Chapter 9
Squares, Cats and Mazes: The Art and Magic of Spatial Complexity

Intelligence in labyrinths
(Michel De Certeau 1984, p. 90)

Abstract Spatial complexity can be playful, surprising and artful: some of its pleasant facets are highlighted in this chapter, as they emerge from the examination of spatial partitions and by means of games playable on boards of squares: chess, go, tic-tac-toe, checkers (among many other spatial games) revealing how charming the spatial complexity of square arrangements can be. Indeed, square maps are scientifically interesting as well as a source of inspiration throughout the ages, from Latin squares and famous modern painters to the mysterious Arnold Cat Maps and video games. Square maps can be both symbols of minimalism in art as well as genitors of highly complex mazes and labyrinths. With innumerable algorithmic challenges pertaining to them, they are a source of entertainment and endowed with a geometric shape perfectly suited for displaying and exploring the puzzling, mystic and aesthetic aspects of spatial complexity.

Keywords Spatial complexity · Spatial Computing · Arnold Cat Map · Spatial games · Mazes · Complexity and Art · Game complexity

9.1 Square Partitions

> "Our physical world not only is described by mathematics, but it is mathematics: a mathematical structure, to be precise"
>
> (Max Tegmark 2014, p. 6)

Creating square grids by intersecting horizontals and verticals at equal lengths is not the only way to partition a given spatial region. Although they are the commonest and by far the easiest to handle numerically, hexagons, triangles, and other shapes can as well be used to tile the plane, or even combinations of shapes (Fig. 9.1). In fact, any rectangular space may also be partitioned by a fixed ratio, such as the golden

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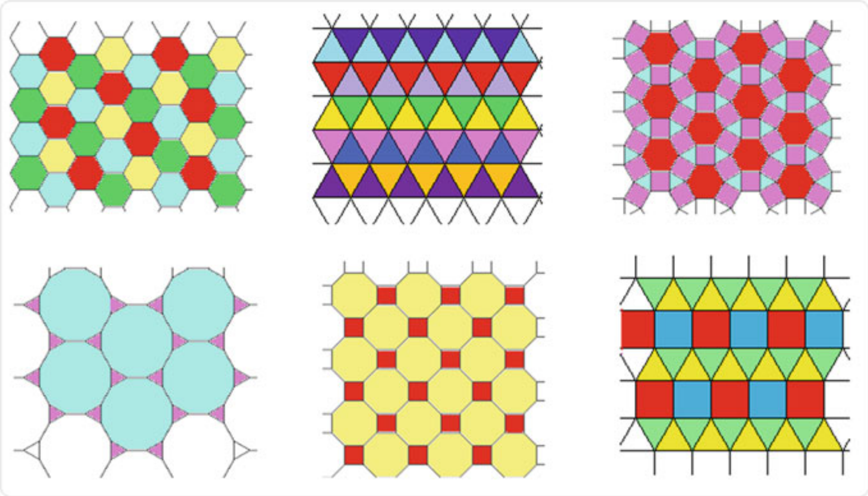


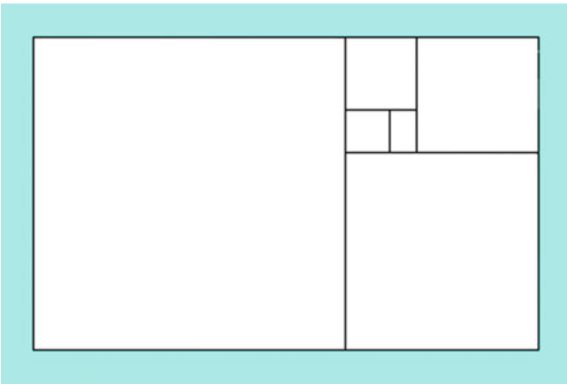
Fig. 9.1 Various spatial partitions, based on simple geometric shapes: triangles, hexagons and combinations of geometric shapes (hexagons, squares and triangles, dodecagons and triangles, octagons and squares etc.)

section, in which any larger rectangle can be made proportional to its adjacent and smaller rectangle, by a factor equal to the *golden section* (Fig. 9.2). Thus, the ratio of the length of the larger rectangle over that of the smaller one is given by

$$\frac{1}{2}(1+\sqrt{5}) = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}} \tag{9.1}$$

and the ratio of any pair of consecutive numbers is given by the Fibonacci sequence (these numbers representing length and width of the rectangle) yielding the golden section, i.e.

