} < 1$ of the matrix $X_\lambda(1,0)$ to construct solutions of (21) with prescribed values on a discrete set of time. The eigenvalues of the matrix $X_\lambda(1,0)$ are μ_0 and μ_0^{-1} with

$b_{11} + b_{22} = \mu_0 + \mu_0^{-1}$. Hence $(b_{11} - \mu_0) + (b_{22} - \mu_0) = -\mu_0 + \mu_0^{-1}$ implies $|b_{11} - \mu_0| + |b_{22} - \mu_0| \geq |(b_{11} - \mu_0) + (b_{22} - \mu_0)| = |\mu_0 - \mu_0^{-1}| > 0$. This leads to

$$\max\{|b_{11} - \mu_0|, |b_{22} - \mu_0|\} \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}| > 0\,.$$

Without loss of generality we can suppose

$$|b_{11} - \mu_0| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}| > 0\,, \quad |b_{22} - \mu_0^{-1}| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}| > 0\,,$$

because of $b_{11} - \mu_0 = -(b_{22} - \mu_0^{-1})$. Further,

$$1 - \frac{b_{21}}{\mu_0^{-1} - b_{22}} \frac{b_{12}}{\mu_0 - b_{11}} = (\mu_0 - \mu_0^{-1}) \frac{1}{b_{22} - \mu_0^{-1}} \neq 0\,.$$

Lemma 3 ([23]) *Let* $W = W(t),\ V = V(t)$ *be two solutions of the equation*

$$w_{tt} + (\lambda\alpha(t) - q(t))\, w = 0$$

with the parameter λ *such that* $b_{21} \neq 0$ *and* $b_{22} \neq \mu_0^{-1}$. *Suppose then that* $W = W(t)$ *takes the initial data*

$$W(0) = 0\,, \quad W_t(0) = 1\,,$$

and that $V = V(t)$ *takes the initial data*

$$V(0) = 1\,, \quad V_t(0) = 0\,.$$

Then for every positive integer number $M \in \mathbb{N}$ *one has*

$$W(M) = \frac{b_{21}}{\mu_0 - \mu_0^{-1}}(\mu_0^M - \mu_0^{-M})\,,$$

$$V(M) = -\mu_0^M \frac{(b_{22} - \mu_0^{-1})}{(\mu_0 - \mu_0^{-1})} + \mu_0^{-M} \frac{b_{21} b_{12}}{(\mu_0 - b_{11})(\mu_0 - \mu_0^{-1})}.$$

For more applications of the Floquet-Lyapunov theory to hyperbolic equations with oscillating coefficients see [14, 16, 20] and the bibliography therein. On the other hand, to study the hyperbolic equations with oscillating coefficients one can apply the so-called method of zones (see, e.g., [5, 6, 21, 25] and the bibliography therein).

5 Proof of Theorem 1: Construction of Blow-up Solution to the Scalar PDE

If condition (5) of Theorem 1 does not hold, then (10) is true, that is, $a > -\infty$ or $b < \infty$.
If $u(t, x)$ is a solution of (16) and takes initial values (12) then the function (17) solves the linear equation (18) and takes initial values

$$v(0, x) = \int_0^{u_0(x)} \exp\left(\int_0^s f(r)dr\right) ds\,, \quad v_t(0, x) = u_1(x) \exp\left(\int_0^{u_0(x)} f(r)dr\right). \tag{23}$$

Now let us choose initial data with the positive numbers $S > 2n$ and M which will be chosen later

$$u_0(x) = \frac{1}{M^S}\chi\left(\frac{x}{M^2}\right) \in C_0^\infty(\mathbb{R}^n),$$
$$u_1(x) = \frac{A}{M^S}\chi\left(\frac{x}{M^2}\right) \exp\left(-\int_0^{u_0(x)} f(r)dr\right) \cos(x \cdot y) \in C_0^\infty(\mathbb{R}^n),$$

where $y \in \mathbb{R}^n$, $|y|^2 = \lambda$, λ is from the instability interval stated by ISIN, while $\chi \in C_0^\infty(R^n)$ is a non-negative cut-off function, $\chi(x) = 1$ when $|x| \leq 1$. The number $A = \pm 1$, which is independent of the large parameter $M \in N$, will be chosen later. Let $u = u(t, x)$ be a classical solution of (16) which takes these initial data. Then the function $v(t, x) = G\big(u(t, x)\big)$ solves Eq. (18) and at $t = 0$ takes values

$$v(0, x) = \int_0^{\frac{1}{M^S}\chi\left(\frac{x}{M^2}\right)} \exp\left(\int_0^s f(r)dr\right) ds \in C_0^\infty(\mathbb{R}^n)\,,$$
$$v_t(0, x) = \frac{A}{M^S}\chi\left(\frac{x}{M^2}\right) \cos(x \cdot y) \in C_0^\infty(\mathbb{R}^n)\,.$$

