

On the Multiplicity of the Zeros of Polynomials with Constrained Coefficients



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Abstract We survey a few recent results focusing on the multiplicity of the zero at 1 of polynomials with constrained coefficients. Some closely related problems and results are also discussed.

Mathematics Subject Classification (2010): 11C08, 41A17, 26C10, 30C15

1 On the Multiplicity of the Zero at 1 of Polynomials with Constrained Coefficients

In [17] and [18], we examined a number of problems concerning polynomials with coefficients restricted in various ways. We were particularly interested in how small such polynomials can be on $[0, 1]$. For example, we proved that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$e^{-c_1\sqrt{n}} \leq \min_{0 \neq Q \in \mathcal{F}_n} \left\{ \max_{x \in [0,1]} |Q(x)| \right\} \leq e^{-c_2\sqrt{n}}$$

for every $n \geq 2$, where \mathcal{F}_n denotes the set of all polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$.

Littlewood considered minimization problems of this variety on the unit disk. His most famous, now solved, conjecture was that the L_1 norm of an element $f \in \mathcal{F}_n$ on the unit circle grows at least as fast as $c \log N$, where N is the number of non-zero coefficients in f and $c > 0$ is an absolute constant.

When the coefficients are required to be integers, the questions have a Diophantine nature and have been studied from a variety of points of view.

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One key to the analysis is the study of the related problem of giving an upper bound for the multiplicity of the zero these restricted polynomials can have at 1. In [17] and [18], we answer this latter question precisely for the class of polynomials of the form

$$Q(x) = \sum_{j=0}^n a_j x^j, \quad |a_j| \leq 1, \quad a_j \in \mathbb{C}, \quad j = 1, 2, \dots, n,$$

with fixed $|a_0| \neq 0$.

Various forms of these questions have attracted considerable study, though rarely have precise answers been possible to give. Indeed, the classical, much studied, and presumably very difficult problem of Prouhet, Tarry, and Escott rephrases as a question of this variety. (Precisely: what is the maximal vanishing at 1 of a polynomial with integer coefficients with l_1 norm $2n$? It is conjectured to be n .)

For $n \in \mathbb{N}$, $L > 0$, and $p \geq 1$, let $\kappa_p(n, L)$ be the largest possible value of k for which there is a polynomial $Q \not\equiv 0$ of the form

$$Q(x) = \sum_{j=0}^n a_j x^j, \quad a_j \in \mathbb{C}, \quad |a_0| \geq L \left(\sum_{j=1}^n |a_j|^p \right)^{1/p},$$

such that $(x - 1)^k$ divides $Q(x)$.

For $n \in \mathbb{N}$ and $L > 0$, let $\kappa_\infty(n, L)$ be the largest possible value of k for which there is a polynomial $Q \not\equiv 0$ of the form

$$Q(x) = \sum_{j=0}^n a_j x^j, \quad a_j \in \mathbb{C}, \quad |a_0| \geq L \max_{1 \leq j \leq n} |a_j|,$$

such that $(x - 1)^k$ divides $Q(x)$.

In [17], we proved that there is an absolute constant $c_3 > 0$ such that

$$\min \left\{ \frac{1}{6} \sqrt{n(1 - \log L)} - 1, n \right\} \leq \kappa_\infty(n, L) \leq \min \left\{ c_3 \sqrt{n(1 - \log L)}, n \right\}$$

for every $n \in \mathbb{N}$ and $L \in (0, 1]$. However, we were far from being able to establish the right result in the case of $L \geq 1$. In [18], we proved the right order of magnitude of $\kappa_\infty(n, L)$ and $\kappa_2(n, L)$ in the case of $L \geq 1$. Our results in [17] and [18] sharpen and generalize the results of Schur [62], Amoroso [1], Bombieri and Vaaler [6], and Hua [49] who gave versions of this result for polynomials with integer coefficients. Our results in [17] have turned out to be related to a number of recent and old publications from a rather wide range of research areas. See [1–16, 18–67], for example. More results on the zeros of polynomials with Littlewood type coefficient constraints may be found in [37]. Markov and Bernstein type inequalities under Erdős type coefficient constraints are surveyed in [36].

For $n \in \mathbb{N}$, $L > 0$, and $q \geq 1$, let $\mu_q(n, L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \left(\sum_{j=1}^n |Q(j)|^q \right)^{1/q}.$$

For $n \in \mathbb{N}$ and $L > 0$, let $\mu_\infty(n, L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \max_{1 \leq j \leq n} |Q(j)|.$$

It is a simple consequence of Hölder’s inequality (see Lemma 3.6 in [42]) that

$$\kappa_p(n, L) \leq \mu_q(n, L)$$

whenever $n \in \mathbb{N}$, $L > 0$, $1 \leq p, q \leq \infty$, and $1/p + 1/q = 1$.