Fig. 9.2 Partitioning rectangular spaces can be made by a fixed ratio. In this case, the ratio is the "golden section"



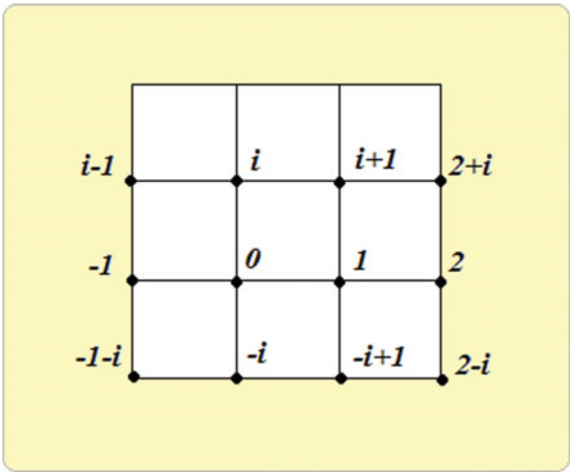


Fig. 9.3 The squares on this 2-dimensional space correspond to points defined by complex numbers with integer real parts, thus partitioning a 2d surface according to the structuring induced by the algebraic integers R_d

$$\frac{89}{55} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} \tag{9.2}$$

Space partitioning may as well be the result of the application of algebraic structures, such as the ring of algebraic integers, R_d. This ring contains all the numbers of the form $a + bs$, where a, b are ordinary integers. When $d = -1$, the ring R_{-1} is the "*ring of Gaussian integers*", that is the ring of complex numbers defined on the complex plain, by a and b integers (Fig. 9.3). When $d = -3$, the ring R_{-3} can represent vertices of equilateral triangles. Notice that four numbers (1, -1, i, $-i$) suffice to define a square in the ring R_{-1} and six numbers in R_{-3} (the numbers 1, -2, $(1 + 3i)/2$, $-(1 + 3i)/2$, $-(1 - 3i)/2$, $(1 - 3i)/2$).

Besides, there also exist iterative schemes for partitioning, making use of pyramidal numbers or fractal patterns. The former are based on the rule that the total number of squares contained in a grid of $m \times m$ unit square is the square "pyramidal number":

$$\frac{m(m+1)(2m+1)}{6} \tag{9.3}$$

In the case of 3×3 maps for instance ($m = 3$), the pyramidal number is 14 (Fig. 9.4). These numbers correspond to alternative coverings of the same map, by varying squares, either non-overlapping (1×1 squares) or overlapping if more than 1×1 cells are used, up to $m \times m$. The overlapping ones are of no apparent usefulness for spatial analyses, so the use of pyramidal numbers is unsuitable for spatial complexity assessments. It is however interesting from the point of view of computational complexity, since identifying squares which correspond to pyramidal numbers is a computationally-hard problem, because it eventually leads to unsolvable diophantine equations (Ma 1985; Anglin 1990).

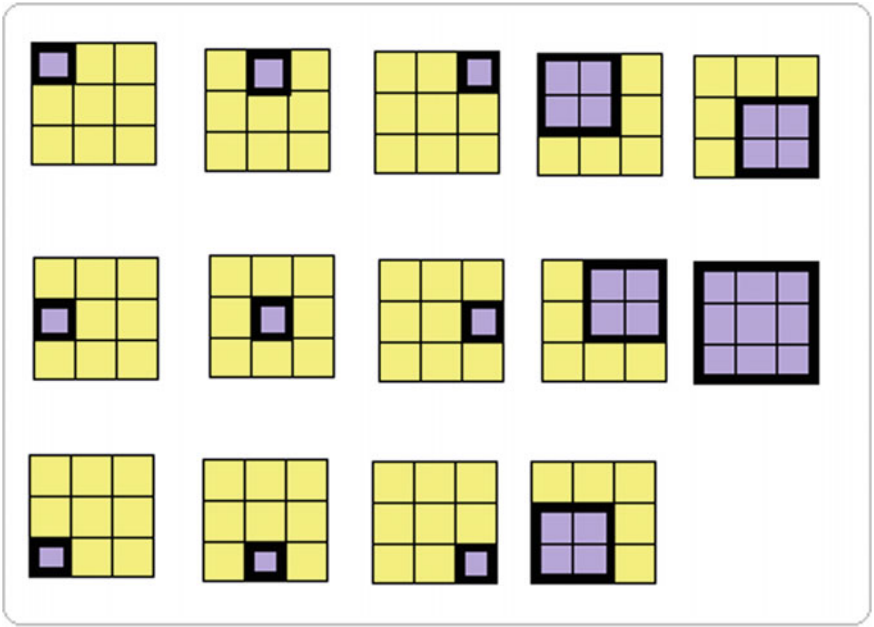


Fig. 9.4 Squares corresponding to the pyramidal number 14 for 3 × 3 binary maps

Another way to divide a square space is to use fractal methods, such as the Sierpinski square (or Cantor gasket) (Fig. 9.5) which, at each step, yields 8 squares of side length 1/3 and therefore has a fractal dimension equal to log8/log3 = 1.89…

Aside of being conceptually closer to the human perception of space however, square partitions have the additional benefit that they can easily emerge by appropriately converting triangular, hexagonal and other symmetric partitions of space to square grids, although this does not preclude deriving parallelogram lattices instead of squares (Fig. 9.6).

