17

Rotating Rings, Discs and Cylinders

Simeon Denis Poisson
(1781-1840)

Simeon Denis Poisson, born on 21 June 1781, was a great French mathematician, geometer and physicist. He worked under two famous mathematicians Pierre-Simon Laplace and Joseph-Louis Lagrange, and Sadi Carnot, who is called the father of thermodynamics, was his one of the famous students. Poisson is most known for applying mathematics to solve problems in electricity and magnetism, mechanics and other areas of physics. He is known for his work on definite integrals, electromagnetic theory and probability. In his Poisson equation, also known as potential theory equation, he corrected the Laplace's second order partial differential equation for potential. He is also known for Poisson's ratio, which is widely used in strength of materials. The Poisson distribution is extremely useful in the analysis of problems relating to radioactivity, traffic and random occurrence of events in time or space. In 1818, he was elected a fellow of the Royal Society and in 1823, a foreign member of the Royal Swedish Academy of Sciences. He is among the 72 people whose names are inscribed on the Eiffel Tower.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- Why is hoop stress also called circumferential stress?
- What is radial stress?
- Why is the area moment of inertia often called the second moment of area?
- What are the centroidal axes?
- Where is parallel-axes theorem used?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_17

17.1 INTRODUCTION

Components such as turbine shafts and discs while rotating at high speeds are subjected to large centrifugal forces, which in turn, produce large stresses that are distributed symmetrically about their axes of rotation. The stress analysis of these components is useful in their safe design so as to prevent their failure.

17.2 ROTATING RING

The force analysis of a thin rotating ring can also be applied to a thin rotating cylinder or rim-type flywheels.

Consider a thin ring or a thin cylinder rotating with a constant angular velocity ω rad/s about its axis as shown in Fig. 17.1.

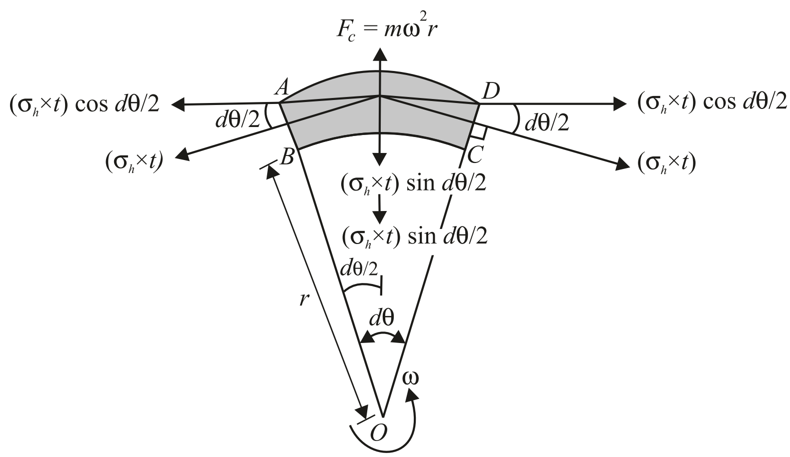


Fig. 17.1 A rotating ring.

Let r = Mean radius of the ring (or cylinder)

t = Thickness of the ring (or cylinder)

ρ = Density of the ring or (cylinder material)

Rotational motion produces centrifugal force on the circumference of the ring or on the walls of the cylinder, which in turn, produces hoop (or circumferential) stress σ_h. Since the thickness is very small, hence there is no variation of the hoop stress along the thickness, that is, the hoop stress may be assumed to be constant.

Now consider a small element *ABCD* of the ring or cylinder making an angle $d\theta$ at the centre as shown in the figure.

Forces on the element

The following three forces are acting on the element *ABCD* :

- The centrifugal force caused due to rotation acting radially outward
- The hoop tension force on the face *AB* caused due to hoop stress σ_h
- The hoop tension force on the face *CD* caused due to hoop stress σ_h

Centrifugal force

Considering unit length of the circumference of the element, the mass m of the element can be obtained as

$$\begin{aligned} m &= \text{Density} \times \text{Volume of the element} \\ &= \text{Density} \times \text{Area of the element} \times \text{Unit length} \\ &= \rho \times r d\theta \times t \times 1 \\ &= \rho r t \, d\theta \qquad \text{...(17.1)} \end{aligned}$$

The centrifugal force is given as

$$\begin{aligned} F_c &= \frac{mV^2}{r} \qquad \text{...(17.2)} \\ &= m\omega^2 r \qquad \text{...(17.3)} \\ &= \rho r^2 \omega^2 t \, d\theta \qquad \text{(on substituting } m\text{)...(17.4)} \end{aligned}$$

where V = Linear velocity $= \omega r$

Hoop tension forces on faces *AB* and *CD*

The hoop tension forces act perpendicular to faces AB and CD are equal but opposite in direction; one is acting in the left direction and another in the right direction. Its magnitude is

$$\sigma_h \times t \times 1 = \sigma_h \times \mathrm{t} \qquad \text{(assuming unit length)}$$

Now the hoop forces are resolved into horizontal and vertical components. The vertical components of the hoop forces acting on faces AB and CD are radially inward and both are equal to

$$\sigma_h \times t \times \sin\frac{d\theta}{2}$$

The horizontal component of the hoop force acting on face AB is directed leftward and the horizontal component on face CD is directed rightward and both are equal to

$$\sigma_h \times t \times \cos\frac{d\theta}{2}$$

Equilibrium of the element

The horizontal components of the hoop tension forces on the faces AB and CD are equal but opposite in direction, hence they cancel each other. Their vertical components are added as they are acting in the same direction.

Considering forces in the vertical direction, we find that the centrifugal force F_c is balanced by the sum of the vertical components of the hoop tension forces on the two faces.

$$F_c = \sigma_h \times t \times \sin\frac{d\theta}{2} + \sigma_h \times t \times \sin\frac{d\theta}{2}$$

$$\rho r^2 \omega^2 t \, d\theta = 2 \times \sigma_h \times t \times \sin\frac{d\theta}{2} \qquad \text{(using equation (17.4))}$$

$$= 2 \times \sigma_h \times t \times \frac{d\theta}{2} \quad \left(\text{as } d\theta \text{ is very small, hence } \sin\frac{d\theta}{2} \simeq \frac{d\theta}{2}\right)$$

which gives

$$\sigma_h = \rho\omega^2 r^2 \qquad \text{...(17.5)}$$

This is the required expression for the hoop stress in a thin rotating ring.

Example 17.1

The thin rim of a 900 mm diameter wheel is made of steel and weighs 7800 kg/m^3. Neglecting the effect of the spokes, how many revolutions per minute may it make, if the hoop stress is not to exceed 150 MPa. Also, find the increase in diameter of the wheel. Take E = 210 GPa.

Solution: Given,

Diameter of the wheel, $d = 900$ mm $= 900 \times 10^{-3}$ m

Density of the rim material, $\rho = 7800$ kg/m^3

Hoop stress, $\sigma_h = 150$ MPa $= 150 \times 10^6$ Pa

Modulus of elasticity, $E = 210$ GPa $= 210 \times 10^9$ Pa

The diameter of the rim is equal to the diameter of the wheel and its radius

$$r = \frac{d}{2} = \frac{900 \times 10^{-3}}{2} = 0.45 \text{ m}$$

The hoop stress is given by using equation (17.5) as

$$\sigma_h = \rho\omega^2 r^2$$

$$150 \times 10^6 = 7800 \times \omega^2 \times (0.45)^2$$

which gives

$$\omega = 308.167 \text{ rad/s}$$

Let N be the number of revolutions per minute, then

$$\omega = \frac{2\pi N}{60}$$

$$308.167 = \frac{2\pi N}{60}$$

which gives

$$N = \frac{308.167 \times 60}{2\pi}$$

$$= 2942.78$$ **Ans.**

Now the hoop strain is given as

$$\epsilon_h = \frac{\sigma_h}{E}$$

$$= \frac{150 \times 10^6}{210 \times 10^9} = 7.143 \times 10^{-4}$$

Hence, the increase in diameter of the wheel is

$$\text{Hoop strain} \times \text{Diameter} = 7.143 \times 10^{-4} \times 900$$

$$= 0.643 \text{ mm}$$ **Ans.**

Example 17.2

A rim-type flywheel of mean diameter 800 mm rotates at a speed of 2000 rpm. If the density of the material of the wheel is 8000 kg/m^3, find the hoop stress developed in the rim due to rotation.

Solution: Given,

Mean diameter of the wheel, $d = 800$ mm

Rotational speed, $N = 2000$ rpm

Density of the wheel material, $\rho = 8000$ kg/m^3

The mean radius of the wheel is given as

$$r = \frac{d}{2} = \frac{800}{2} = 400 \text{ mm}$$

$$= 400 \times 10^{-3} \text{ m}$$

The angular velocity of the wheel is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 2000}{60} = 209.44 \text{ rad/s}$$

Now the hoop stress is given by equation (17.5) as

$$\sigma_h = \rho\omega^2 r^2$$

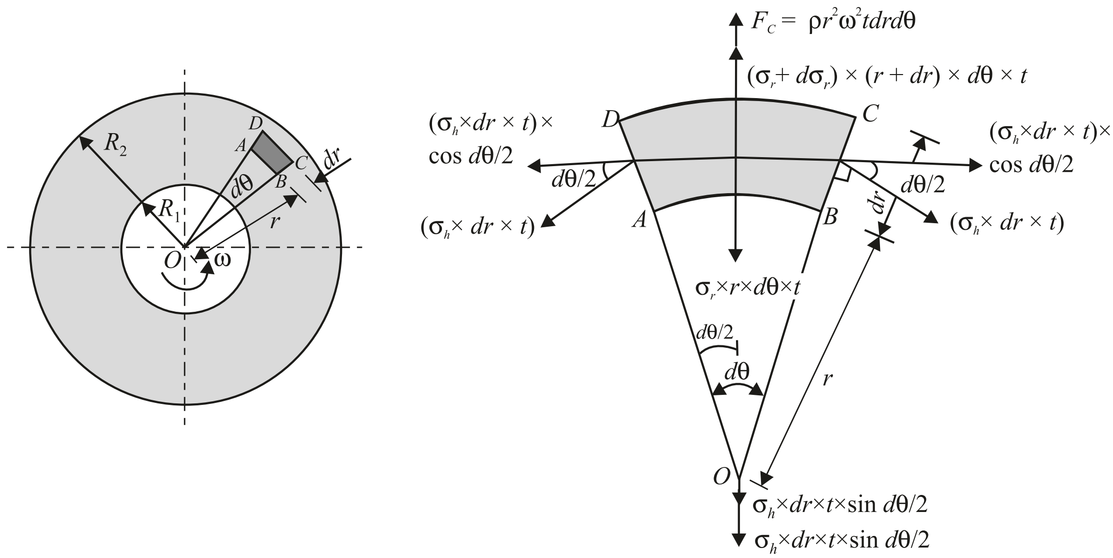
$$= 8000 \times (209.44)^2 \times (400 \times 10^{-3})^2$$

$$= 56.147 \times 10^6 \text{ Pa}$$

$$= 56.147 \text{ MPa}$$ **Ans.**

17.3 ROTATING THIN DISC

Consider a thin disc of inner radius r_1 and outer radius r_2 rotating at angular speed ω rad/s about its axis as shown in Fig. 17.2. The thickness of the disc is negligibly small, so there is no variation of stress across the thickness, and there is no axial stress (longitudinal stress) in the disc. The two stresses acting on the disc include hoop and radial.



(*a*) A rotating disc

(*b*) Forces acting on the element *ABCD*

Fig. 17.2

Let $t =$ Thickness of the disc

$\rho =$ Density of the disc material

Now consider an element *ABCD* of the disc of radial width dr at radius r subtending an angle $d\theta$ at the centre as shown in the figure.

Forces on the element

The following five forces are acting on the element *ABCD*:

- The centrifugal force caused due to rotation
- The radial force on face *AB* caused due to radial stress σ_r
- The radial force on face *CD* caused due to radial stress σ_r
- The hoop tension force on face *AD* caused due to hoop stress σ_h
- The hoop tension force on face *BC* caused due to hoop stress σ_h

Centrifugal force

The volume of the element is

$$(rd\theta) \times dr \times t$$

Now the mass of the element is

$$m = \text{Density} \times \text{Volume of the element}$$

$$= \rho \times (rd\theta) \times dr \times t$$

The centrifugal force acting on the element is

$$\begin{aligned} F_c &= m\omega^2 r \\ &= (\rho \times r d\theta \times dr \times t) \times \omega^2 \times r \\ &= \rho r^2 \omega^2 t \, d\theta dr \end{aligned} \qquad \text{...(17.6)}$$

Hoop tension forces on faces *AD* and *BC*

The hoop tension forces act perpendicular to *AD* and *BC* and are equal to $\sigma_h \times dr \times t$.

Now the hoop forces are resolved into horizontal and vertical components. The vertical components of the hoop forces acting on faces *AD* and *BC* are radially inward and both are equal to

$$\sigma_h \times dr \times t \times \sin\frac{d\theta}{2} = \sigma_h \times dr \times t \times \frac{d\theta}{2} \qquad \left(\text{for small value of } d\theta, \sin\frac{d\theta}{2} \simeq \frac{d\theta}{2}\right)$$

The horizontal component of the hoop force acting on face *AD* is directed leftward and the horizontal component on face *BC* is directed rightward and both are equal to

$$\sigma_h \times dr \times t \times \cos\frac{d\theta}{2} = \sigma_h \times dr \times t \times \frac{d\theta}{2} \qquad \left(\text{for small value of } d\theta, \cos\frac{d\theta}{2} \simeq \frac{d\theta}{2}\right)$$

Radial forces on faces *AB* and *CD*

The radial force on AB is equal to $\sigma_r \times r d\theta \times t$. It acts radially inward.

The radial force on *CD* acting radially outward is

$$(\sigma_r + d\sigma_r) \times (r + dr) \times d\theta \times t$$

Equilibrium of the element

The horizontal components of the hoop tension forces on the faces *AD* and *BC* are equal but opposite in direction, hence they cancel each other. Their vertical components are added as they are acting in the same direction.

