Chapter 12
Complexity of Binary Maps of Primes and Transcendentals

Number became the first principle and this is indefinite and incomprehensible.
The number has in itself all the infinite possible numbers that may come.
And of the numbers the first entity was the unity, which fatherly generated all the other numbers
“Ἀριθμός γέγονε πρῶτος ἀρχή, ὅπερ ἐστίν ἀόριστον ἀκατάληπτον,
ἔχων ἑαυτῶ πάντας τούς ἐπ’ ἄπειρον δυναμένους ἐλθεῖν ἀριθμούς κατά τὸ πλῆθος.
Τῶν δὲ ἀριθμῶν ἀρχή γέγονε καθ’υπόστασιν η πρώτη μονάς,
ήτις ἐστί μονάς ἄρσην γεννῶσα πατρικῶς πάντας τούς ἄλλους ἀριθμούς”
(Pythagoras, 580-496 bC)

Abstract Different map sizes and different approximations of π produce interestingly different binary map representations with substantial differences in their spatial complexity. Sometimes, knowing the numbers behind the spatial structures not only explains a map’s structure, but can also be used to predict how a map might look like if it extended in space (such is the case of transcendental numbers). Assigning black cells to prime numbers produces square binary maps of primes-and-composites from which it can be seen that: (a) even-numbered binary such maps “produce” clumps, while odd-numbered ones do not; (b) C_{P2} decreases with increasing map size n in binary maps of primes-composites; (c) both the author’s C_{P1} and C_{P2} complexity metrics are always higher in odd-numbered binary maps than in even-numbered maps of primes-and-composites.

Keywords Spatial complexity · Prime numbers · Binary maps · Map complexity · Transcendental numbers and Complexity · pi · Number theory and Complexity

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12.1 Numbers Defining Spaces

> "So Nature deals with us, and takes away our playthings one by one, and by the hand leads us to rest so gently, that we go scarce knowing if we wish to go or stay, being too full of sleep to understand how far the unknown transcends the what we know"
>
> (Henry Wadswarth Longfellow, 1807–1882, "Nature")

The title of this section may at first appear surprising, particularly considering that only very limited research has been hitherto carried out in this particular field. The truth is that besides geometry and topology, another branch of mathematics, number theory, most often passes unnoticed in the study of spatial arrangements. The repercussions of this for spatial scientists may be too early to anticipate, but it is worth noticing them.

Some number-related aspects of spatial complexity emerge from simulations and experiments with binary maps. They can be interesting not only in terms of mathematics, but also for future research, as some of them may prove valuable for gaining some deeper insights in spatial complexity. Such number-theoretic considerations may relate to various kinds of numbers (integers, reals, transcendentals, complex etc.). The emergence of π in "Buffon's needle problem" for instance (a spatial probability problem which is examined in another chapter) leads us to question whether this is a unique case of number-theoretical interest emerging from a problem of spatial probability, or it might as well imply that number theory is essential to understand spatial complexity. Before opting for the first case, we should rather consider a few more facts, since, surprisingly, despite the fact that some problems of spatial combinatorics have revealed the presence of π in their solutions, this seems to have passed more or less unnoticed.

Counting the number of ways a square can be covered by domino tilings can be illustrating: according to Matousek (2010, p. 85) there are 12,988,816 possible tilings of the 8×8 chessboard by 2×1 rectangle dominoes (Fig. 12.1), while the formula giving the number of domino tilings of an mxn chessboard involves trigonometric functions and yet, yields integers as results (Matusek 2010, p.93):

$$\sqrt{\prod_{k=1}^{m}\prod_{l=1}^{n}\left(2\cos\frac{k\pi}{m+1}+2i\cos\frac{\pi l}{n+1}\right)} \tag{12.1}$$