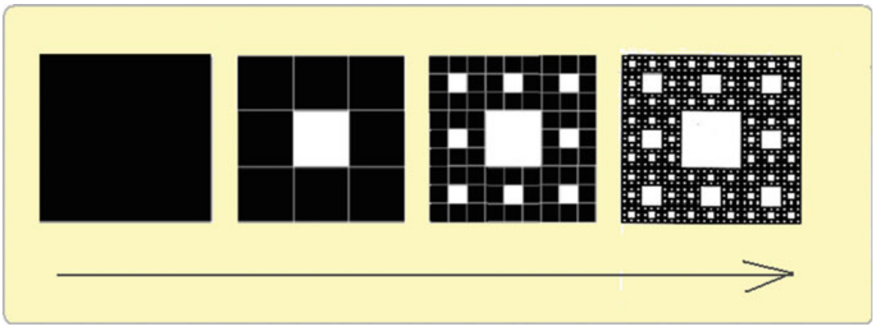


Fig. 9.5 The Sierpinski square is a fractal object dividing the square at every step in more squares, eventually ending up with a "dust" of isolated points around the central square

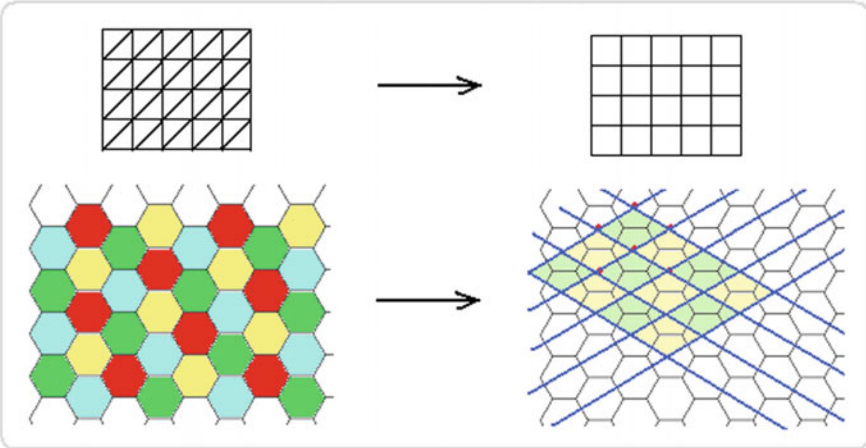


Fig. 9.6 Hexagonal, triangular (and other) symmetric partitions of the plane can easily be transformed to correspond to parallelogram or square grids

Given these, it has become perhaps more clear why our digital technologies rely so much on square arrays. It is thus understandable why maps of square areas are ideally suited for the analysis of spatial complexity (Papadimitriou 2002, 2009, 2012, 2013). There has never been a period of human history in which squares ruled everyday life more than they do now: pixels are squares, and so are digital screens of mobile devices, televisions, computers, and many other essential electronic devises; and all these are outlets displaying spatial complexity. But we are not the first ones to be fascinated by the power of square arrays.

9.2 Squares, Minimalism and Art

Between two words, you have to choose the lesser
"Entre deux mots, il faut choisir le moindre"
(Paul Valéry, 1871–1945, "Tel quel", 1929)

The power of squares in understanding spatial extents has been widely recognized across cultures and civilizations (the square as a sacred form is encountered in the four arms of Vishnu or Shiva, the Tibetan mandalas, the Kaaba cube of Mecca, etc.) and square arrangements have long been sources of inspiration and puzzlement for artists and thinkers. Perhaps nowhere is this more explicit than in the case of "magic squares", of which an example is the 4 × 4 magic square depicted in Albrecht Durer's famous "Melancholia" gravure (1514). The sum of rows of this magic square is 34 that is as much as the sum of columns, as the sum of diagonals and the sum of its four corner cells too:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

In 1693, de la Loubere gave a method for calculating magic squares for any odd size. Likewise, the "diabolical squares" are those of which both negative and positive diagonals produce the same sums. The oldest diabolic magic square was found inscribed in India (12th b.C.):

15	10	3	6
4	5	16	9
14	11	2	7
1	8	13	12

But there is more to art than puzzles and "magic" tricks. Spatial complexity means, signifies, creates meanings, or diffuses meanings. For this reason, it poses as an ideal ground for matching mathematics with art. Besides, as Hilbert said (in 1922) "In the beginning was the sign" in his "The new grounding of Mathematics: First Report" (as reported by Ewald 2001).

In the nineteenth century, the mathematician-writer Abbott (1838–1926) begun his celebrated story "Flatland" (written in 1884) by exclaiming "How franticly I square my talk!". In this famous fictitious two-dimensional story, Abbott (1991) wrote an (unreal) correspondence between human beings and (essentially)…spatial complexity. In the class-sensitive period that this novel was written, the various inhabitants of "Flatland" were probably imagined by the author to correspond to increasing spatial complexity (although he did not specify this), according to his own personal criteria: straight lines would "correspond" to women, triangles to soldiers and "lowest classes of workmen", equilateral triangles or equal-sided triangles to middle class and squares to "professional men and gentlemen". Hexagons are reserved for the nobility and circles for priests. Observing the attribution of shapes to social classes, it easily follows that the higher the social class, the higher the spatial complexity.

