D. K. Singh

Strength of Materials

4th Edition

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Dedicated
to my
parents

Preface to the Fourth Edition

The fourth edition of the book is presented before you. The new edition includes four new chapters and has increased by about 300 pages. Now the book covers the subject in totality and takes care of both the basics and the advanced topics of the strength of materials. Besides including four new chapters, three existing chapters have been rewritten to make them more relevant and meaningful. This has further increased the scope of the book.

I thankfully acknowledge the support and encouragement from two of my best ever friends, Jai Shankar and Shakil Ahmad. My parents and in-laws especially Babuji and Mummi are gratefully remembered for their blessings. My lovely wife Alka and daughters Shivangi and Shalvi deserve special mention for giving their affectionate support that keeps me always elevated and in high spirit.

Finally, I appreciate the response from the readers of the book, which has helped tremendously to further improve the book. I hope readers will continue to give their valuable suggestions and remarks in future also. I promise to honour their suggestions.

New Delhi **D. K. Singh**

Preface to the Third Edition

In this revised and augmented third edition, many new solved problems have been added especially in the chapters 1, 2, 5, 6, 7, 10 and 13. Simultaneously, a large number of model objective questions have also been added to test the understanding of the students from the viewpoints of competitive examinations. The biggest advantage of this book is its simplicity in expressing the subject matter in the most simplified manner. The material covered is so designed that any beginner can follow it easily, and get a complete picture of the subject after having gone through the material covered in the text.

I deeply appreciate the many comments and suggestions that I received from the users of the previous editions of this book. I may be contacted on my email for any further suggestion, comment or criticism.

NSIT, New Delhi **D. K. Singh**

Preface to the Second Edition

The thoroughly revised edition of the book 'Strength of Materials' is in your hand. This new edition has one more chapter on Mechanical Testing of Materials, which further increases the scope of the book, while retaining the flavour of the first edition. A sincere attempt has been made to make the book error-free, but still can't be claimed in totality.

I hope, readers will enjoy this new edition with the same spirit. Any suggestion for the improvement of the book will be gratefully acknowledged.

D. K. Singh

Preface to the First Edition

It gives me immense pleasure to present this book before you. There are many books on this subject and each claims the best one. I can't make such claim but can assure you that you will really find this book interesting and useful. There are thirteen chapters in all and each chapter has sufficient number of solved examples for easy understanding of the subject. A number of multiple choice questions has been added at the end of each chapter, making the book useful for competitive examinations.

I am thankful to all the people who helped in the compilation of this book including my publisher Mr. Sunil Saxena and Mr. Jai Raj Kapoor. Thanks are also due to my parents and in-laws for their continuous encouragement.

Finally, I want to thank my wife, Alka, for her continuing encouragement, support, and affection and my daughters, Shalu and Sheelu, for making the atmosphere lively and workable.

D.K. Singh

New Delhi
March 2007.

Strength of Materials

Fourth Edition

By **D. K. Singh**

What is new in the Fourth Edition?

- Four new chapters have been added (Chapter 14: Combined Loadings, Chapter 15: Unsymmetrical Bending and Shear Centre, Chapter 16: Fixed Beams and Chapter 17: Rotating Rings, Discs and Cylinders).
- Biographies of the famous personalities related to the concerned chapters have been included.
- Chapter 13 is renamed as Plane Trusses.
- Chapter 1 (Simple Stresses and Strains), Chapter 3 (Centroid and Moment of Inertia) and Chapter 10 (Theory of Elastic Failure) have been rewritten.
- A few new solved examples have been added in the chapters 1, 3 and 13.
- Few more Multiple Choice Questions have been added.
- The Fourth Edition has some advanced topics on the subject.
- The Fourth Edition is augmented by about 300 pages.
- Short Answer Questions have been added to every chapter of the book.
- The entire text has been thoroughly revised and updated to eliminate the possible errors left out in the previous editions of the book.
- The scope of the book has further increased.

Contents

About the Author

Dr. D. K. Singh is a Professor and Head of Mechanical Engineering at the Netaji Subhas University of Technology, New Delhi, India. He has an extremely rich experience of both teaching and research for nearly three decades. He has contributed over 45 papers to various journals of national and international repute. He has also participated and presented papers at several national and international conferences. He has also authored 12 books. He is a life member of the Indian Society for Technical Education (ISTE), New Delhi.

1 Simple Stresses and Strains

Thomas Young
(1773-1829)

Thomas Young, born on 13 July 1773, was an English physician, physicist and polymath. By the age of fourteen, he is said to be well acquainted with many languages like Latin, Greek, French, Italian, Hebrew, Arabic and Persian, and hence was nicknamed the English Leonardo. He made notable contributions to the fields of vision, light, solid mechanics, energy, philosophy, language, musical harmony and Egyptology. He discovered the cause of astigmatism, a vision defect, in 1801. The Young's modulus (E), an elastic constant and a measure of the rigidity of materials, is named after him. He is known for the wave theory of light and the double-slit experiment in which he established the transverse wave nature of light. He was also an Egyptologist who helped decipher the Rosetta stone. He developed the Young temperament, a method for tuning musical instruments.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- What is gauge length?
- Why do the stress-strain curves for ductile and brittle materials differ?
- Why is Poisson's ratio so important?
- How are the elastic constants related to each other?
- Why is the factor of safety an important design consideration?

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1.1 INTRODUCTION

The reaction of materials to the action of external forces is indicated by the mechanical properties of materials. These properties are primarily related to the elastic and plastic behaviour, as well as, the over-all strength and fracture characteristics of materials. The mechanical properties are structure-sensitive. This is in distinct contrast to most of the physical properties, which are structure-insensitives. Mechanical properties of materials play a vital role in the development of structures, machines and other products. These properties are evaluated by conducting destructive testing.

Mechanical properties, also sometimes known as engineering properties, include tensile strength, ductility, compressive strength, fatigue, creep and others. The deformations produced and the stresses induced in the material as a result of application of external loads are the important aspects of mechanical properties.

1.2 STRESS-STRAIN CURVES IN TENSION

Tensile strength is defined as the ability of a material to support or carry an applied axial load. The property enables a material to resist being pulled apart. Tensile strength of the material is tested by conducting tension test.

In the tension test, a certain length, known as gauge length (usually 50 mm) of the material is subjected to an axial load as shown in Fig. 1.1 (*a*). The diameter of the cross-section of the material to be tested is generally taken to be 12 mm. The behaviour of the material is shown in the form of a stress-strain curve in Fig. 1.2.

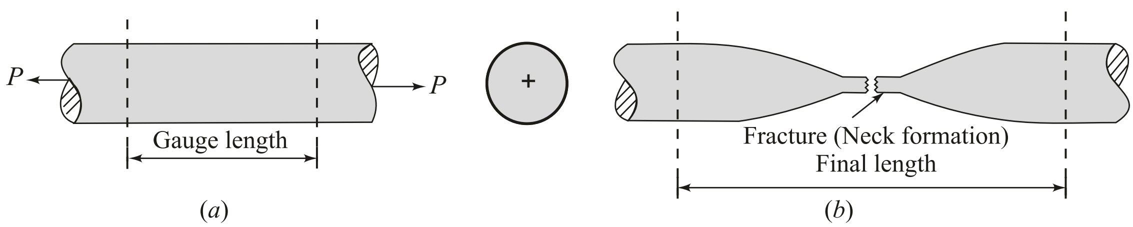


Fig. 1.1 A tension test.

As the load on the test specimen increases, the resisting force also increases. Once the load is removed, the material regains its original shape and size. Hooke's law is strictly valid under this condition, where stress is linearly proportional to strain. This point is known as limit of proportionality, *p*. During this period, the material regains its orignal conditions once the load is removed. Point '*e*' represents elastic limit. Upto '*e*' material recovers its original shape and size but the recovery process is negligibly slow as compared to limit of proportionality. For some materials, these two points *p* and *e* are almost identical, but in most of the cases, the elastic limit is slightly higher.

Elongation beyond the elastic limit becomes unrecoverable and the material tends to stay permanently deformed. This situation is known as plastic deformation. When the load is removed, the specimen retains a permanent change in shape. It means the external load has exceeded the resisting force. An engineer is usually interested in either the elastic or plastic response. Plasticity and elasticity are the properties of critical importance when evaluating a material for manufacturing.

When the elastic limit is exceeded, stress and strain do not increase in a proportional way. Rather increase in strain is more prominent as compared to increase in stress. For some materials, a stress value may be reached where additional strain occurs without further increase in stress. This point is known as yield point and the corresponding stress is called the yield stress. For mild steel (low carbon steel), there are two distinct yield points. The stress at which such plastic deformation first begins is called the upper yield point. Subsequent plastic deformation may occur at a lower stress, called the lower yield point. The upper yield point is very sensitive to rate of loading and accidental bending stress or irregularities in the specimen, so the lower yield point is used for design purposes. But it is the lower yield point which is usually common. As the load and hence the stress is further increased, it reaches its maximum value and then begins to decrease. The maximum stress is called the ultimate stress or tensile strength or ultimate tensile strength, U of the material.

If the specimen is loaded beyond its ultimate strength, neck formation starts. During this period, the cross-sectional are no longer remains uniform in the gage length rather it reduces drastically in the necked region. The stress drops further and the specimen finally fractures in the necked region at the point f as shown in the Fig. 1.2. The material is subjected to neck formation just before failure as is shown in Fig. 1.1(*b*). It is a typical 'cup and cone' type failure of material, where failure occurs on planes at 45° to the specimen axis. The stress at this point is known as fracture or breaking stress.

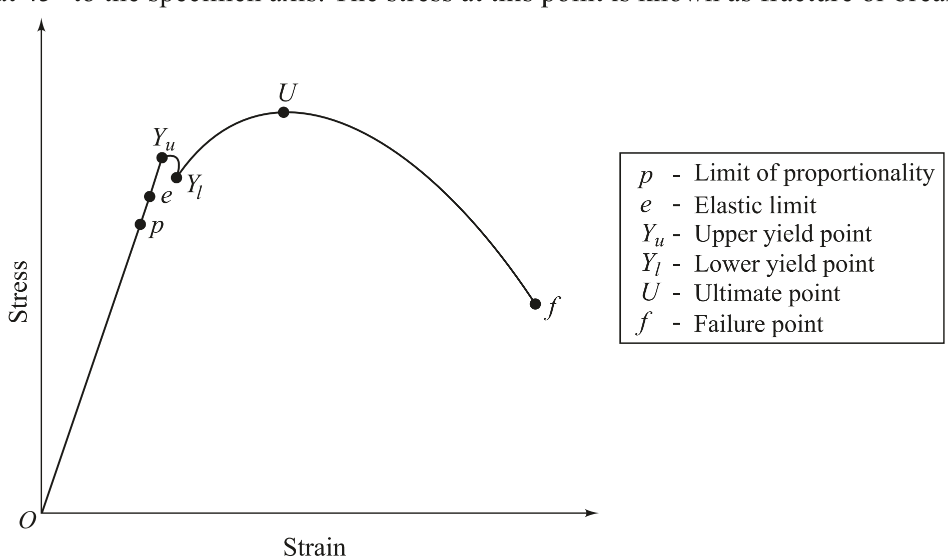


Fig. 1.2 An engineering stress-strain curve for ductile materials.

For most of the materials (for example cast iron), the yield point is not well defined. In such case, the elastic-to-plastic transition is not distinct and the yield point is determined through the use of offset. In the offset method, the strain usually known as offset strain is specified at 0.2% but the values of 0.1% or even 0.02% may also be used when small amounts of plastic deformation could lead to component failure. The point of intersection of a line drawn parallel to the elastic line with the stress-strain curve gives the position of yield point in such cases. If the applied stresses are kept below the 0.2% offset yield strength, the user can be guaranteed that any observed plastic deformation will be less than 0.2% of the original dimension (Fig. 1.3).

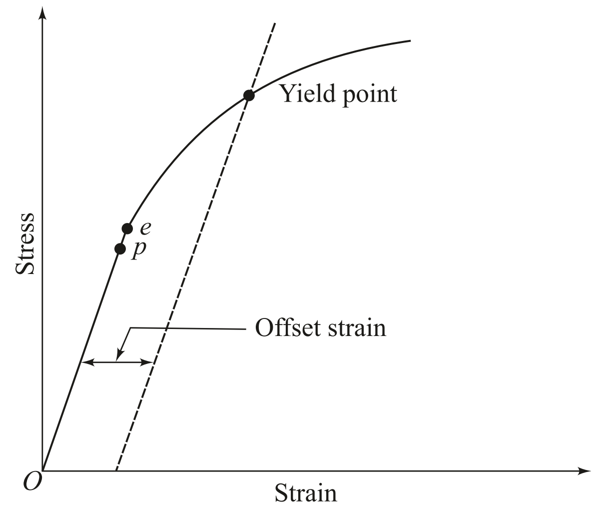


Fig. 1.3 A stress-strain curve for brittle materials.

The modulus of elasticity, also called elastic modulus, denoted by E may be determined as the slope of the linear portion of the stress-strain curve. It is defined as the ratio of longitudinal stress to longitudinal strain.

1.3 TRUE STRESS-STRAIN CURVE

Materials do not obey Hooke's law in plastic region. Fig. 1.2 has been drawn considering the original cross-sectional area of the test specimen. But it is not the actual case. The instantaneous area supporting the load is gradually decreasing as the specimen elongates. Thus, the engineering stress does not represent the actual stress or true stress. Similarly, the true strain differs from engineering strain. These are defined as follows:

$$\text{True stress} = \frac{\text{Load applied}}{\text{Instantaneous area of cross-section}} = \frac{P}{A_f} \qquad \text{... (1.1)}$$

where A_f is not necessarily the cross-sectional area at the point of failure.

$$\text{True strain} = ln\left(\frac{l_f}{l_o}\right) \qquad \text{... (1.2)}$$

where l_f = Instantaneous length

l_o = Original length

True strain is also known as natural or logarithmic strain. For small values of strain, the two strains are approximately equal. But they diverge rapidly as strain increases.

The true stress-strain curve is represented by the equation

$$\sigma = K \,.\, \in^n \qquad \text{... (1.3)}$$

where σ = Stress

$\in$ = Strain

K = A constant, known as strength coefficient

n = Strain-hardening exponent

$\neq 1$ for true stress-strain curve

$= 1$ for engineering stress-strain curve, where K is replaced by E, the modulus of elasticity.

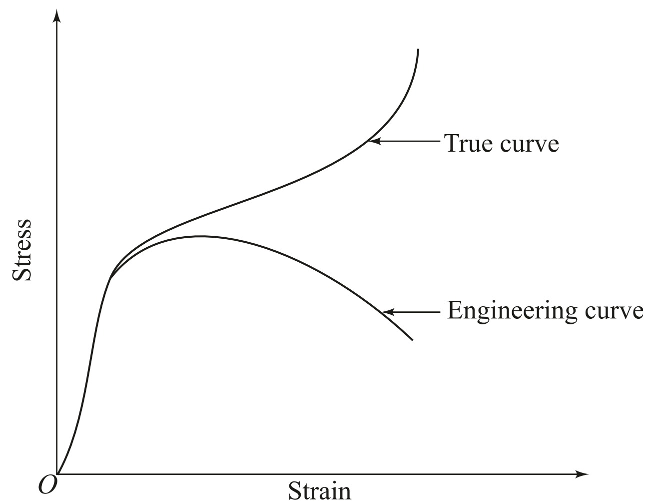


Fig. 1.4 A comparison between engineering and true stress-strain curves.

The higher value of n indicates that material can be stretched uniformly with relative ease before it begins to neck. This is an important consideration in forming operations, where work material is stretched beyond elastic region. An engineering curve and a true curve are compared in Fig. 1.4.

The specimen's necked region is subjected to three-dimensional tensile stresses. This state gives higher stress values than the actual true stress, hence the curve must be corrected downward as shown in Fig. 1.5.

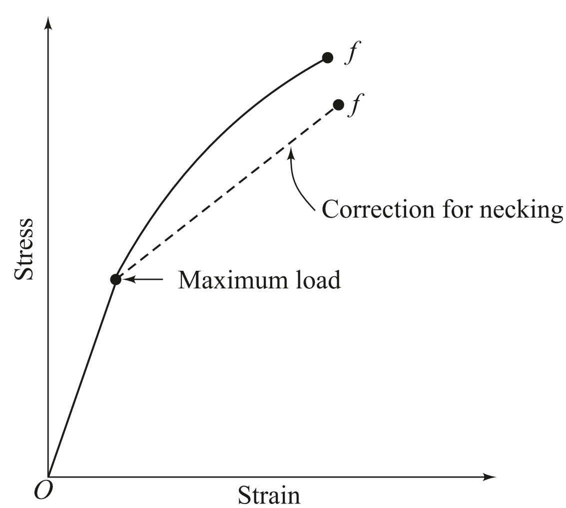


Fig. 1.5 Correction in the stress-strain curve for neck formation.

1.4 POISSON'S RATIO

When a bar is subjected to a tension test, there is an increase in its length in the direction of load applied, but at the same time, there is a contraction of the lateral dimensions in the transverse direction (direction perpendicular to the direction of load applied). These changes in dimensions are related to each other by Poisson's ratio ν, defined as

$$\nu = -\frac{\text{Lateral strain}}{\text{Longitudinal strain}} \qquad \text{... (1.4)}$$

Negative sign indicates that if longitudinal strain increases, lateral strain decreases. For most metals, v is about 0.33, and for cork it is close to zero, which makes cork so useful for stoppers on bottles. In some extreme cases, values like 0.1 (cement concrete) and 0.5 (rubber) have been observed. Its value is same in tension as well as in compression.

1.5 DUCTILITY

The ductility of a material enables it to be easily long stretched without failure and drawn into thin wires. A higher value of ductility indicates that a material can sustain large tensile strain upto the point of rupture. A property very closely related to ductility is malleability, which defines a material's ability to be hammered out in thin sheets. A typical example of a malleable material is lead, which is used in plumbing work for weatherproof seal.

The ductility is usually measured in terms of percentage elongation, which is expressed as

$$\text{Percentage elongation} = \frac{l_f - l_o}{l_o} \times 100 \qquad \text{... (1.5)}$$

where l_f = Final length

l_o = Original length

The ductility is also measured in terms of reduction in area (R_A), which is expressed as

$$R_A = \frac{A_o - A_f}{A_o} \times 100 \qquad \text{... (1.6)}$$

where A_o and A_f represent the original and the final cross-sectional area respectively.

Values for two measures are high in a ductile material and are nil in a non-ductile material. Chalk has zero ductility because it does not stretch at all. R_A can range from 0% (brittle) to 100% (extremely plastic). An arbitrary strain of 0.05 inch/inch is frequently taken as the dividing line between brittle and ductile materials. For relatively ductile materials, the breaking strength is less than the ultimate tensile strength, and necking precedes fracture. For a brittle material, fracture usually occurs before necking, and possibly before the onset of plastic flow.

1.6 ELONGATION PRODUCED IN A TEST SPECIMEN

$$\text{Stress } (\sigma) = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$$

$$\text{Strain } (\in) = \frac{\text{Change in length (elongation)}}{\text{Original length}} = \frac{dl}{l_o}$$

Modulus of elasticity, $$E = \frac{\sigma}{\in} = \left(\frac{P}{A}\right) \Big/ \left(\frac{dl}{l_o}\right) = \frac{Pl_o}{Adl}$$

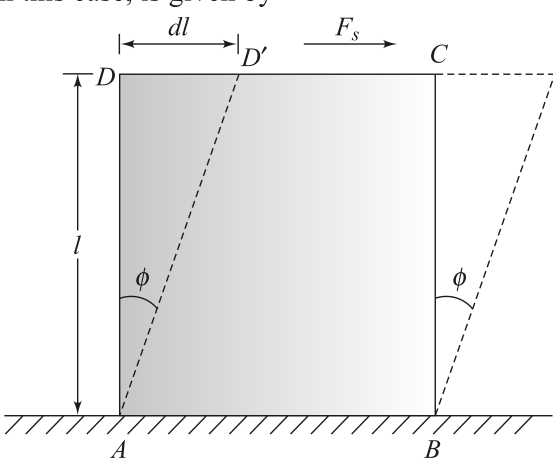
$$dl = \frac{Pl_o}{AE}$$

1.7 SHEAR STRESS AND SHEAR STRAIN

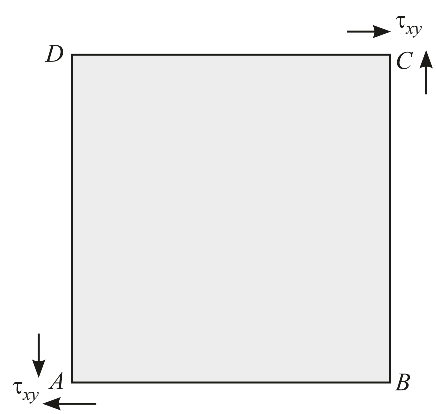
The shear stress is produced as a result of the shear force applied tangential to the surface of a body.

Mathematically, $$\text{Shear stress } \tau = \frac{\text{Shear force}}{\text{Shear area}} = \frac{F_s}{A_s} \qquad \text{... (1.7)}$$

The shear strain is produced by the shear stress, and is measured by the change in the angle, but causes no change in the volume of a body. In Fig. 1.6 (*a*), a block *ABCD* is subjected to a shear force F_s on its upper face *CD,* while the lower face *AB* remains fixed. Due to application of F_s, the body deforms and takes new shape as $ABC'D'$ making an angle ϕ with the vertical. At the same time, the shear stresses are produced on the faces *CD* and *AB*, and complementary shear stresses of equal value but of opposite effect are set up on the faces *AD* and *BC* in order to prevent the rotation of the body. For *xy*-plane, both shear stresses are denoted by the symbol τ_{xy} as shown in Fig. 1.6 (*b*). The shear strain, in this case, is given by



(*a*) A body subjected to a shear force F_s



(*b*) Complementary shear stress on faces *AD* and *BC*

Fig. 1.6

$$\tan\phi \simeq \phi = \frac{DD'}{AD} = \frac{dl}{l} \qquad \text{... (1.8)}$$

The modulus of rigidity or the shear modulus (*G*) is defined as a ratio of the shear stress to the shear strain.

$$G = \frac{\tau}{\phi} \qquad \text{... (1.9)}$$

1.8 VOLUMETRIC STRAIN

The volumetric strain results due to change in the volume of a body. It is a ratio of the change in volume to the original volume, and is also called dilatation.

Mathematically, Volumetric strain, $\in_V = \dfrac{dV}{V_0}$ ... (1.10)

where V_0 = Original volume

dV = Change in volume

The volumetric strain, for a body subjected to three mutually perpendicular normal stresses, is given as

$$\in_V = \in_x + \in_y + \in_z \qquad \text{... (1.11)}$$

where $\in_x$, $\in_y$ and $\in_z$ are the strains produced in x, y and z directions respectively, and are given as

$$\left.\begin{aligned} \in_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\ \in_y &= \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} \\ \text{and} \quad \in_z &= \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \end{aligned}\right\} \qquad \text{... (1.12)}$$

where σ_x, σ_y and σ_z are the normal stresses in x, y and z directions respectively.

where V = Poisson's ratio

E = Modulus of elasticity

The equation (1.11), on substituting equation (1.12), changes to

$$\in_V = \frac{\sigma_x + \sigma_y + \sigma_z}{E}(1 - 2\nu) \qquad \text{... (1.13)}$$

For $\sigma_x = \sigma_y = \sigma_z = \sigma$, the equation (1.13) reduces to

$$\in_V = \frac{3\sigma}{E}(1 - 2\nu) \qquad \text{... (1.14)}$$

For a circular rod of diameter d and length l, the volumetric strain is given as

$$\in_V = 2\in_d + \in_l \qquad \text{... (1.15)}$$

where $\in_d$ = Strain produced in the diameter

$\in_l$ = Strain produced in the length

For a sphere of diameter d, the expression for volumetric strain is

$$\in_V = 3\in_d \qquad \text{... (1.16)}$$

1.9 BULK MODULUS OF ELASTICITY

Consider a cube of side l is subjected to three mutually perpendicular stresses of equal intensity ($\sigma_x = \sigma_y = \sigma_z = \sigma$) as shown in Fig. 1.7.

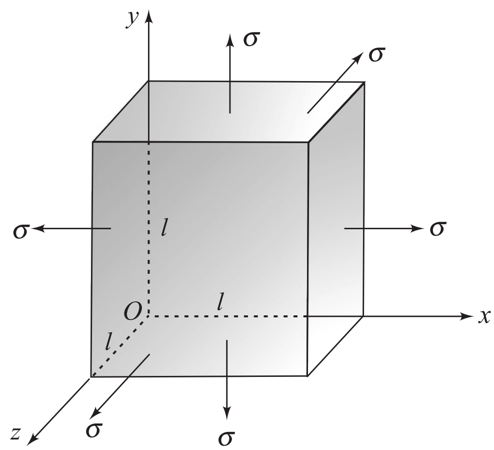


Fig. 1.7

The bulk modulus of elasticity (K) is defined as a ratio of the volumetric stress (σ) or the uniform pressure intensity (p) to the volumetric strain ($\in_V$).

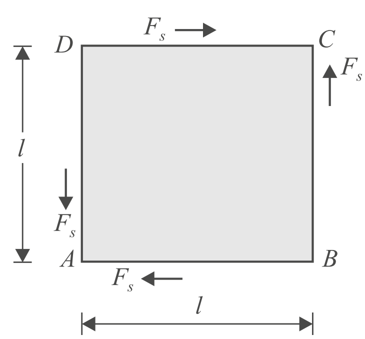
Mathematically, $$K = \frac{\sigma}{\in_V} = \frac{p}{\in_V} \quad \text{... (1.17)}$$

1.10 ELASTIC CONSTANTS AND THEIR RELATIONSHIPS

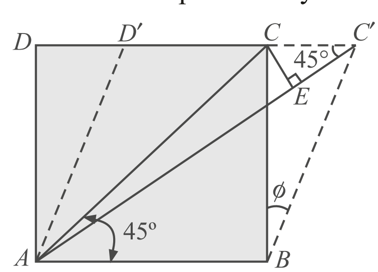
There are three elastic constants, namely elastic modulus (also called Young's modulus or modulus of elasticity, denoted by E), shear modulus (also called modulus of rigidity, denoted by G) and bulk modulus (also called bulk modulus of elasticity, denoted by K). These constants are related to each other through Poisson's ratio (v).

1.10.1 Relationship between *E* and *G*

Consider a cube $ABCD$ of side l being subjected to a shear force F_s as shown, in Fig. 1.8 (a). The shear force produces shear stresses on sides CD and AB and complementary shear stresses on sides



(a) A cube subjected to a shear force F_s



(b) Cube $ABCD$ deformed to $ABC'D'$ due to F_s

Fig. 1.8

AD and BC. The cube gets deformed due to the applied shear force and takes the new shape $ABC'D'$ making an angle with the vertical called the shear strain as shown in Fig. 1.8 (*b*). The diagonal AC elongates to AC' whereas the diagonal BD is shortened.

Drop perpendicular form point C on AC'.

Assume that the strain produced are small so that $< CC'E = 45°$.

Now $$AC = \sqrt{AB^2 + BC^2}$$

$$= \sqrt{BC^2 + BC^2} \qquad \text{(as } AB = BC\text{)}$$

$$= \sqrt{2}\, BC$$

$$\text{Longitudinal strain in } AC = \frac{AC' - AC}{AC}$$

$$= \frac{EC'}{AC} = \frac{CC' \cos 45°}{AC}$$

$$= \frac{CC' \times \frac{1}{\sqrt{2}}}{\sqrt{2} BC} \qquad \text{(on substituting } AC\text{)}$$

$$= \frac{1}{2} \times \frac{CC'}{BC}$$

$$= \frac{\phi}{2} \qquad \text{... (1.18)}$$

where $\frac{CC'}{BC} = \tan \phi = \phi$ (for small value of ϕ)

By definition

$$\text{Shear modulus } G = \frac{\text{Shear stress } \tau}{\text{Shear strain } \phi}$$

or $$\phi = \frac{\tau}{G}$$

Substituting ϕ in equation (1.18), we have

$$\text{Longitudinal strain in } AC = \frac{\phi}{2} = \frac{\tau}{2G} \qquad \text{... (1.19)}$$

Therefore, the longitudinal strain in diagonal AC is one-half of the shear strain, which is tensile in nature. Similarly the longitudinal strain in diagonal BD is one-half of the shear strain, but is compressive in nature.

On replacing the shear stress system by a system of direct stress at 45° as shown in Fig. 1.9, we have tensile direct stress σ_1 along the diagonal AC and compressive direct stress σ_2 along the diagonal BD, and these two stresses are equal in value to the applied shear stresses.

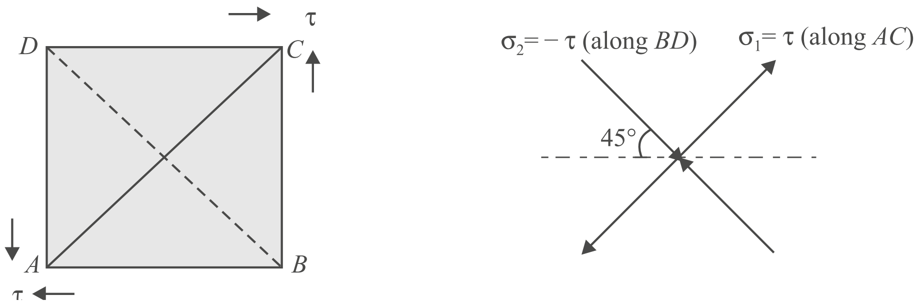


Fig. 1.9 Direct stresses due to shear.

Using the direct stress system for the diagonals, we have

$$\begin{aligned}\text{Strain in } AC &= \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E}\\ &= \frac{\tau}{E} - \nu\frac{(-\tau)}{E}\\ &= \frac{\tau}{E} + \frac{\nu\tau}{E}\\ &= \frac{\tau}{E}(1+\nu) \qquad \text{... (1.20)}\end{aligned}$$

Comparing equations (1.19) and (1.20), we get

$$\frac{\tau}{2G} = \frac{\tau}{E}(1+\nu)$$

Hence, $$E = 2G(1+\nu) \qquad \text{... (1.21)}$$

This is the required relationship between E and G.

1.10.2 Relationship between *E* and *K*

Consider a cube of side l subjected to three equal stresses σ along the three mutually perpendicular directions, that is, $\sigma_x = \sigma_y = \sigma_z = \sigma$ acting on all the sides as shown in Fig. 1.10.

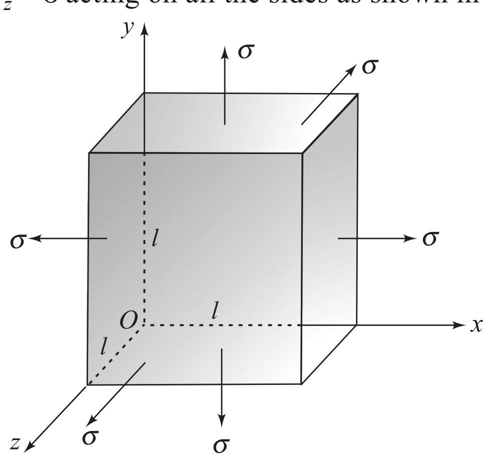


Fig. 1.10 A cube subjected to equal stresses σ on all the sides.

Equal linear strains are produced along any direction because of uniform stress acting on every equal side, that is, $\in_x = \in_y = \in_z$.

Total linear strain produced along any direction

$$= \frac{\sigma}{E} - \nu\frac{\sigma}{E} - \nu\frac{\sigma}{E}$$

$$= \frac{\sigma}{E}(1 - 2\nu) \qquad \text{... (1.22)}$$

Also Volumetric strain = 3 × Linear strain

$$= \frac{3\sigma}{E}(1 - 2\nu) \qquad \text{(using equation (1.22)) ... (1.23)}$$

By definition

$$\text{Bulk modulus } K = \frac{\text{Volumetric stress}}{\text{Volumetric strain}}$$

which gives

$$\text{Volumetric strain} = \frac{\text{Volumetric stress}}{\text{Bulk modulus}} = \frac{\sigma}{K} \qquad \text{... (1.24)}$$

Comparing equations (1.23) and (1.24), we have

$$\frac{\sigma}{K} = \frac{3\sigma}{E}(1 - 2\nu)$$

Hence,

$$E = 3K(1 - 2\nu) \qquad \text{... (1.25)}$$

This is the required relationship between E and K.

1.10.3 Relationship between *G* and *K*

The relationship between E and G is

$$E = 2G(1 + \nu)$$

$$\frac{E}{2G} = 1 + \nu$$

or

$$\nu = \frac{E}{2G} - 1 \qquad \text{... (1.26)}$$

The relationship between E and K is given as

$$E = 3K(1 - 2\nu)$$

$$\frac{E}{3K} = 1 - 2\nu$$

or

$$\nu = \frac{1}{2} - \frac{E}{6K} \qquad \text{... (1.27)}$$

Comparing equations (1.26) and (1.27), we have

$$\frac{E}{2G} = -1 = \frac{1}{2} - \frac{E}{6K} \quad \Rightarrow \quad \frac{E}{2G} + \frac{E}{6K} = 1 + \frac{1}{2}$$

or
$$E\left(\frac{1}{2G} + \frac{1}{6K}\right) = \frac{3}{2} \quad \Rightarrow \quad E\left(\frac{3K+G}{6KG}\right) = \frac{3}{2}$$

Hence,
$$E = \frac{9KG}{(3K+G)} \qquad \text{... (1.28)}$$

This is the required relationship between G and K.

The values of the elastic constants of some materials are given in Table 1.1, Table 1.2 and Table 1.3.

Table 1.1 Young's Modulus (*E*), Elastic Limit (*e*), Maximum Elastic Strain (%) and Tensile Strength of some materials

Materials	Young's modulus (GPa)	Elastic limit (MPa)	Maximum elastic strain (%)	Tensile strength (MPa)
Steel	200	300	0.15	500
Wrought iron	190	170	0.09	330
Copper	120	200	0.16	400
Aluminium	70	180	0.13	200

Table 1.2 Shear Modulus (*G*) of some materials

Materials	*G* (GPa)
Steel	80
Iron	50
Copper	40
Glass	30
Aluminium	25
Wood	10

Table 1.3 Bulk Modulus of Elasticity (*K*) of some materials

Materials	*K* (GPa)
Steel	158
Copper	120
Iron	80
Aluminium	70
Glass	36
Mercury	25
Water	2.2

1.11 FACTOR OF SAFETY

During the design of many structures or components meant for engineering applications, a factor usually called factor of safety, is introduced in their design parameters which ensures their safety with respect to uncertainties of loading conditions, design procedures or production methods. It is denoted by n and is defined as

$$\text{Factor of safety } (n) = \frac{\text{Maximum stress}}{\text{Allowable or working stress}} = \frac{\sigma_{max}}{\sigma_w} \quad \text{... (1.29)}$$

As the yield stress defines the failure criteria for most of the engineering materials undergoing tension, hence it is considered the maximum stress and the factor of safety is also defined as

$$\text{Factor of safety } (n) = \frac{\text{Yield stress}}{\text{Allowable or working stress}} = \frac{\sigma_y}{\sigma_w} \quad \text{... (1.30)}$$

The factor of safety depends upon the risks involved while using the component. Components subjected to dynamic, fluctuating or impact loading situations will require higher factor of safety. A factor of safety of 2 implies that the component is capable of withstanding two times the maximum stress under the normal loading condition. Its typical value lies between 2.5 (for static loading condition) and 10 (for shock loading).

Example 1.1

During a tension test, a mild steel specimen of diameter 12 mm and gauge length 60 mm elongates to 75 mm. The rod can sustain a maximum load of 50 kN but yields at 25 kN and breaks at 30 kN. Find its yield strength, ultimate strength, strength at the point of failure, actual strength at the point of failure when the diameter is reduced to 8 mm. Also, find the percentage elongation and the percentage reduction in area.

Solution: Given,

Initial diameter, $d_1 = 12$ mm

Final diameter, $d_2 = 8$ mm

Initial length, $l_1 = 60$ mm

Final length, $l_2 = 75$ mm

Ultimate load = 50 kN

Yielding load = 25 kN

Breaking load = 30 kN

Initial cross-sectional area, $A_1 = \frac{\pi}{4} d_1^2$

$$= \frac{\pi}{4} \times 12^2 = 113.09 \text{ mm}^2$$

Final cross-sectional area, $A_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 8^2 = 50.26 \text{ mm}^2$

The yield strength is obtained as

$$\sigma_{yp} = \frac{\text{Load at the yield point}}{\text{Initial cross-sectional area}}$$

$$= \frac{25 \times 10^3}{113.09} = 221.06 \text{ N/mm}^2$$ **Ans.**

The ultimate strength is found as

$$\sigma_u = \frac{\text{Maximum load}}{\text{Initial cross-sectional area}}$$

$$= \frac{50 \times 10^3}{113.09} = 442.12 \text{ N/mm}^2$$ **Ans.**

The strength at the failure point is given as

$$\sigma_f = \frac{\text{Load at the failure point}}{\text{Initial cross-sectional area}}$$

$$= \frac{30 \times 10^3}{113.09} = 265.27 \text{ N/mm}^2$$ **Ans.**

The actual strength at the failure point is obtained as

$$\sigma_{fa} = \frac{\text{Load at the failure point}}{\text{Final cross-sectional area}}$$

$$= \frac{30 \times 10^3}{50.26} = 596.9 \text{ N/mm}^2$$ **Ans.**

The percentage elongation is given as

$$\% \text{ elongation} = \frac{l_2 - l_1}{l_1} \times 100$$

$$= \frac{75 - 60}{60} \times 100 = 25\%$$ **Ans.**

The percentage reduction in area is found as

$$R_A = \frac{A_1 - A_2}{A_1} \times 100$$

$$= \frac{113.09 - 50.26}{113.09} \times 100 = 55.55\%$$ **Ans.**

Example 1.2

A 2 m solid steel bar of diameter 100 mm is subjected to a pull of 100 kN. Find the increase in length of the bar. Also, find the stress and the strain produced in the bar. Take $E = 200\ \text{kN/mm}^2$.

Solution: Refer Fig. 1.11.

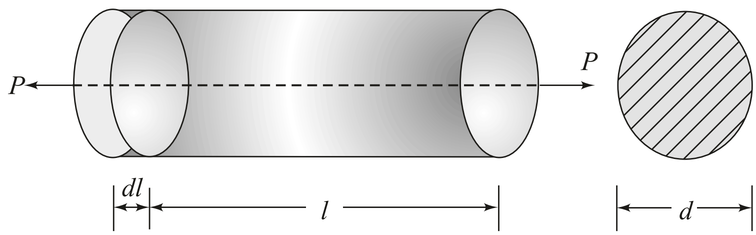


Fig. 1.11

Given, Diameter of the bar, $d = 100$ mm

Length of the bar, $l = 2\ \text{m} = 2 \times 1000 = 2000$ mm

Pull force, $P = 100\ \text{kN} = 10^5$ N

The increase in length, in the direction of load applied, is given as

$$dl = \frac{Pl}{AE}$$

$$= \frac{10^5 \times 2000}{\pi/4 \times 100^2 \times 200 \times 10^3}$$

$$= 0.127\text{mm}$$ **Ans.**

The strain produced in the bar is obtained as

$$\in = \frac{dl}{l} = \frac{0.127\ \text{mm}}{2000\ \text{mm}}$$

$$= 6.36 \times 10^{-5}$$ **Ans.**

The stress induced in the bar is given as

$$\sigma = \in E = 6.36 \times 10^{-5} \times 2 \times 10^5$$

$$= 12.72\ \text{N/mm}^2$$ **Ans.**

Example 1.3

A bar of different cross-sections is subjected to a tensile force of 50 kN (Fig. 1.12). Find the stresses in different sections and the total elongation produced in the bar. Take $E = 200$ kN/mm^2.

Solution: Refer Fig. 1.12.

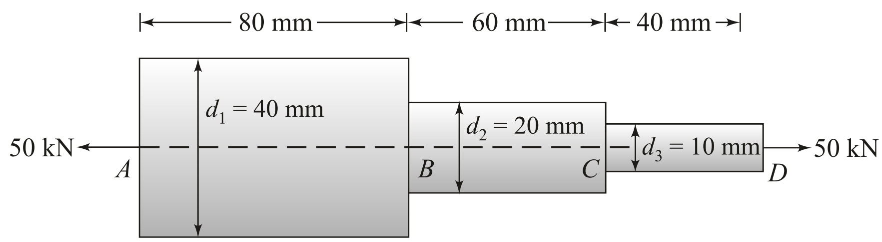


Fig. 1.12

	For part *AB*	For part *BC*	For part *CD*
Length	$l_1 = 80$ mm	$l_2 = 60$ mm	$l_3 = 40$ mm
Diameter	$d_1 = 40$ mm	$d_2 = 20$ mm	$d_3 = 10$ mm

The load acting on each cross-section of the bar is the same, that is, 50 kN.

$$\text{Stress in } AB = \frac{50 \times 10^3}{\pi/4 \times 40^2} = 39.78 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

$$\text{Stress in } BC = \frac{50 \times 10^3}{\pi/4 \times 20^2} = 159.15 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

$$\text{Stress in } CD = \frac{50 \times 10^3}{\pi/4 \times 10^2} = 636.62 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

Hence, the maximum stress is induced in *CD* and the minimum stress in *AB*.

The total elongation of the bar is given as

$$dl = \frac{Pl_1}{A_1E} + \frac{Pl_2}{A_2E} + \frac{Pl_3}{A_3E} = \frac{P}{E}\left[\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3}\right]$$

$$= \frac{50 \times 10^3}{200 \times 10^3}\left[\frac{80}{\pi/4 \times 40^2} + \frac{60}{\pi/4 \times 20^2} + \frac{40}{\pi/4 \times 10^2}\right]$$

$$= 0.191 \text{ mm} \qquad \textbf{Ans.}$$

Example 1.4

A prismatic bar of 10 mm diameter is subjected to different axial forces as shown in Fig. 1.13. Calculate the net change in length of the bar. Take $E = 200$ kN/mm^2.

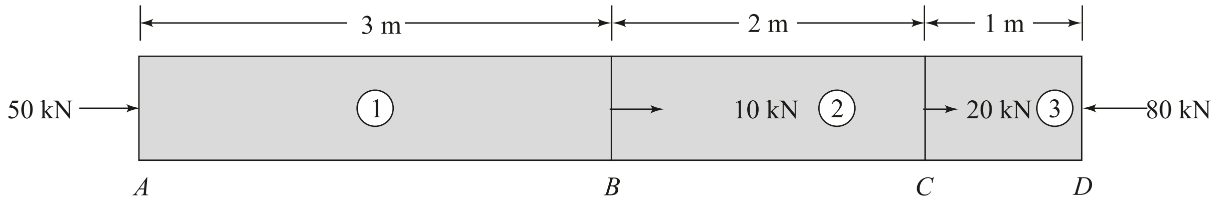


Fig. 1.13

Solution: Refer Fig. 1.13.

	For part *AB*	For part *BC*	For part *CD*
Length	$l_1 = 3$ m	$l_2 = 2$ m	$l_3 = 1$ m

The cross-sectional areas of the three parts are equal.

$$A_1 = A_2 = A_3 = A = \frac{\pi}{4} \times 10^2 = 78.54 \text{ mm}^2$$

The free body diagram of every part of the bar is shown in Fig. 1.14. Each part is subjected to a compressive force.

The change in length of *AB* due to 50 kN is given as

$$dl_1 = -\frac{P_1 l_1}{AE} = -\frac{50 \times 10^3 \times 3 \times 10^3}{78.54 \times 200 \times 10^3} = -9.55 \text{ mm}$$

The change in length of *BC* due to 60 kN is given as

$$dl_2 = -\frac{P_2 l_2}{AE} = -\frac{60 \times 10^3 \times 2 \times 10^3}{78.54 \times 200 \times 10^3} = -7.64 \text{ mm}$$

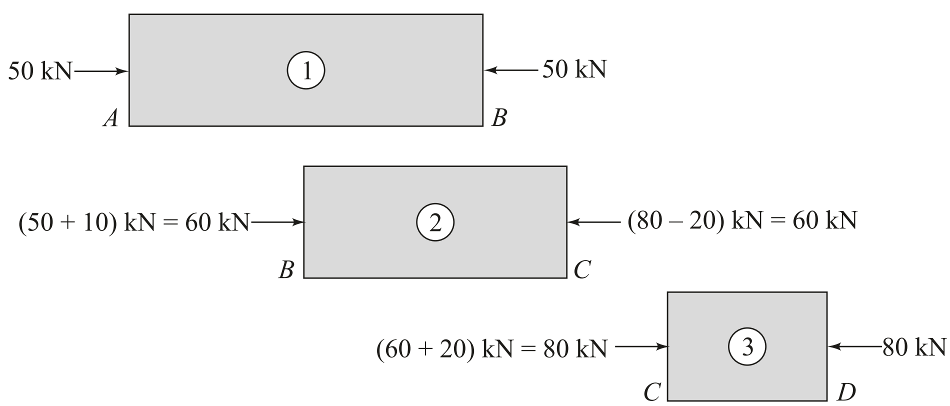


Fig. 1.14

The change in length of CD due to 80 kN is obtained as

$$dl_3 = -\frac{P_3 l_3}{AE} = \frac{80\times10^3\times1\times10^3}{78.54\times200\times10^3} = -\,5.09 \text{ mm}$$

Hence, the total change in length of the bar is given as

$$\begin{aligned} dl &= -\,(dl_1 + dl_2 + dl_3) \\ &= -\,(9.55 + 7.64 + 5.09) \text{ mm} = -\,22.28 \text{ mm} \\ &= 22.28 \text{ mm (Decrease)} \end{aligned}$$

Ans.

Example 1.5

A rod consists of three bars of unequal diameters (Fig. 1.15). Their diameters and lengths are shown in the figure. Find the stress in each bar. Also, find the elongation of the rod. Take $E = 200$ kN/mm^2.

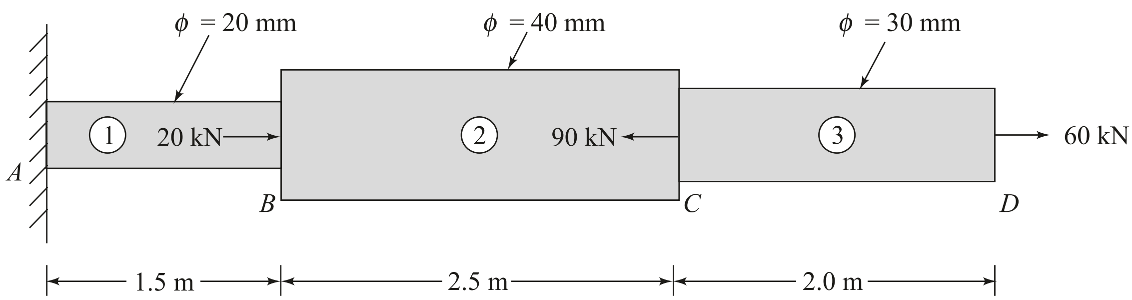


Fig. 1.15

Solution: Refer Fig. 1.15.

	For bar AB	For bar BC	For bar CD
Length	$l_1 = 1.5$ m	$l_2 = 2.5$ m	$l_3 = 2.0$ m
Diameter	$d_1 = 20$ mm	$d_2 = 40$ mm	$d_3 = 30$ mm

The FBD of each bar is shown in Fig. 1.16.

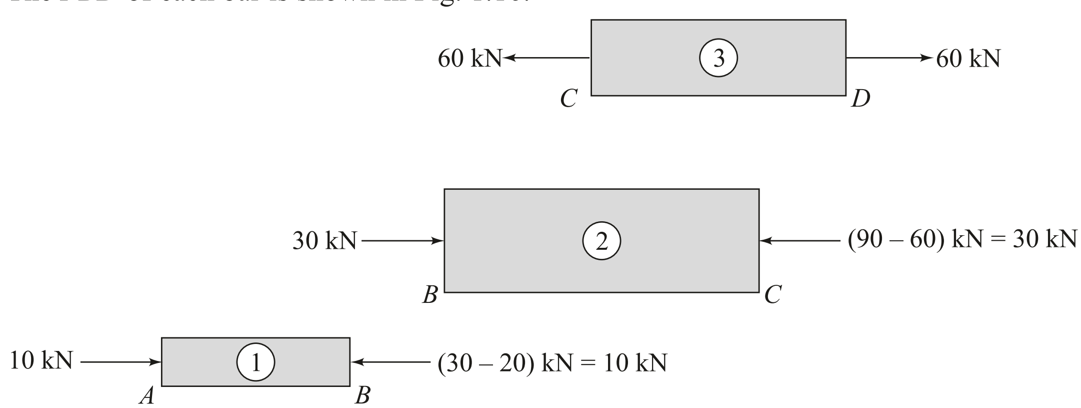


Fig. 1.16

The forces in bars (1) and (2) are compressive and in bar (3) is tensile.

The cross-sectional areas of three bars are found as

$$A_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 20^2 = 314.16 \text{ mm}^2$$

$$A_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 40^2 = 1256.63 \text{ mm}^2$$

and $$A_3 = \frac{\pi}{4} d_3^2 = \frac{\pi}{4} \times 30^2 = 706.86 \text{ mm}^2$$

The stresses induced in three bars are obtained as

$$\sigma_1 = \frac{P_1}{A_1} = \frac{10 \times 10^3}{314.16} = 31.83 \text{ N/mm}^2 \text{ (C)}$$ **Ans.**

$$\sigma_2 = \frac{P_2}{A_2} = \frac{30 \times 10^3}{1256.63} = 23.87 \text{ N/mm}^2 \text{ (C)}$$ **Ans.**

and $$\sigma_3 = \frac{P_3}{A_3} = \frac{60 \times 10^3}{706.86} = 84.88 \text{ N/mm}^2 \text{ (T)}$$ **Ans.**

Hence, the maximum stress is developed in *CD* and the minimum stress in *BC*.

The changes in length of three bars are obtained as

$$dl_1 = -\frac{P_1 l_1}{A_1 E} = -\frac{10 \times 10^3 \times 1.5 \times 10^3}{314.16 \times 200 \times 10^3} = -0.238 \text{ mm}$$

$$dl_2 = -\frac{P_2 l_2}{A_2 E} = -\frac{30 \times 10^3 \times 2.5 \times 10^3}{1256.63 \times 200 \times 10^3} = -0.298 \text{ mm}$$

and $$dl_3 = \frac{P_3 l_3}{A_3 E} = \frac{60 \times 10^3 \times 2 \times 10^3}{706.86 \times 200 \times 10^3} = 0.848 \text{ mm}$$

Hence, the change in length of the rod is given as

$$dl = dl_1 + dl_2 + dl_3$$

$$= (-0.238 - 0.298 + 0.848) \text{ mm}$$

$$= 0.312 \text{ mm (Increase)}$$ **Ans.**

Example 1.6

A 2 m steel bar of diameter 15 mm is subjected to an axial pull of 50 kN. Calculate the change in length, diameter and volume of the bar, if the Poisson's ratio is 0.25. Also, find the workdone in stretching the bar. Take $E = 200\ \text{kN/mm}^2$.

Solution: Given,

Initial diameter of the steel bar, $d = 15$ mm

Initial length of the steel bar, $l = 2\ \text{m} = 2 \times 1000 = 2000$ mm

Pull force, $P = 50\ \text{kN} = 5 \times 10^4$ N

Poisson's ratio, $\nu = 0.25$

The cross-sectional area of the steel bar is

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 15^2 = 176.71\ \text{mm}^2$$

The normal stress produced in the bar is

$$\sigma = \frac{P}{A} = \frac{5 \times 10^4}{176.71} = 282.94\ \text{N/mm}^2$$

The strain produced is

$$\in = \frac{\sigma}{E} = \frac{282.92}{200 \times 10^3} = 1.414 \times 10^{-3}$$

Also $$\in = \frac{\text{Change in length}}{\text{Initial length}} = \frac{\Delta l}{l}$$

or $$\Delta l = \in . l$$

$$= 1.414 \times 10^{-3} \times 2000 = 2.83\ \text{mm}$$ **Ans.**

Poisson's ratio is

$$\nu = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$= \frac{(\text{Change in diameter/Initial diameter})}{(\text{Change in length/Initial length})} = \frac{(\Delta d/d)}{(\Delta l/l)}$$

$$\frac{\Delta d}{d} = \nu \times \frac{\Delta l}{l}$$

$$= 0.25 \times 1.414 \times 10^{-3} = 3.535 \times 10^{-4}$$

or $$\Delta d = d \times 3.535 \times 10^{-4}$$

$$= 15 \times 3.535 \times 10^{-4} = 0.0053\ \text{mm}$$ **Ans.**

The volume of the bar is

$$V = \frac{\pi}{4} d^2 \times l$$

$$= \frac{\pi}{4} \times 15^2 \times 2000 = 353429.17 \text{ mm}^3$$

Now $$\frac{dV}{V} = \in (1 - 2\nu)$$

or $$dV = V \in (1 - 2\nu)$$

$$= 353429.17 \times 1.414 \times 10^{-3} \times (1 - 2 \times 0.25)$$

$$= 249.87 \text{ mm}^3$$ **Ans.**

The workdone in stretching the bar is obtained as

$$W = \frac{1}{2} \times P \times \Delta l$$

$$= \frac{1}{2} \times 5 \times 10^4 \times 2.83 = 70750 \text{ N.mm} = 70.75 \text{ Joules}$$ **Ans.**

Example 1.7

Find the expression for the elongation of a tapered bar of length l whose diameter varies uniformly from d at one end to D at other end, when subjected to an axial pull P with E as modulus of elasticity.