Let $W = W(t)$ be a solution given by Lemma 3. Consider the function

$$V(t, x) = \int_0^{\frac{1}{M^S}} \exp\left(\int_0^s f(r)dr\right) ds + W(t)\frac{b^{n/2}(t)}{b^{n/2}(0)}\frac{A}{M^S} \cos(x \cdot y) \in C^\infty([0, \infty] \times \mathbb{R}^n)\,.$$

Function $V(t, x)$ solves Eq. (18) while

$$V(0, x) = \int_0^{\frac{1}{M^S}} \exp\left(\int_0^s f(r)dr\right) ds, \quad V_t(0, x) = \frac{A}{M^S} \cos(x \cdot y) \text{ for all } x \in \mathbb{R}^n\,.$$

On the other hand for the function $v(t,x)$ we have

$$v(0,x) = \int_0^{\frac{1}{M^S}} \exp\left(\int_0^s f(r)dr\right) ds, \quad v_t(0,x) = \frac{A}{M^S}\cos(x\cdot y) \text{ when } |x| \le M^2.$$

The finite propagation speed in the Cauchy problem (18), (23) implies

$$V(t,x) = v(t,x) \quad \text{in} \quad \Pi_M := [0,M]\times\{x\in\mathbb{R}^n;\, |x|\le M^{3/2}\}$$

for large integer M. Hence

$$v(t,x) = \int_0^{\frac{1}{M^S}} \exp\left(\int_0^s f(r)dr\right) ds + W(t)\frac{b^{n/2}(t)}{b^{n/2}(0)}\frac{A}{M^S}\cos(x\cdot y) \quad \text{in} \quad \Pi_M\,.$$

In particular,

$$v(M,0) = \int_0^{\frac{1}{M^S}} \exp\left(\int_0^s f(r)dr\right) ds + \frac{A}{M^S}\frac{b_{21}}{\mu_0-\mu_0^{-1}}(\mu_0^M - \mu_0^{-M})\,.$$

Assume now that $b<\infty$. Then the global existence of u means

$$v(t,x) = \int_0^{u(t,x)} \exp\left(\int_0^s f(r)dr\right) ds < b \quad \text{for all} \quad t\ge 0,\ x\in\mathbb{R}^n\,. \tag{24}$$

We choose $A=1$, and S such that for M large enough one has (11) for u_0, u_1. On the other hand, there is a number $t(M)\in[0,M]$ such that $v(t(M),0) > b$. This contradicts (24). The case of $a>-\infty$ can be treated in a similar way. The theorem is proved. □

6 Proof of Theorem 2

Assume that the problem has a global solution $(u^1(x,t),\ldots,u^m(x,t))\in C^\infty$ for every initial data $(u_\ell^1(x),\ldots,u_\ell^m(x))\in C^\infty(\mathbb{R}^n)\times\ldots\times C^\infty(\mathbb{R}^n)$, $\ell=0,1$. We are going to prove that the Nakanishi-Ohta condition (5) is fulfilled. Consider the system (8), where $\Gamma^i_{j,k}(u)$ are C^∞ functions satisfying condition (9) and

$$u^i(0,x) = a^i u_0(x), \quad u^i_t(0,x) = a^i u_1(x)\,, \qquad i=1,\ldots m, \quad x\in\mathbb{R}^n\,.$$

Consider also the Cauchy problem (16) for the scalar equation with the initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x)\,, \quad x\in\mathbb{R}^n\,.$$

Then the uniqueness and existence theorem and condition (9) imply

$$u^1(t,x) = a^1 u(t,x),\; u^2(t,x) = a^2 u(t,x),\; \dots,\; u^m(t,x) = a^m u(t,x)$$

for all $x \in \mathbb{R}^n$, $t \geq 0$. Thus we have obtained the existence of the global solution for the Cauchy problem for the nonlinear hyperbolic scalar equation (16).

