Reflecting Random Walks in Curvilinear Wedges

Mikhail V. Menshikov, Aleksandar Mijatović, and Andrew R. Wade

Dedicated to our colleague Vladas Sidoravicius (1963–2019)

Abstract We study a random walk (Markov chain) in an unbounded planar domain bounded by two curves of the form $x_2 = a^+ x_1^{\beta^+}$ and $x_2 = -a^- x_1^{\beta^-}$, with $x_1 \geq 0$. In the interior of the domain, the random walk has zero drift and a given increment covariance matrix. From the vicinity of the upper and lower sections of the boundary, the walk drifts back into the interior at a given angle α^+ or α^- to the relevant inwards-pointing normal vector. Here we focus on the case where α^+ and α^- are equal but opposite, which includes the case of normal reflection. For $0 \leq \beta^+, \beta^- < 1$, we identify the phase transition between recurrence and transience, depending on the model parameters, and quantify recurrence via moments of passage times.

Keywords Reflected random walk · Generalized parabolic domain · Recurrence · Transience · Passage-time moments · Normal reflection · Oblique reflection

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1 Introduction and Main Results

1.1 Description of the Model

We describe our model and then state our main results: see Sect. 1.4 for a discussion of related literature. Write $x \in \mathbb{R}^2$ in Cartesian coordinates as $x = (x_1, x_2)$. For parameters $a^+, a^- > 0$ and $\beta^+, \beta^- \geq 0$, define, for $z \geq 0$, functions $d^+(z) := a^+ z^{\beta^+}$ and $d^-(z) := a^- z^{\beta^-}$. Set

$$\mathscr{D} := \left\{ x \in \mathbb{R}^2 : x_1 \geq 0,\ -d^-(x_1) \leq x_2 \leq d^+(x_1) \right\}.$$

Write $\|\cdot\|$ for the Euclidean norm on $\mathbb{R}^2$. For $x \in \mathbb{R}^2$ and $A \subseteq \mathbb{R}^2$, write $d(x, A) := \inf_{y \in A} \|x - y\|$ for the distance from x to A. Suppose that there exist $B \in (0, \infty)$ and a subset $\mathscr{D}_B$ of $\mathscr{D}$ for which every $x \in \mathscr{D}_B$ has $d(x, \mathbb{R}^2 \setminus \mathscr{D}) \leq B$. Let $\mathscr{D}_I := \mathscr{D} \setminus \mathscr{D}_B$; we call $\mathscr{D}_B$ the *boundary* and $\mathscr{D}_I$ the *interior*. Set $\mathscr{D}_B^{\pm} := \{x \in \mathscr{D}_B : \pm x_2 > 0\}$ for the parts of $\mathscr{D}_B$ in the upper and lower half-plane, respectively.

Let $\xi := (\xi_0, \xi_1, \ldots)$ be a discrete-time, time-homogeneous Markov chain on state-space $S \subseteq \mathscr{D}$. Set $S_I := S \cap \mathscr{D}_I$, $S_B := S \cap \mathscr{D}_B$, and $S_B^{\pm} := S \cap \mathscr{D}_B^{\pm}$. Write $\mathbb{P}_x$ and $\mathbb{E}_x$ for conditional probabilities and expectations given $\xi_0 = x \in S$, and suppose that $\mathbb{P}_x(\xi_n \in S \text{ for all } n \geq 0) = 1$ for all $x \in S$. Set $\Delta := \xi_1 - \xi_0$. Then $\mathbb{P}(\xi_{n+1} \in A \mid \xi_n = x) = \mathbb{P}_x(x + \Delta \in A)$ for all $x \in S$, all measurable $A \subseteq \mathscr{D}$, and all $n \in \mathbb{Z}_+$. In what follows, we will always treat vectors in $\mathbb{R}^2$ as column vectors.

We will assume that ξ has uniformly bounded $p > 2$ moments for its increments, that in S_I it has zero drift and a fixed increment covariance matrix, and that it *reflects* in S_B, meaning it has drift away from $\partial\mathscr{D}$ at a certain angle relative to the inwards-pointing normal vector. In fact we permit perturbations of this situation that are appropriately small as the distance from the origin increases. See Fig. 1 for an illustration.

To describe the assumptions formally, for $x_1 > 0$ let $n^+(x_1)$ denote the inwards-pointing unit normal vector to $\partial\mathscr{D}$ at $(x_1, d^+(x_1))$, and let $n^-(x_1)$ be the corresponding normal at $(x_1, -d^-(x_1))$; then $n^+(x_1)$ is a scalar multiple of $(a^+\beta^+ x_1^{\beta^+ - 1}, -1)$, and $n^-(x_1)$ is a scalar multiple of $(a^-\beta^- x_1^{\beta^- - 1}, 1)$. Let $n^+(x_1, \alpha)$ denote the unit vector obtained by rotating $n^+(x_1)$ by angle α anticlockwise. Similarly, let $n^-(x_1, \alpha)$ denote the unit vector obtained by rotating $n^-(x_1)$ by angle α clockwise. (The orientation is such that, in each case, reflection at angle $\alpha < 0$ is pointing on the side of the normal towards 0.)

We write $\|\cdot\|_{\mathrm{op}}$ for the matrix (operator) norm defined by $\|M\|_{\mathrm{op}} := \sup_u \|Mu\|$, where the supremum is over all unit vectors $u \in \mathbb{R}^2$. We take $\xi_0 = x_0 \in S$ fixed, and impose the following assumptions for our main results.

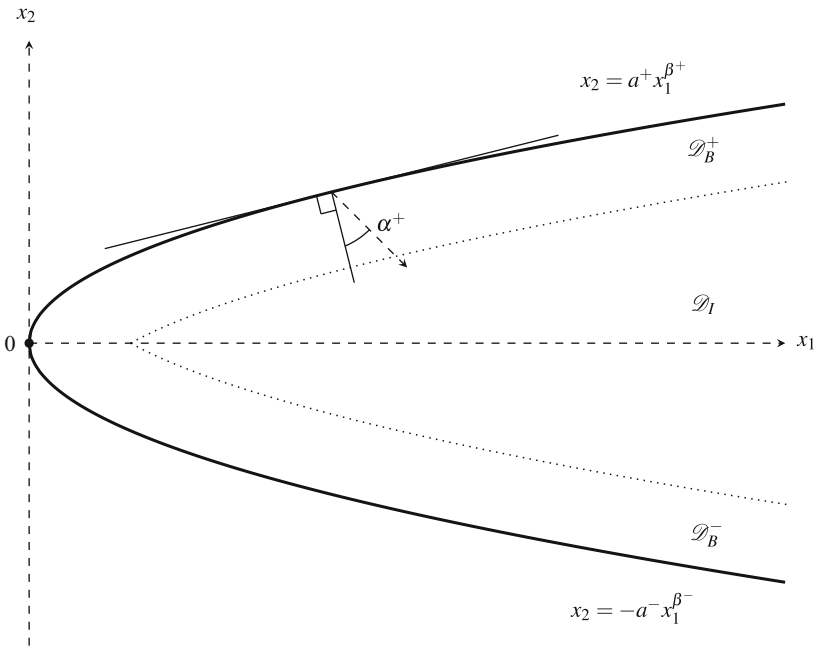

Fig. 1 An illustration of the model parameters, in the case where $\beta^+ = \beta^- \in (0, 1)$

(N) Suppose that $\mathbb{P}_x(\limsup_{n\to\infty} \|\xi_n\| = \infty) = 1$ for all $x \in S$.
(M_p) There exists $p > 2$ such that

$$\sup_{x\in S} \mathbb{E}_x(\|\Delta\|^p) < \infty. \tag{1}$$

(D) We have that $\sup_{x\in S_I:\|x\|\geq r} \|\mathbb{E}_x \Delta\| = o(r^{-1})$ as $r \to \infty$.
(R) There exist angles $\alpha^\pm \in (-\pi/2, \pi/2)$ and functions $\mu^\pm : S_B^\pm \to \mathbb{R}$ with $\liminf_{\|x\|\to\infty} \mu^\pm(x) > 0$, such that, as $r \to \infty$,

$$\sup_{x\in S_B^+:\|x\|\geq r} \|\mathbb{E}_x \Delta - \mu^+(x) n^+(x_1, \alpha^+)\| = O(r^{-1}); \tag{2}$$

$$\sup_{x\in S_B^-:\|x\|\geq r} \|\mathbb{E}_x \Delta - \mu^-(x) n^-(x_1, \alpha^-)\| = O(r^{-1}). \tag{3}$$

(C) There exists a positive-definite, symmetric 2×2 matrix Σ for which

$$\lim_{r\to\infty} \sup_{x\in S_I:\|x\|\geq r} \left\| \mathbb{E}_x(\Delta\Delta^\top) - \Sigma \right\|_{\mathrm{op}} = 0.$$

We write the entries of Σ in (C) as

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{pmatrix}.$$

Here ρ is the asymptotic increment covariance, and, since Σ is positive definite, $\sigma_1 > 0$, $\sigma_2 > 0$, and $\rho^2 < \sigma_1^2 \sigma_2^2$.

To identify the critically recurrent cases, we need slightly sharper control of the error terms in the drift assumption (D) and covariance assumption (C). In particular, we will in some cases impose the following stronger versions of these assumptions:

(D_+) There exists $\varepsilon > 0$ such that $\sup_{x \in S_I : \|x\| \geq r} \| \mathbb{E}_x \, \Delta \| = O(r^{-1-\varepsilon})$ as $r \to \infty$.

(C_+) There exists $\varepsilon > 0$ and a positive definite symmetric 2×2 matrix Σ for which

$$\sup_{x \in S_I : \|x\| \geq r} \left\| \mathbb{E}_x (\Delta \Delta^\top) - \Sigma \right\|_{\text{op}} = O(r^{-\varepsilon}), \text{ as } r \to \infty.$$

Without loss of generality, we may use the same constant $\varepsilon > 0$ for both (D_+) and (C_+).

The non-confinement condition (N) ensures our questions of recurrence and transience (see below) are non-trivial, and is implied by standard irreducibility or ellipticity conditions: see [26] and the following example.

Example 1 Let $S = \mathbb{Z}^2 \cap \mathscr{D}$, and take $\mathscr{D}_B$ to be the set of $x \in \mathscr{D}$ for which x is within unit ℓ_∞-distance of some $y \in \mathbb{Z}^2 \setminus \mathscr{D}$. Then S_B contains those points of S that have a neighbour outside of $\mathscr{D}$, and S_I consists of those points of S whose neighbours are all in $\mathscr{D}$. If ξ is irreducible on S, then (N) holds (see e.g. Corollary 2.1.10 of [26]). If $\beta^+ > 0$, then, for all $\|x\|$ sufficiently large, every point of $x \in S_B^+$ has its neighbours to the right and below in S, so if $\alpha^+ = 0$, for instance, we can achieve the asymptotic drift required by (2) using only nearest-neighbour jumps if we wish; similarly in S_B^-.

Under the non-confinement condition (N), the first question of interest is whether $\liminf_{n \to \infty} \|\xi_n\|$ is finite or infinite. We say that ξ is *recurrent* if there exists $r_0 \in \mathbb{R}_+$ for which $\liminf_{n \to \infty} \|\xi_n\| \leq r_0$, a.s., and that ξ is *transient* if $\lim_{n \to \infty} \|\xi_n\| = \infty$, a.s. The first main aim of this paper is to classify the process into one or other of these cases (which are not a priori exhaustive) depending on the parameters. Further, in the recurrent cases it is of interest to quantify the recurrence by studying the tails (or moments) of return times to compact sets. This is the second main aim of this paper.

In the present paper we focus on the case where $\alpha^+ + \alpha^- = 0$, which we call 'opposed reflection'. This case is the most subtle from the point of view of recurrence/transience, and, as we will see, exhibits a rich phase diagram depending on the model parameters. We emphasize that the model in the case $\alpha^+ + \alpha^- = 0$ is *near-critical* in that both recurrence and transience are possible, depending on the parameters, and moreover (i) in the recurrent cases, return-times to bounded sets

have heavy tails being, in particular, non-integrable, and so stationary distributions will not exist, and (ii) in the transient cases, escape to infinity will be only diffusive. There is a sense in which the model studied here can be viewed as a perturbation of zero-drift random walks, in the manner of the seminal work of Lamperti [19]: see e.g. [26] for a discussion of near-critical phenomena. We leave for future work the case $\alpha^+ + \alpha^- \neq 0$, in which very different behaviour will occur: if $\beta^\pm < 1$, then the case $\alpha^+ + \alpha^- > 0$ gives super-diffusive (but sub-ballistic) transience, while the case $\alpha^+ + \alpha^- < 0$ leads to positive recurrence.

Opposed reflection includes the special case where $\alpha^+ = \alpha^- = 0$, which is 'normal reflection'. Since the results are in the latter case more easily digested, and since it is an important case in its own right, we present the case of normal reflection first, in Sect. 1.2. The general case of opposed reflection we present in Sect. 1.3. In Sect. 1.4 we review some of the extensive related literature on reflecting processes. Then Sect. 1.5 gives an outline of the remainder of the paper, which consists of the proofs of the results in Sects. 1.2–1.3.

1.2 Normal Reflection

First we consider the case of *normal* (i.e., orthogonal) reflection.

Theorem 1 *Suppose that* (N), (M_p), (D), (R), *and* (C) *hold with* $\alpha^+ = \alpha^- = 0$.

(a) Suppose that $\beta^+, \beta^- \in [0, 1)$. *Let* $\beta := \max(\beta^+, \beta^-)$. *Then the following hold.*

(i) If $\beta < \sigma_1^2/\sigma_2^2$, *then* ξ *is recurrent.*
(ii) If $\sigma_1^2/\sigma_2^2 < \beta < 1$, *then* ξ *is transient.*
(iii) If, in addition, (D_+) *and* (C_+) *hold, then the case* $\beta = \sigma_1^2/\sigma_2^2$ *is recurrent.*

(b) Suppose that (D_+) *and* (C_+) *hold, and* $\beta^+, \beta^- > 1$. *Then* ξ *is recurrent.*

Remark 1

(i) Omitted from Theorem 1 is the case when at least one of $\beta^\pm$ is equal to 1, or their values fall each each side of 1. Here we anticipate behaviour similar to [5].
(ii) If $\sigma_1^2/\sigma_2^2 < 1$, then Theorem 1 shows a striking *non-monotonicity* property: there exist regions $\mathscr{D}_1 \subset \mathscr{D}_2 \subset \mathscr{D}_3$ such that the reflecting random walk is recurrent on $\mathscr{D}_1$ and $\mathscr{D}_3$, but transient on $\mathscr{D}_2$. This phenomenon does not occur in the classical case when Σ is the identity: see [28] for a derivation of monotonicity in the case of normally reflecting Brownian motion in unbounded domains in $\mathbb{R}^d$, $d \geq 2$.
(iii) Note that the correlation ρ and the values of a^+, a^- play no part in Theorem 1; ρ will, however, play a role in the more general Theorem 3 below.

Let $\tau_r := \min\{n \in \mathbb{Z}_+ : \|\xi_n\| \le r\}$. Define

$$s_0 := s_0(\Sigma, \beta) := \frac{1}{2}\left(1 - \frac{\sigma_2^2 \beta}{\sigma_1^2}\right). \tag{4}$$

Our next result concerns the moments of τ_r. Since most of our assumptions are asymptotic, we only make statements about r sufficiently large; with appropriate irreducibility assumptions, this restriction could be removed.

Theorem 2 *Suppose that* (N), (M_p), (D), (R), *and* (C) *hold with* $\alpha^+ = \alpha^- = 0$.

(a) Suppose that $\beta^+, \beta^- \in [0, 1)$. *Let* $\beta := \max(\beta^+, \beta^-)$. *Then the following hold.*

(i) If $\beta < \sigma_1^2/\sigma_2^2$, *then* $\mathbb{E}_x(\tau_r^s) < \infty$ *for all* $s < s_0$ *and all* r *sufficiently large, but* $\mathbb{E}_x(\tau_r^s) = \infty$ *for all* $s > s_0$ *and all* x *with* $\|x\| > r$ *for* r *sufficiently large.*

(ii) If $\beta \ge \sigma_1^2/\sigma_2^2$, *then* $\mathbb{E}_x(\tau_r^s) = \infty$ *for all* $s > 0$ *and all* x *with* $\|x\| > r$ *for* r *sufficiently large.*

(b) Suppose that $\beta^+, \beta^- > 1$. *Then* $\mathbb{E}_x(\tau_r^s) = \infty$ *for all* $s > 0$ *and all* x *with* $\|x\| > r$ *for* r *sufficiently large.*

Remark 2

(i) Note that if $\beta < \sigma_1^2/\sigma_2^2$, then $s_0 > 0$, while $s_0 < 1/2$ for all $\beta > 0$, in which case the return time to a bounded set has a heavier tail than that for one-dimensional simple symmetric random walk.

(ii) The transience result in Theorem 1(a)(ii) is essentially stronger than the claim in Theorem 2(a)(ii) for $\beta < \sigma_1^2/\sigma_2^2$, so the borderline (recurrent) case $\beta = \sigma_1^2/\sigma_2^2$ is the main content of the latter.

(iii) Part (b) shows that the case $\beta^\pm > 1$ is critical: no moments of return times exist, as in the case of, say, simple symmetric random walk in $\mathbb{Z}^2$ [26, p. 77].