But this is not the only case that π shows up unexpectedly in spatial analysis. Consider, for instance this problem: Choose one cell of a square map, with an arithmetic regularity, with ever decreasing size. Beginning with an 1×1 square map (that is the entire area) and proceeding to a 2×2 square map (thus ¼ of the map's area has been chosen), choosing one cell again from a 3×3 square map (meaning that 1/9 of this map's area has been selected) and carrying on the same way down to $(1/n) \times (1/n)$ (Fig. 12.2), the total area accumulated from all these choices is tantamount to calculating the following series:

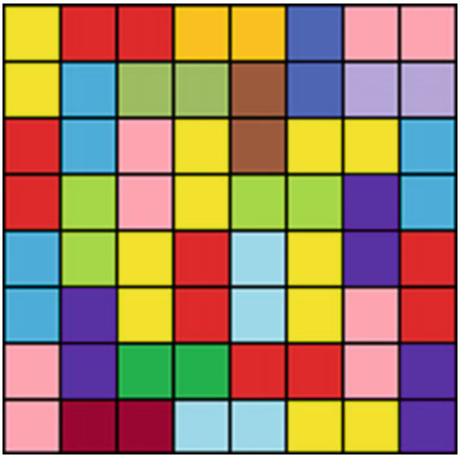

Fig. 12.1 One of the 12,988,816 possible tilings of the 8 × 8 chessboard that can be made by 2 × 1 rectangle dominoes

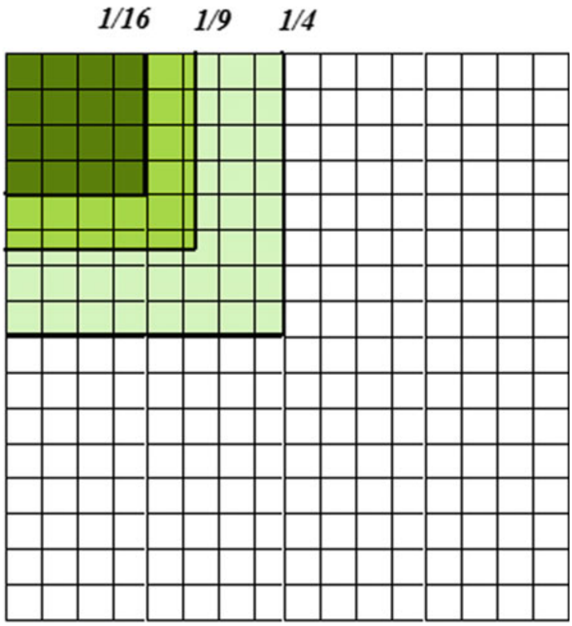

Fig. 12.2 A hint about the possible role of number theory in spatial complexity. Calculating the total area of the sum $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}$ leads to the rather unexpected result $\pi^2/6$, or, otherwise stated, deriving a solution to a spatial problem involving squares leads to a transcendental number which is not usually associated to square shapes

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2} \tag{12.2}$$

The result of this calculation may only be obtained through a "magic" trick due to Euler, and is:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}=\frac{\pi^2}{6}\cong 1.6449340668\ldots \tag{12.3}$$

What is far more interesting to notice here, is that $\pi^2/6$ is a transcendental number. It does seem noteworthy that the solution of a spatial problem *involving squares only* (not circles or other shapes), can *only* be expressed in terms of a transcendental number that is usually associated to a different shape.

But behind the solution to this simple spatial problem might possibly lie some other interesting relations (in terms of number theory) as well, as, for instance, $\pi^2/6$ relates to Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots \tag{12.4}$$

that is one of the most intensely studied functions.