Balancing the forces in the radial direction, we have

$$\sigma_r \times r \times d\theta \times t + \sigma_h \times dr \times t \times \sin\frac{d\theta}{2} + \sigma_h \times dr \times t \times \sin\frac{d\theta}{2}$$
$$= \rho r^2 \omega^2 \, t dr d\theta + (\sigma_r + d\sigma_r) \times (r + dr) \times d\theta \times t$$

$$\sigma_r \times r \times d\theta \times t + 2\sigma_h \times dr \times t \times \frac{d\theta}{2} = \rho r^2 \omega^2 \, t dr d\theta + (\sigma_r + d\sigma_r) \times (r + dr) \times d\theta \times t$$

$$\left(\text{for small value of } d\theta, \sin\frac{d\theta}{2} = \frac{d\theta}{2}\right)$$

Eliminating $d\theta \times t$ from both sides of the equation, we have

$$\sigma_r \times r + \sigma_h \times dr = \rho r^2 \omega^2 \, dr + r\,\sigma_r + \sigma_r\, dr + r d\sigma_r + dr d\sigma_r$$

Eliminating $r\sigma_r$ from both sides and neglecting $drd\sigma_r$ because of its small value, we have

$$\sigma_h \, dr = \rho r^2 \omega^2 \, dr + \sigma_r \, dr + r d\sigma_r$$

$$(\sigma_h - \sigma_r) \, dr = \rho r^2 \omega^2 \, dr + r d\sigma_r$$

Dividing by dr on both sides, we have

$$(\sigma_h - \sigma_r) = \rho r^2 \omega^2 + r \frac{d\sigma_r}{dr} \qquad ...(17.7)$$

Strain in the element

Due to rotation, the radius of the disc increases. Let the radius r changes to $(r + u)$ and dr changes to $(dr + du)$.

Now the hoop strain is given as

$$\in_h = \frac{\text{Final circumference} - \text{Initial circumference}}{\text{Initial circumference}}$$

$$= \frac{2\pi(r+u) - 2\pi r}{2\pi r}$$

$$= \frac{u}{r} \qquad ...(17.8)$$

And the radial strain is given as

$$\in_r = \frac{\text{Final radial witdh} - \text{Initial radial width}}{\text{Initial radial width}}$$

$$= \frac{(dr + du) - dr}{dr}$$

$$= \frac{du}{dr} \qquad ...(17.9)$$

Also

$$\in_h = \frac{\sigma_h}{E} - v\frac{\sigma_r}{E} \qquad ...(17.10)$$

$$\in_r = \frac{\sigma_r}{E} - v\frac{\sigma_h}{E} \qquad ...(17.11)$$

On equating equations (17.8) and (17.10), we have

$$\frac{u}{r} = \frac{\sigma_h}{E} - v\frac{\sigma_r}{E}$$

$$= \frac{1}{E}\,(\sigma_h - v\sigma_r) \qquad ...(17.12)$$

On equating equations (17.9) and (17.11), we have

$$\frac{du}{dr} = \frac{\sigma_r}{E} - v\frac{\sigma_h}{E}$$

$$= \frac{1}{E}(\sigma_r - v\sigma_h) \qquad \text{...(17.13)}$$

From equation (17.12), we get

$$E \times u = r \times (\sigma_h - v\sigma_r)$$

Differentiating w.r.t. r, we have

$$E\frac{du}{dr} = (\sigma_h - v\sigma_r) + \left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right) \times r$$

or $$\frac{du}{dr} = \frac{1}{E}\left[(\sigma_h - v\sigma_r) + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right)\right] \qquad \text{...(17.14)}$$

On equating equations (17.13) and (17.14), we have

$$\sigma_r - v\sigma_h = \sigma_h - v\sigma_r + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right)$$

$$\sigma_r + v\sigma_r = \sigma_h + v\sigma_h + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right)$$

$$\sigma_r(1 + v) = \sigma_h(1 + v) + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right)$$

or $$(\sigma_h - \sigma_r)(1 + v) + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right) = 0 \qquad \text{...(17.15)}$$

On substituting $(\sigma_h - \sigma_r)$ from equation (17.7) in equation (17.15), we have

$$\left(\rho r^2\omega^2 + r\frac{d\sigma_r}{dr}\right)(1 + v) + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right) = 0$$

$$\rho r^2\omega^2 + v\rho r^2\omega^2 + r\frac{d\sigma_r}{dr} + vr\frac{d\sigma_r}{dr} + r\frac{d\sigma_h}{dr} - vr\frac{d\sigma_r}{dr} = 0$$

Eliminating r from the above equation, we have

$$\rho r\omega^2 + v\rho r\omega^2 + \frac{d\sigma_r}{dr} + \frac{d\sigma_h}{dr} = 0$$

$$\rho r\omega^2(1 + v) + \frac{d\sigma_r}{dr} + \frac{d\sigma_h}{dr} = 0$$

or $$\frac{d\sigma_r}{dr} + \frac{d\sigma_h}{dr} = -\rho r\omega^2(1 + v) \qquad \text{...(17.16)}$$

On integration, we have

$$\sigma_r + \sigma_h = -\rho \times \frac{r^2}{2} \times \omega^2 (1 + v) + A$$

$$= \frac{-\rho r^2 \omega^2 (1+v)}{2} + A \qquad \text{...(17.17)}$$

where A is a constant of integration.

Now subtracting equation (17.7) from equation (17.17), we get

$$2\sigma_r = \frac{-\rho r^2 \omega^2 (1+v)}{2} + A - \rho r^2 \omega^2 - r\frac{d\sigma_r}{dr}$$

$$2\sigma_r + r\frac{d\sigma_r}{dr} = -\rho r^2 \omega^2 \left(\frac{1+v}{2} + 1\right) + A$$

$$= -\rho r^2 \omega^2 \left(\frac{1+v+2}{2}\right) + A$$

$$= -\frac{\rho r^2 \omega^2 (3+v)}{2} + A \qquad \text{...(17.18)}$$

Multiplying by r on both sides, we have

$$2 \times r \times \sigma_r + r^2 \frac{d\sigma_r}{dr} = -\frac{\rho r^3 \omega^2 (3+v)}{2} + A \times r$$

$$\frac{d}{dr}(r^2 \times \sigma_r) = -\frac{\rho r^3 \omega^2 (3+v)}{2} + A \times r$$

Integrating both sides, we get

$$r^2 \times \sigma_r = -\frac{\rho \omega^2 (3+v)}{2} \times \frac{r^4}{4} + A \times \frac{r^2}{2} + B$$

where B is another constant of integration.

Dividing by r^2 on both sides, we get

$$\sigma_r = -\frac{\rho \omega^2 r^2 (3+v)}{8} + \frac{A}{2} + \frac{B}{r^2}$$

$$= \frac{A}{2} + \frac{B}{r^2} - \frac{\rho \omega^2 r^2 (3+v)}{8} \qquad \text{...(17.19)}$$

This is the required expression for the radial stress. The constants A and B can be determined by using suitable boundary conditions. Now substituting equation (17.19) in equation (17.17), we get

$$\frac{A}{2} + \frac{B}{r^2} - \frac{\rho \omega^2 r^2 (3+v)}{8} + \sigma_h = -\frac{\rho \omega^2 r^2 (1+v)}{2} + A$$

$$\sigma_h = \frac{\rho\omega^2 r^2 (3+v)}{8} - \frac{\rho\omega^2 r^2 (1+v)}{2} + A - \frac{A}{2} - \frac{B}{r^2}$$

$$= \rho\omega^2 r^2 \left(\frac{3+v}{8} - \frac{1+v}{2}\right) + \frac{A}{2} - \frac{B}{r^2}$$

$$= \rho\omega^2 r^2 \left(\frac{3+v-4-4v}{8}\right) + \frac{A}{2} - \frac{B}{r^2}$$

$$= \rho\omega^2 r^2 \times \frac{(-1-3v)}{8} + \frac{A}{2} - \frac{B}{r^2}$$

$$= -\frac{\rho\omega^2 r^2}{8}(1+3v) + \frac{A}{2} - \frac{B}{r^2}$$

$$= \frac{A}{2} - \frac{B}{r^2} - \frac{\rho\omega^2 r^2 (1+3v)}{8} \qquad \text{...(17.20)}$$

This is the required expression for the hoop stress for a rotating thin disc. The constants A and B can be determined by using suitable boundary conditions.

17.3.1 Hoop and Radial Stresses in a Rotating Solid Disc

For a solid disc, there is no inner radius.

$$R_1 = 0 \text{ and } R_2 = R \text{ (say)}$$

When the value of $r = 0$ is substituted in equations (17.19) and (17.20), we find that the term $\frac{B}{r^2}$ becomes infinity, but the stresses cannot have infinite values. Hence to get finite value of stress, $B = 0$.

Equations (17.19) and (17.20) are now transformed to

$$\sigma_r = \frac{A}{2} - \frac{\rho\omega^2 r^2 (3+v)}{8} \qquad \text{...(17.21)}$$

$$\sigma_h = \frac{A}{2} - \frac{\rho\omega^2 r^2 (1+3v)}{8} \qquad \text{...(17.22)}$$

The boundary conditions is

At the outer radius, where $r = R$, the radial stress is

$$\sigma_r = 0$$

On substituting the boundary condition in equation (17.21), we get

$$\frac{A}{2} = \frac{\rho\omega^2 R^2 (3+v)}{8}$$

Now equation (17.21) on substituting the value of $\frac{A}{2}$ becomes

$$\sigma_r = \frac{\rho\omega^2 R^2 (3+v)}{8} - \frac{\rho\omega^2 r^2 (3+v)}{8}$$

$$= \frac{\rho\omega(3+v)}{8}(R^2 - r^2) \qquad ...(17.23)$$

This is the required expression for the radial stress for a rotating thin solid disc. Equation (17.22) on substituting the value of $\frac{A}{2}$ becomes

$$\sigma_h = \frac{\rho\omega^2 R^2 (3+v)}{8} - \frac{\rho\omega^2 r^2 (1+3v)}{8}$$

$$= \frac{\rho\omega^2}{8}[(3+v)R^2 - (1+3v)r^2] \qquad ...(17.24)$$

This is the required expression of the hoop stress for a rotating thin solid disc.

Hoop and radial stresses at the centre

At the centre of the solid disc, where $r = 0$, both hoop and radial stresses have equal maximum values, given by

$$\sigma_{r_{\max}} = \sigma_{h_{\max}} = \frac{\rho\omega^2 R^2 (3+v)}{8} \qquad ...(17.25)$$

Hoop stress at outer radius

At the outer radius, the radial stress $\sigma_r = 0$, but the hoop stress is not zero.

The value of the hoop stress at the outer radius is obtained by putting $r = R$ in equation (17.24) as

$$\sigma_{h_{r=R}} = \frac{\rho\omega^2}{8}[R^2(3+v) - R^2(1+3v)]$$

$$= \frac{\rho\omega^2}{8}(3R^2 + vR^2 - R^2 - 3vR^2)$$

$$= \frac{\rho\omega^2}{8}(2R^2 - 2vR^2)$$

$$= \frac{\rho\omega^2}{8} \times 2R^2(1-v)$$

$$= \frac{\rho\omega^2 R^2}{4}(1-v) \qquad ...(17.26)$$

The variation of the hoop and radial stresses in a rotating solid disc along the radius is shown in Fig. 17.3.

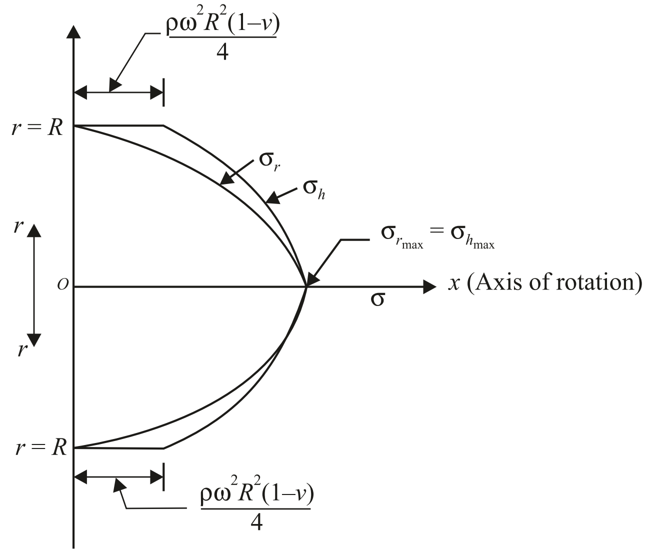


Fig. 17.3 Distribution of the hoop and radial stresses in a rotating solid disc.

17.3.2 Hoop and Radial Stresses in a Rotating Disc with a Central Hole

We have seen in case of a rotating solid disc, the constant of integration B is zero in order to have finite values of radial and hoop stresses. But in case of a disc with a central hole, B is not zero and its value is obtained using suitable conditions.

The radial stress σ_r is zero at both inner and outer radius of the disc.

i.e. at $r = R_1, \quad \sigma_r = 0$

Also at $r = R_2, \quad \sigma_r = 0$

From equation (17.19), we have

$$\sigma_r = \frac{A}{2} + \frac{B}{r^2} - \frac{\rho\omega^2 r^2(3+v)}{8}$$

$$0 = \frac{A}{2} + \frac{B}{R_1^2} - \frac{\rho\omega^2 R_1^2(3+v)}{8} \qquad ...(1)$$

and

$$0 = \frac{A}{2} + \frac{B}{R_2^2} - \frac{\rho\omega^2 R_2^2(3+v)}{8} \qquad ...(2)$$

Subtracting equation (2) from equation (1), we have

$$\frac{B}{R_1^2} - \frac{B}{R_2^2} + \frac{\rho\omega^2 R_2^2(3+v)}{8} - \frac{\rho\omega^2 R_1^2(3+v)}{8} = 0$$

or

$$\frac{B(R_2^2 - R_1^2)}{R_1^2 R_2^2} + \frac{\rho\omega^2(3+v)}{8}[R_2^2 - R_1^2] = 0$$

Eliminating $(R_2^2 - R_1^2)$, we have

$$\frac{B}{R_1^2 R_2^2} + \frac{\rho\omega^2(3+v)}{8} = 0$$

It gives

$$B = -\frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8} \qquad ...(17.27)$$

Now substituting the value of B in equation (1), we get

$$0 = \frac{A}{2} + \frac{1}{R_1^2} \times \left\{ \frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8} \right\} - \frac{\rho\omega^2 R_1^2\,(3+v)}{8}$$

$$= \frac{A}{2} - \frac{(3+v)\rho\omega^2 R_2^2}{8} - \frac{(3+v)\rho\omega^2 R_1^2}{8}$$

$$= \frac{A}{2} - \frac{(3+v)\rho\omega^2}{8}[R_2^2 + R_1^2]$$

It gives

$$A = \frac{(3+v)\rho\omega^2(R_2^2 + R_1^2)}{4} \qquad ...(17.28)$$

Finally the values of A and B are substituted in equation (17.19) to get the expression for the radial stress σ_r as

$$\sigma_r = \frac{1}{2} \times \frac{(3+v)\rho\omega^2(R_2^2 + R_1^2)}{4} + \frac{1}{r^2} \times \left\{ -\frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8} \right\} - \frac{\rho\omega^2 r^2(3+v)}{8}$$

$$= \frac{(3+v)\rho\omega^2(R_2^2 + R_1^2)}{8} - \frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8r^2} - \frac{\rho\omega^2 r^2(3+v)}{8}$$

$$= \frac{(3+v)\rho\omega^2}{8}\left[(R_2^2 + R_1^2) - \frac{R_1^2 R_2^2}{r^2} - r^2 \right] \qquad ...(17.29)$$

This is the required expression for the radial stress for a rotating disc with a central hole.