Solution: Refer Fig. 1.17.

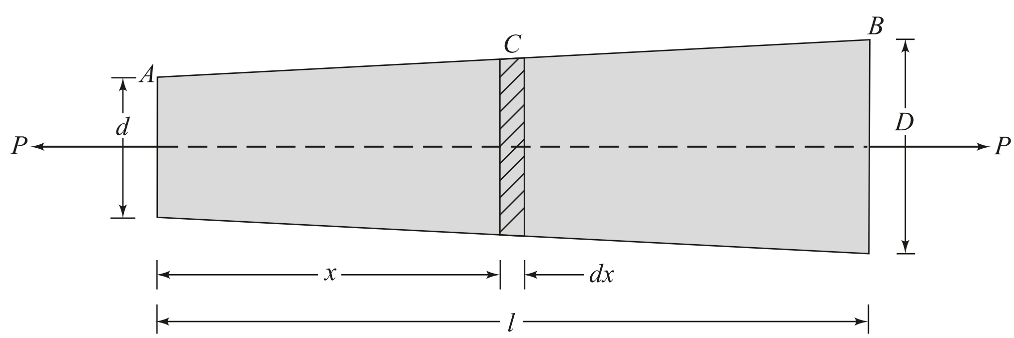


Fig. 1.17

Consider an element of thickness dx at a distance x from A, where diameter is d.

The diameter at C is given as

$$d' = d + \left(\frac{D-d}{l}\right) x$$

The cross-sectional area of the bar at C is

$$A' = \frac{\pi}{4} d'^2 = \frac{\pi}{4}\left[d + \left(\frac{D-d}{l}\right)x\right]^2$$

The elongation produced in the element is

$$dl' = \frac{P \cdot dx}{A'E}$$

The elongation produced in the bar is found as

$$dl = \int_0^l dl' = \int_0^l \frac{4P}{\pi E} \cdot \frac{dx}{\left[d + \frac{D-d}{l} \cdot x\right]^2} = \frac{4Pl}{\pi EDd}$$

Example 1.8

Find the expression for the elongation of a conical bar of length l and base diameter d under its own weight as shown in Fig. 1.18. Take the density of its material as ρ and the modulus of elasticity E.

Solution: Refer Fig. 1.18.

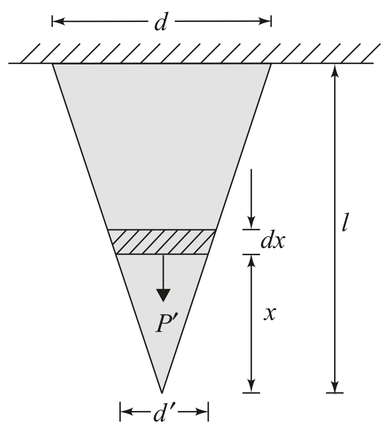


Fig. 1.18

Consider a section of thickness dx at a distance x from the apex of the cone.

The diameter of the bar at the section is

$$d' = \frac{d}{l} \cdot x$$

The cross-sectional area at the section is

$$A' = \frac{\pi}{4} d'^2 = \frac{\pi}{4} \times \frac{d^2 x^2}{l^2}$$

The weight of the cone below the section is

$$P' = \frac{1}{3} \times \text{Area} \times \text{Length} \times \text{Density} \times g$$

$$= \frac{1}{3}\left(\frac{\pi}{4} d'^2\right) \times x \times \rho \times g$$

$$= \frac{\pi d^2 \rho g}{12 l^2} x^3$$

The elongation produced in the length dx is given as

$$dl' = \frac{P' \cdot dx}{A' \cdot E} = \frac{x \rho g}{3E} dx$$

Hence, the total elongation of the bar is obtained as

$$dl = \int_0^l dl' = \int_0^l \frac{x\rho g}{3E} dx = \frac{\rho g l^2}{6E}$$

Example 1.9

In Fig. 1.19, find the force P so that the net decrease in the length of two bars is 0.35 mm. Take

$$E_s = 200 \text{ kN/mm}^2$$

$$E_b = 100 \text{ kN/mm}^2.$$

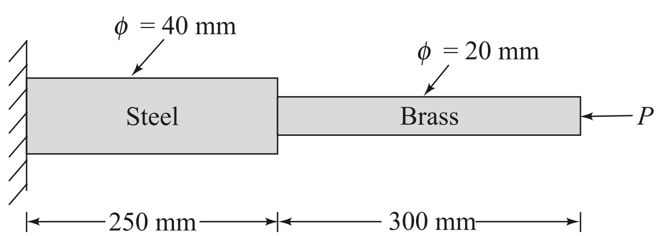


Fig. 1.19

Solution: Given, Length of the steel bar, $l_1 = 250$ mm

Diameter of the steel bar, $d_1 = 40$ mm

Length of the brass bar, $l_2 = 300$ mm

Diameter of the brass bar, $d_2 = 20$ mm

The cross-sectional area of the steel bar, $A_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 40^2 = 1256.63 \text{ mm}^2$

Cross-sectional area of the brass bar, $A_2 = \frac{\pi}{4} \times d_2^2 = \frac{\pi}{4} \times 20^2 = 314.16 \text{ mm}^2$

Force P is acting on both bars. Both bars will contract on the application of P and the net decrease in length of two bars is the sum of two contractions.

$$0.35 = \text{Contraction in steel bar} + \text{Contraction in brass bar}$$

$$= \frac{Pl_1}{A_1 E_1} + \frac{Pl_2}{A_2 E_2}$$

$$= \frac{P \times 250}{1256.63 \times 200 \times 10^3} + \frac{P \times 300}{314.16 \times 100 \times 10^3} = 1.0544 \times 10^{-5} P$$

or $$P = 33194.24 \text{ N}$$ **Ans.**

Example 1.10

A solid steel bar of diameter 60 mm and length 350 mm is placed inside an aluminium cylinder of inside diameter 70 mm and outside diameter 110 mm. The steel bar is shorter than aluminium cylinder by 0.25 mm. A compressive load of 1000 kN is applied on the assembly through two cover plates on its both sides (Fig. 1.20). Find the stresses induced in the cylinder and bar. Take $E_S = 200 \text{ kN/mm}^2$ and $E_{Al} = 70 \text{ kN/mm}^2$.

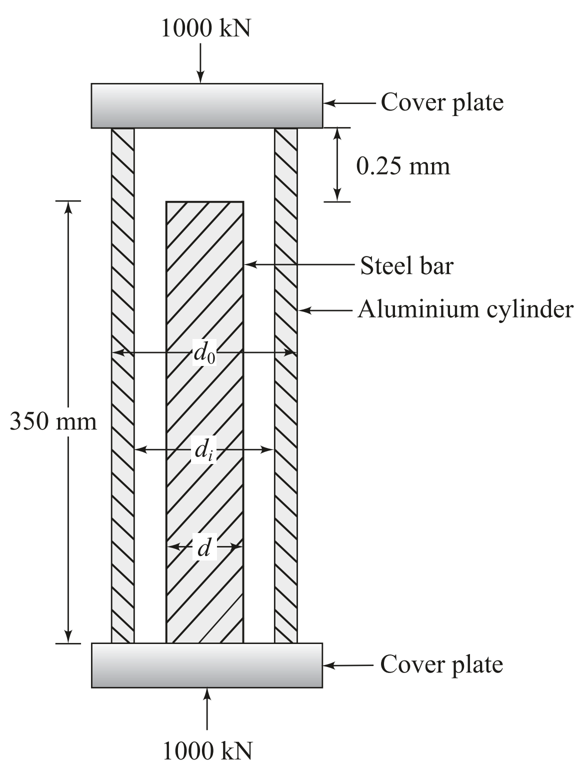


Fig. 1.20

Solution: Given,

Diameter of the steel bar, $d = 60$ mm

Length of the steel bar, $l_S = 350$ mm

Inside diameter of the aluminium cylinder, $d_i = 70$ mm

Outside diameter of the aluminium cylinder, $d_o = 110$ mm

Length of the aluminium cylinder, $l_{Al} = (350 + 0.25)$ mm

$= 350.25$ mm

Let stress in the steel bar $= \sigma_S$

Stress in aluminium cylinder $= \sigma_{Al}$

Assume that the aluminium cylinder contracts by an amount dl on the application of the given load.

The contraction in the steel bar = $(dl - 0.25)$ mm

The cross-sectional area of the steel bar is

$$A_S = \frac{\pi}{4} d^2$$

$$= \frac{\pi}{4} \times 60^2 = 2827.43 \text{ mm}^2$$

The cross-sectional area of the aluminium cylinder is found as

$$A_{Al} = \frac{\pi}{4}(d_o^2 - d_i^2)$$

$$= \frac{\pi}{4}(110^2 - 70^2) = 5654.86 \text{ mm}^2$$

The strain produced in the aluminium cylinder is given as

$$\in_{Al} = \frac{dl}{350.25}$$

The strain produced in the steel bar is given as

$$\in_S = \frac{dl - 0.25}{350}$$

The load on the cover plate is shared by both the steel bar and the aluminium cylinder.

$$1000 \times 10^3 = \sigma_S A_S + \sigma_{Al} A_{Al}$$

$$= E_S \in_S A_S + E_{Al} \in_{Al} A_{Al}$$

$$= 200 \times 10^3 \times \left(\frac{dl - 0.25}{350}\right) \times 2827.43 + 70 \times 10^3 \times \frac{dl}{350.25} \times 5654.86$$

Solving for dl, we get

$$dl = 0.511 \text{ mm}$$

Hence, the stress in the steel bar is obtained as

$$\sigma_S = E_S \in_S$$

$$= 200 \times 10^3 \times \frac{0.511 - 0.25}{350} = 149.14 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

and stress in the aluminium cylinder is obtained as

$$\sigma_{Al} = E_{Al} \in_{Al}$$

$$= 70 \times 10^3 \times \frac{0.511}{350.25} = 102.12 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

Example 1.11

An assembly of a steel bar enclosed in an aluminium tube is compressed between two rigid parallel plates by a force of 600 kN (Fig. 1.21). The diameter of the steel bar is 60 mm, length of both steel bar and aluminium tube is 1m. The inside and the outside diameters of the aluminium tube are 70 mm and 110 mm respectively. Find the stresses in the bar and the tube. Take

$$E_S = 200 \text{kN/mm}^2$$

and $$E_{Al} = 70 \text{kN/mm}^2.$$

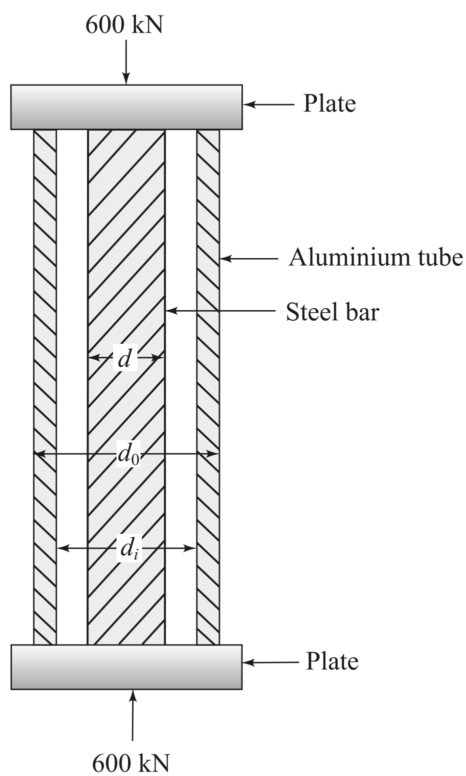


Fig. 1.21

Solution: Refer Fig. 1.21.

Given, Load on the assembly, $P = 600 \times 10^3$ N

Diameter of the steel bar, $d = 60$ mm

Inside diameter of the aluminium tube, $d_i = 70$ mm

Outside diameter of the aluminium tube, $d_o = 110$ mm

Let σ_s = Stress produced in the steel bar

σ_{Al} = Stress produced in the aluminium tube

The cross-sectional area of the steel bar is found as

$$A_s = \frac{\pi}{4} \times d^2$$

$$= \frac{\pi}{4} \times 60^2 = 2827.43 \text{ mm}^2$$

The cross-sectional area of the aluminium tube is found as

$$A_{Al} = \frac{\pi}{4}(d_o^2 - d_i^2) = \frac{\pi}{4}(110^2 - 70^2) = 5654.86 \text{ mm}^2$$

Strain produced in the steel bar = Strain produced in the aluminium tube

$$\frac{\sigma_s}{E_s} = \frac{\sigma_{Al}}{E_{Al}}$$

or

$$\sigma_s = \frac{E_s}{E_{Al}} \cdot \sigma_{Al}$$

$$= \frac{200 \times 10^3}{70 \times 10^3} \times \sigma_{Al} = 2.857\, \sigma_{Al}$$

The applied load is shared by both members.

$$\sigma_s A_s + \sigma_{Al} A_{Al} = 600 \times 10^3$$

$$2.857\sigma_{Al} \times 2827.43 + \sigma_{Al} \times 5654.86 = 600 \times 10^3 \qquad \text{(on substituting } \sigma_s\text{)}$$

On solving, we get $\sigma_{Al} = 43.69$ N/mm^2 **Ans.**

and $\sigma_s = 2.857 \times 43.69 = 124.82$ N/mm^2 **Ans.**

Example 1.12

Two vertical rods of steel and copper each of length 2 m and diameter 10 mm, separated by a distance of 1.5 m, are rigidly fixed to the ceiling (Fig. 1.22). The lower ends of the two rods are connected by a horizontal cross-piece. Where should a load of 15 kN be applied on the cross-piece so that it remains horizontal after being loaded? Take

$$E_s = 200 \text{ kN/mm}^2$$

and

$$E_c = 100 \text{ kN/mm}^2.$$

Solution: Refer Fig. 1.22.

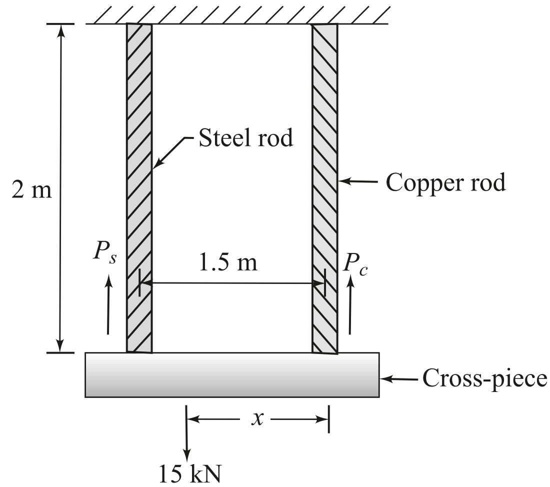


Fig. 1.22

Let the load 15 kN is applied at a distance x from the center of the copper rod and P_s and P_c are the loads shared by the steel and copper rod respectively.

Now
$$15 \times 10^3 = P_s + P_c$$
$$= \sigma_s A_s + \sigma_c A_c \quad \text{... (1)}$$

But
$$A_s = A_c = \frac{\pi}{4} \times 10^2$$
$$= 78.54 \text{ mm}^2$$

On substituting the area in equation (1), we have

$$15 \times 10^3 = (\sigma_s + \sigma_c)\, 78.54$$

or
$$(\sigma_s + \sigma_c) = 190.98 \text{ N/mm}^2 \quad \text{... (2)}$$

Since the cross-piece remains horizontal, hence equal strains are produced in both rods.

$$\frac{\sigma_s}{E_s} = \frac{\sigma_c}{E_c}$$

or
$$\sigma_s = \frac{E_s}{E_c} \times \sigma_c$$

$$= \frac{200 \times 10^3}{100 \times 10^3} \times \sigma_c = 2\sigma_c$$

Using this relation in equation (2), we get

$$2\sigma_c + \sigma_c = 190.98$$

or $\sigma_c = 63.66 \text{ N/mm}^2$

and $\sigma_s = 2\sigma_c$

$= 2 \times 63.66$

$= 127.32 \text{ N/mm}^2$

The load shared by the steel rod is

$$P_s = \sigma_s A_s$$

$$= 10 \text{ kN}$$

and $P_c = (15 - 10) \text{ kN}$

$= 5 \text{ kN}$

Taking moments of the loads about the centre of the copper rod, we have

$$P_s \times 1.5 = 15 \times x$$

or $$x = \frac{10 \times 1.5}{15} = 1\text{m}$$

Hence, the load on the cross-piece should be applied at a distance of 1m from the centre of the copper rod. **Ans.**

Example 1.13

Two steel rods and one brass rod, each of 30 mm diameter are arranged vertically to take a load of 25 kN as shown in Fig. 1.23. Take $E_s = 200 \text{ kN/mm}^2$ and $E_b = 100 \text{ kN/mm}^2$.

Solution: Refer Fig. 1.23.

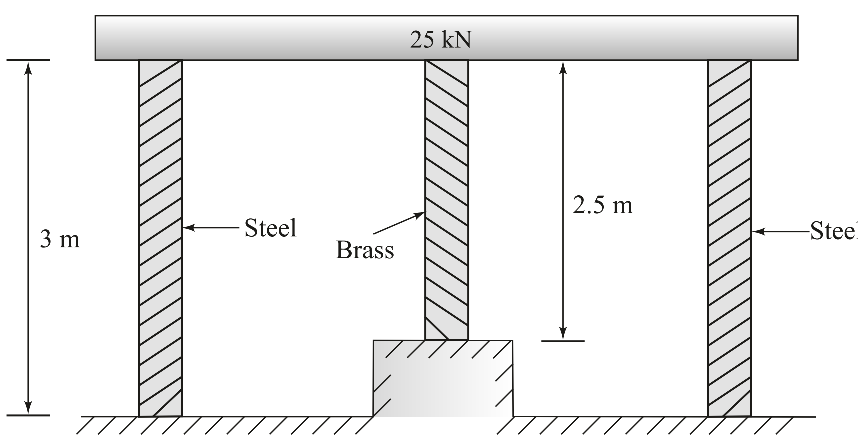


Fig. 1.23

Given,

Diameter of each rod, $d = 30$ mm

Length of the steel rod, $l_s = 3$ m

Length of the brass rod, $l_b = 2.5$ m

Area of the steel rod, A = Area of the brass rod

$$= \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 30^2 = 706.86 \text{ mm}^2$$

Let σ_s = Stress in the steel rod

σ_b = Stress in the brass rod

$\in_s$ = Strain produced in the steel rod

$\in_b$ = Strain produced in the brass rod

When the load acts, all the three rods reduce in their lengths by an equal amount.

Decrease in length of the steel rod = Decrease in length of the brass rod

$$\frac{\sigma_s\, l_s}{E_s} = \frac{\sigma_b\, l_b}{E_b}$$

$$\frac{\sigma_s \times 3 \times 1000}{200 \times 10^3} = \frac{\sigma_b \times 2.5 \times 1000}{100 \times 10^3}$$

or $\sigma_s = 1.67\sigma_b$

The given load is shared by the three rods.

$$2\sigma_s \,.\, A + \sigma_b \,.\, A = 25 \times 10^3$$

$$(2 \times 1.67\sigma_b + \sigma_b) \times 706.86 = 25 \times 10^3 \qquad \text{(on substituting } \sigma_s\text{)}$$

On solving, we get

$\sigma_b = 8.16$ N/mm^2 **Ans.**

and $\sigma_s = 13.6$ N/mm^2 **Ans.**

Example 1.14

The steel bolt shown in Fig. 1.24 has a thread pitch of 1.6 mm. If the nut is initially tightened up by hand so as to cause no stress in the copper spacing tube, calculate the stress induced in the tube and in the bolt, if a spanner is then used to turn the nut through 90°. Take $E_c = 100$ GPa and $E_s = 209$ GPa.

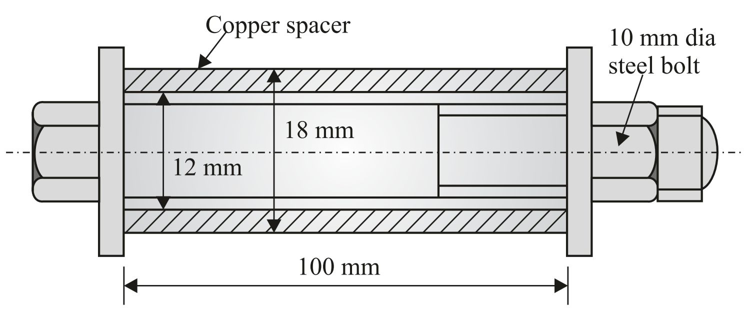


Fig. 1.24

Solution: Given,

Diameter of the steel bolt, $d_s = 10$ mm

Inside diameter of the copper tube, $d_i = 12$ mm

Outside diameter of the copper tube, $d_0 = 18$ mm

Length of the tube = Length of the bolt,

$$l = 100 \text{ mm}$$

Modulus of elasticity for steel, $E_s = 209$ GPa $= 209 \times 10^9$ Pa

Modulus of elasticity for copper, $E_c = 100$ GPa $= 100 \times 10^9$ Pa

Pitch of the thread, $p = 1.6$ mm

Area of the bolt is
$$A_s = \frac{\pi}{4} d_s^2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ mm}^2$$

Area of the tube is
$$A_c = \frac{\pi}{4}(d_0^2 - d_i^2) = \frac{\pi}{4}(18^2 - 12^2) = 141.37 \text{ mm}^2$$

The tensile force acting on the bolt is equal to the compressive force in the tube. At the same time, the sum (Δl) of increase in the length of the bolt (Δl_s) and decrease in the length of the tube (Δl_c) is equal to the axial displacement of the nut.

$$\Delta l = \Delta l_s + \Delta l_c = p \times \frac{90°}{360°}$$

$$= 1.6 \times \frac{1}{4} = 0.4 \text{ mm}$$

or
$$Pl\left[\frac{1}{A_s E_s} + \frac{1}{A_c E_c}\right] = 0.4$$
(where P is force, tensile or compressive.)

or
$$P \times 100\left[\frac{1}{78.54 \times 10^{-6} \times 209 \times 10^9} + \frac{1}{141.37 \times 10^{-6} \times 100 \times 10^9}\right] = 0.4$$

Solving for P, we get

$$P = 30395.14 \text{ N}$$

Hence, stress in the bolt is

$$\sigma_s = \frac{P}{A_s} = \frac{30395.14}{78.54} = 387 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

and stress in the tube is

$$\sigma_c = \frac{P}{A_c} = \frac{30395.14}{141.37}$$

$$= 215 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

Example 1.15

Find the maximum permissible value of load P for the riveted joint shown in Fig. 1.25, if the allowable yield shear strength of the rivet material is 100 MPa. Rivets are 20 mm in diameter.

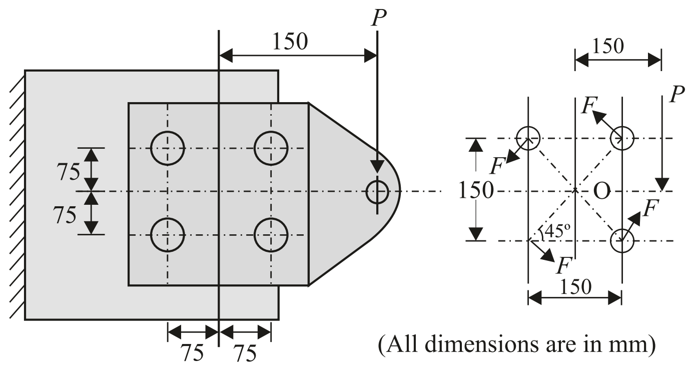


Fig. 1.25

Solution: Given,

Yield strength of rivet material,

$$\sigma_{yp} = 100 \text{ MPa} = 100 \times 10^6 \text{ Pa}$$

Diameter of the rivet, $d = 20$ mm

Number of rivets, $n = 4$

If F be the force in a rivet, and let the rivet be in single shear, then

$$P \times 100 = n \times F \times 75\sqrt{2}$$

$$= 4 \times \left(\frac{\pi}{4} d^2 \times \sigma_{yp}\right) \times 75\sqrt{2}$$

$$= 4 \times \frac{\pi}{4} (20 \times 10^{-3})^2 \times 100 \times 10^6 \times 75\sqrt{2}$$

Hence, $P = 88857.66$ N **Ans.**

Example 1.16

A circular rod of diameter 25 mm and length 1.2 m is subjected to a tensile force of 30 kN. The increase in length of the rod is 0.25 mm and the decrease in diameter is 0.002 mm. Calculate Poisson's ratio and the three elastic constants.

Solution: Given,

Diameter of the rod, $D = 25$ mm

Length of the rod, $l = 1.2$ m

Tension force, $F = 30$ kN

Increase in length of the rod, $dl = 0.25$ mm

Decrease in diameter of the rod $dD = 0.002$ mm

Let v be Poisson's ratio.

The cross-sectional area A of the rod is found as

$$A = \frac{\pi}{4}D^2 = \frac{\pi}{4}\times(25)^2$$

$$= 490.87 \text{ mm}^2$$

Calculation of modulus of elasticity *E*

The longitudinal stress σ is given as

$$\sigma = \frac{F}{A} = \frac{30\times10^3}{490.87}$$

$$= 61.11 \text{ N/mm}^2$$

The longitudinal strain $\in$ is given as

$$\in = \frac{dl}{l} = \frac{0.25\,\text{mm}}{1.2\times1000\,\text{mm}} = 2.08\times10^{-4}$$

Now Modulus of elasticity $E = \dfrac{\sigma}{\in} = \dfrac{61.11}{2.08\times10^{-4}}$

$$= 293798.08 \text{ N/mm}^2$$ **Ans.**

Calculation of Poisson's ratio *v*

$$\text{Lateral Strain} = \frac{dD}{D} = \frac{0.002 \text{ mm}}{25 \text{ mm}} = 8\times10^{-5}$$

Hence, Poisson's ratio $v = \dfrac{\text{Lateral strain}}{\text{Longitudinal strain}}$

$$= \frac{8\times10^{-5}}{2.08\times10^{-4}} = 0.384$$ **Ans.**

Calculation of modulus of rigidity *G*

The relationship between E and G is given as

$$E = 2\,G\,(1 + v)$$

or

$$G = \frac{E}{2(1+\nu)}$$

$$= \frac{293798.08}{2(1+0.384)}$$

$$= 106140.92 \text{ N/mm}^2$$ **Ans.**

Calculation of bulk modulus of elasticity *K*

The relationship between E and K is given as

$$E = 3\,K\,(1 - 2\,v)$$

or

$$K = \frac{E}{3(1-2v)}$$

$$= \frac{293798.08}{3\{1-(2\times 0.384)\}}$$

$$= 422123.68 \text{ N/mm}^2$$ **Ans.**

1.12 THERMAL STRESS AND STRAIN

Materials expand on heating and contract on cooling. This heating and cooling changes the dimensions of an object, thereby producing strain in the given material. Such strain is called thermal strain.

When the deformation due to temperature change is restricted, stresses are induced in the material. Such stress is called thermal stress.

1.12.1 Thermal Stress and Strain in a Simple Bar

Consider a bar of length l_o fixed at its both ends (Fig. 1.26).

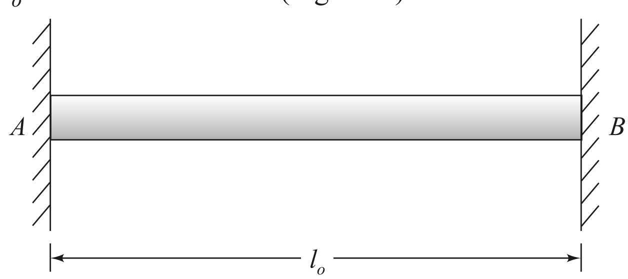


Fig. 1.26

Let T_f = Final temperature

T_i = Initial temperature

ΔT = Change in temperature $= T_f - T_i$

α = Coefficient of thermal expansion

E = Modulus of elasticity of the bar material

dl = Change in length

Suppose that the bar is heated, hence its temperature rises. If the bar were free to expand on heating, then its length at the increased temperature is given as

$$l_t = l_o (1 + \alpha \, \Delta T) \qquad \text{... (1.31)}$$

where l_t = Length at t°C

l_o = Length at room temperature

$$\frac{l_t - l_o}{l_o} = \alpha \, \Delta \mathrm{T}$$

or
$$\in_t = \alpha \, \Delta \mathrm{T} \qquad \text{... (1.32)}$$

where $\in_t$ = Thermal strain

$$= \frac{l_t - l_o}{l_o}$$

$$= \frac{dl}{l_o}$$

Now
$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{\sigma_t}{\in_t}$$

where σ_t = Thermal stress induced in the bar

or
$$\sigma_t = \in_t E$$

$$= E\alpha \, \Delta \mathrm{T} \qquad \text{(on substituting } \in_t\text{) ... (1.33)}$$

1.12.2 Thermal Stress and Strain in a Compound Bar

A compound or composite bar is made of two or more similar or different materials fixed at their ends by welding or riveting. If the two materials are same, then the compound bar has equal expansion or contraction for both, otherwise different expansion or contraction will result. The material with higher coefficient of thermal expansion expands more than the other.

Consider a compound bar consisting of two rods (1) and (2) as shown in Fig. 1.27 is heated.

Let l_o = Initial length of the compound bar (equal for both rods)

α_1 = Coefficient of thermal expansion of the rod (1)

α_2 = Coefficient of thermal expansion of the rod (2)

E_1 = Modulus of elasticity of the rod (1)

E_2 = Modulus of elasticity of the rod (2)

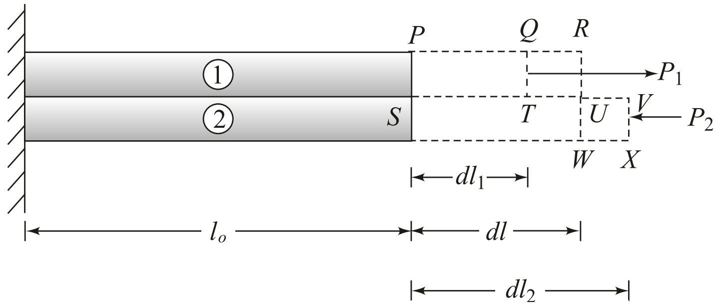


Fig. 1.27

A_1 = Cross-sectional area of the rod (1)

A_2 = Cross-sectional area of the rod (2)

ΔT = Rise in temperature

dl = Elongation of the compound bar

Assuming $\alpha_2 > \alpha_1$, it means that rod (2) will expand more as compared to rod (1). If the two rods were free to expand, then

Increase in length of the rod (1), $dl_1 = l_o \alpha_1 \, \Delta T$ ... (1.34)

Increase in length of the rod (2), $dl_2 = l_o \alpha_2 \, \Delta T$ ... (1.35)

Since the two rods are connected at their ends, hence on heating, the compound bar will expand to an intermediate position RW as shown in Fig. 1.27. To get this intermediate position, the rod (1) is being pulled by the rod (2) by a force P_1 and the rod (2) is being pushed by the rod (1) by a force P_2. Because of no external force acting on the compound bar, the two forces are equal in magnitude. The stress in the rod (1) is given as

$$\sigma_{t_1} = \left[\frac{(dl - dl_1)}{l_o}\right] E_1 \quad \text{... (1.36)}$$

The corresponding force is

$$P_1 = \left[\frac{dl - dl_1}{l_o}\right] E_1 . A_1 \quad \text{... (1.37)}$$

The stress in the rod (2) is given as

$$\sigma_{t_2} = \left[\frac{dl_2 - dl}{l_o}\right] E_2 \quad \text{... (1.38)}$$

The corresponding force is

$$P_2 = \left[\frac{dl_2 - dl}{l_o}\right] E_2 . A_2 \quad \text{... (1.39)}$$

But $\quad P_1 = P_2$

or $\quad \left[\frac{dl - dl_1}{l_o}\right] E_1 A_1 = \left[\frac{dl_2 - dl}{l_o}\right] E_2 A_2$

Using equations (1.34) and (1.35), we have

$$(dl - l_o\alpha_1 \, \Delta T)\, E_1\, A_1 = (l_o\alpha_2\, \Delta T - dl)\, E_2\, A_2$$

$$dl\,(A_1\, E_1 + A_2\, E_2) = l_o\, \Delta T\,(\alpha_2\, A_2\, E_2 + \alpha_1\, A_1\, E_1)$$

Hence,

$$dl = \frac{l_o \Delta T\,(\alpha_1\, A_1\, E_1 + \alpha_2\, A_2\, E_2)}{A_1\, E_1 + A_2\, E_2} \qquad \text{... (1.40)}$$

This is the expression for the elongation produced in the compound bar.

The thermal strain is given as

$$\in_t = \frac{dl}{l_o} = \frac{\Delta T\,(\alpha_1\, A_1\, E_1 + \alpha_2\, A_2\, E_2)}{A_1\, E_1 + A_2\, E_2} \qquad \text{... (1.41)}$$

Example 1.17

A 3 m bar is initially at a temperature of 24°C. It is heated to raise its temperature to 80°C. Estimate the expansion of the bar. If the expansion is not allowed, find the stress in the bar. Take $E = 200$ kN/mm^2 and $\alpha = 1.2 \times 10^{-5}$/°C.

Solution: Given,

Initial length of the bar, $l_o = 3 \text{ m} = 3 \times 10^3 \text{ mm}$

Initial temperature, $T_i = 24°C$

Final temperature, $T_f = 80°C$

The rise in temperature ΔT is given as

$$\Delta T = T_f - T_i$$

$$= (80 - 24)°C = 56°C$$

The expansion of the bar is found by using equation (1.31).

$$dl = l_o\, \alpha\, \Delta T$$

$$= 3 \times 10^3 \times 1.2 \times 10^{-5} \times 56$$

$$= 2.016 \text{ mm}$$ **Ans.**

The thermal stress induced in the bar is given by equation (1.33).

$$\sigma_t = E\alpha\, \Delta T$$

$$= 200 \times 10^3 \times 1.2 \times 10^{-5} \times 56$$

$$= 134.4 \text{ N/mm}^2$$ **Ans.**

Example 1.18

A composite bar is fixed between two supports (Fig. 1.28). If the temperature of the bar is raised from 25°C to 75°C, find the stresses induced in each rod by assuming (*a*) if the supports do not yield and (*b*) if the supports yield by 0.25 mm.

Take $E_1 = 200 \text{ kN/mm}^2$, $\alpha_1 = 1.2 \times 10^{-5}/°\text{C}$

$E_2 = 30 \text{ kN/mm}^2$, $\alpha_2 = 1.8 \times 10^{-5}/°\text{C}$

$E_3 = 100 \text{ kN/mm}^2$, $\alpha_3 = 1.6 \times 10^{-5}/°\text{C}$.

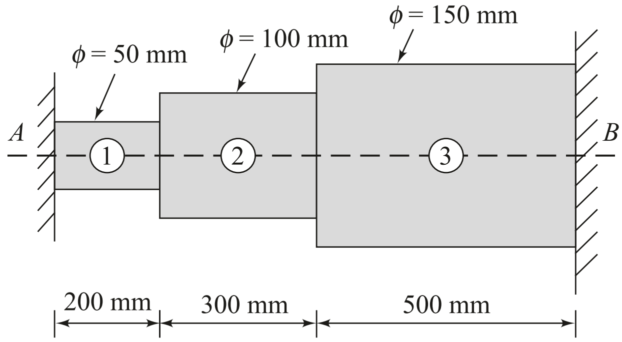


Fig. 1.28

Solution: The composite bar *AB* consists of three rods (1), (2) and (3).

Given, Length of the rod (1), $l_1 = 200$ mm

Length of the rod (2), $l_2 = 300$ mm

Length of the rod (3), $l_3 = 500$ mm

Diameter of the rod (1), $d_1 = 50$ mm

Diameter of the rod (2), $d_2 = 100$ mm

Diameter of the rod (3), $d_3 = 150$ mm

Rise in temperature, $\Delta T = T_f - T_i = 75 - 25 = 50°\text{C}$

The cross-sectional areas of the three rods are found as

$$A_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4}(50)^2 = 1963.5 \text{ mm}^2$$

$$A_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4}(100)^2 = 7853.98 \text{ mm}^2$$

$$A_3 = \frac{\pi}{4} d_3^2 = \frac{\pi}{4}(150)^2 = 17671.46 \text{ mm}^2$$

If the composite bar were free to expand, then the individual expansion of each rod is given as

$$dl_1 = l_1 \alpha_1 \Delta T$$
$$= 200 \times 1.2 \times 10^{-5} \times 50 = 0.12 \text{ mm}$$

$$dl_2 = l_2 \alpha_2 \Delta T$$
$$= 300 \times 1.8 \times 10^{-5} \times 50 = 0.27 \text{ mm}$$

$$dl_3 = l_3\,\alpha_3\,\Delta T$$
$$= 500 \times 1.6 \times 10^{-5} \times 50 = 0.4 \text{ mm}$$

Hence, the total expansion of the composite bar would have been

$$dl = dl_1 + dl_2 + dl_3 = 0.12 + 0.27 + 0.4 = 0.79 \text{ mm} \qquad \text{... (1)}$$

(*a*) When the supports do not yield, the expansion of the composite bar is prevented and hence, compressive stresses are induced in it.

Let P = Compressive force in composite bar
= Compressive force in each rod

Stresses, strains and elongations in the rods (1), (2) and (3) are given as

	rod (1)	rod (2)	rod (3)
Stress	$\sigma_1 = \dfrac{P}{A_1}$	$\sigma_2 = \dfrac{P}{A_2}$	$\sigma_3 = \dfrac{P}{A_3}$
	$= \dfrac{P}{1963.5}$ N/mm^2	$= \dfrac{P}{7853.98}$ N/mm^2	$= \dfrac{P}{17671.46}$ N/mm^2
Strain	$\in_1 = \dfrac{P}{A_1 E_1}$	$\in_2 = \dfrac{P}{A_2 E_2}$	$\in_3 = \dfrac{P}{A_3 E_3}$
	$= \dfrac{P}{3.927\times 10^8}$	$= \dfrac{P}{2.356\times 10^8}$	$= \dfrac{P}{1.767\times 10^9}$
Elongation	$dl'_1 = \in_1 l_1$	$dl'_2 = \in_2 l_2$	$dl'_3 = \in_3 l_3$
	$= \dfrac{200P}{3.927\times 10^8}$	$= \dfrac{300P}{2.356\times 10^8}$	$= \dfrac{500P}{1.767\times 10^9}$

The total elongation of the composite bar is found as

$$dl'_1 + dl'_2 + dl'_3 = dl$$

$$\frac{200P}{3.927\times 10^8} + \frac{300P}{2.356\times 10^8} + \frac{500P}{1.767\times 10^9} = 0.79 \qquad \text{(using equation (1))}$$

$$2.06568 \times 10^{-6} P = 0.79$$

Solving for P, we get $P = 382454.58$ N

Now the stresses induced in each rod are calculated as

$$\sigma_1 = \frac{P}{A_1} = \frac{382454.58}{1963.5} = 194.78 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

$$\sigma_2 = \frac{P}{A_2} = \frac{382454.58}{7853.98} = 48.7 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

and $$\sigma_3 = \frac{P}{A_3} = \frac{382454.58}{17671.46} = 21.64 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

(*b*) When the supports at the ends yield by 0.25 mm, the net elongation produced is given as

$$dl' = (dl_1 + dl_2 + dl_3) - 0.25$$

$$= 0.79 - 0.25 = 0.54 \text{ mm}$$

Hence, $2.06568 \times 10^{-6} P = 0.54$

Solving for P, we get $P = 261424.65$ N

Stresses in the rods are:

$$\sigma_1 = \frac{261424.65}{1963.5} = 133.14 \text{ N/mm}^2$$ **Ans.**

$$\sigma_2 = \frac{261424.65}{7853.98} = 33.28 \text{ N/mm}^2$$ **Ans.**

and $$\sigma_3 = \frac{261424.65}{17671.46} = 14.79 \text{ N/mm}^2$$ **Ans.**

Hence, in both cases, the maximum stress is induced in rod (1) and the minimum stress in rod (3).

Example 1.19

A weight of 900 kN is supported by three pillars of cross-section 600 mm^2 each (Fig. 1.29). The pillars are so adjusted that at a temperature of 20°C each pillar carries equal load. Find the stress in each pillar at 150°C. Take

$E_1 = 200$ kN/mm^2, $\alpha_1 = 1.6 \times 10^{-5}$/°C

$E_2 = 30$ kN/mm^2, $\alpha_2 = 1.2 \times 10^{-5}$/°C

$E_3 = 100$ kN/mm^2, $\alpha_3 = 1.8 \times 10^{-5}$/°C.

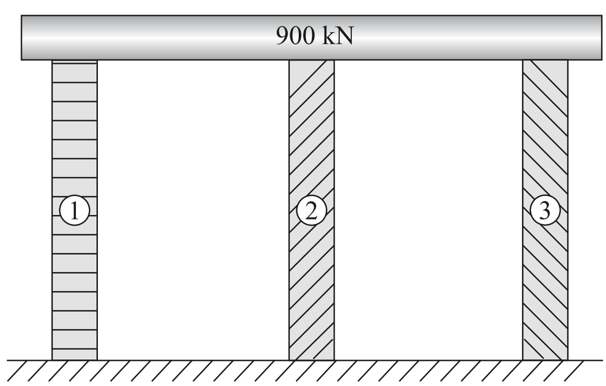


Fig. 1.29

Solution: Given,

Load to be supported, $P = 900 \text{ kN} = 900 \times 10^3 \text{ N}$

Cross-sectional area of each rod $= A_1 = A_2 = A_3 = \text{A} = 600 \text{ mm}^2$

Initial temperature, $T_i = 20°\text{C}$

Final temperature, $T_f = 150°\text{C}$

Rise in temperature, $\Delta T = (150 - 20)°\text{C} = 130°\text{C}$

On heating, the pillars will try to expand but the weight kept on them will prevent their expansion, thereby inducing stresses in them.

Initially *i.e.*, at 20°C, stresses in each pillar are same, given by

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma = \frac{900 \times 10^3}{3 \times 600} = 500 \text{ N/mm}^2 \text{ (Compressive)}$$

Because of equal length and equal weight shared by each pillar, equal strains are produced in them, say $\in$. But because of heating, strain in each pillar changes and takes a new value.

$$\text{Net strain in rod (1)} = \in - \alpha_1 \Delta T$$

$$\text{Net strain in rod (2)} = \in - \alpha_2 \Delta T$$

$$\text{Net strain in rod (3)} = \in - \alpha_3 \Delta T$$

Due to heating, each pillar carries some different load but the total load remains the same. Hence

$$(\in - \alpha_1 \Delta T) A_1 E_1 + (\in - \alpha_2 \Delta T) A_2 E_2 + (\in - \alpha_3 \Delta T) A_3 E_3 = 900 \times 10^3$$

$$[(\in - \alpha_1 \Delta T) E_1 + (\in - \alpha_1 \Delta T) E_2 + (\in - \alpha_3 \Delta T) E_3] A = 900 \times 10^3$$

(Because the pillars are of equal cross-sectional areas)

or

$$[(\in - 1.6 \times 10^{-5} \times 130) \times 200 \times 10^3 + (\in - 1.2 \times 10^{-5} \times 130) \times 30 \times 10^3 + (\in - 1.8 \times 10^{-5} \times 130) \times 100 \times 10^3] \times 600 = 900 \times 10^3$$

Solving for $\in$, we get

$$\in = 6.65 \times 10^{-3}$$

Hence, stress in pillar (1) is

$$\sigma_1 = (\in - \alpha_1 \Delta T) E_1$$
$$= (6.65 \times 10^{-3} - 1.6 \times 10^{-5} \times 130) \times 200 \times 10^3 = 914 \text{ N/mm}^2$$ **Ans.**

Stress in pillar (2) is

$$\sigma_2 = (\in - \alpha_2 \Delta T) E_2$$
$$= (6.65 \times 10^{-3} - 1.2 \times 10^{-5} \times 130) \times 30 \times 10^3 = 152.7 \text{ N/mm}^2$$ **Ans.**

Stress in pillar (3) is

$$\sigma_3 = (\in - \alpha_3 \Delta T) E_3$$
$$= (6.65 \times 10^{-3} - 1.8 \times 10^{-5} \times 130) \times 100 \times 10^3 \text{ N/mm}^2$$
$$= 431 \text{ N/mm}^2$$ **Ans.**

All the stresses are compressive in nature.

Example 1.20

A steel bolt of diameter 10 mm passes through a brass tube of internal diameter 15 mm and external diameter 25 mm as shown in Fig. 1.30.

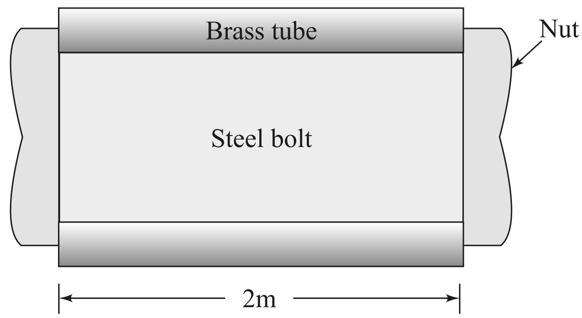


Fig. 1.30

The bolt is tightened by a nut so that the length of tube is reduced by 1.5 mm. If the temperature of the assembly is raised by 40°C, estimate the stresses in the bolt and the tube before and after heating. Take

$$E_s = 200 \text{ kN/mm}^2, \quad \alpha_s = 1.2 \times 10^{-5}/°\text{C}$$

$$E_b = 100 \text{ kN/mm}^2, \quad \alpha_b = 1.9 \times 10^{-5}/°\text{C}.$$

Solution: Refer Fig. 1.30. Given,

Diameter of the steel bolt, $d = 10$ mm

Internal diameter of the brass tube, $d_i = 15$ mm

External diameter of the brass tube, $d_0 = 25$ mm

Length of the brass tube, $l = 2 \text{ m} = 2 \times 1000 = 2000$ mm

Rise in temperature, $\Delta T = 40°\text{C}$

The cross-sectional area of steel bolt is

$$A_s = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ mm}^2$$

The cross-sectional area of brass tube is found as

$$A_b = \frac{\pi}{4} (d_0^2 - d_i^2) = \frac{\pi}{4} (25^2 - 15^2) = 314.16 \text{ mm}^2$$

During tightening of nut on the bolt, the stresses produced in bolt and tube are tensile and compressive respectively. The tensile force in the bolt is equal to the compressive force in the tube, because no external force is acting on the assembly.

$$\sigma_s A_s = \sigma_b A_b$$

$$\sigma_s \times 78.54 = \sigma_b \times 314.16$$

or

$$\sigma_s = 4\sigma_b$$

Stresses before heating

Reduction in tube length, $\Delta l = 1.5$ mm

But $$\Delta l = \frac{\sigma_b l}{E_b}$$

$$1.5 = \frac{\sigma_b \times 2000}{100 \times 10^3}$$

or $\sigma_b = 75$ N/mm^2 **Ans.**

and $\sigma_s = 4\sigma_b = 300$ N/mm^2 **Ans.**

Stresses after heating

When the assembly is heated, on account of higher value of α_b, brass tube will expand more. But due to prevention of expansion, the thermal stresses are induced in the assembly. The total strain in the assembly before heating is equal to the net strain produced in the assembly after heating.

$$\in_b + \in_s = (\alpha_b - \alpha_s)\,\Delta T$$

$$\frac{s_b}{E_b} + \frac{s_s}{E_s} = (\alpha_b - \alpha_s)\,\Delta T$$

$$\frac{\sigma_b}{100 \times 10^3} + \frac{4\sigma_b}{200 \times 10^3} = (1.9 - 1.2) \times 10^{-5} \times 40$$

Solving for σ_b, we get

$\sigma_b = 9.34$ N/mm^2 (Compressive) **Ans.**

and $\sigma_s = 4 \times 9.34 = 37.36$ N/mm^2 (Tensile) **Ans.**

Example 1.21

A 5 m circular rod of diameter 100 mm at one end and 200 mm at other end is fixed between two supports as shown in Fig. 1.31. Determine the stress induced, if the temperature of the rod is raised by 70°C. Take $E = 200$ kN/mm^2 and $\alpha = 1.2 \times 10^{-5}$/°C.

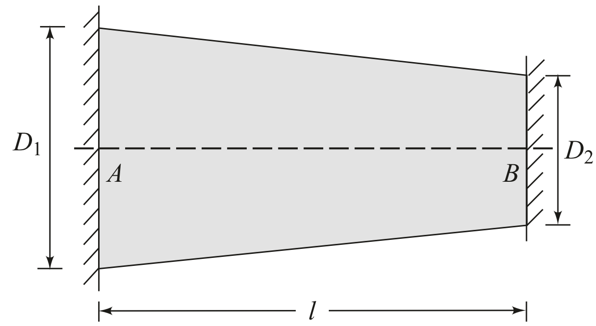


Fig. 1.31

Solution: Refer Fig. 1.31. *AB* is the circular rod of length *l*.

Given,

Diameter at *A*, $D_1 = 200$ mm

Diameter at *B*, $D_2 = 100$ mm

If ends of the rod were not fixed, then on heating, the rod length would have increased by *dl*, given by

$$dl = l\alpha\,\Delta T \quad \text{... (1)}$$

If a compressive force *P* is applied, the contraction achieved is given by

$$dl = \frac{4Pl}{\pi E D_1 D_2} \quad \text{... (2)}$$

Equating two equations (1) and (2), we have

$$l\alpha\,\Delta T = \frac{4Pl}{\pi E D_1 D_2}$$

or

$$P = \frac{\alpha\,\Delta T\,\pi\,E\,D_1\,D_2}{4}$$

$$= \frac{1.2\times10^{-5}\times70\times\pi\times200\times10^3\times200\times100}{4} = 5.27\times10^6\text{ N}$$

The maximum stress is induced at the smaller end, given by

$$\sigma_{max} = \frac{P}{\text{Cross-sectional area at }B} = \frac{P}{(\pi/4)D_2^2} = \frac{5.27\times10^6}{\pi/4\times100^2}$$

$$= 671\text{ N/mm}^2\text{ (Compressive)}$$ **Ans.**

Example 1.22

Determine the temperature rise necessary to induce buckling in a 1 m long circular rod of diameter 40 mm as shown in Fig. 1.32. Assume the rod to be pinned at its ends, and the coefficient of thermal expansion as 20×10^{-6}/°C. The rod is heated uniformly. Take $E = 200$ GPa.

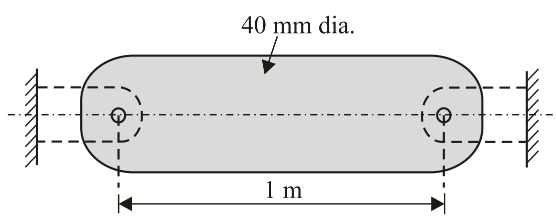


Fig. 1.32

Solution: Given,

Length of the rod, $l = 1$ m

Diameter of the rod, $d = 40\text{ mm} = 40 \times 10^{-3}$ m

Coefficient of thermal expansion,

$$\alpha = 20 \times 10^{-6}/°\text{C}$$

The cross-sectional area of the rod is given as

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times (40 \times 10^{-3})^2 = 1.256 \times 10^{-3} \text{ m}^2$$

The thermal strain is given as

$$\begin{aligned} \epsilon_t &= \alpha \, \Delta T && (\Delta T \text{ is rise in temperature}) \\ &= 20 \times 10^{-6} \, \Delta T \end{aligned}$$

The thermal stress is given as

$$\begin{aligned} \sigma_t &= \epsilon_t \cdot E \\ &= 20 \times 10^{-6} \, \Delta T \cdot E \end{aligned}$$

A force P equal to $\sigma_t \cdot A$, where A is the cross-sectional area of the rod, is acting on the rod, if its ends are prevented from expansion. According to Euler's formula, when both ends of the rod are hinged, the critical load is given as

$$P = \frac{\pi^2 EI}{l^2}$$

or $$20 \times 10^{-6} \, \Delta T \cdot E \times 1.256 \times 10^{-3} = \frac{\pi^2 E \times \frac{\pi}{64} \times (40 \times 10^{-3})^4}{1^2}$$

Solving for ΔT, we get $\Delta T = 49.37°\text{C}$ **Ans.**

SHORT ANSWER QUESTIONS

1. What is a prismatic bar? What benefit does it offer?
2. Why is limit of proportionality so called?
3. How does elastic limit differ from limit of proportionality?
4. What does the zero value of Poisson's ratio indicate?
5. What is offset strain? What is its significance?
6. Why is the shear modulus always smaller than the elastic modulus?
7. Why is the actual stress always greater than the engineering stress?
8. Why is factor of safety used in engineering design?
9. What does the yielding of a material mean?
10. What is working stress? How does it differ from ultimate stress?
11. How is shear strain different from direct strain? What is the effect of shear strain on the volume of a body?
12. Why is mild steel the most important material for engineering applications?