Besides, square divisions of space constitute a recurrent and classic theme in visual arts, encountered within various artistic currents. One of the founders of the De Stijl movement for instance, Theo Van Doesburg (18831931), presented four black and progressively enlarging squares in his "Arithmetic Composition" (1929–1930) (Bridgeman Art Library, Switzerland). In 1906, Henri Matisse created his famous painting "Luxury" (Gallery Orsai, Paris), depicting a calm and pleasant space, giving the impression of being composed from small pixels.

The devotion to squares however, is characteristic of *minimalism* in art. Within the context of the Russian constructivism, Alexander Rodchenko (1891–1956) painted some completely monochrome square paintings, Piet Mondrian (1872–1944), with his 1935 picture titled “Composition C: Yellow, Red and Blue” presented three squares (a red, a blue and a yellow) in a black grid of white colours. In another painting, he presented a plain red square titled “Pure red color” (1921), Supposedly, he was affected by the dutch theosophic school of “plastic mathematics”, which contrasted horizontal and vertical lines to curves.

But probably the most exquisite representative of the links between square maps and art was Kazimir Malevich (1878–1935). His famous “Black Square in a White Font” (1915) is, as its title suggests, nothing but a big black square with a white border around it, apparently inviting the viewer to reflect on the mystery of binary square arrangements. The painting “Quadrilateral” (1914) or “Black Square” is explained by the painter himself in his “*Suprematism*” manifesto. His “Suprematist Elements: Squares” (1923) consisted of two black squares with a beige backdrop. Further, in his “Suprematist Composition” titled “Red Square and Black Square”, Malevich presented a black square and a tilted red square. In interpreting this painting, Altieri (2001) contended that the red square’s tilt posed a geometric challenge to the system of coordinates established by the black square. In the context of Malevich’s “Suprematism”, squares signify feelings and white domains the void.

Similarly, Joseph Albers (1888–1976) presented two red squares in beige ground in his “Homage to the Square” (1961). Perhaps even more characteristically, Piet Mondrian’s works display sets of lines intersecting orthogonally forming square arrangements. In these remarkable cases, spatial complexity was intentionally kept to a minimum in two ways: not only there was one color only (or two), but the spatial shape was also the simplest convex shape to describe algorithmically: the square.

Square grids consisting of squares painted with different colors are representative of “concrete art”, i.e. the “Polychrome of pure colors” (1956) by Karl Gerstner (1930–2017) who used painted cubes of plexiglas to print various multicolored square maps. Conceptually very similar was the “arte programmata”, in which binary orthogonal geometric features and patterns are used with non-repetitive patterning, i.e. with the works of Gianfranco Chiavatti (1936–2011). But, the charm of squares is not confined to art only.

9.3 Mazes, Labyrinths and Spatial Games

> (Ariadne) gave Theseus a string of which the one end he attached to the labyrinth’s gate and when he found the Minotaur at the labyrinth’s end he killed him by smiting him with his fists, then made his way out of the labyrinth by following the string again (back to the gate)
>
> “λίνον εἰσιόντι Θησεῖ δίδωσι: τοῦτο ἐξάψας Θησεὺς τῆς θύρας
>
> ἐφελκόμενος εἰσῄει. καταλαβὼν δὲ Μινώταυρον
>
> ἐν ἐσχάτῳ μέρει τοῦ λαβυρίνθου παίων πυγμαῖς ἀπέκτεινεν,
>
> ἐφελκόμενος δὲ τὸ λίνον πάλιν ἐξῄει”
>
> (Apollodorus, “Epitome”, 1.9)

Mazes probably qualify for the title of the “temples of spatial complexity”. Deeply impressing humans throughout the ages, they constitute the most characteristic example of how spatial complexity can be useful for the creation of games. From king Minos’ famous “labyrinth” in ancient Greece, to the floor of the Chartres cathedral, mazes have been created in gardens, palaces and public areas, all over the world. Some are famous, such as the maze of the gardens of Schönbrunn Palace in Austria, some are particularly large, as the Gardens Shopping Mall in Dubai (currently the world’s largest indoor maze) and the Samsø Labyrinten in Denmark (the world’s largest maze, with an area of 60,000 m^2).

Nowadays, several algorithms have been devised for *generating mazes* (i.e. Prim’s, Kruskal’s, Sidewinder, Aldous-Broder, Binary Tree, Eller’s Recursive Backtracker, Wilson’s, Growing tree, Hunt and Kill, Growing Forest). Equivalently, there are algorithms for *solving mazes* (Pledge algorithm, Recursive backtracker, Chain algorithm, Dead and Filler, Tremaux’s algorithm, Wall follower, Cul-de-sac filler, Blind alley filler, Blind Eye Sealer, Shortest Path Finder etc.). The reader may find a rich literature documenting these algorithms, but presenting them here analytically is beyond the scope of this book.