To find the expression for the hoop stress, the values of A and B are substituted in equation (17.20).

$$\sigma_h = \frac{A}{2} - \frac{B}{r^2} - \frac{\rho\omega^2 r^2(1+3v)}{8}$$

$$= \frac{1}{2} \times \frac{(3+v)\rho\omega^2(R_2^2 + R_1^2)}{4} - \frac{1}{r^2} \times \left\{ -\frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8} \right\} - \frac{\rho\omega^2 r^2(1+3v)}{8}$$

$$= \frac{(3+v)\rho\omega^2(R_2^2+R_1^2)}{8} + \frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8r^2} - \frac{\rho\omega^2 r^2(1+3v)}{8}$$

$$= \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1^2) + \frac{(3+v)R_1^2R_2^2}{r^2} - (1+3v)r^2\right] \qquad ...(17.30)$$

This is the required expression for the hoop stress for a rotating disc with a central hole.

Maximum hoop stress (Hoop stress at inner radius)

Equation (17.30) suggests that with increase in the value of radius r, the hoop stress σ_h decreases and vice versa. Hence, σ_h is maximum when r is minimum, that is, when r approaches to R_1. Substituting $r = R_1$ in equation (17.30), we have

$$\sigma_{h_{max}} = \sigma_{h_{r=R_1}} = \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1^2) + \frac{(3+v)R_1^2R_2^2}{R_1^2} - R_1^2(1+3v)\right]$$

$$= \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1^2) + (3+v)R_2^2 - R_1^2(1+3v)\right]$$

$$= \frac{\rho\omega^2}{8}\left[3R_2^2 + 3R_1^2 + vR_2^2 + vR_1^2 + 3R_2^2 + vR_2^2 - R_1^2 - 3vR_1^2\right]$$

$$= \frac{\rho\omega^2}{8}\left[6R_2^2 + 2vR_2^2 + 2R_1^2 - 2vR_1^2\right]$$

$$= \frac{\rho\omega^2}{8}\left[2R_2^2(3+v) + 2R_1^2(1-v)\right]$$

$$= \frac{\rho\omega^2}{4}\left[(3+v)R_2^2 + (1-v)R_1^2\right] \qquad ...(17.31)$$

This is the required expression for the maximum hoop stress for a rotating disc with a central hole, which occurs at inner radius of the disc.

When R_1 approaches $R_2 = r$, we have from equation (17.31)

$$\sigma_h = \frac{\rho\omega^2}{4}[(3+v)r^2 + (1-v)r^2]$$

$$= \frac{\rho\omega^2}{4}[3r^2 + vr^2 + r^2 - vr^2] = \frac{\rho\omega^2}{4} \times 4r^2$$

$$= \rho\omega^2 r^2$$

The above expression is same as equation (17.5), which applies to a thin rotating ring or a thin rotating cylinder.

Hoop stress at outer radius

For hoop stress at outer radius, put $r = R_2$ in equation (17.30).

$$\sigma_{h_{r=R_2}} = \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1^2)+\frac{(3+v)R_1^2R_2^2}{R_2^2}-R_2^2(1+3v)\right]$$

$$= \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1^2)+(3+v)R_1^2-R_2^2(1+3v)\right]$$

$$= \frac{\rho\omega^2}{8}\left[3R_2^2+3R_1^2+vR_2^2+vR_1^2+3R_1^2+vR_1^2-R_2^2-3vR_2^2\right]$$

$$= \frac{\rho\omega^2}{8}\left[2R_2^2-2vR_2^2+6R_1^2+2vR_1^2\right]$$

$$= \frac{\rho\omega^2}{8}\left[2R_2^2(1-v)+2R_1^2(3+v)\right]$$

$$= \frac{\rho\omega^2}{4}\left[(1-v)R_2^2+(3+v)R_1^2\right] \qquad \text{...(17.32)}$$

Maximum radial stress

To find the position of the maximum radial stress, we differentiate equation (17.29) with respect to r and equate it to zero.

$$\frac{d\sigma_r}{dr} = 0$$

$$\frac{d}{dr}\left[\frac{(3+v)\,\rho\omega^2}{8}\left\{(R_2^2+R_1^2)-\frac{R_1^2R_2^2}{r^2}-r^2\right\}\right] = 0$$

$$\frac{(3+v)\,\rho\omega^2}{8}\left[0-\frac{(-2)}{r^3}R_1^2R_2^2-2r\right] = 0$$

$$\frac{(3+v)\,\rho\omega^2}{8}\left[\frac{2R_1^2R_2^2}{r^3}-2r\right] = 0$$

which gives

$$r = \sqrt{R_1R_2} \qquad \text{...(17.33)}$$

Substituting the value of r in equation (17.29), we get

$$\sigma_{r_{\max}} = \frac{(3+v)\rho\omega^2}{8}\left[(R_2^2+R_1^2)-\frac{R_1^2R_2^2}{R_1R_2}-R_1R_2\right]$$

$$= \frac{(3+v)\rho\omega^2}{8}\left[R_2^2+R_1^2-2R_1R_2\right]$$

$$= \frac{(3+v)\rho\omega^2}{8}(R_2-R_1)^2 \qquad \text{...(17.34)}$$

This is the required expression for the maximum radial stress for a rotating disc with a central hole.

17.3.3 Hoop and Radial Stresses in a Rotating Disc with a Pin Hole at the Centre

In this case, R_1 tends to zero. Substituting $R_1 = 0$ and $R_2 = R$ in equations (17.31) and (17.34), we get the expressions for the maximum hoop and radial stresses for a rotating disc with a central pin hole.

$$\sigma_{h_{\max}} = \frac{(3+v)\rho\omega^2 R^2}{4} \qquad \text{...(17.35)}$$

and

$$\sigma_{r_{\max}} = \frac{(3+v)\rho\omega^2 R^2}{8} \qquad \text{...(17.36)}$$

Comparing equations (17.35) and (17.36), we get

$$\sigma_{h_{\max}} = 2 \times \sigma_{r_{\max}} \qquad \text{...(17.37)}$$

Also, when we compare equation (17.35) with equation (17.25), we observe that the maximum hoop stress for a rotating disc with a central pin hole is twice the maximum hoop stress for a rotating solid disc.

Example 17.3

A steel disc of diameter 800 mm rotates at 2500 rpm. Calculate the hoop and radial stresses developed at the centre and outer radius of the disc. The Poisson's ratio is 0.25 and the density of the disc material is 7800 kg/m^3.

Solution: Given,

Radius of the disc, $R = \dfrac{800}{2} = 400$ mm

$= 400 \times 10^{-3}$ m

Rotational speed, N = 2500

Poisson's ratio, $v = 0.25$

Density of the disc material, $\rho = 7800$ kg/m^3

The angular speed of the disc is given as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 2500}{60} = 261.8 \text{ rad/s}$$

Hoop stress at the centre

From equation (17.25), the hoop stress at the centre of the disc is maximum, and is given as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2 R^2 (3+v)}{8}$$

$$= \frac{7800 \times (261.8)^2 \times (400 \times 10^{-3})^2 \times (3 + 0.25)}{8}$$

$$= 34.75 \times 10^6 \text{ N/m}^2$$

$$= 34.75 \text{ MPa} \qquad \textbf{Ans.}$$

Hoop stress at outer radius

From equation (17.26), the hoop stress at the outer radius is given as

$$\sigma_{h_{r=R}} = \frac{\rho\omega^2 R^2 (1 - v)}{4}$$

$$= \frac{7800 \times (261.8)^2 \times (400 \times 10^{-3})^2 \times (1 - 0.25)}{4}$$

$$= 16.04 \times 10^6 \text{ N/m}^2$$

$$= 16.04 \text{ MPa} \qquad \textbf{Ans.}$$

Radial stress at the centre

The radial stress at the centre is maximum, and is equal to the hoop stress at the centre.

$$\sigma_{r_{\max}} = \sigma_{h_{\max}} = 34.75 \text{ MPa} \qquad \textbf{Ans.}$$

Radial stress at outer radius

The radial stress at the outer radius of the disc is zero.

$$\sigma_{r_{r=R}} = 0 \qquad \textbf{Ans.}$$

Example 17.4

A steel disc of diameter 250 mm has a central hole of diameter 50 mm, and rotates at 5000 rpm. Calculate the hoop stresses developed at the inner and outer radius of the disc. The Prisson's ratio is 0.25 and the density of the disc material is 7800 kg/m^3.

Solution: Given,

Outer radius of the disc, $R_2 = \frac{250}{2} = 125 \text{ mm} = 125 \times 10^{-3}$ m

Inner radius of the disc, $R_1 = \frac{50}{2} = 25 \text{ mm} = 25 \times 10^{-3}$ m

Rotation speed, N = 5000 rpm

Poisson's ratio, v = 0.25

Density of the disc material, ρ = 7800 kg/m^3

The angular speed of the disc is given as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 5000}{60} = 523.6 \text{ rad/s}$$

Hoop stress at inner radius

Substituting the values of ρ, ω, v, R_1 and R_2 in equation (17.31), we get the hoop stress at the inner radius of the disc as

$$\sigma_{h_{r=R_1}} = \frac{\rho\omega^2}{4}[(3+v)R_2^2 + (1-v)R_1^2]$$

$$= \frac{7800 \times (523.6)^2}{4} \times [(3+0.25) \times (125 \times 10^{-3})^2$$

$$+ (1-0.25) \times (25 \times 10^{-3})^2]$$

$$= 27.4 \times 10^6 \text{ N/m}^2$$

$= 27.4$ MPa **Ans.**

This is also the maximum value of the hoop stress, which occurs at the inner radius of the disc.

Hoop stress at outer radius

Substituting the values of ρ, ω, v, R_1 and R_2 in equation (17.32), we get the hoop stress at the outer radius of the disc as

$$\sigma_{h_{r=R_2}} = \frac{\rho\omega^2}{4}[(1-v)R_2^2 + (3+v)R_1^2]$$

$$= \frac{7800 \times (523.6)^2}{4} \times [(1 - 0.25) \times (125 \times 10^{-3})^2 + (3 + 0.25)$$

$$\times (25 \times 10^{-3})^2]$$

$$= 7.35 \times 10^6 \text{ N/m}^2$$

$= 7.35$ MPa **Ans.**

Example 17.5

A circular saw of thickness 5 mm and diameter 800 mm is secured upon a shaft of diameter 120 mm. The saw material has the density of 8100 kg/m^3 and the Prisson's ratio is 0.3. Calculate the permissible speed of the saw, if the allowable hoop stress is 250 MPa. Also, find the maximum value of the radial stress in the saw.

Solution: Given,

Inner radius of the saw, $R_1 = \frac{120}{2} = 60 \text{ mm} = 60 \times 10^{-3}$ m

$$\text{Outer radius of the saw, } R_2 = \frac{800}{2} = 400 \text{ mm} = 400 \times 10^{-3} \text{ m}$$

$$\text{Density of the saw material, } \rho = 8100 \text{ kg/m}^3$$

$$\text{Poisson's ratio, } v = 0.3$$

$$\text{Maximum hoop stress, } \sigma_{h_{\max}} = 250 \text{ MPa} = 250 \times 10^6 \text{ Pa}$$

From equation (17.31), the maximum hoop stress is given as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{4}[(3+v)R_2^2 + (1-v)R_1^2]$$

Substituting values of the given parameters in the above equation, we have

$$250 \times 10^6 = \frac{8100\times\omega^2}{4}[(3+0.3)\times(400\times10^{-3})^2 + (1-0.3)\times(60\times10^{-3})^2]$$

$$= 2025\ \omega^2\ [3.3 \times (400\times10^{-3})^2 + 0.7 \times (60 \times 10^{-3})^2]$$

$$= 1074.303\ \omega^2$$

or $$\omega^2 = \frac{250\times10^6}{1074.303}$$

which gives $$\omega = 482.4 \text{ rad/s}$$

Now $$\omega = \frac{2\pi N}{60}$$

or $$N = \frac{\omega\times60}{2\pi} = \frac{482.4\times60}{2\pi} = 4606.6 \text{ rpm}$$ **Ans.**

From equation (17.34), the maximum radial stress is given as

$$\sigma_{r_{\max}} = \frac{(3+v)\rho\omega^2}{8}(R_2 - R_1)^2$$

$$= \frac{(3+0.3)\times8100\times(482.4)^2\times\{(400-60)\times10^{-3}\}^2}{8}$$

$$= 89.88 \times 10^6 \text{ N/m}^2 = 89.88 \text{ MPa}$$ **Ans.**

Example 17.6

A thin disc of diameter 900 mm has a central hole of diameter 100 mm. Calculate the maximum hoop stress developed in the disc, if the maximum radial stress is 25 MPa. The Poisson's ratio is 0.25.

Solution: Given,

$$\text{Outer radius of the disc, } R_2 = \frac{900}{2} = 450 \text{ mm}$$

$$= 450 \times 10^{-3} \text{ m}$$

$$\text{Inner radius of the disc, } R_1 = \frac{100}{2} = 50 \text{ mm}$$

$$= 50 \times 10^{-3} \text{ m}$$

$$\text{Poisson's ratio, } v = 0.25$$

$$\text{Maximum radial stress, } \sigma_{r_{\max}} = 25 \text{ MPa}$$

$$= 25 \times 10^6 \text{ N/m}^2$$

From equation (17.34), the maximum radial stress is given as

$$\sigma_{r_{\max}} = \frac{(3+v)\rho\omega^2}{8}(R_2 - R_1)^2$$

$$250 \times 10^6 = \frac{(3+0.25)\rho\omega^2}{8} \times \{(450 - 50) \times 10^{-3}\}^2$$

$$= 0.065\ \rho\omega^2$$

or

$$\rho\omega^2 = \frac{25 \times 10^6}{0.065}$$

$$= 3.84615 \times 10^8 \qquad ...(1)$$

Now the maximum hoop stress, using equation (17.31), is given as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{4}[(3+v)R_2^2 + (1-v)R_1^2]$$

$$= \frac{3.84615 \times 10^8}{4}\ [(3 + 0.25) \times (450 \times 10^{-3})^2 + (1 - 0.25) \times (50 \times 10^{-3})^2]$$

$$= 63.46 \times 10^6 \text{ N/m}^2$$

$$= 63.46 \text{ MPa}$$ **Ans.**

Example 17.7

A steel disc of diameter 300 mm has a central hole of diameter 100 mm and it rotates at 4000 rpm. Taking Poisson's ratio as 0.3 and the density of the disc material to be 7800 kg/m^3, find the following parameters :

(*a*) the hoop stress at the inner and outer radius of the disc

(*b*) the radius at which the radial stress is maximum and

(*c*) the maximum radial stress.