In [42], we have found the size of $\kappa_p(n, L)$ and $\mu_q(n, L)$ for all $n \in \mathbb{N}$, $L > 0$, and $1 \leq p, q \leq \infty$. The result about $\mu_\infty(n, L)$ is due to Coppersmith and Rivlin, [27], but our proof presented in [42] is completely different and much shorter even in that special case. Another short proof of the Coppersmith–Rivlin inequality is presented in [41].

Our results in [17] may be viewed as finding the size of $\kappa_\infty(n, L)$ and $\mu_1(n, L)$ for all $n \in \mathbb{N}$ and $L \in (0, 1]$.

Our results in [18] may be viewed as finding the size of $\kappa_\infty(n, L)$, $\mu_1(n, L)$, $\kappa_2(n, L)$, and $\mu_2(n, L)$ for all $n \in \mathbb{N}$ and $L > 0$.

Our main results in [42] are stated below.

Theorem 1 *Let $p \in (1, \infty]$ and $q \in [1, \infty)$ satisfy $1/p + 1/q = 1$. There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\sqrt{n}(c_1 L)^{-q/2} - 1 \leq \kappa_p(n, L) \leq \mu_q(n, L) \leq \sqrt{n}(c_2 L)^{-q/2} + 2$$

for every $n \in \mathbb{N}$ and $L > 1/2$, and

$$c_3 \min \left\{ \sqrt{n(-\log L)}, n \right\} \leq \kappa_p(n, L) \leq \mu_q(n, L) \leq c_4 \min \left\{ \sqrt{n(-\log L)}, n \right\} + 4$$

for every $n \in \mathbb{N}$ and $L \in (0, 1/2]$. Here, $c_1 := 1/53$, $c_2 := 40$, $c_3 := 2/7$, and $c_4 := 13$ are the appropriate choices.

Theorem 2 *There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \sqrt{n(1-L)} - 1 \leq \kappa_1(n, L) \leq \mu_\infty(n, L) \leq c_2 \sqrt{n(1-L)} + 1$$

for every $n \in \mathbb{N}$ and $L \in (1/2, 1]$, and

$$c_3 \min \left\{ \sqrt{n(-\log L)}, n \right\} \leq \kappa_1(n, L) \leq \mu_\infty(n, L) \leq c_4 \min \left\{ \sqrt{n(-\log L)}, n \right\} + 4$$

for every $n \in \mathbb{N}$ and $L \in (0, 1/2]$. Note that $\kappa_1(n, L) = \mu_\infty(n, L) = 0$ for every $n \in \mathbb{N}$ and $L > 1$. Here, $c_1 := 1/5$, $c_2 := 1$, $c_3 := 2/7$, and $c_4 := 13$ are the appropriate choices.

Note that in [39], extending a result of Totik and Varjú in [66], we showed that if the modulus of a monic polynomial P of degree at most n , with complex coefficients, on the unit circle of the complex plane is at most $1 + o(1)$ uniformly, then the multiplicity of each zero of P outside the open unit disk is $o(n^{1/2})$. Equivalently, if a polynomial P of degree at most n , with complex coefficients and constant term 1, has modulus at most $1 + o(1)$ uniformly on the unit circle, then the multiplicity of each zero of P in the closed unit disk is $o(n^{1/2})$. These observations were obtained in [39] as a consequence of our “one-sided” improvement of an old Erdős–Turán Theorem in [43]. Namely in [39], we proved that if the zeros of

$$P(z) := \sum_{j=0}^n a_j z^j, \quad a_j \in \mathbb{C}, \quad a_0 a_n \neq 0$$

are denoted by

$$z_j = r_j \exp(i\varphi_j), \quad r_j > 0, \quad \varphi_j \in [0, 2\pi), \quad j = 1, 2, \dots, n,$$

then for every $0 \leq \alpha < \beta \leq 2\pi$, we have

$$\sum_{j \in I_1(\alpha, \beta)} 1 - \frac{\beta - \alpha}{2\pi} n \leq 16\sqrt{n \log R_1},$$

and

$$\sum_{j \in I_2(\alpha, \beta)} 1 - \frac{\beta - \alpha}{2\pi} n \leq 16\sqrt{n \log R_2},$$

where

$$R_1 := |a_n|^{-1} \|P\|, \quad R_2 := |a_0|^{-1} \|P\|,$$

and

$$I_1(\alpha, \beta) := \{j : \alpha \leq \varphi_j \leq \beta, r_j \geq 1\}, \quad I_2(\alpha, \beta) := \{j : \alpha \leq \varphi_j \leq \beta, r_j \leq 1\}.$$