MULTIPLE CHOICE QUESTIONS

1. Poisson's ratio is defined as the ratio of

(*a*) longitudinal strain to lateral strain (*b*) lateral strain to longitudinal strain

(*c*) axial stress to shear stress (*d*) axial stress to bending stress.

2. The maximum possible theoretical value of Poisson's ratio is

(*a*) 0.25 (*b*) 0.35 (*c*) 0.50 (*d*) 1.0.

3. For a 12 mm diameter steel rod test specimen, the suitable gauge length is

(*a*) 24 mm (*b*) 36 mm (*c*) 72 mm (*d*) 60 mm.

4. The stress produced on a surface normal to the load applied is called

(*a*) shear stress (*b*) bending stress (*c*) normal stress (*d*) none of these.

5. The deformation of a uniform section bar subjected to an axial pull P is given as

(*a*) $\frac{Pl}{AE}$ (*b*) $\frac{2Pl}{AE}$ (*c*) $\frac{Pl}{2AE}$ (*d*) $\frac{Pl}{3AE}$.

6. The tensile load results in

(*a*) contraction (*b*) elongation (*c*) bending (*d*) twisting.

7. The factor of safety is a ratio of

(*a*) shear stress to working stress (*b*) bending stress to shear stress

(*c*) ultimate stress to working stress (*d*) working stress to ultimate stress.

8. The relationship between E and G is

(*a*) $E = 2G(1 - v)$ (*b*) $E = 2G(1 + v)$

(*c*) $E = 2G(1 - 2v)$ (*d*) $E = 2\,G(1 + 2v)$.

9. The relationship between E and K is

(*a*) $E = 3\,K(1 - 2v)$ (*b*) $E = 3\,K(1 + 2v)$

(*c*) $E = 2\,K(1 - 2v)$ (*d*) $E = 2\,K(1 + 2v)$.

10. The relationship among E, G and K is

(*a*) $E = \frac{3KG}{2K + G}$ (*b*) $E = \frac{9KG}{3K + G}$ (*c*) $E = \frac{5KG}{2K + G}$ (*d*) $G = \frac{9EK}{3E + K}$.

11. The shear stress

(*a*) acts normal to the surface (*b*) acts tangential to the surface

(*c*) is equal to the tensile stress (*d*) is equal to the compressive stress.

12. The modulus of rigidity is defined as a ratio of

(*a*) shear strain to volumetric strain (*b*) shear stress to shear strain

(*c*) normal stress to shear strain (*d*) normal stress to linear strain.

13. If brass has higher coefficient of thermal expansion than steel, it means that

(*a*) both brass and steel have equal expansion

(*b*) steel expands more than brass

(*c*) brass expands more than steel

(*d*) expansion is not related to coefficient of thermal expansion.

14. The thermal expansion of a material varies

(*a*) directly proportional to coefficient of thermal expansion

(*b*) inversely proportional to coefficient of thermal expansion

(*c*) directly proportional to the square of coefficient of thermal expansion

(*d*) none of these.

15. The thermal stress depends on

(*a*) temperature rise or fall and the modulus of elasticity of the material

(*b*) coefficient of thermal expansion/contraction and temperature rise/fall

(*c*) temperature rise/fall, coefficient of thermal expansion/contraction and modulus of elasticity of material

(*d*) none of these.

16. During the tightening of a nut on a bolt, the stress induced in the bolt is

(*a*) compressive (*b*) shear (*c*) tensile (*d*) bending.

17. The length of a bar at 15°C is 150 mm. Its length at 50°C, if the coefficient of thermal expansion of its material is 1.2×10^{-5}/°C, is given by

(*a*) 151.02 mm (*b*) 150.063 mm (*c*) 150.036 mm (*d*) 150.36 mm.

ANSWERS

1. (*b*)	**2.** (*c*)	**3.** (*d*)	**4.** (*c*)	**5.** (*a*)	**6.** (*b*)	**7.** (*c*)	**8.** (*b*)
9. (*a*)	**10.** (*b*)	**11.** (*b*)	**12.** (*b*)	**13.** (*c*)	**14.** (*a*)	**15.** (*c*)	**16.** (*c*)
17. (*b*).							

EXERCISES

1. A 3 m solid rectangular bar of cross-section 10 mm × 15 mm is subjected to a compressive force of 150 kN. What is the change in length of the bar? Also, find the strain and stress produced in the bar. Take $E = 2 \times 10^5$ N/mm^2.

(*Ans.* 15 mm (Decrease), 0.005, 1000 N /mm^2).

2. A square section rod of length l and side D at one end tapers to square section of side d at the other end. Find the elongation produced when subjected to an axial pull P.

$$\left(Ans.\ \frac{Pl}{EDd}\right).$$

3. A 1.5 m circular rod tapers uniformly from 40 mm diameter at one end to 20 mm diameter at other end. Find the elongation produced, if it is subjected to an axial pull of 200 kN. Take $E = 2 \times 10^5$ N/mm^2.

(*Ans.* 2.39 mm).

4. A rod rigidly fixed at its left end consists of three bars of unequal diameters (Fig, 1.33). Find the stress in each bar of the rod. Take $E = 2 \times 10^5$ N/mm^2.

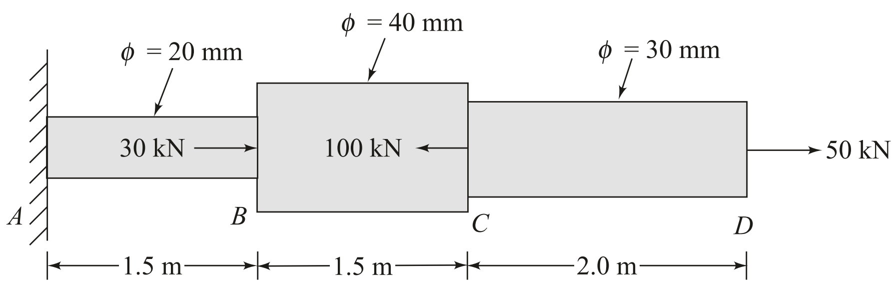


Fig. 1.33

(*Ans.* $\sigma_{AB} = 63.66$ N/mm^2

$\sigma_{BC} = 39.78$ N/mm^2

$\sigma_{CD} = 70.73$ N/mm^2).

5. A 3 m steel rod of diameter 25 mm is placed inside a brass tube of the same length and inside and outside diameters of 25 mm and 35 mm respectively (Fig. 1.34). What is the deformation of the rod and the tube when a force of 50 kN is applied on them through a rigid plate? Take

$E_s = 2 \times 10^5$ N/mm^2

and $E_b = 1 \times 10^5$ N/mm^2.

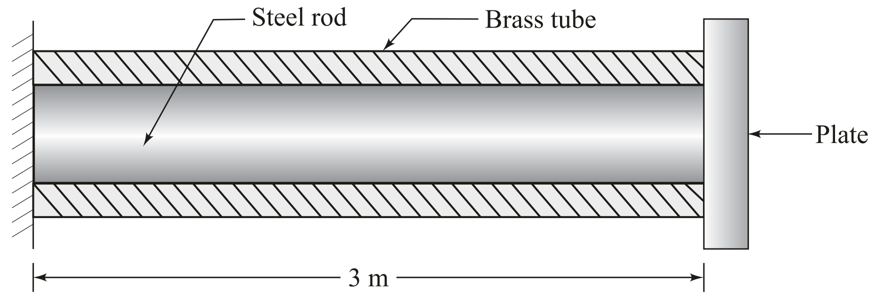


Fig. 1.34

(*Ans.* 1.032 mm, deformations in the rod and the tube are equal).

6. Two cylindrical rods one of steel and the other of brass are joined at *D* and restrained by rigid supports at *A* and *B*. For the loading given in Fig. 1.35 and assuming $E_s = 2 \times 10^5$ N/mm^2 and $E_b = 1.05 \times 10^5$ N/mm^2, find (*a*) the reactions at *A* and *B* and (*b*) the deflection of point *D*.

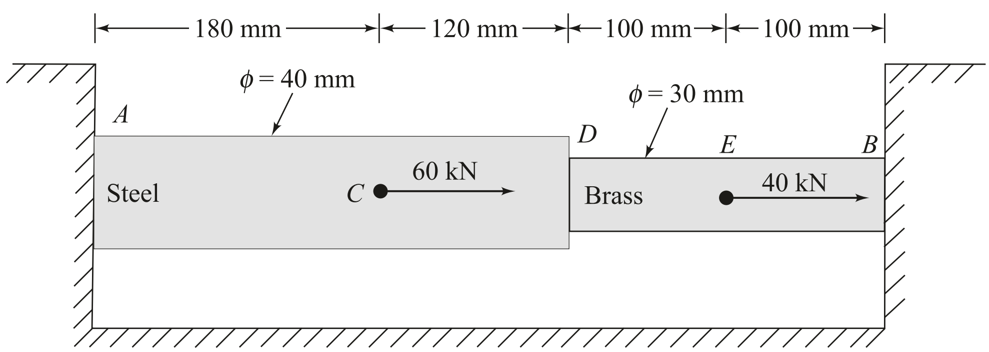


Fig. 1.35

(*Ans.* (*a*) R_A = 62.8 kN (←), R_B = 37.2 *kN* (←)
(*b*) 0.0463 mm (→)).

7. A composite bar made of brass and steel is fixed between two supports (Fig. 1.36). If the temperature is increased by 80°C, find the stresses induced in the steel and brass section assuming (*a*) if the supports do not yield and (*b*) if the supports yield by 0.15 mm.

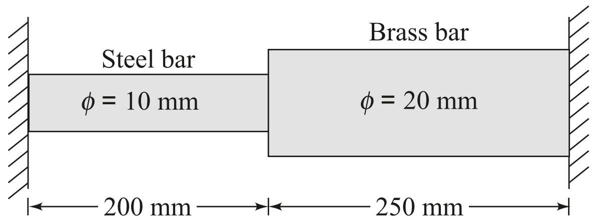


Fig. 1.36

Take $E_s = 2 \times 10^5$ N/mm^2

$E_b = 1 \times 10^5$ N/mm^2

$\alpha_s = 1.2 \times 10^{-5}$ N/°C

$\alpha_b = 1.9 \times 10^{-5}$ N/°C.

(*Ans.* (*a*) 352 N/mm^2, 88 N/mm^2
(*b*) 259.7 N/mm^2, 64.92 N/mm^2).

8. A composite bar made of two steel sections is rigidly fixed at the top. The gap between its lower end and the rigid support is 0.2 mm (Fig. 1.37). Estimate the stresses in the two sections, if the bar is heated through 80°C. Take $E = 2 \times 10^5$ N/mm^2 and $\alpha = 1.2 \times 10^{-5}$/°C.

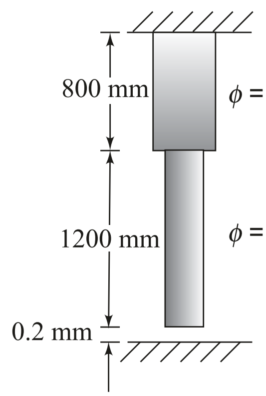


Fig. 1.37

(*Ans.* 61.42 N/mm^2, 245.71 N/mm^2).

9. A steel rod of 20 mm diameter passes through a brass tube of 30 mm inside diameter and 40 mm outside diameter. Two nuts one on each side are tightened until a stress of 10 N/mm^2 is induced in the rod. Find the stresses in the rod and the tube, if the assembly is heated through 60°C. Take

$E_s = 2 \times 10^5$ N/mm^2, $\alpha_s = 1.2 \times 10^{-5}$/°C

$E_b = 0.8 \times 10^5$ N/mm^2, $\alpha_b = 1.9 \times 10^{-5}$/°C.

(*Ans.* 44.58 N/mm^2 (Tensile), 25.47 N/mm^2 (Compressive)).

10. A 1.5 m circular bar of tapered section is rigidly fixed at both ends (Fig. 1.38). Find the maximum stress induced in the bar, if its temperature is increased by 50°C. Take

$E = 2 \times 10^5$ N/mm^2 and $\alpha = 1.2 \times 10^{-5}$/°C.

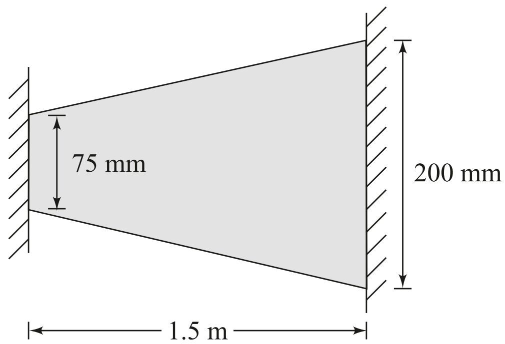


Fig. 1.38

(*Ans.* σ_{max} = 320 N/mm^2 (Compressive)).

11. A compound bar made of a steel rod and a brass rod is rigidly fixed at the two ends (Fig. 1.39). Find the stresses in the two rods, if the assembly is heated through 50°C. Take

$$E_s = 2 \times 10^5 \text{ N/mm}^2,\ \alpha_s = 1.2 \times 10^{-5}/°\text{C}$$
$$E_b = 1 \times 10^5 \text{ N/mm}^2,\ \alpha_b = 1.9 \times 10^{-5}/°\text{C}.$$

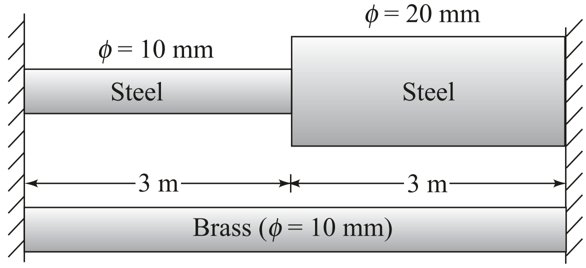


Fig. 1.39

(*Ans.* Stress in brass = 25.45 N/mm^2
Stress in steel = 12.72 N/mm^2).

12. A thin-walled cylinder with closed ends and a thin-walled sphere of the same diameter and same wall thickness are put under the same internal pressure. Find the ratio of the change in diameter of the cylinder to the change in diameter of the sphere.

(*Ans.* $(2 - v)/(1 - v)$).

13. A 75 mm diameter compound bar is constructed by shrinking a circular brass bush onto the outside of a 50 mm diameter solid steel rod. The compound bar is now subjected to an axial compressive load of 160 kN. Determine the load carried by the steel rod and the brass bush, and the compressive stress set up in each material. Take E_s = 210 GPa and E_b = 100 GPa

(*Ans.* 100.3 kN, 59.7 kN, 51.1 MPa, 24.3 MPa).

❑❑❑

2 Principal Stresses

Christian Otto Mohr
(1835-1918)

Christian Otto Mohr, born on 8 October 1835, was a German civil engineer. He designed some famous bridges while working on railroad engineering. He had keen interest in the theories of mechanics and the strength of materials. In 1867, he became professor of mechanics at Stuttgart Polytechnic, and in 1873 at Dresden Polytechnic. In 1874, Mohr formalised the idea of a statically indeterminate structure. In 1882, he developed the famous graphical method for analysing stress, known as Mohr's circle, and used it to propose an early theory of strength based on shear stress. He also developed the Williot-Mohr diagram for truss displacements and the Maxwell-Mohr method for analysing statically indeterminate structures, the former method can also be used to determine the displacement of truss nodes and forces acting on each member. The Maxwell-Mohr method is also referred to as the virtual force method for redundant trusses.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- What is a principal plane?
- What are the principal stresses?
- How is the maximum shear stress related to the principal stresses?
- Why is Mohr's circle important?
- What does the radius of Mohr's circle indicate?

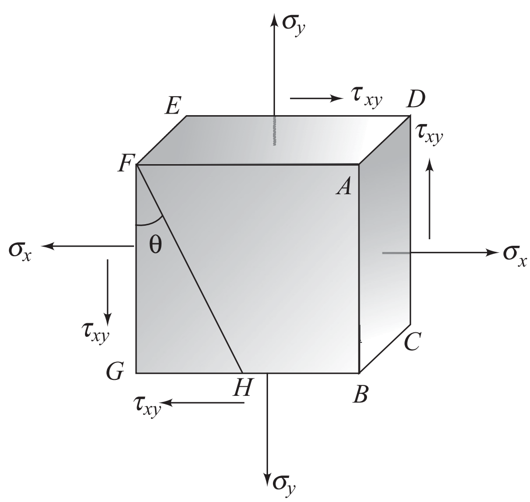
D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_2

2.1 INTRODUCTION

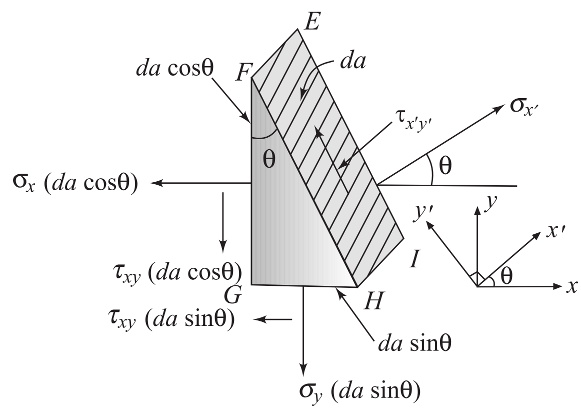
Stresses and strains produced due to uniaxial loading are discussed in the first chapter. These are the most simple and idealized cases. In those cases, planes on which stresses are induced, are normal to the direction of load applied. But in the actual practice, plane is not always in such condition, rather it is oblique to the load applied and hence stresses and strains are not simple in nature and the effect of normal stress and shear stress are considered simultaneously for such analysis. Now the plane is under combined or compound stress condition. We will discuss the most general case, when the body is subjected to two normal stresses and a shear stress, that is, it is a plane stress condition.

2.2 STRESSES ON AN INCLINED PLANE (PRINCIPAL PLANES AND PRINCIPAL STRESSES)

Consider a rectangular element of unit thickness being subjected to two normal stresses (tensile) in x and y directions and a shear stress τ_{xy} (Fig. 2.1). The inclined plane is *HIEF*. Let da be the area of the inclined plane making an angle θ with the vertical plane.



(*a*) Rectangular element subjected to various stresses



(*b*) Forces acting on different planes

Fig. 2.1

Area of plane	FG	$= da\cos\theta$
Area of plane	GH	$= da\sin\theta$
Normal force acting on the plane	FG	$= \sigma_x\, da\cos\theta$
Normal force acting on the plane	GH	$= \sigma_y\, da\sin\theta$
Shear force acting on the plane	FG	$= \tau_{xy}\, da\cos\theta$
Shear force acting on the plane	GH	$= \tau_{xy}\, da\sin\theta$

For inclined plane *HIEF*

Normal force acting on the plane	$= \sigma_{x'}\, da$
Shear force acting on the plane	$= \tau_{x'y'}\, da$

Considering the equilibrium of forces normal to the inclined plane and tangential to it, we have

$$\sigma_{x'} da - (\sigma_x\, da \cos\theta) \cos\theta - (\sigma_y\, da \sin\theta) \sin\theta - (\tau_{xy}\, da \cos\theta) \sin\theta - (\tau_{xy}\, da \sin\theta) \cos\theta = 0 \quad \text{... (2.1)}$$

$$\tau_{x'y'} da + (\sigma_x\, da \cos\theta) \sin\theta - (\sigma_y\, da \sin\theta) \cos\theta - (\tau_{xy}\, da \cos\theta) \cos\theta + (\tau_{xy}\, da \sin\theta) \sin\theta = 0 \quad \text{...(2.2)}$$

From equation (2.1), we get

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

$$= \sigma_x \left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y \left(\frac{1-\cos 2\theta}{2}\right) + \tau_{xy} \sin 2\theta$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \quad \text{... (2.3)}$$

From equation (2.2), we have

$$\tau_{x'y'} = (\sigma_y - \sigma_x) \sin\theta \cos\theta + \tau_{xy} (\cos^2\theta - \sin^2\theta)$$

$$= -(\sigma_x - \sigma_y) \sin\theta \cos\theta + \tau_{xy} (\cos^2\theta - \sin^2\theta)$$

$$= -\frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \quad \text{... (2.4)}$$

The resultant stress on the inclined plane is

$$\sigma_r = \sqrt{\sigma_{x'}^2 + \tau_{x'y'}^2} \quad \text{... (2.5)}$$

Principal plane is a plane of zero shear stress. There are two principal planes at any arbitrary point within a material under plane stress condition. These planes are mutually perpendicular to each other and have only normal stresses.

For plane *HIEF* to be a principal plane

$$\tau_{x'y'} = 0$$

$$\frac{\sigma_x - \sigma_y}{2} \sin 2\theta_p - \tau_{xy} \cos 2\theta_p = 0 \qquad \text{(denoting } \theta \text{ by } \theta_p \text{ for principal plane)}$$

$$\frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta_p = \tau_{xy} \cos 2\theta_p$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \text{... (2.6)}$$

Since $\tan 2\theta_p = \tan(180° + 2\theta_p)$

Hence, there are two planes for which equation (2.6) holds good. One plane is located at

$\theta_{p_1} = \dfrac{1}{2} \tan^{-1}\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$ and the other plane is separated by 90°, that is, at $\theta_{p_2} = \theta_{p_1} + 90°$.

From equation (2.6), we have

$$\sin 2\theta_p = \frac{2\tau_{xy}}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{\pm\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Substituting $\sin 2\theta_p$ and $\cos 2\theta_p$ in equation (2.3), we have

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left[\pm\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}\right] + \tau_{xy}\left[\pm\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}\right]$$

$$= \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}\left[\frac{(\sigma_x - \sigma_y)^2}{2} + 2\tau_{xy}^2\right]$$

$$= \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}\left[\frac{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}{2}\right]$$

$$= \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

The stress with maximum value is known as major principal stress, given by

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2} \qquad \text{... (2.7)}$$

The stress with minimum value is known as minor principal stress, given by

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2} \qquad \text{... (2.8)}$$

Maximum shear stress

For maximum shear stress, differentiate equation (2.4) w.r.t. θ and equate it to zero.

$$\frac{d(\tau_{x'y'})}{d\theta} = 0$$

$$\left(\frac{\sigma_x - \sigma_y}{2}\right) 2\cos 2\theta_s - \tau_{xy}\,(-\sin 2\theta_s)\,.\,2 = 0$$

(Denoting θ by θ_s for shear stress)

$$(\sigma_x - \sigma_y)\cos 2\theta_s + 2\tau_{xy}\sin 2\theta_s = 0$$

or $$\tan 2\theta_s = -\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right) \qquad \text{...(2.9)}$$

Hence, $$\sin 2\theta_s = \pm\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

and $$\cos 2\theta = \pm\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Substituting $\sin 2\theta_s$ and $\cos 2\theta_s$ in equation (2.4), we have

$$\tau_{\max} = \pm\frac{\sigma_x - \sigma_y}{2}\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \pm \tau_{xy}\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$= \pm\frac{(\sigma_x - \sigma_y)^2}{2\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \pm \tau_{xy}\frac{2\tau_{xy}^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau}}$$

$$= \pm\frac{1}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}\left[\frac{(\sigma_x - \sigma_y)^2}{2} + 2\tau_{xy}^2\right]$$

$$= \pm\frac{1}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}\left[\frac{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}{2}\right]$$

$$= \pm\frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2} \qquad \text{... (2.10)}$$

Using equations (2.7) and (2.8), we get the maximum shear stress in terms of principal stresses as

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} \qquad \text{... (2.11)}$$

Hence, the maximum shear strtess is one-half the difference between the maximum and minimum principal stresses, and it occurs on the planes inclined at 45° to the principal planes.

Example 2.1

For a plane stress condition shown in Fig. 2.2, find the following parameters:

(*a*) the principal stresses

(*b*) the maximum shear stress

(*c*) the corresponding normal stress

(*d*) the position of the principal planes and

(*e*) the plane of maximum shear stress.

Solution: Refer Fig. 2.2.

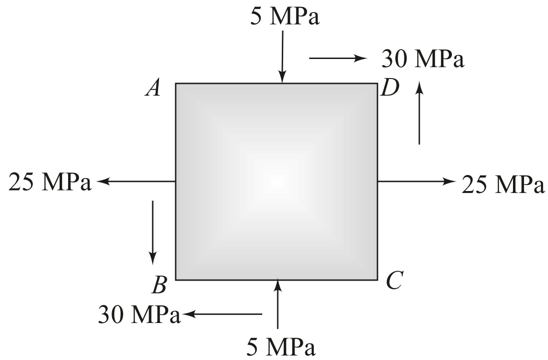


Fig. 2.2

$$\sigma_x = 25 \text{ MPa}$$
$$\sigma_y = -5\text{MPa}$$
$$\tau_{xy} = 30 \text{ MPa}$$

(*a*) The principal stresses are given as

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$= \frac{25 + (-5)}{2} \pm \frac{\sqrt{\{25 - (-5)\}^2 + 4 \times 30^2}}{2} = 10 \pm 15\sqrt{5}$$

Hence, the major principal stress, $\sigma_1 = (10 + 15\sqrt{5})$ MPa $= 43.54$ MPa

The minor principal stress, $\sigma_2 = (10 - 15\sqrt{5})$ MPa $= -23.54$ MPa **Ans.**

(*b*) The maximum shear stress is

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{43.54 - (-23.54)}{2} = 33.54 \text{ MPa}$$ **Ans.**

(*c*) The normal stress is the average stress acting on all the four faces.

$$\sigma_{av} = \frac{\sigma_x + \sigma_y}{2} = \frac{25 - 5}{2} = 10 \text{ MPa}$$ **Ans.**

(*d*) Using equation (2.6), we have

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2 \times 30}{25 - (-5)} = \frac{60}{30} = 2$$

$$2\theta_p = \tan^{-1}(2) = 63.4^\circ \text{ and } 180^\circ + 63.4^\circ = 243.4^\circ$$

or $$\theta_p = 31.7^\circ \text{ and } 121.7^\circ$$ **Ans.**

These are the angles made by the principal planes with the plane *AB*.

(*e*) If θ_s be the angle made by the planes of maximum shear stress with *AB*, then

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \qquad \text{(using equation (2.9))}$$

$$= -\frac{25-(-5)}{2\times 30} = -\frac{1}{2}$$

$$2\theta_s = -26.5^{\circ}$$

$$\theta_s = -13.25^{\circ}$$

Negative sign signifies that the angle is measured in the clockwise direction. Hence, the actual angle (in the anticlockwise direction) is

$$\theta_s = 76.72^{\circ} \text{ and } 166.72^{\circ} \qquad \textbf{Ans.}$$

Example 2.2

Show that the sum of the normal stresses on any set of two perpendicular planes at a point in a strained material is constant.

Solution: Refer Fig. 2.3.

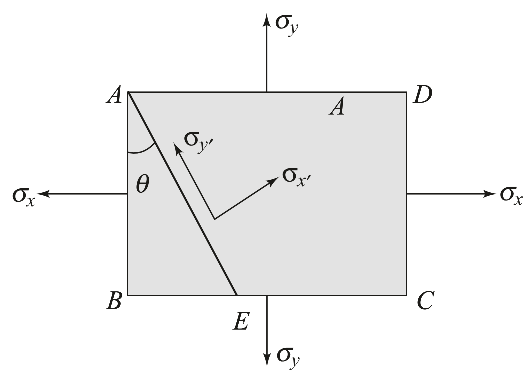


Fig. 2.3

ABCD is the rectangular element on which two principal stresses σ_x and σ_y are acting. An inclined plane *AE* is making an angle θ with the plane *AB*.

The principal stress acting on the plane *AE* is

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta$$

The principal stress on another plane at $(\theta + 90^{\circ})$ is

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(180^{\circ} + 2\theta)$$

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta \qquad [\cos(180^{\circ} + 2\theta) = -\cos 2\theta]$$

Now $\sigma_1 + \sigma_2 = \sigma_x + \sigma_y = \text{Constant}$

Example 2.3

The principal stresses at a point across two perpendicular planes are 60 MPa and 50 MPa. Find the normal, tangential and resultant stress and its obliquity on a plane at 20° with the major principal plane.

Solution: Refer Fig. 2.4 (*a*).

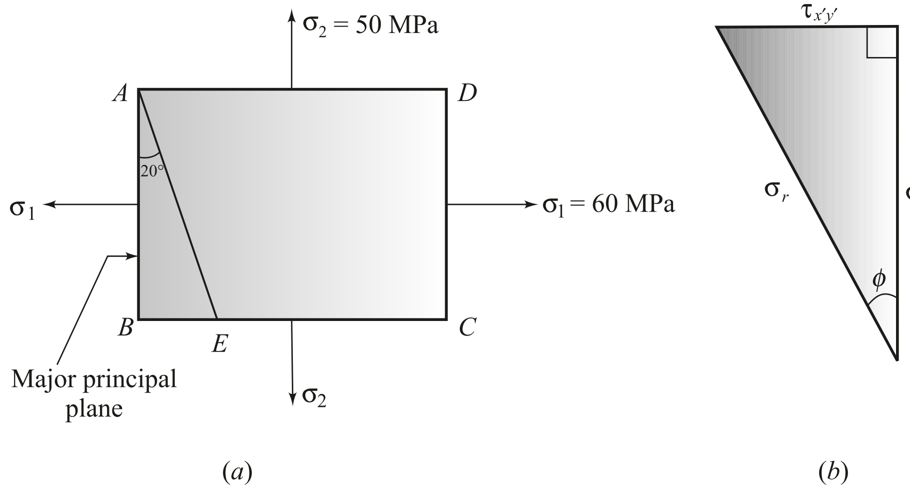


Fig. 2.4

AB is the major principal plane. Given,

$$\sigma_1 = 60 \text{ MPa}$$
$$\sigma_2 = 50 \text{ MPa}$$
$$\theta = 20^{\circ}$$

The normal stress on the inclined plane *AE* is

$$\sigma_{x'} = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 40^{\circ} + 0 \qquad \text{(using equation (2.3))}$$

(because no shear stress is acting)

$$= \frac{60+50}{2} + \frac{60-50}{2} \cos 40^{\circ} = 58.83 \text{ MPa} \qquad \textbf{Ans.}$$

The tangential stress on the plane *AE* is given as

$$\tau_{x'y'} = \frac{\sigma_1 - \sigma_2}{2} \sin 40^{\circ} \qquad \text{(using equation (2.4))}$$

$$= \frac{60-50}{2} \sin 40^{\circ} = 3.21 \text{ MPa} \qquad \textbf{Ans.}$$

The resultant stress is

$$\sigma_r = \sqrt{\sigma_{x'}^2 + \tau_{x'y'}^2}$$

$$= \sqrt{(58.83)^2 + (3.21)^2} = 58.91 \text{ MPa} \qquad \textbf{Ans.}$$

If ϕ is the angle of obliquity (Refer Fig. 2.4 (*b*)), then

$$\tan\phi = \frac{\tau_{x'y'}}{\sigma_{x'}} = \frac{3.21}{58.83} = 0.054$$

or $\phi = 3.12°$ **Ans.**

Example 2.4

At a certain point in a strained material, there are two mutually perpendicular planes. The normal stresses acting on them are 80 MPa tensile and 30 MPa compressive. If the major principal stress is 100 MPa tensile, find the following parameters:

(*a*) the shear stress acting on two planes

(*b*) the minor principal stress and

(*c*) the maximum shear stress at the point.

Solution: Refer Fig. 2.5.

Given, $\sigma_x = 80$ MPa

$\sigma_y = -30$ MPa

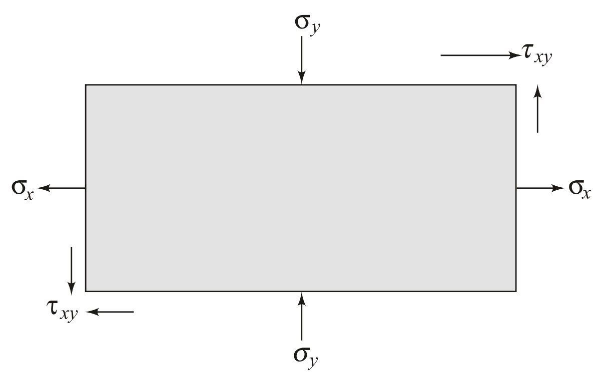


Fig. 2.5

Let σ_1 and σ_2 be the major and the minor principal stresses and τ_{xy}, the shear stress acting on the two planes.

$$\sigma_1 = 100 \text{ MPa (Given)}$$

(*a*) The major principal stress is given by

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$100 = \frac{80 - 30}{2} + \frac{\sqrt{(80+30)^2 + 4\tau_{xy}^2}}{2}$$

or $\tau_{xy} = 51$ MPa **Ans.**

(*b*) The minor principal stress is given as

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$= \frac{80 - 30}{2} - \frac{\sqrt{(80+30)^2 + 4 \times 51^2}}{2} = -\,50\text{MPa}$$

= 50 MPa (Compressive) **Ans.**

(*c*) The maximum shear stress is given as

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{100 - (-50)}{2} = 75 \text{ MPa}$$ **Ans.**

Example 2.5

At a point in a strained material, the two normal stresses acting on two mutually perpendicular planes are 180 MPa tensile and 50 MPa compressive. A shear stress of 20 MPa is also acting on these plnaes. Find the normal stress, the tangential stress and the resultant stress on a plane inclined at 30° to the plane of compressive stress. Also, find the angle of obliquity.

Solution: Refer Fig. 2.6.

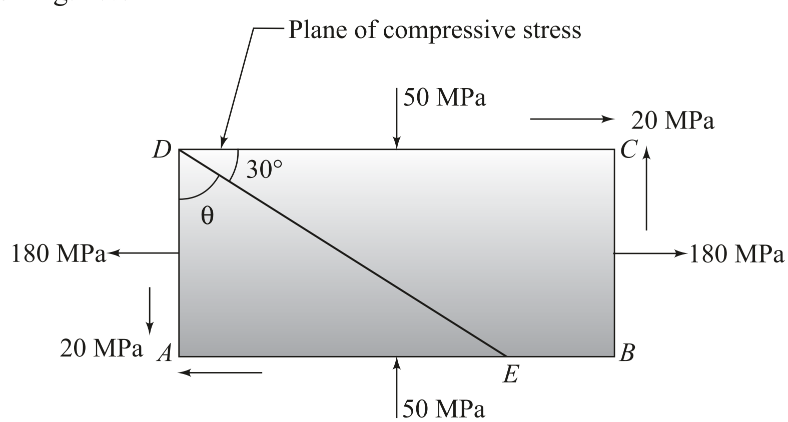


Fig. 2.6

CD is the plane of compressive stress and *DE* is the inclined plane.

Given,

$$\sigma_x = 180 \text{ MPa}$$
$$\sigma_y = -\,50 \text{ MPa}$$
$$\tau_{xy} = 20 \text{ MPa}$$
$$\theta = 90° - 30° = 60°$$

The normal stress on the plane *DE* is given as

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \qquad \text{(using equation (2.3))}$$

$$= \frac{180 - 50}{2} + \frac{180 + 50}{2} \cos 120° + 20 \times \sin 120°$$

= 24.82 MPa **Ans.**

The shear stress on the plane DE is given as

$$\tau_{x'y'} = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta - \tau_{xy} \cos 2\theta \qquad \text{(using equation (2.4))}$$

$$= \frac{180 + 50}{2} \sin 120^\circ - 20 \times \cos 120^\circ = 109.6 \text{ MPa} \qquad \textbf{Ans.}$$

The resultant stress on the plane DE is given as

$$\sigma_r = \sqrt{\sigma_{x'}^2 + \tau_{x'y'}^2} \qquad \text{(using equation (2.5))}$$

$$= \sqrt{(24.82)^2 + (109.6)^2} = 112.37 \text{ MPa} \qquad \textbf{Ans.}$$

The angle of obliquity ϕ is given by

$$\tan \phi = \frac{\tau_{x'y'}}{\sigma_{x'}} = \frac{109.6}{24.82} = 4.41$$

or $$\phi = 77.24^\circ \qquad \textbf{Ans.}$$

Example 2.6

At a certain point within a strained material, the two normal stresses acting on two mutually perpendicular planes are 60 MPa tensile and 30 MPa compressive. The maximum principal stress is limited to 100 MPa. Find the shear stress on the planes. Also, find the maximum shear stress at the point.

Solution: Refer Fig. 2.7.

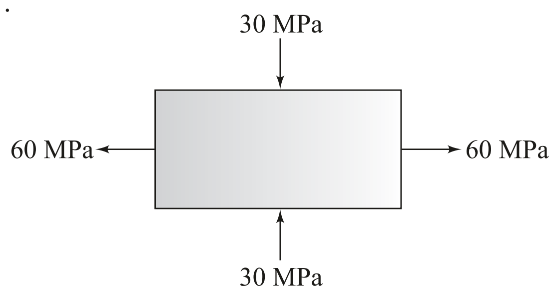


Fig. 2.7

Given,
$$\sigma_x = 60 \text{ MPa}$$
$$\sigma_y = -30 \text{ MPa}$$
$$\sigma_1 = 100 \text{ MPa}$$

The major principal stress is given as

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$100 = \frac{60 - 30}{2} + \frac{\sqrt{(60 + 30)^2 + 4\tau_{xy}^2}}{2}$$

On solving, we get $$\tau_{xy} = 72.11 \text{ MPa} \qquad \textbf{Ans.}$$

The maximum shear stress is given as

$$\tau_{max} = \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2} \qquad \text{(using equation (2.10))}$$

$$= \frac{\sqrt{(60+30)^2 + 4 \times 5200}}{2} = 85 \text{ MPa} \qquad \textbf{Ans.}$$

Example 2.7

In a strained material, the state of stress at a point is given below:

$$\sigma_x = 40 \text{ MPa}, \sigma_y = 25 \text{ MPa and } \tau_{xy} = 15 \text{ MPa}$$

Find the following parameters:

(*a*) the principal stresses on two mutually perpendicular planes at the point
(*b*) the maximum shear stress
(*c*) the principal stress planes
(*d*) the planes of maximum shear stress and
(*e*) the normal stress and shear stress on the planes of maximum shear stress.

Solution: Refer Fig. 2.8.

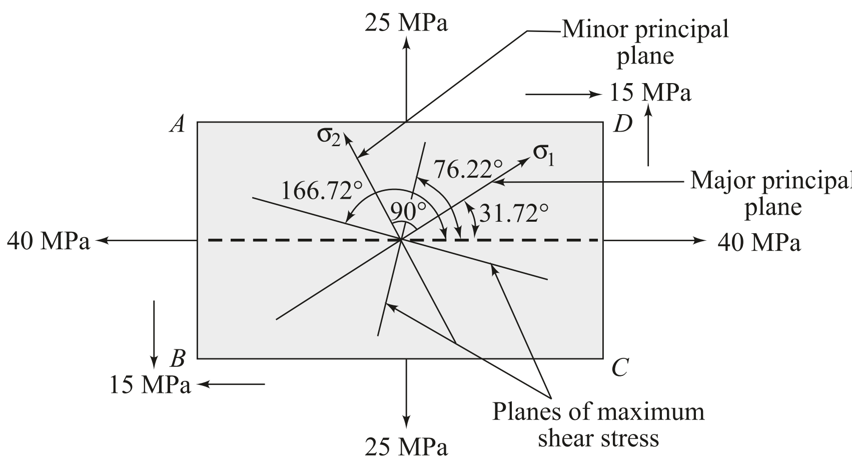


Fig. 2.8

(*a*) The major principal stress is given as

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$= \frac{40+25}{2} + \frac{\sqrt{(40-25)^2 + 4 \times 15^2}}{2} = 49.27 \text{ MPa} \qquad \textbf{Ans.}$$

The minor principal stress is given as

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$= \frac{40 + 25}{2} - \frac{\sqrt{(40-25)^2 + 4 \times 15^2}}{2} = 15.73 \text{ MPa} \qquad \textbf{Ans.}$$

(*b*) The maximum shear stress is obtained as

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{49.27 - 15.73}{2} = 16.77 \text{ MPa} \qquad \textbf{Ans.}$$

(*c*) The principal planes are given by

$$\tan 2\theta_p = \frac{\sigma_1 - \sigma_2}{2} \qquad \text{(using equation (2.6))}$$

$$= \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 2$$

$$2\theta_p = 63.44^\circ \text{ and } 243.44^\circ$$

or $\theta_p = 31.72^\circ$ and 121.72°

Hence, $\theta_{p_1} = 31.72^\circ$

and $\theta_{p_2} = 121.72^\circ$

(*d*) The planes of maximum shear stress are given by

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \qquad \text{(using equation (2.9))}$$

$$= -\frac{40 - 25}{2 \times 15} = -0.5$$

$$2\theta_s = 153.44^\circ \text{ and } 333.44^\circ$$

or $\theta_s = 76.72^\circ$ and 166.72°

The angle between maximum shear stress plane and the direction of σ_x is 76.72°. The normal stress on the planes of maximum shear stress is given as

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta_s + \tau_{xy} \sin 2\theta_s$$

(using equation (2.3))

$$= \frac{40 + 25}{2} + \frac{40 - 25}{2} \cos (153.44^\circ) + 15 \sin (153.44^\circ)$$

$$= 32.5 \text{ MPa} \qquad \textbf{Ans.}$$

The shear stress on the planes of maximum shear stress is given as

$$\tau_{x'y'} = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta_s - \tau_{xy} \cos 2\theta_s \qquad \text{(using equation (2.4))}$$

$$= \frac{40 - 25}{2} \sin (153.44^\circ) - 15 \cos (153.44^\circ)$$

$$= 16.77 \text{ MPa}$$ **Ans.**

The principal planes and the planes of maximum shear stress are shown in Fig. 2.8.

Example 2.8

At a section in a beam the tensile stress due to bending is 50 N/mm^2 and there is a shear stress of 20 N/mm^2. Determine the magnitude and direction of the principal stresses and calculate the maximum shear stress.

Solution: Given,

Bending stress, $\sigma_b = 50$ N/mm^2

Shear stress, $\tau = 20$ N/mm^2

The major principal stress is given as

$$\sigma_1 = \frac{\sigma_b}{2} + \frac{\sqrt{\sigma_b^2 + 4\tau^2}}{2}$$

$$= \frac{50}{2} + \frac{\sqrt{50^2 + 4 \times 20^2}}{2}$$

$$= 57.01 \text{ N/mm}^2 \text{ (Tensile)}$$ **Ans.**

The minor principal stress is given as

$$\sigma_2 = \frac{\sigma_b}{2} - \frac{\sqrt{\sigma_b^2 + 4\tau^2}}{2}$$

$$= \frac{50}{2} - \frac{\sqrt{50^2 + 4 \times 20^2}}{2} = -7.01 \text{ N/mm}^2$$

$$= 7.01 \text{ N/mm}^2 \text{ (Compressive)}$$ **Ans.**

The angle made by the principal planes with the vertical section of the beam is given as

$$\tan 2\theta_p = \frac{2\tau}{\sigma_1 - \sigma_2}$$

$$= \frac{2 \times 20}{57.01 - (-7.01)} = 0.625$$

Hence, $\theta_p = 16^\circ$ and 106° **Ans.**

The maximum shear stress is

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{57.01-(-7.01)}{2}$$

$$= 32.01 \text{ N/mm}^2$$ **Ans.**

Example 2.9

At a point in an elastic material, the stresses on three mutually perpendicular planes are as follow:
First plane: 50 MPa tensile and 40 MPa shear
Second plane: 30 MPa compressive and 40 MPa shear
Third plane: No stress
Find the following parameters:
(*a*) the magnitude and positions of the principal stresses
(*b*) the position of planes on which the maximum shear stress acts and
(*c*) the normal and shear stresses on the planes of maximum shear stress.

Solution: Refer Fig. 2.9.

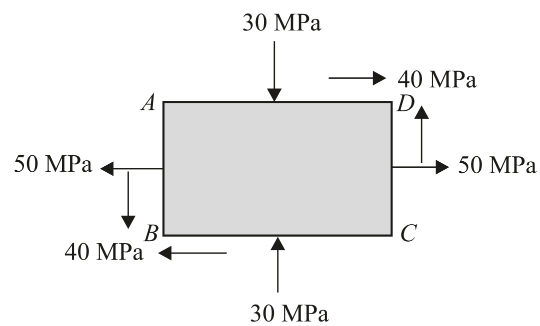


Fig. 2.9

Given, $\sigma_x = 50$ MPa

$\sigma_y = -30$ MPa

$\tau_{xy} = 40$ MPa

(*a*) The principal stresses are given as

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$= \frac{50-30}{2} \pm \frac{\sqrt{(50+30)^2 + 4\times 40^2}}{2}$$

$$= (10 \pm 56.56) \text{ MPa}$$

Hence, the major principal stress is

$$\sigma_1 = 10 + 56.56$$

$$= 66.56 \text{ MPa (Tensile)}$$ **Ans.**

and, the minor principal stess is

$$\sigma_2 = 10 - 56.56 = -46.56 \text{ MPa}$$

$$= 46.56 \text{ MPa (Compressive)}$$ **Ans.**

Position of principal planes

The angle made by the principal planes with the vertical section is given as

$$\tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$= \frac{2 \times 40}{50 - (-30)} = \frac{80}{80} = 1$$

Hence, $\theta_P = 22.5^\circ$ and 112.5° **Ans.**

(*b*) If θ_s be the angle made by the planes of maximum shear stress with *AB*, then

$$\tan 2\theta_s = -\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right)$$

$$= -\left(\frac{50+30}{2 \times 40}\right) = -1$$

or $\theta_s = -22.5^\circ$

Negative sign signifies that the angle is measured in the clockwise direction. Hence, the actual angle (in the anticlockwise direction) is

$\theta_s = 67.5^\circ$ and 157.5° **Ans.**

The normal stresses on the planes of maximum shear stress are:

For $\theta = 67.5^\circ$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= \frac{50-30}{2} + \frac{50+30}{2}\cos 135^\circ + 40 \times \sin 135^\circ$$

$$= 10 - 28.28 + 28.28$$

$= 10$ MPa **Ans.**

For $\theta = 157.5^\circ$

$$\sigma_{x'} = \frac{50-30}{2} + \frac{50+30}{2}\cos 315^\circ + 40 \times \sin 315^\circ$$

$$= 10 + 28.28 - 28.28$$

$= 10$ MPa **Ans.**

The shear stresses on the planes of maximum shear stress are:

For $\theta = 67.5^\circ$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$= -\frac{50+30}{2}\sin 135^\circ + 40 \times \cos 135^\circ$$

$$= -28.28 - 28.28$$

$$= -56.56 \text{ MPa}$$ **Ans.**

For $\theta = 157.5^\circ$

$$\tau_{x'y'} = -\frac{50+30}{2}\sin 315^\circ + 40 \times \cos 315^\circ$$

$$= 28.28 + 28.28$$

$$= 56.56 \text{ MPa}$$ **Ans.**

Example 2.10

A hollow shaft of 40 mm outer diameter and 25 mm inner diameter is subjected to a twisting moment of 120 N.m., bending moment of 800 N.m, and axial thrust of 10 kN. Calculate the maximum compressive and shear stresses.

Solution: Refer Fig. 2.10.

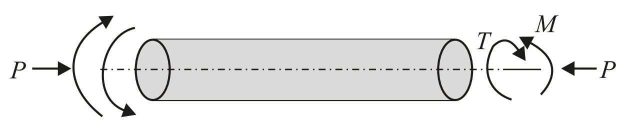


Fig. 2.10

Given, Outer diameter of the hollow shaft, d_o = 40 mm

Inner diameter of the hollow shaft, d_i = 25 mm

Twisting moment, T = 120 N.m

Bending moment, M = 800 N.m

Axial thrust, P = 10 kN

The bending stress is

$$\sigma_b = \frac{M}{I} \times y \qquad \text{(using bending equation)}$$

where I = Moment of inertia of the shaft cross-section about the neutral axis

$$= \frac{\pi}{64}(d_0^4 - d_i^4)$$

$$y = \text{Distance of the outermost fibre from the neutral axis}$$

$$= \frac{d_0}{2}$$

Now

$$\sigma_b = \frac{M}{\frac{\pi}{64}(d_0^4 - d_i^4)} \times \frac{d_0}{2}$$

$$= \frac{32 M d_0}{\pi(d_0^4 - d_i^4)}$$

$$= \frac{32 \times 800 \times 40 \times 10^{-3}}{\pi[(40 \times 10^{-3})^4 - (25 \times 10^{-3})]} \times \frac{1}{10^6} \text{ MPa}$$

$$= 150.25 \text{ MPa}$$

The shear stress is

$$\tau = \frac{T}{J} \times \frac{d_0}{2} \qquad \text{(using torsion equation)}$$

$$= \frac{16 T d_0}{\pi(d_0^4 - d_i^4)} \qquad \left[\begin{array}{l} J = \text{Polar moment of inertia} \\ \quad \text{of the shaft cross-section} \\ \quad = \frac{\pi}{32}(d_0^4 - d_i^4) \end{array}\right]$$

$$= \frac{16 \times 120 \times 40 \times 10^{-3}}{\pi[(40 \times 10^{-3})^4 - (25 \times 10^{-3})^4]} \times \frac{1}{10^6} \text{ MPa}$$

$$= 11.27 \text{ MPa}$$

The direct stress (compressive) is

$$\sigma = -\frac{P}{A} \qquad \left[\begin{array}{l} A = \text{Cross-section area of the hollow shaft} \\ \quad = \frac{\pi}{4}(d_0^2 - d_i^2) \end{array}\right]$$

$$= -\frac{10 \times 10^{-3}}{\frac{\pi}{4}[(40 \times 10^{-3})^2 - (25 \times 10^{-3})^2]} \times \frac{1}{10^6} \text{ MPa}$$

$$= -13.06 \text{ MPa}$$

The maximum tensile stress along the axis of the shaft is

$$\sigma_b + \sigma = 150.25 + (-13.06) = 137.19 \text{ MPa}$$

and the maximum compressive stress $= 150.25 + 13.06$

$$= 163.31 \text{ MPa}$$

Since 163.31 MPa > 137.19, hence 163.31 MPa (σ') is used in the expression for principal stresses.

The principal stresses are given as

$$\sigma_{1,2} = \frac{\sigma'}{2} \pm \frac{\sqrt{\sigma'^2 + 4\tau^2}}{2}$$

$$= \frac{163.31}{2} \pm \frac{\sqrt{(163.31)^2 + 4 \times (11.27)^2}}{2}$$

$$= (81.65 \pm 82.43) \text{ MPa}$$

Hence, $\sigma_1 = 81.65 + 82.43$

$$= 164.08 \text{ MPa}$$

and $\sigma_2 = 81.65 - 82.43 = -0.78 \text{ MPa}$

Hence, the maximum compressive stress is

$$\sigma_2 = 0.78 \text{ MPa}$$ **Ans.**

The maximum shear stress is obtained as

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{164.08 - (-0.78)}{2}$$

$$= 82.43 \text{ MPa}$$ **Ans.**

Example 2.11

Fig. 2.11 shows a hollow shaft of 150 mm external diameter and 80 mm internal diameter. At its free end a pulley of 500 mm diameter is rigidly fixed. A force of 25 kN is applied tangential to the pulley as shown in the figure. Determine the principal stresses and the absolute maximum shear stress at point *A*, located 1 m from the free end, and at the top shaft surface.

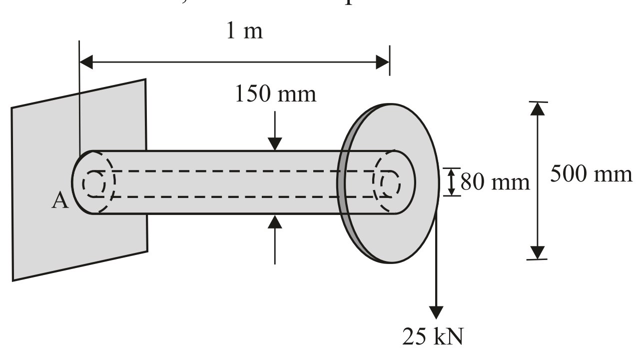


Fig. 2.11

Solution: Refer Fig. 2.11.