1.3 Opposed Reflection

We now consider the more general case where $\alpha^+ + \alpha^- = 0$, i.e., the two reflection angles are equal but opposite, relative to their respective normal vectors. For $\alpha^+ = -\alpha^- \ne 0$, this is a particular example of *oblique* reflection. The phase transition in β now depends on ρ and α in addition to σ_1^2 and σ_2^2. Define

$$\beta_c := \beta_c(\Sigma, \alpha) := \frac{\sigma_1^2}{\sigma_2^2} + \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2}\right)\sin^2\alpha + \frac{\rho}{\sigma_2^2}\sin 2\alpha. \tag{5}$$

The next result gives the key properties of the critical threshold function β_c which are needed for interpreting our main result.

Proposition 1 *For a fixed, positive-definite Σ such that $|\sigma_1^2 - \sigma_2^2| + |\rho| > 0$, the function $\alpha \mapsto \beta_c(\Sigma, \alpha)$ over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is strictly positive for $|\alpha| \le \pi/2$, with two stationary points, one in $(-\frac{\pi}{2}, 0)$ and the other in $(0, \frac{\pi}{2})$, at which the function takes its maximum/minimum values of*

$$\frac{1}{2} + \frac{\sigma_1^2}{2\sigma_2^2} \pm \frac{1}{2\sigma_2^2}\sqrt{\left(\sigma_1^2 - \sigma_2^2\right)^2 + 4\rho^2}. \tag{6}$$

The exception is the case where $\sigma_1^2 - \sigma_2^2 = \rho = 0$, when $\beta_c = 1$ is constant.

Here is the recurrence classification in this setting.

Theorem 3 *Suppose that* (N), (M_p), (D), (R), *and* (C) *hold with $\alpha^+ = -\alpha^- = \alpha$ for $|\alpha| < \pi/2$.*

(a) Suppose that $\beta^+, \beta^- \in [0, 1)$. Let $\beta := \max(\beta^+, \beta^-)$. Then the following hold.

(i) If $\beta < \beta_c$, then ξ is recurrent.
(ii) If $\beta > \beta_c$, then ξ is transient.
(iii) If, in addition, (D_+) *and* (C_+) *hold, then the case $\beta = \beta_c$ is recurrent.*

(b) Suppose that (D_+) *and* (C_+) *hold, and $\beta^+, \beta^- > 1$. Then ξ is recurrent.*

Remark 3

(i) The threshold (5) is invariant under the map $(\alpha, \rho) \mapsto (-\alpha, -\rho)$.
(ii) For fixed Σ with $|\sigma_1^2 - \sigma_2^2| + |\rho| > 0$, Proposition 1 shows that β_c is non-constant and has exactly one maximum and exactly one minimum in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Since $\beta_c(\Sigma, \pm\frac{\pi}{2}) = 1$, it follows from uniqueness of the minimum that the minimum is strictly less than 1, and so Theorem 3 shows that there is always an open interval of α for which there is transience.
(iii) Since $\beta_c > 0$ always, recurrence is certain for small enough β.
(iv) In the case where $\sigma_1^2 = \sigma_2^2$ and $\rho = 0$, then $\beta_c = 1$, so recurrence is certain for all $\beta^+, \beta^- < 1$ and all α.
(v) If $\alpha = 0$, then $\beta_c = \sigma_1^2/\sigma_2^2$, so Theorem 3 generalizes Theorem 1.

Next we turn to passage-time moments. We generalize (4) and define

$$s_0 := s_0(\Sigma, \alpha, \beta) := \frac{1}{2}\left(1 - \frac{\beta}{\beta_c}\right), \tag{7}$$

with β_c given by (5). The next result includes Theorem 2 as the special case $\alpha = 0$.

Theorem 4 *Suppose that* (N), (M_p), (D), (R), *and* (C) *hold with $\alpha^+ = -\alpha^- = \alpha$ for $|\alpha| < \pi/2$.*

(a) Suppose that $\beta^+, \beta^- \in [0, 1)$. *Let* $\beta := \max(\beta^+, \beta^-)$. *Then the following hold.*

(i) If $\beta < \beta_c$, *then* $s_0 \in (0, 1/2]$, *and* $\mathbb{E}_x(\tau_r^s) < \infty$ *for all* $s < s_0$ *and all* r *sufficiently large, but* $\mathbb{E}_x(\tau_r^s) = \infty$ *for all* $s > s_0$ *and all* x *with* $\|x\| > r$ *for* r *sufficiently large.*

(ii) If $\beta \geq \beta_c$, *then* $\mathbb{E}_x(\tau_r^s) = \infty$ *for all* $s > 0$ *and all* x *with* $\|x\| > r$ *for* r *sufficiently large.*

(b) Suppose that $\beta^+, \beta^- > 1$. *Then* $\mathbb{E}_x(\tau_r^s) = \infty$ *for all* $s > 0$ *and all* x *with* $\|x\| > r$ *for* r *sufficiently large.*

1.4 Related Literature

The stability properties of reflecting random walks or diffusions in unbounded domains in $\mathbb{R}^d$ have been studied for many years. A pre-eminent place in the development of the theory is occupied by processes in the quadrant $\mathbb{R}^2_+$ or quarter-lattice $\mathbb{Z}^2_+$, due to applications arising in queueing theory and other areas. Typically, the process is assumed to be maximally homogeneous in the sense that the transition mechanism is fixed in the interior and on each of the two half-lines making up the boundary. Distinct are the cases where the motion in the interior of the domain has *non-zero* or *zero drift*.

It was in 1961, in part motivated by queueing models, that Kingman [18] proposed a general approach to the non-zero drift problem on $\mathbb{Z}^2_+$ via Lyapunov functions and Foster's Markov chain classification criteria [14]. A formal statement of the classification was given in the early 1970s by Malyshev, who developed both an analytic approach [22] as well as the Lyapunov function one [23] (the latter, Malyshev reports, prompted by a question of Kolmogorov). Generically, the classification depends on the drift vector in the interior and the two boundary reflection angles. The Lyapunov function approach was further developed, so that the bounded jumps condition in [23] could be relaxed to finiteness of second moments [10, 27, 29] and, ultimately, of first moments [13, 30, 33]. The analytic approach was also subsequently developed [11], and although it seems to be not as robust as the Lyapunov function approach (the analysis in [22] was restricted to nearest-neighbour jumps), when it is applicable it can yield very precise information: see e.g. [15] for a recent application in the continuum setting. Intrinsically more complicated results are available for the non-zero drift case in $\mathbb{Z}^3_+$ [24] and $\mathbb{Z}^4_+$ [17].

The recurrence classification for the case of *zero-drift* reflecting random walk in $\mathbb{Z}^2_+$ was given in the early 1990s in [6, 12]; see also [13]. In this case, generically, the classification depends on the increment covariance matrix in the interior as well as the two boundary reflection angles. Subsequently, using a semimartingale approach

extending work of Lamperti [19], passage-time moments were studied in [5], with refinements provided in [2, 3].

Parallel continuum developments concern reflecting Brownian motion in wedges in $\mathbb{R}^2$. In the zero-drift case with general (oblique) reflections, in the 1980s Varadhan and Williams [31] had showed that the process was well-defined, and then Williams [32] gave the recurrence classification, thus preceding the random walk results of [6, 12], and, in the recurrent cases, asymptotics of stationary measures (cf. [4] for the discrete setting). Passage-time moments were later studied in [7, 25], by providing a continuum version of the results of [5], and in [2], using discrete approximation [1]. The non-zero drift case was studied by Hobson and Rogers [16], who gave an analogue of Malyshev's theorem in the continuum setting.

For domains like our $\mathscr{D}$, Pinsky [28] established recurrence in the case of reflecting Brownian motion with normal reflections and standard covariance matrix in the interior. The case of general covariance matrix and oblique reflection does not appear to have been considered, and neither has the analysis of passage-time moments. The somewhat related problem of the asymptotics of the *first exit time* τ_e of planar Brownian motion from domains like our $\mathscr{D}$ has been considered [8, 9, 20]: in the case where $\beta^+ = \beta^- = \beta \in (0, 1)$, then $\log \mathbb{P}(\tau_e > t)$ is bounded above and below by constants times $-t^{(1-\beta)/(1+\beta)}$: see [20] and (for the case $\beta = 1/2$) [9].

1.5 Overview of the Proofs

The basic strategy is to construct suitable Lyapunov functions $f : \mathbb{R}^2 \to \mathbb{R}$ that satisfy appropriate semimartingale (i.e., drift) conditions on $\mathbb{E}_x[f(\xi_1) - f(\xi_0)]$ for x outside a bounded set. In fact, since the Lyapunov functions that we use are most suitable for the case where the interior increment covariance matrix is $\Sigma = I$, the identity, we first apply a linear transformation T of $\mathbb{R}^2$ and work with $T\xi$. The linear transformation is described in Sect. 2. Of course, one could combine these two steps and work directly with the Lyapunov function given by the composition $f \circ T$ for the appropriate f. However, for reasons of intuitive understanding and computational convenience, we prefer to separate the two steps.

Let $\beta^{\pm} < 1$. Then for $\alpha^+ = \alpha^- = 0$, the reflection angles are both pointing essentially vertically, with an asymptotically small component in the positive x_1 direction. After the linear transformation T, the reflection angles are no longer almost vertical, but instead are almost opposed at some oblique angle, where the deviation from direct opposition is again asymptotically small, and in the positive x_1 direction. For this reason, the case $\alpha^+ = -\alpha^- = \alpha \neq 0$ is not conceptually different from the simpler case where $\alpha = 0$, because after the linear transformation, both cases are oblique. In the case $\alpha \neq 0$, however, the details are more involved as both α and the value of the correlation ρ enter into the analysis of the Lyapunov functions, which is presented in Sect. 3, and is the main technical work of the paper. For $\beta^{\pm} > 1$, intuition is provided by the case of reflection in the half-plane (see e.g. [32] for the Brownian case).

Once the Lyapunov function estimates are in place, the proofs of the main theorems are given in Sect. 4, using some semimartingale results which are variations on those from [26]. The appendix contains the proof of Proposition 1 on the properties of the threshold function β_c defined at (5).

2 Linear Transformation

The inwards pointing normal vectors to $\partial\mathscr{D}$ at $(x_1, d^{\pm}(x_1))$ are

$$n^{\pm}(x_1) = \frac{1}{r^{\pm}(x_1)} \begin{pmatrix} a^{\pm}\beta^{\pm}x_1^{\beta^{\pm}-1} \\ \mp 1 \end{pmatrix}, \text{ where } r^{\pm}(x_1) := \sqrt{1 + (a^{\pm})^2(\beta^{\pm})^2 x_1^{2\beta^{\pm}-2}}.$$

Define

$$n_{\perp}^{\pm}(x_1) := \frac{1}{r^{\pm}(x_1)} \begin{pmatrix} \pm 1 \\ a^{\pm}\beta^{\pm}x_1^{\beta^{\pm}-1} \end{pmatrix}.$$

Recall that $n^{\pm}(x_1, \alpha^{\pm})$ is the unit vector at angle $\alpha^{\pm}$ to $n^{\pm}(x_1)$, with positive angles measured anticlockwise (for n^{+}) or clockwise (for n^{-}). Then (see Fig. 2 for the case of n^{+}) we have $n^{\pm}(x_1, \alpha^{\pm}) = n^{\pm}(x_1)\cos\alpha^{\pm} + n_{\perp}^{\pm}(x_1)\sin\alpha^{\pm}$, so

$$n^{\pm}(x_1, \alpha^{\pm}) = \frac{1}{r^{\pm}(x_1)} \begin{pmatrix} \sin\alpha^{\pm} + a^{\pm}\beta^{\pm}x_1^{\beta^{\pm}-1}\cos\alpha^{\pm} \\ \mp\cos\alpha^{\pm} \pm a^{\pm}\beta^{\pm}x_1^{\beta^{\pm}-1}\sin\alpha^{\pm} \end{pmatrix}.$$

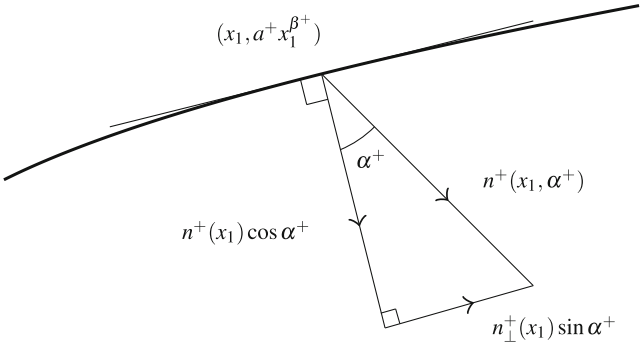

Fig. 2 Diagram describing oblique reflection at angle $\alpha^+ > 0$

In particular, if $\alpha^+ = -\alpha^- = \alpha$,

$$n^{\pm}(x_1, \alpha^{\pm}) = \frac{1}{r^{\pm}(x_1)} \begin{pmatrix} \pm \sin\alpha + a^{\pm}\beta^{\pm} x_1^{\beta^{\pm}-1} \cos\alpha \\ \mp \cos\alpha + a^{\pm}\beta^{\pm} x_1^{\beta^{\pm}-1} \sin\alpha \end{pmatrix} =: \begin{pmatrix} n_1^{\pm}(x_1, \alpha^{\pm}) \\ n_2^{\pm}(x_1, \alpha^{\pm}) \end{pmatrix}. \tag{8}$$

Recall that $\Delta = \xi_1 - \xi_0$. Write $\Delta = (\Delta_1, \Delta_2)$ in components.

Lemma 1 *Suppose that* (R) *holds, with* $\alpha^+ = -\alpha^- = \alpha$ *and* $\beta^+, \beta^- \geq 0$. *If* $\beta^{\pm} < 1$, *then, for* $x \in S_B^{\pm}$, *as* $\|x\| \to \infty$,

$$\mathbb{E}_x \Delta_1 = \pm\mu^{\pm}(x) \sin\alpha + a^{\pm}\beta^{\pm}\mu^{\pm}(x) x_1^{\beta^{\pm}-1} \cos\alpha + O(\|x\|^{2\beta^{\pm}-2}) + O(\|x\|^{-1}); \tag{9}$$

$$\mathbb{E}_x \Delta_2 = \mp\mu^{\pm}(x) \cos\alpha + a^{\pm}\beta^{\pm}\mu^{\pm}(x) x_1^{\beta^{\pm}-1} \sin\alpha + O(\|x\|^{2\beta^{\pm}-2}) + O(\|x\|^{-1}). \tag{10}$$

If $\beta^{\pm} > 1$, *then, for* $x \in S_B^{\pm}$, *as* $\|x\| \to \infty$,

$$\mathbb{E}_x \Delta_1 = \mu^{\pm}(x) \cos\alpha \pm \frac{\mu^{\pm}(x) \sin\alpha}{a^{\pm}\beta^{\pm}} x_1^{1-\beta^{\pm}} + O(x_1^{2-2\beta^{\pm}}) + O(\|x\|^{-1}); \tag{11}$$

$$\mathbb{E}_x \Delta_2 = \mu^{\pm}(x) \sin\alpha \mp \frac{\mu^{\pm}(x) \cos\alpha}{a^{\pm}\beta^{\pm}} x_1^{1-\beta^{\pm}} + O(x_1^{2-2\beta^{\pm}}) + O(\|x\|^{-1}). \tag{12}$$

Proof Suppose that $x \in S_B^{\pm}$. By (2), we have that $\|\mathbb{E}_x \Delta - \mu^{\pm}(x) n^{\pm}(x_1, \alpha^{\pm})\| = O(\|x\|^{-1})$. First suppose that $0 \leq \beta^{\pm} < 1$. Then, $1/r^{\pm}(x_1) = 1 + O(x_1^{2\beta^{\pm}-2})$, and hence, by (8),

$$n_1^{\pm}(x_1, \alpha^{\pm}) = \pm \sin\alpha + a^{\pm}\beta^{\pm} x_1^{\beta^{\pm}-1} \cos\alpha + O(x_1^{2\beta^{\pm}-2});$$
$$n_2^{\pm}(x_1, \alpha^{\pm}) = \mp \cos\alpha + a^{\pm}\beta^{\pm} x_1^{\beta^{\pm}-1} \sin\alpha + O(x_1^{2\beta^{\pm}-2}).$$

Then, since $\|x\| = x_1 + o(x_1)$ as $\|x\| \to \infty$ with $x \in \mathscr{D}$, we obtain (9) and (10).

On the other hand, if $\beta^{\pm} > 1$, then

$$\frac{1}{r^{\pm}(x_1)} = \frac{x_1^{1-\beta^{\pm}}}{a^{\pm}\beta^{\pm}} + O(x_1^{3-3\beta^{\pm}}),$$

and hence, by (8),

$$n_1^{\pm}(x_1, \alpha^{\pm}) = \cos\alpha \pm \frac{\sin\alpha}{a^{\pm}\beta^{\pm}} x_1^{1-\beta^{\pm}} + O(x_1^{2-2\beta^{\pm}});$$
$$n_2^{\pm}(x_1, \alpha^{\pm}) = \sin\alpha \mp \frac{\cos\alpha}{a^{\pm}\beta^{\pm}} x_1^{1-\beta^{\pm}} + O(x_1^{2-2\beta^{\pm}}).$$

The expressions (11) and (12) follow. □

It is convenient to introduce a linear transformation of $\mathbb{R}^2$ under which the asymptotic increment covariance matrix Σ appearing in (C) is transformed to the identity. Define

$$T := \begin{pmatrix} \frac{\sigma_2}{s} & -\frac{\rho}{s\sigma_2} \\ 0 & \frac{1}{\sigma_2} \end{pmatrix}, \text{ where } s := \sqrt{\det \Sigma} = \sqrt{\sigma_1^2\sigma_2^2 - \rho^2};$$

recall that $\sigma_2, s > 0$, since Σ is positive definite. The choice of T is such that $T\Sigma T^{\top} = I$ (the identity), and $x \mapsto Tx$ leaves the horizontal direction unchanged. Explicitly,

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{\sigma_2}{s} x_1 - \frac{\rho}{s\sigma_2} x_2 \\ \frac{1}{\sigma_2} x_2 \end{pmatrix}. \tag{13}$$

Note that T is positive definite, and so $\|Tx\|$ is bounded above and below by positive constants times $\|x\|$. Also, if $x \in \mathscr{D}$ and $\beta^+, \beta^- < 1$, the fact that $|x_2| = o(x_1)$ means that Tx has the properties (i) $(Tx)_1 > 0$ for all x_1 sufficiently large, and (ii) $|(Tx)_2| = o(|(Tx)_1|)$ as $x_1 \to \infty$. See Fig. 3 for a picture.