As well known, the value of the Riemann ζ function of a positive integer n is defined as:

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n} \tag{12.5}$$

and $\pi^2/6$ is exactly the value of Riemann zeta for the integer 2:

$$\zeta(2) = \frac{\pi^2}{6} \cong 1.6449340668\ldots \tag{12.6}$$

A short digression may be useful here, to recall that the Riemann zeta function $\zeta(s)$ is a function of a complex variable $s = \sigma + it$ which can also be written as a converging infinite series for all complex numbers s with real part greater than 1:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots \tag{12.7}$$

This function is meromorphic on the whole complex s-plane, but for real numbers x, it relates to the gamma function:

$$\zeta(x) = \left(\frac{1}{\Gamma(x)}\right)\left(\int_0^{\infty} \frac{u^{x-1}}{e^u - 1} du\right). \tag{12.8}$$

Two characteristic values are $\zeta(0) = -1/2$ and $\zeta(1) = \infty$, with some of its values relating to other known numbers and constants: the harmonic numbers, the Euler-Mascheroni constant γ, and, most surprisingly, to physical quantities, such as the critical temperature of the Einstein-Bose condensates $\zeta(3/2) = 2.612$, the integration of Planck's law to derive the Stefan-Boltzmann law in Physics: $\zeta(4) = \pi^4/90 = 1.0823\ldots$ among several other relations. Given these, it makes sense to question whether any more relations between different types of numbers (transcendental, primes, reals, complex etc.) and spatial complexity might exist. With partial series we have that up to a given n, the total area of squares is:

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{12.9}$$

but the first term of this equation is equal to the $\zeta(-2)$ of the Riemann zeta function, and hence, the sum of the series

$$\zeta(-2) = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 \tag{12.10}$$

also results from calculations that entail trigonometric functions again:

$$\zeta(-2) = -\frac{1}{56}\int_0^\infty \frac{(1+t^2)\cos\left(-2\tan^{-1}(t)\right)}{\cosh(\pi t/2)}dt \tag{12.11}$$

or even e:

$$\zeta(-2) = \frac{1}{6} + 2\int_0^\infty \frac{(1+t^2)\sin\left(-2\tan^{-1}(t)\right)}{e^{2\pi t}-1}dt \tag{12.12}$$

(where $\tan^{-1}$ is the inverse tangent function and cosh is the hyperbolic cosine function). Equivalently, the sum of increasingly larger volumes up to infinity is $\zeta(-3) = 0.0083333\ldots$

12.2 Complexity of Binary Square Maps of Primes and Composites

> Good definitely scattered among the figures of evil. Anamorphosis of good. Evil definitely scattered among the figures of good. Anamorphosis of evil
>
> (Jean Baudrillard 2005, p.142)

The key question tackled here is: "What is the spatial complexity of binary maps if prime numbers are represented as colored cells on square maps?"

Binary maps resulting from the allocation of primes on $n =$ odd-numbered square maps display diagonal alignments, so cells sharing a common side are uncommon (Fig. 12.3).

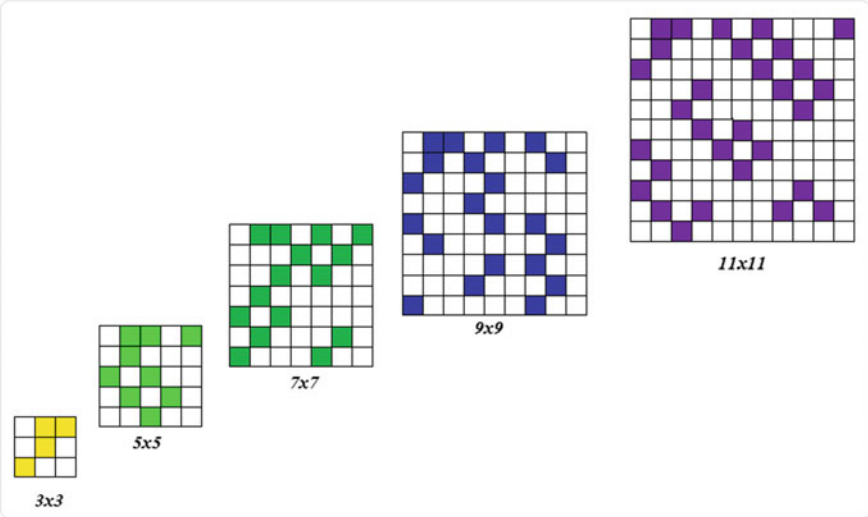

Fig. 12.3 Binary maps resulting from n=odd-numbered maps, with prime numbers allocated as darker cells on progressively larger maps. Notice the diagonal patterns emerging from these maps and the relative absence of clumps

Notice that the maps with n = odd have only one clump, even as their size increases from $n = 9$ to $n = 121$. On the contrary, when the primes are allocated on even-numbered maps, they produce more clumps with size of at least two cells (Fig. 12.4).