Solution: Given,

$$\text{Outer radius of the disc, } R_2 = \frac{300}{2} = 150 \text{ mm} = 150 \times 10^{-3} \text{ m}$$

$$\text{Inner radius of the disc, } R_1 = \frac{100}{2} = 50 \text{ mm} = 50 \times 10^{-3} \text{ m}$$

Poisson's ratio, $v = 0.3$

Rotational speed, $N = 4000$ rpm

Density of the disc material, $\rho = 7800\ \text{kg/m}^3$

The angular speed of the disc is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 4000}{60} = 418.88\ \text{rad/s}$$

(*a*) **Hoop stress at inner radius**

The hoop stress at the inner radius is the maximum value of the hoop stress, and is given by equation (17.31) as

$$\sigma_{h_{r=R_1}} = \sigma_{h_{\max}} = \frac{\rho\omega^2}{4}[(3+v)R_2^2 + (1-v)R_1^2]$$

$$= \frac{7800 \times (418.88)^2}{4} \times [(3 + 0.3) \times (150 \times 10^{-3})^2 + (1 - 0.3) \times (50 \times 10^{-3})^2]$$

$$= 26 \times 10^6\ \text{N/m}^2$$

$$= 26\ \text{MPa}$$ **Ans.**

Hoop stress at outer radius

The hoop stress at the outer radius is given by equation (17.32) as

$$\sigma_{h_{r=R_2}} = \frac{\rho\omega^2}{4}\left[(1-v)R_2^2 + (3+v)R_1^2\right]$$

$$= \frac{7800 \times (418.88)^2}{4} \times [(1 - 0.3) \times (150 \times 10^{-3})^2 + (3 + 0.3) \times (50 \times 10^{-3})^2]$$

$$= 8.211 \times 10^6\ \text{N/m}^2$$

$$= 8.211\ \text{MPa}$$ **Ans.**

(*b*) The radius at which the radial stress is maximum, is given by equation (17.33) as

$$r = \sqrt{R_1 R_2}$$

$$= \sqrt{(50 - 10^{-3}) \times (150 \times 10^{-3})}$$

$$= 0.0866\ \text{m}$$

$$= 86.6\ \text{mm}$$ **Ans.**

(*c*) The maximum radial stress is given by equation (17.34) as

$$\sigma_{r_{max}} = \frac{(3+v)\rho\omega^2}{8}(R_2 - R_1)^2$$

$$= \frac{(3+0.3)\times 7800\times(418.88)^2}{8} \times \{(150 - 50) \times 10^{-3}\}^2$$

$$= 5.645 \times 10^6 \text{ N/m}^2$$

$$= 5.645 \text{ MPa}$$ **Ans.**

Example 17.8

A circular disc of outside diameter 500 mm has a central hole and rotates at a uniform speed about an axis through its centre. The diameter of the hole is such that the maximum stress due to rotation is 85% of that in a thin ring whose mean diameter is also 500 mm. If both disc and ring are made of the same material and rotate at the same speed, determine (*a*) the diameter of the central hole and (*b*) the speed of rotation, if the allowable stress in the disc is 90 MPa. Take the Poisson's ratio of 0.3 and the density of both disc and ring material as 7800 kg/m^3.

Solution: Given,

Mean radius of the thin ring, $r = \frac{500}{2}$ mm

$= 250 \times 10^{-3}$ m

Outside radius of the disc, $R_2 = \frac{500}{2}$ mm

$= 250 \times 10^{-3}$ m

Density of the disc and ring material,

$\rho = 7800$ kg/m^3

Poisson's ratio $v = 0.3$

Maximum hoop stress in the disc,

$\sigma_{h_{max}} = 90$ MPa

$= 90 \times 10^6$ Pa

Let σ_h be the maximum hoop stress in the thin ring.

Given $\sigma_{h_{max}} = 0.85 \times \sigma_h$

Hence $$\sigma_h = \frac{\sigma_{h_{max}}}{0.85}$$

For thin ring

The hoop stress in the thin ring is given as

$$\sigma_h = \rho\omega^2 r^2$$ (using equation (17.5))

$$\frac{\sigma_{h_{\max}}}{0.85} = \rho\omega^2 r^2 \qquad \text{(on substituting } \sigma_h\text{)}$$

$$\frac{90 \times 10^6}{0.85} = 7800 \times \omega^2 \times (250 \times 10^{-3})^2$$

$$\omega^2 = \frac{90 \times 10^6}{7800 \times 0.85 \times (250 \times 10^{-3})^2}$$

$$= 217194.6$$

which gives

$$\omega = 466 \text{ rad/s}$$

Now the rotational speed N is given as

$$N = \frac{60\omega}{2\pi}$$

$$= \frac{60 \times 466}{2\pi}$$

$$= 4450 \text{ rpm}$$ **Ans.**

For hollow disc

The maximum hoop stress is given as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{4}[(3+v)R_2^2 + (1-v)R_1^2] \qquad \text{(using equation (17.31))}$$

$$90 \times 10^6 = \frac{7800 \times 217194.6}{4} \times [(3 + 0.3) \times (250 \times 10^{-3})^2 + (1 - 0.3)\, R_1^{\,2}]$$

$$= 4.23 \times 10^8 \times [3.3 \times (250 \times 10^{-3})^2 + 0.7 \times R_1^{\,2}]$$

$$\frac{90 \times 10^6}{4.23 \times 10^8} = 0.20625 + 0.7\, R_1^{\,2}$$

$$0.2128 = 0.20625 + 0.7\, R_1^{\,2}$$

$$R_1^{\,2} = \frac{0.2128 - 0.20625}{0.7} = 9.36 \times 10^{-3}$$

which gives

$$R_1 = 0.09674 \text{ m}$$

$$= 96.74 \text{ mm}$$

Hence, the diameter of the central hole $= 2 \times R_1$

$$= 2 \times 96.74$$

$$= 193.5 \text{ mm}$$ **Ans.**

Example 17.9

A steel disc of uniform thickness and of diameter 800 mm has a pin hole at the center. Calculate the maximum hoop stress developed in the disc, if it rotates at 3000 rpm. The Poisson's ratio is 0.25 and the density of the disc material is 7800 kg/m^3.

Solution: Given,

$$\text{Radius of the disc, } R = \frac{800}{2} = 400 \text{ mm}$$

$$= 400 \times 10^{-3} \text{ m}$$

$$\text{Rotational speed, } N = 3000 \text{ rpm}$$

$$\text{Poisson's ratio, } v = 0.25$$

$$\text{Density of the disc material, } \rho = 7800\ \pi\text{kg/m}^3$$

The angular speed of the disc is found as

Now
$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 3000}{60} = 314.16 \text{ rad/s}$$

From equation (17.35), the maximum hoop stress is given as

$$\sigma_{h_{max}} = \frac{(3+v)\rho\omega^2 R^2}{4}$$

$$= \frac{(3+0.25)\times 7800 \times (314.16)^2 \times (400\times 10^{-3})^2}{4}$$

$$= 10^8 \text{ Pa}$$

$$= 100 \text{ MPa}$$ **Ans.**

Example 17.10

A thin uniform steel disc of diameter 500 mm rotates at 2000 rpm. Calculate the maximum principal stress induced in the disc and also plot the distribution of the hoop stress and the radial stress along the radius of the disc. Take Poisson's ratio as 0.3 and the density of the disc material is equal to 7800 kg/m^3.

Solution: Given,

$$\text{Radius of the disc, } R = \frac{500}{2} = 250 \text{ mm}$$

$$= 250 \times 10^{-3} \text{ m}$$

Rotational speed, N = 2000 rpm

Poisson's ratio, v = 0.3.

Density of the disc material, ρ = 7800 kg/m^3.

The angular speed of the disc is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 2000}{60}$$

$$= 209.44 \text{ rad/s}$$

The maximum hoop stress is also the maximum principal stress, which can be obtained by using equation (17.25) as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2 R^2 (3+v)}{8}$$

$$= \frac{7800 \times (209.44)^2 \times (250 \times 10^{-3})^2 \times (3+0.3)}{8}$$

$$= 8.821 \times 10^6 \text{ N/m}^2$$

$$= 8.821 \text{ MPa}$$ **Ans.**

Distribution of the hoop stress

The hoop stress is given by equation (17.24) as

$$\sigma_h = \frac{\rho\omega^2}{8} [(3 + v) R^2 - (1 + 3v) r^2]$$

Now we select different values of the radius r and determine the corresponding hoop stresses using the above equation. The distribution of the stresses is shown in Table 17.1.

Table 17.1 Distribution of the hoop stress

r (mm)	0	50	100	150	200	250
σ_h (MPa)	8.821	8.618	8.00	7.00	5.570	3.742

Distribution of the radial stress

The radial stress is given by equation (17.23) as

$$\sigma_r = \frac{\rho\omega^2 (3+v)}{8} (R^2 - r^2)$$

The values of the radial stresses corresponding to the selected values of the radius are determined using the above equation, which are shown in Table 17.2.

Table 17.2 Distribution of the radial stress

r (mm)	0	50	100	150	200	250
σ_r (MPa)	8.821	8.468	7.409	5.645	3.175	0

Plotting of the hoop and radial stresses

The values of the radius are plotted on x-axis and the values of the hoop and radial stresses on y-axis. The resulting curves for the two stresses are shown in Fig. 17.4.

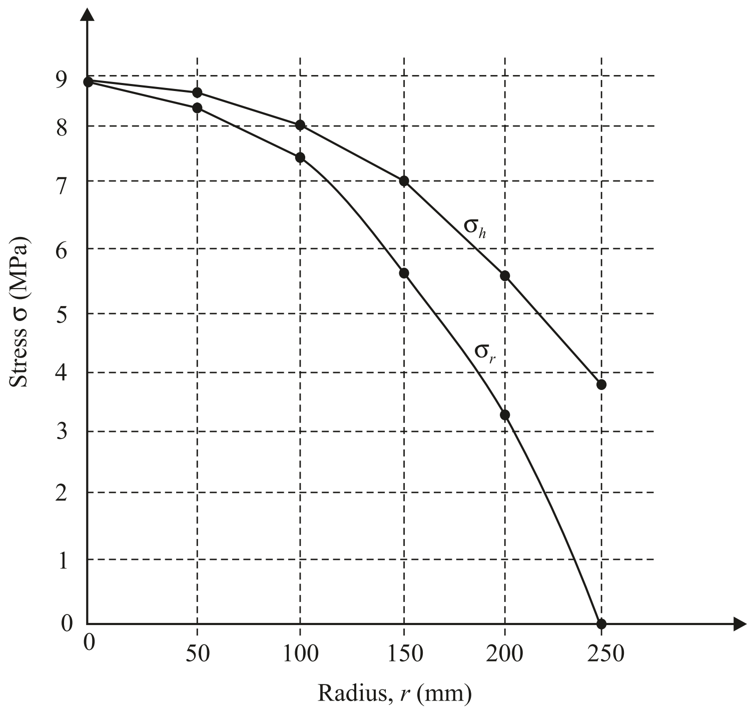


Fig. 17.4

Example 17.11

A steel disc of diameter 400 mm has a central hole of diameter 100 mm and rotates at 8000 rpm. Taking Poisson's ratio as 0.3 and the density of the disc material to be 7800 kg/m^3, plot the distribution of the hoop and radial stresses along the radius of the disc.

Solution: Given,

$$\text{Outer radius of the disc, } R_2 = \frac{400}{2} = 200 \text{ mm} = 200 \times 10^{-3} \text{ m}$$

$$\text{Inner radius of the disc, } R_1 = \frac{100}{2} = 50 \text{ mm}$$

$$= 50 \times 10^{-3} \text{ m}$$

$$\text{Poisson's ratio, } v = 0.3$$

$$\text{Rotational speed, } N = 8000 \text{ rpm}$$

$$\text{Density of the disc material, } \rho = 7800 \text{ kg/m}^3$$

The angular speed of the disc is obtained as

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 8000}{60} = 837.76 \text{ rad/s}$$

Distribution of the hoop stress

The hoop stress is given by equation (17.30) as

$$\sigma_h = \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1)^2 + \frac{(3+v)R_1^2R_2^2}{r^2} - (1+3v)r^2\right]$$

Now we select different values of the radius r and determine the corresponding hoop stresses using the above equation. The distribution of the stresses is shown in Table 17.3.

Table 17.3 Distribution of the hoop stress

r (mm)	50	100	150	200
σ_h (MPa)	183.05	105.55	76.75	49.61

Distribution of the radial stress

The radial stress is given by equation (17.29) as

$$\sigma_r = \frac{(3+v)\rho\omega^2}{8}\left[(R_2^2+R_1^2) - \frac{R_1^2R_2^2}{r^2} - r^2\right]$$

The values of the radial stresses corresponding to the selected values of the radius are determined using the above equation, which are shown in Table 17.4. The radius at which radial stress is maximum is $\sqrt{R_1R_2} = \sqrt{(50\times10^{-3})\times(200\times10^{-3})} = 0.1$ m = 100 mm.

Table 17.4 Distribution of the radial stress

r (mm)	50	100	150	200
σ_r (MPa)	0	50.80	35.13	0

Plotting of the hoop and radial stresses

The values of the radius are plotted on x-axis and the values of the hoop and radial stresses on y-axis. The resulting curves for the two stresses are shown in Fig. 17.5.

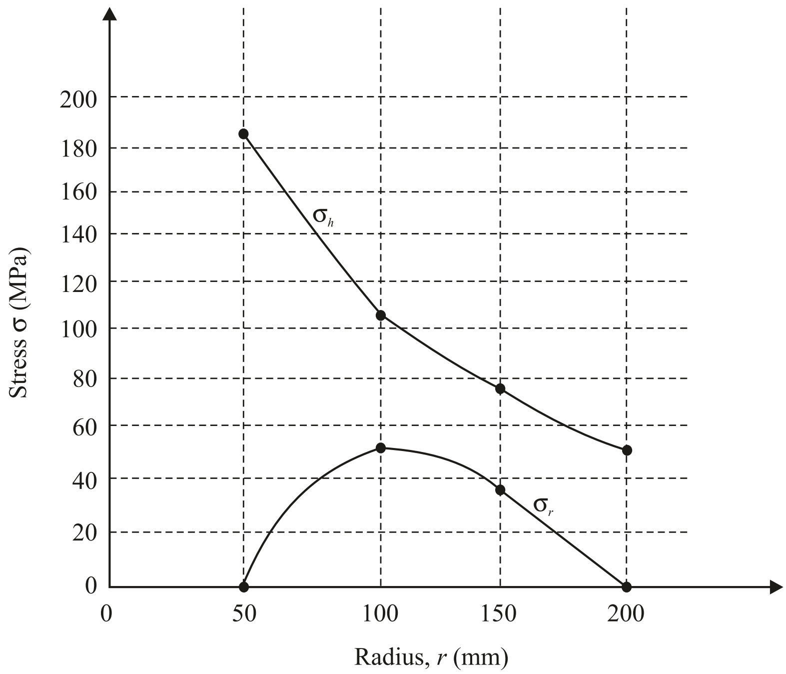


Fig. 17.5

17.4 ROTATING DISC OF UNIFORM STRENGTH

In case of a disc with uniform thickness, the hoop and radial stresses are not uniform and they vary along the radius of the disc. On the other hand, a disc of uniform strength has equal values of hoop and radial stresses at every radius. Thickness of such a disc is not uniform and varies along the axis. Analysis of such a disc is useful in the design of turbine blades rotating at high speeds and are required to be subjected to constant stress conditions to prevent their premature failure.