The next result describes the increment moment properties of the process under the transformation T. For convenience, we set $\tilde{\Delta} := T\Delta$ for the transformed increment, with components $\tilde{\Delta}_i = (T\Delta)_i$.

Lemma 2 *Suppose that* (D), (R), *and* (C) *hold, with* $\alpha^+ = -\alpha^- = \alpha$, *and* $\beta^+, \beta^- \geq 0$. *Then, if* $\|x\| \to \infty$ *with* $x \in S_I$,

$$\|\mathbb{E}_x \tilde{\Delta}\| = o(\|x\|^{-1}), \text{ and } \left\|\mathbb{E}_x(\tilde{\Delta}\tilde{\Delta}^{\top}) - I\right\|_{\text{op}} = o(1). \tag{14}$$

If, in addition, (D_+) *and* (C_+) *hold with* $\varepsilon > 0$, *then, if* $\|x\| \to \infty$ *with* $x \in S_I$,

$$\|\mathbb{E}_x \tilde{\Delta}\| = O(\|x\|^{-1-\varepsilon}), \text{ and } \left\|\mathbb{E}_x(\tilde{\Delta}\tilde{\Delta}^{\top}) - I\right\|_{\text{op}} = O(\|x\|^{-\varepsilon}). \tag{15}$$

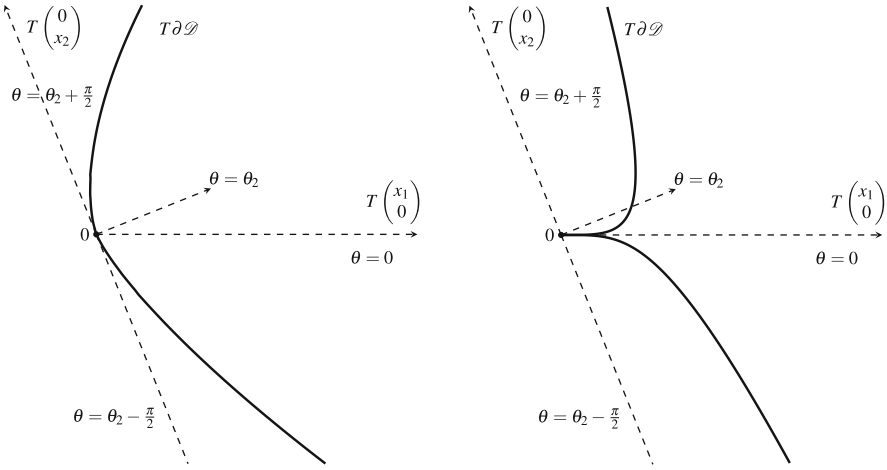

Fig. 3 An illustration of the transformation T with $\rho > 0$ acting on a domain $\mathscr{D}$ with $\beta^+ = \beta^- = \beta$ for $\beta \in (0, 1)$ (*left*) and $\beta > 1$ (*right*). The angle θ_2 is given by $\theta_2 = \arctan(\rho/s)$, measured anticlockwise from the positive horizontal axis

If $\beta^\pm < 1$, *then, as* $\|x\| \to \infty$ *with* $x \in S_B^\pm$,

$$\mathbb{E}_x\,\tilde{\Delta}_1 = \pm\frac{\sigma_2\mu^\pm(x)}{s}\sin\alpha \pm \frac{\rho\mu^\pm(x)}{s\sigma_2}\cos\alpha + \frac{\sigma_2 a^\pm\beta^\pm\mu^\pm(x)}{s}x_1^{\beta^\pm-1}\cos\alpha - \frac{\rho a^\pm\beta^\pm\mu^\pm(x)}{s\sigma_2}x_1^{\beta^\pm-1}\sin\alpha + O(\|x\|^{2\beta^\pm-2}) + O(\|x\|^{-1}); \tag{16}$$

$$\mathbb{E}_x\,\tilde{\Delta}_2 = \mp\frac{\mu^\pm(x)}{\sigma_2}\cos\alpha + \frac{a^\pm\beta^\pm\mu^\pm(x)}{\sigma_2}x_1^{\beta^\pm-1}\sin\alpha + O(\|x\|^{2\beta^\pm-2}) + O(\|x\|^{-1}). \tag{17}$$

If $\beta^\pm > 1$, *then, as* $\|x\| \to \infty$ *with* $x \in S_B^\pm$,

$$\mathbb{E}_x\,\tilde{\Delta}_1 = \frac{\sigma_2\mu^\pm(x)}{s}\cos\alpha - \frac{\rho\mu^\pm(x)}{s\sigma_2}\sin\alpha \pm \frac{\sigma_2\mu^\pm(x)}{a^\pm\beta^\pm s}x_1^{1-\beta^\pm}\sin\alpha \pm \frac{\rho\mu^\pm(x)}{a^\pm\beta^\pm s\sigma_2}x_1^{1-\beta^\pm}\cos\alpha + O(x_1^{2-2\beta^\pm}) + O(\|x\|^{-1}); \tag{18}$$

$$\mathbb{E}_x\,\tilde{\Delta}_2 = \frac{\mu^\pm(x)}{\sigma_2}\sin\alpha \mp \frac{\mu^\pm(x)}{a^\pm\beta^\pm\sigma_2}x_1^{1-\beta^\pm}\cos\alpha + O(x_1^{2-2\beta^\pm}) + O(\|x\|^{-1}). \tag{19}$$

Proof By linearity,

$$\mathbb{E}_x\, \tilde{\Delta} = T\, \mathbb{E}_x\, \Delta, \tag{20}$$

which, by (D) or (D$_+$), is, respectively, $o(\|x\|^{-1})$ or $O(\|x\|^{-1-\varepsilon})$ for $x \in S_I$. Also, since $T\Sigma T^\top = I$, we have

$$\mathbb{E}_x(\tilde{\Delta}\tilde{\Delta}^\top) - I = T\, \mathbb{E}_x(\Delta\Delta^\top)T^\top - I = T\left(\mathbb{E}_x(\Delta\Delta^\top) - \Sigma\right)T^\top.$$

For $x \in S_I$, the middle matrix in the last product here has norm $o(1)$ or $O(\|x\|^{-\varepsilon})$, by (C) or (C$_+$). Thus we obtain (14) and (15). For $x \in S_B^{\pm}$, the claimed results follow on using (20), (13), and the expressions for $\mathbb{E}_x\, \Delta$ in Lemma 1. □

3 Lyapunov Functions

For the rest of the paper, we suppose that $\alpha^+ = -\alpha^- = \alpha$ for some $|\alpha| < \pi/2$. Our proofs will make use of some carefully chosen functions of the process. Most of these functions are most conveniently expressed in polar coordinates.

We write $x = (r, \theta)$ in polar coordinates, with angles measured relative to the positive horizontal axis: $r := r(x) := \|x\|$ and $\theta := \theta(x) \in (-\pi, \pi]$ is the angle between the ray through 0 and x and the ray in the Cartesian direction $(1, 0)$, with the convention that anticlockwise angles are positive. Then $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$.

For $w \in \mathbb{R}$, $\theta_0 \in (-\pi/2, \pi/2)$, and $\gamma \in \mathbb{R}$, define

$$h_w(x) := h_w(r, \theta) := r^w \cos(w\theta - \theta_0), \text{ and } f_w^\gamma(x) := (h_w(Tx))^\gamma, \tag{21}$$

where T is the linear transformation described at (13). The functions h_w were used in analysis of processes in wedges in e.g. [5, 21, 29, 31]. Since the h_w are harmonic for the Laplacian (see below for a proof), Lemma 2 suggests that $h_w(T\xi_n)$ will be approximately a martingale in S_I, and the choice of the geometrical parameter θ_0 gives us the flexibility to try to arrange things so that the level curves of h_w are incident to the boundary at appropriate angles relative to the reflection vectors. The level curves of h_w cross the horizontal axis at angle θ_0: see Fig. 4, and (33) below. In the case $\beta^{\pm} < 1$, the interest is near the horizontal axis, and we take θ_0 to be such that the level curves cut $\partial\mathscr{D}$ at the reflection angles (asymptotically), so that $h_w(T\xi_n)$ will be approximately a martingale also in S_B. Then adjusting w and γ will enable us to obtain a supermartingale with the properties suitable to apply some Foster–Lyapunov theorems. This intuition is solidified in Lemma 4 below, where we show that the parameters w, θ_0, and γ can be chosen so that $f_w^\gamma(\xi_n)$ satisfies an appropriate supermartingale condition outside a bounded set. For the case $\beta^{\pm} < 1$, since we only need to consider $\theta \approx 0$, we could replace these harmonic functions in polar coordinates by suitable polynomial approximations in Cartesian components,

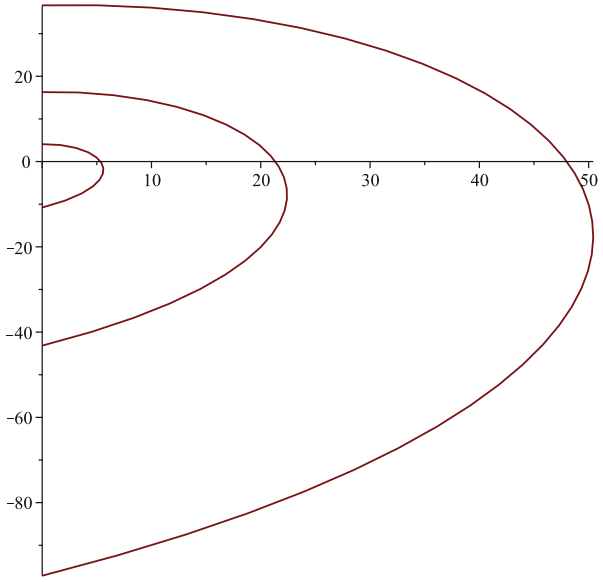

Fig. 4 Level curves of the function $h_w(x)$ with $\theta_0 = \pi/6$ and $w = 1/4$. The level curves cut the horizontal axis at angle θ_0 to the vertical

but since we also want to consider $\beta^{\pm} > 1$, it is convenient to use the functions in the form given. When $\beta^{\pm} > 1$, the recurrence classification is particularly delicate, so we must use another function (see (57) below), although the functions at (21) will still be used to study passage time moments in that case.

If $\beta^+, \beta^- < 1$, then $\theta(x) \to 0$ as $\|x\| \to \infty$ with $x \in \mathscr{D}$, which means that, for any $|\theta_0| < \pi/2$, $h_w(x) \geq \delta \|x\|^w$ for some $\delta > 0$ and all $x \in S$ with $\|x\|$ sufficiently large. On the other hand, for $\beta^+, \beta^- > 1$, we will restrict to the case with $w > 0$ sufficiently small such that $\cos(w\theta - \theta_0)$ is bounded away from zero, uniformly in $\theta \in [-\pi/2, \pi/2]$, so that we again have the estimate $h_w(x) \geq \delta \|x\|^w$ for some $\delta > 0$ and all $x \in \mathscr{D}$, but where now $\mathscr{D}$ is close to the whole half-plane (see Remark 4). In the calculations that follow, we will often use the fact that $h_w(x)$ is bounded above and below by a constant times $\|x\|^w$ as $\|x\| \to \infty$ with $x \in \mathscr{D}$.

We use the notation $D_i := \frac{\mathrm{d}}{\mathrm{d}x_i}$ for differentials, and for $f : \mathbb{R}^2 \to \mathbb{R}$ write Df for the vector with components $(Df)_i = D_i f$. We use repeatedly

$$D_1 r = \cos\theta, \ D_2 r = \sin\theta, \ D_1\theta = -\frac{\sin\theta}{r}, \ D_2\theta = \frac{\cos\theta}{r}. \tag{22}$$

Define

$$\theta_1 := \theta_1(\Sigma, \alpha) := \arctan\left(\frac{\sigma_2^2}{s}\tan\alpha + \frac{\rho}{s}\right) \in (-\pi/2, \pi/2). \tag{23}$$

For $\beta^{\pm} > 1$, we will also need

$$\theta_2 := \theta_2(\Sigma) := \arctan\left(\frac{\rho}{s}\right) \in (-\pi/2, \pi/2), \tag{24}$$

and $\theta_3 := \theta_3(\Sigma, \alpha) \in (-\pi, \pi)$ for which

$$\sin\theta_3 = \frac{s\sin\alpha}{\sigma_2 d}, \text{ and } \cos\theta_3 = \frac{\sigma_2^2\cos\alpha - \rho\sin\alpha}{\sigma_2 d}, \tag{25}$$

where

$$d := d(\Sigma, \alpha) := \sqrt{\sigma_2^2\cos^2\alpha - 2\rho\sin\alpha\cos\alpha + \sigma_1^2\sin^2\alpha}. \tag{26}$$

The geometric interpretation of θ_1, θ_2, and θ_3 is as follows.

- The angle between $(0, \pm 1)$ and $T(0, \pm 1)$ has magnitude θ_2. Thus, if $\beta^{\pm} < 1$, then θ_2 is, as $x_1 \to \infty$, the limiting angle of the transformed inwards pointing normal at x_1 relative to the vertical. On the other hand, if $\beta^{\pm} > 1$, then θ_2 is, as $x_1 \to \infty$, the limiting angle, relative to the horizontal, of the inwards pointing normal to $T\partial\mathscr{D}$. See Fig. 3.
- The angle between $(0, -1)$ and $T(\sin\alpha, -\cos\alpha)$ is θ_1. Thus, if $\beta^{\pm} < 1$, then θ_1 is, as $x_1 \to \infty$, the limiting angle between the vertical and the transformed reflection vector. Since the normal in the transformed domain remains asymptotically vertical, θ_1 is in this case the limiting reflection angle, relative to the normal, after the transformation.
- The angle between $(1, 0)$ and $T(\cos\alpha, \sin\alpha)$ is θ_3. Thus, if $\beta^{\pm} > 1$, then θ_3 is, as $x_1 \to \infty$, the limiting angle between the horizontal and the transformed reflection vector. Since the transformed normal is, asymptotically, at angle θ_2 relative to the horizontal, the limiting reflection angle, relative to the normal, after the transformation is in this case $\theta_3 - \theta_2$.

We need two simple facts.

Lemma 3 *We have (i)* $\inf_{\alpha\in[-\frac{\pi}{2},\frac{\pi}{2}]} d(\Sigma, \alpha) > 0$, *and (ii)* $|\theta_3 - \theta_2| < \pi/2$.

Proof For (i), from (26) we may write

$$d^2 = \sigma_2^2 + \left(\sigma_1^2 - \sigma_2^2\right)\sin^2\alpha - \rho\sin 2\alpha. \tag{27}$$

If $\sigma_1^2 \neq \sigma_2^2$, then, by Lemma 11, the extrema over $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ of (27) are

$$\sigma_2^2 + \frac{\sigma_1^2 - \sigma_2^2}{2}\left(1 \pm \sqrt{1 + \frac{4\rho^2}{(\sigma_1^2 - \sigma_2^2)^2}}\right).$$

Hence

$$d^2 \geq \frac{\sigma_1^2 + \sigma_2^2}{2} - \frac{1}{2}\sqrt{(\sigma_1^2 - \sigma_2^2)^2 + 4\rho^2},$$

which is strictly positive since $\rho^2 < \sigma_1^2\sigma_2^2$. If $\sigma_1^2 = \sigma_2^2$, then $d^2 \geq \sigma_2^2 - |\rho|$, and $|\rho| < |\sigma_1\sigma_2| = \sigma_2^2$, so d is also strictly positive in that case.