As a result of the fact that even-numbered maps produce vertical arrangements of black cells, while odd-numbered maps display alignments of black cells along diagonal directions, it is expected that odd-shaped maps would create higher spatial complexity, since they generate more dissimilar contacts between cells.

This can be verified from calculations of C_{P2} for maps 3×3 up to 10×10, in that the formula relating C_{P2} and map size n grows with a trigonometric pattern, in which *odd-numbered maps have higher complexity than even-numbered ones* (Fig. 12.5):

$$C_{P2} = 0.785n + \tan(0.04828n) - \tan(\tan(0.04459n)) \tag{12.13}$$

with correlation coefficient = 0.99973885

As may easily be verified, *even-numbered maps "produce" clumps while odd-numbered ones do not*. Indeed, the clump number B remains constant (equal to 1) in 3×3, 5×5, 7×7 etc. maps, but increases with map size in even-numbered maps. It can be verified that odd-numbered maps "keep" the number of clumps B to a minimum (counting as a clump a conglomerate of two cells or more, either horizontally or vertically), so a question raises as to the relationship of B with the number of primes p (corresponding to the entropy of the map) in n = even numbered maps. For square binary maps 4×4 up to 16×16 with n = even, a trigonometric

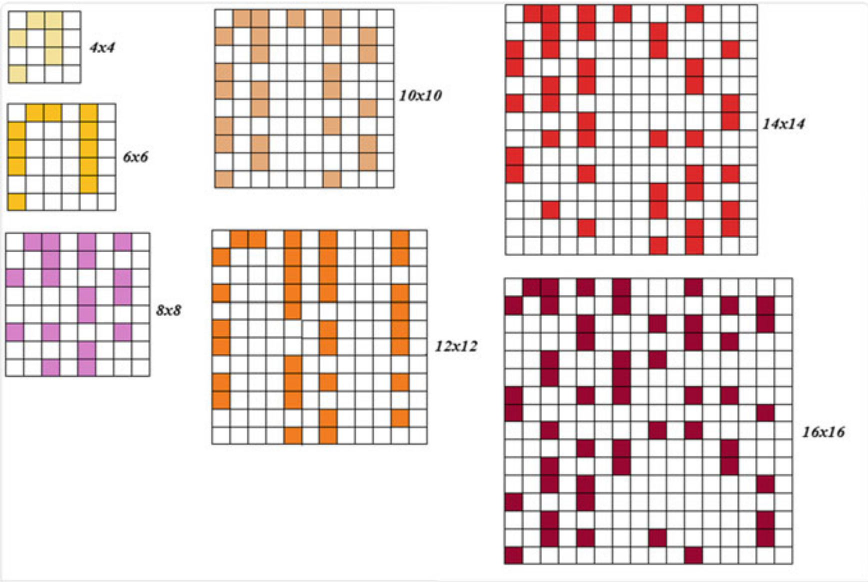

Fig. 12.4 Binary maps resulting from prime numbers allocated on progressively larger maps, but always on map sizes with n = even. Notice the presence of clumps (of at least two colored cells) and the presence of entire columns without any colored cell at all