Consider a rotating disc of uniform strength which is subjected to equal hoop and radial stresses, that is, $\sigma_h = \sigma_r = \sigma$ and the stresses do not vary with radius. Now consider an element *ABCD* of radial width dr of the disc at a radius r from the axis of rotation making an angle $d\theta$ at the center as shown in Fig. 17.6.

Let t = Thickness of the disc at radius r

$t + dt$ = Thickness of the disc at radius $(r + dr)$

t_o = Thickness of the disc at radius $r = 0$, that is, at the axis of rotation

ω = Angular speed of the disc

σ = Equal hoop and radial stresses

ρ = Density of the disc material

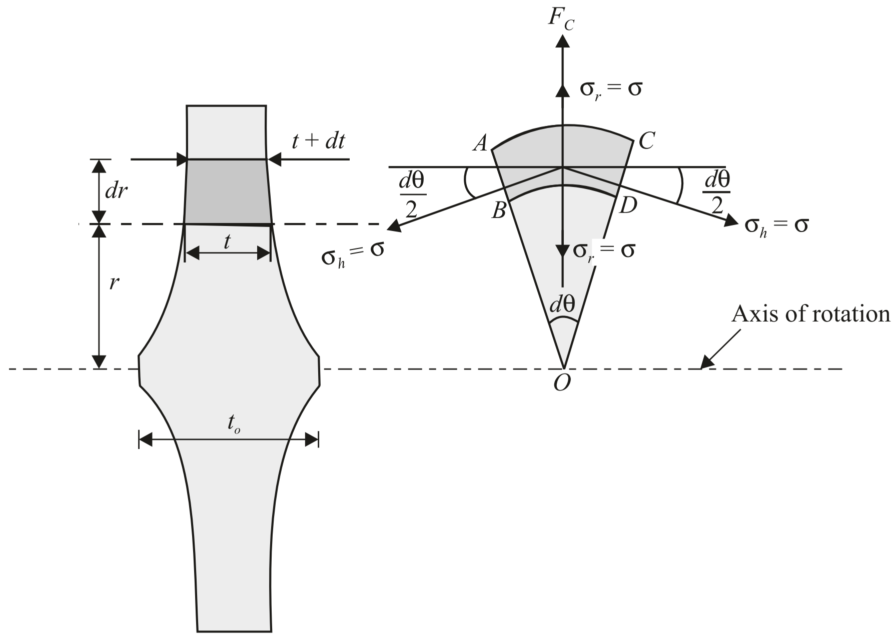


Fig. 17.6

Forces on the element

The following forces are acting on the element *ABCD* :

- The centrifugal force caused due to rotation acting radially outward
- The hoop tension force on the face *AB* caused due to hoop stress σ
- The hoop tension force on the face *CD* caused due to hoop stress σ
- The radial force on the face *BD* caused due to radial stress σ
- The radial force on the face *AC* caused due to radial stress σ

Centrifugal force

The mass m of the element can be obtained as

$$\begin{aligned} m &= \text{Density} \times \text{Volume of the element} \\ &= \rho \times (rd\theta \times dr \times t) \\ &= \rho rtd\theta dr \qquad \text{...(17.38)} \end{aligned}$$

The centrifugal force is given as

$$\begin{aligned} F_c &= \frac{mV^2}{r} \\ &= m\omega^2 r \qquad (\text{as } V = \omega r) \\ &= (\rho rtd\theta dr \times \omega^2 r) \qquad (\text{on substituting } m) \\ &= \rho t\omega^2 r^2 d\theta dr \qquad \text{...(17.39)} \end{aligned}$$

Hoop tension forces on faces *AB* and *CD*

The hoop tension forces act perpendicular to *AB* and *CD* and are equal to $\sigma_h \times dr \times t$.

Now the hoop forces are resolved into horizontal and vertical components. The vertical components of the hoop forces acting on faces *AB* and *CD* are radially inward and both are equal to $\sigma \times dr \times t \times \sin\dfrac{d\theta}{2}$ The horizontal component of the hoop forces acting on face *AB* is directed leftward and the horizontal component on face *CD* is directed rightward, and both are equal to $\sigma \times dr \times t \times \cos\dfrac{d\theta}{2}$

Radial force on face *BD*

Radial force on $BD = \sigma \times rd\theta \times t$, and its acts radially inward.

Radial force on face *AC*

Radial force on $AC = \sigma \times (r + dr)d\theta \times (t + dt)$, and it acts radially outward.

Equilibrium of the element

The horizontal components of the hoop tension forces on the faces *AB* and *CD* are equal but opposite in direction, hence they cancel each other. The vertical components are added as they are acting in the same direction (radially inward).

Considering forces in the radial direction, we have

$$\sigma \times rd\theta \times t + \sigma \times dr \times t \times \sin\frac{d\theta}{2} + \sigma \times dr \times t \times \sin\frac{d\theta}{2} = F_c + \sigma \times (r + dr)\, d\theta \times (t + dt)$$

or $$\sigma \times rd\theta \times t + 2\sigma \times dr \times t \times \sin\frac{d\theta}{2} = F_c + \sigma \times (r + dr)\, d\theta \times (t + dt)$$

Substituting the value of F_c from equation (17.39) in the above equation and equating $\sin\dfrac{d\theta}{2} = \dfrac{d\theta}{2}$ as $d\theta$ is very small, we get

$$\sigma \times rd\theta \times t + 2\sigma \times dr \times t \times \frac{d\theta}{2} = \rho t\omega^2 r^2 d\theta dr + \sigma rtd\theta + \sigma rd\theta dt + \sigma tdrd\theta + \sigma drd\theta dt$$

Eliminating $d\theta$ from both sides, we get

$$\sigma \times r \times t + \sigma tdr = \rho t\omega^2 r^2 dr + \sigma rt + \sigma rdt + \sigma tdr + \sigma drdt$$

Now eliminating σrt and σtdr from both sides and neglecting ($drdt$) as being the product of two smaller quantities, we find

$$0 = \rho t\omega^2 r^2 dr + \sigma rdt$$

or $$-\sigma rdt = \rho t\omega^2 r^2 dr$$

$$\frac{dt}{t} = -\frac{\rho\omega^2 rdr}{\sigma}$$

On integration, we have

$$\log_e t = -\frac{\rho\omega^2 r^2}{2\sigma} + \log_e A \qquad \text{...(17.40)}$$

where $\log_e A$ is a constant of integration.

The boundary condition is

when $r = 0$

$t = t_o$

Substituting the boundary condition in equation (17.40), we get

$$\log_e t_o = \log_e A$$

which gives $A = t_o$

Equation (17.40) on substituting A becomes

$$\log_e t = -\frac{\rho\omega^2 r^2}{2\sigma} + \log_e t_o$$

$$\log_e t - \log_e t_o = -\frac{\rho\omega^2 r^2}{2\sigma}$$

$$\log_e \left(\frac{t}{t_o}\right) = -\frac{\rho\omega^2 r^2}{2\sigma}$$

$$\frac{t}{t_o} = e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

which gives

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)} \qquad \text{...(17.41)}$$

This is the required expression for the thickness of the disc, which varies according to the given value of the radius r.

Example 17.12

A turbine rotor is to be designed for uniform strength for a tensile stress of 150 MPa . The rotor runs at 6000 rpm and its thickness at the centre is 90 mm. If the density of the material of the rotor is 7800 kg/m^3, determine the thickness of the rotor at a radius of 400 mm.

Solution: Given,

Uniform stress, $\sigma = 150$ MPa

$= 150 \times 10^6$ Pa

Rotational speed, $N = 6000$ rpm

Thickness of the rotor at the center, $t_o = 90$ mm

$= 90 \times 10^{-3}$ m

Density of the rotor material, ρ= 7800 kg/m^3

Radius at the required thickness, r = 400 mm

$$= 400 \times 10^{-3} \text{ m}$$

The angular speed of the rotor is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 6000}{60}$$

$$= 628.32 \text{ rad/s}$$

From equation (17.41), the expression for the thickness is given as

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

$$= 90 \times 10^{-3} \times e^{\left(-\frac{7800\times(628.32)^2\times(400\times10^{-3})^2}{2\times150\times10^6}\right)}$$

$$= 90 \times 10^{-3} \times e^{-1.642}$$

$$= 90 \times 10^{-3} \times 0.193$$

$$= 0.01742 \text{ m}$$

$$= 17.42 \text{ mm}$$ **Ans.**

Example 17.13

The minimum thickness of a steam turbine rotor is 10 mm at a radius of 200 mm and is required to be designed for uniform strength under rotational conditions for a stress of 180 MPa. It runs at 10,000 rpm and its material weighs 7800 kg/m^3. Determine the thickness of the rotor at a radius of 40 mm.

Solution: Given,

Rotational speed of the rotor, N = 10,000 rpm

Radius at the desired thickness, r = 40 mm

$$= 40 \times 10^{-3} \text{ m}$$

Uniform stress, σ = 180 MPa

$$= 180 \times 10^6 \text{ Pa}$$

Density of the rotor material, ρ = 7800 kg/m^3

The angular speed of the rotor is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 10{,}000}{60}$$

$$= 1047.2 \text{ rad/s}$$

The thickness expression is given as

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

At $r = 200$ mm, $t = 10$ mm. Substituting these values in the above equation, we have

$$10 \times 10^{-3} = t_o \times e^{\left(-\frac{7800 \times (1047.2)^2 \times (200 \times 10^{-3})^2}{2 \times 180 \times 10^6}\right)}$$

$$= t_o \times e^{-0.9504}$$

$$= t_o \times 0.3866$$

which gives

$$t_o = 0.02586 \text{ m}$$

$$= 25.86 \text{ mm}$$

Again using the thickness equation and substituting the value of t_o, we have

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

$$= 0.02586 \times e^{\left(-\frac{7800 \times (1047.2)^2 \times (40 \times 10^{-3})^2}{2 \times 180 \times 10^6}\right)}$$

$$= 0.02586 \times e^{-0.038}$$

$$= 0.02586 \times 0.9627$$

$$= 0.0249 \text{ m}$$

$$= 24.9 \text{ mm}$$ **Ans.**

Example 17.14

A steam turbine rotor is 160 mm diameter below the blade ring and 5 mm thick, and runs at 30,000 rpm. If the material of the rotor weighs 7800 kg/m^3 and the allowable stress is 160 MPa, what is the thickness of the rotor at a radius of 60 mm and at the centre? Assume uniform strength condition.

Solution: Given,

Rotational speed of the rotor,

$$N = 30{,}000 \text{ rpm}$$

Radius at the desired thickness, $r = 60$ mm

$$= 60 \times 10^{-3} \text{ m}$$

Uniform stress, $\sigma = 160$ MPa

$$= 160 \times 10^6 \text{ Pa}$$

Density of the rotor material, $\rho = 7800$ kg/m^3

The angular speed of the rotor is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 30{,}000}{60} = 3141.6 \text{ rad/s}$$

The thickness expressions is given as

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

where t_o is thickness of the rotor at the center, that is, at $r = 0$.

Now at $r = 80$ mm, $t = 5$ mm

Using these values in the above equation, we have

$$5 \times 10^{-3} = t_o \times e^{\left(-\frac{7800 \times (3141.6)^2 \times (80 \times 10^{-3})^2}{2 \times 160 \times 10^6}\right)}$$

$$= t_o \times e^{-1.53966}$$

$$= t_o \times 0.2144$$

which gives

$$t_o = 0.0233 \text{ m}$$

$$= 23.3 \text{ mm}$$ **Ans.**

Again using the thickness equation for $r = 60$ mm, we have

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

$$= 23.3 \times 10^{-3} \times e^{\left(-\frac{7800 \times (3141.6)^2 \times (60 \times 10^{-3})^2}{2 \times 160 \times 10^6}\right)}$$

$$= 23.3 \times 10^{-3} \times e^{-0.866}$$

$$= 23.3 \times 10^{-3} \times 0.4206 = 9.8 \times 10^{-3} \text{ m}$$

$$= 9.8 \text{ mm}$$ **Ans.**

Example 17.15

A steel turbine disc is to be designed so that between radii of 250 mm and 400 mm, the radial and hoop stresses are required to be constant at 60 MPa, when running at 3000 rpm. If the axial thickness is 12 mm at the outer edge of this zone, what should it be at the inner edge? The density of the disc material is 7800 kg/m^3.

Solution: Given,

Uniform stress, $\sigma = 60$ MPa

$= 60 \times 10^6$ Pa

Rotational speed, $N = 3000$ rpm

Density of the disc material, $\rho = 7800 \text{ kg/m}^3$.

The angular speed of the disc is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 3000}{60}$$

$$= 314.16 \text{ rad/s}$$

The thickness expression is given as

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

Now at $r = 400$ mm, $t = 12$ mm. Substituting these values in the above equation, we have

$$12 \times 10^{-3} = t_o \times e^{\left(-\frac{7800 \times (314.16)^2 \times (400 \times 10^{-3})^2}{2 \times 60 \times 10^6}\right)}$$

$$= t_o \times e^{-1.0264}$$

$$= t_o \times 0.3583$$

which gives

$$t_o = 0.0335 \text{ m}$$

$$= 33.5 \text{ mm}$$

Again using thickness equation for $r = 250$ mm, we have

$$t = 0.0335 \times e^{\left(-\frac{7800 \times (314.16)^2 \times (250 \times 10^{-3})^2}{2 \times 60 \times 10^6}\right)}$$

$$= 0.0335 \times e^{-0.40095}$$

$$= 0.0335 \times 0.6697$$

$$= 0.0224 \text{ m}$$

$$= 22.4 \text{ mm}$$ **Ans.**

17.5 ROTATING LONG CYLINDER

The force analysis of a rotating thick cylinder is similar to that of a rotating thin disc except the introduction of axial stress. Hence three stresses acting on a rotating thick cylinder include hoop stress (σ_h), radial stress (σ_r) and axial stress, also called longitudinal stress (σ_l). It is assumed that the transverse sections of the cylinder remain plane even at high speeds of rotation, which implies that longitudinal strain is constant. At the same time, all the three stresses acting on the cylinder are principal stresses.

Consider a small element *ABCD* of the cylinder at a distance r and of radial thickness dr subtending an angle $d\theta$ at the center as shown in Fig. 17.7.