For (ii), we use the fact that $\cos(\theta_3 - \theta_2) = \cos\theta_3 \cos\theta_2 + \sin\theta_3 \sin\theta_2$, where, by (24), $\sin\theta_2 = \frac{\rho}{\sigma_1\sigma_2}$ and $\cos\theta_2 = \frac{s}{\sigma_1\sigma_2}$, and (25), to get $\cos(\theta_3 - \theta_2) = \frac{s}{\sigma_1 d}\cos\alpha > 0$. Since $|\theta_3 - \theta_2| < 3\pi/2$, it follows that $|\theta_3 - \theta_2| < \pi/2$, as claimed. □

We estimate the expected increments of our Lyapunov functions in two stages: the main term comes from a Taylor expansion valid when the jump of the walk is not too big compared to its current distance from the origin, while we bound the (smaller) contribution from big jumps using the moments assumption (M_p). For the first stage, let $B_b(x) := \{z \in \mathbb{R}^2 : \|x - z\| \leq b\}$ denote the (closed) Euclidean ball centred at x with radius $b \geq 0$. We use the multivariable Taylor theorem in the following form. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is thrice continuously differentiable in $B_b(x)$. Recall that $Df(x)$ is the vector function whose components are $D_i f(x)$. Then, for $y \in B_b(x)$,

$$f(x+y) = f(x) + \langle Df(x), y\rangle + y_1^2 \frac{D_1^2 f(x)}{2} + y_2^2 \frac{D_2^2 f(x)}{2} + y_1 y_2 D_1 D_2 f(x) + R(x, y), \tag{28}$$

where, for all $y \in B_b(x)$, $|R(x, y)| \leq C\|y\|^3 R(x)$ for an absolute constant $C < \infty$ and

$$R(x) := \max_{i,j,k} \sup_{z \in B_b(x)} \left| D_i D_j D_k f(z) \right|.$$

For dealing with the large jumps, we observe the useful fact that if $p > 2$ is a constant for which (1) holds, then for some constant $C < \infty$, all $\delta \in (0, 1)$, and all $q \in [0, p]$,

$$\mathbb{E}_x \left[\|\Delta\|^q \mathbf{1}\{\|\Delta\| \geq \|x\|^\delta\} \right] \leq C\|x\|^{-\delta(p-q)}, \tag{29}$$

for all $\|x\|$ sufficiently large. To see (29), write $\|\Delta\|^q = \|\Delta\|^p \|\Delta\|^{q-p}$ and use the fact that $\|\Delta\| \geq \|x\|^\delta$ to bound the second factor.

Here is our first main Lyapunov function estimate.

Lemma 4 *Suppose that* (M_p), (D), (R), *and* (C) *hold, with* $p > 2$, $\alpha^+ = -\alpha^- = \alpha$ *for* $|\alpha| < \pi/2$, *and* $\beta^+, \beta^- \geq 0$. *Let* $w, \gamma \in \mathbb{R}$ *be such that* $2 - p < \gamma w < p$. *Take*

$\theta_0 \in (-\pi/2, \pi/2)$. *Then as* $\|x\| \to \infty$ *with* $x \in S_I$,

$$\mathbb{E}[f_w^\gamma(\xi_{n+1}) - f_w^\gamma(\xi_n) \mid \xi_n = x] = \frac{\gamma(\gamma-1)}{2} w^2 (h_w(Tx))^{\gamma-2} \|Tx\|^{2w-2} + o(\|x\|^{\gamma w-2}). \tag{30}$$

We separate the boundary behaviour into two cases.

(i) If $0 \le \beta^\pm < 1$, *take* $\theta_0 = \theta_1$ *given by* (23). *Then, as* $\|x\| \to \infty$ *with* $x \in S_B^\pm$,

$$\mathbb{E}[f_w^\gamma(\xi_{n+1}) - f_w^\gamma(\xi_n) \mid \xi_n = x] = \gamma w \|Tx\|^{w-1} (h_w(Tx))^{\gamma-1} \frac{a^\pm \mu^\pm(x) \sigma_2 \cos\theta_1}{s \cos\alpha} \left(\beta^\pm - (1-w)\beta_c\right) x_1^{\beta^\pm - 1} + o(\|x\|^{w\gamma+\beta^\pm-2}), \tag{31}$$

where β_c *is given by* (5).

(ii) If $\beta^\pm > 1$, *suppose that* $w \in (0, 1/2)$ *and* $\theta_0 = \theta_0(\Sigma, \alpha, w) = \theta_3 - (1-w)\theta_2$, *where* θ_2 *and* θ_3 *are given by* (24) *and* (25), *such that* $\sup_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} |w\theta - \theta_0| < \pi/2$. *Then, with* $d = d(\Sigma, \alpha)$ *as defined at* (26), *as* $\|x\| \to \infty$ *with* $x \in S_B^\pm$,

$$\mathbb{E}[f_w^\gamma(\xi_{n+1}) - f_w^\gamma(\xi_n) \mid \xi_n = x] = \gamma w \|Tx\|^{w-1} (h_w(Tx))^{\gamma-1} \frac{d\mu^\pm(x)}{s} \left(\cos((1-w)(\pi/2)) + o(1)\right). \tag{32}$$

Remark 4 We can choose $w > 0$ small enough so that $|\theta_3 - (1-w)\theta_2| < \pi/2$, by Lemma 3(ii), and so if $\theta_0 = \theta_3 - (1-w)\theta_2$, we can always choose $w > 0$ small enough so that $\sup_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} |w\theta - \theta_0| < \pi/2$, as required for the $\beta^\pm > 1$ part of Lemma 4.

Proof (of Lemma 4) Differentiating (21) and using (22) we see that

$$D_1 h_w(x) = w r^{w-1} \cos\left((w-1)\theta - \theta_0\right), \text{ and}$$
$$D_2 h_w(x) = -w r^{w-1} \sin\left((w-1)\theta - \theta_0\right). \tag{33}$$

Moreover,

$$D_1^2 h_w(x) = w(w-1) r^{w-2} \cos\left((w-2)\theta - \theta_0\right) = -D_2^2 h_w(x),$$

verifying that h_w is harmonic. Also, for any i, j, k, $|D_i D_j D_k h_w(x)| = O(r^{w-3})$. Writing $h_w^\gamma(x) := (h_w(x))^\gamma$, we also have that $D_i h_w^\gamma(x) = \gamma h_w^{\gamma-1}(x) D_i h_w(x)$, that

$$D_i D_j h_w^\gamma(x) = \gamma h_w^{\gamma-1}(x) D_i D_j h_w(x) + \gamma(\gamma-1) h_w^{\gamma-2}(x) (D_i h_w(x))(D_j h_w(x)),$$

and $|D_i D_j D_k h_w^\gamma(x)| = O(r^{\gamma w-3})$. We apply Taylor's formula (28) in the ball $B_{r/2}(x)$ together with the harmonic property of h_w, to obtain, for $y \in B_{r/2}(x)$,

$$\begin{aligned} h_w^\gamma(x+y) = h_w^\gamma(x) + \gamma\langle Dh_w(x), y\rangle h_w^{\gamma-1}(x) + \frac{\gamma(\gamma-1)}{2}\langle Dh_w(x), y\rangle^2 h_w^{\gamma-2}(x) \\ + \gamma\left(\frac{(y_1^2-y_2^2)D_1^2 h_w(x)}{2} + y_1 y_2 D_1 D_2 h_w(x)\right) h_w^{\gamma-1}(x) \\ + R(x,y), \end{aligned} \tag{34}$$

where $|R(x,y)| \le C\|y\|^3\|x\|^{\gamma w-3}$, using the fact that $h_w(x)$ is bounded above and below by a constant times $\|x\|^w$.

Let $E_x := \{\|\Delta\| < \|x\|^\delta\}$, where we fix a constant δ satisfying

$$\frac{\max\{2, \gamma w, 2-\gamma w\}}{p} < \delta < 1; \tag{35}$$

such a choice of δ is possible since $p > 2$ and $2 - p < \gamma w < p$. If $\xi_0 = x$ and E_x occurs, then $Tx + \tilde{\Delta} \in B_{r/2}(Tx)$ for all $\|x\|$ sufficiently large. Thus, conditioning on $\xi_0 = x$, on the event E_x we may use the expansion in (34) for $h_w^\gamma(Tx+\tilde{\Delta})$, which, after taking expectations, yields

$$\begin{aligned} \mathbb{E}_x\left[(f_w^\gamma(\xi_1) - f_w^\gamma(\xi_0))\mathbf{1}_{E_x}\right] &= \gamma\,(h_w(Tx))^{\gamma-1}\,\mathbb{E}_x\left[\langle Dh_w(Tx), \tilde{\Delta}\rangle \mathbf{1}_{E_x}\right] \\ &+ \gamma\,(h_w(Tx))^{\gamma-1}\left[\frac{D_1^2 h_w(Tx)\,\mathbb{E}_x\left[(\tilde{\Delta}_1^2 - \tilde{\Delta}_2^2)\mathbf{1}_{E_x}\right]}{2} + D_1 D_2 h_w(Tx)\,\mathbb{E}_x\left[\tilde{\Delta}_1\tilde{\Delta}_2\mathbf{1}_{E_x}\right]\right] \\ &+ \frac{\gamma(\gamma-1)}{2}\,(h_w(Tx))^{\gamma-2}\,\mathbb{E}_x\left[\langle Dh_w(Tx), \tilde{\Delta}\rangle^2 \mathbf{1}_{E_x}\right] + \mathbb{E}_x\left[R(Tx, \tilde{\Delta})\mathbf{1}_{E_x}\right]. \end{aligned} \tag{36}$$

Let $p' = p \wedge 3$, so that (1) also holds for $p' \in (2, 3]$. Then, writing $\|\tilde{\Delta}\|^3 = \|\tilde{\Delta}\|^{p'}\|\tilde{\Delta}\|^{3-p'}$,

$$\mathbb{E}_x\left[|R(Tx, \tilde{\Delta})|\mathbf{1}_{E_x}\right] \le C\|x\|^{\gamma w-3+(3-p')\delta}\,\mathbb{E}_x\left[\|\tilde{\Delta}\|^{p'}\right] = o(\|x\|^{\gamma w-2}),$$

since $(3-p')\delta < 1$. If $x \in S_I$, then (14) shows $|\,\mathbb{E}_x\langle Dh_w(Tx), \tilde{\Delta}\rangle| = o(\|x\|^{w-2})$, so

$$\mathbb{E}_x\left|\langle Dh_w(Tx), \tilde{\Delta}\rangle \mathbf{1}_{E_x}\right| \le C\|x\|^{w-1}\,\mathbb{E}_x(\|\Delta\|\mathbf{1}_{E_x^c}) + o(\|x\|^{w-2}).$$

Note that, by (35), $\delta > \frac{2}{p} > \frac{1}{p-1}$. Then, using the $q = 1$ case of (29), we get

$$\mathbb{E}_x\left|\langle Dh_w(Tx), \tilde{\Delta}\rangle \mathbf{1}_{E_x}\right| = o(\|x\|^{w-2}). \tag{37}$$

A similar argument using the $q = 2$ case of (29) gives

$$\mathbb{E}_x \left[\langle Dh_w(Tx), \tilde{\Delta} \rangle^2 \mathbf{1}_{E_x^c} \right] \le C \|x\|^{2w-2-\delta(p-2)} = o(\|x\|^{2w-2}).$$

If $x \in S_I$, then (14) shows that $\mathbb{E}_x(\tilde{\Delta}_1^2 - \tilde{\Delta}_2^2)$ and $\mathbb{E}_x(\tilde{\Delta}_1 \tilde{\Delta}_2)$ are both $o(1)$, and hence, by the $q = 2$ case of (29) once more, we see that $\mathbb{E}_x[|\tilde{\Delta}_1^2 - \tilde{\Delta}_2^2| \mathbf{1}_{E_x}]$ and $\mathbb{E}_x[|\tilde{\Delta}_1 \tilde{\Delta}_2| \mathbf{1}_{E_x}]$ are both $o(1)$. Moreover, (14) also shows that

$$\begin{aligned} \mathbb{E}_x \langle Dh_w(Tx), \tilde{\Delta} \rangle^2 &= \mathbb{E}_x \left((Dh_w(Tx))^\top \tilde{\Delta} \tilde{\Delta}^\top Dh_w(Tx) \right) \\ &= (Dh_w(Tx))^\top Dh_w(Tx) + o(\|x\|^{2w-2}) \\ &= (D_1 h_w(Tx))^2 + (D_2 h_w(Tx))^2 + o(\|x\|^{2w-2}). \end{aligned}$$

Putting all these estimates into (36) we get, for $x \in S_I$,

$$\begin{aligned} \mathbb{E}_x \left[(f_w^\gamma(\xi_1) - f_w^\gamma(\xi_0)) \mathbf{1}_{E_x} \right] = \frac{\gamma(\gamma-1)}{2} (h_w(Tx))^{\gamma-2} \left((D_1 h_w(Tx))^2 + (D_2 h_w(Tx))^2 \right) \\ + o(\|x\|^{\gamma w - 2}). \end{aligned} \tag{38}$$

On the other hand, given $\xi_0 = x$, if $\gamma w \ge 0$, by the triangle inequality,

$$\begin{aligned} \left| f_w^\gamma(\xi_1) - f_w^\gamma(x) \right| &\le \|T\xi_1\|^{\gamma w} + \|Tx\|^{\gamma w} \le 2\big(\|T\xi_1\| + \|Tx\|\big)^{\gamma w} \\ &\le 2\big(2\|Tx\| + \|\tilde{\Delta}\|\big)^{\gamma w}. \end{aligned} \tag{39}$$

It follows from (39) that $|f_w^\gamma(\xi_1) - f_w^\gamma(x)| \mathbf{1}_{E_x^c} \le C \|\Delta\|^{\gamma w/\delta}$, for some constant $C < \infty$ and all $\|x\|$ sufficiently large. Hence

$$\mathbb{E}_x \left| (f_w^\gamma(\xi_1) - f_w^\gamma(\xi_0)) \mathbf{1}_{E_x^c} \right| \le C \, \mathbb{E}_x \left[\|\Delta\|^{\gamma w/\delta} \mathbf{1}_{E_x^c} \right].$$

Since $\delta > \frac{\gamma w}{p}$, by (35), we may apply (29) with $q = \frac{\gamma w}{\delta}$ to get

$$\mathbb{E}_x \left| (f_w^\gamma(\xi_1) - f_w^\gamma(\xi_0)) \mathbf{1}_{E_x^c} \right| = O(\|x\|^{\gamma w - \delta p}) = o(\|x\|^{\gamma w - 2}), \tag{40}$$

since $\delta > \frac{2}{p}$. If $w\gamma < 0$, then we use the fact that f_w^γ is uniformly bounded to get

$$\mathbb{E}_x \left| (f_w^\gamma(\xi_1) - f_w^\gamma(\xi_0)) \mathbf{1}_{E_x^c} \right| \le C \mathbb{P}_x(E_x^c) = O(\|x\|^{-\delta p}),$$

by the $q = 0$ case of (29). Thus (40) holds in this case too, since $\gamma w > 2 - \delta p$ by choice of δ at (35). Then (30) follows from combining (38) and (40) with (33).

Next suppose that $x \in S_B$. Truncating (34), we see that for all $y \in B_{r/2}(x)$,

$$h_w^\gamma(x + y) = h_w^\gamma(x) + \gamma \langle Dh_w(x), y \rangle h_w^{\gamma-1}(x) + R(x, y), \tag{41}$$

where now $|R(x, y)| \le C\|y\|^2\|x\|^{\gamma w-2}$. It follows from (41) and (M_p) that

$$\mathbb{E}_x\left[(f_w^\gamma(\xi_1) - f_w^\gamma(\xi_0))\mathbf{1}_{E_x}\right] = \gamma h_w^{\gamma-1}(Tx)\,\mathbb{E}_x\left[\langle Dh_w(Tx), \tilde{\Delta}\rangle \mathbf{1}_{E_x}\right] + O(\|x\|^{\gamma w-2}).$$

By the $q = 1$ case of (29), since $\delta > \frac{1}{p-1}$, we see that $\mathbb{E}_x[\langle Dh_w(Tx), \tilde{\Delta}\rangle \mathbf{1}_{E_x^c}] = o(\|x\|^{w-2})$, while the estimate (40) still applies, so that

$$\mathbb{E}_x\left[f_w^\gamma(\xi_1) - f_w^\gamma(\xi_0)\right] = \gamma h_w^{\gamma-1}(Tx)\,\mathbb{E}_x\langle Dh_w(Tx), \tilde{\Delta}\rangle + O(\|x\|^{\gamma w-2}). \tag{42}$$

From (33) we have

$$Dh_w(Tx) = w\|Tx\|^{w-1}\begin{pmatrix}\cos((1-w)\theta(Tx) + \theta_0)\\ \sin((1-w)\theta(Tx) + \theta_0)\end{pmatrix}. \tag{43}$$

First suppose that $\beta^\pm < 1$. Then, by (13), for $x \in S_B^\pm$, $x_2 = \pm a^\pm x_1^{\beta^\pm} + O(1)$ and

$$\sin\theta(Tx) = \pm\frac{sa^\pm}{\sigma_2^2}x_1^{\beta^\pm-1} + O(x_1^{2\beta^\pm-2}) + O(x_1^{-1}).$$

Since $\arcsin z = z + O(z^3)$ as $z \to 0$, it follows that

$$\theta(Tx) = \pm\frac{sa^\pm}{\sigma_2^2}x_1^{\beta^\pm-1} + O(x_1^{2\beta^\pm-2}) + O(x_1^{-1}).$$

Hence

$$\cos((1-w)\theta(Tx) + \theta_0) = \cos\theta_0 \mp (1-w)\frac{sa^\pm}{\sigma_2^2}x_1^{\beta^\pm-1}\sin\theta_0 + O(x_1^{2\beta^\pm-2}) + O(x_1^{-1});$$

$$\sin((1-w)\theta(Tx) + \theta_0) = \sin\theta_0 \pm (1-w)\frac{sa^\pm}{\sigma_2^2}x_1^{\beta^\pm-1}\cos\theta_0 + O(x_1^{2\beta^\pm-2}) + O(x_1^{-1}).$$

Then (43) with (16) and (17) shows that

$$\mathbb{E}_x\langle Dh_w(Tx), \tilde{\Delta}\rangle = w\|Tx\|^{w-1}\frac{\mu^\pm(x)\cos\theta_0\cos\alpha}{s\sigma_2}\left(\pm A_1 + (a^\pm A_2 + o(1))x_1^{\beta^\pm-1}\right), \tag{44}$$

where, for $|\theta_0| < \pi/2$, $A_1 = \sigma_2^2\tan\alpha + \rho - s\tan\theta_0$, and

$$A_2 = \sigma_2^2\beta^\pm - \rho\beta^\pm\tan\alpha - (1-w)s\tan\theta_0\tan\alpha - (1-w)\frac{s\rho}{\sigma_2^2}\tan\theta_0 + s\beta^\pm\tan\theta_0\tan\alpha - (1-w)\frac{s^2}{\sigma_2^2}.$$

Now take $\theta_0 = \theta_1$ as given by (23), so that $s \tan\theta_0 = \sigma_2^2 \tan\alpha + \rho$. Then $A_1 = 0$, eliminating the leading order term in (44). Moreover, with this choice of θ_0 we get, after some further cancellation and simplification, that

$$A_2 = \frac{\sigma_2^2 \left(\beta^{\pm} - (1-w)\beta_c\right)}{\cos^2\alpha},$$

with β_c as given by (5). Thus with (44) and (42) we verify (31).