Given,

External diameter of the hollow shaft,

$$d_0 = 150 \text{ mm}$$

Internal diameter of the hollow shaft,

$$d_i = 80 \text{ mm}$$

Length of the hollow shaft, $l = 1$ m

The twisting moment on the shaft is

$$T = (25 \times 10^3) \times \frac{500}{2} \times 10^{-3} \text{ N.m}$$

$$= 6250 \text{ N.m}$$

The bending moment on the shaft is

$$M = (25 \times 10^3) \times 1 \text{ N.m}$$

$$= 25000 \text{ N.m}$$

The maximum bending stress induced in the shaft is

$$\sigma_b = \frac{32 M d_0}{\pi(d_0^4 - d_i^4)} \quad \text{(using bending equation)}$$

$$= \frac{32 \times 25000 \times 150 \times 10^{-3}}{\pi[(150 \times 10^{-3})^4 - (80 \times 10^{-3})^4]} \times \frac{1}{10^6} \text{ MPa}$$

$$= 82.09 \text{ MPa}$$

Shear stress induced in the shaft is

$$\tau = \frac{16 T d_0}{\pi(d_0^4 - d_i^4)} \quad \text{(using torsion equation)}$$

$$= \frac{16 \times 6250 \times 150 \times 10^{-3}}{\pi[(150 \times 10^{-3})^4 - (80 \times 10^{-3})^4]} \times \frac{1}{10^6} \text{ MPa}$$

$$= 10.26 \text{ MPa}$$

The principal stresses are given as

$$\sigma_{1,2} = \frac{\sigma_b}{2} \pm \frac{\sqrt{\sigma_b^2 + 4\tau^2}}{2}$$

$$= \frac{82.09}{2} \pm \frac{\sqrt{(82.09)^2 + 4 \times (10.26)^2}}{2}$$

$$= (41.04 \pm 42.30) \text{ MPa}$$

Hence, σ_1 = Major principal stress

$$= 41.04 + 42.30$$

$$= 83.34 \text{ MPa}$$ **Ans.**

and σ_2 = Minor principal stress

$$= 41.01 - 42.30$$

$$= -1.26 \text{ MPa}$$

Negative sign associated with σ_2 is indicative of its compressive nature. **Ans.**

The maximum shear stress is

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{83.34 - (-1.26)}{2}$$

$$= 42.3 \text{ MPa}$$ **Ans.**

2.3 MOHR'S CIRCLE OF PLANE STRESS

Mohr's circle is useful for problems involving plane stresses to find the stresses on an inclined plane graphically or alternatively to find an inclined plane with the stresses given. It was developed by a famous German civil engineer Otto Mohr (1835–1918). The results obtained by using this method are very close to that being obtained by analytical method.

Consider a rectangular element of a material being subjected to biaxial stress condition as shown in Fig. 2.12.

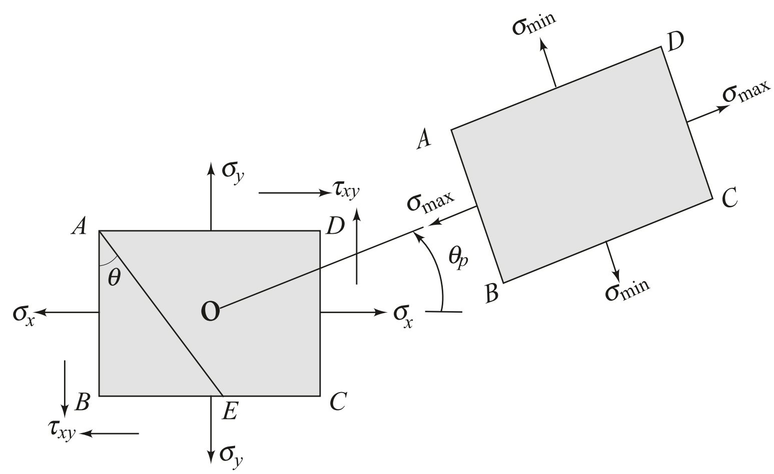


Fig. 2.12

Let σ_x = Normal tensile stress on planes *AB* and *CD*

σ_y = Normal tensile stress on planes *AD* and *BC*

θ = Angle made by inclined plane *AE* with *AB*

τ_{xy} = Shear stress on all the planes *AB*, *BC*, *CD* with *DA*

$\sigma_{x'}$ = Normal stress on inclined plane *AE*

$\tau_{x'y'}$ = Shear stress on inclined plane *AE*

Construction of Mohr's Circle (Refer Fig. 2.13)

- A proper scale is chosen to represent the stresses.
- Normal stresses are taken on the abscissa the x-axis and shear stresses on the ordinate the y-axis).
- To represent σ_x and σ_y, two points M and N are located on x-axis such that $ON = \sigma_x$ and $OM = \sigma_y$, and $\sigma_x > \sigma_y$.
- Normals are dropped at the points M and N above and below x-axis such that $MY = +\tau_{xy}$ and $NX = -\tau_{xy}$.
- Points X and Y are joined together to get a straight line XY. This line intersects x-axis at C. Taking C as centre and CX or CY as radius, draw a circle. The resulting circle is called Mohr's circle. The circle cuts x-axis at two points K and L.

$$OL = \sigma_{max} = \sigma_1 = \text{Maximum (major) principal stress}$$

$$OK = \sigma_{min} = \sigma_2 = \text{Minimum(minor) principal stress}$$

- The centre C of the Mohr's circle is located at $\dfrac{\sigma_x + \sigma_y}{2}$.
- $\angle XCL$ (measured anticlockwise) $= 2\theta_p$, half of this angle *i.e.*, θ_p is defined as the angle made by the major principal plane with plane AB of the element under consideration in Fig. 2.12.
- The radius of the Mohr's circle gives the maximum shear stress.

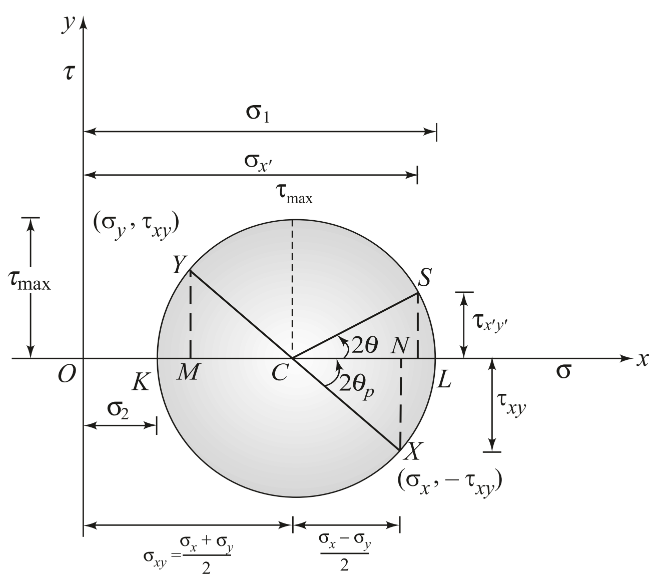


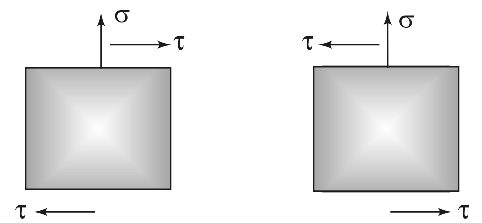
Fig. 2.13 The Mohr's circle.

- To know the position of the inclined plane AE, a point S is taken on the circle such that $\angle LCS = 2\theta$ (measured anticlockwise). The perpendicular distance from S on x-axis gives shear stress τ'_{xy} and the distance of this perpendicular from y-axis gives normal stress $\sigma_{x'}$ on the inclined plane.

  The complete sequence of construction of the Mohr's circle is shown in Fig. 2.13.

Sign Conventions

- The tensile stresses are considered positive and hence they are plotted on right side of the origin.
- The compressive stresses are considered negative, hence they are plotted on left side of the origin.
- If the shear stress acting on a face has the tendency to rotate the element clockwise, then it is considered to be positive and is taken above *x*-axis (Fig. 2.14 (*a*)).



(*a*) Positive shear (*b*) Negative shear

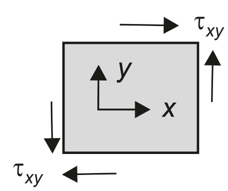
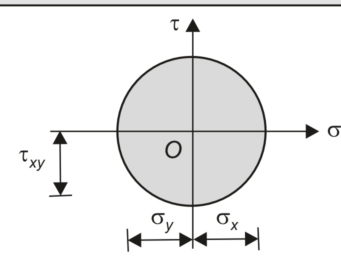
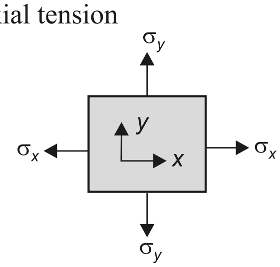
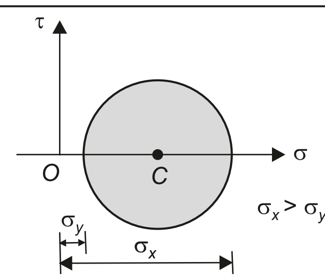
Fig. 2.14

- If the shear stress acting on a face has the tendency to rotate the element counterclockwise, then it is considered to be negative and is taken below *x*-axis (Fig. 2.14 (*b*)).

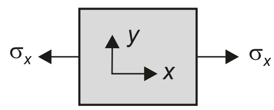
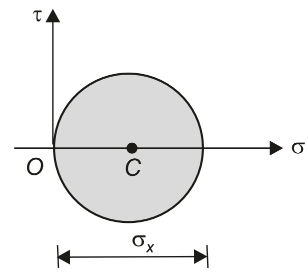
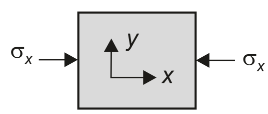
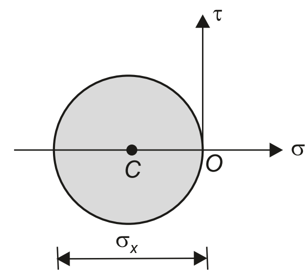
Example 2.12

Draw Mohr's circle for a two-dimensional stress field subjected to (*a*) pure shear (*b*) pure biaxial tension (*c*) pure uniaxial tension and (*d*) pure uniaxial compression.

Solution:

Conditions	***Mohr's Circle***
(*a*) Pure shear 	 σ_x and σ_y are equal in magnitude.
(*b*) Pure biaxial tension 	

Contd...

(*c*) Pure uniaxial tension 	
(*d*) Pure uniaxial compression 	

Example 2.13

Using normal and shear stresses given in Example 2.1,

(*a*) draw the Mohr's circle

(*b*) find the principal stresses

(*c*) find the maximum shear stress and

(*d*) find the position of principal planes.

Solution: (*a*) Refer Fig. 2.15.

Selection of Scale

Take $0.5 \text{ cm} = 5 \text{ MPa}$

Hence, $\sigma_y = 5 \text{ MPa} = 0.5 \text{ cm}$

$\sigma_x = 25 \text{ MPa} = 2.5 \text{ cm}$

$\tau_{xy} = 30 \text{ MPa} = 3.0 \text{ cm}$

The Mohr's circle is shown in Fig. 2.16.

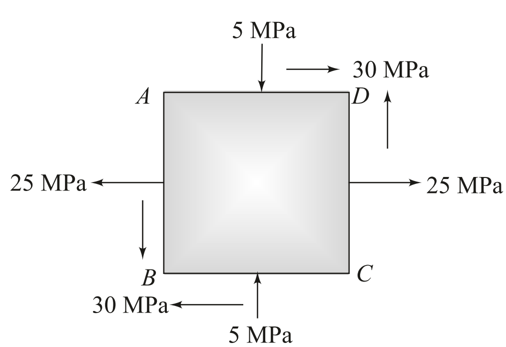


Fig. 2.15

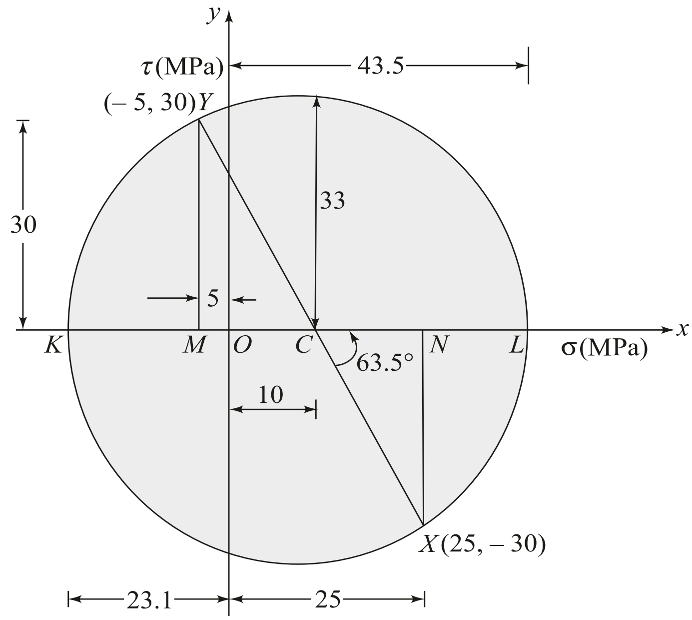


Fig. 2.16

(*b*) From Fig. 2.16, we have

$$\sigma_{max} = \sigma_1 = 4.35 \text{ cm} = 43.5 \text{ MPa}$$ **Ans.**

$$\sigma_{min} = \sigma_2 = -2.31 \text{ cm} = -23.1 \text{ MPa}$$ **Ans.**

$$= 23.1 \text{ MPa (Compressive)}$$ **Ans.**

(*c*) The maximum shear stress is

$$\tau_{max} = 3.3 \text{ cm} = 33 \text{ MPa}$$ **Ans.**

(*d*) The position of principal planes are given as:

$$2\theta_p = 63.5°$$

or $$\theta_{p_1} = 31.75°$$

and $$2\theta_p = 243.5°$$

or $$\theta_{p_2} = 121.75°$$ **Ans.**

Example 2.14

At a point in the cross-section of a loaded member, the maximum principal stress is 15 N/mm^2 tensile and maximum shear stress of 8 N/mm^2. Using Mohr's circle, find (*a*) the magnitude and nature of direct stress on the plane of maximum shear stress (*b*) the state of stress on a plane, making an angle 30º with the plane of maximum principal stress.

Solution: **Selection of Scale**

Take 1 cm = 5 N/mm^2

The Mohr's circle is shown in Fig. 2.17.

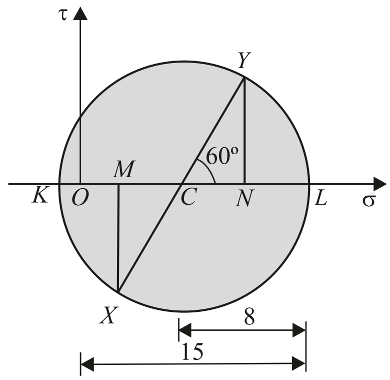


Fig. 2.17

$$
\begin{aligned}
OL &= \sigma_1 = \text{Maximum principal stress} \\
&= 15 \text{ N/mm}^2 \\
CL &= \tau_{max} = \text{Maximum shear stress} \\
&= \text{Radius of Mohr's circle} \\
&= 8 \text{ N/mm}^2
\end{aligned}
$$

The direct stresses on the plane of maximum shear stress are:

$$OC = \sigma_x = 1.4 \text{ cm} = 7 \text{ N/mm}^2 \text{ (Tensile)}$$

$$OK = \sigma_y = -0.2 \text{ cm} = -1 \text{ N/mm}^2 \text{ (Compressive)}$$ **Ans.**

The stresses on a plane making an angle 30º with the plane of maximum principal stress are:

$$ON = \sigma_x = 2.2 \text{ cm} = 11 \text{ N/mm}^2 \text{ (Tensile)}$$

$$OM = \sigma_y = 0.6 \text{ cm} = 3 \text{ N/mm}^2 \text{ (Tensile)}$$

$$YN = \tau_{xy} = 1.4 \text{ cm} = 7 \text{ N/mm}^2$$ **Ans.**

Example 2.15

A circle of 100 mm diameter is drawn on a mild steel plate before it is subjected to direct tensile stresses of 80 N/mm^2 and 20 N/mm^2 in two mutually perpendicular directions and a shear stress of 40 N/mm^2. Find the major and minor axes of the ellipse formed as a result of deformation of the circle. Assume, E = 200 GPa and Poisson’s ratio as 0.25.

Solution: Refer Fig. 2.18.

Given, Diameter of the circle on the plate, d = 100 mm

Tensile stress in x-direction is $\sigma_x = 80 \text{ N/mm}^2$

Tensile stress in y-direction is $\sigma_y = 20\ \text{N/mm}^2$

Shear stress, $\tau_{xy} = 40\ \text{N/mm}^2$

Modulus of elasticity, $E = 200\ \text{GPa} = 200 \times 10^9\ \text{Pa}$

Poisson's ratio $v = 0.25$

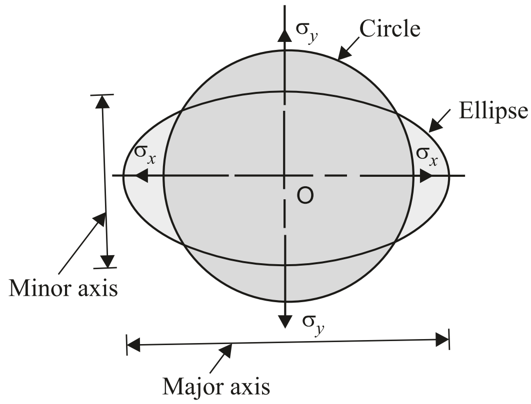


Fig. 2.18

The principal stresses are given as

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$= \frac{80 + 20}{2} \pm \frac{\sqrt{(80 - 20)^2 + 4 \times 40^2}}{2}$$

$$= (50 \pm 50)\ \text{N/mm}^2$$

Hence, $\sigma_1 = 50 + 50 = 100\ \text{N/mm}^2$

and $\sigma_2 = 50 - 50 = 0$

Strain produced in the direction of σ_1 is $\in_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \frac{100 \times 10^6\ \text{Pa}}{200 \times 10^9\ \text{Pa}}$

$= 5 \times 10^{-4}$

Strain produced in the direction of σ_2 is $\in_2 = \frac{\sigma_2}{E} - \nu\frac{\sigma_1}{E}$

$= -1.25 \times 10^{-4}$

Change in diameter of the circle along σ_1 is $\Delta d_1 = \in_1 d$

$= 5 \times 10^{-4} \times 100 = 0.05\ \text{mm}$

Change in diameter of the circle along σ_2 is $\Delta d_2 = \in_2 d$

$= -1.25 \times 10^{-4} \times 100 = -0.0125\ \text{mm}$

As a result of the stresses applied in x and y directions, the circle takes the form of an ellipse (Fig. 2.18). The diameter of the circle increases along σ_1 and decreases along σ_2.

Hence, major diameter of the ellipse $= (100 + 0.05)$ mm

$= 100.05$ mm **Ans.**

and minor diameter of the ellipse $= (100 - 0.0125)$ mm

$= 99.9875$ mm **Ans.**

Example 2.16

A thin cylinder with closed ends has an internal diameter of 50 mm and a wall thickness of 2.5 mm. It is subjected to an axial pull of 10 kN and a torque of 500 N.m, while under an internal pressure of 6 MN/m^2.

(*a*) Determine the principal stresses in the tube and the maximum shear stress.

(*b*) Represent the stress configuration on a square element taken in the load direction with direction and magnitude indicated (schematic).

(*c*) Sketch the Mohr's stress circle.

Solution: Given,

Inside diameter of the cylinder, $d = 50$ mm

Wall thickness of the cylinder, $t = 2.5$ mm

Axial pull, $P = 10$ kN

Torque applied, $T = 500$ N.m

Internal pressure, $p = 6$ MN/m^2

The cross-sectional area of the cylinder is

$$A = \pi d \times t = \pi \times 50 \times 10^{-3} \times 2.5 \times 10^{-3}$$

$$= 3.927 \times 10^{-4}\ \text{m}^2$$

(*a*) The hoop stress is given as

$$\sigma_h = \frac{pd}{2t} = \frac{6 \times 50 \times 10^{-3}}{2 \times 2.5 \times 10^{-3}} = 60\ \text{MPa}$$

The longitudinal stress is

$$\sigma_l = \frac{\sigma_h}{2} = \frac{60}{2} = 30\ \text{MPa}$$

The direct stress due to axial pull is

$$\sigma = \frac{P}{A} = \frac{10 \times 10^3}{3.927 \times 10^{-4}} \times \frac{1}{10^6}\text{MPa} = 25.46\ \text{MPa}$$

The longitudinal stress acts along the axis of the cylinder and the hoop stress acts perpendicular to it. The direct stress also acts along the axis of the cylinder. Hence, the total stress acting along the axis is

$$\sigma_l + \sigma = 30 + 25.46 = 55.46 \text{ MPa} = \sigma_x \text{ (say)}$$

and $\sigma_h = \sigma_y$ (say)

The shear stress is

$$\tau_{xy} = \frac{2T}{\pi d^2 t} = \frac{2 \times 500}{\pi \times (50 \times 10^{-3})^2 \times (2.5 \times 10^{-3})} \times \frac{1}{10^6} \text{ MPa} = 50.93 \text{ MPa}$$

The principal stresses are given as

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$= \frac{55.46 + 60}{2} \pm \frac{\sqrt{(55.46 - 60)^2 + 4 \times (50.93)^2}}{2}$$

$$= (57.73 \pm 50.98) \text{ MPa}$$

Hence, σ_1 = Major principal stress

$= 57.73 + 50.98 = 108.71$ MPa **Ans.**

and σ_2 = Minor principal stress

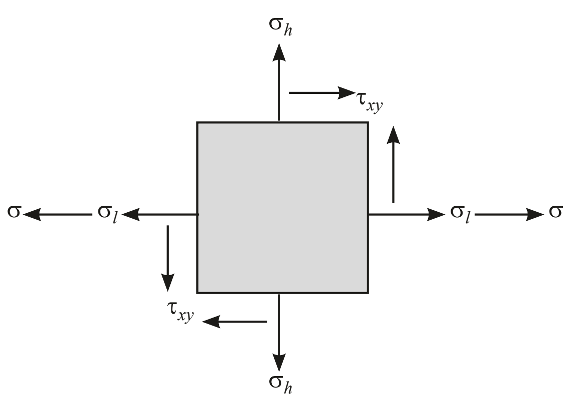
$= 57.73 - 50.98 = 6.75$ MPa **Ans.**

The maximum shear stress is obtained as

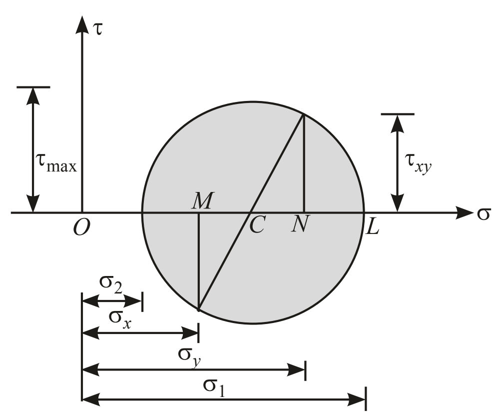
$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{108.71 - 6.75}{2} = 50.98 \text{ MPa}$$ **Ans.**

(*b*) and (*c*)

The stress configuration on a square element and the related Mohr's circle is shown in Fig. 2.19.



(*a*) Stress configuration on a square element



(*b*) Mohr's circle

Fig. 2.19

Example 2.17

A circular shaft is subjected to combined loads of bending M and torque T. With the help of Mohr's circle diagram, represent the stresses on an element of the shaft surface. From this diagram or by calculation, find the maximum shear stress due to the combined effect of these gradually applied loads of M and T.

Solution: The bending stress is

$$\sigma_b = \frac{32M}{\pi d^3} \qquad \text{(using bending equation)}$$

M = Bending moment

d = Diameter of the shaft

The shear stress is

$$\tau = \frac{16M}{\pi d^3} \qquad \text{(using torsion equation)}$$

The principal stresses are given as

$$\sigma_{1,2} = \frac{\sigma_b}{2} \pm \frac{\sqrt{\sigma_b^2 + 4\tau^2}}{2}$$

$$= \frac{16}{\pi d^3}(M \pm \sqrt{M^2 + T^2})$$

The maximum shear stress is obtained as

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{\sqrt{\sigma_b^2 + 4\tau^2}}{2}$$

$$= \frac{16M}{\pi d^3}\sqrt{M^2 - T^2}$$

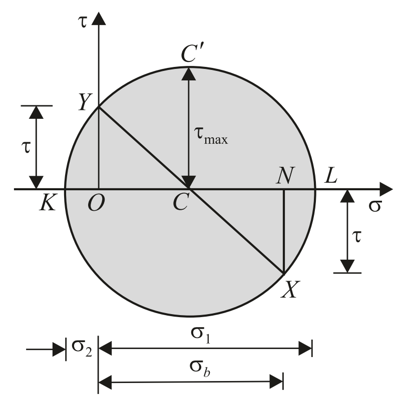


Fig. 2.20

The Mohr's circle is shown in Fig. 2.20.

$ON = \sigma_b$

$OL = \sigma_1$

$OK = \sigma_2$

$OY = NX = \tau$

$CC' = \tau_{max}$

SHORT ANSWER QUESTIONS

1. What is principal stress? How many principal stresses are there for a plane stress condition?
2. What is a principal plane? What is its significance?
3. Who invented Mohr's circle?
4. What does the radius of Mohr's circle indicate?
5. How is the plane of maximum shear stress located?
6. When is the shear stress positive or negative?

MULTIPLE CHOICE QUESTIONS

1. The stresses are said to be compound, when
   (*a*) normal and shear stresses are acting simultaneously
   (*b*) torsion and bending stresses are acting simultaneously
   (*c*) normal and bending stresses are acting simultaneously
   (*d*) bending and stresses are acting simultaneously.

2. The principal planes are the planes of
   (*a*) maximum shear stress
   (*b*) minimum shear stress
   (*c*) maximum normal stress
   (*d*) zero shear stress.

3. The principal stresses are given as
   (*a*) $\dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\dfrac{(\sigma_x + \sigma_y)^2 - 4\tau_{xy}^2}{2}}$
   (*b*) $\dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\dfrac{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}{2}}$
   (*c*) $\dfrac{\sigma_x - \sigma_y}{2} \pm \sqrt{\dfrac{(\sigma_x + \sigma_y)^2 + 4\tau_{xy}^2}{2}}$
   (*d*) $\dfrac{\sigma_x - \sigma_y}{2} \pm \sqrt{\dfrac{(\sigma_x + \sigma_y)^2 - 4\tau_{xy}^2}{2}}$.

4. The principal stresses are basically
   (*a*) shear stresses
   (*b*) bending stresses
   (*c*) normal stresses
   (*d*) none of these.

5. The planes of maximum shear stress are located at which of the following angle to the principal planes?
   (*a*) 90°
   (*b*) 45°
   (*c*) 60°
   (*d*) 30°.

6. The maximum shear stress is equal to

(*a*) $\pm \dfrac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$ (*b*) $\pm \dfrac{\sqrt{(\sigma_x + \sigma_y)^2 + 4\tau_{xy}^2}}{2}$

(*c*) $\pm \dfrac{\sqrt{(\sigma_x - \sigma_y)^2 - 4\tau_{xy}^2}}{2}$ (*d*) $\pm \dfrac{\sqrt{(\sigma_x + \sigma_y)^2 - 4\tau_{xy}^2}}{2}$.

7. The principal planes are separated by

(*a*) 180° (*b*) 45° (*c*) 90° (*d*) 60°.

8. The maximum shear stress is equal to

(*a*) one-half of the algebraic difference of the principal stresses
(*b*) the algebraic difference of the principal stresses
(*c*) the sum of the principal stresses
(*d*) the difference of the principal stresses.

9. For uniaxial loading condition, the maximum shear stress is equal to

(*a*) uniaxial stress (*b*) two times the uniaxial stress
(*c*) three times the uniaxial stress (*b*) one-half of uniaxial stress.

10. For a complex stress system, the total number of principal planes is

(*a*) two (*b*) four (*c*) three (*d*) none of these.

11. The radius of the Mohr's circle indicates the

(*a*) maximum principal stress (*b*) minimum principal stress
(*c*) maximum shear stress (*d*) minimum shear stress.

12. In case one principal stress is zero, the other principal stress is equal to

(*a*) maximum shear stress (*b*) two times the maximum shear stress
(*c*) three times the maximum shear stress (*d*) none of these.

ANSWERS

1. (*a*) **2.** (*d*) **3.** (*b*) **4.** (*c*) **5.** (*b*) **6.** (*a*) **7.** (*c*) **8.** (*a*)
9. (*d*) **10.** (*c*) **11.** (*c*) **12.** (*b*).

EXERCISES

1. A rectangular element is subjected to a tensile stress of 60 MPa, a compressive stress of 20 MPa and a shear stress of 30 MPa. Find (*a*) the principal stresses, (*b*) the maximum shear stress and (*c*) the location of the principal planes.
(*Ans.* (*a*) 70 MPa, – 30 MPa; (*b*) 50 MPa; (*c*) 18.4° and 108.4°).

2. The stresses at a point on two perpendicular planes *BC* and *AC* are as shown in Fig. 2.21. Determine the position of the plane *AB* such that the shear stress on it is equal to zero. What will be the normal (principal) stresses on such plane?

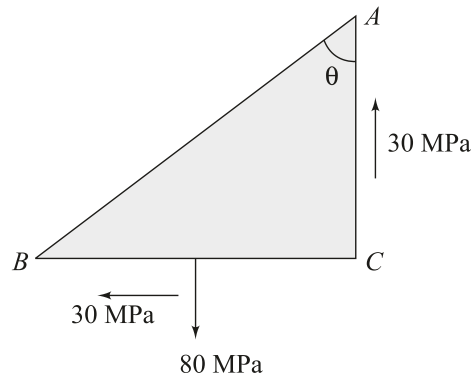


Fig. 2.21

(*Ans.* θ = 18.43°, 90 MPa (Tensile), 10 MPa (Compressive)).

3. The principal stresses at a point within a strained material are 90 MPa and 70 MPa as shown in Fig. 2.22. Find the normal, the tangential and the resultant stress on a plane inclined at 30° to the axis of the major principal stress.

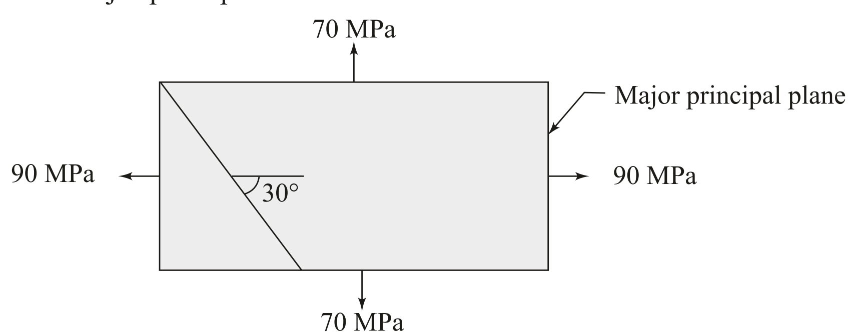


Fig. 2.22

(*Ans.* 75 MPa, 8.66 MPa, 75.5 MPa).

4. At a certain point in a strained material, two normal stresses, one a tensile stress of 100 MPa and another a compressive stress of 90 MPa are applied at two mutually perpendicular planes. If maximum direct stress is limited to 150 MPa (tensile), find the shear stress on the planes on which two normal stresses are acting. (*Ans.* 109.54 MPa).

5. For a biaxial stress system, $\sigma_x = 65$ MPa, $\sigma_y = 75$ MPa and $\tau_{xy} = 40$ MPa. Find (*a*) the principal stresses and their planes and (*b*) the maximum shear stresses and the planes on which they occur. (*Ans.* (*a*) 110.31 MPa, 29.68 MPa, 48.56°, 138.56°
(*b*) 40.315 MPa, 3.56°, 93.56°).

6. At a point in a strained material, the resultant intensity of stress across a plane is 100 MPa tensile inclined at 45° to its normal. The normal component of stress intensity across the plane at right angle is 30 MPa compression. Find the position of principal planes and stresses across them. Also, find the value of maximum shear at that point.
(*Ans.* 27.25°, 117.25°, 100.7 MPa, – 66.35 MPa, 86.6 MPa).

7. At a point in a loaded specimen, the principal stresses acting on two mutually perpendicular planes are 60 MPa and 40 MPa, both being compressive. Determine the resultant stress acting on a plane inclined at 60° measured clockwise to the plane on which the larger of the normal stresses is acting. (*Ans.* 45.82 MPa).

8. A steel shaft is subjected to a torque of 20 kN·m and a bending moment of 10 kN·m. The diameter of the shaft is 100 mm. Calculate the maximum and the minimum principal stresses and the maximum shear stress in the shaft at its surface.
(*Ans.* 165 MPa, – 63 MPa, 114 MPa).

9. At a point in a strained material, the vertical shear stress is 15 MPa and the horizontal tensile stress is 25 MPa. Using Mohr's circle method, find the principal stresses and direction of principal planes. (*Ans.* 32 MPa, – 7 MPa, 25°, 115°).

10. At a point in a material, there is a horizontal tensile stress of 270 MPa, a vertical tensile stress of 130 MPa and shearing stress of 40 MPa downward on left. With the aid of Mohr's circle or otherwise, find out the maximum and the minimum principal stresses and the planes on which they act. Determine also the maximum shearing stress in magnitude and direction.
(*Ans.* 280.62 MPa, 119.38 MPa, 14.87°, 104.87°, 80.62 MPa, 59.87°, 149.87°).

11. For a plane stress condition in which $\sigma_x = 140$ MPa, $\sigma_y = 20$ MPa and $\tau_{xy} = -60$ MPa, find the principal stresses and the locations of the principal planes.
(*Ans.* 165 MPa, –5 MPa, – 22.5°, – 112.5°).

12. For the plane stress condition shown in Fig. 2.23, find σ_x and the position of the principal planes, if the maximum principal stress is – 7MPa.

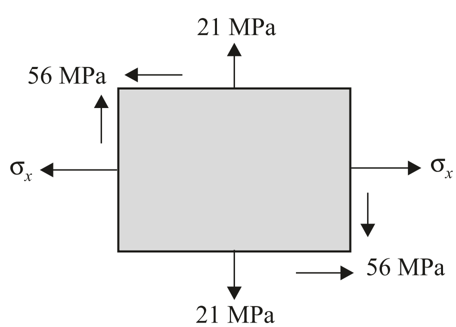


Fig. 2.23

(*Ans.* 105 MPa, – 26.6º, – 116.6°).

13. For the plane stress condition shown in Fig. 2.24, find the principal stresses and the position of the principal planes.

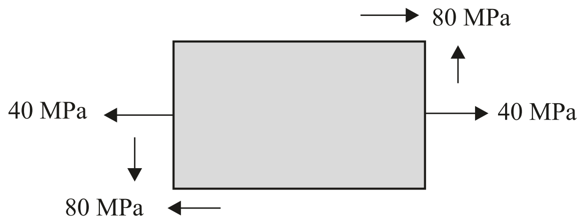


Fig. 2.24

(*Ans.* 102.5 MPa, – 62.5 MPa, 36°, 126°).

14. A tensile stress σ_1 and a shear stress τ act on a given plane of a material. Show that the principal stresses are always of opposite sign. If an additional tensile stress σ_2 acts on a plane perpendicular to that of σ_1, find the condition that both principal stresses may be of the same sign.

(*Ans.* $\tau = \sqrt{(\sigma_1\sigma_2)}$).

❑❑❑

3 Centroid and Moment of Inertia

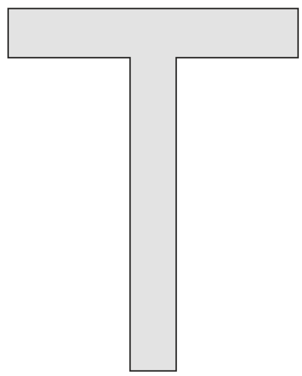


Figure shows an I-section, which is one of the most widely used cross-sections for beams.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- What is centroid?
- How do centre of gravity and centroid differ?
- Why is the area moment of inertia often called the second moment of area?
- What is centroidal axis?
- Where is parallel-axes theorem used?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_3

3.1 CENTRE OF GRAVITY

The centre of gravity of a body is the point where the entire weight of the body is assumed to be concentrated. In other words, it is the point where the weight of the body acts. It may or may not be located within the body and its position depends upon the shape of the body. It is denoted by *G*.

If the mass is uniformly scattered in the body, then the body can be assumed to be made of many elemental masses $m_1, m_2, m_3, \ldots$. Suppose these elemental masses are located at distances $x_1, x_2, x_3, \ldots$ from *y*-axis and $y_1, y_2, y_3, \ldots$ from *x*-axis, then centre of gravity of the body is defined by $\bar{x}$ and $\bar{y}$, given by

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum mx}{\sum m} \quad \ldots (3.1)$$

and

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum my}{\sum m} \quad \ldots (3.2)$$

3.2 FIRST MOMENT OF AREA AND CENTROID

The first moment of a plane area defines the centroid of the area. The centroid is defined as the centre of gravity of a plane area, where the entire area is assumed to be concentrated. Consider an area *A* located in the *xy*-plane as shown in Fig. 3.1.

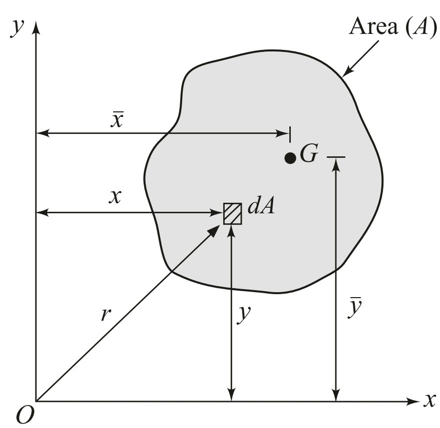


Fig. 3.1 Plane area *A* with the centroid *G*.

Further we consider a small element of area *dA* in the plane area *A* with a centroid having coordinates (*x*, *y*) relative to the *x* and *y* axes.

The first moment of the area *A* with respect to the *x*-axis is defined as

$$Q_x = \int_A y \, dA \quad \ldots (3.3)$$

Similarly, the first moment of the area *A* with respect to the *y*-axis is defined as

$$Q_y = \int_A x \, dA \quad \ldots (3.4)$$

The values of Q_x and Q_y may be positive, negative or zero, depending on the location of the origin O of the coordinate axes. In SI units, Q_x and Q_y are expressed in m^3, and their dimension is $[L^3]$.

The centroid G of the area A is defined as the point in the xy-plane that has coordinates

$$\bar{x} = \frac{Q_y}{A} \text{ and } \bar{y} = \frac{Q_x}{A} \qquad \text{... (3.5)}$$

The first moment of an area has the following properties:

- If the centroid of the area coincides with the origin of the xy coordinate system for which $\bar{x} = \bar{y} = 0$, then $Q_x = Q_y = 0$.
- For an area having an axis of symmetry, the centroid of the area lies on that axis.
- If the area can be subdivided into simple geometric shapes like rectangles, triangles, circles etc. of areas A_1, A_2, A_3, ..., then equations (3.3) and (3.4) can be expressed in the following forms:

$$Q_x = \sum\left(\int_{A_i} y dA\right) = \Sigma(Q_x)_i = \Sigma\, A_i \bar{y}_i \qquad \text{... (3.6)}$$

$$Q_y = \sum\left(\int_{A_i} x dA\right) = \Sigma(Q_y)_i = \Sigma\, A_i \bar{x}_i \qquad \text{... (3.7)}$$

where $(\bar{x}_i, \bar{y}_i)$ are the coordinates of the centroid of area A_i.

For the purpose of illustration, consider an area A consisting of a number of elemental areas a_1, a_2, a_3, ... Their locations w.r.t. xy-plane are shown in Fig. 3.2.

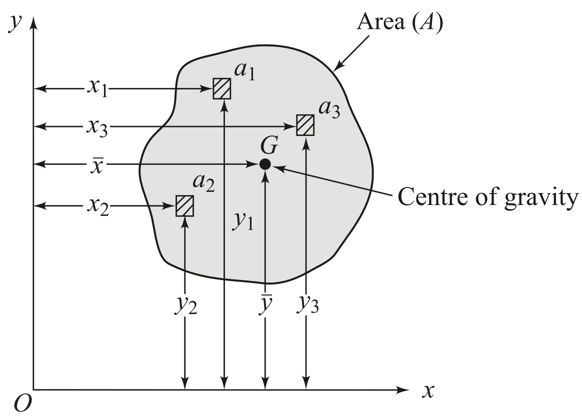


Fig. 3.2 Location of the centroid.

$$A = a_1 + a_2 + a_3 + \ldots$$
$$= \sum a$$

Let $\bar{x}$ = Distance of the centroid of the area A from the y-axis

$\bar{y}$ = Distance of the centroid of the area A from the x-axis

Using the moment principle of area, we have

$$\bar{x} = \frac{a_1\, x_1 + a_2\, x_2 + a_3\, x_3 + \ldots}{a_1 + a_2 + a_3 + \ldots}$$

$$= \frac{\sum ax}{\sum a}$$

$$= \frac{\sum ax}{A} \qquad ...(3.8)$$

and $$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3 + \ldots}{a_1 + a_2 + a_3 + \ldots}$$

$$= \frac{\sum ay}{\sum a}$$

$$= \frac{\sum ay}{A} \qquad ...(3.9)$$

$\bar{x}$ and $\bar{y}$ taken together gives the position of the centroid of the area A.

Example 3.1

Find the centroid of a *T*-section shown in Fig. 3.3.

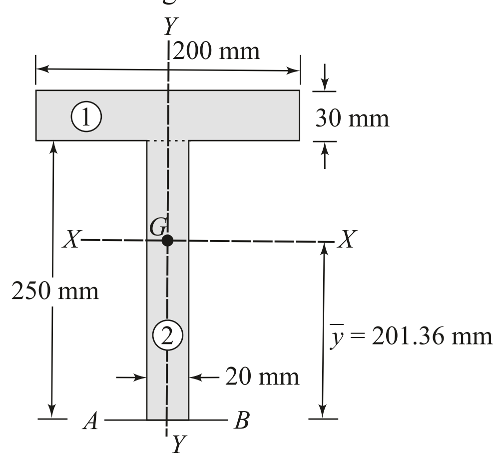


Fig. 3.3

Solution: Refer Fig. 3.3. The entire *T*-section is divided two parts (1) and (2). The section is symmetrical about *YY* axis, hence the centroid will lie somewhere on this line. Let *AB* is chosen as the reference line.

For part (1)

Area $a_1 = 200 \times 30 = 6 \times 10^3 \text{ mm}^2$

Distance of its *C.G.* from *AB*

$$y_1 = \left(250 + \frac{30}{2}\right) = 265 \text{ mm}$$

For part (2)

Area $a_2 = 250 \times 20 = 5 \times 10^3 \text{ mm}^2$

Distance of its *C.G.* from *AB*

$$y_2 = \frac{250}{2} = 125 \text{ mm}$$

By definition
$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$$= \frac{(6 \times 10^3 \times 265) + (5 \times 10^3 \times 125)}{(6 \times 10^3) + (5 \times 10^3)} = 201.36 \text{ mm}$$

and $\bar{x} = 0$

Hence, the centroid of the given section lies at a distance of 201.36 mm from *AB* on line *YY*. **Ans.**

Example 3.2

Find the centroid of an *I*-section shown in Fig. 3.4.

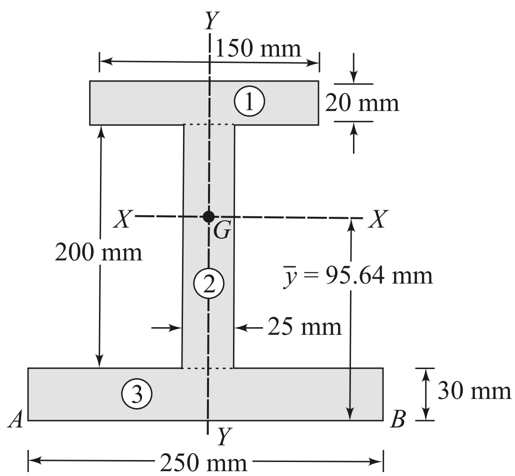


Fig. 3.4

Solution: Refer Fig. 3.4. The entire section is divided into three parts (1), (2) and (3). The section is symmetrical about *YY* axis, hence centroid of the section will lie on it. Let *AB* be the reference line.

For part (1)

Area $a_1 = 150 \times 20 = 3 \times 10^3 \text{ mm}^2$

Distance of its *C.G.* from *AB*

$$y_1 = \left(30 + 200 + \frac{20}{2}\right) = 240 \text{ mm}$$

For part (2)

Area $a_2 = 200 \times 25 = 5 \times 10^3 \text{ mm}^2$

Distance of its *C.G.* from *AB*

$$y_2 = \left(30 + \frac{200}{2}\right) = 130 \text{ mm}$$

For part (3)

Area $a_3 = 250 \times 30 = 7.5 \times 10^3 \text{ mm}^2$

Distance of its *C.G.* from *AB*

$$y_3 = \frac{30}{2} = 15\text{mm}$$

By definition

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

$$= \frac{(3\times 10^3 \times 240) + (5\times 10^3 \times 130) + (7.5\times 10^3 \times 15)}{(3\times 10^3) + (5\times 10^3) + (7.5\times 10^3)} = 95.64 \text{ mm}$$

and $\bar{x} = 0$

Hence, the centroid of the given section lies at a distance of 95.64 mm from *AB*. **Ans.**

Example 3.3

Find the centroid of the angle section (*L*-section) shown in Fig. 3.5.

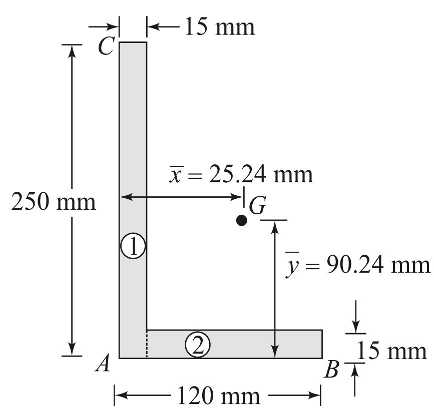


Fig. 3.5

Solution: Refer Fig. 3.5. Take *AB* and *AC* as the reference axes. The given section is divided into two parts (1) and (2).

For part (1)

Area $a_1 = 250 \times 15 = 3750 \text{ mm}^2$

Distance of its $C.G.$ from AC

$$x_1 = \frac{15}{2} = 7.5 \text{ mm}$$

Distance of its $C.G.$ from AB

$$y_1 = \frac{250}{2} = 125 \text{ mm}$$

For part (2)

Area $a_2 = (120 - 15) \times 15$

$$= 1575 \text{ mm}^2$$

Distance of its $C.G.$ from AC

$$x_2 = \left[15 + \left(\frac{120 - 15}{2}\right)\right]$$

$$= 67.5 \text{ mm}$$

Distance of its $C.G.$ from AB

$$y_2 = \frac{15}{2} = 7.5 \text{ mm}$$

By definition

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2}$$

$$= \frac{(3750 \times 7.5) + (1575 \times 67.5)}{3750 + 1575}$$

$$= 25.24 \text{ mm}$$

and $$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$$= \frac{(3750 \times 125) + (1575 \times 7.5)}{3750 + 1575}$$

$$= 90.24 \text{ mm}$$

Hence, the centroid of the given section is located at a distance of 25.24 mm from AC and at a distance of 90.24 mm from AB. **Ans.**

Example 3.4

Find the centroid of the channel section shown in Fig. 3.6.

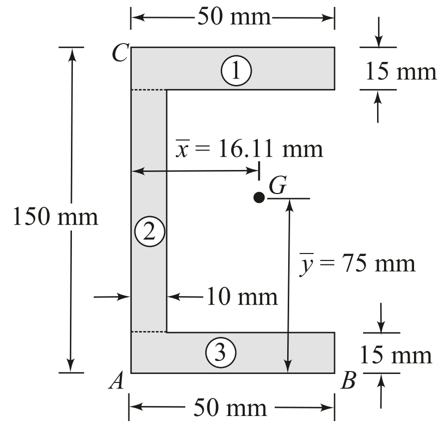


Fig. 3.6

Solution: Refer Fig. 3.6. The channel section is divided into three parts (1), (2) and (3). Let *AB* and *AC* be the axes of reference.

For part (1)

Area $a_1 = 50 \times 15 = 750 \text{ mm}^2$

Distance of its *C.G.* from *AC*

$$x_1 = \frac{50}{2} = 25 \text{ mm}$$

Distance of its *C.G.* from *AB*

$$y_1 = \left[(150 - 15) + \frac{15}{2}\right] = 142.5 \text{ mm}$$

For part (2)

Area $a_2 = [150 - (15 \times 2)] \times 10$

$= 1200 \text{ mm}^2$

Distance of its *C.G.* from *AC*

$$x_2 = \frac{10}{2} = 5 \text{ mm}$$

Distance of its *C.G.* from *AB*

$$y_2 = \frac{150}{2} = 75 \text{ mm}$$

For part (3)

Area $a_3 = 50 \times 15 = 750 \text{ mm}^2$

Distance of its *C.G.* from *AC*

$$x_3 = \frac{50}{2} = 25 \text{ mm}$$

Distance of its *C.G.* from *AB*

$$y_3 = \frac{15}{2} = 7.5 \text{ mm}$$

By definition

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3}$$

$$= \frac{(750 \times 25) + (1200 \times 5) + (750 \times 25)}{750 + 1200 + 750} = 16.11 \text{ mm}$$

and

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

$$= \frac{(750 \times 142.5) + (1200 \times 75) + (750 \times 7.5)}{750 + 1200 + 750} = 75 \text{ mm}$$

Hence, the centroid of the section lies at a distance of 16.11 mm from *AC* and at a distance of 75 mm from *AB*. **Ans.**

Example 3.5

A circular part of diameter 70 mm is cut out from a circular plate of diameter 200 mm as shown in Fig. 3.7. Find the centroid of the remaining part.

Solution: Refer Fig. 3.7. Let *AB* and *CD* be the axes of reference.

G = Centroid of the circular plate

G' = Centroid of the circular part being cut (shaded area)

G'' = Centroid of the remaining part

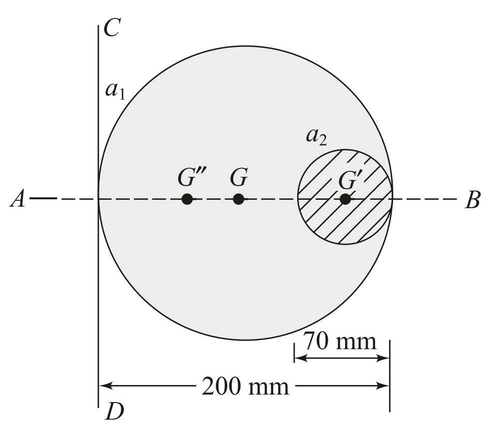


Fig. 3.7

For circular plate

Area of the plate, $a_1 = \frac{\pi}{4} \times 200^2 = 3.141 \times 10^4 \text{ mm}^2$

Its *C.G.* lies on *AB* *i.e.*, $y_1 = 0$ and the distance of the *C.G.* from *CD* is

$$x_1 = \frac{200}{2} = 100 \text{ mm}$$

For part being cut

Area of the shaded portion, $a_2 = \frac{\pi}{4} \times 70^2 = 3.848 \times 10^3 \text{ mm}^2$

Its *C.G.* from *CD*

$$x_2 = \left[(200 - 70) + \frac{70}{2}\right] = 165 \text{ mm}$$

For remaining part

The distance of the *C.G.* of the remaining part from CD is

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2}$$

$$= \frac{(3.141 \times 10^4 \times 100) - (3.848 \times 10^3 \times 165)}{3.141 \times 10^4 - 3.848 \times 10^3}$$

$$= 90.92 \text{ mm}$$

and $\bar{y} = 0$

Hence, the centroid of the remaining part lies on *XX* line at a distance of 90.92 mm from *CD*. **Ans.**

Example 3.6

Find the centroid of the section shown in Fig. 3.8.

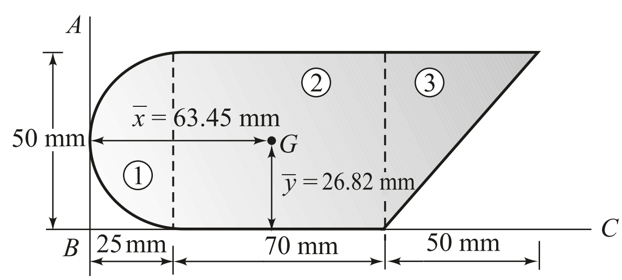


Fig. 3.8

Solution: Refer Fig. 3.8. The entire section is divided into three parts (1), (2) and (3). Take *AB* and *BC* as the reference lines.