Here, $\|P\|$ denotes the maximum modulus of P on the closed unit disk of the complex plane. For better constants in the Erdős–Turán Theorem in [43], see the

recent paper [64] by Soundararajan, who also offers a very elegant new approach to prove the Erdős–Turán Theorem in [43].

2 Remarks and Problems

A question we have not considered in [42] is if there are examples of n , L , and p for which the values of $\kappa_p(n, L)$ are significantly smaller if the coefficients are required to be rational (perhaps together with other restrictions). The same question may be raised about $\mu_q(n, L)$. As the conditions on the coefficients of the polynomials in Theorems 1 and 2 are homogeneous, assuming rational coefficients and integer coefficients lead to the same results. Four special classes of interest are

$$\mathcal{F}_n := \left\{ Q : Q(z) = \sum_{j=0}^n a_j z^j, \quad a_j \in \{-1, 0, 1\} \right\},$$

$$\mathcal{N}_n := \left\{ Q : Q(z) = \sum_{j=0}^n a_j z^j, \quad a_j \in \{0, 1\} \right\},$$

$$\mathcal{L}_n := \left\{ Q : Q(z) = \sum_{j=0}^n a_j z^j, \quad a_j \in \{-1, 1\} \right\},$$

and

$$\mathcal{K}_n := \left\{ Q : Q(z) = \sum_{j=0}^n a_j z^j, \quad a_j \in \mathbb{C}, \quad |a_j| = 1 \right\}.$$

Elements of \mathcal{F}_n are often called Borwein polynomials of degree at most n . Elements of \mathcal{N}_n are often called Newman polynomials of degree at most n . Elements of \mathcal{L}_n are often called Littlewood polynomials of degree n . Elements of \mathcal{K}_n are often called unimodular or Kahane polynomials of degree n . In [17], we proved the following result.

Theorem 3 *Let $p \leq n$ be a prime. Suppose $Q \in \mathcal{F}_n$ and Q has exactly k zeros at 1 and exactly m zeros at a primitive p th root of unity. Then*

$$p(m + 1) \geq k \frac{\log p}{\log(n + 1)}.$$

The proof of Theorem 3 is so simple that we reproduce it here.

Proof of Theorem 3 Let

$$\xi_j := \exp\left(\frac{2\pi i j}{p}\right), \quad j = 1, 2, \dots, p-1.$$

Let $Q \in \mathcal{F}_n$ be of the form

$$Q(x) = (x-1)^k R(x),$$

where R is a polynomial of degree at most $n-k$ with integer coefficients. Then, for every integer $m \leq k$, we have

$$Q^{(m)}(x) = (x-1)^{k-m} S(x),$$

where S is a polynomial of degree at most $n-k$ with integer coefficients. Hence,

$$K := \prod_{j=1}^{p-1} Q^{(m)}(\xi_j) = \prod_{j=1}^{p-1} (\xi_j - 1)^{k-m} \prod_{j=1}^{p-1} S(\xi_j) =: p^{k-m} N,$$

where both K and N are integers by the fundamental theorem of symmetric polynomials. Further,

$$|K| \leq \prod_{j=1}^{p-1} (n+1)n^m \leq (n+1)^{(p-1)(m+1)}.$$

Hence, $K \neq 0$ implies

$$p^{k-m} \leq (n+1)^{(p-1)(m+1)},$$

that is,

$$k-m \leq \frac{(p-1)(m+1) \log(n+1)}{\log p},$$

and the result follows. \square

The following three problems arise naturally, and they have been already raised in [10], for example.

Problem 1 How many zeros can a polynomial $0 \neq Q \in \mathcal{F}_n$ have at 1?

Problem 2 How many zeros can a polynomial $Q \in \mathcal{L}_n$ have at 1?

Problem 3 How many zeros can a polynomial $Q \in \mathcal{K}_n$ have at 1?

The case when $p = \infty$ and $L = 1$ in our Theorem 1 gives that every $0 \neq Q \in \mathcal{F}_n$, every $Q \in \mathcal{L}_n$, and every $Q \in \mathcal{K}_n$ can have at most $cn^{1/2}$ zeros at 1 with an absolute constant $c > 0$, but one may expect better results by utilizing the additional pieces of information on their coefficients.