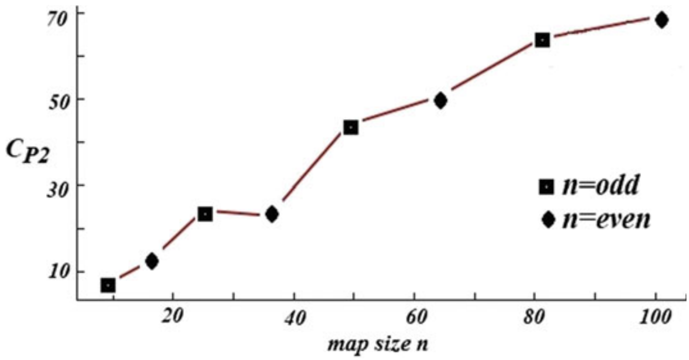

Fig. 12.5 Odd-numbered maps have higher C_{LP} complexity than even-numbered ones (measurements only on maps from $n = 3 \times 3$ to $n = 10 \times 10$)

relationship is a best fit for clumpiness B values with respect to p:

$$B = 0.292p - 0.4872\tan(0.5739p) - 0.6177\cos(0.5234 + p) \qquad (12.14)$$

with goodness of fit = 0.99978491 and correlation coefficient = 0.99989552.

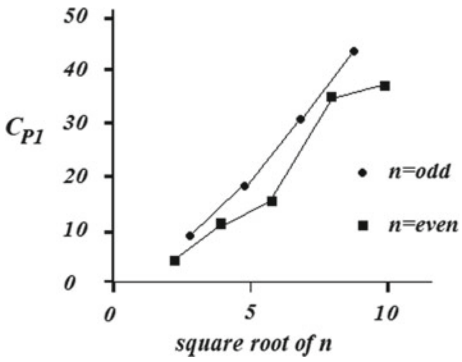

Fig. 12.6 The C_{P1} complexity of compressed strings of odd-numbered binary maps of primes-and-composites is constantly higher than that of even-numbered maps (map sizes 2×2 up to 10×10)

Yet, the calculation of C_{P1}-complexity of square maps of sizes 2×2 up to 10×10 yields differing results for strings for even-numbered and odd-numbered maps (Fig. 12.6). In fact, the *C_{P1}-complexity is always higher for odd-numbered binary maps.*

Expecting the same to hold true for map sizes larger than 10×10, it might be a strong indication (not a proof) that *sequences of odd-numbered square binary maps of primes-and-composites are "able" to "generate" higher spatial complexity than even-numbered ones.*

Some spatial aggregates that appear in even-numbered maps of primes-and-composites may be due to the presence of twin primes. Allocating primes as black cells along a line does not reveal any interesting pattern. But it does reveal striking patterns when carried out over a 2d space. The "*Ulam spiral*" is a known such pattern, but it can not reveal differences in spatial complexity as allocations of primes on square maps do (as, i.e. evidence from small maps suggests here, clumps tend to appear in even-numbered grids).

But all square binary maps of primes with size n higher than 3×3 have $r_{\max} > p$ and, consequently, *the larger maps with number of primes equal to the map's maximum entropy class are the 3×3 maps.*

With these considerations, some questions for future research arise from the spatial allocations of prime numbers on square maps:

– Is there an optimal square size for n (with $n =$ even), for which the number of clumps is maximized in binary maps of primes?
– What are the biggest clumps that can be created in binary maps of primes-and-composites, with size up to a specified value of n?
– Is there an upper barrier to clump size, whatever the value of n?

We currently do not possess answers to these questions. But answering them might give us hints about the ways that number theory underlies spatial complexity and the reverse: the ways by which spatial complexity may reveal (and lead to) number-theoretical problems. Understanding the relationships between prime numbers and spatial complexity can have repercussions in other fields related to spatial complexity.

For instance. knots can be "*primes*" or "composites" (nontrivial knots), just as integers are and the number of prime knots with n crossings increases fast: 0, 0, 1, 1, 2, 3, 7, 21, 49, 165, 552, 2176, 9988, 46,972, 253,293, 1,388,705 etc.