For part (1)

Area $a_1 = \frac{1}{2} \times \frac{\pi}{4} \times 50^2 = 982.14 \text{ mm}^2$

Distance of its *C.G.* from *AB*

$$x_1 = 25 - \frac{4 \times 25}{3\pi} = 14.39 \text{ mm}$$

Distance of its *C.G.* from *BC*

$$y_1 = \frac{50}{2} = 25 \text{ mm}$$

For part (2)

Area $a_2 = 70 \times 50 = 3500 \text{ mm}^2$

Distance of its *C.G.* from *AB*

$$x_2 = \left(25 + \frac{70}{2}\right) = 60 \text{ mm}$$

Distance of its *C.G.* from *BC*

$$y_2 = \frac{50}{2} = 25 \text{ mm}$$

For part (3)

Area $a_3 = \frac{1}{2} \times 50 \times 50 = 1250 \text{ mm}^2$

Distance of its *C.G.* from *AB*

$$x_3 = \left(25 + 70 + \frac{50}{3}\right) = 111.66 \text{ mm}$$

Distance of its *C.G.* from *BC*

$$y_3 = \frac{100}{3} = 33.33 \text{ mm}$$

By definition

$\bar{x}$ = Distance of the centriod of the given section from *AB*

$$= \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3}$$

$$= \frac{(981.74 \times 14.39) + (3500 \times 60) + (1250 \times 111.66)}{981.74 + 3500 + 1250} = 63.45 \text{ mm}$$

and $\bar{y}$ = Distance of the centriod of the given section from *BC*

$$= \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

$$= \frac{(981.74 \times 25) + (3500 \times 25) + (1250 \times 33.33)}{981.74 + 3500 + 1250} = 26.82 \text{ mm}$$

Hence, the centroid of the given section lies at a distance of 63.45 mm from *AB* and at a distance of 26.82 mm from *BC*. **Ans.**

Example 3.7

Find the centroid of the area bounded by the x-axis, the line $x = a$ and the parabola $y^2 = kx$ as shown in Fig. 3.9.

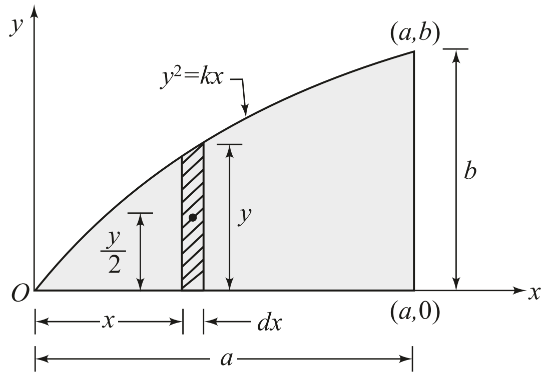


Fig. 3.9

Solution: Consider a vertical elementary strip of thickness dx at a distance x from the y-axis as shown in Fig. 3.9.

The area of the strip is $dA = y\,dx$

The centroid of the strip is located at distance of $y' = \dfrac{y}{2}$ from the x-axis.

The distance of the centroid of the whole area from the x-axis is given as

$$\bar{y} = \frac{\sum dA \,.\, y'}{\sum dA} = \frac{\int y dx \,.\, \frac{y}{2}}{A} = \frac{\int \frac{y^2 dx}{2}}{A}$$

$$= \frac{\int_0^a Kx\,dx}{2A} \qquad (y^2 = kx)$$

$$= \frac{K}{2A}\left(\frac{x^2}{2}\right)_0^a$$

$$= \frac{Ka^2}{4A}$$

$$= K.\frac{a^2}{4}\cdot\frac{1}{A}$$

$$= \frac{b^2}{a}\cdot\frac{a^2}{4}\cdot\frac{3b}{8}$$

$$= \frac{3b}{8}$$

Now consider a horizontal strip of thickness dy at a distance x from the y-axis as shown in Fig. 3.10.

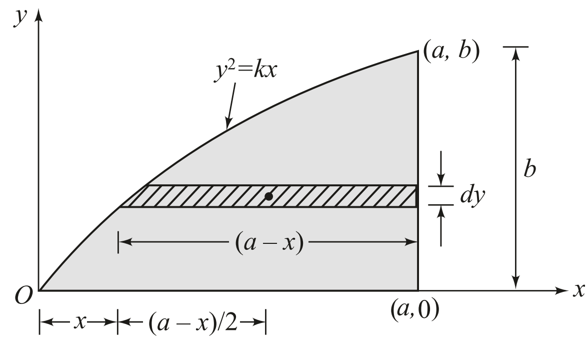


Fig. 3.10

The length of the strip $= (a - x)$

The area of the strip is $dA = (a - x)\, dy$

The centroid of the strip is located at a distance of $x' = \left(x + \dfrac{a-x}{2}\right) = \dfrac{a+x}{2}$ from the y-axis.

The distance of the centroid of the whole area from the y-axis is given as

$$\bar{x} = \frac{\sum dA \,.\, x'}{\sum dA}$$

$$= \frac{\int_0^b (a-x)\, dy \,.\, \dfrac{a+x}{2}}{A}$$

$$= \frac{1}{2A}\int_0^b (a^2 - x^2)\, dy$$

$$= \frac{1}{2A}\int_0^b \left\{a^2 - \left(\frac{y^2}{k}\right)^2\right\} dy \qquad (y^2 = kx)$$

$$= \frac{1}{2A}\left[\int_0^b a^2\, dy - \int_0^b \frac{y^4}{K^2}\, dy\right]$$

$$= \frac{1}{2A}\left[(a^2\, y)_0^b - \left(\frac{y^5}{5K^2}\right)_0^b\right]$$

$$= \frac{1}{2A}\left[a^2b - \frac{b^5}{5\left(\frac{b^4}{a^2}\right)}\right]$$

When $y = b$, then $x = a$, hence

$$K = \frac{y^2}{x} = \frac{b^2}{a}$$

Hence, $$\bar{x} = \frac{1}{2A}\left[a^2b - \frac{a^2b}{5}\right] = \frac{1}{2A}\cdot\frac{4}{5}a^2b$$

$$= \frac{1}{2\left(\frac{2ab}{3}\right)}\cdot\frac{4}{5}a^2b = \frac{3a}{5}$$

Hence, the centroid is located at the point ($\bar{x}$, $\bar{y}$), that is, $\left(\frac{3}{5}a, \frac{3}{8}b\right)$. **Ans.**

Example 3.8

Find the coordinates of the centroid of a quarter-ellipse shown in Fig. 3.11, using direct integration method.

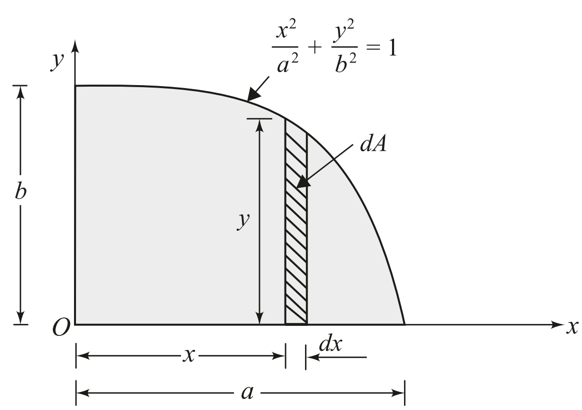


Fig. 3.11

Solution: The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are the semi-major and semi-minor axes of the ellipse.

From the equation of ellipse, we have

$$x = \frac{a}{b}\sqrt{b^2 - y^2} \qquad ...(1)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad ...(2)$$

Now consider a vertical strip of height y and width dx at a distance x from the y-axis as shown in Fig. 3.11.

Area of the strip $\quad dA = y \cdot dx$

The distance of the centroid of the strip from the y-axis is

$$\bar{x} = \frac{\int x dA}{\int dA}$$

$$= \frac{\int_0^a x.y dx}{\int_0^a y dx}$$

$$= \frac{\int_0^a x \cdot \frac{b}{a}\sqrt{a^2 - x^2}\, dx}{\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx} \qquad \text{(on substituting } y \text{ from equation (2))}$$

or
$$\bar{x} = \frac{\int_0^a x\sqrt{a^2 - x^2}\, dx}{\int_0^a \sqrt{a^2 - x^2}\, dx} \qquad ...(3)$$

Let $\quad x = a \sin\theta$ then $dx = a \cos\theta\, d\theta \qquad ...(4)$

when $\quad x = 0,\ \theta = 0$

$\quad x = a,\ \theta = \frac{\pi}{2}$

Equation (3) can now be expressed as

$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}} a\sin\theta \cdot \sqrt{(a^2 - a^2\sin^2\theta)} \cdot a\cos\theta d\theta}{\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta d\theta}$$

$$= \frac{\int_0^{\frac{\pi}{2}} a\sin\theta \cdot a\cos\theta \cdot a\cos\theta\, d\theta}{\int_0^{\frac{\pi}{2}} a\cos\theta \cdot a\cos\theta\, d\theta}$$

$$= \frac{\int_0^{\frac{\pi}{2}} a^3 \sin\theta \cos^2\theta\, d\theta}{\int_0^{\frac{\pi}{2}} a^2 \cos^2\theta\, d\theta}$$

$$= a \cdot \frac{\int_0^{\frac{\pi}{2}} \sin\theta (1 - \sin^2\theta)\, d\theta}{\int_0^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta} \qquad (\cos 2\theta = 2\cos^2\theta - 1)$$

$$= 2a \cdot \frac{\int_0^{\frac{\pi}{2}} (\sin\theta - \sin^3\theta)\, d\theta}{\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta}$$

$$= 2a \cdot \frac{\int_0^{\frac{\pi}{2}} \sin\theta - \left(\frac{3\sin\theta - \sin 3\theta}{4}\right) d\theta}{\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta}$$

$$= 2a \cdot \frac{\int_0^{\frac{\pi}{2}} \left(\sin\theta - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4}\right) d\theta}{\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta}$$

$$= 2a \cdot \frac{\left[-\cos\theta + \frac{3\cos\theta}{4} - \frac{\cos 3\theta}{12}\right]_0^{\frac{\pi}{2}}}{\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}}$$

$$= 2a \cdot \frac{\left(-\cos\frac{\pi}{2} + \frac{3}{4}\cos\frac{\pi}{2} - \frac{1}{12}\cos\frac{3\pi}{2} + \cos 0 - \frac{3}{4}\cos 0 + \frac{1}{12}\cos 0\right)}{\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi - \sin 0 - \frac{1}{2}\sin 0\right)}$$

$$= 2a \cdot \frac{\left(-0 + 0 - 0 + 1 - \frac{3}{4} + \frac{1}{12}\right)}{\left(\frac{\pi}{2} + 0 - 0 - 0\right)} = 2a \cdot \frac{\left(\frac{4}{12}\right)}{\left(\frac{\pi}{2}\right)}$$

or $$\bar{x} = \frac{4a}{3\pi}$$

Similarly, the distance of the centroid of the strip from the x-axis is

$$\bar{y} = \frac{\int \left(\frac{y}{2}\right) dA}{\int dA}$$

As the centroid of the area dA lies a distance $(y/2)$ from the x-axis.

or $$\bar{y} = \frac{\int_0^a \left(\frac{y}{2}\right) y \cdot dx}{\int_0^a y dx} \qquad (dA = ydx)$$

$$= \frac{1}{2} \cdot \frac{\int_0^a y^2 dx}{\int_0^a y dx}$$

$$= \frac{1}{2} \cdot \frac{\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\, dx}{\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx} \qquad \text{(on substituting } y \text{ from equation (2))}$$

$$= \frac{b}{2a} \cdot \frac{\int_0^a (a^2 - x^2)\, dx}{\int_0^a \sqrt{a^2 - x^2}\, dx}$$

$$= \frac{b}{2a} \cdot \frac{\int_0^{\frac{\pi}{2}} (a^2 - a^2 \sin^2 \theta)\, a \cos\theta \, d\theta}{\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \sin^2 \theta}\, a \cos\theta \, dx} \qquad \text{(using equation (4))}$$

$$= \frac{b}{2a} \cdot \frac{\int_0^{\frac{\pi}{2}} a^2 \cos^2 \theta \cdot a \cos\theta \, d\theta}{\int_0^{\frac{\pi}{2}} a \cos\theta \cdot a \cos\theta \, d\theta}$$

$$= \frac{b}{2} \cdot \frac{\int_0^{\frac{\pi}{2}} \cos^3 \theta \, d\theta}{\int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta}$$

$$= \frac{b}{2} \cdot \frac{\int_0^{\frac{\pi}{2}} \left(\frac{\cos 3\theta + 3\cos\theta}{4} \right) d\theta}{\int_0^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta} \qquad (\cos 3\theta = 4\cos^3\theta - 3\cos\theta)$$

$$= \frac{b}{4} \cdot \frac{\int_0^{\frac{\pi}{2}} (\cos 3\theta + 3\cos\theta)\, d\theta}{\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta}$$

$$= \frac{b}{4} \cdot \frac{\left[\frac{\sin 3\theta}{3} + 3\sin\theta \right]_0^{\frac{\pi}{2}}}{\left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}}}$$

$$= \frac{b}{4} \cdot \frac{\left(\frac{1}{3} \times \sin\frac{3\pi}{2} + 3\sin\frac{\pi}{2} - \frac{\sin 0}{3} - 3\sin 0 \right)}{\left(\frac{\pi}{2} + \frac{1}{2} \times \sin\pi - 0 - \frac{\sin 0}{2} \right)}$$

$$= \frac{b}{4} \cdot \frac{\left(\frac{1}{3} \times (-1) + 3 \times 1 - 0 - 0\right)}{\left(\frac{\pi}{2} + \frac{1}{2} \times 0 - 0 - 0\right)}$$

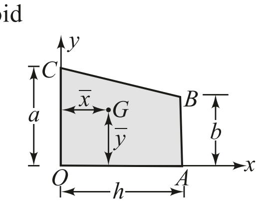
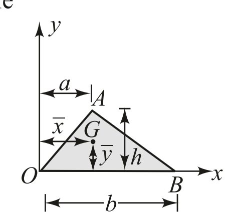
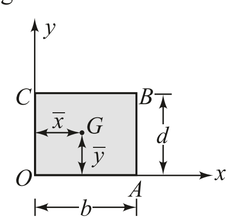
$$= \frac{b}{4} \cdot \frac{\left(-\frac{1}{3} + 3\right)}{\frac{\pi}{2}} = \frac{b}{4} \cdot \frac{\left(\frac{8}{3}\right)}{\left(\frac{\pi}{2}\right)}$$

or $$\bar{y} = \frac{4b}{3\pi}$$

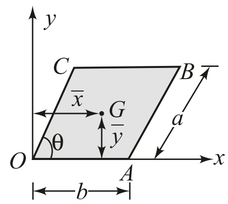
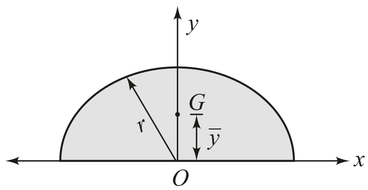
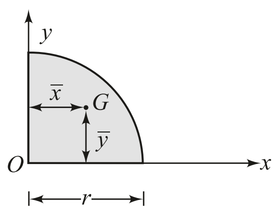
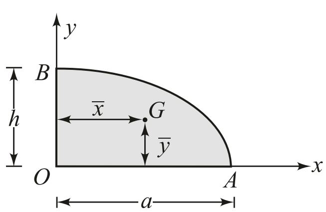
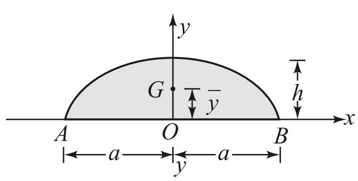
Hence, the centroid of the quarter-ellipse is ($\bar{x}$, $\bar{y}$), that is, $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$.

The centroids of some important geometries are given in Table 3.1.

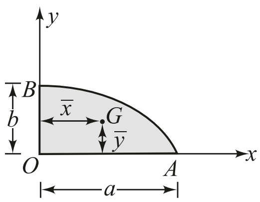
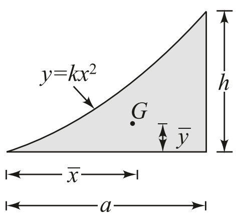
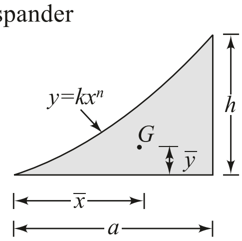
Table 3.1

Figure	***Centroid***	***Area***
Trapezoid 	$\bar{x} = \frac{h(a+2b)}{3(a+b)}$ $\bar{y} = \frac{a^2+ab+b^2}{3(a+b)}$	$\frac{(a+b)h}{2}$
Triangle 	$\bar{x} = \frac{(a+b)}{3}$ $\bar{y} = \frac{h}{3}$	$\frac{1}{2}bh$
Rectangle 	$\bar{x} = \frac{b}{2}$ $\bar{y} = \frac{d}{2}$	bd

Contd....

Parallelogram 	$\bar{x} = \dfrac{(a\cos\theta + b)}{2}$ $\bar{y} = \dfrac{a\sin\theta}{2}$	$ab \sin\theta$
Semi-circle 	$\bar{x} = 0$ $\bar{y} = \dfrac{4r}{3\pi}$	$\dfrac{\pi r^2}{2}$
Quarter-circle 	$\bar{x} = \dfrac{4r}{3\pi}$ $\bar{y} = \dfrac{4r}{3\pi}$	$\dfrac{\pi r^2}{4}$
Semi-parabola 	$\bar{x} = \dfrac{3}{8}a$ $\bar{y} = \dfrac{2}{5}h$	$\dfrac{2ah}{3}$
Parabola 	$\bar{x} = 0$ $\bar{y} = \dfrac{2}{5}h$	$\dfrac{4ah}{3}$

Contd....

Quarter-ellipse 	$\bar{x} = \dfrac{4a}{3\pi}$ $\bar{y} = \dfrac{4b}{3\pi}$	$\dfrac{\pi ab}{4}$
Parabolic segment (2nd degree) 	$\bar{x} = \dfrac{3}{4}a$ $\bar{y} = \dfrac{3}{10}h$	$\dfrac{ah}{3}$
General spander 	$\bar{x} = \left(\dfrac{n+1}{n+2}\right)a$ $\bar{y} = \left(\dfrac{n+1}{2n+1}\right)\dfrac{h}{2}$	$\dfrac{ah}{(n+1)}$

3.3 MOMENT OF INERTIA

A body rotating about an axis in a reference frame resists any change in its rotational motion (angular velocity). On account of this property, the body is said to have a moment of inertia or rotational inertia about that axis.

The moment of inertia is of two types:

- Mass moment of inertia
- Area moment of inertia, also called Second moment of area

3.3.1 Mass Moment of Inertia

If a particle of mass m is located at a distance r from an axis of rotation, then its moment of inertia I about that axis is given as

$$I = mr^2 \qquad ...(3.10)$$

If m_1, m_2, m_3, ... are the masses of the various particles constituting the body and r_1, r_2, r_3 ... are their distances from the axis of rotation (Fig. 3.12), then the moment of inertia of the body about that axis is given as

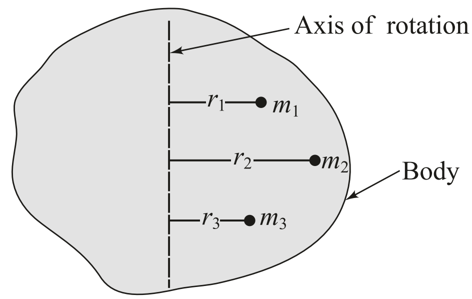


Fig. 3.12

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots = \Sigma mr^2 \quad \text{... (3.11)}$$

If the mass is continuously distributed in the body, then

$$I = \int r^2 \, dm \quad \text{... (3.12)}$$

where dm is the mass of an infinitesimally small element of the body at a distance r from the axis of rotation.

3.3.2 Radius of Gyration w.r.t. Mass Moment of Inertia

The radius of gyration, K, of a body is defined as the radial distance from the given axis of rotation, the square of which on being multiplied by the total mass of the body gives the moment of inertia of the body about that axis. Hence,

$$I = MK^2 = \Sigma mr^2 \quad \text{... (3.13)}$$

where M is the total mass of the body.

From Equation (3.13), we have

$$K = \sqrt{\frac{I}{M}} \quad \text{... (3.14)}$$

3.3.3 Second Moment of Area

The moments of inertia of a plane area, with respect to the xy-axes as shown in Fig. 3.1, are defined as

$$I_x = \int y^2 dA \quad \text{... (3.15)}$$

and

$$I_y = \int x^2 dA \quad \text{... (3.16)}$$

where x and y are the coordinates of the differential elements of area dA. Because dA is multiplied by the square of the distance, the moments of inertia are also called second moments of the area. Further, the integrals I_x and I_y are also referred to as rectangular moments of inertia, as they are computed from the rectangular coordinates of the element dA. While each integral is actually a double integral, it is possible in many applications to select elements of area dA in the shape of thin horizontal or vertical strips, and thus reduce the computations to integration in a single variable. The moments of inertia are always positive and have SI units of m^4, and their dimension is $[L^4]$.

3.3.4 Radius of Gyration w.r.t. Second Moment of Area

The radius of gyration of an area about an axis is defined as its distance from the given axis, the square of which on being multiplied by the total area of the body gives the second moment of inertia about that axis. It is given as

$$I = AK^2$$

or

$$K = \sqrt{\frac{I}{A}} \quad \text{... (3.17)}$$

For the x-axis

$$K_x = \sqrt{\frac{I_x}{A}} \quad \text{... (3.18)}$$

For the y-axis

$$K_y = \sqrt{\frac{I_y}{A}} \quad \text{... (3.19)}$$

The concept of the radius of gyration is useful in the analysis of behaviour of structural members like beams, columns etc.

3.4 PRODUCT OF INERTIA

The product of inertia of a plane area is also called the product second moment of area or the product moment of inertia or simply the product of inertia. It is defined with respect to a set of perpendicular axes lying in the plane of the area. With respect to the xy-axes as shown in Fig. 3.1, the product moment of inertia is defined as

$$I_{xy} = \int_A xy\,dA \quad \text{... (3.20)}$$

As each element of the area dA is multiplied by the product of its coordinates, hence the product of inertia can be positive, negative or zero. It is important to note that product of inertia of an area is zero with respect to any pair of axes in which one axis (either x or y-axis) is an axis of symmetry of the area.

As most of the structural members used in bending applications consist of cross-sections of a combination of rectangles, the value of the product of inertia for such sections is determined by the addition of the I_{xy} value for each rectangle. It has SI unit of m^4 and dimension $[L^4]$.

3.5 PRINCIPAL AXES AND PRINCIPAL MOMENTS OF INERTIA

Principal axes are the axes about which the product of inertia of the cross-section of a beam is zero. All plane sections, whether they have an axis of symmetry or not, have two mutually perpendicular axes about which the product of inertia is zero. Simple bending of a beam takes place about a principal axis, and the moments are applied in a plane parallel to one such axis. The moments of inertia of an area about the principal axes are called the principal moments of inertia, and their maximum and minimum values are respectively referred to as the maximum principal moment of inertia and the minimum principal moment of inertia.

The expression for the principal moments of inertia can be obtained using the expression for the principal stresses given below.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{... (3.21)}$$

Substituting $\sigma_x = I_x$, $\sigma_y = I_y$ and $\tau_{xy} = I_{xy}$ in equation (3.21), we can write the expression for the principal moments of inertia as

$$I_1, I_2 = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \quad \text{... (3.22)}$$

where

I_1 = Maximum principal moment of inertia

I_2 = Minimum principal moment of inertia

I_x = Moment of inertia of the section about the x-axis

I_y = Moment of inertia of the section about the y-axis.

I_{xy} = Product moment of inertia of the section with respect to x and y axes.

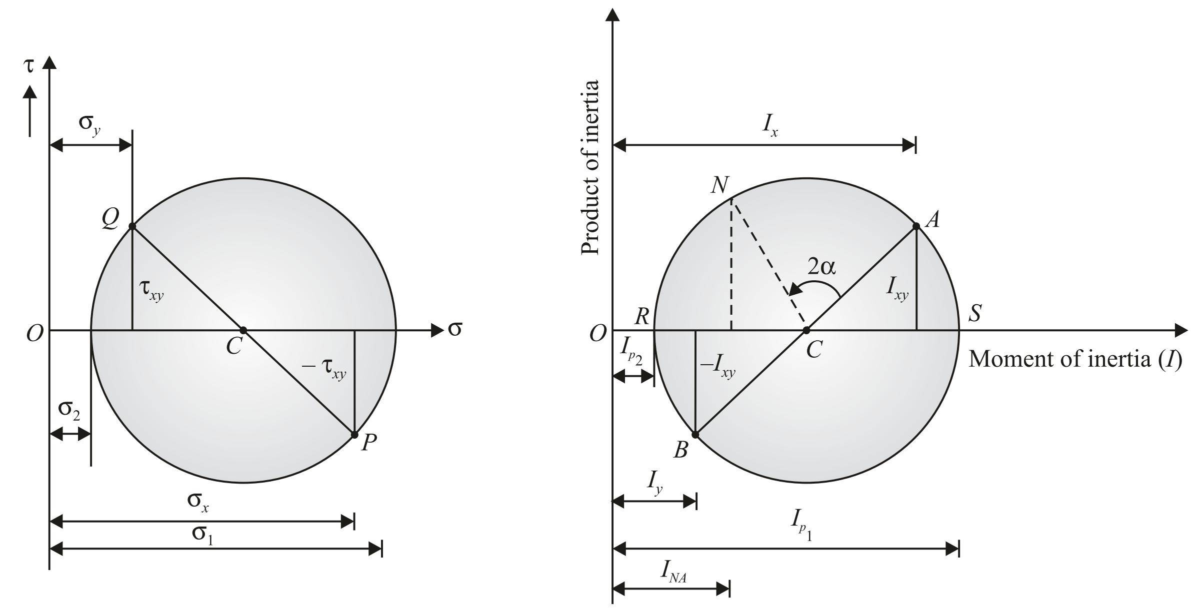
The orientation of the principal axes is defined as

$$\tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y} \quad \text{... (3.23)}$$

The two values of θ_p differ by 90° and define the two perpendicular principal axes, of which one corresponds to the maximum moment of inertia and the other corresponds to the minimum moment of Inertia. There are no shear stresses on the principal planes, hence the product of inertia is zero with respect to the principal axes.

3.6 MOHR'S CIRCLE FOR SECOND MOMENTS OF AREA

The transformation equations for the normal and shear stresses for a plane stress condition can be represented in a graphical form, known as Mohr's circle, as discussed in the chapter of principal stresses. The coordinates of each point on the circle correspond to the normal and shear stresses acting on specific set of perpendicular planes. Mohr's circle can also be applied to the second moments of area to represent the transformation equations for the second moments of area and the product of inertia of area, and the coordinates of each point on the circle represent the moment of inertia and the product of inertia with respect to a specific set of perpendicular axes. For constructing the circle (Fig. 3.13(*b*)), the second moments of area are plotted on the horizontal axis (the x-axis) and the product of inertia on the vertical axis (the y-axis). Two points, say A and B are considered. Point A with coordinates (I_x, I_{xy}) is plotted above the horizontal axis, if I_{xy} is positive. Similarly, point B with its coordinates $(I_y, -I_{xy})$ is plotted below the horizontal axis. The two points A and B are now joined by a straight line, and a circle is constructed with this line (AB) as the diameter. The resulting circle is called Mohr's circle for second moments of area with centre C. OR (I_{P_2}) and OS (I_{P_1}) are the principal second moments of area with their minimum and maximum values respectively. To find the second moment of area about some other axis, say the neutral axis (NA), which is inclined at an angle α to the x-axis, a point N is chosen on the circle such that $\angle ACN = 2\alpha$ in the counterclockwise direction. The horizontal coordinate of N gives the value of second moment of area about the neutral axis, that is, I_{NA}.



(*a*) Mohr's circle for a plane stress condition (*b*) Mohr's circle for second moments of area

Fig. 3.13 Comparison of Mohr's circles of plane stress and second moment of area.

On comparing Mohr's circles for plane stress and second moments of area as shown in Fig. 3.13, we find that τ_{xy} is associated with $-I_{xy}$. The point *A* with its coordinates (I_x, I_{xy}) in Fig. 3.13 (*b*) is plotted above the horizontal axis, if I_{xy} is positive. In contrast, the corresponding point *P* with its coordinates (σ_x, $-\tau_{xy}$) in Fig. 3.13 (*a*) is plotted below the horizontal axis for positive τ_{xy}. Otherwise, the procedure to construct Mohr's circle for the second moments of area is identical to Mohr's circle for plane stress.

3.7 PARALLEL-AXES THEOREM

The parallel-axes theorem is used to find the moment of inertia of a plane area about an axis parallel to its centroidal axis. Consider an area *A* with its centroid *G* having coordinates ($\bar{x}$, $\bar{y}$) as shown in Fig. 3.14. The horizontal centroidal axis is the x'-axis, which is parallel to the *x*-axis and the vertical centroidal axis is the y'-axis, which is parallel to the *y*-aixs. Further an element of area *dA* is considered, where *x* and *y* coordinates are defined as

$$x = \bar{x} + x'$$

$$y = \bar{y} + y'$$

where $\bar{x}$ is the *x*-coordinate of the centroid *G*, which represents the perpendicular distance between two vertical parallel axes, namely the *y*-axis and the y'-axis. Similarly, $\bar{y}$ is the *y*-coordinate of the centroid *G*, which represents the perpendicular distance between two horizontal parallel axes, namely the *x*-axis and the x'-axis.

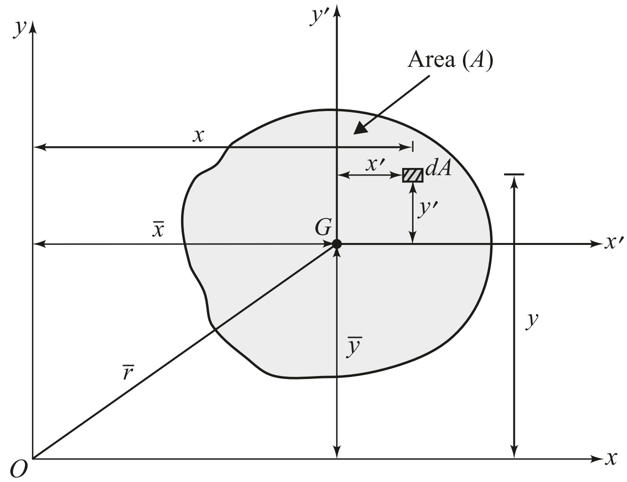


Fig. 3.14

The mathematical statements of the parallel-axes theorem for the moment of inertia are now expressed as

$$I_x = \overline{I}_x + A\overline{y}^2 \quad \text{(for the } x\text{-axis and the } x'\text{-axis)} \qquad \text{... (3.24)}$$

$$I_y = \overline{I}_y + A\overline{x}^2 \quad \text{(for the } y\text{-axis and the } y'\text{-axis)} \qquad \text{... (3.25)}$$

where $\overline{I}_x = I_{x'}$ = Moment of inertia about the centroidal x-axis, that is, the x'-axis

$\overline{I}_y = I_{y'}$ = Moment of inertia about the centroidal y-axis, that is, the y'-axis

The parallel-axes theorem for the product of inertia of an area can be obtained by substituting x and y in the expression for the product of inertia, that is,

$$I_{xy} = \int xy dA$$

$$= \int (\overline{x} + x')(\overline{y} + y')dA$$

$$= \int x'y'dA + \overline{x}\int y'dA + \overline{y}\int x'dA + \overline{x}\,\overline{y}\int dA$$

where $\int x'y'dA = \overline{I}_{xy} = I_{x'y'}$ = Product of inertia of the cross-sectional area with respect to the centroidal axes x any y, that is, the axes x' and y'.

$\int y'dA = \int x'dA = 0$ (as they are the first moment of area about the centroidal axes x and y.)

and $\int dA = A$

Hence, equation of the product of inertia of the area reduces to

$$I_{xy} = \overline{I}_{xy} + A\,\overline{x}\,\overline{y} \qquad \text{... (3.26)}$$

Equation (3.26) represents the mathematical statement of the parallel-axes theorem for the product of inertia of an area.

Now, the polar moment of inertia of an area is expressed as

$$J_o = I_x + I_y \qquad \text{... (3.27)}$$

where J_o = Polar moment of inertia about the origin O.

On substituting I_x and I_y from equations (3.24) and (3.25), we get

$$J_o = (\overline{I}_x + A\overline{y}^2) + (\overline{I}_y + A\overline{x}^2)$$

$$= \overline{I}_x + \overline{I}_y + A(\overline{x}^2 + \overline{y}^2)$$

Hence,

$$J_o = \overline{J}_G + A\overline{r}^2 \qquad \text{... (3.28)}$$

where $\overline{J}_G = \overline{I}_x + \overline{I}_y$ = Polar moment of inertia of the area about the centroid G

$\overline{r}^2 = \overline{x}^2 + \overline{y}^2$, and its square root $\overline{r} = \sqrt{\overline{x}^2 + \overline{y}^2}$ represents the distance between O and G as shown in Fig. 3.14.

Equation (3.28) represents the mathematical statement of the parallel-axes theorem for the polar moment of inertia of an area.

3.8 MOMENT OF INERTIA OF A RECTANGULAR SECTION

Consider a rectangular section $OABC$ (Fig. 3.15).

Let b = Width of the section

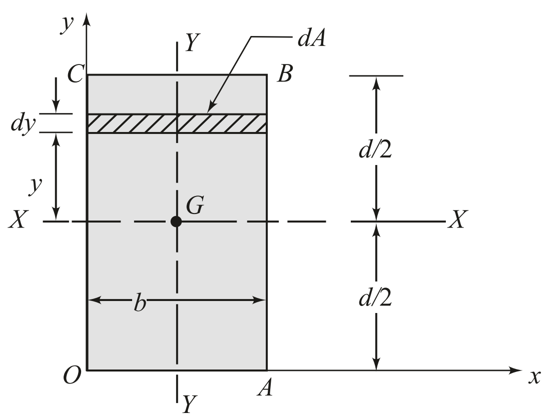


Fig. 3.15

d = Depth of the section

XX = Horizontal centroidal axis

YY = Vertical centroidal axis

Consider a strip (parallel to the x-axis) of thickness dy at a distance y from the XX-axis.

Area of the strip = $dA = bdy$

Moment of inertia of the strip about the XX-axis = $dA \times y^2$

$$= by^2\, dy$$

The moment of inertia of the entire section about the XX-axis is found as

$$I_{XX} = \int_{-d/2}^{d/2} by^2\, dy$$

$$= b\left(\frac{y^3}{3}\right)_{-d/2}^{d/2}$$

$$= \frac{b}{3}\left[\left(\frac{d}{2}\right)^3 - \left(-\frac{d}{2}\right)^3\right]$$

$$= \frac{b}{3}\left(\frac{d^3}{8} + \frac{d^3}{8}\right)$$

$$= \frac{bd^3}{12} \qquad \text{... (3.29)}$$

The moment of inertia about the YY-axis is given as

$$I_{YY} = \frac{db^3}{12} \qquad \text{... (3.30)}$$

The moment of inertia about the x-axis is given as

$$I_x = I_{XX} + Ah^2$$

$$= \frac{bd^3}{12} + (bd) \times \left(\frac{d}{2}\right)^2 \qquad \left(h = \frac{d}{2}\right)$$

$$= \frac{bd^3}{3} \qquad \text{... (3.31)}$$

The moment of inertia about the y-axis is given as

$$I_y = I_{YY} + Ah^2$$

$$= \frac{db^3}{12} + (bd) \times \left(\frac{b}{2}\right)^2 \qquad \left(h = \frac{b}{2}\right)$$

$$= \frac{db^3}{3} \qquad \text{... (3.32)}$$

3.9 MOMENT OF INERTIA OF A SOLID CIRCULAR SECTION

First method

Consider a circular section of radius r (Fig. 3.16).

An elemental strip of thickness dy is considered at a distance y from the x-axis.

Area of the strip, $dA = 2r \cos\theta \cdot dy$

Now $\quad y = r\sin\theta$

$\quad dy = r\cos\theta\, d\theta$

Hence, $\quad dA = 2r\cos\theta \cdot r\cos\theta\, d\theta$

$\quad = 2r^2\cos^2\theta d\theta$

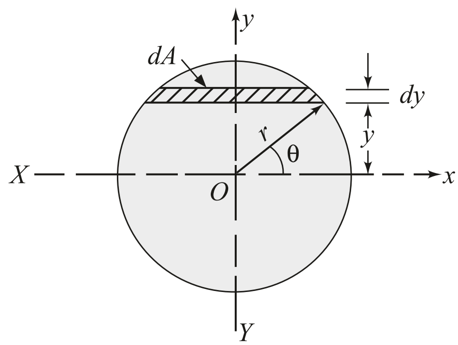


Fig. 3.16

The moment of inertia of the section about the centroidal axis XX is given as

$$I_{XX} = \int y^2\, dA$$

$$= \int_{-\pi/2}^{\pi/2} (r\sin\theta)^2 \,.\, 2r^2\cos^2\theta\, d\theta$$

$$= 4r^4 \int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta$$

$$= \frac{\pi r^4}{4} = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi d^4}{64} \qquad (d = 2r) \quad \text{... (3.33)}$$

where d is the diameter of the circular section.

Similarly, $\quad I_{YY} = \dfrac{\pi d^4}{64}$ ... (3.34)

Also $\quad I_x = I_{XX}$ and $I_y = I_{YY}$

and $\quad I_z = \dfrac{\pi d^4}{32}$ ... (3.35)

Second method

Consider an elemental ring of thickness dx at a radius x (Fig. 3.17).

Area of the elemental ring,

$$dA = 2\pi x dx$$

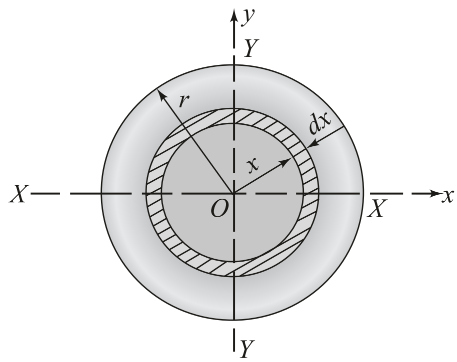


Fig. 3.17

The moment of inertia of the ring about an axis perpendicular to the plane of the circular section (z-axis), called polar moment of inertia, is given by

$$\begin{aligned} I_{z_{\text{ring}}} &= \text{Area of the ring} \times \text{Radius}^2 \\ &= 2\pi x dx \times x^2 \\ &= 2\pi x^3 dx \end{aligned}$$

The polar moment of inertia of the entire section is given as

$$\begin{aligned} I_z &= \int_0^r I_{z_{\text{ring}}} \\ &= \int_0^r 2\pi x^3 dx \\ &= 2\pi \left(\frac{x^4}{4}\right)_0^r = \frac{\pi r^4}{2} \\ &= \frac{\pi d^4}{32} \qquad \left(r = \frac{d}{2}\right) \end{aligned}$$

Now

$$\begin{aligned} I_z &= I_{XX} + I_{YY} \\ &= 2I_{XX} \qquad (I_{XX} = I_{YY}) \end{aligned}$$

$$I_{XX} = \frac{I_z}{2} = \frac{\pi d^4}{64} = I_x \qquad \text{... (3.36)}$$

Similarly,

$$I_{YY} = \frac{\pi d^4}{64} = I_y \qquad \text{... (3.37)}$$

3.10 MOMENT OF INERTIA OF A HOLLOW CIRCULAR SECTION

Refer Fig. 3.18.

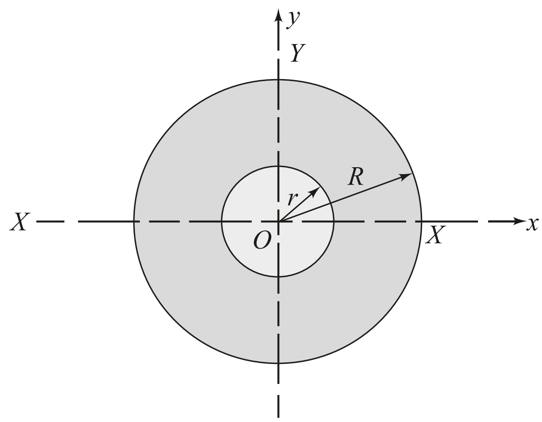


Fig. 3.18

$$I_{XX} = I_{YY} \qquad \text{(also } I_x = I_y,\ I_x = I_{XX} \text{ and } I_y = I_{YY})$$

$$= \frac{\pi}{64} D^4 - \frac{\pi}{64} d^4$$

$$= \frac{\pi}{64} (D^4 - d^4) \qquad \text{... (3.38)}$$

and
$$I_z = \frac{\pi}{32} (D^4 - d^4) \qquad \text{... (3.39)}$$

where
$$D = 2R$$
$$d = 2r$$

3.11 MOMENT OF INERTIA OF A SEMI-CIRCLE

Refer Fig. 3.19.

$$I_x = \frac{\pi}{128} d^4 \qquad \text{... (3.40)}$$

Also
$$I_y = \frac{\pi}{128} d^4 \qquad \text{... (3.41)}$$

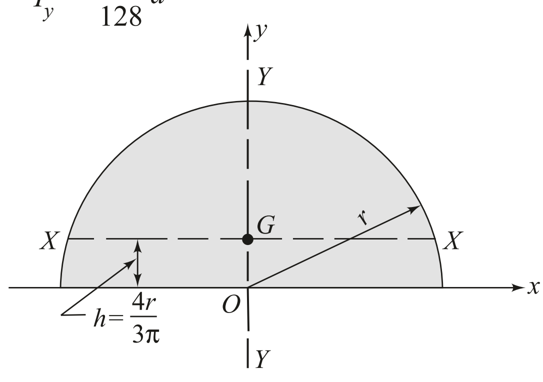


Fig. 3.19

and $$I_Z = \frac{\pi d^4}{64} \qquad \text{... (3.42)}$$

where $$d = 2r$$

$$I_x = I_{XX} + Ah^2 \qquad \text{(using parallel-axes theorem)}$$

or $$I_{XX} = I_x - Ah^2$$

$$= \frac{\pi}{128} d^4 - \frac{1}{2}\left(\frac{\pi}{4} d^2\right) \times \left(\frac{4}{3\pi} \times \frac{d}{2}\right)^2$$

$$= 0.00686 d^4 = 0.11 r^4 \qquad \text{... (3.43)}$$

3.12 MOMENT OF INERTIA OF A QUARTER-CIRCLE

Refer Fig. 3.20.

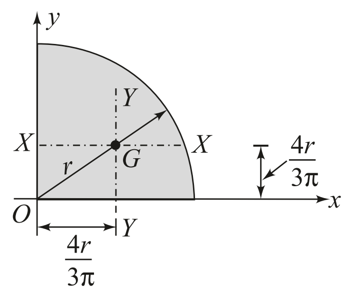


Fig. 3.20

$$I_x = \frac{1}{2} \times \left(\frac{\pi}{128} d^4\right)$$

$$= \frac{\pi}{256} d^4 = I_y \qquad \text{... (3.44)}$$

$$I_x = I_{XX} + Ah^2 \qquad \text{(using parallel-axes theorem)}$$

$$I_{XX} = I_x - Ah^2$$

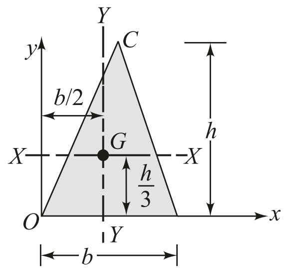
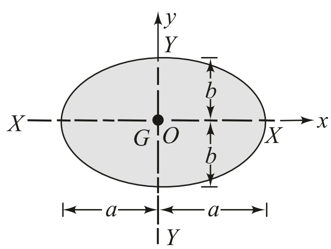
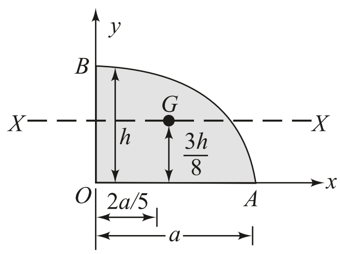
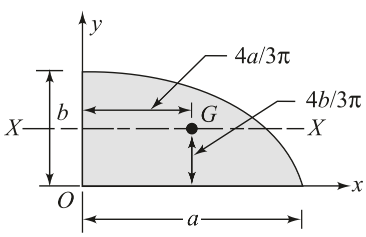
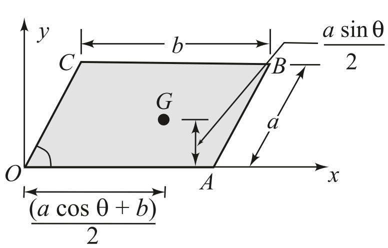
$$= \frac{\pi}{256} d^4 - \frac{1}{4} \times \left(\frac{\pi}{4} d^2\right) \times \left(\frac{4}{3\pi} \times \frac{d}{2}\right)^2$$

$$= 0.\ 00343 d^4 = 0.0549 r^4 = I_{YY} \qquad \text{... (3.45)}$$

and $$I_z = \frac{\pi d^4}{128} \qquad \text{...(3.46)}$$

The second moments of area of some more plane figures are given in Table 3.2.

Table 3.2

<table>
<tr><th>Figure</th><th>Second moment of area</th></tr>
<tr><td>Triangle

</td><td>$$I_{XX} = \frac{bh^3}{36}$$ $$I_{YY} = \frac{hb^3}{48}$$ $$I_x = \frac{bh^3}{12}$$</td></tr>
<tr><td>Ellipse

</td><td>$$I_x = \frac{\pi}{4} ba^3$$ $$I_y = \frac{\pi}{4} ba^3$$ $$I_z = \frac{\pi ab}{4}(a^2 + b^2)$$</td></tr>
<tr><td>Parabola

</td><td>$$I_x = \frac{2ah^3}{15}$$ $$I_y = \frac{16a^3h}{15}$$</td></tr>
<tr><td>Quarter-ellipse

</td><td>$$I_x = \frac{\pi ab^3}{16}$$ $$I_y = \frac{\pi ba^3}{16}$$</td></tr>
<tr><td>Parallelogram

</td><td>$$I_x = \frac{a^2 b \sin^3 \theta}{3}$$ $$I_y = \frac{ab}{6 \sin \theta} \times$$ $$(2a^2 \cos^2 \theta + 3ab \cos \theta + 2b^2)$$</td></tr>
</table>

Example 3.9

Find the second moment of area of an L-section about the centroidal axes XX and YY as shown in Fig. 3.21.

Solution: Refer Fig. 3.21. $\bar{x}$ and $\bar{y}$ are given as

$$\bar{x} = 25.24 \text{ mm}$$

and $$\bar{y} = 90.24 \text{ mm}$$

The moment of inertia of the section about the XX-axis is the sum of the moments of inertia of the two parts about the same axis, and is given by using theorem of parallel-axes as

$$I_{XX} = I_{XX_1} + I_{XX_2}$$

$$= \left[\frac{1}{12} \times 15 \times 250^3 + 250 \times 15 \times \left(\frac{250}{2} - 90.24\right)^2\right]$$

$$+ \left[\frac{1}{12} \times (120 - 15) \times 15^3 + \{(120 - 15) \times 15\} \times \left(90.24 - \frac{15}{2}\right)^2\right]$$

$$= 30.47 \times 10^6 \text{ mm}^4$$ **Ans.**

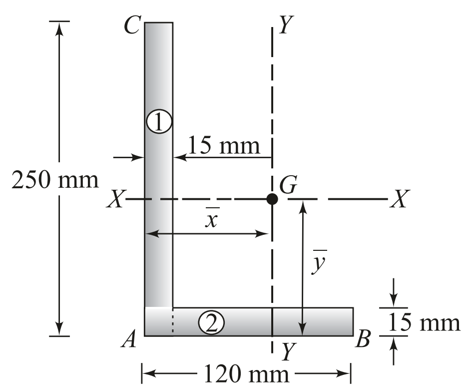


Fig. 3.21

The moment of inertia of the section about the YY-axis is given as

$$I_{YY} = I_{YY_1} + I_{YY_2}$$

$$= \left[\frac{1}{12} \times 250 \times 15^3 + 250 \times 15 \times \left(25.24 - \frac{15}{2}\right)^2\right]$$

$$+ \left[\frac{1}{12} \times 15 \times 105^3 + 105 \times 15 \times (67.5 - 25.24)^2\right]$$

$$= 55.1 \times 10^5 \text{ mm}^4$$ **Ans.**

Example 3.10

Find the second moment of area of a *T*-section shown in Fig. 3.22 about the centroidal axes *XX* and *YY*.

Solution: Refer Fig. 3.22. The section is symmetrical about the *YY*-axis, hence

$$\bar{x} = 0$$

and $$\bar{y} = 201.36 \text{ mm}$$

Using theorem of parallel-axes, the moment of inertia of the section about the *XX*-axis is given as

$$I_{XX} = I_{XX_1} + I_{XX_2}$$

$$= \left[\frac{1}{12} \times 200 \times 30^3 + 200 \times 30 \times (265 - 201.36)^2\right]$$

$$+ \left[\frac{1}{12} \times 20 \times 250^3 + 250 \times 20 \times (201.36 - 125)^2\right]$$

$$= 7.99 \times 10^7 \text{ mm}^4$$ **Ans.**

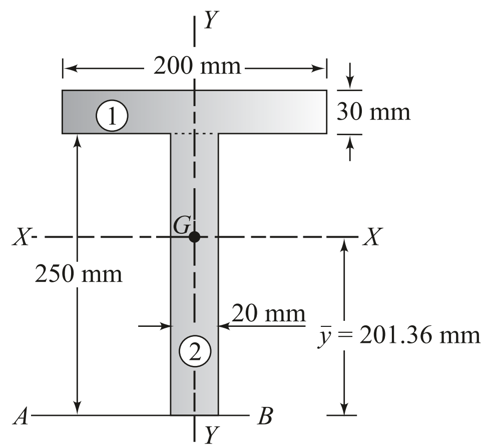


Fig. 3.22

The moment of inertia of the section about the *YY*-axis is given as

$$I_{YY} = I_{YY_1} + I_{YY_2}$$

$$= \left(\frac{1}{12} \times 30 \times 200^3 + \frac{1}{12} \times 250 \times 20^3\right) = 2.016 \times 10^7 \text{ mm}^4$$ **Ans.**

Example 3.11

Find the second moment of area of an *I*-section shown in Fig. 3.23 about the centroidal axes *XX* and *YY*.

Solution: Refer Fig. 3.23.

The section is symmetrical about the *YY*-axis, hence

$$\bar{x} = 0$$

and $$\bar{y} = 95.64 \text{ mm}$$

Using theorem of parallel-axes, the moment of inertia of the section about the XX-axis is given as

$$I_{XX} = I_{XX_1} + I_{XX_2}$$

$$= \left[\frac{1}{12}\times 150\times 20^3 + 150\times 20\times (240-95.64)^2\right] + \left[\frac{1}{12}\times 25\times 200^3 + 200\times 25\times (130-95.64)^2\right]$$

$$+ \left[\frac{1}{12}\times 250\times 30^3 + 250\times 30\times (95.64-15)^2\right]$$

$$= 7.55 \times 10^7 \text{ mm}^4$$ **Ans.**

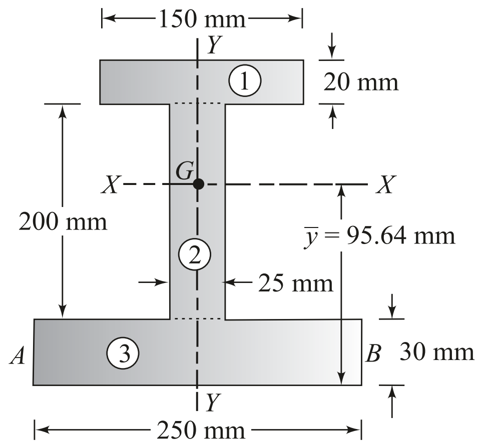


Fig. 3.23

The moment of inertia of the section about the YY-axis is given as

$$I_{YY} = I_{YY_1} + I_{YY_2}$$

$$= \left[\frac{1}{12}\times 20\times 150^3 + \frac{1}{12}\times 200\times 25^3 + \frac{1}{12}\times 30\times 250^3\right] = 4.49 \times 10^7 \text{ mm}^4$$ **Ans.**

Since $I_{XX} > I_{YY}$, hence the section is more stronger about the XX-axis.

Example 3.12

Find the second moment of area of the shaded section shown in Fig. 3.24 about its centroidal axis parallel to the base.

Solution: The given section is symmetrical about the YY-axis and is equivalent to a section made of two triangles and one square minus one semi-circle as shown in Fig. 3.25. Let XX and YY be the centroidal axes.