From a computational complexity perspective however, it is interesting to notice that two maze problems, the “rolling block” and “Alice” mazes have been shown to be *PSPACE*-complete (Holzer and Jakobi 2012). But many other *spatial games* (and video-games) with maze-like forms (such as the games Lemmings, Loder Runner, Mindbender, Skweek, Starcraft, Tron and the famous Pac-Man) are all *NP*-hard (Viglietta 2013).

Spatial games fascinated people since the early antiquity. The game “Go”, invented in China two millennia b.C., based on a 19 × 19 square board, can host as many as 10^{768} possible games and “future conflicts may resemble the oriental game of Go more than the western game of chess" (Arquilla and Ronfeldt 2001, p. 2). Recreations with spatial complexity involve a wide range of spatial games that are notoriously difficult to play, precisely due to their very large number of combinations, i.e. there are 6,670,903,752,021,072,936,960 possible configurations of sudoku (Stewart 2008), while “Eternity-II puzzle” (a game invented in 2007 and played on a 16 × 16 grid) has 1.115×10^{557} possible configurations (Pickover 2009).

Besides these, there are old games challenging the player to discover possible square allocations of numbers, complying to certain rules. Latin squares is one such, Sudoku is another, in which the player is expected to assign positive integers to cells of a big 9 × 9 square composed of 3 × 3 squares, so that in no column or row of the big square appears anyone of the numbers 1–9 twice. A 9 × 9 map needs few steps only to check whether a solution is valid (exactly 81 steps), but the number of steps required to search for a solution if it is not known beforehand is impossibly high: 6.6×10^{27} (Aron 2011). Most spatial games essentially draw their complexity from the breadth of possible spatial combinations of cells on a board. Chess and checkers are only two such games (and most well known), among many others: Laska, Lanrik, Kriegspiel, Zetan, Chancellor’s chess, Satrange, Japanese chess, Marseille chess, Alice, Kamikaze, Ming Mang, Hazami Sogi, etc. The size of game boards for spatial games varies depending on the game, but it is usually a square (i.e. 8 × 8 in chess,

10×10 for Snakes and Ladders, 15×15 for Scrabble, 18×18 for Go), although it can also extend in higher than two dimensions.

The computational complexity classes also vary: chess is *EXPTIME*-complete (Fraenkel and Lichtenstein 1981), as are checkers (Robson 1984) and "Go" (Robson 1983). These games can have a time duration that is exponential with respect to the size of their playing board. The game "Reversi" (or "Othello") playable on a square board is *PSPACE*-complete (Iwata and Kasai 1994). In fact, even the simplest of all spatial games, the "tic-tac-toe" with its 9 cells, is *PSPACE*-complete (Reisch 1980), making it an excellent example of how high spatial complexity may emerge from very simple spatial arrangements. Similarly, the game "Tetris" has been shown (Demaine et al. 2002) to be "intractable" for the human mind ("*NP*-complete") and sudoku is *NP*-hard (verifiable in polynomial number of steps, but solvable in exponentially high number of steps).

Several famous spatial problems have been examined in chess. For instance, Euler's "Knight's Tour Problem" (1759) asks for the tour of a knight over the board passing once through all the chessboard's squares. It has a solution for the 8×8 chessboard but not for the 4×4 chessboard. "*Schwenk's theorem*" characterizes the rectangular boards that can support a knight's tours and defines that (Stewart 2010) a $m \times k$ parallelogram chessboard supports a knight's tour unless either (a) m and k are both odd, (b) m equals 1, 2, 3, 4, or (c) $m = 3$ and $k = 4$ or $k = 6$ or $k = 8$.

Chess on Klein surfaces is spatially more complex than common planar 2d chess (Fig. 9.7), so calculations of movements of chess pieces on this surface presented by Watkins (2004) are interesting to see how more complex formulas emerge depending on whether the piece "king" moves on a Klein surface.

The number of kings required to cover a Klein $m \times m$ chessboard, depends on two calculations Watkins (2004):

Fig. 9.7 A chessboard on a Klein surface

$$\left\lceil \frac{m}{6} \right\rceil \left\lceil \frac{2m}{3} \right\rceil \tag{9.4}$$

But if the chessboard surface is asymmetric (non-square, that is $m \times k$), then the complexity of the previous calculations increases to (Watkins 2004):

$$\left\{ \begin{array}{ll} \left\lceil \frac{m}{6} \right\rceil \left\lceil \frac{2k}{3} \right\rceil - \left\lceil \frac{k-1}{3} \right\rceil & m = 1, 2, 3 \bmod 6 \\ \left\lceil \frac{m}{6} \right\rceil \left\lceil \frac{2k}{3} \right\rceil & m = 4, 5, 6 \bmod 6 \end{array} \right\} \tag{9.5}$$

Again, spatial asymmetry induces increases in spatial complexity. By far the most important problem in chess mathematics however, is the "*Covering Problem*", consisting in the determination of the number of pieces of a particular type of movement (i.e. kings, queens, knights, rooks etc.) required to cover a square chessboard. Nine kings are necessary to cover the 8 × 8 chessboard and the same can be done with 8 bishops or 8 rooks. For queens (whose movement is the most far-reaching over the chessboard), the "*Spencer-Welch theorem*" defines the number of queens required to "cover" the chessboard. Some mathematical chess problems have also been been studied over 3d and 4d chessboards (Gibbins 1944; Jelliss and Marlow 1987; DeMaio 2007; Kumar 2008).