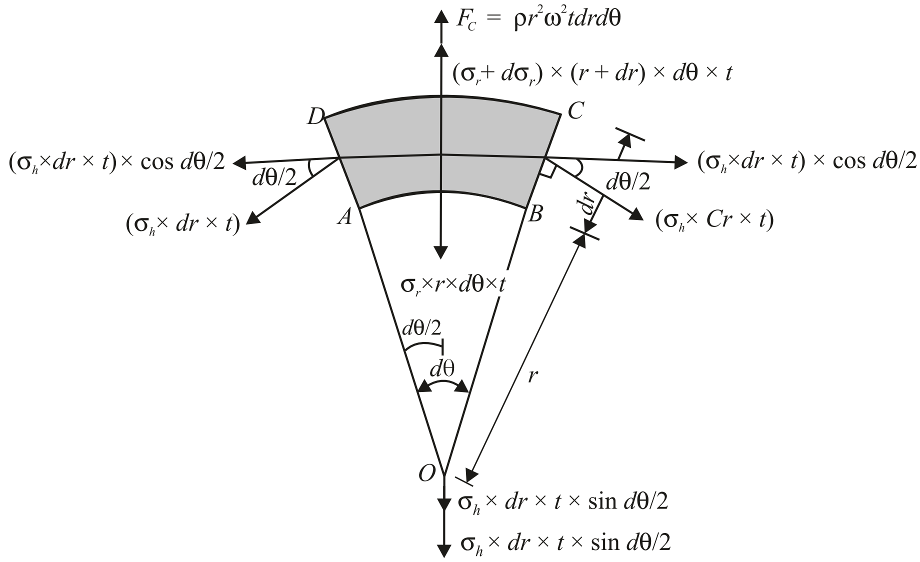


Fig. 17.7

Let $\in_h$ = Hoop strain, also called circumferential strain

$\in_r$ = Radial strain

$\in_l$ = Longitudinal strain

ω = Angular speed of rotation of the cylinder

v = Poisson's ratio

E = Modulus of elasticity of the cylinder

The strains produced by various stresses are obtained as

$$\in_h = \frac{\sigma_h}{E} - v\frac{\sigma_r}{E} - v\frac{\sigma_l}{E}$$

$$= \frac{1}{E}\,[\sigma_h - v\,(\sigma_r + \sigma_l)] \qquad ...(17.42)$$

$$\in_r = \frac{\sigma_r}{E} - v\frac{\sigma_h}{E} - v\frac{\sigma_l}{E}$$

$$= \frac{1}{E}\,[\sigma_r - v\,(\sigma_h + \sigma_l)] \qquad ...(17.43)$$

and

$$\in_l = \frac{\sigma_l}{E} - v\frac{\sigma_h}{E} - v\frac{\sigma_r}{l}$$

$$= \frac{1}{E}\,[\sigma_l - v\,(\sigma_h + \sigma_r)] \qquad ...(17.44)$$

Due to rotation, the radius of the cylinder increases. Let the radius r changes to $(r + u)$ and dr changes to $(dr + du)$.

Now the hoop strain is also expressed as

$$\in_h = \frac{2\pi(r+u) - 2\pi r}{2\pi r}$$

$$= \frac{u}{r} \quad ...(17.45)$$

The radial strain is also expressed as

$$\in_r = \frac{(dr + du) - dr}{dr}$$

$$= \frac{du}{dr} \quad ...(17.46)$$

Comparing equations (17.42) and (17.45), we have

$$\in_h = \frac{u}{r} = \frac{1}{E}\ [\sigma_h - v\ (\sigma_r + \sigma_l)]$$

which gives

$$Eu = r\ [\sigma_h - v\ (\sigma_r + \sigma_l)] \quad ...(17.47)$$

Comparing equations (17.43) and (17.46), we have

$$\in_r = \frac{du}{dr} = \frac{1}{E}\ [\sigma_r - v\ (\sigma_h + \sigma_l)] \quad ...(17.48)$$

Differentiating equation (17.47) with respect to r, we get

$$\frac{du}{dr} = \frac{1}{E}\ [\sigma_h - v\ (\sigma_r + \sigma_l)] + \frac{1}{E}\left[r\frac{d\sigma_h}{dr} - vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right)\right] \quad ...(17.49)$$

Comparing equations (17.48) and (17.49), we get

$$\sigma_r - v\ (\sigma_h + \sigma_l) = \sigma_h - v\ (\sigma_r + \sigma_l) + r\frac{d\sigma_h}{dr} - vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right)$$

$$\sigma_r - v\sigma_h - v\sigma_l = \sigma_h - v\sigma_r - v\sigma_l + r\frac{d\sigma_h}{dr} - vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right)$$

Cancelling $v\sigma_l$ from both sides of the equation and rearranging the terms, we get

$$\sigma_r + v\sigma_r = \sigma_h + v\sigma_h + r\frac{d\sigma_h}{dr} - vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right)$$

$$\sigma_r\ (1 + v) = \sigma_h\ (1 + v) + r\frac{d\sigma_h}{dr} - vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right)$$

$$(1 + v)(\sigma_r - \sigma_h) = r\frac{d\sigma_h}{dr} - vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right)$$

$$(1 + v)(\sigma_r - \sigma_h) - r\frac{d\sigma_h}{dr} + vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right) = 0 \quad ...(17.50)$$

Since $\in_l$ is constant, hence from equation (17.44), we have

$$\in_l = \frac{1}{E}[\sigma_l - v(\sigma_h + \sigma_r)] = \text{Constant}$$

or $\sigma_l - v(\sigma_h + \sigma_r) = \text{Constant}$ (as E is a constant)

Differentiating with respect to r, we have

$$\frac{d\sigma_l}{dr} - v\left(\frac{d\sigma_h}{dr} + \frac{d\sigma_r}{dr}\right) = 0$$

Multiplying all the terms by r, we get

$$r\frac{d\sigma_l}{dr} - vr\left(\frac{d\sigma_h}{dr} + \frac{d\sigma_r}{dr}\right) = 0$$

or

$$r\frac{d\sigma_l}{dr} = vr\left(\frac{d\sigma_h}{dr} + \frac{d\sigma_r}{dr}\right) \quad ...(17.51)$$

From equation (17.50), we have

$$(1 + v)(\sigma_r - \sigma_h) - r\frac{d\sigma_h}{dr} + vr\frac{d\sigma_r}{dr} + vr\frac{d\sigma_l}{dr} = 0$$

Substituting equation (17.51) in the above equation, we get

$$(1 + v)(\sigma_r - \sigma_h) - r\frac{d\sigma_h}{dr} + vr\frac{d\sigma_r}{dr} + v^2r\left(\frac{d\sigma_h}{dr} + \frac{d\sigma_r}{dr}\right) = 0$$

$$(1 + v)(\sigma_r - \sigma_h) - r\frac{d\sigma_h}{dr} + vr\frac{d\sigma_r}{dr} + v^2r\frac{d\sigma_h}{dr} + v^2r\frac{d\sigma_r}{dr} = 0$$

$$(1 + v)(\sigma_r - \sigma_h) - r\frac{d\sigma_h}{dr}(1 - v^2) + vr\frac{d\sigma_r}{dr}(1 + v) = 0$$

Eliminating $(1 + v)$ from all the terms, we get

$$(\sigma_r - \sigma_h) - r\frac{d\sigma_h}{dr}(1 - v) + vr\frac{d\sigma_r}{dr} = 0$$

$$(\sigma_h - \sigma_r) + r(1 - v)\frac{d\sigma_h}{dr} - vr\frac{d\sigma_r}{dr} = 0 \quad ...(17.52)$$

The equilibrium equation of the element can be obtained in a similar manner as in case of a rotating thin disc, which is given by equation (17.7) as

$$(\sigma_h - \sigma_r) = \rho r^2\omega^2 + r\frac{d\sigma_r}{dr} \quad \text{(from equation (17.7))}$$

Substituting the above in equation (17.52), we get

$$\rho r^2\omega^2 + r\frac{d\sigma_r}{dr} + r(1-v)\frac{d\sigma_h}{dr} - vr\frac{d\sigma_r}{dr} = 0$$

Eliminating r from all the terms and simplifying, we have

$$\rho r\omega^2 + \frac{d\sigma_r}{dr}(1-v) + (1-v)\frac{d\sigma_h}{dr} = 0$$

or

$$\frac{d\sigma_h}{dr} + \frac{d\sigma_r}{dr} = -\frac{\rho r\omega^2}{(1-v)} \qquad \text{...(17.53)}$$

Integration of equation (17.53) gives

$$\sigma_h + \sigma_r = -\frac{\rho r^2\omega^2}{2(1-v)} + A \qquad \text{...(17.54)}$$

where A is a constant of integration.

Now subtracting equation (17.7) from equation (17.54), we get

$$\sigma_h + \sigma_r - \sigma_h + \sigma_r = -\frac{\rho r^2\omega^2}{2(1-v)} + A - \rho r^2\omega^2 - r\frac{d\sigma_r}{dr}$$

$$2\sigma_r + r\frac{d\sigma_r}{dr} = -\rho r^2\omega^2\left[\frac{1}{2(1-v)} + 1\right] + A$$

$$= -\rho r^2\omega^2\left[\frac{1+2-2v}{2(1-v)}\right] + A$$

$$= -\frac{\rho r^2\omega^2}{2}\left(\frac{3-2v}{1-v}\right) + A$$

Multiplying by r on both sides, we get

$$2r\sigma_r + r^2\frac{d\sigma_r}{dr} = -\frac{\rho r^3\omega^2}{2}\left(\frac{3-2v}{1-v}\right) + Ar$$

$$\frac{d}{dr}(r^2 \times \sigma_r) = -\frac{\rho r^3\omega^2}{2}\left(\frac{3-2v}{1-v}\right) + Ar \qquad \text{...(17.55)}$$

On integration, we have

$$r^2 \times \sigma_r = -\frac{\rho r^4\omega^2}{8}\left(\frac{3-2v}{1-v}\right) + \frac{Ar^2}{2} + B$$

where B is another constant of integration.

Dividing throughout by r^2, we get

$$\sigma_r = \frac{A}{2} + \frac{B}{r^2} - \frac{\rho r^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right) \qquad \text{...(17.56)}$$

This is the required expression for the radial stress for a rotating thick cylinder. The constants *A* and *B* can be determined by using suitable boundary conditions.

Substituting equation (17.56) in equation (17.54), we can obtain the value of σ_h.

$$\sigma_h + \frac{A}{2} + \frac{B}{r^2} - \frac{\rho r^2 \omega^2}{8}\left(\frac{3-2v}{1-v}\right) = -\frac{\rho r^2\omega^2}{2(1-v)} + A$$

or

$$\begin{aligned}\sigma_h &= A - \frac{A}{2} - \frac{B}{r^2} + \frac{\rho r^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right) - \frac{\rho r^2\omega^2}{2(1-v)} \\ &= \frac{A}{2} - \frac{B}{r^2} + \frac{\rho r^2\omega^2}{2(1-v)}\left(\frac{3-2v}{4} - 1\right) \\ &= \frac{A}{2} - \frac{B}{r^2} + \frac{\rho r^2\omega^2}{2\,(1-v)}\left(\frac{3-2v-4}{4}\right) \\ &= \frac{A}{2} - \frac{B}{r^2} - \frac{\rho r^2\omega^2}{8}\left(\frac{1+2v}{1-v}\right)\end{aligned} \quad ...(17.57)$$

This is the required expression for the hoop stress for a rotating thick cylinder. The constants *A* and *B* can be determined by using suitable boundary conditions.

17.5.1 Hoop and Radial Stresses in a Rotating Solid Cylinder or a Solid Shaft

When we put $r = 0$ in equations (17.56) and (17.57), we see that stresses become infinite. Hence for the meaningful values of the two stresses, the constant *B* has to be zero. Now the stresses are expressed as

$$\sigma_r = \frac{A}{2} - \frac{\rho r^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right) \quad ...(17.58)$$

and

$$\sigma_h = \frac{A}{2} - \frac{\rho r^2\omega^2}{8}\left(\frac{1+2v}{1-v}\right) \quad ...(17.59)$$

These are the expressions for the radial and hoop stresses respectively at the centre of a rotating solid cylinder.

At the surface of the cylinder, where $r = R_2$ (= *R* say),

$$\sigma_r = 0$$

Substituting σ_r in equation (17.58), we have

$$0 = \frac{A}{2} - \frac{\rho R^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

which gives

$$\frac{A}{2} = \frac{\rho R^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

On putting $\frac{A}{2}$ in equation (17.58), we have

$$\sigma_r = \frac{\rho R^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right) - \frac{\rho r^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left(R^2 - r^2\right) \quad ...(17.60)$$

This is the required expression for the radial stress for any value of r.

The expression for the hoop stress on substituting the value of $\frac{A}{2}$ in equation (17.59) becomes

$$\sigma_h = \frac{\rho\omega^2 R^2}{8}\left(\frac{3-2v}{1-v}\right) - \frac{\rho\omega^2 r^2}{8}\left(\frac{1+2v}{1-v}\right) \quad ...(17.61)$$

Maximum radial stress

The radial stress is maximum at the centre of the cylinder, that is, at $r = 0$. Putting $r = 0$ in equation (17.60), we get the expression for the maximum radial stress as

$$\sigma_{r_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right) \quad ...(17.62)$$

Maximum hoop stress

The hoop stress is also maximum at the centre of cylinder, that is, at $r = 0$. Putting $r = 0$ in equation (17.61), we get the expression for the maximum hoop stress as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right) \quad ...(17.63)$$

Hence,

$$\sigma_{r_{\max}} = \sigma_{h_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right) \quad ...(17.64)$$

17.5.2 Hoop and Radial Stresses in a Rotating Hollow Cylinder

For a hollow cylinder

$$\sigma_r = 0 \quad \text{at} \quad r = R_1$$

Also

$$\sigma_r = 0 \quad \text{at} \quad r = R_2$$

Substituting these boundary conditions in equation (17.56), we have

$$0 = \frac{A}{2} + \frac{B}{R_1^2} - \frac{\rho\omega^2 R_1^2}{8}\left(\frac{3-2v}{1-v}\right) \quad \text{(where } r = R_1\text{)} \quad ...(1)$$

and

$$0 = \frac{A}{2} + \frac{B}{R_2^2} - \frac{\rho\omega^2 R_2^2}{8}\left(\frac{3-2v}{1-v}\right) \quad \text{(where } r = R_2\text{)} \quad ...(2)$$

Subtracting equation (2) from equation (1), we have

$$B\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)+\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_2^2-R_1^2) = 0$$

$$B\left(\frac{R_2^2-R_1^2}{R_1^2R_2^2}\right)+\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_2^2-R_1^2) = 0$$

Eliminating $(R_2^2-R_1^2)$, we have

$$\frac{B}{R_1^2R_2^2} = -\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

which gives

$$B = -\frac{\rho\omega^2R_1^2R_2^2}{8}\left(\frac{3-2v}{1-v}\right) \qquad ...(17.65)$$

Substituting the value of B in equation (1), we have

$$0 = \frac{A}{2}+\frac{1}{R_1^2}\times\left[-\frac{\rho\omega^2R_1^2R_2^2}{8}\left(\frac{3-2v}{1-v}\right)\right]-\frac{\rho\omega^2R_1^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{A}{2}-\frac{\rho\omega^2R_2^2}{8}\left(\frac{3-2v}{1-v}\right)-\frac{\rho\omega^2R_1^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{A}{2}-\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2)$$

which gives

$$A = \frac{\rho\omega^2}{4}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2) \qquad ...(17.66)$$

Now Substituting the values of A and B in equation (17.56), we get the expression for the radial stress as

$$\sigma_r = \frac{1}{2}\times\frac{\rho\omega^2}{4}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2)+\frac{1}{r^2}\times\left[-\frac{\rho\omega^2R_1^2R_2^2}{8}\left(\frac{3-2v}{1-v}\right)\right]$$

$$-\frac{\rho r^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2)-\frac{\rho\omega^2R_1^2R_2^2}{8r^2}\left(\frac{3-2v}{1-v}\right)-\frac{\rho r^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[R_1^2+R_2^2-\frac{R_1^2R_2^2}{r^2}-r^2\right] \qquad ...(17.67)$$

This is the required expression for the radial stress for a long rotating hollow cylinder.