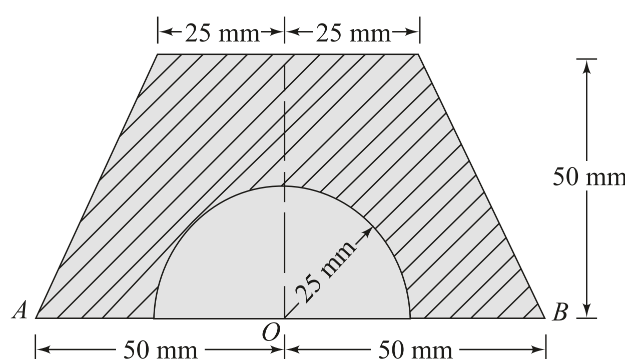


Fig. 3.24

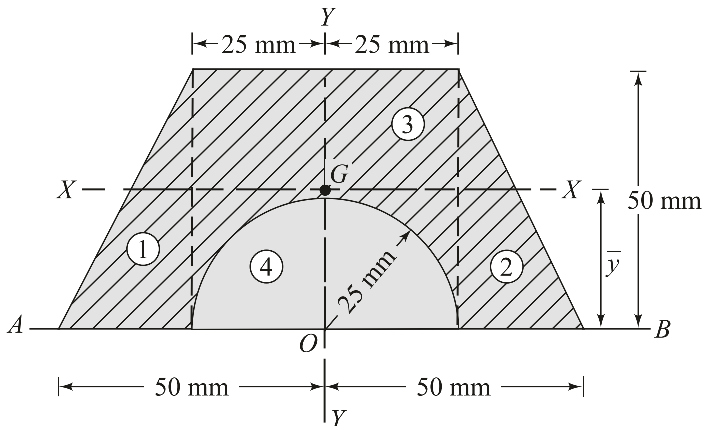


Fig. 3.25

Here $\bar{x} = 0$

Calculation of $\bar{y}$

Part **(1)**	***Part*** **(2)**	***Part*** **(3)**	***Part*** **(4)**
Area, $a_1 = \frac{1}{2} \times 25 \times 50$ $= 625\ \text{mm}^2$	$a_2 = 625\ \text{mm}^2$	$a_3 = 50 \times 50$ $= 2500\ \text{mm}^2$	$a_4 = \frac{1}{2} \times \pi \times 25^2$ $= 981.74\ \text{mm}^2$
Distance of *C.G.* from *AB* $y_1 = \frac{50}{3}$ $= 16.67$ mm	$y_2 = \frac{50}{3}$ $= 16.67$ mm	$y_3 = \frac{50}{2}$ $= 25$ mm	$y_4 = \frac{4 \times 25}{3\pi}$ $= 10.61$ mm

The distance of the centroid of the section from *AB* is given as

$$\bar{y} = \frac{a_1\, y_1 + a_2\, y_2 + a_3\, y_3 - a_4\, y_4}{a_1 + a_2 + a_3 - a_4}$$

$$= \frac{(625 \times 16.67) + (625 \times 16.67) + (2500 \times 25) - (981.74 \times 10.61)}{625 + 625 + 2500 - 981.74}\ \text{mm}$$

$$= 26.34\ \text{mm}$$

The second moment of area of the section about the *XX*-axis is found by using parallel-axes theorem.

$$I_{XX} = I_{XX_1} + I_{XX_2} + I_{XX_3}$$

$$= 2 \times \left[\frac{25\times 50^3}{36} + \frac{1}{2}\times 25\times 50\times\left(26.34 - \frac{50}{3}\right)^2\right] + \left[\frac{50\times 50^3}{12} + 50\times 50\times(26.34-25)^2\right]$$

$$- \left[0.11\times 25^4 + \frac{\pi\times 25^2}{2}\times\left(26.34 - \frac{4\times 25}{3\pi}\right)^2\right] \text{mm}^4$$

$$= 5.3\times 10^5 \text{ mm}^4$$ **Ans.**

Example 3.13

Find the second moment of area of the section shown in Fig. 3.26 about its centroidal axes.

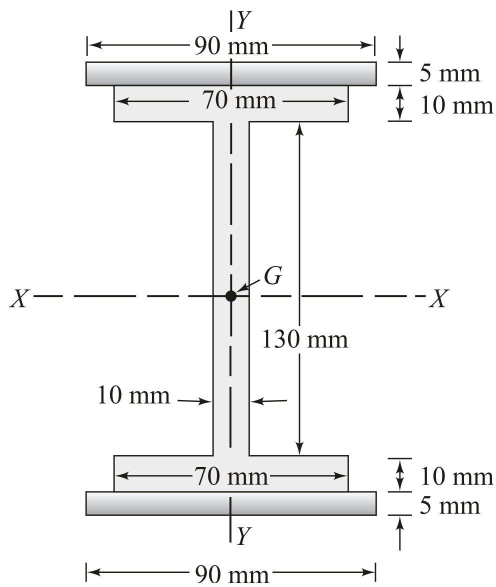


Fig. 3.26

Solution: The section is symmetrical about both XX and YY axes.

The second moment of area of the section about the XX-axis is given as

$I_{XX} = I_{XX}$ for I-section + I_{XX} for two plates

$$= \left[\left(\frac{1}{12}\times 70\times 10^3 + 70\times 10\times 70^2\right) + \left(\frac{1}{12}\times 10\times 130^3 + 0\right) + \left(\frac{1}{12}\times 70\times 10^3 + 70\times 10\times 70^2\right)\right]$$

$$+ 2\left[\frac{1}{12}\times 90\times 5^3 + 90\times 5\times\left(65+10+\frac{5}{2}\right)^2\right]$$

$$= 1.41\times 10^7 \text{ mm}^4$$ **Ans.**

The second moment of area of the section about the YY-axis is given as

$I_{YY} = I_{YY}$ for I-section + I_{YY} for two plates

$$= \left[2\left(\frac{1}{12}\times 10\times 70^3\right) + \frac{1}{12}\times 130\times 10^3\right] + 2\left[\frac{1}{12}\times 5\times 90^3\right] = 1.19\times 10^6 \text{ mm}^4$$ **Ans.**

Example 3.14

Find the expressions for the moments of inertia of a rectangle shown in Fig. 3.27 about the *x*-axis, the *y*-axis, the horizontal centroidal axis (*XX*) and the vertical centroidal axis (*YY*).

Solution:

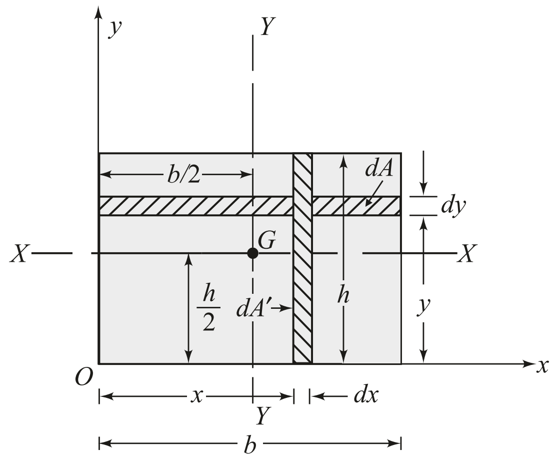


Fig. 3.27

An elementary area dA (horizontal) is considered as shown in Fig. 3.27.

The moment of inertia of the rectangle about the x-axis is given as

$$I_x = \int y^2 \, dA$$

$$= \int_o^h y^2 \cdot b \, dy \qquad \text{(where } dA = b \cdot dy\text{)}$$

$$= b \int_o^h y^2 \, dy$$

$$= b \left[\frac{y^3}{3} \right]_o^h = \frac{bh^3}{3} \qquad \textbf{Ans.}$$

For moment of inertia of the rectangle about the y-axis, we consider a vertical elementary area dA'.

Hence,
$$I_y = \int x^2 \, dA'$$

$$= \int_o^b x^2 \cdot h \, dx$$

$$= h \int_0^b x^2 \, dx = h \left[\frac{x^3}{3} \right]_o^b = \frac{hb^3}{3} \qquad \textbf{Ans.}$$

For the moment of inertia of the rectangle about its horizontal centroidal axis (I_{XX}), we use parallel-axes theorem as

$$I_x = I_{XX} + Ad^2$$

where d = Perpendicular distance between two axes

Hence,

$$I_{XX} = I_x - Ad^2$$

$$= \frac{bh^3}{3} - bh \cdot \left(\frac{h}{2}\right)^2$$

$$= \frac{bh^3}{3} - \frac{bh^3}{4} = \frac{bh^3}{12} \quad \textbf{Ans.}$$

Similarly, the moment of inertia of the rectangle about its vertical centroidal axis (I_{YY}) is given as

$$I_{YY} = I_y - Ad^2$$

$$= \frac{hb^3}{3} - b \cdot h \cdot \left(\frac{b}{2}\right)^2$$

$$= \frac{hb^3}{3} - \frac{hb^3}{4} = \frac{hb^3}{12} \quad \textbf{Ans.}$$

Example 3.15

Find the expressions for the moments of inertia of a circle about its diametral axes.

Solution: Refer Fig. 3.28.

Consider an elementary area dA.

$$dA = rd\theta \cdot dr$$

The moment of inertia of the circle about its horizontal diameter (the horizontal centroidal axis) is given as

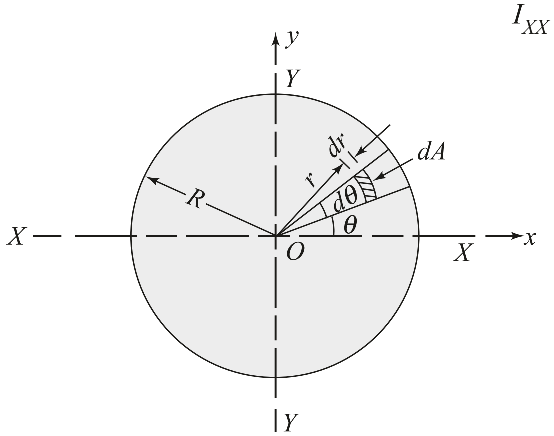


FIG. 3.28

$$I_{XX} = \int y^2\, dA$$

$$= \int\int (r \sin\theta)^2 \cdot rd\theta \cdot dr$$

$$= \int_o^R \int_o^{2\pi} \sin^2\theta \cdot d\theta \cdot r^3 dr$$

$$= \int_o^R \int_o^{2\pi} \left(\frac{1 - \cos 2\theta}{2}\right) d\theta \cdot r^3 dr \quad \left(\sin^2\theta = \frac{1 - \cos 2\theta}{2}\right)$$

$$= \frac{1}{2}\int_o^R \left[\int_o^{2\pi} d\theta - \int_o^{2\pi} \cos 2\theta\, d\theta\right] \cdot r^3 dr$$

$$= \frac{1}{2}\int_o^R \left[\theta - \frac{\sin 2\theta}{2}\right]_o^{2\pi} \cdot r^3\, dr \;=\; \frac{1}{2}\int_o^R 2\pi \cdot r^3 dr$$

$$= \pi \int_o^R r^3 dr \;=\; \pi\left(\frac{r^4}{4}\right)_o^R \;=\; \frac{\pi R^4}{4} \qquad \textbf{Ans.}$$

The moment of inertia of the circle about its vertical diameter (the vertical centroidal axis) is given as

$$I_{YY} = \int x^2\, dA$$

$$= \iint (r\cos\theta)^2 \cdot r d\theta \cdot dr \;=\; \int_o^{2\pi}\int_o^R r^3 dr \cdot \cos^2\theta\, d\theta$$

$$= \int_o^{2\pi}\int_o^R r^3 dr \cdot \left(\frac{1+\cos 2\theta}{2}\right) d\theta \qquad \left(\cos^2\theta = \frac{1+\cos 2\theta}{2}\right)$$

$$= \int_o^{2\pi} \frac{R^4}{4}\cdot\frac{1}{2}\cdot(1+\cos 2\theta)\, d\theta \;=\; \frac{R^4}{8}\left[\int_o^{2\pi} d\theta + \int_o^{2\pi}\cos 2\theta \cdot d\theta\right]$$

$$= \frac{R^4}{8}\left[\theta + \frac{\sin 2\theta}{2}\right]_o^{2\pi} \;=\; \frac{R^4}{8}\cdot 2\pi \;=\; \frac{\pi R^4}{4} \qquad \textbf{Ans.}$$

Hence, $$I_{XX} = I_{YY} = \frac{\pi R^4}{4} = \frac{\pi D^4}{64} \qquad (D = 2R)$$

Example 3.16

Find the expressions for the moments of inertia of a triangle shown in Fig. 3.29 about its base, the horizontal centroidal axis and the horizontal axis passing through its vertex.

Solution: Refer Fig. 3.29.

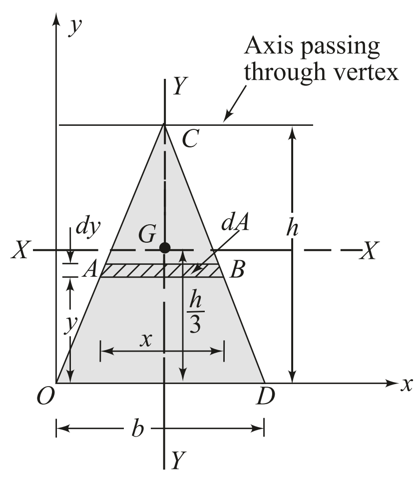


Fig. 3.29

Consider a triangle of base b and height h. G represents its centroid.

An elemental area dA, parallel to the base of the triangle, is considered.

$$dA = x \cdot dy \quad \text{... (1)}$$

From similar triangles OCD and ABC, we have

$$\frac{b}{x} = \frac{h}{(h-y)}$$

$$x = \frac{b(h-y)}{h}$$

On substituting x in equation (1), we get

$$dA = \frac{b(h-y)}{h} \cdot dy$$

Now the moment of inertia of the triangle about its base is given as

$$I_x = \int y^2 dA$$

$$= \int_o^h y^2 \cdot \frac{b(h-y)}{h} \cdot dy = \frac{b}{h}\left[h\int_o^h y^2 dy - \int_o^h y^3 dy\right]$$

$$= \frac{b}{h}\left[h\left(\frac{y^3}{3}\right)_o^h - \left(\frac{y^4}{4}\right)_o^h\right] = \frac{b}{h}\left(h \cdot \frac{h^3}{3} - \frac{h^4}{4}\right)$$

$$= \frac{bh^3}{3} - \frac{bh^3}{4} = \frac{bh^3}{12}$$ **Ans.**

Using parallel-axes theorem, we have

$$I_x = I_{XX} + A.h'^2$$

where h' = Perpendicular distance between the x-axis and the XX-axis

A = Area of the triangle

Now $$I_{XX} = I_x - A.h'^2$$

$$= \frac{bh^3}{12} - \frac{1}{2} \times b \times h \times \left(\frac{h}{3}\right)^2$$

$$= \frac{bh^3}{12} - \frac{bh^3}{18} = \frac{3bh^3 - 2bh^3}{36} = \frac{bh^3}{36}$$ **Ans.**

The moment of inertia of the triangle about the axis passing through its vertex I_V, using parallel-axes theorem, is given by

$$I_V = I_{XX} + Ah'^2$$

where h' = Perpendicular distance between the axis passing through vertex and the XX-axis

$$= \frac{2h}{3}$$

$$I_V = \frac{bh^3}{36} + \frac{1}{2} \times b \times h \times \left(\frac{2h}{3}\right)^2$$

$$= \frac{bh^3}{36} + \frac{2}{9} bh^3$$

$$= \frac{bh^3 + 8\,bh^3}{36} = \frac{bh^3}{4}$$ **Ans.**

Example 3.17

Find the moment of inertia of the area shown in Fig. 3.30 about the x-axis.

Solution:

Consider an elemental area dA parallel to the x-axis as shown in the figure.

$$dA = (30 - x)\,dy$$

The moment of inertia of the area about the x-axis is given as

$$I_x = \int y^2 dA$$

$$= \int_o^{30} y^2\,(30 - x)\,dy$$

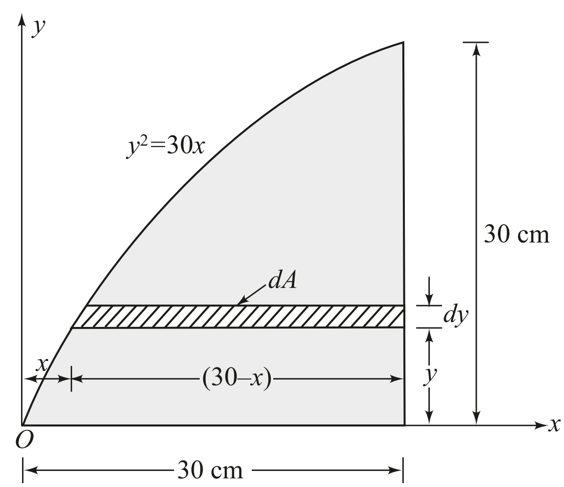


Fig. 3.30

$$= 30 \int_0^{30} y^2 dy - \int_0^{30} y^2 \cdot \frac{y^2}{30} \cdot dy \qquad \left(x = \frac{y^2}{30}\right)$$

$$= 30\left(\frac{y^3}{3}\right)_0^{30} - \frac{1}{30}\left(\frac{y^5}{5}\right)_0^{30} = 270000 - 162000 = 108000 \text{ cm}^4$$ **Ans.**

Example 3.18

Find the moment of inertia of the shaded section shown in Fig. 3.31 about its centroidal axes.

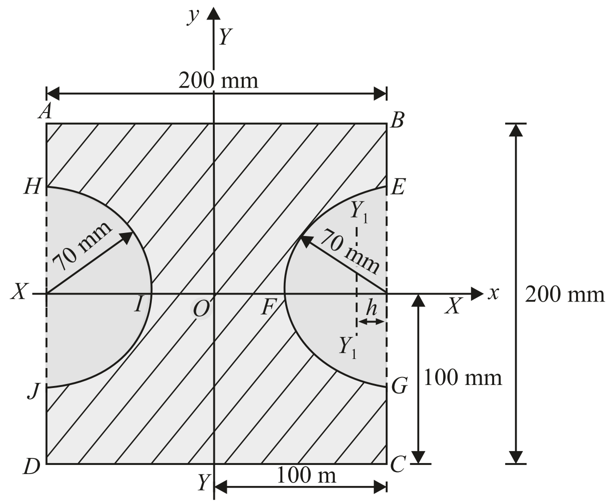


Fig. 3.31

Solution:

The section is symmetrical about both x and y axes, hence these two axes are also the centroidal axes with O as origin as well as the centroid of the section. Hence, we are required to find I_x and I_y.

Diameter of each semi-circle = 2 × 70 = 140 mm

Calculation of I_x

The moment of inertia of the shaded section about the horizontal centroidal axis is given as

$$I_{x_{\text{section}}} = I_{XX_{\text{square } ABCD}} - I_{XX_{\text{semi-circle } EFG}} - I_{XX_{\text{semi-circle } HIJ}}$$

$$= \frac{1}{12}\times(200)^4 - 2\times\frac{\pi}{128}\times(140)^4 \qquad \left(I_{XX_{\text{semi-circle } EFG}} = I_{XX_{\text{semi-circle } HIJ}} = \frac{1}{2}\times\frac{\pi}{64}\times(140)^4\right)$$

$$= 1.14 \times 10^8 \text{ mm}^4$$ **Ans.**

Calculation of I_y

$$I_{YY_{\text{square}}} = \frac{1}{12}\times(200)^4$$

$$= 1.34 \times 10^8 \text{ mm}^4$$

The moment of inertia of the semi-circle EFG about its diameter EG is

$$I_{EG} = \frac{1}{2}\times\frac{\pi\times(140)^4}{64} = 9.43 \times 10^6 \text{ mm}^4$$

The distance of the centroid of the semi-circle *EFG* from *EG* is

$$h = \frac{4r}{3\pi} \qquad (r = \text{Radius of the semi-circle})$$

$$= \frac{4 \times 70}{3\pi} = 29.7 \text{ mm}$$

The area of the semi-circle *EFG* is

$$A = \frac{\pi r^2}{2}$$

$$= \frac{\pi \times (70)^2}{2} = 7696.9 \text{ mm}^2$$

The moment of inertia of the semi-circle *EFG* about its vertical centroidal axis Y_1Y_1, using parallel-axes theorem, is given as

$$I_{EG} = I_{Y_1Y_1} + Ah^2 \qquad (h = \text{Distance between } EG \text{ and } Y_1Y_1)$$

or

$$I_{Y_1Y_1} = I_{EG} - Ah^2$$

$$= 9.43 \times 10^6 - 7696.9 \times (29.7)^2$$

$$= 2.64 \times 10^6 \text{ mm}^4$$

Now the moment of inertia of the semi-circle *EFG* about the *y*-axis, that is, the vertical centroidal axis *YY*, is given as

$$I_{YY_{EFG}} = I_{Y_1Y_1} + Ah_1^2 \qquad (h_1 = \text{Distance between } YY \text{ and } Y_1Y_1)$$

$$= 2.64 \times 10^6 + 7696.9 \times (100 - 29.7)^2 \qquad (h_1 = 100 - h)$$

$$= 4.06 \times 10^7 \text{ mm}^4$$

Also

$$I_{YY_{HIJ}} = I_{YY_{EFG}} = 4.06 \times 10^7 \text{ mm}^4$$

Finally, the moment of inertia of the shaded section about the vertical centroidal axis is given as

$$I_{YY_{\text{section}}} = I_{YY_{\text{square}}} - I_{YY_{\text{semi-circles}}}$$

$$= (1.34 \times 10^8 - 2 \times 4.06 \times 10^7) \text{ mm}^4$$

$$= 5.28 \times 10^7 \text{ mm}^4 \qquad \textbf{Ans.}$$

Example 3.19

Calculate the product of inertia I_{xy} for the plane area (an angle section) with respect to the axes x and y as shown in Fig. 3.32

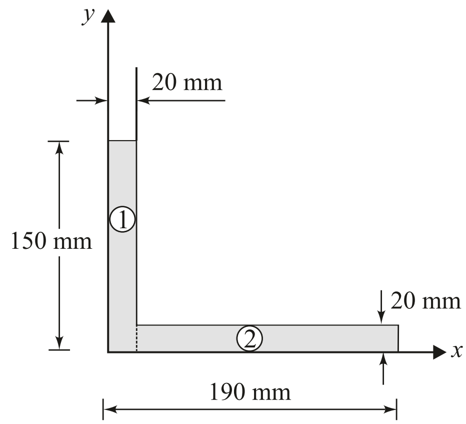


Fig. 3.32

Solution:

The angle section consists of two rectangles (1) and (2) having dimensions of 20 mm × 150 mm and 170 mm × 20 mm respectively.

The product of inertia I_{xy} for each rectangle is calculated separately and then they are added to get I_{xy} for the angle section, that is,

$$(I_{xy})_{\text{section}} = (I_{xy})_1 + (I_{xy})_2$$

I_{xy} for rectangle (1)

$$(I_{xy})_1 = (\bar{I}_{xy})_1 + A_1\bar{x}_1\bar{y}_1 \qquad \text{(using parallel-axes theorem)}$$

where $(\bar{I}_{xy})_1$ is the product of inertia of the rectangle about its own centroidal axes, and $\bar{x}_1$ and $\bar{y}_1$ are the distances of its centroid from y and x axes respectively.

Since the rectangle is symmetrical about its both centroidal axes, hence

$$(\bar{I}_{xy})_1 = 0$$

Hence, $$(I_{xy})_1 = 0 + (150 \times 20) \times (10) \times (75) = 2.25 \times 10^6 \text{ mm}^4$$

I_{xy} for rectangle (2)

It is also symmetrical about its both centroidal axes, hence its

$$(\bar{I}_{xy})_2 = 0$$

Now $$(I_{xy})_2 = (\bar{I}_{xy})_2 + A_2\bar{x}_2\bar{y}_2$$
$$= 0 + (170 \times 20) \times (105) \times (10) = 3.57 \times 10^6 \text{ mm}^4$$

Hence, $$(I_{xy})_{\text{section}} = (I_{xy})_1 + (I_{xy})_2$$
$$= (2.25 \times 10^6 + 3.57 \times 10^6) \text{ mm}^4$$
$$= 5.82 \times 10^6 \text{ mm}^4$$ **Ans.**

Example 3.20

Determine the product of inertia I_{xy} of the Z-section with respect to the axes x and y as shown in Fig. 3.33.

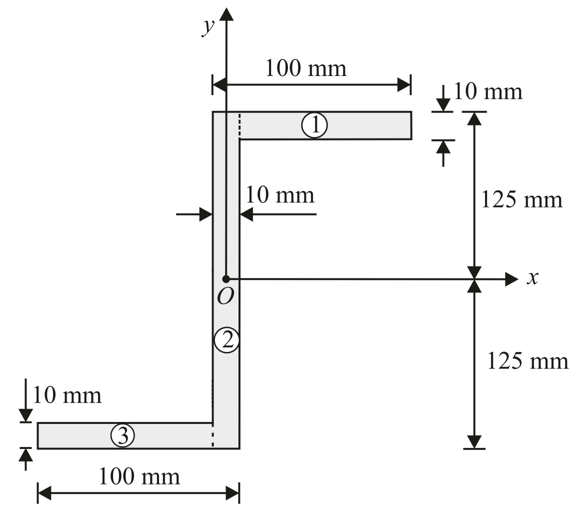


Fig. 3.33

Solution:

The Z-section is split into three rectangles (1), (2) and (3) having cross-sections (90 mm × 10 mm), (10 mm × 250 mm) and (90 mm × 10 mm) respectively, and their areas are found as

$$A_1 = 90 \times 10 = 900 \text{ mm}^2$$
$$A_2 = 10 \times 250 = 2500 \text{ mm}^2$$
$$A_3 = 90 \times 10 = 900 \text{ mm}^2$$

The product of inertia I_{xy} of each rectangle is calculated separately and they are added to get I_{xy} for the Z-section, that is,

$$(I_{xy})_{\text{section}} = (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$$

I_{xy} for rectangle (1)

$$(I_{xy})_1 = (\bar{I}_{xy})_1 + A_1\,\bar{x}_1\,\bar{y}_1$$

where $(\bar{I}_{xy})_1$ is the product of inertia of the rectangle about its own centroidal axes, and $\bar{x}_1$ and $\bar{y}_1$ are the distances of its centroid from y and x axes respectively.

Since the rectangle (1) is symmetrical about its own centroidal axes, hence $(\bar{I}_{xy})_1 = 0$

Now $\quad (I_{xy})_1 = 0 + (90 \times 10) \times (50) \times (120) \quad$ (using parallel-axes theorem)

$$= 5.4 \times 10^6 \text{ mm}^4$$

I_{xy} for rectangle (2)

Since the rectangle (2) is also symmetrical about its centroidal axes, so its $(\bar{I}_{xy})_2 = 0$. Also, the centroid O of the Z-section coincides with the centroid of the rectangle, that is, $\bar{x}_2 = 0$ and $\bar{y}_2 = 0$.

Hence, $$(I_{xy})_2 = (\bar{I}_{xy})_2 + A_2\,\bar{x}_2\,\bar{y}_2 \qquad \text{(using parallel-axes theorem)}$$
$$= 0 + (10 \times 250) \times (0) \times (0)$$
$$= 0$$

I_{xy} for rectangle (3)

The rectangle (3) is also symmetrical about its own centroidal axes, so its $(\bar{I}_{xy})_3 = 0$.

Hence, $$(I_{xy})_3 = (\bar{I}_{xy})_3 + A_3\,\bar{x}_3\,\bar{y}_3 \qquad \text{(using parallel-axes theorem)}$$
$$= 0 + (90 \times 10) \times (-50) \times (-120)$$
$$= 5.4 \times 10^6 \text{ mm}^4$$

Therefore, the product of inertia I_{xy} of the entire Z-section is calculated as

$$(I_{xy})_{\text{section}} = (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3$$
$$= 5.4 \times 10^6 + 0 + 5.4 \times 10^6$$
$$= 10.8 \times 10^6 \text{ mm}^4$$ **Ans.**

SHORT ANSWER QUESTIONS

1. What is the difference between centre of gravity and centroid?
2. What is the significance of radius of gyration?
3. What is centroidal axis? Do the centroidal axis same as the neutral axis?
4. Why is the area moment of inertia also called the second moment of area?
5. What is product of inertia? What is its value about a principal axis?
6. What is the use of parallel-axes theorem?
7. What is principal moment of inertia?

MULTIPLE CHOICE QUESTIONS

1. The centroid of a hollow cone of height h and radius r placed on its base lies at the following distance.

(*a*) $\frac{h}{3}$ from the base (*b*) $\frac{h}{4}$ from the base

(*c*) $\frac{2h}{3}$ from the base (*d*) $\frac{2h}{3}$ from the apex.

2. The centroid of a solid cone of height h and radius r placed on its base lies at the following distance.

(*a*) $\frac{h}{3}$ from the base (*b*) $\frac{h}{4}$ from the base

(*c*) $\frac{2h}{3}$ from the base (*d*) $\frac{2h}{3}$ from the apex.

3. The centroid of a semi-circle of radius r lies at the following distance from its base.

(*a*) $\frac{3\neq}{4r}$ (*b*) $\frac{4r}{3\pi}$ (*c*) $\frac{4\pi}{3r}$ (*d*) $\frac{3r}{4\pi}$.

4. The centroid of an equilateral triangle of side l lies at the following distance (perpendicular) from any side.

(*a*) $\frac{2}{\sqrt{3}}l$ (*b*) $\frac{2\sqrt{3}}{l}$ (*c*) $\frac{l}{2\sqrt{3}}$ (*d*) $\frac{\sqrt{3}}{2}l$.

5. The centroid of a right-angled triangle with base b and height h is

(*a*) $\left(\frac{2}{\sqrt{3}}b, \frac{1}{\sqrt{3}}h\right)$ (*b*) $\left(\frac{b}{3}, \frac{h}{2}\right)$ (*c*) $\left(\frac{b}{3}, \frac{h}{3}\right)$ (*d*) $\left(\frac{1}{\sqrt{3}}b, \frac{1}{\sqrt{3}}h\right)$.

6. The moment of inertia of a triangular section of base b and height h about the centroidal axis parallel to its base is

(*a*) $\frac{b^3h}{36}$ (*b*) $\frac{b^3h}{18}$ (*c*) $\frac{bh^3}{36}$ (*d*) $\frac{bh^3}{18}$.

7. The moment of inertia of a square section of side a about the centroidal axis parallel to its side is

(*a*) $\frac{a^4}{4}$ (*b*) $\frac{a^4}{12}$ (*c*) $\frac{a^4}{8}$ (*d*) $\frac{a^4}{16}$.

ANSWERS

1. (*a*) **2.** (*b*) **3.** (*b*) **4.** (*c*) **5.** (*c*) **6.** (*c*) **7.** (*b*).

EXERCISES

1. Find the centroid of the *Z*-section shown in Fig. 3.34.

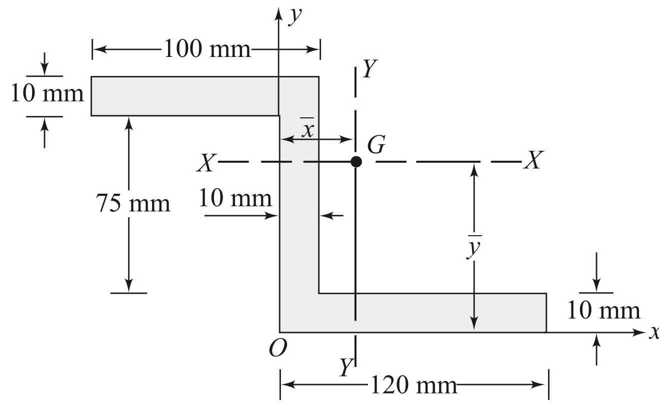


Fig. 3.34

(*Ans.* $\bar{x}$ = 39.23 mm, $\bar{y}$ = 44.62 mm).

2. Find the centroid of the channel section shown in Fig. 3.35.

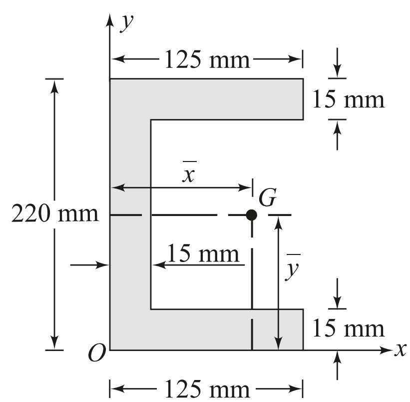


Fig. 3.35

(*Ans.* $\bar{x}$ = 38.75 mm, $\bar{y}$ = 110 mm).

3. Find the centroid of the angle section shown in Fig. 3.36.

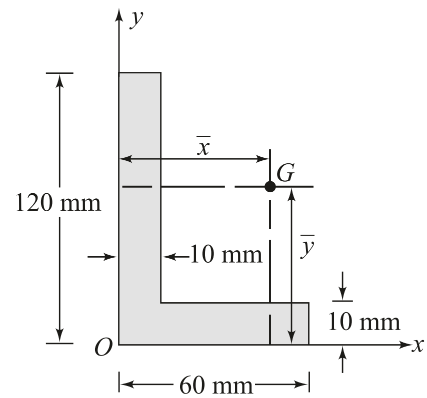


Fig. 3.36

(*Ans.* $\bar{x}$ = 13.82 mm, $\bar{y}$ = 43.82 mm).

4. Find the centroid of the shaded portion of the section shown in Fig. 3.37 after a square is cut out from it.

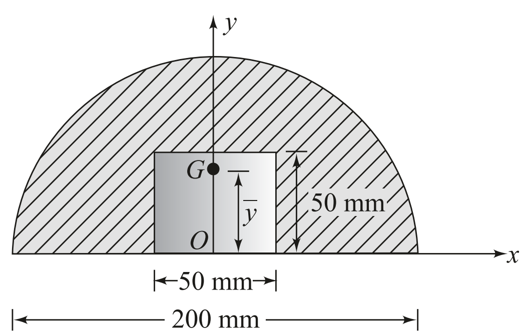


Fig. 3.37

(*Ans.* $\bar{y}$ = 45.74 mm).

5. Find the centroid and the area moment of inertia of the *I*-section shown in Fig. 3.38 about its centroidal axes.

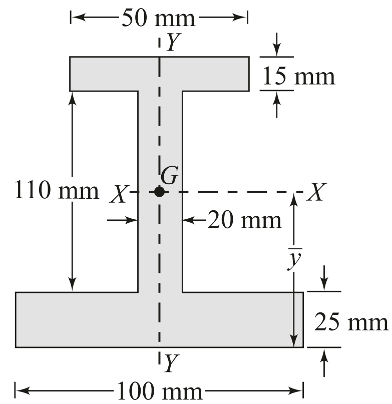


Fig. 3.38

(*Ans.* $\bar{x} = 0$, $\bar{y} = 45.64$ mm, $I_{XX} = 1.47 \times 10^7$ mm^4, $I_{YY} = 2.31 \times 10^6$ mm^4).

6. Find the coordinates of the centroid of a quarter-ellipse shown in Fig. 3.39, using direct integration method.

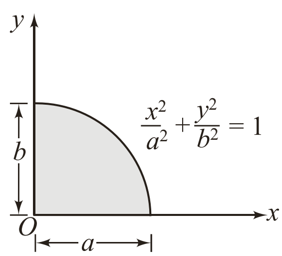


Fig. 3.39

(*Ans.* $\bar{x} = \dfrac{4a}{3\pi}$, $\bar{y} = \dfrac{4b}{3\pi}$).

7. Find the centroid of the shaded area formed by a straight line $y = mx$ and a curve $y = kx^2$ as shown in Fig. 3.40, using direct integration method.

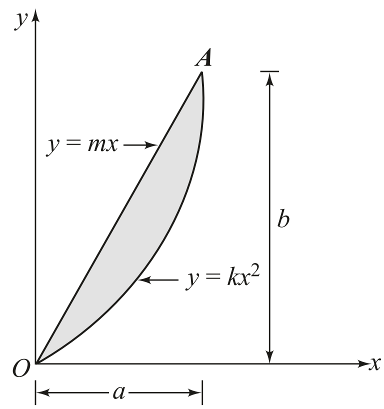


Fig. 3.40

(*Ans.* $\bar{x} = \dfrac{a}{2}$, $\bar{y} = \dfrac{2}{5}b$).

8. Determine the second moment of area of a triangle of base b and height h about its centroidal axes and base.

(*Ans.* $I_{XX} = \dfrac{bh^3}{36}$, $I_{YY} = \dfrac{hb^3}{48}$, $I_{\text{base}} = \dfrac{bh^3}{12}$).

9. Determine the second moment of area of a semi-circle of radius r about its horizontal centroidal axis. (*Ans.* $0.11r^4$).

10. Find the moment of inertia of the shaded area shown in Fig. 3.41 about the x-axis and the horizontal centroidal axis.

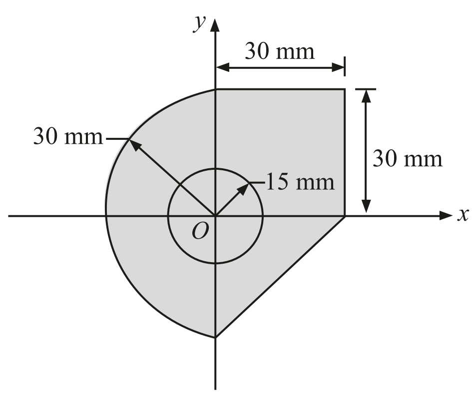


Fig. 3.41

(*Ans.* 6.16×10^5 mm^4, 5.76×10^5 mm^4).

11. Find the product of inertia I_{xy} of the *I*-section shown in Fig. 3.42. .

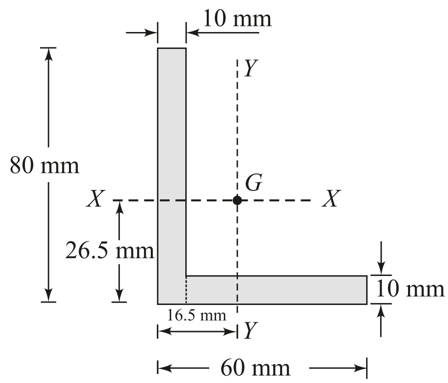


Fig. 3.42

(*Ans.* -3.23×10^5 mm^4).

12. Determine the product of inertia I_{xy} of the *Z*-section shown in Fig. 3.43.

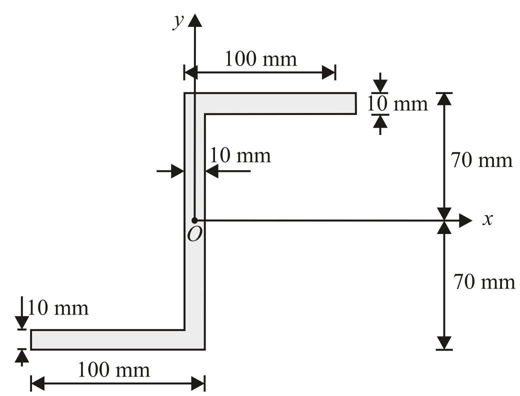


Fig. 3.43

(*Ans.* 5.85×10^6 mm^4).

❑❑❑

4 Shear Forces and Bending Moments in Beams

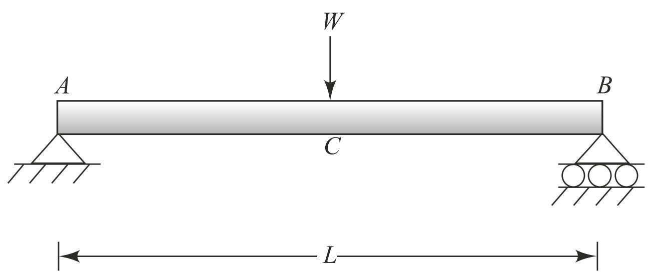


Figure shows a simple beam, one of the widely used statically determinate beams loaded with a central point load *W*.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- What is the use of shear force and bending moment diagrams?
- Why is the positive bending moment often called sagging?
- What is point of contraflexure?
- How are the shear force and the bending moment related?
- How is the nature of a bending moment curve decided?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_4

4.1 WHAT IS A BEAM?

Beam is a structural member designed to support vertical loads. Floors of a building are supported on beams. Many shafts of machinery act simultaneously as torsion members and as beams. Application of load produces shear and bending in the beam. When loads are not acting at right angles, they will also produce axial forces in the beam.

Beams are usually long, straight prismatic bars placed horizontally. There are two important points in the design of a beam; one is the determination of shear forces and bending moments as a result of the application of external loads on the beam and the other is the selection of the suitable cross-sections of the beam to resist shearing forces and bending moments.

4.2 CLASSIFICATION OF BEAMS

Beams are classified according to the supports provided, and are discussed below.

- A *simple beam* is supported at its two ends. One end is hinge supported and the other end is roller supported.

  A pin or hinge support is capable of resisting a force acting in any direction of the plane, that is, it prevents horizontal or vertical movement but does not prevent rotation, hence the beam can rotate in the plane. Such support has two reaction force components, one in the horizontal and another in the vertical direction, but has no moment reaction.

  A roller allows free movement in a horizontal plane and hence has no reaction in the horizontal direction, but prevents vertical movement. At the same time, it does not prevent rotation, hence the beam is free to rotate. Hence, it has only one reaction force in the vertical direction and no moment reaction. Roller and pin supports taken together are termed as simple supports.
- An *overhanging beam* is one whose certain portion extends beyond the support on one or both sides.
- A *continuous beam* has more than two supports.
- A *cantilever beam* is fixed at one end and free at the other end.
- A *fixed beam* is confined between two supports.

  Fixed support can resist a force in any direction as well as a moment or a couple. It is in-built at the end and is prevented from rotating.
- A *propped cantilever beam* has a suitable support in order to prevent its deflection.

Simply supported, overhanging and cantilever beams are categorized as statically determinate beams while continuous, fixed and propped cantilever as statically indeterminate beams. It should be noted that the reactions will be determinate if the supports involve only three unknowns. If more unknowns are involved, then the reactions will be statically indeterminate because the methods of statics are not sufficient to determine the reactions, and the properties of the beam with regard to its resistance to bending must be taken into consideration. The shear force and bending moment diagrams are drawn for statically determinate beams only.

Different types of beams are shown in Fig. 4.1. In figures (*a*), (*b*), and (*c*), beams are hinge supported at *A* and roller supported at *B*, *C*, and *B* and C respectively, whereas in figures (*d*), (*e*), and (*f*), beams are fixed at *A*, *A* and *B*, and *A* respectively.

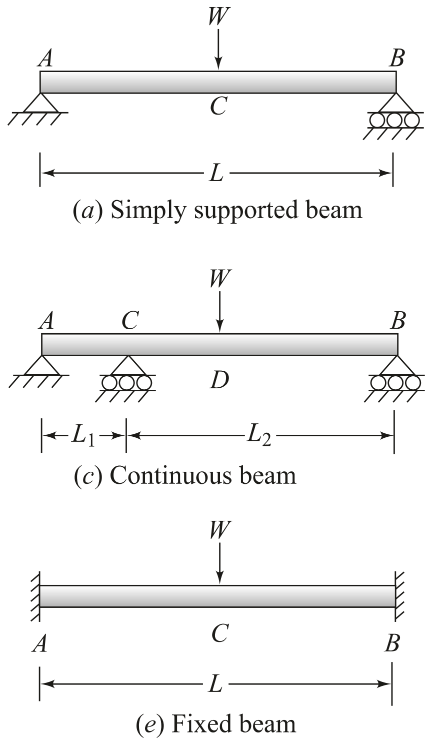


Fig. 4.1 Types of beams.

4.3 TYPES OF LOADINGS

- A *concentrated load* is acting at a point and hence it is also called a point load (Fig. 4.2(*a*)). It is expressed in newtons (N) in *SI* units.
- A *distributed load* acts over a finite length of the beam (Fig. 4.2(*b*)). It is specified by the intensity of loading per unit length, say *w* N/m in *SI* units. When it has a constant value for a certain part of the beam, it is said to be uniformly distributed over that part of the beam (*udl*).

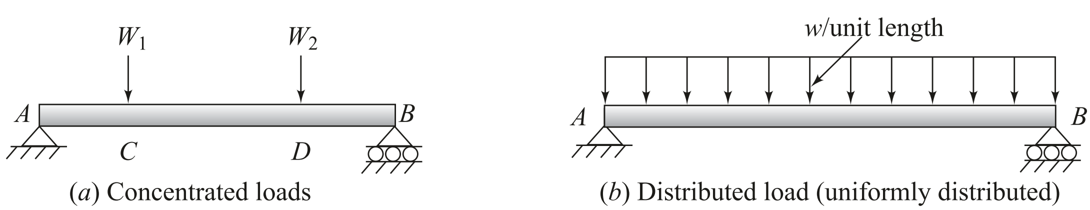
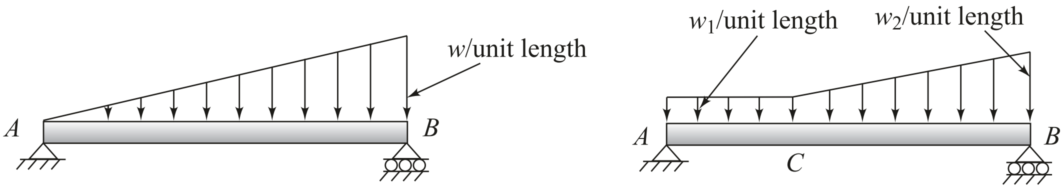


Fig. 4.2 Types of loadings.

Uniformly varying load implies the increase or decrease of loading intensity at a constant rate along the length of the beam. Distributed load may be represented by a parabolic, cubic or a higher order curve for non-uniformly varying load. A uniform load is shown by a ractangular distribution, a uniformly-varying load by a triangle and the combination by a trapezium drawn on the beam (Fig. 4.2 (*b*) and Fig. 4.3).



(*a*) Triangular distribution (*b*) Trapezium distribution

Fig. 4.3 Types of uniformly-varying loads.

4.4 CALCULATION OF BEAM REACTIONS

The beam reactions are calculated by using the three fundamental equations of static equilibrium.

$$\Sigma F_x = 0$$

$$\Sigma F_y = 0$$

and $$\Sigma M_z = 0$$

The beam is usually placed horizontally, so it is the x-axis and the load on the beam is placed vertically, so it is the y-axis.

4.5 SHEAR FORCES IN A BEAM

Consider a section XX of the beam at its certain distance (Fig. 4.4). The part to the left of the section is in equilibrium under the action of three forces: the vertical reaction force at A, that is, R_A, the external load W_1, and force V induced at the section. For the equilibrium of the part to the right of the section, an opposite force V acts at the section, so that its equilibrium is decided by the reaction force at B *i.e.*, R_B, the external load W_2, and the force V. The force V is called vertical shear force. Numerically, the shear force is equal to the algebraic sum of all the vertical components of the external forces to the left or to the right of the section. Whether the right-hand segment or the left-hand segment is used to determine the shear force at a section is immaterial and only arithmetical simplicity governs.

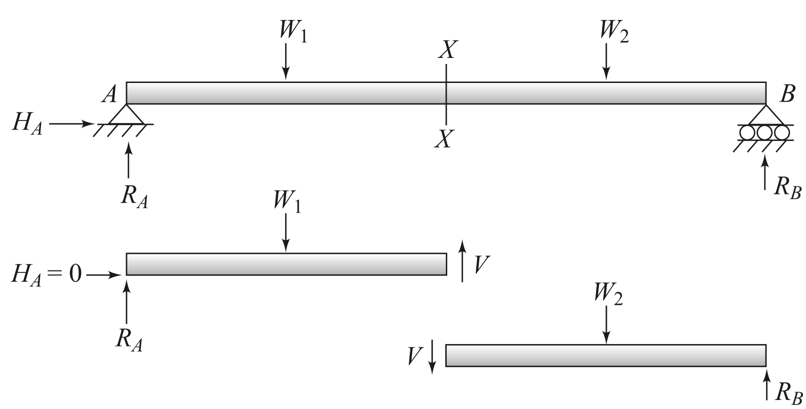


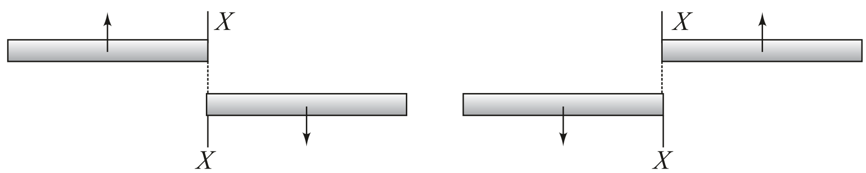
Fig. 4.4 Shear force in a beam.

4.6 BENDING MOMENTS IN A BEAM

When a load is applied on the beam, bending moment is developed which tends to bend the beam in the plane of the loads. To counteract the moment caused by external load, internal resisting moment is produced to satisfy the static equilibrium condition for moment. Magnitude of the internal resisting moment is equal to the external moment. The bending moment at a section is equal to the algebraic sum of moments to the left or to the right of the section.

4.7 SIGN CONVENTIONS FOR SHEAR FORCE AND BENDING MOMENT

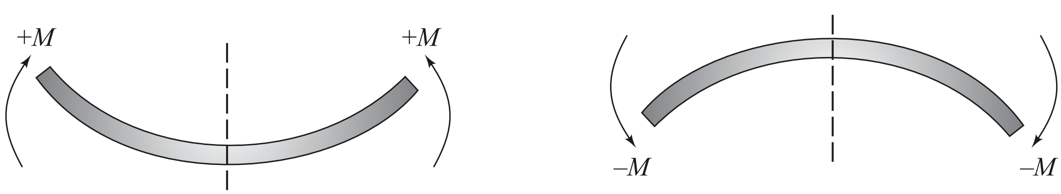
(*a*) **For shear force:** If the portion of the beam to the left of the section moves upward and the portion to the right of the section comes downward, then it is due to positive shear. Reverse of it is due to negative shear (Fig. 4.5).



(*a*) Positive shear (*b*) Negative shear

Fig. 4.5

(*b*) **For bending moment:** When bending moment causes upward concavity in the beam, it is termed as positive B.M. (sagging). As a result of positive B.M., the upper part of the beam is in compression and the lower part in tension, thereby increasing the length of the bottom surface and decreasing the length of the top surface of the beam. On the other hand, if bending moment produces downward concavity or produces compression in the lower part and tension in the upper part of the beam's cross-section, then it is called a negative bending moment (hogging). (Fig. 4.6).



(*a*) Positive B.M. (Sagging) (*b*) Negative B.M. (Hogging)

Fig. 4.6

4.8 SHEAR FORCE AND BENDING MOMENT DIAGRAMS (SFD AND BMD)

Shear force and bending moment diagrams are the pictorial representation of shear forces and bending moments respectively. The axis of the beam is shown by a horizontal line and values of shear force and bending moment are shown on it as vertical lines according to a suitable scale. Positive values are shown above the axis and negative values below the axis of the beam. All the values are then joined by a straight line or a curve depending upon the nature of loading.

Between two point loads, the bending moment distribution is shown by a straight line. If there is uniformly distributed load between two points, then the curve is parabola of second or higher order.

The bending moment M is maximum or minimum when $\frac{dM}{dx} = 0$. Thus at sections where shear force is zero or changes its sign (from maximum to minimum), the B.M. is either maximum or minimum. There may be a point in the bending moment diagram, where the bending moment is zero and the sign of the bending moments are changed. This point is called inflection point or point of contraflexure.

The shear force diagram consists of horizontal straight lines in case of point loads and inclined straight lines in case of uniformly distributed loads. The corresponding portions in the bending moment diagram are inclined straight lines and parabolic curves. The shear force diagram has parabolic curves and bending moment diagram has cubic parabolic curves for uniformly varying loads.

4.9 POINT OF CONTRAFLEXURE

It is also called point of inflection. It is the point where the curvature (of the deflection curve) and bending moment change signs. The bending moment and the curvature of the deflection curve both are zero at this point. However, a point where both bending moment and curvature (deflection) are zero is not necessarily a point of contraflexure as they may be zero without changing signs at that point.

4.10 SFD AND BMD FOR CANTILEVER BEAMS

4.10.1 Cantilever Beam carrying a Point Load at its Free End

Refer Fig. 4.7 (*a*). *AB* is the length of the beam.

Calculations for shear forces

Consider a section *XX* of the beam at a distance x from the free end B.

Shear force at the section is $V = + W$ and it is independent of the distance x. It remains constant throughout the length of the cantilever (Fig. 4.7 (*b*)).

Calculations for bending moments

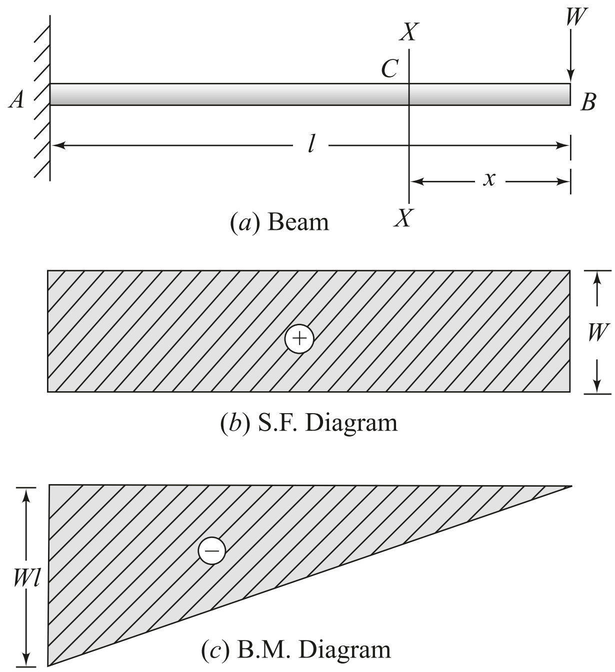
Bending moment at the section is

$$M_x = - Wx \text{ (hogging)}$$

Bending moment at the free end, $M_B = 0$ (for $x = 0$)

Bending moment at the fixed end, $M_A = - Wl$ (for $x = l$)

The equation of B.M. at the section represents a straight line, hence the bending moments at A and B are connected by a straight line (Fig. 4.7 (*c*)).



(a) Beam

(b) S.F. Diagram

(c) B.M. Diagram

Fig. 4.7

Alternative method (for bending moment)

A section of the beam may be selected at a distance x from the fixed end (Fig. 4.8).