Another chess-like spatial game is John Conway's "*Game of Life*" that can be played on square boards and provides useful insights into how self-organisation can emerge in space. One of its variants, the "Garden of Eden", of size $5k \times 5k$, produces a large number of configurations (Berelkamp et al. 2004):

$$(2^{25} - 1)^{k^2} \tag{9.6}$$

and, given adequate time for self-replication, spatial patterns eventually emerge. The "Game of Life" begins with 3 × 3 sub-squares, to which simple rules apply, depending on whether the cell is occupied by a digital entity or not. The rules define how many entities are required in the 3 × 3 sub-square in order for that entity to survive or reproduce at the next time step and in this way, artificial ecosystems can be created *in silica*, which led to the exciting research field of "*Artificial Life*" that aims to simulate life-like behaviors and processes by using computer-made (artificial) animals and plants.

9.4 The Arnold Cat Map

> God is sufficiently wise and powerful to mix the many into one and to dissolve again the one into many. But there is no man, nor will ever be, who will be able to do this
>
> "ὅτι θεὸς μὲν τὰ πολλὰ εἰς ἓν συγκεραννύναι καὶ πάλιν
>
> ἐξ ἑνὸς εἰς πολλὰ διαλύειν ἱκανῶς ἐπιστάμενος ἅμα καὶ δυνατός,
>
> ἀνθρώπων δὲ οὐδεὶς οὐδέτερα τούτων ἱκανὸς οὔτε ἔστι νῦν οὔτε εἰς αὖθίς ποτε ἔσται"
>
> (Plato, 428–348 b.C., "Timaeus", 68d)

Or, may be, not? Entropy implies irreversibility: whatever is will never be the same again. This is what Physics says. Physics exploits Mathematics but has no much room for magic. But there is plenty of room in Mathematics for unexpected truths and bewildering results.

Occasionally, mathematics may give the impression of a touch of "magic": the "Arnold Cat Map" (ACM) is one such a case and it is interesting to examine it here in the context of spatial complexity, because it shows unexpected properties of 2d maps (although it can be expected to extend to 3d volumes also). The ACM is a discrete map transformation of an image converting it into another, and iteratively into another, so that after successive iterations, the final image that eventually appears is completely identical to the original. It was invented (or discovered?) by Vladimir Arnold and as he used a cat's face to show the power of the mapping, it was since called Arnold's Cat Map (Arnold and Avez 1968).

This simple yet almost magical transformation rearranges the position of each map cell and repositions it elsewhere on the image, according to a predefined (and unchanging) rule. After a number of iterations, the cell returns to the same position as it initially was and it therefore contributes (along with all other cells, which have been transformed according to the same rule) to a reproduction of the original image again after all the iterations have been performed (Fig. 9.8).

For their amazing behaviours, ACMs have applications in cryptography and steganography (data encoding in images). They have positive Kolmogorov-Sinai

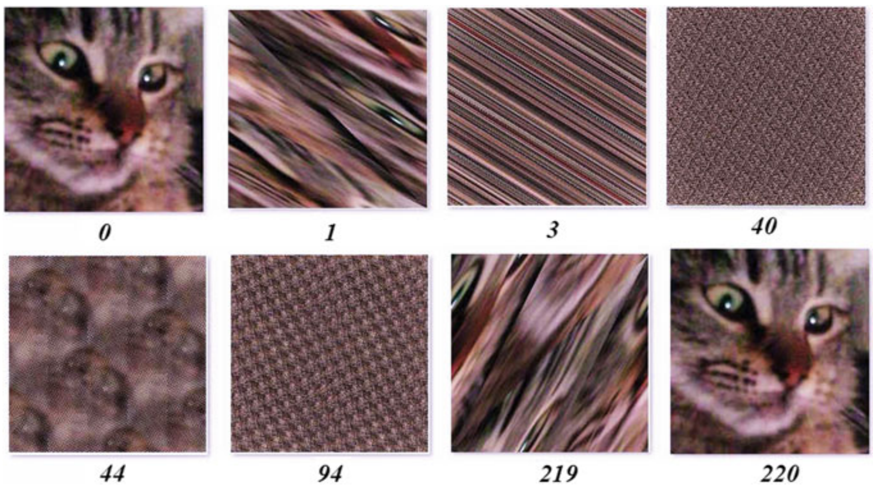


Fig. 9.8 The Arnold cat map transformations of the image of Liuliuta. Numbers beneath each image show the iteration number of the Arnold Cat Map transformation. Soon after the second iteration, the image has lost all its resemblance to the original cat's image and it looks chaotic at the 40th. Oddly, at the 44th iteration, ghost-like features of the original image reappear but do not last. Eventually, after 219 iterations, the 220th suddenly produces exactly the original again: "Cats have nine lives"

entropy (Lichtenberg and Lieberman 1992) and lie at the heart of classical dynamical chaos (Chirikov 1979; Kornfeld et al. 1982).