Substituting the values of A and B in equation (17.57), we get the expression for the hoop stress as

$$\begin{aligned}\sigma_h &= \frac{1}{2}\times\frac{\rho\omega^2}{4}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2)-\frac{1}{r^2}\times\left[-\frac{\rho\omega^2 R_1^2 R_2^2}{8}\left(\frac{3-2v}{1-v}\right)\right] \\ &\qquad -\frac{\rho r^2\omega^2}{8}\left(\frac{1+2v}{1-v}\right) \\ &= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2)+\frac{\rho\omega^2 R_1^2 R_2^2}{8r^2}\left(\frac{3-2v}{1-v}\right)-\frac{\rho r^2\omega^2}{8}\left(\frac{1+2v}{1-v}\right) \\ &= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[R_1^2+R_2^2+\frac{R_1^2R_2^2}{r^2}\right]-\frac{\rho\omega^2 r^2}{8}\left(\frac{1+2v}{1-v}\right) \qquad \text{...(17.68)}\end{aligned}$$

This is the required expression for the hoop stress for a long rotating hollow cylinder.

Maximum hoop stress

Equation (17.68) suggests that the hoop stress is maximum where r is minimum, that is, at $r = R_1$. Putting $r = R_1$ in equation (17.68), we obtain the value of the maximum hoop stress for the hollow cylinder as

$$\begin{aligned}\sigma_{h_{\max}} &= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[R_1^2+R_2^2+\frac{R_1^2R_2^2}{R_1^2}\right]-\frac{\rho\omega^2R_1^2}{8}\left(\frac{1+2v}{1-2v}\right) \\ &= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2+2R_2^2)-\frac{\rho\omega^2R_1^2}{8}\left(\frac{1+2v}{1-v}\right) \qquad \text{...(17.69)}\end{aligned}$$

Maximum radial stress

To find the location of the maximum radial stress, we differentiate equation (17.67) with respect to r and equate it to zero.

$$\frac{d\sigma_r}{dr} = 0$$

$$\frac{d}{dr}\left[\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left(R_1^2+R_2^2-\frac{R_1^2R_2^2}{r^2}-r^2\right)\right] = 0$$

$$\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[0+0-\frac{R_1^2R_2^2\times(-2)}{r^3}-2r\right] = 0$$

or

$$\frac{2R_1^2R_2^2}{r^3}-2r = 0$$

which gives

$$r = \sqrt{R_1R_2} \qquad \text{...(17.70)}$$

Hence, the radial stress is maximum at $r = \sqrt{R_1 R_2}$. Substituting this value in equation (17.67), we obtain the maximum value of the radial stress for a hollow cylinder as

$$\sigma_{r_{max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left(R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{R_1 R_2} - R_1 R_2\right)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2 + R_2^2 - 2R_1 R_2)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_2 - R_1)^2 \qquad ...(17.71)$$

This is the required expression for the maximum radial stress for a hollow cylinder.

Example 17.16

Determine the maximum hoop stress in a long cast iron solid cylinder of diameter 400 mm, which rotates at 2000 rpm about its axis. It weighs 7200 kg/m^3 and the Poisson's ratio is 0.3.

Solution: Given,

Radius of the solid cylinder, $R = \frac{400}{2}$ mm

$= 200 \times 10^{-3}$ m

Rotational speed, N = 2000 rpm

Density of the cylinder material,

ρ = 7200 kg/m^3

Poisson's ratio, v = 0.3

The angular speed of the cylinder is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 2000}{60}$$

$= 209.44$ rad/s

The maximum hoop stress is obtained using equation (17.63) as

$$\sigma_{h_{max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{7200 \times (209.44)^2 \times (200 \times 10^{-3})^2}{8} \times \left(\frac{3 - 2\times 0.3}{1-0.3}\right)$$

$= 5.41 \times 10^6$ Pa

$= 5.41$ MPa **Ans.**

Example 17.17

Calculate the maximum hoop stress in a long hollow cylinder of inside diameter 40 mm and outside diameter 200 mm rotating at 3000 rpm. The density of the cylinder material is 7800 kg/m^3 and the Poisson's ratio is 0.25.

Solution: Given,

Inside radius of the hollow cylinder, $R_1 = \dfrac{40}{2}$ mm

$= 20 \times 10^{-3}$ m

Outside radius of the hollow cylinder, $R_2 = \dfrac{200}{2}$ mm

$= 100 \times 10^{-3}$ m

Rotational speed, $N = 3000$ rpm

Density of the cylinder material, $\rho = 7800$ kg/m^3

Poisson's ratio, $v = 0.25$

The angular speed of the cylinder is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 3000}{60}$$

$$= 314.16 \text{ rad/s}$$

The maximum hoop stress is obtained using equation (17.69) as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2 + 2R_2^2) - \frac{\rho\omega^2 R_1^2}{8}\left(\frac{1+2v}{1-v}\right)$$

$$= \frac{7800 \times (314.16)^2}{8} \times \left(\frac{3-2\times 0.25}{1-0.25}\right) \times \left[(20\times 10^{-3})^2 + 2\times(100\times 10^{-3})^{-2}\right]$$

$$- \frac{7800\times(314.16)^2\times(20\times 10^{-3})^2}{8} \times \left(\frac{1+2\times 0.25}{1-0.25}\right)$$

$= 6543578.3 - 76983.3$

$= 6466595$ Pa

$= 6.46$ MPa **Ans.**

Example 17.18

Compare the peripheral velocities for the same maximum intensity of stress of (*a*) a solid cylinder (*b*) a solid thin disc and (*c*) a thin ring, if they are made of the same material. Take velocity of the ring as unity and the Poisson's ratio 0.3.

Solution:

For a solid cylinder

The maximum hoop stress is

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right) \qquad \text{(using equation (17.64))}$$

Since the peripheral velocity is

$$V = \omega R$$

Hence, the hoop stress equation for the solid cylinder is

$$\sigma_{h_{\max}} = \frac{\rho V_1^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{\rho V_1^2}{8} \times \left(\frac{3-2\times 0.3}{1-0.3}\right)$$

$$= 0.428\ \rho\ V_1^2$$

or

$$V_1^2 = \frac{\sigma_{h_{\max}}}{0.428\rho}$$

$$= 2.33 \times \frac{\sigma_{h_{\max}}}{\rho} \qquad ...(1)$$

For a solid thin disc

The maximum hoop stress is

$$\sigma_{h_{\max}} = \frac{\rho\omega^2 R^2(3+v)}{8} \qquad \text{(using equation (17.25))}$$

$$= \frac{\rho V_2^2\,(3+v)}{8} \qquad (\text{as } V_2 = \omega R)$$

$$= \frac{\rho V_2^2 \times (3+0.3)}{8}$$

$$= 0.4125\ \rho\ V_2^2$$

or

$$V_2^2 = \frac{\sigma_{h_{\max}}}{0.4125\rho}$$

$$= 2.42 \times \frac{\sigma_{h_{\max}}}{\rho} \qquad ...(2)$$

For a thin ring

The maximum hoop stress is

$$\sigma_{h_{\max}} = \rho\omega^2 R^2 \qquad \text{(using equation (17.5))}$$

$$= \rho V_3^2 \qquad (\text{as } V_3 = \omega R)$$

or
$$V_3^2 = \frac{\sigma_{h_{\max}}}{\rho} \qquad ...(3)$$

Hence,

$$V_1^2 : V_2^2 : V_3^2 = 2.33 : 2.42 : 1$$

which gives

$$V_1 : V_2 : V_3 = 1.526 : 1.556 : 1$$ **Ans.**

Example 17.19

A long solid cylinder of diameter 500 mm is rotating at 3500 rpm. Taking Poisson's ratio as 0.3 and the density of the cylinder material to be 7500 kg/m^3, find (*a*) the maximum stress developed in the cylinder and (*b*) plot the distribution of the hoop and radial stresses along the radius of the cylinder.

Solution: Given,

Radius of the cylinder, $R = \dfrac{500}{2}$ mm $= 250 \times 10^{-3}$ m

Rotational speed, $N = 3500$ rpm

Poisson's ratio, $v = 0.3$

Density of the cylinder material, $\rho = 7500$ kg/m^3

The angular speed of the cylinder is obtained as

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 3500}{60} = 366.52 \text{ rad/s}$$

The maximum hoop and radial stresses are equal and are given by equation (17.64) as

$$\sigma_{h_{\max}} = \sigma_{r_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{7500 \times (366.52)^2 \times (250 \times 10^{-3})^2}{8} \times \left(\frac{3 - 2\times 0.3}{1-0.3}\right)$$

$$= 26.98 \times 10^6 \text{ Pa} = 26.98 \text{ MPa}$$ **Ans.**

Distribution of the hoop stress

The hoop stress is given by equation (17.61) as

$$\sigma_h = \frac{\rho\omega^2 R^2}{8}\left(\frac{3-2v}{1-v}\right) - \frac{\rho\omega^2 r^2}{8}\left(\frac{1+2v}{1-v}\right)$$

Now we select different values of the radius r and determine the corresponding hoop stresses using the above equation. The distribution of the hoop stresses are shown in Table 17.5.

Table 17.5 Distribution of the hoop stress

r (mm)	0	50	100	150	200	250
σ_h (MPa)	26.98	26.26	24.10	20.50	15.47	8.99

Distribution of the radial stress

The radial stress is given by equation (17.60) as

$$\sigma_r = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R^2 - r^2)$$

The values of the radial stresses corresponding to the selected values of the radius r are determined using the above equation, which are shown in Table 17.6.

Table 17.6 Distribution of the radial stress

r (mm)	0	50	100	150	200	250
σ_r (MPa)	26.98	25.90	22.67	17.27	9.71	0

Plotting of the hoop and radial stresses

The values of the radius are plotted on x-axis and the values of the hoop and radial stresses on y-axis. The resulting curves for the two stresses are shown in Fig. 17.8.

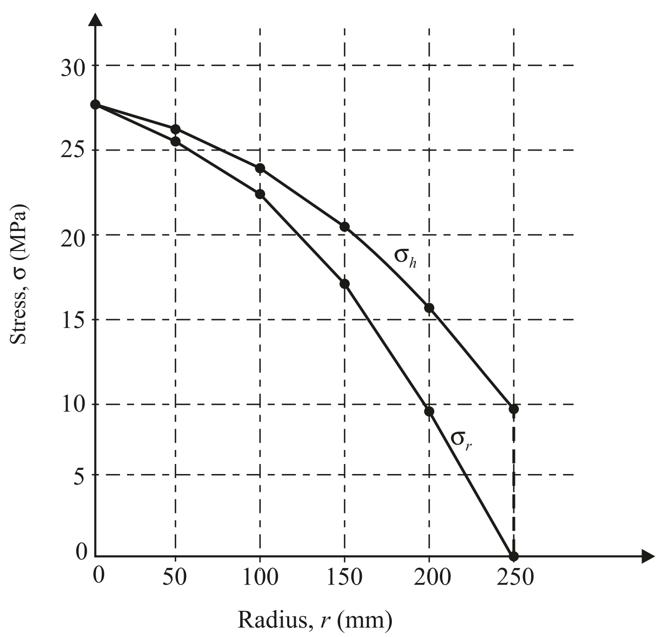


Fig. 17.8

Example 17.20

A long hollow cast iron cylinder of inside diameter 50 mm and outside diameter 300 mm is rotating at 6000 rpm. Taking Poisson's ratio as 0.3 and density of the cylinder material to be 7200 kg/m^3, find (*a*) the maximum hoop stress (*b*) the radius at which the radial stress is maximum (*c*) the maximum radial stress and (*d*) plot the distribution of the hoop and radial stresses along the radius of the cylinder.

Solution: Given,

$$\text{Inside radius of the hollow cylinder, } R_1 = \frac{50}{2} \text{ mm}$$

$$= 25 \times 10^{-3} \text{ m}$$

$$\text{Outside radius of the hollow cylinder, } R_2 = \frac{300}{2} \text{ mm}$$

$$= 150 \times 10^{-3} \text{ m}$$

Poisson's ratio, v = 0.3

Density of the cylinder material, ρ = 7200 kg/m^3

Rotational speed, N = 3000

The angular speed of the cylinder is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 6000}{60}$$

$$= 628.32 \text{ rad/s}$$

(*a*) The maximum hoop stress is obtained using equation (17.69) as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2 + 2R_2^2) - \frac{\rho\omega^2 R_1^2}{8}\left(\frac{1+2v}{1-v}\right)$$

$$= \frac{7200 \times (628.32)^2}{8} \times \left(\frac{3-2\times 0.3}{1-0.3}\right) \times [(25\times 10^{-3})^2 + 2\times(150\times 10^{-3})^2]$$

$$- \frac{7200\times(628.32)^2}{8} \times \left(25\times 10^{-3}\right)^2 \times \left(\frac{1+2\times 0.3}{1-0.3}\right)$$

$$= 55580232 - 507582.03$$

$$= 55072650 \text{ Pa}$$

$$= 55.07 \text{ MPa}$$ **Ans.**

(b) The radius at which the radial stress is maximum, is given by equation (17.70) as

$$r = \sqrt{R_1 \times R_2}$$

$$= \sqrt{25 \times 10^{-3} \times 150 \times 10^{-3}}$$

$$= 0.0612 \text{ m}$$

$$= 61.2 \text{ mm}$$ **Ans.**

(c) The maximum radial stress is given by equation (17.71) as

$$\sigma_{r_{max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_2 - R_1)^2$$

$$= \frac{7200 \times (628.32)^2}{8} \times \left(\frac{3-2\times 0.3}{1-0.3}\right) \times (150\times 10^{-3} - 25 \times 10^{-3})^2$$

$$= 19034326 \text{ Pa}$$

$$= 19.03 \text{ MPa}$$ **Ans.**

(d) **Distribution of the hoop stress**

The hoop stress is given by equation (17.68) as

$$\sigma_h = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[R_1^2 + R_2^2 + \frac{R_1^2 R_2^2}{r^2}\right] - \frac{\rho\omega^2 r^2}{8}\left(\frac{1+2v}{1-v}\right)$$

Now we select different values of the radius r and determine the corresponding hoop stresses using the above equation. The distribution of the hoop stresses is shown in Table 17.7.