12.3 Square Maps from Transcendental Numbers

> God makes the world by calculating, but his calculations never work out exactly and this inexactitude or injustice in the result, this irreducible inequality, forms the condition of the world. The world "happens", while God calculates; if the calculation were exact, there would be no world
>
> (Gilles Deleuze 2010, p. 280)

In 1874, Cantor proved that the algebraic numbers are countable, but the real numbers are uncountable. The transcendental numbers are uncountably infinite. But *how* can they be identified? And, transposing this line of thought to spatial analysis of square maps, how might these uncountably infinite many transcendentals correspond to spatial forms? It would make sense to assume that some spatial patterns may correspond to numbers lurking almost imperceptible and this raises questions about possible limits to perceiving and understanding spatial complexity. Let us consider a simple example, by considering a string of binary square maps, of increasing size, corresponding to the following strings (from left to right):

011111011 . . .
0111110110011101 . . .
0111110110011101110011001 . . .
011111011001110111001100101100111111 . . .

At first sight, none of these strings (or, ultimately, the last string, which comprises all the previous ones) seems to reveal any known pattern. But they conceal one: if the digits 1,2,3,4,5 of another string are mapped to 0 s and the digits 6,7,8,9,0 are mapped to 1 s, the following correspondence between the two strings is derived:

0.011111011001110111001100101100111111
|| | . . .
0.207879576350761908546955619834978770

The latter string corresponds to the decimal digits of the number i^i, since

$$i^i = e^{i \ln i} = e^{-\left(\frac{\pi}{2}+2k\pi\right)} = 0.207\ldots \tag{12.15}$$

Expectedly, arbitrarily many digits of the binary string can be isolated and converted to a spatial representation (Fig. 12.7).

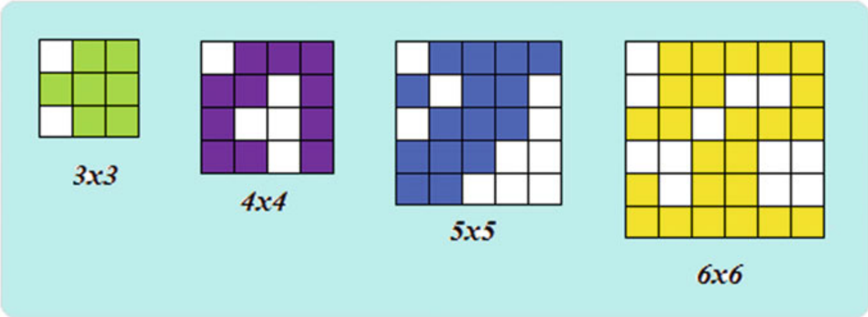

Fig. 12.7 Changing patterns and clusters of colored and white cells can mislead human perception: in fact, all these square binary maps are generated by the same number: they represent the first decimal digits of the number $i^i = 0.207\ldots$ with the decimals 1,2,3,4,5 represented by white cells and the decimals 6,7,8,9,0 by colored cells

Interestingly, following *Bellard's formula*, a binary description of π can be calculated:

$$\pi = \frac{1}{2^6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\left(-\frac{2^5}{4n+1}-\frac{1}{4n+3}+\frac{2^8}{10n+1}\right.$$
$$\left.-\frac{2^6}{10n+3}-\frac{2^2}{10n+5}-\frac{2^2}{10n+7}+\frac{1}{10n+9}\right) \tag{12.16}$$

which yields a binary string for $\pi = 3.141$:

$$11.001001000011100101011000000100000110001001001101110100101111100\ldots$$

but if the precision of π were allowed to increase to two more decimals, that is 3.14159, then the binary string is altered after the 13th decimal:

$$11.0010010000111111001111100000000110111000011001101110010000111101\ldots$$

Thus, depending on the *precision* of the description of π, Bellard's formula produces different binary strings. As in other cases, precision-dependence brings about significant changes in spatial representations (which, in turn, results in differences in spatial complexity).

If we take, for instance, the first 49 digits of 3.14… and arrange them in a 7×7 map (Fig. 12.8), we would produce a completely different map than if we took the first 49 digits with the higher precision $\pi = 3.14159$. In turn, this means there are significant differences in the spatial complexities of the two maps.