Now we turn to the scalar equation of (16), where $f(u)$, $b(t)$ are C^∞ functions and $f(u)$ is from condition (9). The Hopf-Cole-Nakanishi-Ohta transformation converted equation of (16) into the linear wave equation for v defined by (18). For a and b defined by (20), there exists the inverse H (19) of G since $G \in C^2(\mathbb{R})$ and $G' > 0$. We choose initial data

$$u_0(x) = 0\,, \qquad u_1(x) = 1\,,$$

then

$$v(0) = 0, \quad v_t(0) = 1$$

and

$$v_{tt} - n\frac{\dot b(t)}{b(t)} v_t = 0\,.$$

The explicit formula for the solution v implies

$$\int_0^{u(t)} \exp\left(\int_0^s f(r)dr\right) ds = v(t) = b^{-n}(0)\int_0^t b^n(\tau)\,d\tau \to \pm\infty \quad \text{as} \quad t \to \pm\infty\,.$$

Hence condition (5) is fulfilled. The theorem is proved. □

7 Proof of Lemma 1

In some chart the geodesic satisfies the system of equations

$$\frac{d^2u^i}{ds^2}(s) + \sum_{j,k=1}^{m} \Gamma^i_{jk}(u^1(s),\dots,u^m(s))\frac{du^j}{ds}(s)\frac{du^k}{ds}(s) = 0 \quad \text{for all} \quad i = 1,\dots,m\,.$$

For the smooth geodesic lying in the segment I of the straight line $\mathbb{L} = \{(a_1 t,\dots,a_m t)\,|\,t \in \mathbb{R}\}$ of the Riemannian manifold M we have $u^1(s) = a_1 u(s),\; \dots, u^m(s) = a_m u(s)$ for all $s \in [c,d]$ and

$$\left(\frac{du}{ds}(s)\right)^2 \sum_{j,k=1}^{m} \Gamma^i_{jk}(a_1u(s),\dots,a_m u(s))a_j a_k = -a_i\frac{d^2u}{ds^2}(s) \quad \text{for all} \quad s \in [c,d],$$

$i = 1, \ldots, m$. The constant speed property of geodesics imply

$$\left(\frac{du}{ds}(s)\right)^2 \sum_{j,k=1}^{m} h_{kj}(a_1 u(s), \ldots, a_m u(s)) a_j a_k = constant\,.$$

Consequently, the function $du(s)/ds$ has no zeros and we can set

$$\widetilde{f}(s) = -\frac{d^2 u}{ds^2}(s)\left(\frac{du}{ds}(s)\right)^{-2} \quad \text{and} \quad f(u(s)) = \widetilde{f}(s)\,,$$

since the function $u = u(s)$ has an inverse. On the other hand such geodesic covers the segment $I \subseteq \mathbb{L}$ with the parameter $t = u(s)$, and (9) follows.

Conversely, suppose that (9) holds. We can assume that $I = \{(a_1 t, \ldots, a_m t) \mid t \in [1, b]\}$. Then for the point $(a_1, \ldots, a_m) \in I$ we can solve the Cauchy problem for the scalar equation

$$\frac{d^2 u}{ds^2}(s) + f(u(s))\left(\frac{du}{ds}(s)\right)^2 = 0 \tag{25}$$

with the initial condition

$$u(0) = 1\,, \quad \frac{du}{ds}(0) = \widehat{\xi}, \quad \text{where} \quad \widehat{\xi}^2 = \left(\sum_{j,k=1} h_{kj}(a_1, \ldots, a_m) a_j a_k\right)^{-1}.$$

Further, since the point $(a_1 u(s), \ldots, a_m u(s))$ belongs to the segment I for all sufficiently small s, the relation (25) together with (9) implies

$$a_i \frac{d^2 u}{ds^2}(s) + \left(\sum_{j,k=1}^{m} \Gamma^i_{j,k}(a_1 u(s), \ldots, a_m u(s)) a_j a_k\right)\left(\frac{du}{ds}(s)\right)^2 = 0\,.$$

Thus, $(u_1(s), \ldots, u_m(s)) = (a_1 u(s), \ldots, a_m u(s))$ is a geodesic. The existence and uniqueness theorem for the system of ordinary differential equations guarantees that two geodesics with a common point and equal tangent at that point must coincide. Hence, the geodesic covers the segment $I \subseteq \mathbb{L}$. The lemma is proved. □

Remark 4 The Poincaré half-plane model (see, e.g., [14]) possesses vertical half-lines which are distinguished geodesics. Another interesting example of a Lorentzian manifold that possesses half-lines, which are distinguished geodesics is the Schwarzschild spacetime in the Eddington-Finkelstein coordinates (see, e.g., [15, Sec. 8.3]).

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