Finally suppose that $\beta^{\pm} > 1$, and restrict to the case $w \in (0, 1/2)$. Let $\theta_2 \in (-\pi/2, \pi/2)$ be as given by (24). Then if $x = (0, x_2)$, we have $\theta(Tx) = \theta_2 - \frac{\pi}{2}$ if $x_2 < 0$ and $\theta(Tx) = \theta_2 + \frac{\pi}{2}$ if $x_2 > 0$ (see Fig. 3). It follows from (13) that

$$\theta(Tx) = \theta_2 \pm \frac{\pi}{2} + O(x_1^{1-\beta^{\pm}}), \text{ for } x \in S_B^{\pm},$$

as $\|x\| \to \infty$ (and $x_1 \to \infty$). Now (43) with (18) and (19) shows that

$$\begin{aligned}\mathbb{E}_x \langle Dh_w(Tx), \tilde{\Delta}\rangle = w\|Tx\|^{w-1} \frac{\mu^{\pm}(x)}{s\sigma_2}\Big(&\sigma_2^2 \cos\alpha \cos\left((1-w)\theta(Tx) + \theta_0\right) \\ &- \rho \sin\alpha \cos\left((1-w)\theta(Tx) + \theta_0\right) \\ &+ s \sin\alpha \sin\left((1-w)\theta(Tx) + \theta_0\right) + O(x_1^{1-\beta^{\pm}})\Big). \end{aligned} \tag{45}$$

Set $\phi := (1-w)\frac{\pi}{2}$. Choose $\theta_0 = \theta_3 - (1-w)\theta_2$, where $\theta_3 \in (-\pi, \pi)$ satisfies (25). Then we have that, for $x \in S_B^{\pm}$,

$$\begin{aligned}\cos\left((1-w)\theta(Tx) + \theta_0\right) &= \cos\left(\theta_3 \pm \phi\right) + O(x_1^{1-\beta^{\pm}}) \\ &= \cos\phi\cos\theta_3 \mp \sin\phi\sin\theta_3 + O(x_1^{1-\beta^{\pm}}). \end{aligned} \tag{46}$$

Similarly, for $x \in S_B^{\pm}$,

$$\sin\left((1-w)\theta(Tx) + \theta_0\right) = \cos\phi\sin\theta_3 \pm \sin\phi\cos\theta_3 + O(x_1^{1-\beta^{\pm}}). \tag{47}$$

Using (46) and (47) in (45), we obtain

$$\mathbb{E}_x \langle Dh_w(Tx), \tilde{\Delta}\rangle = w\|Tx\|^{w-1} \frac{\mu^{\pm}(x)}{s\sigma_2} \left(A_3 \cos\phi \mp A_4 \sin\phi + o(1)\right),$$

where

$$\begin{aligned} A_3 &= \left(\sigma_2^2 \cos\alpha - \rho \sin\alpha\right)\cos\theta_3 + s\sin\alpha\sin\theta_3 \\ &= \sigma_2 d \cos^2\theta_3 + \sigma_2 d \sin^2\theta_3 = \sigma_2 d, \end{aligned}$$

by (25), and, similarly,

$$A_4 = \left(\sigma_2^2 \cos\alpha - \rho \sin\alpha\right) \sin\theta_3 - s \sin\alpha \cos\theta_3 = 0.$$

Then with (42) we obtain (32). □

In the case where $\beta^+, \beta^- < 1$ with $\beta^+ \neq \beta^-$, we will in some circumstances need to modify the function f_w^γ so that it can be made insensitive to the behaviour near the boundary with the smaller of β^+, β^-. To this end, define for $w, \gamma, \nu, \lambda \in \mathbb{R}$,

$$F_w^{\gamma,\nu}(x) := f_w^\gamma(x) + \lambda x_2 \|Tx\|^{2\nu}. \tag{48}$$

We state a result for the case $\beta^- < \beta^+$; an analogous result holds if $\beta^+ < \beta^-$.

Lemma 5 *Suppose that* (M_p), (D), (R), *and* (C) *hold, with* $p > 2$, $\alpha^+ = -\alpha^- = \alpha$ *for* $|\alpha| < \pi/2$, *and* $0 \le \beta^- < \beta^+ < 1$. *Let* $w, \gamma \in \mathbb{R}$ *be such that* $2 - p < \gamma w < p$. *Take* $\theta_0 = \theta_1 \in (-\pi/2, \pi/2)$ *given by* (23). *Suppose that*

$$\gamma w + \beta^- - 2 < 2\nu < \gamma w + \beta^+ - 2.$$

Then as $\|x\| \to \infty$ *with* $x \in S_I$,

$$\begin{aligned}&\mathbb{E}[F_w^{\gamma,\nu}(\xi_{n+1}) - F_w^{\gamma,\nu}(\xi_n) \mid \xi_n = x] \\ &\quad = \frac{1}{2}\gamma(\gamma-1)(w^2 + o(1))(h_w(Tx))^{\gamma-2}\|Tx\|^{2w-2}.\end{aligned} \tag{49}$$

As $\|x\| \to \infty$ *with* $x \in S_B^+$,

$$\begin{aligned}&\mathbb{E}[F_w^{\gamma,\nu}(\xi_{n+1}) - F_w^{\gamma,\nu}(\xi_n) \mid \xi_n = x] \\ &\quad = \gamma w \|Tx\|^{w-1} (h_w(Tx))^{\gamma-1} \frac{a^+\mu^+(x)\sigma_2 \cos\theta_1}{s\cos\alpha} \left(\beta^+ - (1-w)\beta_c\right) x_1^{\beta^+-1} \\ &\qquad + o(\|x\|^{w\gamma+\beta^+-2}).\end{aligned} \tag{50}$$

As $\|x\| \to \infty$ *with* $x \in S_B^-$,

$$\mathbb{E}[F_w^{\gamma,\nu}(\xi_{n+1}) - F_w^{\gamma,\nu}(\xi_n) \mid \xi_n = x] = \lambda \|Tx\|^{2\nu} \left(\mu^-(x)\cos\alpha + o(1)\right). \tag{51}$$

Proof Suppose that $0 \le \beta^- < \beta^+ < 1$. As in the proof of Lemma 4, let $E_x = \{\|\Delta\| < \|x\|^\delta\}$, where $\delta \in (0, 1)$ satisfies (35). Set $v_\nu(x) := x_2\|Tx\|^{2\nu}$. Then, using Taylor's formula in one variable, for $x, y \in \mathbb{R}^2$ with $y \in B_{r/2}(x)$,

$$\|x+y\|^{2\nu} = \|x\|^{2\nu}\left(1 + \frac{2\langle x, y\rangle + \|y\|^2}{\|x\|^2}\right)^\nu = \|x\|^{2\nu} + 2\nu\langle x, y\rangle \|x\|^{2\nu-2} + R(x, y),$$

where $|R(x,y)| \le C\|y\|^2\|x\|^{2\nu-2}$. Thus, for $x \in S$ with $y \in B_{r/2}(x)$ and $x+y \in S$,

$$\begin{aligned} v_\nu(x+y) - v_\nu(x) &= (x_2+y_2)\|Tx+Ty\|^{2\nu} - x_2\|Tx\|^{2\nu} \\ &= y_2\|Tx\|^{2\nu} + 2\nu x_2\langle Tx, Ty\rangle\|Tx\|^{2\nu-2} + 2\nu y_2\langle Tx, Ty\rangle\|Tx\|^{2\nu-2} \\ &\quad + R(x,y), \end{aligned} \tag{52}$$

where now $|R(x,y)| \le C\|y\|^2\|x\|^{2\nu+\beta^+-2}$, using the fact that both $|x_2|$ and $|y_2|$ are $O(\|x\|^{\beta^+})$. Taking $x = \xi_0$ and $y = \Delta$ so $Ty = \tilde{\Delta}$, we obtain

$$\begin{aligned} \mathbb{E}_x\left[(v_\nu(\xi_1) - v_\nu(\xi_0))\mathbf{1}_{E_x}\right] &= \|Tx\|^{2\nu}\,\mathbb{E}_x\left[\Delta_2\mathbf{1}_{E_x}\right] + 2\nu x_2\|Tx\|^{2\nu-2}\,\mathbb{E}_x\left[\langle Tx, \tilde{\Delta}\rangle\mathbf{1}_{E_x}\right] \\ &\quad + 2\nu\|Tx\|^{2\nu-2}\,\mathbb{E}\left[\Delta_2\langle Tx, \tilde{\Delta}\rangle\mathbf{1}_{E_x}\right] \\ &\quad + \mathbb{E}\left[R(x,\Delta)\mathbf{1}_{E_x}\right]. \end{aligned} \tag{53}$$

Suppose that $x \in S_I$. Similarly to (37), we have $\mathbb{E}_x[\langle Tx, \tilde{\Delta}\rangle\mathbf{1}_{E_x}] = o(1)$, and, by similar arguments using (29), $\mathbb{E}[\Delta_2\mathbf{1}_{E_x}] = o(\|x\|^{-1})$, $\mathbb{E}_x\,|\Delta_2\langle Tx, \tilde{\Delta}\rangle\mathbf{1}_{E_x^c}| = o(\|x\|)$, and $\mathbb{E}_x\,|R(x,\Delta)\mathbf{1}_{E_x}| = o(\|x\|^{2\nu-1})$, since $\beta^+ < 1$. Also, by (13),

$$\begin{aligned} \mathbb{E}_x(\Delta_2\langle Tx, \tilde{\Delta}\rangle) &= \sigma_2\,\mathbb{E}_x(\tilde{\Delta}_2\langle Tx, \tilde{\Delta}\rangle) \\ &= \sigma_2(Tx)_1\,\mathbb{E}_x(\tilde{\Delta}_1\tilde{\Delta}_2) + \sigma_2(Tx)_2\,\mathbb{E}_x(\tilde{\Delta}_2^2). \end{aligned}$$

Here, by (14), $\mathbb{E}_x(\tilde{\Delta}_1\tilde{\Delta}_2) = o(1)$ and $\mathbb{E}_x(\tilde{\Delta}_2^2) = O(1)$, while $\sigma_2(Tx)_2 = x_2 = O(\|x\|^{\beta^+})$. Thus $\mathbb{E}_x(\Delta_2\langle Tx, \tilde{\Delta}\rangle) = o(\|x\|)$. Hence also

$$\mathbb{E}_x\left[\Delta_2\langle Tx, \tilde{\Delta}\rangle\mathbf{1}_{E_x}\right] = o(\|x\|).$$

Thus from (53) we get that, for $x \in S_I$,

$$\mathbb{E}_x\left[(v_\nu(\xi_1) - v_\nu(\xi_0))\mathbf{1}_{E_x}\right] = o(\|x\|^{2\nu-1}). \tag{54}$$

On the other hand, since $|v_\nu(x+y) - v_\nu(x)| \le C(\|x\| + \|y\|)^{2\nu+\beta^+}$ we get

$$\mathbb{E}_x\left[|v_\nu(\xi_1) - v_\nu(\xi_0)|\mathbf{1}_{E_x^c}\right] \le C\,\mathbb{E}_x\left[\|\Delta\|^{(2\nu+\beta^+)/\delta}\mathbf{1}_{E_x^c}\right].$$

Here $2\nu + \beta^+ < 2\nu + 1 < \gamma w < \delta p$, by choice of ν and (35), so we may apply (29) with $q = (2\nu+\beta^+)/\delta$ to get

$$\mathbb{E}_x\left[|v_\nu(\xi_1) - v_\nu(\xi_0)|\mathbf{1}_{E_x^c}\right] = O(\|x\|^{2\nu+\beta^+-\delta p}) = o(\|x\|^{2\nu-1}), \tag{55}$$

since $\delta p > 2$, by (35). Combining (54), (55) and (30), we obtain (49), provided that $2\nu - 1 < \gamma w - 2$, which is the case since $2\nu < \gamma w + \beta^+ - 2$ and $\beta^+ < 1$.

Now suppose that $x \in S_B^{\pm}$. We truncate (52) to see that, for $x \in S$ with $y \in B_{r/2}(x)$ and $x + y \in S$,

$$v_\nu(x + y) - v_\nu(x) = y_2 \|Tx\|^{2\nu} + R(x, y),$$

where now $|R(x, y)| \le C\|y\|\|x\|^{2\nu+\beta^{\pm}-1}$, using the fact that for $x \in S_B^{\pm}$, $|x_2| = O(\|x\|^{\beta^{\pm}})$. It follows that, for $x \in S_B^{\pm}$,

$$\mathbb{E}_x\left[(v_\nu(\xi_1) - v_\nu(\xi_0))\mathbf{1}_{E_x}\right] = \|Tx\|^{2\nu}\,\mathbb{E}_x\left[\Delta_2 \mathbf{1}_{E_x}\right] + O(\|x\|^{2\nu+\beta^{\pm}-1}).$$

By (29) and (35) we have that $\mathbb{E}[|\Delta_2|\mathbf{1}_{E_x^c}] = O(\|x\|^{-\delta(p-1)}) = o(\|x\|^{-1})$, while if $x \in S_B^{\pm}$, then, by (10), $\mathbb{E}_x\,\Delta_2 = \mp\mu^{\pm}(x)\cos\alpha + O(\|x\|^{\beta^{\pm}-1})$. On the other hand, the estimate (55) still applies, so we get, for $x \in S_B^{\pm}$,

$$\mathbb{E}_x[v_\nu(\xi_1) - v_\nu(\xi_0)] = \mp\|Tx\|^{2\nu}\mu^{\pm}(x)\cos\alpha + O(\|x\|^{2\nu+\beta^{\pm}-1}). \tag{56}$$

If we choose ν such that $2\nu < \gamma w + \beta^+ - 2$, then we combine (56) and (31) to get (50), since the term from (31) dominates. If we choose ν such that $2\nu > \gamma w + \beta^- - 2$, then the term from (56) dominates that from (31), and we get (51). □

In the critically recurrent cases, where $\max(\beta^+, \beta^-) = \beta_c \in (0, 1)$ or $\beta^+, \beta^- > 1$, in which no passage-time moments exist, the functions of polynomial growth based on h_w as defined at (21) are not sufficient to prove recurrence. Instead we need functions which grow more slowly. For $\eta \in \mathbb{R}$ let

$$h(x) := h(r, \theta) := \log r + \eta\theta, \text{ and } \ell(x) := \log h(Tx), \tag{57}$$

where we understand $\log y$ to mean $\max(1, \log y)$. The function h is again harmonic (see below) and was used in the context of reflecting Brownian motion in a wedge in [31]. Set

$$\eta_0 := \eta_0(\Sigma, \alpha) := \frac{\sigma_2^2 \tan\alpha + \rho}{s}, \text{ and } \eta_1 := \eta_1(\Sigma, \alpha) := \frac{\sigma_1^2 \tan\alpha - \rho}{s}. \tag{58}$$

Lemma 6 *Suppose that* (M_p), (D_+), (R), *and* (C_+) *hold, with* $p > 2$, $\varepsilon > 0$, $\alpha^+ = -\alpha^- = \alpha$ *for* $|\alpha| < \pi/2$, *and* $\beta^+, \beta^- \ge 0$. *For any* $\eta \in \mathbb{R}$, *as* $\|x\| \to \infty$ *with* $x \in S_I$,

$$\mathbb{E}[\ell(\xi_{n+1}) - \ell(\xi_n) \mid \xi_n = x] = -\frac{1 + \eta^2 + o(1)}{2\|Tx\|^2(\log\|Tx\|)^2}. \tag{59}$$

If $0 \le \beta^{\pm} < 1$, *take* $\eta = \eta_0$ *as defined at* (58). *Then, as* $\|x\| \to \infty$ *with* $x \in S_B^{\pm}$,

$$\mathbb{E}[\ell(\xi_{n+1}) - \ell(\xi_n) \mid \xi_n = x] = \frac{\sigma_2^2 a^{\pm} \mu^{\pm}(x)}{s^2 \cos\alpha} \frac{1}{\|Tx\|^2 \log \|Tx\|} \left((\beta^{\pm} - \beta_{\mathrm{c}}) x_1^{\beta^{\pm}} + O(\|x\|^{2\beta^{\pm}-1}) + O(1) \right). \tag{60}$$

If $\beta^{\pm} > 1$, *take* $\eta = \eta_1$ *as defined at* (58). *Then as* $\|x\| \to \infty$ *with* $x \in S_B^{\pm}$,

$$\mathbb{E}[\ell(\xi_{n+1}) - \ell(\xi_n) \mid \xi_n = x] = \frac{\mu^{\pm}(x)}{s^2 \cos\alpha} \frac{x_1}{\|Tx\|^2 \log \|Tx\|} \left(\sigma_1^2 \sin^2\alpha + \sigma_2^2 \cos^2\alpha - \frac{\sigma_1^2}{\beta^{\pm}} - \rho \sin 2\alpha + o(1) \right). \tag{61}$$