The general formula transforming the position of a cell located at (x, y) to another position on the map is:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & p \\ q & pq+1 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \bmod \left(\sqrt{n}\right) \tag{9.7}$$

where n is the size of the square map (thus the root is a positive integer), k is the number of iteration (a positive integer), p and q are the parameters of the ACM (some positive integers).

Since the determinant of the transformation matrix equals to 1, the map is area-preserving and the final image is identical to the initial.

As an example, consider the case of an 124×124 map, with parameters $p = q = 1$. The ACM thus is:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k + y_k \\ x_k + 2y_k \end{bmatrix} \bmod (124) \tag{9.8}$$

The first iteration of a cell described by coordinates $(x, y) = (8, 6)$ yields $(x, y) = (14, 20)$. The second, $(x, y) = (34, 54)$. In this way, after visiting the positions (88, 18) (106, 0), (106, 106), (88, 70), (34, 104), (14, 118), (8, 2), (10, 12), (22, 34), (56, 90), (22, 112), (10, 122), and eventually, the 15th iteration yields a transition from the position (132, 254) to the original place of the cell: (8, 6).

This shows how the Arnold Cat Map circulates a cell around the image and then returns it back to its original position. Apparently, as this process is valid for one cell, it simultaneously applies to all the image's cells. Hence, after some iterations, all cells have returned back to their original positions.

Noticeably, the simplest ACM is when $p = q = 1$ and this eventually entails the golden section, because the Lyapunov characteristic exponents of the ACM

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \tag{9.9}$$

are given by the equation

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 1-u & 1 \\ 1 & 2-u \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \tag{9.10}$$

which leads to

$$u^2 - 3u + 1 = 0. \tag{9.11}$$

Hence

$$u = \frac{3 \pm \sqrt{5}}{2} \tag{9.12}$$

which, if plugged into

$$\begin{bmatrix} 1-u & 1 \\ 1 & 2-u \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9.13}$$

yields

$$y = \left(\frac{1+\sqrt{5}}{2}\right)x = \varphi x \tag{9.14}$$

A somewhat similar behavior results from the *chaotic* "*Chebyshev Map*", described by the equation

$$x(n+1) = \cos\left(\frac{k}{\cos(x(n))}\right) \tag{9.15}$$

where $k(n)$ is the modulo of

$$\left\lfloor \frac{x(n)+1}{2} \right\rfloor$$

The Arnold Cat Map can be applied to 3d objects also (Chen et al. 2004):

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \mathbf{A}\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} \text{ mod } (n)$$

where

$$\mathbf{A} = \begin{bmatrix} 1+a_x a_z b_y & a_z & a_y + a_x a_z + a_x a_y a_z b_y \\ b_z + a_x b_y + a_x a_z b_y b_z & 1 + a_z b_z & a_x + a_y b_z + a_x a_y a_z b_y b_z + a_x a_z b_z + a_x a_y b_y \\ a_x b_x b_y + b_y & b_x & 1 + a_x b_x + a_y b_y + a_x a_y b_x b_y \end{bmatrix} \tag{9.16}$$

with all parameters alpha and beta being positive integers (it can be verified that **A** has determinant equal to 1).

Despite its random and chaotic appearance, ACM is an invertible, ergodic and structurally stable type of Anosov diffeomorphisms, essentially a homeomorphism of a closed surface preserving the two-dimensional Lebesgue measure and has the "*Poincaré Recurrence Theorem*" inbuilt into it. This theorem guarantees ergodicity for all dynamical systems (under the condition that the system is Hamiltonian and preserves its volume in the phase space).

But the completely accurate reproduction of the image after successive iterations (despite the fact that each and all cells seem randomly transposed) is not the only enigmatic behavior of ACMs. There is yet another, perhaps even more intriguing phenomenon, and this has to do with the still poorly understood relationship between map size and number of iterations. For instance, for $p = q = 1$, while the 100×100 map needs as many as 150 iterations to bring back any cell at its original position, the slightly larger 101×101 map needs only 25 iterations, the 124×124 only 15, but the 150×150 needs 300. So simple map transformations acting on 2d square maps display complex associations with the map size. This inevitably leads us to hypothesize that some map sizes might be endowed with some peculiar properties, but we don't know which ones these map sizes are.