Bending moment at the section is, $M_x = -W(l - x)$

Bending moment at the fixed end A, $M_A = -W(l - 0) = -Wl$ (for $x = 0$)

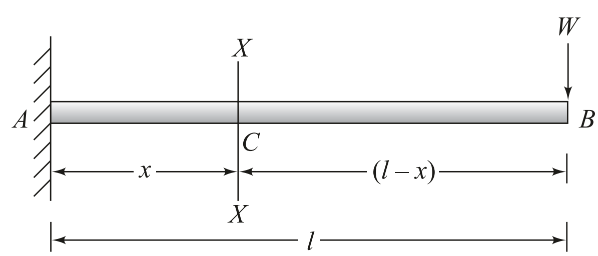


Fig. 4.8

Bending moment at the free end B, $M_B = -W(l - l) = 0$ (for $x = l$)

4.10.2 Cantilever Beam carrying Uniformly Distributed Load (udl) throughout the Span

Refer Fig. 4.9 (*a*).

Consider a section XX of the beam at C, a distance x from the free end B (Fig. 4.9 (*a*)).

Calculations for shear forces

Shear force at the section is $V = + wx$

Shear force at the free end B, $V_B = 0$ (for $x = 0$)

Shear force at the fixed end A, $V_A = + wl$ (for $x = l$)

The equation of the shear force at the section represents a straight line, hence the variation of the shear force between A and B is shown by a straight line (Fig. 4.9 (*b*)).

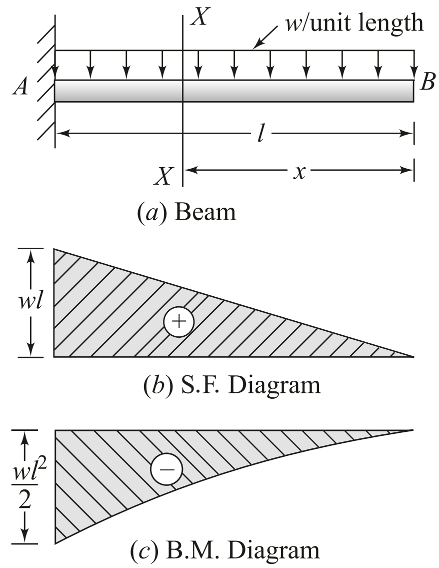


Fig. 4.9

Calculations for bending moments

Bending moment at the section is

$$M_x = - wx \cdot \frac{x}{2} = - \frac{wx^2}{2}$$

Bending moment at B, $M_B = 0$ (for $x = 0$)

Bending moment at A, $M_A = -\frac{wa^2}{2}$ (for $x = l$)

The equation of the bending moment at the section represents a parabola, hence the bending moment variation is parabolic between A and B (Fig. 4.9 (*c*)).

Alternative method

Calculations for shear forces

Consider a section XX of the beam at a distance x from the fixed end A (Fig. 4.10).

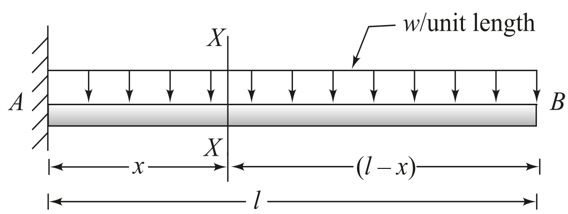


Fig. 4.10

Shear force at the section is

$$V = + w\,(l - x)$$

Shear force at the free end, $V_B = + w\,(l - l) = 0$ (for $x = l$)

Shear force at the fixed end, $V_A = + wl$ (for $x = 0$)

Calculations for bending moments

Bending moment at the section is

$$M_x = - w\,(l - x) \cdot \frac{(l - x)}{2} = - \frac{w(l - x)^2}{2}$$

Bending moment at the free end is

$$M_B = \frac{-w(l - l)^2}{2} = 0 \quad (\text{for } x = l)$$

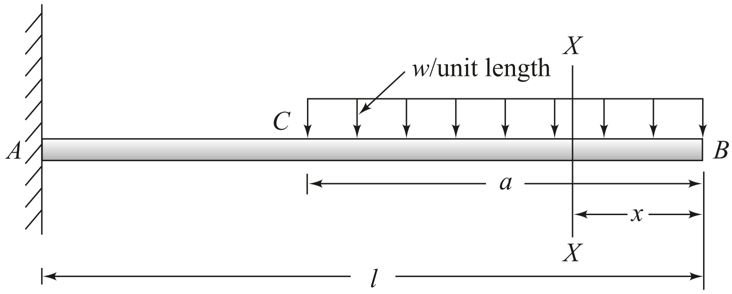
Bending moment at the fixed end is

$$M_A = \frac{w(l-0)^2}{2} = -\frac{wl^2}{2} \quad \text{(for } x = 0\text{)}$$

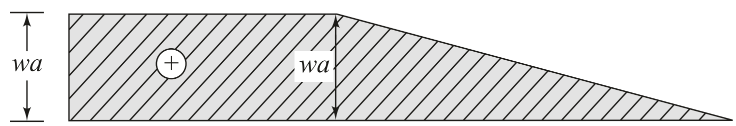
4.10.3 Cantilever Beam carrying Uniformly Distributed Load over a certain Length from the Free End

Refer Fig. 4.11. (*a*).

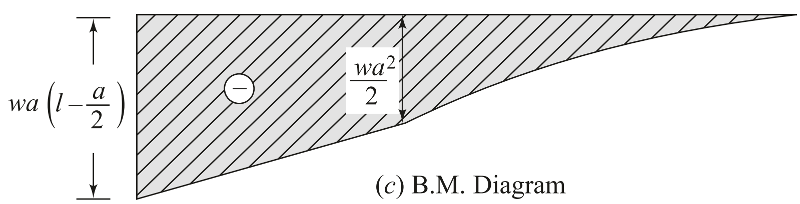
Consider a section *XX* of the beam at a distance *x* from the free end *B* (Fig. 4.11 (*a*)).



(*a*) Beam



(*b*) S.F. Diagram



(*c*) B.M. Diagram

Fig. 4.11

Calculations for shear forces

Shear force at the section is

$$V = +wx$$

Shear force at the free end, $V_B = 0$ (for $x = 0$)

Shear force at *C* is

$$V_C = +wa \quad \text{(for } x = a\text{)}$$

The shear force between *A* and *C* remains constant at *wa*. The shear force variation between *B* and *C* is shown by an inclined straight line and between *A* and *C* by a horizontal straight line (Fig. 4.11 (*b*)).

Calculations for bending moments

Bending moment at the section is

$$M_x = -wx \cdot \frac{x}{2} = -\frac{wx^2}{2}$$

Bending moment at free end, $M_B = 0$ (for $x = 0$)
Bending moment at C is

$$M_C = -\frac{wa^2}{2} \quad \text{(for } x = a\text{)}$$

Bending moment at the fixed end is

$$M_A = -wa\left(l - \frac{a}{2}\right)$$

The variation of the bending moment between B and C is shown by a parabolic curve and between A and C by an inclined straight line (Fig. 4.11 (*c*)).

4.10.4 Cantilever Beam carrying Uniformly Distributed Load over a certain Length from the Fixed End

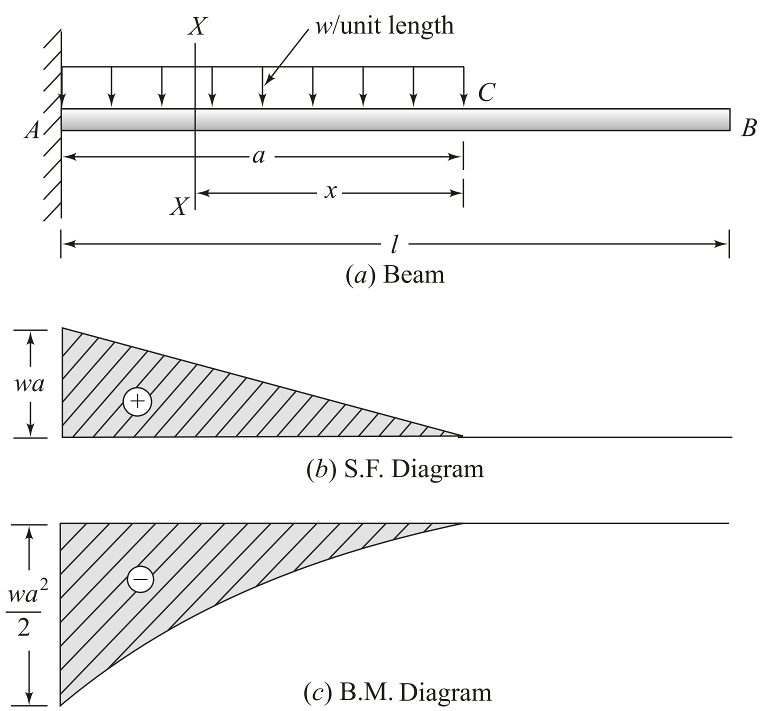
Consider a section XX of the beam at a distance x from C (Fig. 4.12 (*a*)).

Calculations for shear forces

Shear force at the section is

$$V = +wx$$

Shear force at C, $V_C = 0$ (for $x = 0$)



(*a*) Beam

(*b*) S.F. Diagram

(*c*) B.M. Diagram

Fig. 4.12

Shear force at A, $V_A = wa$ (for $x = a$)

Shear force variation between A and C is shown by an inclined straight line and there is no shear force for the portion BC (Fig. 4.12 (b)).

Calculations for bending moments

Bending moment at the section is

$$M_x = -wx \cdot \frac{x}{2} = -\frac{wx^2}{2}$$

Bending moment at C, $M_C = 0$ (for $x = 0$)

Bending moment at A, $M_A = -\frac{wa^2}{2}$ (for $x = a$)

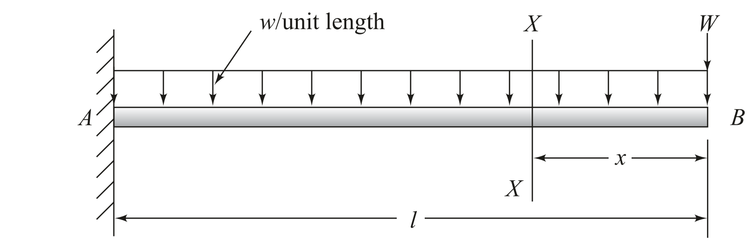
The variation of the bending moment between A and C is parabolic and there is no bending moment on BC (Fig. 4.12 (c)).

4.10.5 Cantilever Beam carrying Uniformly Distributed Load over its Entire Span and a Point Load at its Free End

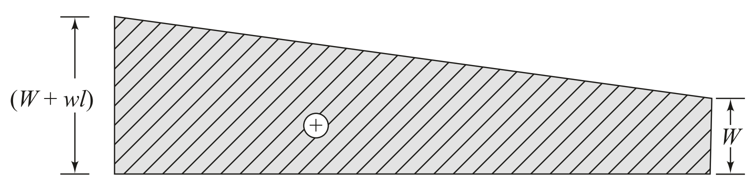
Refer Fig. 4.13 (a).

Calculations for shear forces

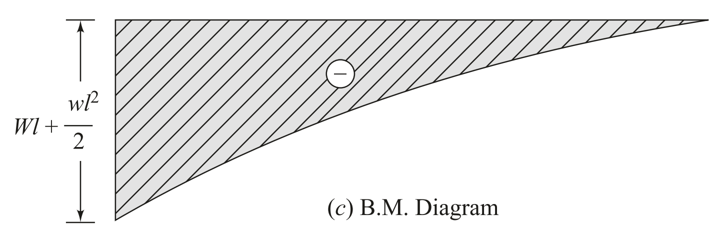
Consider a section XX of the beam at a distance x from the free end B (Fig. 4.13 (a)).



(a) Beam



(b) S.F. Diagram



(c) B.M. Diagram

Fig. 4.13

Shear force at the section is

$$V = +(wx + W)$$

Shear force at B is

$$V_B = +(w\,.\,0 + W) = +W \quad (\text{for } x = 0)$$

Shear force at the fixed end A is

$$V_A = +(wl + W) \quad (\text{for } x = l)$$

The *SFD* is shown in Fig. 4.13 (*b*).

Calculations for bending moments

Bending moment at the section is

$$M_x = -\left(Wx + w\cdot x\cdot\frac{x}{2}\right)$$

$$= -\left(Wx + \frac{wx^2}{2}\right)$$

Bending moment at B is

$$M_B = -\left(W\cdot 0 + \frac{w\cdot 0^2}{2}\right) = 0 \quad (\text{for } x = 0)$$

Bending moment at A is

$$M_A = -\left(Wl + \frac{wl^2}{2}\right) \quad (\text{for } x = l)$$

Since the equation for bending moment at the section represents a parabola, hence the bending moment variation between A and B is parabolic (Fig. 4.13 (*c*)).

4.10.6 Cantilever Beam carrying several Point Loads

Let us consider that three point loads W_1, W_2 and W_3 are acting at respective distances of l_1, l_2 and l from the fixed end A (Fig. 4.14 (*a*)).

Part *BD*

Consider a section XX of the beam between B and D, at a distance x_1 from free end B. Shear force at the section is $V = +W_3$ and it remains constant between B and D.

Bending moment at the section is

$$M_x = -W_3 x_1$$

Bending moment at B, $M_B = 0 \quad (\text{for } x_1 = 0)$

Bending moment at D, $M_D = -W_3\,(l - l_2)$ (for $x_1 = l - l_2$)

The variation of the shear force between B and D is shown by a horizontal straight line and that of bending moment by an inclined straight line.

Part *CD*

Consider a section XX of the beam between C and D, at a distance x_2 from B.

Shear force at the section is

$$V = +(W_2 + W_3), \text{ and it remains constant between } C \text{ and } D.$$

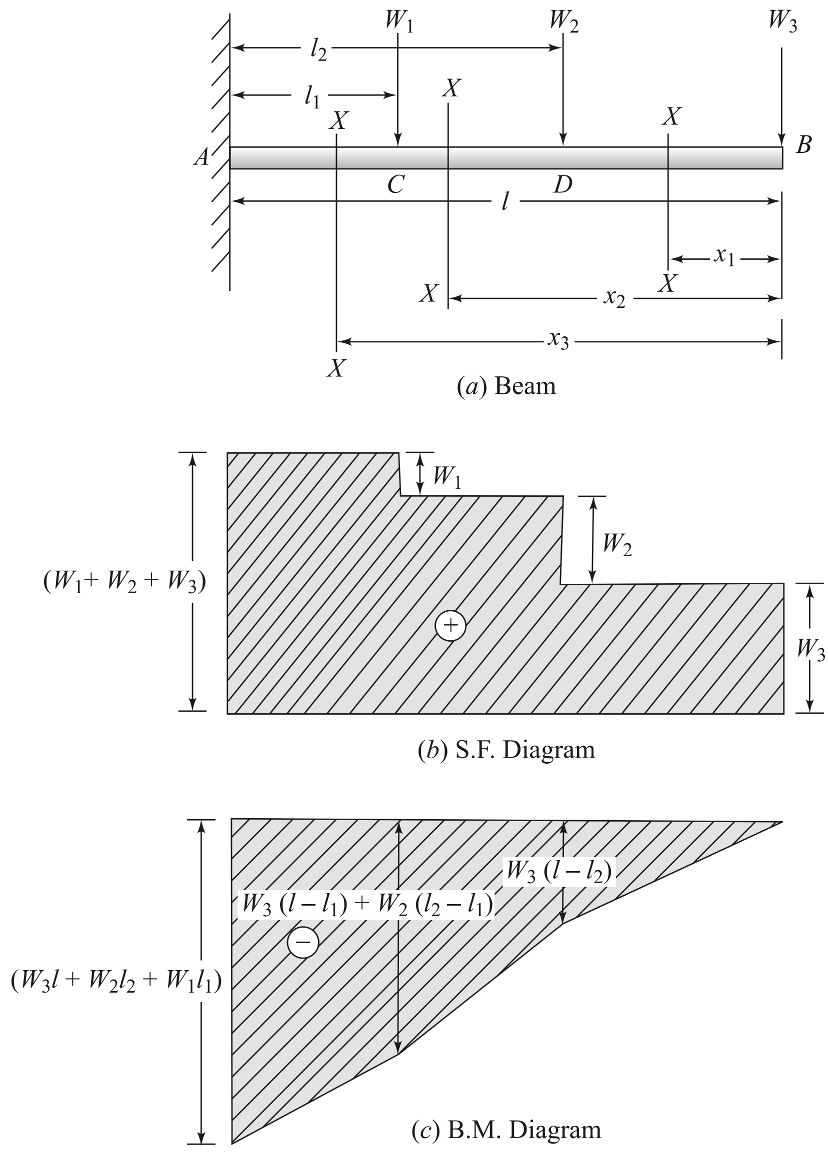


Fig. 4.14

Bending moment at the section is

$$M_x = -\,[W_3\, x_2 + W_2\, \{x_2 - (l - l_2)\}]$$

Bending moment at C is

$$M_C = -\,[W_3\,(l - l_1) + W_2\,\{l - l_1 - l + l_2\}] \quad (\text{for } x_2 = l - l_1)$$

$$= -\,[W_3\,(l - l_1) + W_2\,(l_2 - l_1)]$$

Part *AC*

Consider a section *XX* of the beam between *A* and *C*, at a distance x_3 from *B*.

Shear force at the section is

$$V = +(W_1 + W_2 + W_3), \text{ and it remains constant between } A \text{ and } C.$$

Bending moment at the section is

$$M_x = -[W_3 x_3 + W_2 \{x_3 - (l - l_2)\} + W_1 \{x_3 - (l - l_1)\}]$$

Bending moment at *A* is

$$M_A = -[W_3 l + W_2 \{l - l + l_2\} + W_1 \{l - l + l_1\}] \quad (\text{for } x_3 = l)$$

$$= -[W_3 l + W_2 l_2 + W_1 l_1]$$

The *SFD* and the *BMD* are shown in Fig. 4.14 (*b*) and (*c*) respectively.

4.10.7 Cantilever Beam carrying Uniformly Varying Load

The load on the cantilever varies uniformly from zero at free end to *w*/unit length at the fixed end (Fig. 4.15 (*a*)).

Calculations for shear forces

Consider a section *XX* of the beam at a distance *x* from the free end *B*.

Δs *ABC* and *BDE* are compared to find the rate of loading at the section.

$$\frac{AC}{DE} = \frac{AB}{BD}$$

or

$$DE = \frac{AC.BD}{AB} = \frac{w.x}{l}$$

Shear force at the section is

$$V = +\text{ Triangular load } BDE$$

$$= +\frac{1}{2} \times \text{base} \times \text{height}$$

$$= +\frac{1}{2} \times x \times \frac{wx}{l} = +\frac{wx^2}{2l}$$

Shear force at *B*, $F_B = 0$ (for $x = 0$)

Shear force at *A* is

$$V_A = +\frac{wx^2}{2l} \quad (\text{for } x = l)$$

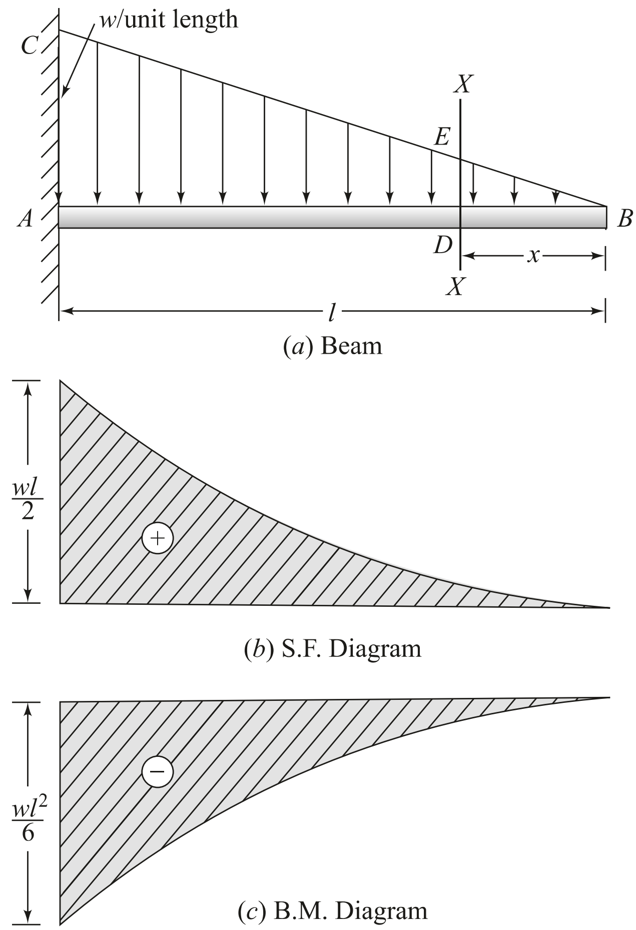


Fig. 4.15

Calculations for bending moments

Bending moment at the section is

$$M_x = -\text{ Triangular load } BDE \times \frac{x}{3}$$

$$= -\frac{wx^2}{2l} \times \frac{x}{3} = -\frac{wx^3}{6l}$$

Bending moment at the free end, $M_B = 0$ (for $x = 0$)

Bending moment at the fixed end A is

$$M_A = -\frac{wl^2}{6} \quad (\text{for } x = l)$$

The variation of the shear force between A and B is shown by a parabolic curve and of the bending moment by a cubic curve (Fig. 4.15 (b) and Fig. 4.15 (c) respectively).

Example 4.1

Draw the shear force and bending moment diagrams for a cantilever beam loaded as shown in Fig. 4.16 (*a*).

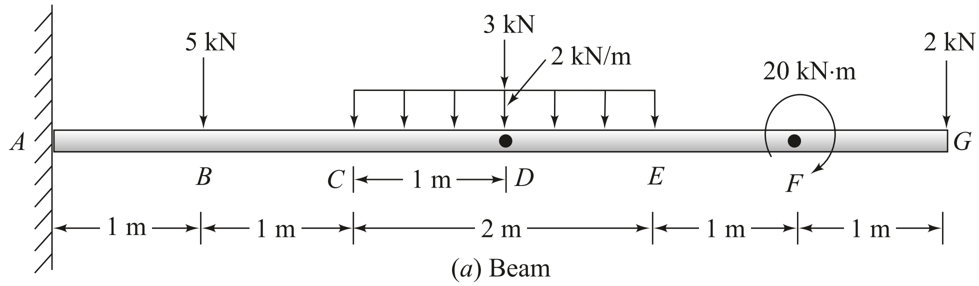
Solution: **Calculations for shear forces**

Shear force at G is

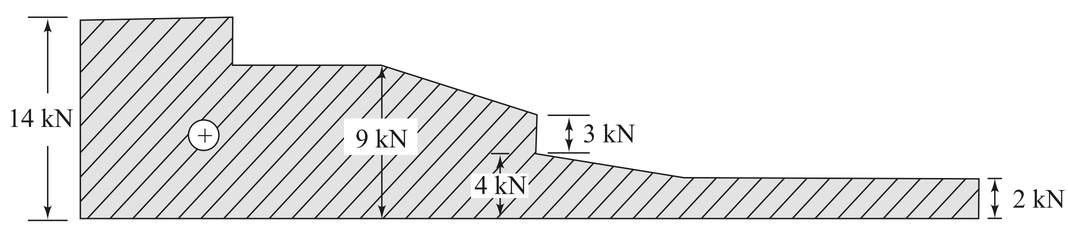
$$V_G = +2 \text{ kN, and it remains constant upto } E.$$

Shear force just to the right of D

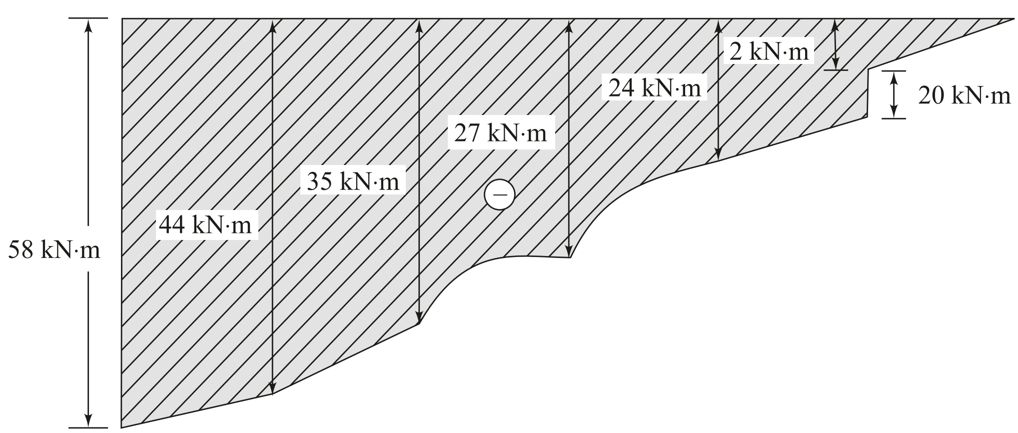
$$= +2 + (2 \times 1) = +4 \text{ kN}$$



(*a*) Beam



(*b*) S.F. Diagram



(*c*) B.M. Diagram

Fig. 4.16

Shear force just to the left of D

$$= +2 + (2 \times 1) + 3 = +7 \text{ kN}$$

Shear force at C is

$$V_C = +2 + (2 \times 2) + 3$$

$= +9$ kN, and it remains constant upto B.

Shear force just to the left of B

$$= +2 + (2 \times 2) + 3 + 5$$

$= +14$ kN, and it remains constant upto A.

The shear force variation between C and D, and between D and E are shown by inclined straight lines (Fig. 4.16 (*b*)).

Calculations for bending moments

Bending moment at G is

$$M_G = 0$$

Bending moment just to the right of F

$$= -(2 \times 1) = -2 \text{ kN·m}$$

Bending moment just to the left of F

$= -(2 + 20)$ (since a couple of magnitude 20 kN·m is acting at F)

$$= -22 \text{ kN·m}$$

Bending moment at E is

$$M_E = -[2 \times (1 + 1) + 20] = -24 \text{ kN·m}$$

Bending moment at D is

$$M_D = -\left[2 \times 3 + 20 + 2 \times 1 \times \frac{1}{2}\right] = -27 \text{ kN·m}$$

Bending moment at C is

$$M_C = -\left[2 \times 4 + 20 + 2 \times 2 \times \frac{2}{2} + 3 \times 1\right] = -35 \text{ kN·m}$$

Bending moment at B is

$$M_B = -\left[2 \times 5 + 20 + 2 \times 2 \times \left(1 + \frac{2}{2}\right) + 3 \times 2\right] = -44 \text{ kN·m}$$

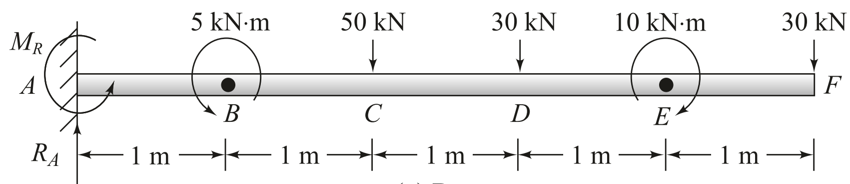
Bending moment at A is

$$M_A = -\left[2 \times 6 + 20 + 2 \times 2 \times \left(2 + \frac{2}{2}\right) + 3 \times 3 + 5 \times 1\right] = -58 \text{ kN·m}$$

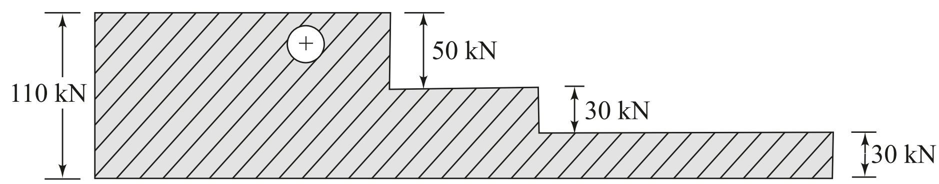
The *BMD* is shown in Fig. 4.16 (*c*).

Example 4.2

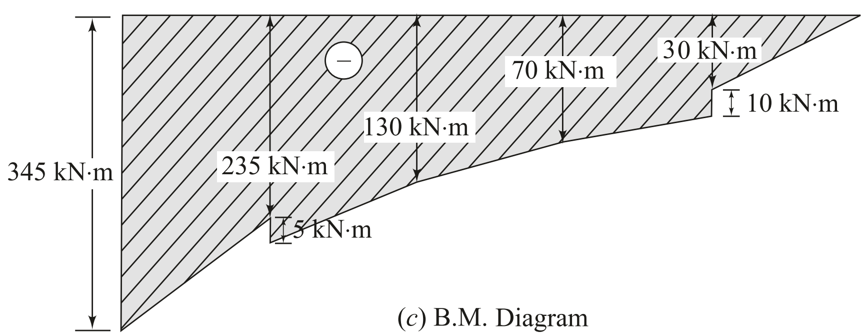
Find the reaction at the fixed end of the cantilever loaded as shown in Fig. 4.17. (*a*). Draw also shear force and bending moment diagrams.



(*a*) Beam



(*b*) S.F. Diagram



(*c*) B.M. Diagram

Fig. 4.17

Solution: **Reaction at *A***

Total downward load on the beam

$$= 50 + 30 + 30 = 110 \text{ kN}$$

Hence, the vertical upward reaction at the fixed end A

$$= 110 \text{ kN}$$

Taking moments of forces about A, the resultant moment on the beam is given as

$$M_R = -30 \times 5 - 10 \text{ (Couple at } E) - 30 \times 3 - 50 \times 2 + 5 \text{ (Couple at } B)$$
$$= -345 \text{ kN·m (Clockwise)}$$

Hence, a reaction moment of 345 kN·m (anticlockwise) will act at the fixed end A along with a vertical reaction of 110 kN.

Calculations for shear forces

Shear force between D and F = 30 kN

Shear force just to the left of D = 60 kN, and it remains constant upto C.

Shear force just to the left of C = 110 kN, and it remains constant upto A.

The *SFD* is shown in Fig. 4.17 (*b*).

Calculations for bending moments

Bending moment at F is

$$M_F = 0$$

Bending moment just to the right of E = − 30 kN·m

Bending moment just to the left of E = − (30 + 10) = − 40 kN·m

(Since a couple of magnitude 10 kN·m is acting at E)

Bending moment at D is

$$M_D = -(30 \times 2 + 10) = -70 \text{ kN·m}$$

Bending moment at C is

$$M_C = -(30 \times 3 + 10 + 30 \times 1) = -130 \text{ kN·m}$$

Bending moment just to the right of B

$$= -(30 \times 4 + 10 + 30 \times 2 + 50 \times 1) = -240 \text{ kN·m}$$

Bending moment just to the left of B

$= -240 + 5 = -235$ kN·m (Since another couple of magnitude 5 kN·m is acting at B)

Bending moment at A is

$$M_A = -(30 \times 5 + 10 + 30 \times 3 + 50 \times 2 - 5) = -345 \text{ kN·m}$$

The *BMD* is shown in Fig. 4.17 (*c*).

4.11 SFD AND BMD FOR SIMPLY SUPPORTED BEAMS

4.11.1 Simply Supported Beam carrying a Central Point Load

A simply supported beam AB of length l with a point load W acting at its centre is shown in Fig. 4.18 (*a*).

Reactions at *A* and *B*

To find the reactions at the two supports A and B, take moments of the forces about A.

$$R_B \times l = W \times \frac{l}{2}$$

or

$$R_B = \frac{W}{2} (\uparrow)$$

Also

$$R_A + R_B = W = \text{Load on the beam}$$

or $$R_A = W - R_B$$

$$= \frac{W}{2} \ (\uparrow)$$

Calculations for shear forces

Consider a section *XX* of the beam in *BC*, at a distance *x* from *B*.

Shear force at the section is

$$V = -\frac{W}{2}$$

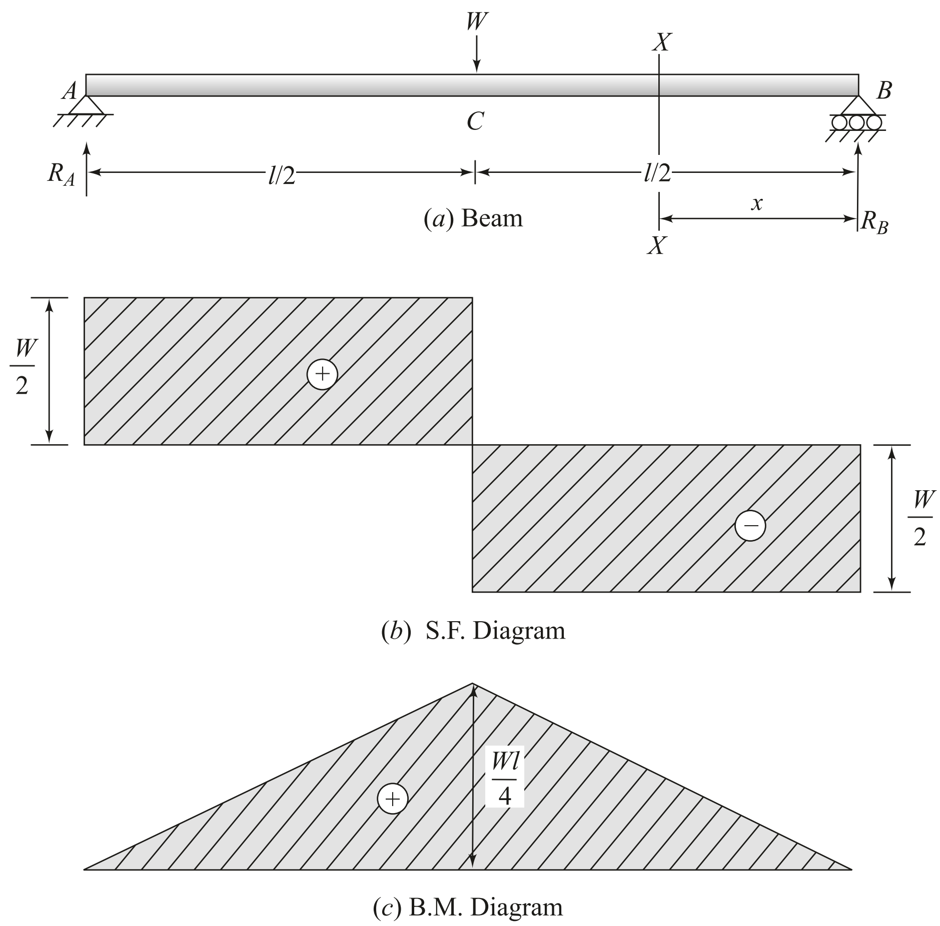


Fig. 4.18

Shear force just to the right of C is $-\dfrac{W}{2}$, and it remains constant for the portion BC.

Shear force just to the left of C is $-\dfrac{W}{2} + W = +\dfrac{W}{2}$, and it remains constant for the portion AC.

Calculations for bending moments

Bending moment at the section is

$$M_x = +\frac{W}{2}\cdot x$$

Bending moment at B, $\quad M_B = 0 \quad$ (for $x = 0$)

Bending moment at C, $\quad M_C = \frac{W}{2}\cdot\frac{l}{2} \quad$ (for $x = \frac{l}{2}$)

$$= +\frac{Wl}{4}$$

Bending moment at A is

$$M_A = +\frac{W}{2}\cdot l - W\cdot\frac{l}{2} = 0$$

The bending moments between B and C, and between A and C are joined by inclined straight lines because *B.M.* at the section represents a straight line.

The *SFD* and the *BMD* are shown in Fig. 4.18 (*b*) and (*c*) respectively.

4.11.2 Simply Supported Beam carrying an Eccentric Point Load

A simply supported beam AB is carrying a point load W at a distance a from A and b from B (Fig. 4.19 (*a*)).

Reactions at A and B

Taking moments of the forces about A, we have

$$R_B \times l = W \times a$$

or $$R_B = \frac{Wa}{l} \ (\uparrow)$$

But $$R_A + R_B = W$$

or $$R_A = W - R_B$$

$$= \frac{Wb}{l} \ (\uparrow)$$

Calculations for shear forces

Shear force between B and C is $-\frac{Wa}{l}\cdot$

Shear force just to the left of C is

$$-\frac{Wa}{l} + W = \frac{Wb}{l}$$, and it remains constant for the portion AC.

Calculations for bending moments

The bending moments at A and B are zero.

Bending moment at C is

$$M_C = R_B \times b$$

$$= \frac{Wab}{l}$$

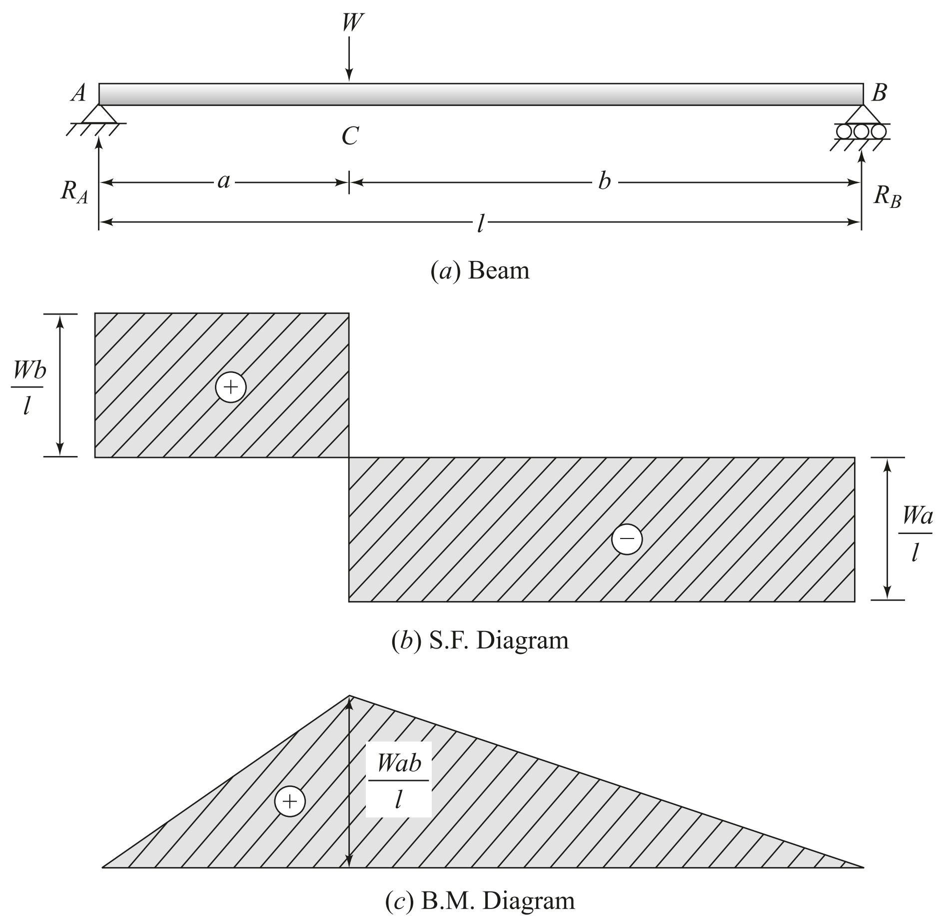


Fig. 4.19

The *SFD* and the *BMD* are shown in Fig. 4.19 (b) and Fig. 4.19 (c) respectively.

4.11.3 Simply Supported Beam carrying Uniformly Distributed Load (*udl*) over its Entire Span

The intensity of loading on the beam is w/unit length (Fig. 4.20 (*a*)).

Reactions at *A* and *B*

Taking moments of the forces about A, we have

$$R_B \times l = w \times l \times \frac{l}{2}$$

(Total load equivalent to *udl* is $w \times l$ acting at its centre of gravity *i.e.* at a distance $\frac{l}{2}$ from A.)

or $$R_B = \frac{wl}{2} \ (\uparrow)$$

Now $$R_A + R_B = wl = \text{Total load on the beam}$$

or $$R_A = \frac{wl}{2} \ (\uparrow)$$

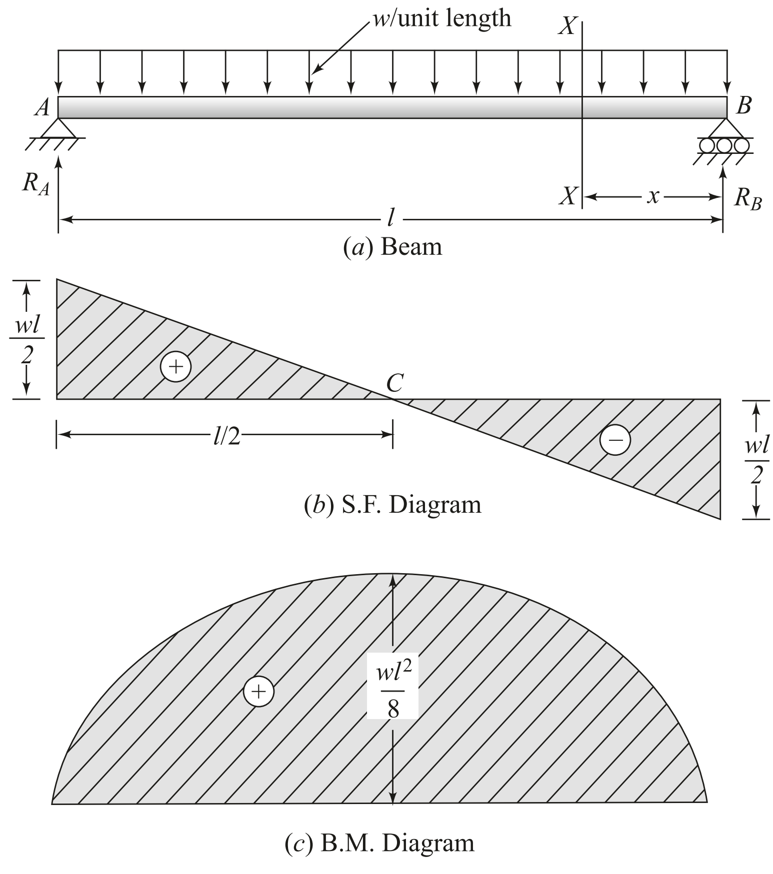


Fig. 4.20

Calculations for shear forces

Consider a section XX of the beam at a distance x from B.

Shear force at the section is

$$V = -R_B + wx = -\frac{wl}{2} + wx$$

Shear force at B, $$V_B = -\frac{wl}{2} \quad (\text{for } x = 0)$$

Shear force at A, $$V_A = -\frac{wl}{2} + wl \quad (\text{for } x = l) \ = +\frac{wl}{2}$$

Shear force at midpoint C of the beam

$$= -\frac{wl}{2} + \frac{wl}{2} = 0$$

The *SFD* is shown in Fig. 4.20 (*b*).

Calculations for bending moments

Bending moment at the section is

$$M_x = R_B\, x - wx \times \frac{x}{2}$$

$$= \frac{wl}{2}x - \frac{wx^2}{2}$$

Bending moment at B, $M_B = 0 \quad (\text{for } x = 0)$

Bending moment at A is

$$M_A = \frac{wl}{2}l - \frac{wl^2}{2} \quad (\text{for } x = l)$$

$$= 0$$

Bending moment at midpoint C of the beam is

$$M_C = \frac{wl}{2}\cdot\frac{l}{2} - \frac{w}{2}\left(\frac{l}{2}\right)^2 \quad (\text{for } x = \frac{l}{2})$$

$$= \frac{wl^2}{8}$$

The equation of the bending moment at the section represents a parabola, hence the bending moment curve is parabolic (Fig. 4.20 (*c*)).

4.11.4 Simply Supported Beam carrying Uniformly Varying Load which varies from Zero at Each End to *w* per unit length at the Midpoint

The beam is of length l and carries w/unit length of load at its midpoint C (Fig. 4.21 (*a*)).

Reactions at *A* and *B*

Taking moments of the forces about A, we have

$$R_B \times l = \frac{wl}{2} \cdot \frac{l}{2}$$

(Since total load on the beam = $\frac{wl}{2}$, and it acts at a distance $\frac{l}{2}$ from A.)

or $$R_B = \frac{wl}{4} (\uparrow)$$

But $$R_A + R_B = \frac{wl}{2}$$

or $$R_A = \frac{wl}{2} - \frac{wl}{4} = \frac{wl}{4} (\uparrow)$$

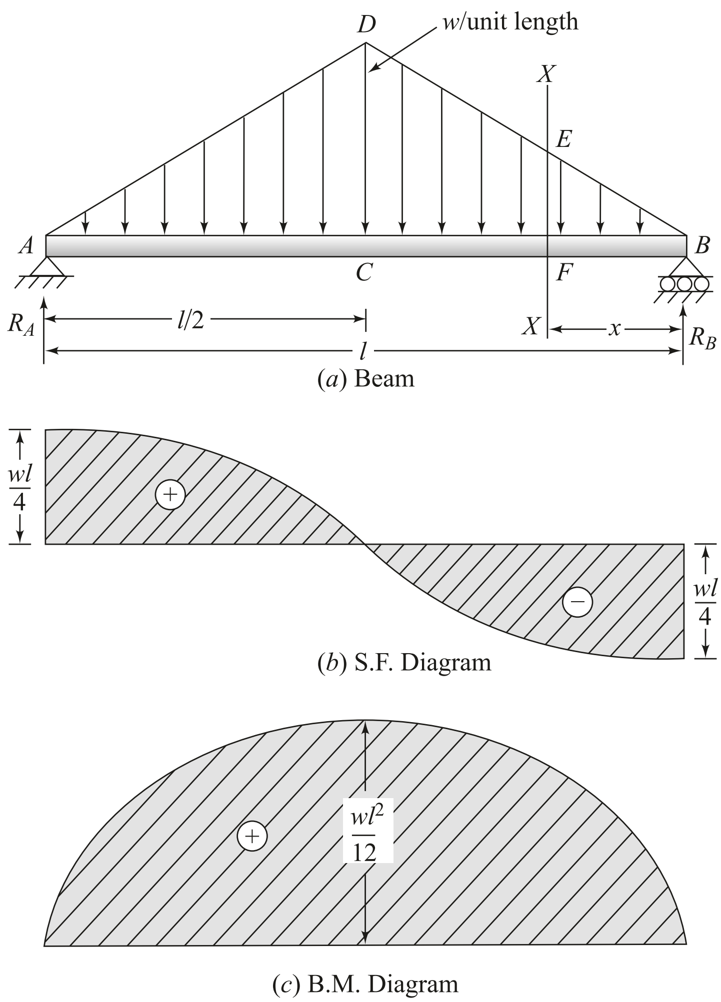


Fig. 4.21

Calculations for shear forces

Consider a section XX of the beam in BC at a distance x from B.

Shear force at the section is

$$V = -R_B + \text{Triangular load } BEF$$

Compare Δs BCD and BFE.

$$\frac{CD}{BC} = \frac{EF}{FB}$$

$$EF = \frac{CD \cdot FB}{BC}$$

$$= \frac{wx}{\left(\frac{l}{2}\right)} = \frac{2wx}{l}$$

The triangular load $BEF = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times x \times \frac{2wx}{l} = \frac{wx^2}{l}$$

Hence, shear force at the section is

$$V = -\frac{wl}{4} + \frac{wx^2}{l}$$

Shear force at B, $V_B = -\frac{wl}{4}$ (for $x = 0$)

Shear force at C is $V_C = -\frac{wl}{4} + \frac{w}{l}\left(\frac{l}{2}\right)^2$ (for $x = \frac{l}{2}$)

$$= -\frac{wl}{4} + \frac{w}{l} \cdot \frac{l^2}{4} = 0$$

Shear force at A is $V_A = -\frac{wl}{4} + \frac{wl}{2} = +\frac{wl}{4}$

The shear force diagram is parabolic, since the equation of the shear force at the section represents a parabola (Fig. 4.21 (*b*)).

Calculations for bending moments

Bending moment at the section is

$$M_x = R_B\, x - \text{Triangular load } BEF \times \frac{x}{3}$$

$$= \frac{wl}{4}x - \frac{wx^2}{l} \cdot \frac{x}{3} = \frac{wl}{4}x - \frac{wx^3}{3l}$$

Bending moments at A and B are:

$$M_A = 0$$

$$M_B = 0$$

For bending moment to be maximum

$$\frac{dM_x}{dx} = 0$$

$$\frac{wl}{4} - \frac{wx^2}{l} = 0$$

It gives the value of $x = \frac{l}{2}$, negative value has no significance.

Hence, the bending moment is maximum at $x = \frac{l}{2}$.

Substituting $x = \frac{l}{2}$ in the equation for bending moment at the section to get maximum $B.M.$, we have

$$M_{\max} = M_C = \frac{wl}{4} \cdot \frac{l}{2} - \frac{w}{3l} \cdot \left(\frac{l}{2}\right)^3 = + \frac{wl^2}{12}$$

The BMD is shown in Fig. 4.21 (c).

4.11.5 Simply Supported Beam carrying Uniformly Varying Load which varies from Zero at One End to *w* per unit length at Other End

Refer Fig. 4.22 (a). Load at A is zero and at B is w per unit length.

Reactions at *A* and *B*

Taking moments of the forces about A, we have

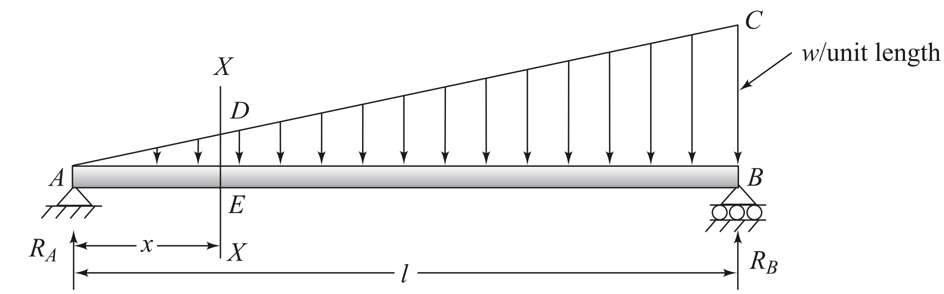
$$R_B \times l = \frac{w \times l}{2} \times \frac{2l}{3}$$

(Since the centre of gravity of a triangle lies at a distance $\frac{2l}{3}$ from the vertex.)

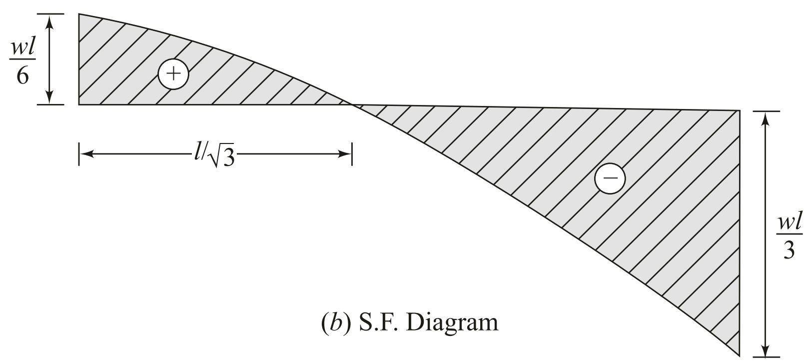
or
$$R_B = \frac{wl}{3} (\uparrow)$$

and
$$R_A + R_B = \frac{wl}{2}$$

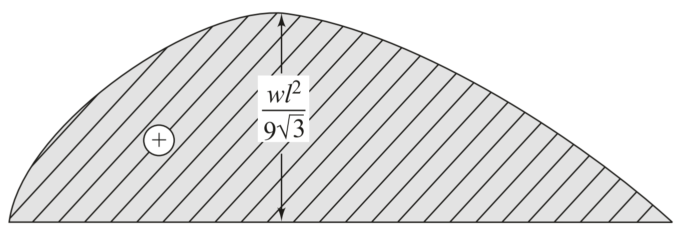
or
$$R_A = \frac{wl}{2} - \frac{wl}{3} = \frac{wl}{6} (\uparrow)$$



(a) Beam



(b) S.F. Diagram



(c) B.M. Diagram

Fig. 4.22

Calculations for shear forces

Consider a section XX of the beam at a distance x from A. Shear force at the section is

$$V = + R_A - \text{Triangular load } ADE$$

Compare Δs ABC and ADE.

$$\frac{BC}{AB} = \frac{DE}{AE}$$

$$DE = \frac{BC \cdot AE}{AB} = \frac{wx}{l}$$

Hence, shear force at the section is

$$V = +\frac{wl}{6} - \frac{1}{2} \times x \times \frac{wx}{l} = +\frac{wl}{6} - \frac{wx^2}{2l}$$

Shear force at A is

$$V_A = \frac{wl}{6} \text{ (for } x = 0)$$

Shear force at B is

$$V_B = +\frac{wl}{6} - \frac{w}{2l} \cdot l^2 \quad (\text{for } x = l) = -\frac{wl}{6}$$

To find x, equate F_x to zero.

We get $x = \frac{l}{\sqrt{3}}$, ignoring negative value.

The variation of the shear force is parabolic, since the shear force equation at the section represents a parabola (Fig. 4.22 (*b*)).