Proof Given $\eta \in \mathbb{R}$, for $r_0 = r_0(\eta) = \exp(\mathrm{e} + |\eta|\pi)$, we have from (58) that both h and $\log h$ are infinitely differentiable in the domain $\mathscr{R}_{r_0} := \{x \in \mathbb{R}^2 : x_1 > 0,\ r(x) > r_0\}$. Differentiating (58) and using (22) we obtain, for $x \in \mathscr{R}_{r_0}$,

$$D_1 h(x) = \frac{1}{r} (\cos\theta - \eta \sin\theta), \text{ and } D_2 h(x) = \frac{1}{r} (\sin\theta + \eta \cos\theta). \tag{62}$$

We verify that h is harmonic in $\mathscr{R}_{r_0}$, since

$$D_1^2 h(x) = \frac{\eta \sin 2\theta}{r^2} - \frac{\cos 2\theta}{r^2} = -D_2^2 h(x).$$

Also, for any i, j, k, $|D_i D_j D_k h(x)| = O(r^{-3})$. Moreover, $D_i \log h(x) = (h(x))^{-1} D_i h(x)$,

$$D_i D_j \log h(x) = \frac{D_i D_j h(x)}{h(x)} - \frac{(D_i h(x))(D_j h(x))}{(h(x))^2},$$

and $|D_i D_j D_k \log h(x)| = O(r^{-3}(\log r)^{-1})$. Recall that $Dh(x)$ is the vector function whose components are $D_i h(x)$. Then Taylor's formula (28) together with the harmonic property of h shows that for $x \in \mathscr{R}_{2r_0}$ and $y \in B_{r/2}(x)$,

$$\log h(x+y) = \log h(x) + \frac{\langle Dh(x), y \rangle}{h(x)} + \frac{(y_1^2 - y_2^2) D_1^2 h(x)}{2h(x)} + \frac{y_1 y_2 D_1 D_2 h(x)}{h(x)} - \frac{\langle Dh(x), y \rangle^2}{2(h(x))^2} + R(x, y), \tag{63}$$

where $|R(x,y)| \le C\|y\|^3\|x\|^{-3}(\log\|x\|)^{-1}$ for some constant $C < \infty$, all $y \in B_{r/2}(x)$, and all $\|x\|$ sufficiently large. As in the proof of Lemma 4, let $E_x = \{\|\Delta\| < \|x\|^\delta\}$ for $\delta \in (\frac{2}{p}, 1)$. Then applying the expansion in (63) to $\log h(Tx+\tilde{\Delta})$, conditioning on $\xi_0 = x$, and taking expectations, we obtain, for $\|x\|$ sufficiently large,

$$\mathbb{E}_x\left[(\ell(\xi_1)-\ell(\xi_0))\mathbf{1}_{E_x}\right] = \frac{\mathbb{E}_x\left[\langle Dh(Tx),\tilde{\Delta}\rangle \mathbf{1}_{E_x}\right]}{h(Tx)} + \frac{D_1^2 h(Tx)\,\mathbb{E}_x\left[(\tilde{\Delta}_1^2-\tilde{\Delta}_2^2)\mathbf{1}_{E_x}\right]}{2h(Tx)}$$
$$+ \frac{D_1D_2h(Tx)\,\mathbb{E}_x\left[\tilde{\Delta}_1\tilde{\Delta}_2\mathbf{1}_{E_x}\right]}{h(Tx)} - \frac{\mathbb{E}_x\left[\langle Dh(Tx),\tilde{\Delta}\rangle^2\mathbf{1}_{E_x}\right]}{2(h(Tx))^2} + \mathbb{E}_x\left[R(Tx,\tilde{\Delta})\mathbf{1}_{E_x}\right]. \tag{64}$$

Let $p' \in (2,3]$ be such that (1) holds. Then

$$\mathbb{E}_x\left|R(Tx,\tilde{\Delta})\mathbf{1}_{E_x}\right| \le C\|x\|^{-3+(3-p')\delta}\,\mathbb{E}_x(\|\Delta\|^{p'}) = O(\|x\|^{-2-\varepsilon'}),$$

for some $\varepsilon' > 0$.

Suppose that $x \in S_I$. By (15), $\mathbb{E}_x(\tilde{\Delta}_1\tilde{\Delta}_2) = O(\|x\|^{-\varepsilon})$ and, by (29), $\mathbb{E}_x|\tilde{\Delta}_1\tilde{\Delta}_2\mathbf{1}_{E_x^c}| \le C\,\mathbb{E}[\|\Delta\|^2\mathbf{1}_{E_x^c}] = O(\|x\|^{-\varepsilon'})$, for some $\varepsilon' > 0$. Thus $\mathbb{E}_x(\tilde{\Delta}_1\tilde{\Delta}_2\mathbf{1}_{E_x}) = O(\|x\|^{-\varepsilon'})$. A similar argument gives the same bound for $\mathbb{E}_x[(\tilde{\Delta}_1^2-\tilde{\Delta}_2^2)\mathbf{1}_{E_x}]$. Also, from (15) and (62), $\mathbb{E}_x(\langle Dh(Tx),\tilde{\Delta}\rangle) = O(\|x\|^{-2-\varepsilon})$ and, by (29), $\mathbb{E}_x|\langle Dh(Tx),\tilde{\Delta}\rangle\mathbf{1}_{E_x^c}| = O(\|x\|^{-2-\varepsilon'})$ for some $\varepsilon' > 0$. Hence $\mathbb{E}_x[\langle Dh(Tx),\tilde{\Delta}\rangle\mathbf{1}_{E_x}] = O(\|x\|^{-2-\varepsilon'})$. Finally, by (15) and (62),

$$\begin{aligned}\mathbb{E}_x\langle Dh(Tx),\tilde{\Delta}\rangle^2 &= \mathbb{E}_x\left((Dh(Tx))^\top\tilde{\Delta}\tilde{\Delta}^\top Dh(Tx)\right)\\ &= (Dh(Tx))^\top Dh(Tx) + O(\|x\|^{-2-\varepsilon})\\ &= (D_1h(Tx))^2 + (D_2h(Tx))^2 + O(\|x\|^{-2-\varepsilon}),\end{aligned}$$

while, by (29), $\mathbb{E}_x|\langle Dh(Tx),\tilde{\Delta}\rangle^2\mathbf{1}_{E_x^c}| = O(\|x\|^{-2-\varepsilon'})$. Putting all these estimates into (64) gives

$$\mathbb{E}_x\left[(\ell(\xi_1)-\ell(\xi_0))\mathbf{1}_{E_x}\right] = -\frac{(D_1h(Tx))^2+(D_2h(Tx))^2}{2(h(Tx))^2} + O(\|x\|^{-2-\varepsilon'}),$$

for some $\varepsilon' > 0$. On the other hand, for all $\|x\|$ sufficiently large, $|\ell(x+y)-\ell(x)| \le C\log\log\|x\| + C\log\log\|y\|$. For any $p > 2$ and $\delta \in (\frac{2}{p}, 1)$, we may (and do) choose $q > 0$ sufficiently small such that $\delta(p-q) > 2$, and then, by (29),

$$\begin{aligned}\mathbb{E}_x\left[(\ell(\xi_1)-\ell(\xi_0))\mathbf{1}_{E_x^c}\right] &\le C\,\mathbb{E}_x\left[\|\Delta\|^q\mathbf{1}_{E_x^c}\right]\\ &= O(\|x\|^{-\delta(p-q)}) = O(\|x\|^{-2-\varepsilon'}),\end{aligned} \tag{65}$$

for some $\varepsilon' > 0$. Thus we conclude that

$$\mathbb{E}_x\left[\ell(\xi_1) - \ell(\xi_0)\right] = -\frac{(D_1 h(Tx))^2 + (D_2 h(Tx))^2}{2(h(Tx))^2} + O(\|x\|^{-2-\varepsilon'}),$$

for some $\varepsilon' > 0$. Then (59) follows from (62).

Next suppose that $x \in S_B$. Truncating (63), we have for $x \in \mathscr{R}_{2r_0}$ and $y \in B_{r/2}(x)$,

$$\log h(x+y) = \log h(x) + \frac{\langle Dh(x), y\rangle}{h(x)} + R(x,y),$$

where now $|R(x,y)| \le C\|y\|^2\|x\|^{-2}(\log\|x\|)^{-1}$ for $\|x\|$ sufficiently large. Hence

$$\mathbb{E}_x\left[(\ell(\xi_1) - \ell(\xi_0))\mathbf{1}_{E_x}\right] = \frac{\mathbb{E}_x\left[\langle Dh(Tx), \tilde{\Delta}\rangle \mathbf{1}_{E_x}\right] + O(\|x\|^{-2})}{h(Tx)}.$$

Then by (65) and the fact that $\mathbb{E}_x\,|\langle Dh(Tx), \tilde{\Delta}\rangle \mathbf{1}_{E_x^c}| = O(\|x\|^{-2-\varepsilon'})$ (as above),

$$\mathbb{E}_x\left[\ell(\xi_1) - \ell(\xi_0)\right] = \frac{\mathbb{E}_x\left[\langle Dh(Tx), \tilde{\Delta}\rangle\right] + O(\|x\|^{-2})}{h(Tx)}. \tag{66}$$

From (62) we have

$$Dh(x) = \frac{1}{\|x\|^2}\begin{pmatrix} x_1 - \eta x_2 \\ x_2 + \eta x_1\end{pmatrix}, \text{ and hence } Dh(Tx) = \frac{1}{\|Tx\|^2}\begin{pmatrix} \frac{\sigma_2}{s}x_1 - \frac{\rho}{s\sigma_2}x_2 - \frac{\eta}{\sigma_2}x_2 \\ \frac{1}{\sigma_2}x_2 + \frac{\eta\sigma_2}{s}x_1 - \frac{\eta\rho}{s\sigma_2}x_2\end{pmatrix},$$

using (13). If $\beta^{\pm} < 1$ and $x \in S_B^{\pm}$, we have from (16) and (17) that

$$\begin{aligned}&\mathbb{E}_x\langle Dh(Tx), \tilde{\Delta}\rangle \\ &= \frac{\mu^{\pm}(x)}{s^2}\frac{1}{\|Tx\|^2}\Big\{a^{\pm}\Big[\left(s\eta(\beta^{\pm}-1) - \rho(1+\beta^{\pm})\right)\sin\alpha + \left(\sigma_2^2\beta^{\pm} - \sigma_1^2\right)\cos\alpha\Big]x_1^{\beta^{\pm}} \\ &\quad \pm \Big[\sigma_2^2\sin\alpha + (\rho - s\eta)\cos\alpha\Big]x_1 + O(x_1^{2\beta^{\pm}-1}) + O(1)\Big\}.\end{aligned}$$

Taking $\eta = \eta_0$ as given by (58), the $\pm x_1$ term vanishes; after simplification, we get

$$\mathbb{E}_x\langle Dh(Tx), \tilde{\Delta}\rangle = \frac{\sigma_2^2 a^{\pm}\mu^{\pm}(x)}{\|Tx\|^2 s^2\cos\alpha}\left(\left(\beta^{\pm} - \beta_c\right)x_1^{\beta^{\pm}} + O(x_1^{2\beta^{\pm}-1}) + O(1)\right). \tag{67}$$

Using (67) in (66) gives (60).

On the other hand, if $\beta^{\pm} > 1$ and $x \in S_B^{\pm}$, we have from (18) and (19) that

$$\begin{aligned}\mathbb{E}_x\langle Dh(Tx), \tilde{\Delta}\rangle &= \frac{\mu^{\pm}(x)}{s^2}\frac{1}{\|Tx\|^2}\left\{\frac{1}{\beta^{\pm}}\Big[\big(s\eta(\beta^{\pm}-1)-\rho(1+\beta^{\pm})\big)\sin\alpha+\Big(\sigma_2^2\beta^{\pm}-\sigma_1^2\Big)\cos\alpha\Big]x_1\right.\\ &\quad\left.\pm a^{\pm}\Big[\sigma_1^2\sin\alpha-(\rho+s\eta)\cos\alpha\Big]x_1^{\beta^{\pm}}+O(x_1^{2-\beta^{\pm}})+O(1)\right\}.\end{aligned}$$

Taking $\eta = \eta_1$ as given by (58), the $\pm x_1^{\beta^{\pm}}$ term vanishes, and we get

$$\mathbb{E}_x\langle Dh(Tx), \tilde{\Delta}\rangle = \frac{\mu^{\pm}(x)}{s^2\cos\alpha}\frac{x_1}{\|Tx\|^2}\left(\sigma_1^2\sin^2\alpha+\sigma_2^2\cos^2\alpha-\frac{\sigma_1^2}{\beta^{\pm}}-\rho\sin 2\alpha+o(1)\right),$$

as $\|x\| \to \infty$ (and $x_1 \to \infty$). Then using the last display in (66) gives (61). □

The function ℓ is not by itself enough to prove recurrence in the critical cases, because the estimates in Lemma 6 do not guarantee that ℓ satisfies a supermartingale condition for all parameter values of interest. To proceed, we modify the function slightly to improve its properties near the boundary. In the case where $\max(\beta^+, \beta^-) = \beta_c \in (0, 1)$, the following function will be used to prove recurrence,

$$g_\gamma(x) := g_\gamma(r, \theta) := \ell(x) + \frac{\theta^2}{(1+r)^\gamma},$$

where the parameter η in ℓ is chosen as $\eta = \eta_0$ as given by (58).

Lemma 7 *Suppose that* (M_p), (D_+), (R), *and* (C_+) *hold, with* $p > 2$, $\varepsilon > 0$, $\alpha^+ = -\alpha^- = \alpha$ *for* $|\alpha| < \pi/2$, *and* $\beta^+, \beta^- \in (0, 1)$ *with* $\beta^+, \beta^- \le \beta_c$. *Let* $\eta = \eta_0$, *and suppose*

$$0 < \gamma < \min(\beta^+, \beta^-, 1-\beta^+, 1-\beta^-, p-2).$$

Then as $\|x\| \to \infty$ *with* $x \in S_I$,

$$\mathbb{E}[g_\gamma(\xi_{n+1}) - g_\gamma(\xi_n) \mid \xi_n = x] = -\frac{1+\eta^2+o(1)}{2\|Tx\|^2(\log\|Tx\|)^2}. \tag{68}$$

Moreover, as $\|x\| \to \infty$ *with* $x \in S_B^{\pm}$,

$$\mathbb{E}[g_\gamma(\xi_{n+1}) - g_\gamma(\xi_n) \mid \xi_n = x] \le -2a^{\pm}\mu^{\pm}(x)(\cos\alpha + o(1))\|x\|^{\beta^{\pm}-2-\gamma}. \tag{69}$$

Proof Set $u_\gamma(x) := u_\gamma(r,\theta) := \theta^2(1+r)^{-\gamma}$, and note that, by (22), for $x_1 > 0$,

$$D_1 u_\gamma(x) = -\frac{2\theta\sin\theta}{r(1+r)^\gamma} - \frac{\gamma\theta^2\cos\theta}{(1+r)^{1+\gamma}},\ D_2 u_\gamma(x) = \frac{2\theta\cos\theta}{r(1+r)^\gamma} - \frac{\gamma\theta^2\sin\theta}{(1+r)^{1+\gamma}},$$

and $|D_i D_j u_\gamma(x)| = O(r^{-2-\gamma})$ for any i, j. So, by Taylor's formula (28), for all $y \in B_{r/2}(x)$,

$$u_\gamma(x+y) = u_\gamma(x) + \langle Du_\gamma(x), y\rangle + R(x,y),$$

where $|R(x,y)| \le C\|y\|^2\|x\|^{-2-\gamma}$ for all $\|x\|$ sufficiently large. Once more define the event $E_x = \{\|\Delta\| < \|x\|^\delta\}$, where now $\delta \in (\frac{2+\gamma}{p}, 1)$. Then

$$\mathbb{E}_x\left[(u_\gamma(\xi_1) - u_\gamma(\xi_0))\mathbf{1}_{E_x}\right] = \mathbb{E}_x\left[\langle Du_\gamma(x), \Delta\rangle \mathbf{1}_{E_x}\right] + O(\|x\|^{-2-\gamma}).$$

Moreover, $\mathbb{E}_x\,|\langle Du_\gamma(x), \Delta\rangle \mathbf{1}_{E_x^c}| \le C\|x\|^{-1-\gamma}\,\mathbb{E}_x(\|\Delta\|\mathbf{1}_{E_x^c}) = O(\|x\|^{-2-\gamma})$, by (29) and the fact that $\delta > \frac{2}{p} > \frac{1}{p-1}$. Also, since u_γ is uniformly bounded,

$$\mathbb{E}_x\left[|u_\gamma(\xi_1) - u_\gamma(\xi_0)|\mathbf{1}_{E_x^c}\right] \le C\mathbb{P}_x(E_x^c) = O(\|x\|^{-p\delta}),$$

by (29). Since $p\delta > 2+\gamma$, it follows that

$$\mathbb{E}_x\left[u_\gamma(\xi_1) - u_\gamma(\xi_0)\right] = \mathbb{E}_x\langle Du_\gamma(x), \Delta\rangle + O(\|x\|^{-2-\gamma}). \tag{70}$$

For $x \in S_I$, it follows from (70) and (D_+) that $\mathbb{E}_x[u_\gamma(\xi_1) - u_\gamma(\xi_0)] = O(\|x\|^{-2-\gamma})$, and combining this with (59) we get (68).