References

Abbott, E. A. (1991). *Flatland: A romance of many dimensions*. Princeton, N.J.: Princeton University Press.

Altieri, C. (2001). Representation, representativeness and non-representational Art. In C. Shukla (Ed.), *Art and representation: Contributions to contemporary aesthetics* (pp. 243–261). Westport, CT: Praeger.

Anglin, W. S. (1990). The square pyramid puzzle. *The American Mathematical Monthly, 97*(2), 120–124.

Arnold, V., & Avez, A. (1968). *Ergodic problems in classical mechanics*. New York: Benjamin.

Aron, J. (2011, June 4). The world's hardest problem: There's a solution that will change everything. *New Scientist, 2815*, 37–41.

Arquilla, J., & Ronfeldt, D. (2001). The advent of netwars (revisited). In J. Arquilla, & D. Ronfeldt (Eds.), *Networks and Netwars*. Santa Monica, CA: Rand.

Berelkamp, E. R., Conway, J. H., & Guy, R. K. (2004). *Winning ways for your Mathematical Plays* (Vol. 4). Wellsley, MA: A.K. Peters.

Chen, G., Mao, Y., & Chui, C. (2004). A symmetric image encryption scheme based on 3d Chaotic Cat Maps. *Chaos, Solitons and Fractals, 21,* 749–761.

Chirikov, B. V. (1979). A universal instability of many-dimensional oscillator systems. *Physics Reports, 52,* 263.

De Certeau, M. (1984). *The practice of everyday life*. Berkeley, CA: University of California Press.

Demaine, E. D., Hohenberger, S., Liben-Nowell, D. (2002, October 21). *Tetris is hard, even to approximate*. arXiv:cs/0210020v1.

DeMaio, J. (2007). Which chessboards have a closed knight's tour within the cube? *The Electronic Journal of Combinatorics, 14,* R32.

Ewald, K. C. (2001). The neglect of aesthetics in landscape planning in Switzerland. *Landscape and Urban Planning, 54*(1), 255–266.

Fraenkel, A., & Lichtenstein, D. (1981). Computing a perfect strategy for $n \times n$ chess requires time exponential in *n. Journal of Combinatorial Theory A, 31,* 199–214.

Gibbins, N. M. (1944, May). Chess in 3 and 4 dimensions, *Mathematical Gazette*, 46–50.

Holzer, M., & Jakobi, S. (2012). On the complexity of rolling block and Alice mazes. *Lecture Notes in Computer Science, 7288,* 210–222.

Iwata, S., & Kasai, T. (1994). The Othello game on an $n \times n$ board is PSPACE-complete. *Theoretical Computer Science, 123*(2), 329–340.

Jelliss, G. P., & Marlow, T. W. (1987). 3d tours. *Chessics, 2*(29/30), 162.

Kornfeld, I. P., Fomin, S. V., & Sinai, Ya. G. (1982). *Ergodic theory*. New York: Springer.

Kumar, A. (2008). Magic knight's tours for chess in three dimensions. *Mathematical Gazette, 92,* 111–115.

Lichtenberg, A., & Lieberman, M. (1992). *Regular and chaotic dynamics*. New York: Springer.

Ma, D. G. (1985). An elementary proof of the solution to the diophantine equation $6y^2 = x(x + 1)(2x + 1)$. *Sichuan Daxue Xuebao, 4,* 107–116.

Papadimitriou, F. (2002). Modelling Indicators and indices of landscape complexity: An approach using GIS. *Ecological Indicators, 2,* 17–25.

Papadimitriou, F. (2009). Modelling spatial landscape complexity using the Levenshtein algorithm. *Ecological Informatics, 4*(1), 51–58.

Papadimitriou, F. (2012). The algorithmic complexity of landscapes. *Landscape Research, 37*(5), 599–611.

Papadimitriou, F. (2013). Mathematical modelling of land use and landscape complexity with ultrametric topology. *Journal of Land Use Science, 8*(2), 234–254.

Pickover, C. (2009). *The Maths book*. London: Sterling.

Reisch, S. (1980). Gobang ist PSPACE-vollstandig (Gomoku is PSPACE-complete). *Acta Informatica, 13,* 5966.

Robson, J. M. (1983). The complexity of go. In *Information Processing; Proceedings of IFIP Congress* (pp. 413–417).

Robson, J. M. (1984). N by N checkers is Exptime complete. *SIAM Journal on Computing, 13*(2), 252–267.

Stewart, I. (2008). *Professor Stewart's cabinet of mathematical curiosities*. London: Profile Books.

Stewart, I. (2010). *Cows in the Maze and other Mathematical explorations*. Oxford: Oxford University Press.

Tegmark, M. (2014). *Our Mathematical universe*. New York: Vintage.

Viglietta, G. (2013). *Gaming is a hard job, but someone has to do it*. arXiv:1201.4995v5.

Watkins, J. J. (2004). *Across the board: The Mathematics of chessboard problems*. Princeton, NJ: Princeton University Press.