It was observed in [17] that for every integer $n \geq 2$ there is a $Q \in \mathcal{F}_n$ having at least $c(n/\log n)^{1/2}$ zeros at 1 with an absolute constant $c > 0$. This is a simple pigeon hole argument. However, as far as we know, closing the gap between $cn^{1/2}$ and $c(n/\log n)^{1/2}$ in Problem 1 is an open and most likely very difficult problem.

It is proved in [11] that every polynomial P of the form

$$P(z) = \sum_{j=0}^n a_j z^j, \quad |a_0| = 1, \quad |a_j| \leq 1, \quad a_j \in \mathbb{C}$$

has at most $c\sqrt{n}$ zeros inside any polygon with vertices on the unit circle ∂D , where c depends only on the polygon. One of the main results of [19] gives explicit estimates for the number and location of zeros of polynomials with bounded coefficients. Namely, if

$$\delta_n := 33\pi \frac{\log n}{\sqrt{n}} \leq 1,$$

then every polynomial P of the form

$$P(z) = \sum_{j=0}^n a_j z^j, \quad |a_0| = |a_n| = 1, \quad |a_j| \leq 1, \quad a_j \in \mathbb{C}$$

has at least $8\sqrt{n} \log n$ zeros in any disk with center on the unit circle and radius δ_n . More on Littlewood polynomials may be found in [7, 37], for example.

As far as Problem 2.3 is concerned, Boyd [23] showed that for $n \geq 3$ every $Q \in \mathcal{L}_n$ has at most

$$\frac{c(\log n)^2}{\log \log n} \tag{2.1}$$

zeros at 1, and this is the best known upper bound even today. Boyd’s proof is very clever and up to an application of the Prime Number Theorem is completely elementary. It is reasonable to conjecture that for $n \geq 2$ every $Q \in \mathcal{L}_n$ has at most $c \log n$ zeros at 1. It is easy to see that for every integer $n \geq 2$ there are $Q_n \in \mathcal{L}_n$ with at least $c \log n$ zeros at 1 with an absolute constant $c > 0$. Indeed, the polynomials P_k defined by

$$P_k(z) = \prod_{j=0}^k (z^{2^j} - 1), \quad k = 1, 2, \dots$$

has degree $2^{k+1} - 1$ and a zero of multiplicity $k + 1$ at 1. By using Boyd's elegant method, it is easy to prove also that if M_k denotes the largest possible multiplicity that a zero of a $P \in \mathcal{L}_k$ can have at 1 and (C_k) is an arbitrary sequence of positive integers tending to ∞ , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \in \{1, 2, \dots, n\} : M_k \geq C_k\}| = 0.$$

This was proposed as a problem in the Monthly [40] in 2009, and a few people have solved it.

As far as Problem 3 is concerned, one may suspect that for $n \geq 2$ every $Q \in \mathcal{K}_n$ has at most $c \log n$ zeros at 1. However, just to see if Boyd's bound (2.1) holds for every $Q \in \mathcal{K}_n$ seems quite challenging and beyond reach at the moment.

Problem 4 How many zeros can a polynomial $P \in \mathcal{F}_n$ have at α if $|\alpha| \neq 1$ and $\alpha \neq 0$? Can it have as many as we want?

Problem 5 How many zeros can a polynomial $P \in \mathcal{L}_n$ have at α if $|\alpha| \neq 1$ and $\alpha \neq 0$? Can it have as many as we want?

The Mahler measure

$$M_0(P) := \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{it})| dt \right)$$

is defined for bounded measurable functions P defined on the unit circle. It is well known that

$$M_0(P) := \lim_{q \rightarrow 0^+} M_q(P),$$

where, for $q > 0$,

$$M_q(P) := \left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{it})|^q dt \right)^{1/q}.$$

It is a simple consequence of the Jensen formula that

$$M_0(P) = |c| \prod_{k=1}^n \max\{1, |z_k|\}$$

for every polynomial of the form

$$P(z) = c \prod_{k=1}^n (z - z_k), \quad c, z_k \in \mathbb{C}.$$

Lehmer’s conjecture is a problem in number theory raised by Derrick Henry Lehmer. The conjecture asserts that there is an absolute constant $\mu > 1$ such that for every polynomial P with integer coefficients satisfying $P(0) \neq 0$ we have either $M_0(P) = 1$ (that is, P is monic and has all its zeros on the unit circle) or $M_0(P) \geq \mu$.