Let $\beta = \max(\beta^+, \beta^-) < 1$. For $x \in S$, $|\theta(x)| = O(r^{\beta-1})$ as $\|x\| \to \infty$, so (70) gives

$$\mathbb{E}_x[u_\gamma(\xi_1) - u_\gamma(\xi_0)] = \frac{2\theta\cos\theta\,\mathbb{E}_x\,\Delta_2}{\|x\|(1+\|x\|)^\gamma} + O(\|x\|^{2\beta-3-\gamma}) + O(\|x\|^{-2-\gamma}).$$

If $x \in S_B^\pm$ then $\theta = \pm a^\pm(1+o(1))x_1^{\beta^\pm-1}$ and, by (10), $\mathbb{E}_x\,\Delta_2 = \mp\mu^\pm(x)\cos\alpha + o(1)$, so

$$\mathbb{E}_x[u_\gamma(\xi_1) - u_\gamma(\xi_0)] = -2a^\pm\mu^\pm(x)(\cos\alpha + o(1))\|x\|^{\beta^\pm-2-\gamma}. \tag{71}$$

For $\eta = \eta_0$ and $\beta^+, \beta^- \le \beta_c$, we have from (60) that

$$\mathbb{E}_x[\ell(\xi_1) - \ell(\xi_0)] \le \frac{1}{\|Tx\|^2\log\|Tx\|}\left(O(\|x\|^{2\beta^\pm-1}) + O(1)\right).$$

Combining this with (71), we obtain (69), provided that we choose γ such that $\beta^{\pm} - 2 - \gamma > 2\beta^{\pm} - 3$ and $\beta^{\pm} - 2 - \gamma > -2$, that is, $\gamma < 1 - \beta^{\pm}$ and $\gamma < \beta^{\pm}$. □

In the case where $\beta^+, \beta^- > 1$, we will use the function

$$w_\gamma(x) := \ell(x) - \frac{x_1}{(1 + \|x\|^2)^\gamma},$$

where the parameter η in ℓ is now chosen as $\eta = \eta_1$ as defined at (58). A similar function was used in [6].

Lemma 8 *Suppose that* (M_p), (D_+), (R), *and* (C_+) *hold, with* $p > 2$, $\varepsilon > 0$, $\alpha^+ = -\alpha^- = \alpha$ *for* $|\alpha| < \pi/2$, *and* $\beta^+, \beta^- > 1$ *Let* $\eta = \eta_1$, *and suppose that*

$$\frac{1}{2} < \gamma < \min\left(1 - \frac{1}{2\beta^+}, 1 - \frac{1}{2\beta^-}, \frac{p-1}{2}\right).$$

Then as $\|x\| \to \infty$ *with* $x \in S_I$,

$$\mathbb{E}[w_\gamma(\xi_{n+1}) - w_\gamma(\xi_n) \mid \xi_n = x] = -\frac{1 + \eta^2 + o(1)}{2\|Tx\|^2(\log \|Tx\|)^2}. \tag{72}$$

Moreover, as $\|x\| \to \infty$ *with* $x \in S_B^{\pm}$,

$$\mathbb{E}[w_\gamma(\xi_{n+1}) - w_\gamma(\xi_n) \mid \xi_n = x] = -\frac{\mu^{\pm}(x)\cos\alpha + o(1)}{\|x\|^{2\gamma}}. \tag{73}$$

Proof Let $q_\gamma(x) := x_1(1 + \|x\|^2)^{-\gamma}$. Then

$$D_1 q_\gamma(x) = \frac{1}{(1 + \|x\|^2)^\gamma} - \frac{2\gamma x_1^2}{(1 + \|x\|^2)^{1+\gamma}}, \quad D_2 q_\gamma(x) = -\frac{2\gamma x_1 x_2}{(1 + \|x\|^2)^{1+\gamma}},$$

and $|D_i D_j q_\gamma(x)| = O(\|x\|^{-1-2\gamma})$ for any i, j. Thus by Taylor's formula, for $y \in B_{r/2}(x)$,

$$q_\gamma(x + y) - q_\gamma(x) = \langle Dq_\gamma(x), y\rangle + R(x, y),$$

where $|R(x, y)| \le C\|y\|^2\|x\|^{-1-2\gamma}$ for $\|x\|$ sufficiently large. Once more let $E_x = \{\|\Delta\| < \|x\|^\delta\}$, where now we take $\delta \in (\frac{1+2\gamma}{p}, 1)$. Then

$$\mathbb{E}_x\left[(q_\gamma(\xi_1) - q_\gamma(\xi_0))\mathbf{1}_{E_x}\right] = \mathbb{E}_x\left[\langle Dq_\gamma(x), \Delta\rangle \mathbf{1}_{E_x}\right] + O(\|x\|^{-1-2\gamma}).$$

Moreover, we get from (29) that $\mathbb{E}_x\,|\langle Dq_\gamma(x),\,\Delta\rangle \mathbf{1}_{E_x^c}| = O(\|x\|^{-2\gamma-\delta(p-1)})$, where $\delta(p-1) > 2\gamma > 1$, and, since q_γ is uniformly bounded for $\gamma > 1/2$,

$$\mathbb{E}_x\left[(q_\gamma(\xi_1) - q_\gamma(\xi_0))\mathbf{1}_{E_x^c}\right] = O(\|x\|^{-p\delta}),$$

where $p\delta > 1 + 2\gamma$. Thus

$$\mathbb{E}_x\left[q_\gamma(\xi_1) - q_\gamma(\xi_0)\right] = \mathbb{E}_x\langle Dq_\gamma(x),\,\Delta\rangle + O(\|x\|^{-1-2\gamma}). \tag{74}$$

If $x \in S_I$, then (D_+) gives $\mathbb{E}_x\langle Dq_\gamma(x),\,\Delta\rangle = O(\|x\|^{-1-2\gamma})$ and with (59) we get (72), since $\gamma > 1/2$. On the other hand, suppose that $x \in S_B^{\pm}$ and $\beta^{\pm} > 1$. Then $\|x\| \geq c x_1^{\beta^{\pm}}$ for some $c > 0$, so $x_1 = O(\|x\|^{1/\beta^{\pm}})$. So, by (74),

$$\mathbb{E}_x[q_\gamma(\xi_1) - q_\gamma(\xi_0)] = \frac{\mathbb{E}_x\,\Delta_1}{(1+\|x\|^2)^\gamma} + O\left(\|x\|^{\frac{1}{\beta^{\pm}}-1-2\gamma}\right).$$

Moreover, by (11), $\mathbb{E}_x\,\Delta_1 = \mu^{\pm}(x)\cos\alpha + o(1)$. Combined with (61), this yields (73), provided that $2\gamma \leq 2 - (1/\beta^{\pm})$, again using the fact that $x_1 = O(\|x\|^{1/\beta^{\pm}})$. This completes the proof. □

4 Proofs of Main Results

We obtain our recurrence classification and quantification of passage-times via Foster–Lyapunov criteria (cf. [14]). As we do not assume any irreducibility, the most convenient form of the criteria are those for discrete-time adapted processes presented in [26]. However, the recurrence criteria in [26, §3.5] are formulated for processes on $\mathbb{R}_+$, and, strictly, do not apply directly here. Thus we present appropriate generalizations here, as they may also be useful elsewhere. The following recurrence result is based on Theorem 3.5.8 of [26].

Lemma 9 *Let $X_0, X_1, \ldots$ be a stochastic process on $\mathbb{R}^d$ adapted to a filtration $\mathscr{F}_0, \mathscr{F}_1, \ldots$. Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be such that $f(x) \to \infty$ as $\|x\| \to \infty$, and $\mathbb{E}\,f(X_0) < \infty$. Suppose that there exist $r_0 \in \mathbb{R}_+$ and $C < \infty$ for which, for all $n \in \mathbb{Z}_+$,*

$$\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathscr{F}_n] \leq 0, \text{ on } \{\|X_n\| \geq r_0\};$$
$$\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathscr{F}_n] \leq C, \text{ on } \{\|X_n\| < r_0\}.$$

Then if $\mathbb{P}(\limsup_{n\to\infty} \|X_n\| = \infty) = 1$, we have $\mathbb{P}(\liminf_{n\to\infty} \|X_n\| \leq r_0) = 1$.

Proof By hypothesis, $\mathbb{E}\, f(X_n) < \infty$ for all n. Fix $n \in \mathbb{Z}_+$ and let $\lambda_n := \min\{m \geq n : \|X_m\| \leq r_0\}$ and, for some $r > r_0$, set $\sigma_n := \min\{m \geq n : \|X_m\| \geq r\}$. Since $\limsup_{n\to\infty} \|X_n\| = \infty$ a.s., we have that $\sigma_n < \infty$, a.s. Then $f(X_{m\wedge\lambda_n\wedge\sigma_n})$, $m \geq n$, is a non-negative supermartingale with $\lim_{m\to\infty} f(X_{m\wedge\lambda_n\wedge\sigma_n}) = f(X_{\lambda_n\wedge\sigma_n})$, a.s. By Fatou's lemma and the fact that f is non-negative,

$$\mathbb{E}\, f(X_n) \geq \mathbb{E}\, f(X_{\lambda_n\wedge\sigma_n}) \geq \mathbb{P}(\sigma_n < \lambda_n) \inf_{y:\|y\|\geq r} f(y).$$

So

$$\mathbb{P}\left(\inf_{m\geq n} \|X_m\| \leq r_0\right) \geq \mathbb{P}(\lambda_n < \infty) \geq \mathbb{P}(\lambda_n < \sigma_n) \geq 1 - \frac{\mathbb{E}\, f(X_n)}{\inf_{y:\|y\|\geq r} f(y)}.$$

Since $r > r_0$ was arbitrary, and $\inf_{y:\|y\|\geq r} f(y) \to \infty$ as $r \to \infty$, it follows that, for fixed $n \in \mathbb{Z}_+$, $\mathbb{P}(\inf_{m\geq n} \|X_m\| \leq r_0) = 1$. Since this holds for all $n \in \mathbb{Z}_+$, the result follows. □

The corresponding transience result is based on Theorem 3.5.6 of [26].

Lemma 10 *Let $X_0, X_1, \ldots$ be a stochastic process on $\mathbb{R}^d$ adapted to a filtration $\mathscr{F}_0, \mathscr{F}_1, \ldots$. Let $f : \mathbb{R}^d \to \mathbb{R}_+$ be such that $\sup_x f(x) < \infty$, $f(x) \to 0$ as $\|x\| \to \infty$, and $\inf_{x:\|x\|\leq r} f(x) > 0$ for all $r \in \mathbb{R}_+$. Suppose that there exists $r_0 \in \mathbb{R}_+$ for which, for all $n \in \mathbb{Z}_+$,*

$$\mathbb{E}[f(X_{n+1}) - f(X_n) \mid \mathscr{F}_n] \leq 0, \text{ on } \{\|X_n\| \geq r_0\}.$$

Then if $\mathbb{P}(\limsup_{n\to\infty} \|X_n\| = \infty) = 1$, we have that $\mathbb{P}(\lim_{n\to\infty} \|X_n\| = \infty) = 1$.

Proof Since f is bounded, $\mathbb{E}\, f(X_n) < \infty$ for all n. Fix $n \in \mathbb{Z}_+$ and $r_1 \geq r_0$. For $r \in \mathbb{Z}_+$ let $\sigma_r := \min\{n \in \mathbb{Z}_+ : \|X_n\| \geq r\}$. Since $\mathbb{P}(\limsup_{n\to\infty} \|X_n\| = \infty) = 1$, we have $\sigma_r < \infty$, a.s. Let $\lambda_r := \min\{n \geq \sigma_r : \|X_n\| \leq r_1\}$. Then $f(X_{n\wedge\lambda_r})$, $n \geq \sigma_r$, is a non-negative supermartingale, which converges, on $\{\lambda_r < \infty\}$, to $f(X_{\lambda_r})$. By optional stopping (e.g. Theorem 2.3.11 of [26]), a.s.,

$$\sup_{x:\|x\|\geq r} f(x) \geq f(X_{\sigma_r}) \geq \mathbb{E}[f(X_{\lambda_r}) \mid \mathscr{F}_{\sigma_r}] \geq \mathbb{P}(\lambda_r < \infty \mid \mathscr{F}_{\sigma_r}) \inf_{x:\|x\|\leq r_1} f(x).$$

So

$$\mathbb{P}(\lambda_r < \infty) \leq \frac{\sup_{x:\|x\|\geq r} f(x)}{\inf_{x:\|x\|\leq r_1} f(x)},$$

which tends to 0 as $r \to \infty$, by our hypotheses on f. Thus,

$$\mathbb{P}\left(\liminf_{n\to\infty} \|X_n\| \leq r_1\right) = \mathbb{P}\left(\cap_{r\in\mathbb{Z}_+} \{\lambda_r < \infty\}\right) = \lim_{r\to\infty} \mathbb{P}(\lambda_r < \infty) = 0.$$

Since $r_1 \geq r_0$ was arbitrary, we get the result. □

Now we can complete the proof of Theorem 3, which includes Theorem 1 as the special case $\alpha = 0$.

Proof (of Theorem 3) Let $\beta = \max(\beta^+, \beta^-)$, and recall the definition of β_c from (5) and that of s_0 from (7). Suppose first that $0 \leq \beta < 1 \wedge \beta_c$. Then $s_0 > 0$ and we may (and do) choose $w \in (0, 2s_0)$. Also, take $\gamma \in (0, 1)$; note $0 < \gamma w < 1$. Consider the function f_w^γ with $\theta_0 = \theta_1$ given by (23). Then from (30), we see that there exist $c > 0$ and $r_0 < \infty$ such that, for all $x \in S_I$,

$$\mathbb{E}[f_w^\gamma(\xi_{n+1}) - f_w^\gamma(\xi_n) \mid \xi_n = x] \leq -c\|x\|^{\gamma w - 2}, \text{ for all } \|x\| \geq r_0. \tag{75}$$

By choice of w, we have $\beta - (1 - w)\beta_c < 0$, so (31) shows that, for all $x \in S_B^\pm$,

$$\mathbb{E}[f_w^\gamma(\xi_{n+1}) - f_w^\gamma(\xi_n) \mid \xi_n = x] \leq -c\|x\|^{\gamma w - 2 + \beta^\pm},$$

for some $c > 0$ and all $\|x\|$ sufficiently large. In particular, this means that (75) holds throughout S. On the other hand, it follows from (39) and (M_p) that there is a constant $C < \infty$ such that

$$\mathbb{E}[f_w^\gamma(\xi_{n+1}) - f_w^\gamma(\xi_n) \mid \xi_n = x] \leq C, \text{ for all } \|x\| \leq r_0. \tag{76}$$

Since $w, \gamma > 0$, we have that $f_w^\gamma(x) \to \infty$ as $\|x\| \to \infty$. Then by Lemma 9 with the conditions (75) and (76) and assumption (N), we establish recurrence.

Next suppose that $\beta_c < \beta < 1$. If $\beta^+ = \beta^- = \beta$, we use the function f_w^γ, again with $\theta_0 = \theta_1$ given by (23). We may (and do) choose $\gamma \in (0, 1)$ and $w < 0$ with $w > -2|s_0|$ and $\gamma w > w > 2 - p$. By choice of w, we have $\beta - (1 - w)\beta_c > 0$. We have from (30) and (31) that (75) holds in this case also, but now $f_w^\gamma(x) \to 0$ as $\|x\| \to \infty$, since $\gamma w < 0$. Lemma 10 then gives transience when $\beta^+ = \beta^-$.

Suppose now that $\beta_c < \beta < 1$ with $\beta^+ \neq \beta^-$. Without loss of generality, suppose that $\beta = \beta^+ > \beta^-$. We now use the function $F_w^{\gamma,\nu}$ defined at (48), where, as above, we take $\gamma \in (0, 1)$ and $w \in (-2|s_0|, 0)$, and we choose the constants λ, ν with $\lambda < 0$ and $\gamma w + \beta^- - 2 < 2\nu < \gamma w + \beta^+ - 2$. Note that $2\nu < \gamma w - 1$, so $F_w^{\gamma,\nu}(x) = f_w^\gamma(x)(1 + o(1))$. With $\theta_0 = \theta_1$ given by (23), and this choice of ν, Lemma 5 applies. The choice of γ ensures that the right-hand side of (49) is eventually negative, and the choice of w ensures the same for (50). Since $\lambda < 0$, the right-hand side of (51) is also eventually negative. Combining these three estimates shows, for all $x \in S$ with $\|x\|$ large enough,

$$\mathbb{E}[F_w^{\gamma,\nu}(\xi_{n+1}) - F_w^{\gamma,\nu}(\xi_n) \mid \xi_n = x] \leq 0.$$

Since $F_w^{\gamma,\nu}(x) \to 0$ as $\|x\| \to \infty$, Lemma 10 gives transience.

Of the cases where $\beta^+, \beta^- < 1$, it remains to consider the borderline case where $\beta = \beta_c \in (0, 1)$. Here Lemma 7 together with Lemma 9 proves recurrence. Finally,

if $\beta^+, \beta^- > 1$, we apply Lemma 8 together with Lemma 9 to obtain recurrence. Note that both of these critical cases require (D_+) and (C_+). □

Next we turn to moments of passage times: we prove Theorem 4, which includes Theorem 2 as the special case $\alpha = 0$. Here the criteria we apply are from [26, §2.7], which are heavily based on those from [5].