The largest black block on the left map is 13 while on the right one it is 8 (61% lower). Yet, the white blocks are comparable: the largest white block on the left is

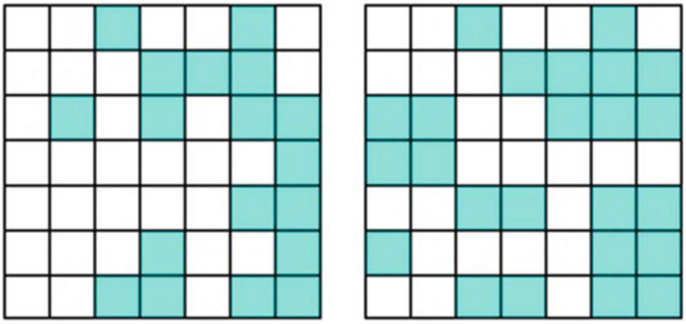

Fig. 12.8 The spatial representation of the first 49 digits of $\pi = 3.141$ arranged on a 7×7 binary map is different than the representation of the same number of digits of $\pi = 3.141\mathbf{59}$ (right)

27 (it covers more than half of the 7×7 map's area) while the one on the right is 22 (that is 81% as large). Also, the number of patches on the left map is 4, while on the map on the right it is 7 (75% higher). The C_{P1} complexity of the left map is 37 (with the compression $\lambda = W^2BW^2B$), while the C_{P1} complexity of the right map is 32 (with the compression $\lambda = WB^2$). The C_{P2} complexity of the left map is 34 and of the right is 38 and hence the differences in spatial complexity range from +17% to −13.5% for only two decimals of π.

Hence, *different configurations producing different spatial complexities, may all be derived from the same transcendental number.* And this can not be perceived initially without knowing the underlying mathematical process or property that has generated the spatial configuration. Moreover, knowing the transcendental number behind a map's spatial structure, not only explains the spatial allocation, but can also be used to predict a map's configuration, if enlarged to extend in space.

Recalling that any real decimal number x belonging to the interval [0,1] can be represented by a continuous fraction and using continued fractions to encode spatial properties, it is easy to verify that different encodings of the same spatial elements may produce varying descriptions of spatial complexity and this without changing the scale of observation. But, identifying an optimal encoding is a computationally hard undertaking in both binary and multicolored maps. Continued fractions however, are "base-invariant", meaning that some numbers which may appear random eventually present unexpected "beautiful" patterns. One such is the "golden section" (the first 50 digits are given here):

$$1.6180339887498948482045868343656381177203091798057 6\ldots,$$

This apparently "messy" series of decimals surprisingly corresponds to the simplest continued fraction possible:

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}}$$

Another such example is $\sqrt{2} = 1.4142135623730950488\ldots$ This number also appears to have a random allocation of decimals, but as a continued fraction, it reveals a very simple pattern:

$$[1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots\ldots].$$

Similarly, the transcendental number e appears random in its decimals

$$2.71828182845904523536028747135266249775724709369995\ldots$$

but in continued fraction form it presents a logical procession of even numbers separated after every two 1 s:

$$\begin{gathered}[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10,\\ 1, 1, 12, 1, 1, 14, 1, 1, 16, 1, 1, 18, 1, 1,\\ 20, 1, 1, 22, 1, 1, 24, 1, 1, 26, 1, 1, \ldots.]\end{gathered}$$

Simple patterns with continued fractions have other numbers also with infinite decimals; in example π which is equal to to:

$$\cfrac{4}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\ldots}}}}}}$$

or

$$3+\cfrac{1^2}{6+\cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\cfrac{9^2}{6+\ldots}}}}}$$

These examples show that there may exist a hidden structure in a spatial form and this structure may correspond to some number, but we have no clue as to how to identify the number behind the structure.

References

Baudrillard, J. (2005). *The intelligence of evil or the lucidity pact*. London: Bloomsbury.
Deleuze, G. (2010). *Difference and Repetition.* London: Continuum.
Matousek, J. (2010). *33 Miniatures.* Providence: American Mathematical Society.