The smallest known Mahler measure greater than 1 is taken for the “Lehmer’s polynomial”

$$P(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1,$$

for which

$$M_0(P) = 1.176280818 \dots$$

It is widely believed that this example represents the true minimal value: that is,

$$\mu = 1.176280818 \dots$$

in Lehmer’s conjecture.

In 1973, Pathiaux [57] proved that if Q is an irreducible polynomial with integer coefficients and $M_0(Q) < 2$, then there exists a polynomial $P \in \mathcal{F}_n$ such that Q divides P . A remark at the end of this paper notes that the proof may be modified to establish the same result for reducible polynomials. Mignotte [52] found a simpler proof of this statement for irreducible polynomials Q with integer coefficients and derived an upper bound on the degree of P in terms of the degree of Q and $M_0(Q)$. His proof may also be extended to the reducible case. These results were generalized and strengthened by Bombieri and Vaaler in [6], as an application of their improved formulation of Siegel’s lemma.

Similarly, it is a simple counting argument to show that if $k \geq 2$ is an integer, the monic polynomial Q has only integer coefficients, and $M_0(Q) < k$, then there is a polynomial P with integer coefficients in $[-k + 1, k - 1]$ such that Q divides P . See the hint to E.8 on page 23 of [7].

The result of Pathiaux [57] leads us to the following observations.

Remark 1 If

$$Q(z) := (z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1)^4,$$

then $M_0(Q) = (1.176280818 \dots)^4 < 2$, hence there is a polynomial $P \in \mathcal{F}_n$ such that Q divides P .

Remark 2 If Lehmer’s conjecture is false, then the answer to Problem 4 is yes. Indeed, if Lehmer’s conjecture is false, then for every $1 < \mu < 2$ there is a monic polynomial Q such that $1 < M_0(Q) \leq \mu$, so if $k := \lfloor \log 2 / \log \mu \rfloor - 1$, then Q^k divides a polynomial $P \in \mathcal{F}_n$.

Remark 3 It remains open whether or not a polynomial $P \in \mathcal{F}_n$ with $P(0) = 1$ can have a zero α of multiplicity at least 5 outside the unit circle.

To find examples of Newman polynomials with constant term 1 and with at least one repeated zero outside the unit circle had been asked by Odlyzko and Poonen [56]. This question was later answered by Mossinghoff [54] who found examples of several such polynomials with repeated zeros outside the unit circle.

To find examples of Littlewood polynomials with at least one repeated zero outside the unit circle is also a very interesting problem. It is easy to see that such Littlewood polynomials must have odd degree. Drungilas, Jankauskas, and Šiurys [29] have found a Littlewood polynomial P of degree 195 such that $(x^3 - x + 1)^2$ divides P . See more in [29, 30, 34, 48].

We close this section by a version of an old and hard unsolved problem known as the already mentioned Tarry–Escott Problem.

Problem 6 Let $N \in \mathbb{N}$ be fixed. Let $a(N)$ be the smallest value of m for which there is a polynomial $P \in \cup_{n=1}^{\infty} \mathcal{F}_n$ with exactly m non-zero terms in it and with a zero at 1 with multiplicity at least N . Prove or disprove that $a(N) = 2N$.

To prove that $a(N) \geq 2N$ is simple. The fact that $a(N) \leq 2N$ is known for $N = 1, 2, \dots, 12$, but the problem is open for every $N \geq 13$. In 1999, S. Chen found the first ideal solution with $N \geq 12$:

$$\begin{aligned} &0^k + 11^k + 24^k + 65^k + 90^k + 129^k + 173^k + 212^k + 237^k + 278^k + 291^k + 302^k \\ &= 3^k + 5^k + 30^k + 57^k + 104^k + 116^k + 186^k + 198^k + 245^k + 272^k + 297^k + 299^k, \end{aligned}$$

valid for all $k = 1, 2, \dots, 11$.

The best known upper bound for $a(N)$ in general seems to be $a(N) \leq cN^2 \log N$ with an absolute constant $c > 0$. See [21]. Even improving this (like dropping the factor $\log N$) would be a significant achievement. Note that for every integer $n \geq 2$ there is a polynomial $Q \in \mathcal{F}_n$ having at least $c(n/\log n)^{1/2}$ zeros at 1 with an absolute constant $c > 0$. This was observed in [17] based on a simple counting argument. The inequality $a(N) \leq cN^2 \log N$ with an absolute constant $c > 0$ follows simply from this.

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