Proof (of Theorem 4) Again let $\beta = \max(\beta^+, \beta^-)$. First we prove the existence of moments part of (a)(i). Suppose that $0 \le \beta < 1 \wedge \beta_c$, so s_0 as defined at (7) satisfies $s_0 > 0$. We use the function f_w^γ, with $\gamma \in (0, 1)$ and $w \in (0, 2s_0)$ as in the first part of the proof of Theorem 3. We saw in that proof that for these choices of γ, w we have that (75) holds for all $x \in S$. Rewriting this slightly, using the fact that $f_w^\gamma(x)$ is bounded above and below by constants times $\|x\|^{\gamma w}$ for all $\|x\|$ sufficiently large, we get that there are constants $c > 0$ and $r_0 < \infty$ for which

$$\mathbb{E}[f_w^\gamma(\xi_{n+1}) - f_w^\gamma(\xi_n) \mid \xi_n = x] \le -c(f_w^\gamma(x))^{1-\frac{2}{\gamma w}}, \text{ for all } x \in S \text{ with } \|x\| \ge r_0. \tag{77}$$

Then we may apply Corollary 2.7.3 of [26] to get $\mathbb{E}_x(\tau_r^s) < \infty$ for any $r \ge r_0$ and any $s < \gamma w/2$. Taking $\gamma < 1$ and $w < 2s_0$ arbitrarily close to their upper bounds, we get $\mathbb{E}_x(\tau_r^s) < \infty$ for all $s < s_0$.

Next suppose that $0 \le \beta \le \beta_c$. Let $s > s_0$. First consider the case where $\beta^+ = \beta^-$. Then we consider f_w^γ with $\gamma > 1$, $w > 2s_0$ (so $w > 0$), and $0 < w\gamma < 2$. Then, since $\beta - (1-w)\beta_c = \beta_c - \beta + (w - 2s_0)\beta_c > 0$, we have from (30) and (31) that

$$\mathbb{E}[f_w^\gamma(\xi_{n+1}) - f_w^\gamma(\xi_n) \mid \xi_n = x] \ge 0, \tag{78}$$

for all $x \in S$ with $\|x\|$ sufficiently large. Now set $Y_n := f_w^{1/w}(\xi_n)$, and note that Y_n is bounded above and below by constants times $\|\xi_n\|$, and $Y_n^{\gamma w} = f_w^\gamma(\xi_n)$. Write $\mathscr{F}_n = \sigma(\xi_0, \xi_1, \ldots, \xi_n)$. Then we have shown in (78) that

$$\mathbb{E}[Y_{n+1}^{\gamma w} - Y_n^{\gamma w} \mid \mathscr{F}_n] \ge 0, \text{ on } \{Y_n > r_1\}, \tag{79}$$

for some r_1 sufficiently large. Also, from the $\gamma = 1/w$ case of (30) and (31),

$$\mathbb{E}[Y_{n+1} - Y_n \mid \mathscr{F}_n] \ge -\frac{B}{Y_n}, \text{ on } \{Y_n > r_2\}, \tag{80}$$

for some $B < \infty$ and r_2 sufficiently large. (The right-hand side of (31) is still eventually positive, while the right-hand-side of (30) will be eventually negative if $\gamma < 1$.) Again let $E_x = \{\|\Delta\| < \|x\|^\delta\}$ for $\delta \in (0, 1)$. Then from the $\gamma = 1/w$ case of (41),

$$\left| f_w^{1/w}(\xi_1) - f_w^{1/w}(\xi_0) \right|^2 \mathbf{1}_{E_x} \le C\|\Delta\|^2,$$

while from the $\gamma = 1/w$ case of (39) we have

$$\left| f_w^{1/w}(\xi_1) - f_w^{1/w}(\xi_0) \right|^2 \mathbf{1}_{E_x^{\mathrm{c}}} \le C \|\Delta\|^{2/\delta}.$$

Taking $\delta \in (2/p, 1)$, it follows from (M_p) that for some $C < \infty$, a.s.,

$$\mathbb{E}[(Y_{n+1} - Y_n)^2 \mid \mathscr{F}_n] \le C. \tag{81}$$

The three conditions (79)–(81) show that we may apply Theorem 2.7.4 of [26] to get $\mathbb{E}_x(\tau_r^s) = \infty$ for all $s > \gamma w/2$, all r sufficiently large, and all $x \in S$ with $\|x\| > r$. Hence, taking $\gamma > 1$ and $w > 2s_0$ arbitrarily close to their lower bounds, we get $\mathbb{E}_x(\tau_r^s) = \infty$ for all $s > s_0$ and appropriate r, x. This proves the non-existence of moments part of (a)(i) in the case $\beta^+ = \beta^-$.

Next suppose that $0 \le \beta^+, \beta^- \le \beta_{\mathrm{c}}$ with $\beta^+ \ne \beta^-$. Without loss of generality, suppose that $0 \le \beta^- < \beta^+ = \beta \le \beta_{\mathrm{c}}$. Then $0 \le s_0 < 1/2$. We consider the function $F_w^{\gamma,\nu}$ given by (48) with $\theta_0 = \theta_1$ given by (23), $\lambda > 0$, $w \in (2s_0, 1)$, and $\gamma > 1$ such that $\gamma w < 1$. Also, take ν for which $\gamma w + \beta^- - 2 < 2\nu < \gamma w + \beta^+ - 2$. Then by choice of γ and w, we have that the right-hand sides of (49) and (50) are both eventually positive. Since $\lambda > 0$, the right-hand side of (51) is also eventually positive. Thus

$$\mathbb{E}[F_w^{\gamma,\nu}(\xi_{n+1}) - F_w^{\gamma,\nu}(\xi_n) \mid \xi_n = x] \ge 0,$$

for all $x \in S$ with $\|x\|$ sufficiently large. Take $Y_n := (F_w^{\gamma,\nu}(\xi_n))^{1/(\gamma w)}$. Then we have shown that, for this Y_n, the condition (79) holds. Moreover, since $\gamma w < 1$ we have from convexity that (80) also holds. Again let $E_x = \{\|\Delta\| < \|x\|^{\delta}\}$. From (41) and (52),

$$\left| F_w^{\gamma,\nu}(x + y) - F_w^{\gamma,\nu}(x) \right| \le C \|y\| \|x\|^{\gamma w - 1},$$

for all $y \in B_{r/2}(x)$. Then, by another Taylor's theorem calculation,

$$\left| \left(F_w^{\gamma,\nu}(x + y)\right)^{1/(\gamma w)} - \left(F_w^{\gamma,\nu}(x)\right)^{1/(\gamma w)} \right| \le C \|y\|,$$

for all $y \in B_{r/2}(x)$. It follows that $\mathbb{E}_x[(Y_1 - Y_0)^2 \mathbf{1}_{E_x}] \le C$. Moreover, by a similar argument to (40), $|Y_1 - Y_0|^2 \le C\|\Delta\|^{2\gamma w/\delta}$ on E_x^{c}, so taking $\delta \in (2/p, 1)$ and using the fact that $\gamma w < 1$, we get $\mathbb{E}_x[(Y_1 - Y_0)^2 \mathbf{1}_{E_x^{\mathrm{c}}}] \le C$ as well. Thus we also verify (81) in this case. Then we may again apply Theorem 2.7.4 of [26] to get $\mathbb{E}_x(\tau_r^s) = \infty$ for all $s > \gamma w/2$, and hence all $s > s_0$. This completes the proof of (a)(i).

For part (a)(ii), suppose first that $\beta^+ = \beta^- = \beta$, and that $\beta_{\mathrm{c}} \le \beta < 1$. We apply the function f_w^{γ} with $w > 0$ and $\gamma > 1$. Then we have from (30) and (31) that (78) holds. Repeating the argument below (78) shows that $\mathbb{E}_x(\tau_r^s) = \infty$ for all

$s > \gamma w/2$, and hence all $s > 0$. The case where $\beta^+ \neq \beta^-$ is similar, using an appropriate $F_w^{\gamma,\nu}$. This proves (a)(ii).

It remains to consider the case where $\beta^+, \beta^- > 1$. Now we apply f_w^γ with $\gamma > 1$ and $w \in (0, 1/2)$ small enough, noting Remark 4. In this case (30) with (32) and Lemma 3 show that (78) holds, and repeating the argument below (78) shows that $\mathbb{E}_x(\tau_r^s) = \infty$ for all $s > 0$. This proves part (b). □

Appendix: Properties of the Threshold Function

For a constant $b \neq 0$, consider the function

$$\phi(\alpha) = \sin^2 \alpha + b \sin 2\alpha.$$

Set $\alpha_0 := \frac{1}{2} \arctan(-2b)$, which has $0 < |\alpha_0| < \pi/4$.

Lemma 11 *There are two stationary points of ϕ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. One of these is a local minimum at α_0, with*

$$\phi(\alpha_0) = \frac{1}{2}\left(1 - \sqrt{1 + 4b^2}\right) < 0.$$

The other is a local maximum, at $\alpha_1 = \alpha_0 + \frac{\pi}{2}$ if $b > 0$, or at $\alpha_1 = \alpha_0 - \frac{\pi}{2}$ if $b < 0$, with

$$\phi(\alpha_1) = \frac{1}{2}\left(1 + \sqrt{1 + 4b^2}\right) > 1.$$

Proof We compute $\phi'(\alpha) = \sin 2\alpha + 2b \cos 2\alpha$ and $\phi''(\alpha) = 2 \cos 2\alpha - 4b \sin 2\alpha$. Then $\phi'(\alpha) = 0$ if and only if $\tan 2\alpha = -2b$. Thus the stationary values of ϕ are $\alpha_0 + k\frac{\pi}{2}$, $k \in \mathbb{Z}$. Exactly two of these values fall in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, namely α_0 and α_1 as defined in the statement of the lemma. Also

$$\phi''(\alpha_0) = 2 \cos 2\alpha_0 - 4b \sin 2\alpha_0 = \left(2 + 8b^2\right) \cos 2\alpha_0 > 0,$$

so α_0 is a local minimum. Similarly, if $|\delta| = \pi/2$, then $\sin 2\delta = 0$ and $\cos 2\delta = -1$, so

$$\phi''(\alpha_0 + \delta) = -\cos 2\alpha_0 + 4b \sin 2\alpha_0 = -\phi''(\alpha_0),$$

and hence the stationary point at α_1 is a local maximum. Finally, to evaluate the values of ϕ at the stationary points, note that

$$\cos 2\alpha_0 = \frac{1}{\sqrt{1 + 4b^2}}, \text{ and } \sin 2\alpha_0 = \frac{-2b}{\sqrt{1 + 4b^2}},$$

and use the fact that $2\sin^2\alpha_0 = 1 - \cos 2\alpha_0$ to get $\phi(\alpha_0)$, and that $2\cos^2\alpha_0 = \cos 2\alpha_0 + 1$ to get $\phi(\alpha_1) = \cos^2\alpha_0 - b\sin 2\alpha_0 = 1 - \phi(\alpha_0)$. □

Proof (of Proposition 1) By Lemma 11 (and considering separately the case $\sigma_1^2 = \sigma_2^2$) we see that the extrema of $\beta_c(\Sigma, \alpha)$ over $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ are

$$\frac{\sigma_1^2 + \sigma_2^2}{2\sigma_2^2} \pm \frac{1}{2\sigma_2^2}\sqrt{\left(\sigma_2^2 - \sigma_1^2\right)^2 + 4\rho^2},$$

as claimed at (6). It remains to show that the minimum is strictly positive, which is a consequence of the fact that

$$\sigma_1^2 + \sigma_2^2 - \sqrt{\left(\sigma_1^2 + \sigma_2^2\right)^2 - 4\left(\sigma_1^2\sigma_2^2 - \rho^2\right)} > 0,$$

since $\rho^2 < \sigma_1^2\sigma_2^2$ (as Σ is positive definite). □

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References

1. Aspandiiarov, S.: On the convergence of 2-dimensional Markov chains in quadrants with boundary reflection. Stochastics Stochastics Rep. **53**, 275–303 (1995)
2. Aspandiiarov, S., Iasnogorodski, R.: Tails of passage-times and an application to stochastic processes with boundary reflection in wedges. Stochastic Process. Appl. **66**, 115–145 (1997)
3. Aspandiiarov, S., Iasnogorodski, R.: General criteria of integrability of functions of passage-times for nonnegative stochastic processes and their applications. Theory Probab. Appl. **43**, 343–369 (1999). Translated from Teor. Veroyatnost. i Primenen. **43**, 509–539 (1998) (in Russian)
4. Aspandiiarov, S., Iasnogorodski, R.: Asymptotic behaviour of stationary distributions for countable Markov chains, with some applications. Bernoulli **5**, 535–569 (1999)
5. Aspandiiarov, S., Iasnogorodski, R., Menshikov, M.: Passage-time moments for nonnegative stochastic processes and an application to reflected random walks in a quadrant. Ann. Probab. **24**, 932–960 (1996)
6. Asymont, I.M., Fayolle, G., Menshikov, M.V.: Random walks in a quarter plane with zero drifts: transience and recurrence. J. Appl. Probab. **32**, 941–955 (1995)
7. Balaji, S., Ramasubramanian, S.: Passage time moments for multidimensional diffusions. J. Appl. Probab. **37**, 246–251 (2000)
8. Bañuelos, R., Carroll, T.: Sharp integrability for Brownian motion in parabola-shaped regions. J. Funct. Anal. **218**, 219–253 (2005)
9. Bañuelos, R., DeBlassie, R.D., Smits, R.: The first exit time of a planar Brownian motion from the interior of a parabola. Ann. Probab. **29**, 882–901 (2001)

10. Fayolle, G.: On random walks arising in queueing systems: ergodicity and transience via quadratic forms as Lyapunov functions – Part I. Queueing Syst. **5**, 167–184 (1989)
11. Fayolle, G., Iasnogorodski, R., Malyshev, V.: Random Walks in the Quarter-Plane. 2nd edn. Springer, Berlin (2017)
12. Fayolle, G., Malyshev, V.A., Menshikov, M.V.: Random walks in a quarter plane with zero drifts. I. Ergodicity and null recurrence. Ann. Inst. Henri Poincaré **28**, 179–194 (1992)
13. Fayolle, G., Malyshev, V.A., Menshikov, M.V.: Topics in the Constructive Theory of Countable Markov Chains. Cambridge University Press, Cambridge (1995)
14. Foster, F.G.: On the stochastic matrices associated with certain queuing processes. Ann. Math. Stat. **24**, 355–360 (1953)
15. Franceschi, S., Raschel, K.: Integral expression for the stationary distribution of reflected Brownian motion in a wedge. Bernoulli **25**, 3673–3713 (2019)
16. Hobson, D.G., Rogers, L.C.G.: Recurrence and transience of reflecting Brownian motion in the quadrant. Math. Proc. Cambridge Philos. Soc. **113**, 387–399 (1993)
17. Ignatyuk, I.A., Malyshev, V.A.: Classification of random walks in $\mathbb{Z}_+^4$. Selecta Math. **12**, 129–194 (1993)
18. Kingman, J.F.C.: The ergodic behaviour of random walks. Biometrika **48**, 391–396 (1961)
19. Lamperti, J.: Criteria for stochastic processes II: passage-time moments. J. Math. Anal. Appl. **7**, 127–145 (1963)
20. Li, W.V.: The first exit time of a Brownian motion from unbounded convex domains. Ann. Probab. **31**, 1078–1096 (2003)
21. MacPhee, I.M., Menshikov, M.V., Wade, A.R.: Moments of exit times from wedges for non-homogeneous random walks with asymptotically zero drifts. J. Theoret. Probab. **26**, 1–30 (2013)
22. Malyshev, V.A.: Random Walks, The Wiener-Hopf Equation in a Quarter Plane, Galois Automorphisms. Moscow State University, Moscow (1970) (in Russian)
23. Malyshev, V.A.: Classification of two-dimensional positive random walks and almost linear semimartingales. Soviet Math. Dokl. **13**, 136–139 (1972). Translated from Dokl. Akad. Nauk SSSR **202**, 526–528 (1972) (in Russian)
24. Menshikov, M.V.: Ergodicity and transience conditions for random walks in the positive octant of space. Soviet Math. Dokl. **15**, 1118–1121 (1974). Translated from Dokl. Akad. Nauk SSSR **217**, 755–758 (1974) (in Russian)
25. Menshikov, M., Williams, R.J.: Passage-time moments for continuous non-negative stochastic processes and applications. Adv. Appl. Probab. **28**, 747–762 (1996)
26. Menshikov, M., Popov, S., Wade, A.: Non-homogeneous random walks. Cambridge University Press, Cambridge (2017)
27. Mikhailov, V.A.: Methods of Random Multiple Access, Thesis. Dolgoprudny, Moscow, 1979 (in Russian)
28. Pinsky, R.G.: Transience/recurrence for normally reflected Brownian motion in unbounded domains. Ann. Probab. **37**, 676–686 (2009)
29. Rosenkrantz, W.A.: Ergodicity conditions for two-dimensional Markov chains on the positive quadrant. Probab. Theory Relat. Fields **83**, 309–319 (1989)
30. Vaninskii, K.L., Lazareva, B.V.: Ergodicity and nonrecurrence conditions of a homogeneous Markov chain in the positive quadrant. Probl. Inf. Transm. **24**, 82–86 (1988). Translated from Problemy Peredachi Informatsii **24**, 105–110 (1988) (in Russian)
31. Varadhan, S.R.S., Williams, R.: Brownian motion in a wedge with oblique reflection. Comm. Pure Appl. Math. **38**, 405–443 (1985)
32. Williams, R.J.: Recurrence classification and invariant measures for reflected Brownian motion in a wedge. Ann. Probab. **13**, 758–778 (1985)
33. Zachary, S.: On two-dimensional Markov chains in the positive quadrant with partial spatial homogeneity. Markov Process. Related Fields **1**, 267–280 (1995)