Table 17.7 Distribution of the hoop stress

r (mm)	25	50	75	100	125	150
σ_h (MPa)	55.07	33.00	26.65	21.76	16.58	10.66

Distribution of the radial stress

The radial stress is given by equation (17.67) as

$$\sigma_r = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2\right]$$

The values of the radial stresses corresponding to the selected values of the radius r are determined using the above equation, which are shown in Table 17.8.

Table 17.8 Distribution of the radial stress

r (mm)	25	50	75	100	125	150
σ_r (MPa)	0	18.27	18.27	14.28	8.04	0

Plotting of the hoop and radial stresses

The values of the radius are plotted on x-axis and the values of the hoop and radial stresses on y-axis. The resulting curves for the two stresses are shown in Fig. 17.9.

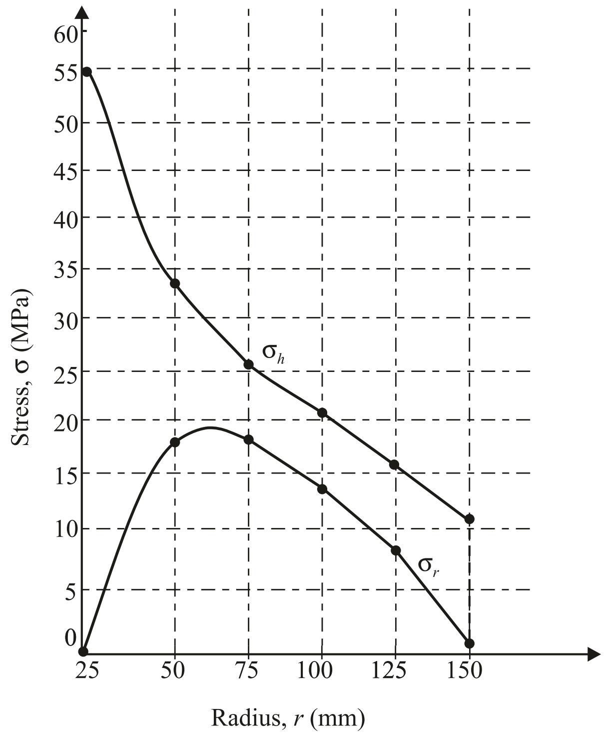


Fig. 17.9

SHORT ANSWER QUESTIONS

1. What are the forces that act on a rotating ring?
2. How do hoop stress produce?
3. Name the stresses that act on a rotating thin disc.
4. What is meant by a disc of uniform strength?
5. How does a rotating disc of uniform thickness differ from a rotating disc of uniform strength?
6. What are the stresses that act on a rotating long cylinder?

MULTIPLE CHOICE QUESTIONS

1. The rotational speed N (rpm) and the angular velocity ω (rad/s) are related as

(*a*) $N = \dfrac{2\pi\omega}{60}$ (*b*) $\omega = \dfrac{2\pi N}{60}$ (*c*) $\omega = \dfrac{\pi N}{60}$ (*d*) $\omega = \dfrac{3\pi N}{60}$.

2. The expression for the hoop stress for a thin rotating ring is given as (ρ = Density, ω = Angular speed and r = Radius)

(*a*) $\rho\omega r^2$ (*b*) $\rho\omega^2 r$ (*c*) $\rho\omega^2 r^2$ (*d*) $\rho^2\omega^2 r^2$.

3. The expression for the hoop stress for a solid rotating disc at any radial distance r is given as (ρ = Density, ω = Angular speed, v = Poisson's ratio, R = Radius).

(*a*) $\dfrac{\rho\omega(3+v)}{8}(R^2 - r^2)$ (*b*) $\dfrac{\rho\omega^2(1+v)}{8}(R^2 - r^2)$

(*c*) $\dfrac{\rho\omega^2}{8}[(3+v)R^2 - (1+3v)r^2]$ (*d*) $\dfrac{\rho\omega^2}{8}[(3+v)R^2 - (1+3v)r^2]$.

4. The expression for the radial stress for a solid rotating disc at any radial distance r is given as (ρ = Density, ω = Angular speed, v = Poisson's Ratio, R = Radius).

(*a*) $\dfrac{\rho\omega(3+v)}{8}(R^2 - r^2)$ (*b*) $\dfrac{\rho\omega^2(1+v)}{8}(R^2 - r^2)$

(*c*) $\dfrac{\rho\omega^2}{8}[(3+v)R^2 - (1+3v)r^2]$ (*d*) $\dfrac{\rho\omega^2}{8}[(3+v)R^2 - (1+3v)r^2]$.

5. The hoop and radial stresses at the centre of a solid rotating disc are expressed as (ρ = Density, ω = Angular speed, R = Radius and v = Poisson's ratio)

(*a*) $\dfrac{\rho\omega^2 R^2(3+v)}{8}, \dfrac{\rho\omega^2 R^2(1+v)}{8}$ (*b*) $\dfrac{\rho\omega^2 R^2(3+v)}{8}, \dfrac{\rho\omega^2 R^2(3+v)}{8}$

(*c*) $\dfrac{\rho\omega^2 R^2(1-v)}{4}, \dfrac{\rho\omega^2 R^2(3+v)}{8}$ (*d*) $\dfrac{\rho\omega^2 R^2(1-v)}{4}, \dfrac{\rho\omega^2 R^2(1+v)}{8}$.

6. The hoop stress at the outer radius of a solid rotating disc is (ρ = Density, ω = Angular speed, r = Radius and v = Poisson's ratio)

(*a*) $\dfrac{\rho\omega^2 R^2(3+v)}{8}$ (*b*) $\dfrac{\rho\omega^2 R^2(1+v)}{4}$ (*c*) $\dfrac{\rho\omega^2 R^2(1-v)}{4}$ (*d*) $\dfrac{\rho\omega^2 R^2(1+v)}{8}$.

7. The maximum radial stress in case of a hollow disc occurs at a radial distance equal to (R_1 = Inner radius and R_2 = Outer radius)

(*a*) $\sqrt{R_1}$ (*b*) $\sqrt{R_2}$ (*c*) $\sqrt{R_1 R_2}$ (*d*) $\sqrt{2R_1 R_2}$.

8. The maximum value of the radial stress for a hollow disc is (ρ = density, ω = Angular speed, v = Poisson's ratio, R_2 = Outer radius and R_1 = Inner radius)

(*a*) $\dfrac{(1+v)\rho\omega^2}{4}(R_2 - R_1)^2$ (*b*) $\dfrac{(3+v)\rho\omega^2}{8}(R_2^2 - R_1^2)^2$

(*c*) $\dfrac{(1+v)\rho\omega^2}{8}(R_2 - R_1)^2$ (*d*) $\dfrac{(3+v)\rho\omega^2}{8}(R_2^2 - R_1^2)^2$.

9. The expressions for the hoop and radial stresses in a rotating disc with a central pin hole are (ρ = Density, v = Poisson's ratio, ω = Angular speed and R = Radius)

(*a*) $\dfrac{(3+v)}{8}\rho\omega^2 R^2, \dfrac{(3+v)\rho\omega^2 R^2}{4}$ (*b*) $\dfrac{(1+v)\rho\omega^2 R^2}{4}, \dfrac{(3+v)\rho\omega^2 R^2}{8}$

(*c*) $\dfrac{(2+v)\rho\omega^2 R^2}{4}, \dfrac{(3-v)\rho\omega^2 R^2}{8}$ (*d*) $\dfrac{(3+v)\rho\omega^2 R^2}{4}, \dfrac{(3+v)\rho\omega^2 R^2}{8}$.

10. Consider the following statements :

1. The radial stress is zero at both inner and outer radius of a hollow rotating disc.
2. Both radial and hoop stresses at the centre of a solid rotating disc are maximum and equal.
3. The radial stress at the outer radius of a solid rotating disc is zero.
4. The maximum hoop stress for a rotating disc with a central pin hole is twice the maximum hoop stress for a rotating solid disc.

Of these statements:

(*a*) 1 and 2 are true (*b*) 1, 2 and 4 are true

(*c*) 2 and 3 are true (*d*) 1, 2, 3 and 4 are true.

11. Consider the following statements about a disc of uniform strength :

1. The hoop and radial stresses do not vary along the radius of the disc.
2. It has maximum thickness at the centre.
3. It has uniform thickness throughout.
4. Its thickness decreases gradually towards its outer edge.

Of these statements :

(*a*) 1 alone is true (*b*) 1, 2 and 4 are true

(*c*) 1 and 2 are true (*d*) 1 and 3 are true.

12. Consider the following statements about a rotating long cylinder :

1. It involves three stresses, namely hoop, radial and axial.
2. The longitudinal strain is constant.
3. All the stresses are principal stresses.
4. The radial stress is zero at the surface of the cylinder

Of these statements :

(*a*) 1 and 2 are true (*b*) 1, 3 and 4 are true

(*c*) 2 and 4 are true (*d*) 1, 2, 3 and 4 are true.

13. The maximum radial stress in case of a solid long rotating cylinder is (ρ = Density, ω = Angular speed, R = Radius, v = Poisson's ratio)

(*a*) $\dfrac{\rho\omega^2 R^2}{4}\left(\dfrac{3-v}{1-v}\right)$ (*b*) $\dfrac{\rho\omega^2 R^2}{4}\left(\dfrac{1-v}{3-2v}\right)$

(*c*) $\dfrac{\rho\omega^2 R^2}{8}\left(\dfrac{3-2v}{1-v}\right)$ (*d*) $\dfrac{\rho\omega^2 R^2}{8}\left(\dfrac{1-v}{3-2v}\right)$.

14. The maximum hoop stress in case of a solid long rotating cylinder is (ρ = Density, ω = Angular speed, R = Radius, v = Poisson's ratio)

(*a*) $\dfrac{\rho\omega^2 R^2}{8}\left(\dfrac{1-v}{3-2v}\right)$ (*b*) $\dfrac{\rho\omega^2 R^2}{8}\left(\dfrac{3-2v}{1-v}\right)$

(*c*) $\dfrac{\rho\omega^2 R^2}{4}\left(\dfrac{3-2v}{1-v}\right)$ (*d*) $\dfrac{\rho\omega^2 R^2}{4}\left(\dfrac{1-v}{3-2v}\right)$.

15. The maximum radial stress in case of a hollow long rotating cylinder is

(*a*) $\dfrac{\rho\omega^2}{4}\left(\dfrac{3-2v}{1-v}\right)(R_2-R_1)^2$ (*b*) $\dfrac{\rho\omega^2}{8}\left(\dfrac{1-v}{3-2v}\right)(R_2-R_1)^2$

(*c*) $\dfrac{\rho\omega^2}{8}\left(\dfrac{3-2v}{1-v}\right)(R_2-R_1)^2$ (*d*) $\dfrac{\rho\omega^2}{8}\left(\dfrac{3-2v}{1-v}\right)(R_2^2-R_1^2)$.

ANSWERS

1. (*b*) **2.** (*c*) **3.** (*c*) **4.** (*a*) **5.** (*b*) **6.** (*c*) **7.** (*c*) **8.** (*b*)

9. (*d*) **10.** (*d*) **11.** (*b*) **12.** (*d*) **13.** (*c*) **14.** (*b*) **15.** (*c*)

EXERCISES

1. A uniform thin disc of diameter 600 mm has a central hole of diameter 100 mm. Determine the maximum hoop stress induced in the disc, if the maximum radial stress is not to exceed 15 MPa. Take Poisson's ratio as 0.25. (*Ans.* 43.48 MPa).

2. A uniform thin disc of diameter 700 mm has a pin hole at the centre. Determine the maximum hoop stress induced in the disc, if it rotates at 3000 rpm. Take Poisson's ratio as 0.3 and the density of the disc material to be 7800 kg/m^3. (*Ans.* 77.8 MPa).

3. A hollow steel disc of uniform thickness has outer diameter 500 mm and inner diameter 200 mm and it rotates at 3000 rpm. Taking Poisson's ratio as 0.3 and the density of the disc material to be 7800 kg/m^3, find (*a*) the maximum hoop and radial stresses induced in the disc and (*b*) the radius at which the radial stress is maximum.

(*Ans.* (*a*) 41.04 MPa, 7.14 MPa (*b*) 158.11 mm).

4. A thin uniform steel disc of diameter 300 mm rotates at 4000 rpm. Calculate the maximum hoop stress induced in the disc and plot the distribution of the hoop and radial stresses along the radius of the disc. Take Poisson's ratio as 0.25 and the density of the disc material equals to 7800 kg/m^3. (*Ans.* 12.5 MPa).

5. Derive the expression for the hoop stress at the outer radius of a solid disc of radius R, which rotates at ω rad/s and is made of material having a density ρ and Poisson's ratio v. Hence, prove that the hoop stress reduces to zero, if the Poisson's ratio tends to unity.

6. A long thick cylinder of inner diameter 150 mm and outer diameter 450 mm rotates at 4000 rpm. Find the hoop stresses at its inner and outer surfaces. Take the Poisson's ratio of 0.3 and the density of the cylinder material as 7470 kg/m^3.

(*Ans.* 57.9 MPa, 11.9 MPa).

7. A steam turbine rotor is 150 mm diameter below the blade ring and 5 mm thick, and runs at 35,000 rpm. What is the thickness of the rotor at a radius of 50 mm and at the centre ? Take the allowable stress of 150 MPa and the density of the rotor material to be 7800 kg/m^3. Assume uniform strength condition.

(*Ans.* 14.9 mm, 35.7 mm).

8. A long solid steel cylinder of diameter 400 mm is rotating at 5000 rpm. Taking Poisson's ratio as 0.3 and the density of the material of the cylinder to be 7800 kg/m^3, find (*a*) the maximum stress developed in the cylinder and (*b*) plot the distribution of the hoop and radial stresses along the radius of the cylinder.

(*Ans.* 36.66 MPa).

9. A long hollow cast iron cylinder of inside diameter 60 mm and outside diameter 300 mm is rotating at 3600 rpm. Taking Poisson's ratio as 0.3 and the density of the material of the cylinder to be 7200 kg/m^3, find the following parameters :

(*a*) the maximum hoop stress

(*b*) the radius at which the radial stress is maximum, and

(*c*) the maximum radial stress.

(*Ans.* (*a*) 19.87 MPa (*b*) 67.08 mm (*c*) 6.31 MPa).

10. Prove that the maximum hoop stress at the centre of a long rotating solid cylinder is given as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2 R^2}{8}\left(\frac{3-2v}{1-v}\right)$$

where ρ = Density of the cylinder material

ω = Angular speed of the cylinder

R = Mean radius of the cylinder

v = Poisson's ratio.

❑❑❑