Calculations for bending moments

Bending moment at the section is

$$M_x = R_A \times x - \text{Triangular load } ADE \times \frac{x}{3}$$

$$= \frac{wl}{6} x - \frac{wx^2}{2l} \cdot \frac{x}{3} = \frac{wl}{6} x - \frac{wx^3}{6l}$$

Bending moment at A, where $x = 0$, is

$$M_A = 0$$

and $$M_B = 0 \text{ (for } x = l)$$

The bending moment is maximum, where shear force is zero, that is, at $x = \frac{l}{\sqrt{3}}$.

Maximum bending moment is

$$M_{\max} = \frac{wl}{6} \cdot \frac{l}{\sqrt{3}} - \frac{w}{6l}\left(\frac{l}{\sqrt{3}}\right)^3$$

$$= \frac{wl^2}{6\sqrt{3}} - \frac{wl^2}{18\sqrt{3}} = \frac{wl^2}{9\sqrt{3}}$$

The bending moment diagram is a cubic curve, because the bending moment equation at the section is a curve of third order (Fig. 4.22 (*c*)).

4.11.6 Simply Supported Beam subjected to a Couple

The couple is acting at point C (Fig. 4.23 (a)).

Reactions at A and B

Take moments of the forces about B.

$$R_A \times l = M$$

or $$R_A = \frac{M}{l}$$

But $$R_A + R_B = 0 \quad \text{or} \quad R_B = -R_A$$

Calculations for shear forces

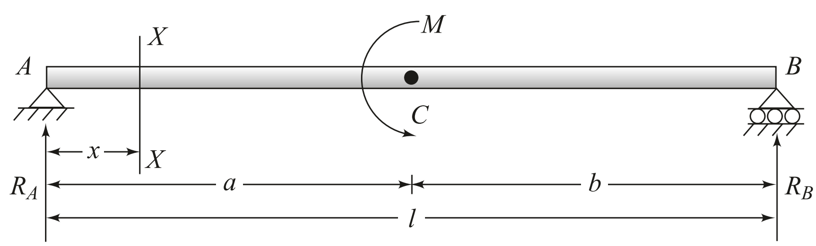
Consider a section of the beam at a distance x from A.

Shear force at the section is

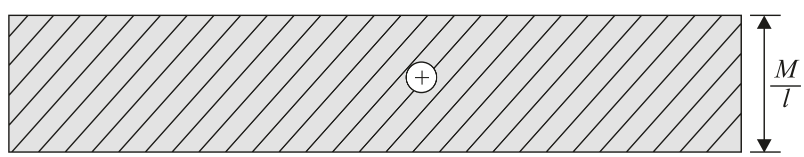
$$V = +R_A$$

The shear force between A and B remains constant at $+\dfrac{M}{l}$.

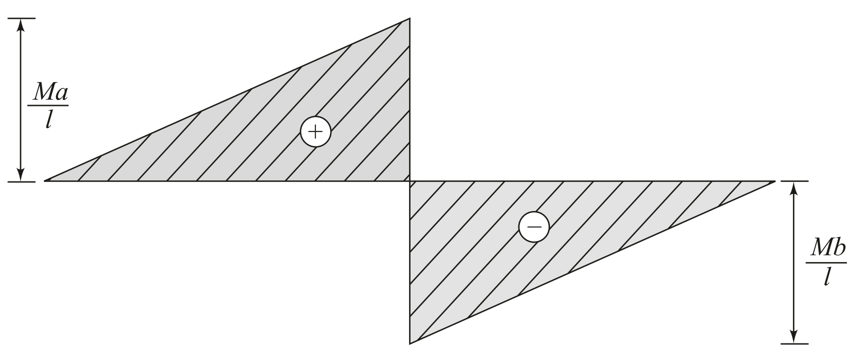
The SFD is shown in Fig. 4.23 (b).



(a) Beam



(b) S.F. Diagram



(c) B.M. Diagram

Fig. 4.23

Calculations for bending moments

Bending moment at the section is

$$M_x = +R_A x = +\frac{M}{l}x$$

Bending moment at A is

$$M_A = 0 \quad (\text{for } x = 0)$$

Bending moment at C is

$$M_C = +\frac{Ma}{l} \quad (\text{for } x = a)$$

Bending moment just to the right of C

$$= \frac{Ma}{l} - M = \frac{M(a-l)}{l} = -\frac{M}{l}(l-a) = -\frac{Mb}{l}$$

Bending moment at B is

$$M_B = R_A l - M = \frac{M}{l}l - M = 0$$

The bending moments between A and C, and between B and C are shown by inclined straight lines (Fig. 4.23 (*c*)).

Example 4.3

Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.24 (*a*).

Solution: **Reactions at *A* and *B***

Take moments of the forces about A.

$$R_B \times 4 = 20 \times 1 + 10 \times 2 \times 3 = 80$$

or $$R_B = \frac{80}{4} = 20 \text{ kN } (\uparrow)$$

and $$R_A + R_B = 20 + (10 \times 2) = 40 \text{ kN}$$

or $$R_A = 20 \text{ kN } (\uparrow)$$

Calculations for shear forces

Consider a section XX of the beam at a distance x from A. The shear force at the section is

$$V = +R_A$$
$$= +20 \text{ kN}$$

The shear force between A and C remains constant at 20 kN.

Shear force just to the right of C

$$= +20 - 20 = 0$$

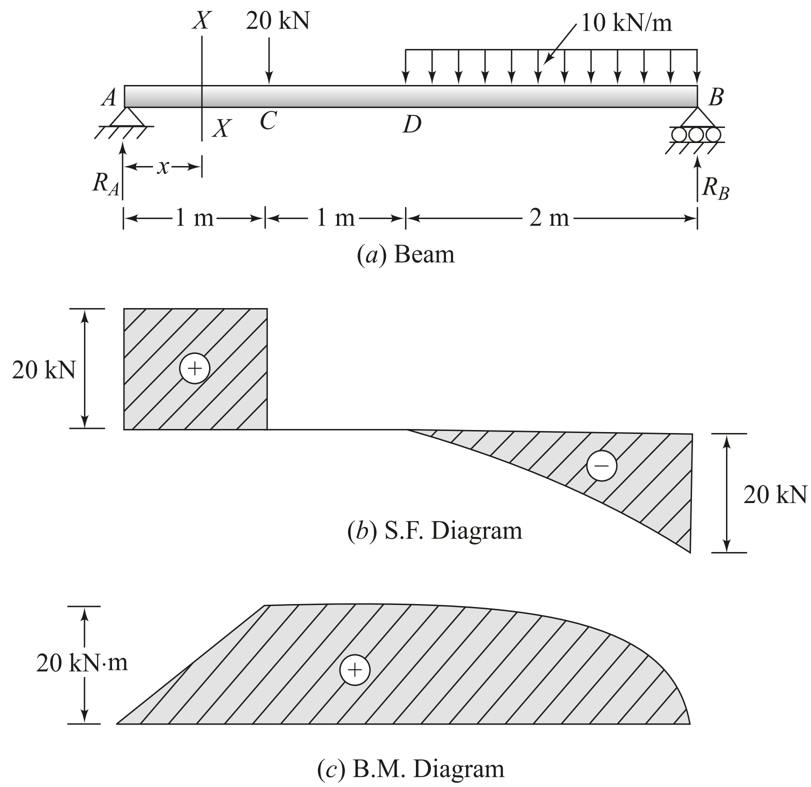


Fig. 4.24

The shear force remains zero for the portion *CD*.

Shear force at *B* is

$$V_B = + 20 - 20 - (10 \times 2)$$

$$= - 20 \text{ kN}$$

The shear force between *A* and *C* is connected by a horizontal straight line and between *B* and *D* by an inclined straight line due to *udl* for this portion (Fig. 4.24 (*b*)).

Calculations for bending moments

Bending moment at the section is

$$M_x = + R_A \cdot x$$

$$= + 20x$$

Bending moment at *A* is

$$M_A = 0 \quad (\text{for } x = 0)$$

Bending moment at *C* is

$$M_C = + 20 \times 1 \quad (\text{for } x = 1 \text{ m})$$

$$= + 20 \text{ kN·m}$$

Bending moment at D is

$$M_D = +20 \times (1+1) - 20 \times 1$$
$$= +20 \text{ kN·m}$$

Bending moment at B is

$$M_B = +20 \times (1+1+2) - 20 \times (1+2) - 10 \times 2 \times \left(\frac{2}{2}\right) = 0$$

The bending moment for AC is shown by an inclined straight line, for CD by a horizontal straight line and for BD by a parabolic curve (Fig. 4.24 (c)).

Example 4.4

Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.25 (a).

Solution: **Reactions at A and D**

Take moments of the forces about A.

$$R_D \times (0.5 + 2 + 2) = 15 \times 2 \times \left(0.5 + \frac{2}{2}\right)$$

or
$$R_D = \left(\frac{45}{4.5}\right) = 10 \text{ kN}(\uparrow)$$

But
$$R_A + R_D = 15 \times 2 = 30 \text{ kN}$$

or
$$R_A = 20 \text{ kN} (\uparrow)$$

Calculations for shear forces

Consider a section XX of the beam at a distance x from A.

Shear force at the section is

$$V = +R_A = +20 \text{ kN}$$

The shear force for the portion AB remains constant at 20 kN.

Shear force at C is

$$V_C = +20 - 15 \times 2 = -10 \text{ kN}$$

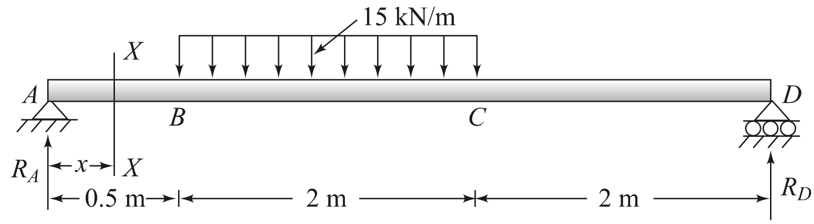
The shear force between C and D remains constant at (–10) kN.

The shear force for AB and CD are shown by two horizontal lines and for BC, by an inclined straight line (Fig. 4.25 (b)).

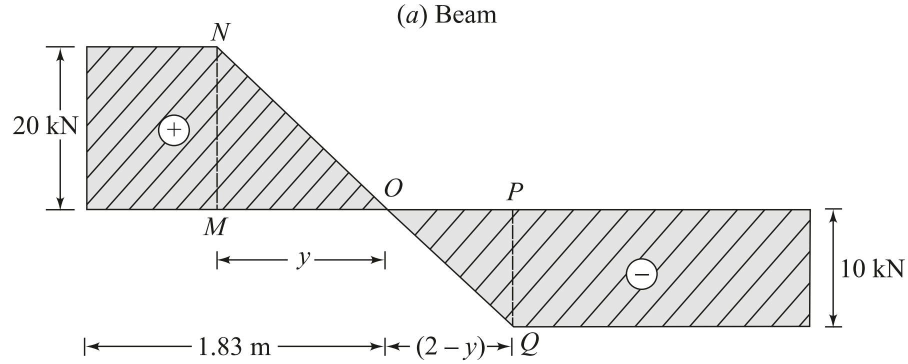
Calculations for bending moments

Bending moment at the section is

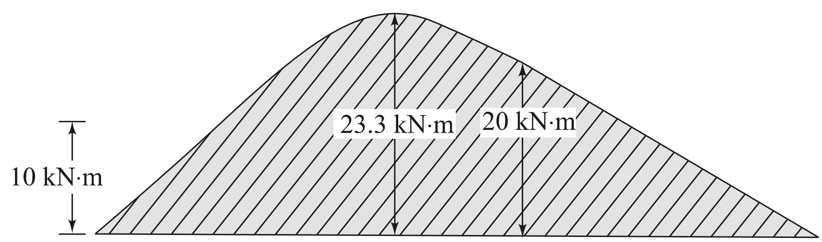
$$M_x = +R_A \cdot x$$
$$= +20x$$



(*a*) Beam



(*b*) S.F. Diagram



(*c*) B.M. Diagram

Fig. 4.25

Bending moment at *A* is

$$M_A = 0 \quad (\text{for } x = 0)$$

Bending moment at *B* is

$$M_B = + R_A \times 0.5$$

$$= + 20 \times 0.5$$

$$= + 10 \text{ kN·m}$$

Bending moment at *C* is

$$M_C = + R_A \times (0.5 + 2) - 15 \times 2 \times \frac{2}{2}$$

$$= + 20 \times 2.5 - 30 \times 1 = + 20 \text{ kN·m}$$

Bending moment at D is

$$M_D = + R_A \times (0.5 + 2 + 2) - 15 \times 2 \times \left(\frac{2}{2} + 2\right) = 0$$

Location of zero shear force

Compare Δs *MNO* and *OPQ*.

$$\frac{MN}{MO} = \frac{PQ}{OP}$$

$$\frac{20}{y} = \frac{10}{(2 - y)}$$

or $y = 1.33$ m

Hence, the shear force is zero at a distance (0.5 + 1.33) m = 1.83 m from A.

The maximum bending moment occurs at O, given by

$$M_O = M_{max}$$

$$= + R_A \times (1.33 + 0.5) - 15 \times 1.33 \times (1.33 \times 0.5) = + 23.4 \text{ kN·m}$$

The *BMD* is shown in Fig. 4.25 (c).

Example 4.5

Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.26 (a).

Solution: **Reactions at A and B**

Take moments of the forces about A.

$$R_B \times 3 = + 70 \times 1 + 20 \times 2 \times 2$$

On solving, we get $R_B = 50$ kN (↑)

But $R_A + R_B = 70 + 20 \times 2 = 110$ kN

or $R_A = 110 - 50 = 60$ kN (↑)

Calculations for shear forces

The shear force between A and C remains constant at $R_A = + 60$ kN.

Shear force just to the right of C

$$= 60 - 70 = - 10 \text{ kN}$$

Shear force at B is

$$V_B = 60 - 70 - 20 \times 2$$

$$= - 50 \text{ kN}$$

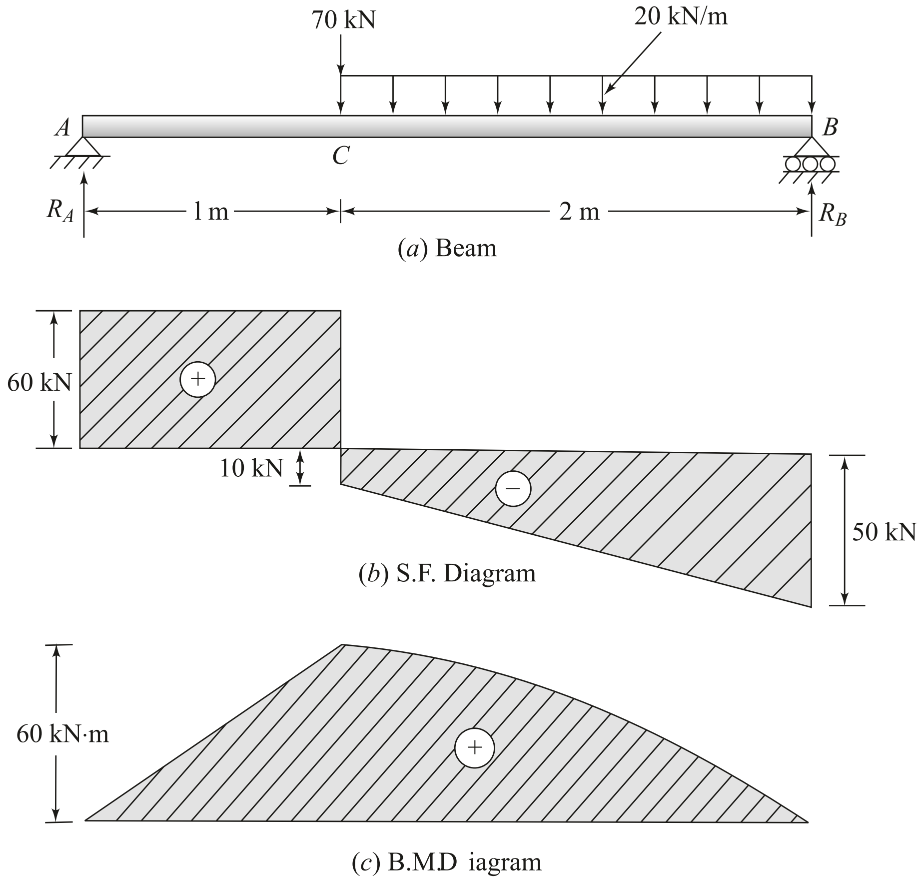


Fig. 4.26

The *SFD* is shown in Fig. 4.26 (*b*).

Calculations for bending moments

The bending moments at *A* and *B* are zero, since the beam is simply supported.

$$M_A = M_B = 0$$

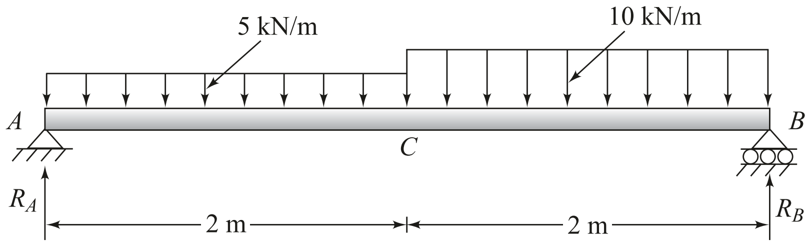
Bending moment at *C* is

$$\begin{aligned} M_C &= +R_A \times 1 \\ &= +60 \times 1 \\ &= 60 \text{ kN·m} \end{aligned}$$

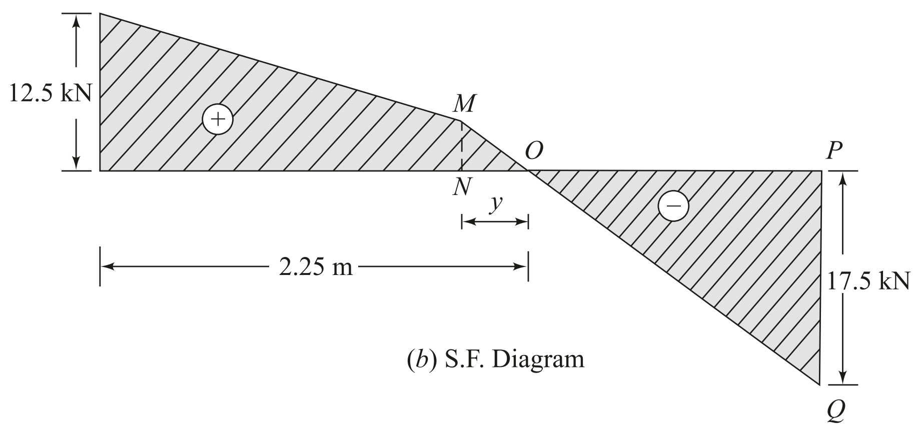
The *BMD* is shown in Fig. 4.26 (*c*).

Example 4.6

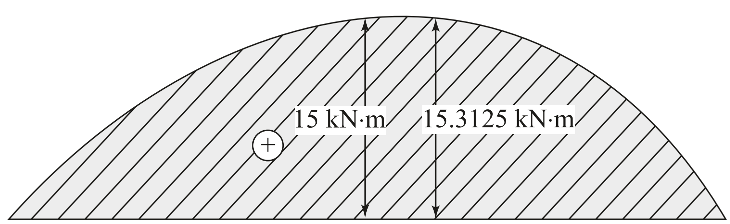
Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.27 (*a*).



(*a*) Beam



(*b*) S.F. Diagram



(*c*) B.M. Diagram

Fig. 4.27

Solution: **Reactions at *A* and *B***

Take moments of the forces about *A*.

$$R_B \times (2 + 2) = 5 \times 2 \times \frac{2}{2} + 10 \times 2 \times \left(2 + \frac{2}{2}\right)$$

On solving, we get $R_B = 17.5 \text{ kN } (\uparrow)$

But $R_A + R_B = 5 \times 2 + 10 \times 2 = 30 \text{ kN}$

or $R_A = 30 - R_B = 30 - 17.5 = 12.5 \text{ kN } (\uparrow)$

Calculations for shear forces

Shear force at A is

$$V_A = + R_A = + 12.5 \text{ kN}$$

Sear force at C is

$$V_C = + R_A - 5 \times 2$$

$$= 12.5 - 10 = 2.5 \text{ kN}$$

Shear force at B is

$$V_B = + R_A - 5 \times 2 - 10 \times 2 = - 17.5 \text{ kN}$$

The *SFD* is shown in Fig. 4.27 (*b*).

Calculations for bending moments

The bending moments at A and B are zero.

$$M_A = M_B = 0$$

Bending moment at C is

$$M_C = + R_A \times 2 - 5 \times 2 \times \frac{2}{2}$$

$$= + 12.5 \times 2 - 5 \times 2 \times 1 = 15 \text{ kN·m}$$

Location of zero shear force

Compare Δs *MNO* and *OPQ*.

$$\frac{MN}{NO} = \frac{PQ}{OP}$$

$$\frac{2.5}{y} = \frac{17.5}{2 - y}$$

Solving for y, we get

$$y = 0.25 \text{ m}$$

Hence, the shear force is zero at a distance

$$(2 + 0.25) \text{ m} = 2.25 \text{ m from } A.$$

The maximum bending moment occurs at O, given by

$$M_O = M_{\max} = + R_A \times (2 + 0.25) - 5 \times 2 \times \left(\frac{2}{2} + 0.25\right) - 10 \times 0.25 \times \frac{0.25}{2}$$

$$= + 12.5 \times 2.25 - 10 \times 1.25 - 5 \times 0.25 \times 0.25 = + 15.3125 \text{ kN·m}$$

The *BMD* is shown in Fig. 4.27 (*c*).

Example 4.7

Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.28 (*a*).

Solution: Refer Fig. 4.28.

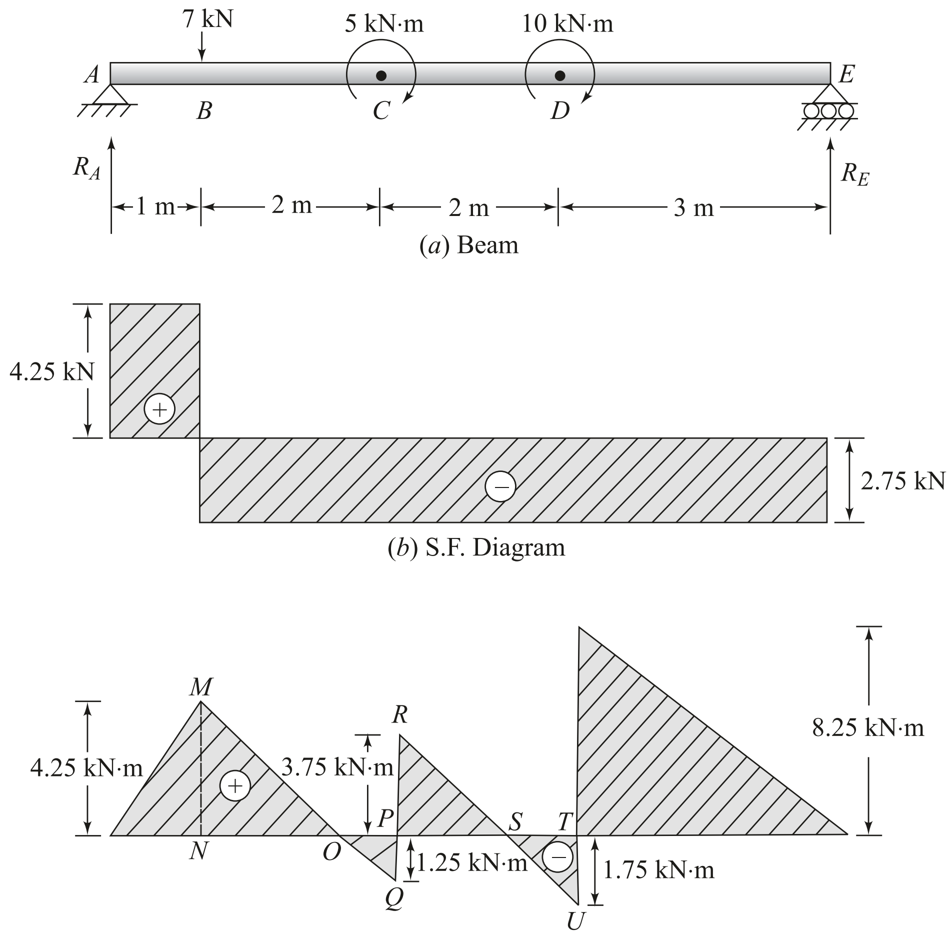


Fig. 4.28

Reactions at *A* and *E*

Take moments of the forces about *A*.

$$R_E \times 8 = 7 \times 1 + 5 + 10$$

or $$R_E = 2.75 \text{ kN } (\uparrow)$$

Now $$R_A + R_E = 7$$

or $$R_A = 4.25 \text{ kN } (\uparrow)$$

Calculations for shear forces

Shear force at A is

$$V_A = +R_A = +4.25 \text{ kN}$$

The shear force between A and B remains constant at + 4.25 kN.

Shear force just to the right of B

$$= +4.25 - 7 = -2.75 \text{ kN}$$

The shear force between B and E remains constant at –2.75 kN.

The *SFD* is shown in Fig 4.28 (*b*).

Calculations for bending moments

Bending moments at A and E are zero, because the beam is simply supported.

$$M_A = M_E = 0$$

Bending moment at B is

$$M_B = +R_A \cdot 1 = 4.25 \text{ kN·m}$$

Bending moment at C is

$$M_C = +R_A \times 3 - 7 \times 2 = -1.25 \text{ kN·m}$$

Bending moment just to the right of $C = -1.25 + 5 = +3.75$ kN.m

Bending moment at D is

$$M_D = +R_A \times 5 - 7 \times 4 + 5 = -1.75 \text{ kN·m}$$

Bending moment just to the right of $D = -1.75 + 10 = 8.25$ kN·m

The *BMD* is shown in Fig. 4.28 (*c*).

Location of point of contraflexure

There are two points where bending moments are zero, one point lies between B and C and other between C and D.

Compare Δs MNO and OPQ.

$$\frac{MN}{NO} = \frac{PQ}{OP}$$

$$\frac{MN}{NO} = \frac{PQ}{NP - NO}$$

$$\frac{4.25}{NO} = \frac{1.25}{2 - NO}$$

or $$NO = 1.54 \text{ m}$$

Hence, the first point of contraflexure lies at a distance of (1 + 1.54) = 2.54 m from A.

Now compare Δs *PRS* and *STU*.

$$\frac{PR}{PS} = \frac{UT}{ST}$$

$$\frac{3.75}{PS} = \frac{1.75}{2 - PS}$$

or $$PS = 1.36 \text{ m}$$

Hence, the second point of contraflexure lies at a distance of

$$(1 + 2 + 1.36) \text{ m} = 4.36 \text{ m from } A.$$

Example 4.8

The loading on a simply supported beam of length 10 m varies gradually from 3 kN/m at one end to 7 kN/m at the other end. Draw the shear force and bending moment diagrams for this beam.

Solution: Refer Fig. 4.29. The given beam is converted into an equivalent beam such that it consists of uniformly distributed load of intensity 3 kN/m throughout its length and a uniformly varying load with zero at *B* to a maximum load of 4 kN/m at *A*.

Reactions at *A* and *B*

Take moments of the forces about *A*.

$$R_B \times 10 = 3 \times 10 \times \frac{10}{2} + \frac{1}{2} \times 4 \times 10 \times \frac{10}{3} = 216.66$$

or $$R_B = 21.66 \text{ kN } (\uparrow)$$

Also $$R_A + R_B = 3 \times 10 + \frac{1}{2} \times 4 \times 10 = 50 \text{ kN}$$

or $$R_A = 28.34 \text{ kN } (\uparrow)$$

Calculations for shear forces

Consider a section *XX* of the beam at a distance *x* from *B*.

The rate of loading (vertical) for the triangular load at the section is $\frac{4x}{10}$.
Shear force at the section is

$$V = -R_B + 3x + \text{Triangular load at the section}$$

$$= -R_B + 3x + \frac{1}{2} \times \frac{4x}{10}\, x = -21.66 + 3x + 0.2x^2$$

Shear force at *B* is

$$V_B = -R_B = -21.66 \text{ kN} \quad (\text{for } x = 0)$$

Shear force at *A* is

$$V_A = -21.66 + 3 \times 10 + 0.2 \times 100 \quad (\text{for } x = 10 \text{ m}) = 28.34 \text{ kN}$$

The *SFD* is shown in Fig. 4.29 (*c*).

Calculations for bending moments

Bending moment at the section is

$$M_x = + R_B x - 3 \times x \times \frac{x}{2} - \frac{1}{2} \times \frac{4x}{10} \times x \times \frac{x}{3} = 21.66x - 1.5x^2 - 0.066x^3$$

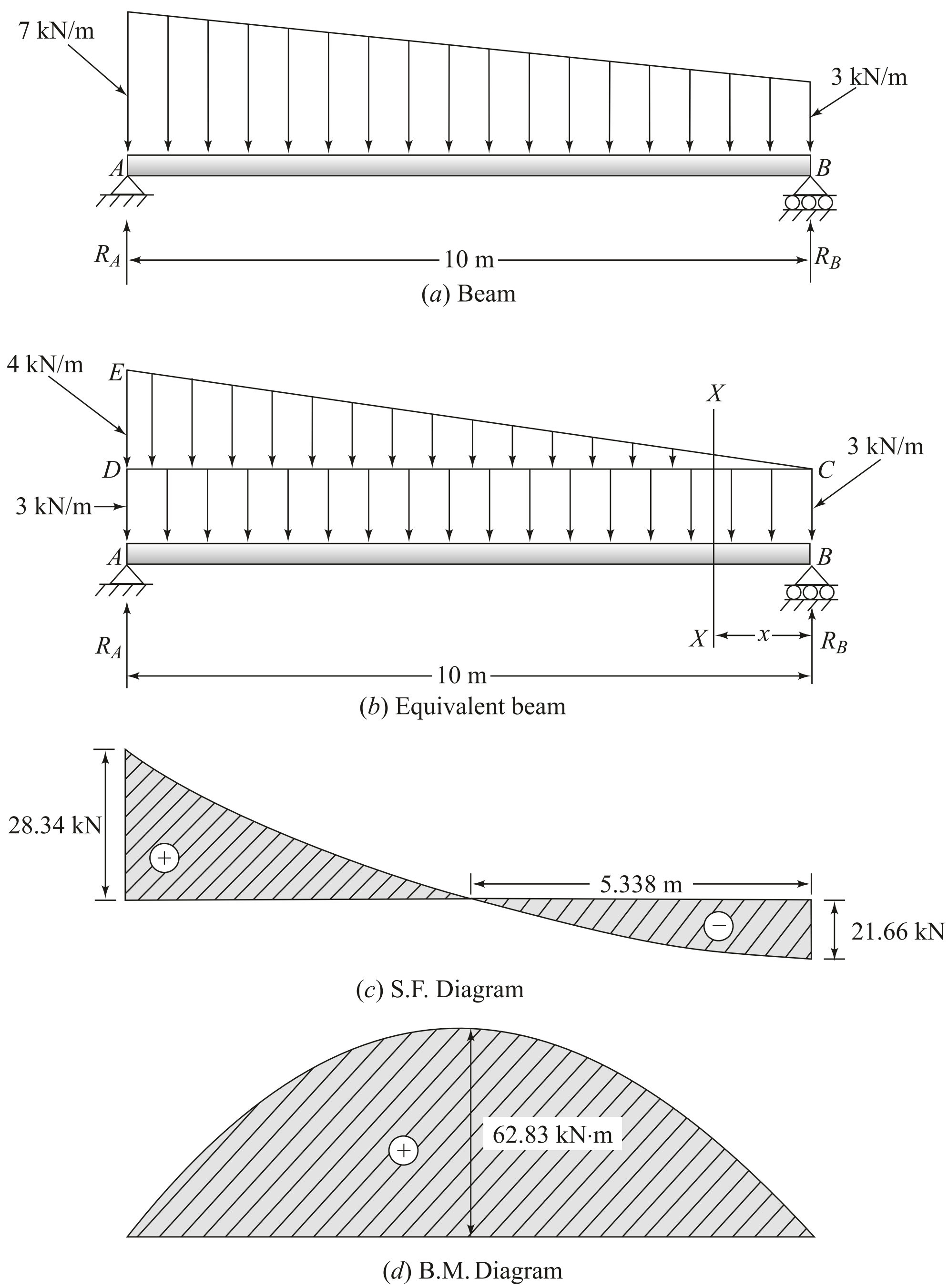


Fig. 4.29

For shear force to be zero

$$\frac{dM_x}{dx} = 0$$

$$21.66 - 3x - 0.198x^2 = 0$$

On solving, we get $x = 5.338$ or -20.5

We accept $x = 5.338$ m (Because $x = -20.5$ m carries no meaning).

Bending moment is maximum at the point, where shear force is zero *i.e.*, at $x = 5.338$ m.

$$M_{\max} = +21.66 \times 5.338 - 1.5 \times 5.338^2 - 0.066 \times 5.338^3 = 62.83 \text{ kN·m}$$

The bending moments at A and B are zero.

$$M_A = M_B = 0$$

The *BMD* is shown in Fig. 4.29 (*d*).

Example 4.9

Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.30 (*a*).

Solution:

Reactions at *A* and *B*

The load of 15 kN is applied on the beam through a bracket of length 1.2 m fixed to it at a distance of 4m from A. Due to this, a load of 15 kN and a bending moment of (15 × 1.2) = 18 kN·m are acting at C.

Take moments of the forces about A.

$$R_B \times 6 + 18 = 15 \times 4$$

or $$R_B = 7 \text{ kN } (\uparrow)$$

Also $$R_A + R_B = 15 \text{ kN}$$

or $$V_A = 8 \text{ kN } (\uparrow)$$

Calculations for shear forces

Shear force at A is

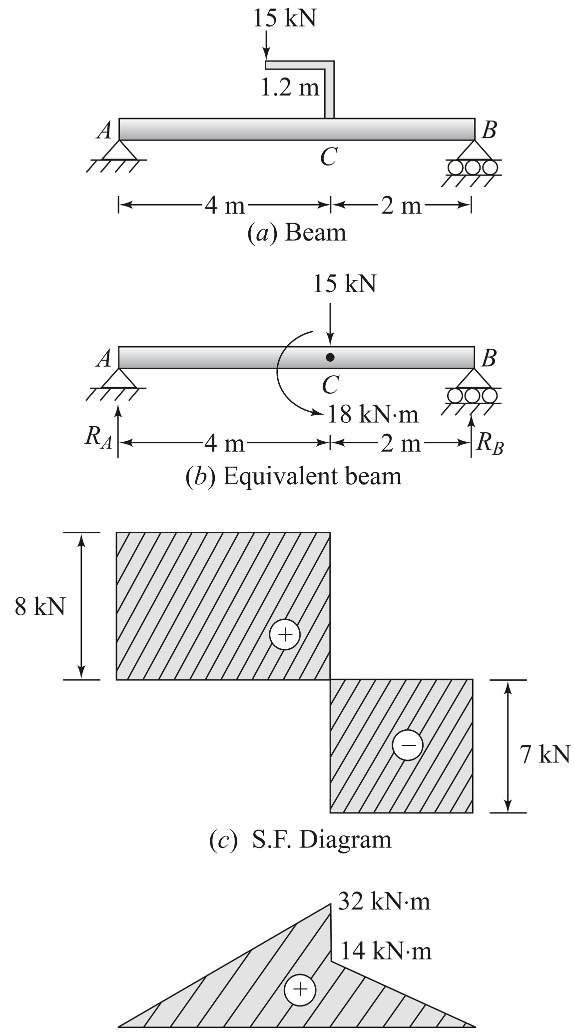
$$V_A = +R_A = 8 \text{ kN}$$

The shear force between A and C remains constant + 8 kN.

The shear force just to the right of $C = +8 - 15 = -7$ kN

The shear force between B and C remains constant at – 7 kN.

The *SFD* is shown in Fig. 4.30 (*c*).



(*a*) Beam

(*b*) Equivalent beam

(*c*) S.F. Diagram

(*d*) B.M. Diagram

Fig. 4.30

Calculations for bending moments

The bending moments at *A* and *B* are zero.

$$M_A = M_B = 0$$

Bending moment just to the left of $C = +R_A \times 4 = +8 \times 4$

$= +32$ kN·m

Bending moment just to the right of $C = +32 - 18$

$= +14$ kN·m

The *BMD* is shown in Fig. 4.30 (*d*).

4.12 RELATIONS AMONG LOAD, SHEAR FORCE AND BENDING MOMENT

AB is a simple beam supported at *A* and *B* and loaded with varying load with intensity *w*/unit length (Fig. 4.31).

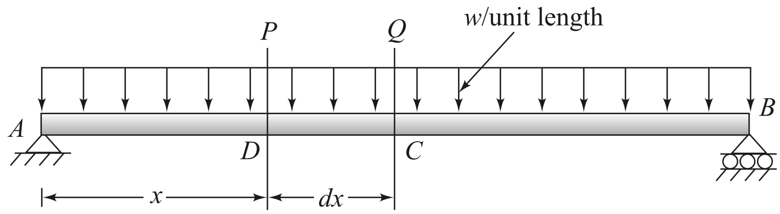


Fig. 4.31

End *A* is chosen as the origin. A section *CDPQ* of length *dx* of beam is considered.

The relationship between load and shear force is given as

$$W = \frac{dV}{dx}$$

i.e., the rate of change of shear force at any section is equal to the rate of loading at that section.

The relationship between shear force and bending moment is given as

$$V = \frac{dM}{dx}$$

It means that the rate of change of bending moment at any section is equal to the shear force at that section.

4.13 SFD AND BMD FOR OVERHANGING BEAMS

4.13.1 Overhanging Beam with equal Overhangs on Each Side and loaded with Point Loads at the Ends

The beam is supported at *B* and *C*. The length of the beam is $(l + 2a)$ [Fig. 4.32 (*a*)].

Reactions at *B* and *C*

Take moments of the forces about *B*.

$$W \times a + R_C \times l = W(l + a)$$

or $$R_C = W(\uparrow)$$

But $$R_B + R_C = W + W = 2W$$

or $$R_B = W(\uparrow)$$

Calculations for shear forces

Consider a section *XX* of the beam at a distance *x* from *A*.

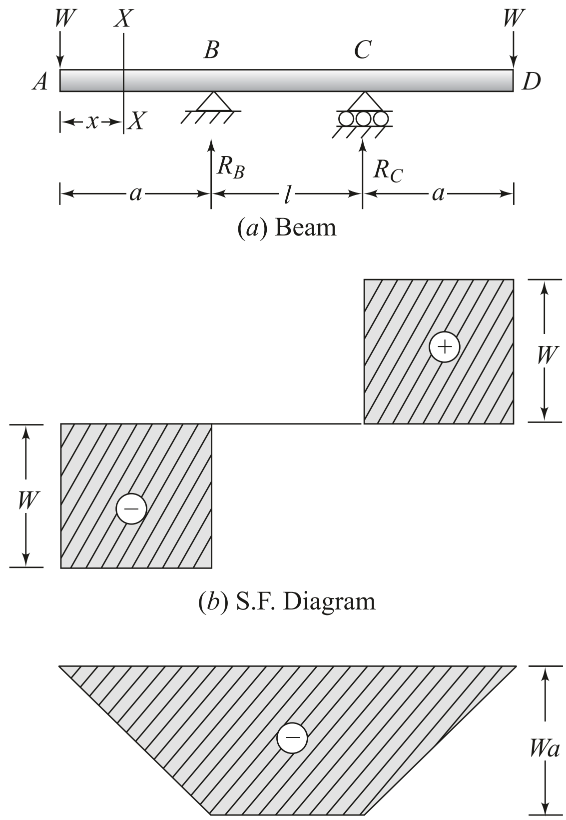
Shear force at the section is

$$V = -W$$

The shear force between A and B remains constant at $-W$.

Shear force just to the right of B

$$= -W + R_B = -W + W = 0$$



(*a*) Beam

(*b*) S.F. Diagram

(*c*) B.M. Diagram

Fig. 4.32

The shear force remains zero between B and C.

$$= \frac{M}{l}\cdot l - M = 0$$

Shear force at C is

$$V_C = 0 + R_C = +W$$

The shear force between C and D remains constant at $+W$.

The shear forces between A and B, and between C and D are connected by horizontal straight lines (Fig. 4.32 (*b*)).

Calculations for bending moments

Bending moment at the section is

$$M_x = -Wx$$

Bending moment at A, where $x = 0$, is

$$M_A = 0$$

Bending moment at B, where $x = a$, is

$$M_B = -Wa$$

Bending moment at C is

$$\begin{aligned} M_C &= -W(a + l) + R_B l \\ &= -Wa - Wl + Wl \\ &= -Wa \end{aligned}$$

Bending moment at D is

$$\begin{aligned} M_D &= -W(a + l + a) + R_B(l + a) + R_C a \\ &= -Wa - Wl - Wa + Wl + Wa + Wa = 0 \end{aligned}$$

The bending moment diagrams between A and B, and between C and D are shown by inclined straight lines (Fig. 4.32 (*c*)).

4.13.2 Overhanging Beam with equal Overhangs on Each Side and loaded with a Uniformly Distributed Load over its Entire Span

Refer Fig. 4.33(*a*).

Reactions at *B* and *C*

Take moments of the forces about B.

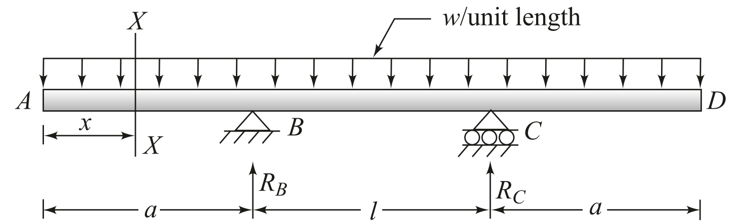
$$w \times a \times \frac{a}{2} + R_C \times l = w(l + a)\,\frac{(l + a)}{2}$$

$$\frac{wa^2}{2} + R_C\, l = \frac{w}{2}\,[l^2 + a^2 + 2la]$$

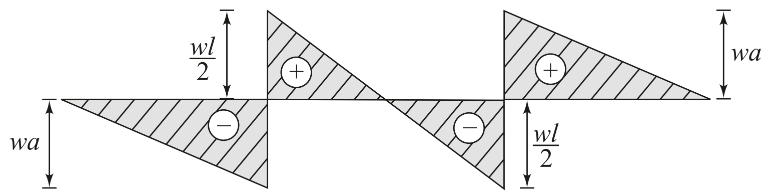
or $$R_C = \frac{w(l + 2a)}{2}\,(\uparrow)$$

But $$R_B + R_C = w(l + 2a)$$

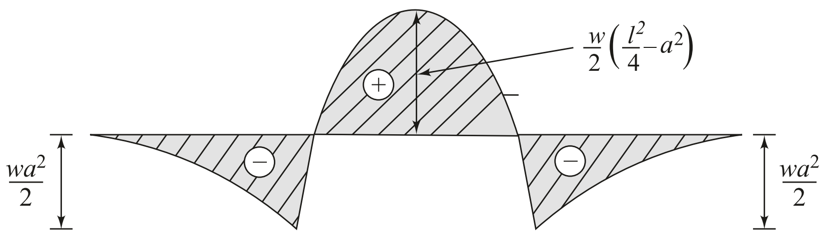
or $$R_B = \frac{w(l + 2a)}{2}\,(\uparrow)$$



(*a*) Beam

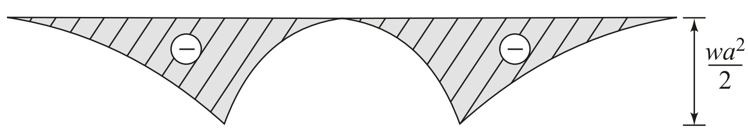


(*b*) S.F. Diagram



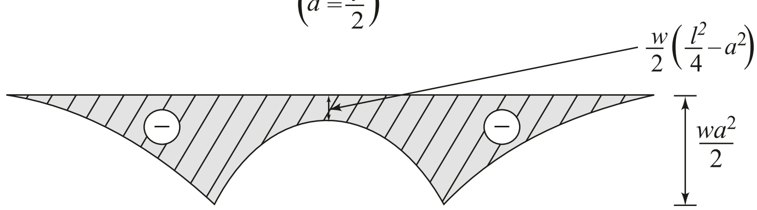
(*c*) B.M. Diagram

$\left(a < \frac{l}{2}\right)$



(*d*) B.M. Diagram

$\left(a = \frac{l}{2}\right)$



(*e*) B.M. Diagram

$\left(a > \frac{l}{2}\right)$

Fig. 4.33

Calculations for shear forces

Consider a section XX of the beam at a distance x from A. Shear force at the section is

$$V = -wx$$

Shear force at A,

$$V_A = 0 \quad (\text{for } x = 0)$$

Shear force at B,

$$V_B = -wa \quad (\text{for } x = a)$$

Shear force just to the right of B

$$= -wa + R_B$$

$$= -wa + \frac{w}{2}(l + 2a)$$

$$= \frac{w}{2}l$$

Shear force at C is

$$V_C = \frac{w}{2}\,l - wl$$

$$= -\frac{wl}{2}$$

Shear force just to the right of C

$$= -\frac{wl}{2} + R_C$$

$$= -\frac{wl}{2} + \frac{w}{2}\,(l + 2a)$$

$$= wa$$

Shear force at D is

$$V_D = wa - wa = 0$$

The SFD is shown in Fig. 4.33 (b).

Calculations for bending moments

Bending moment at the section is

$$M_x = -w \cdot x \cdot \frac{x}{2}$$

Bending moment at A, $\quad M_A = 0 \quad (\text{for } x = 0)$

Bending moment at B is

$$M_B = -w \cdot a \cdot \frac{a}{2}$$

$$= -\frac{wa^2}{2}$$

Bending moment at C is

$$M_C = -w\,(a+l)\,\frac{(a+l)}{2} + R_B \times l$$

$$= -\frac{w}{2}\,(a+l)^2 + \frac{w}{2}\,(l+2a) \times l$$

$$= -\frac{w}{2}a^2$$

Bending moment at D is

$$M_D = 0$$

The shear force is zero at a distance $\left(a + \frac{l}{2}\right)$ from A. It is the position of the maximum bending moment.

The maximum bending moment is given as

$$M_{\max} = -w\left(a+\frac{l}{2}\right)\cdot\left(a+\frac{l}{2}\right)\cdot\frac{1}{2} + R_B\left(a+\frac{l}{2}-a\right)$$

$$= -\frac{w}{2}\left(a+\frac{l}{2}\right)^2 + \frac{w}{2}(l+2a)\cdot\frac{l}{2}$$

$$= \frac{w}{2}\left(\frac{l^2}{4} - a^2\right)$$

Case I

When $a < \frac{l}{2}$, then $M_{\max}$ will be positive.

Case II

When $a = \frac{l}{2}$, then $M_{\max} = 0$

$$\frac{w}{2}\left(\frac{l^2}{4} - a^2\right) = 0$$

or $$a = \pm \frac{l}{2} \qquad \left(\frac{w}{2} \neq 0\right)$$

We accept $$a = +\frac{l}{2}\left[\text{because}\left(-\frac{l}{2}\right)\text{carries no meaning.}\right]$$

Case III

When $a > \frac{l}{2}$, then M_{max} will be negative.

The *BMD* for all the cases are shown in Fig. 4.33 (*c*), (*d*) and (*e*).

Example 4.10

Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.34(*a*).

Solution:

Reactions at *B* and *E*

Take moments of the forces about *B*.

$$50 \times 5 - 70 \times 4 - 3 \times 8 \times \left(4 + 4 + \frac{8}{2}\right) + R_E \times 16 = 0$$

or $$R_E = 19.87 \text{ kN } (\uparrow)$$

Also $$R_B + R_E = 50 + 70 + 3 \times 8$$

$$= 144 \text{ kN}$$

or $$R_B = 124.13 \text{ kN } (\uparrow)$$

Calculations for shear forces

Shear force at *A* is

$$V_A = -50 \text{ kN}$$

The shear force between *A* and *B* remains constant at 50 kN.

Shear force just to the right of *B*

$$= -50 + R_B = -50 + 124.13$$

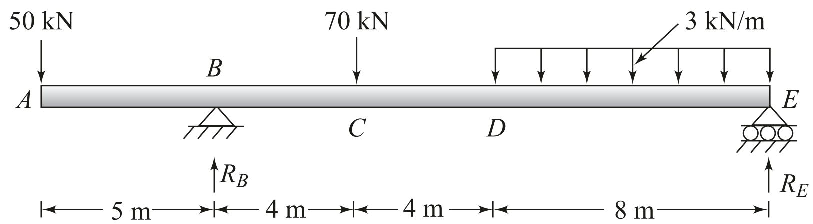
$$= 74.13 \text{ kN}$$

The shear force between *B* and *C* remains constant at + 74.13 kN.

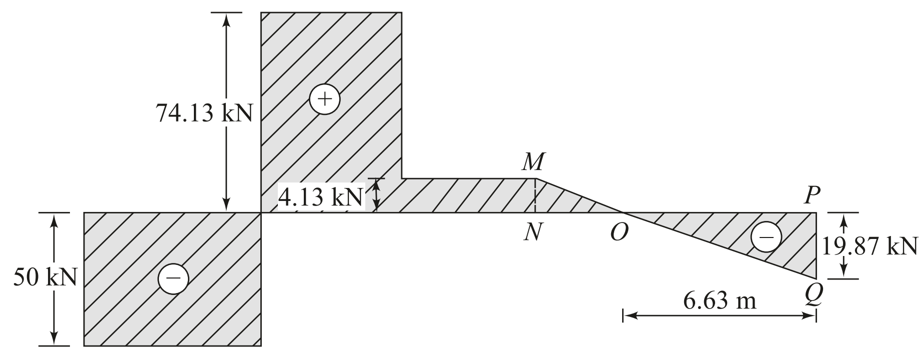
Shear force just to the right of *C*

$$= +74.13 - 70$$

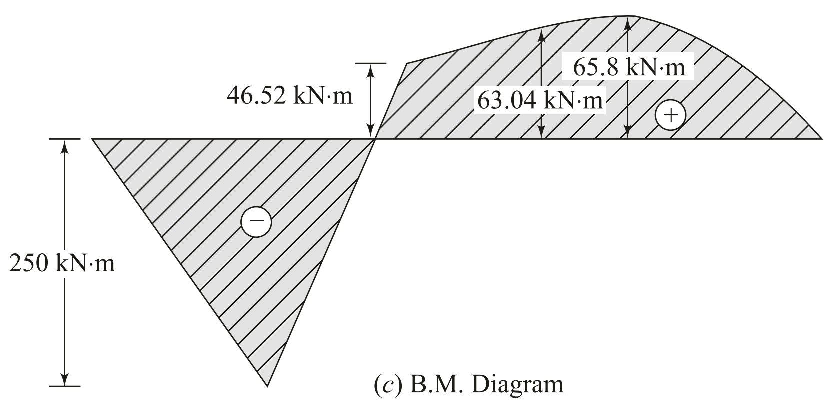
$$= 4.13 \text{ kN}$$



(a) Beam



(b) S.F. Diagram



(c) B.M. Diagram

Fig. 4.34

The shear force between C and D remains constant at + 4.13 kN.

Shear force at E is

$$V_E = +\,4.13 - 3 \times 8 = -\,19.87 \text{ kN}$$

Shear force just to the right of E

$$= -\,19.87 + R_E$$

$$= -\,19.87 + 19.87 = 0$$

The *SFD* is shown in Fig. 4.34 (*b*).

Calculations for bending moments

The bending moments at *A* and *E* are zero.

$$M_A = M_E = 0$$

Bending moment at *B* is

$$M_B = -50 \times 5 = -250 \text{ kN·m}$$

Bending moment at *C* is

$$M_C = -50 \times (5 + 4) + R_B \times 4$$
$$= +46.52 \text{ kN·m}$$

Bending moment at *D* is

$$M_D = -50 \times (5 + 4 + 4) + R_B \times (4 + 4) - 70 \times 4$$
$$= -50 \times 13 + 124.13 \times 8 - 280$$
$$= +63.04 \text{ kN·m}$$

The *BMD* is shown in Fig. 4.34 (*c*).

Location of zero shear force

Compare two triangles *MNO* and *OPQ*.

$$\frac{MN}{NO} = \frac{PQ}{OP}$$

$$\frac{MN}{NO} = \frac{PQ}{NP - NO}$$

$$\frac{4.13}{NO} = \frac{19.87}{8 - NO}$$

or $$NO = 1.37 \text{ m}$$

Hence, the shear force is zero at a distance (8 – 1.37) = 6.63 m from *E*.

The maximum bending moment occurs at *O*, given by

$$M_{\max} = M_O = +R_E \times 6.63 - 3 \times 6.63 \times \frac{6.63}{2}$$

$$= +19.87 \times 6.63 - 3 \times 6.63 \times \frac{6.63}{2}$$
$$= 65.8 \text{ kN·m}$$

The *BMD* is shown in Fig. 4.34 (*c*).

Example 4.11

Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.35(*a*).

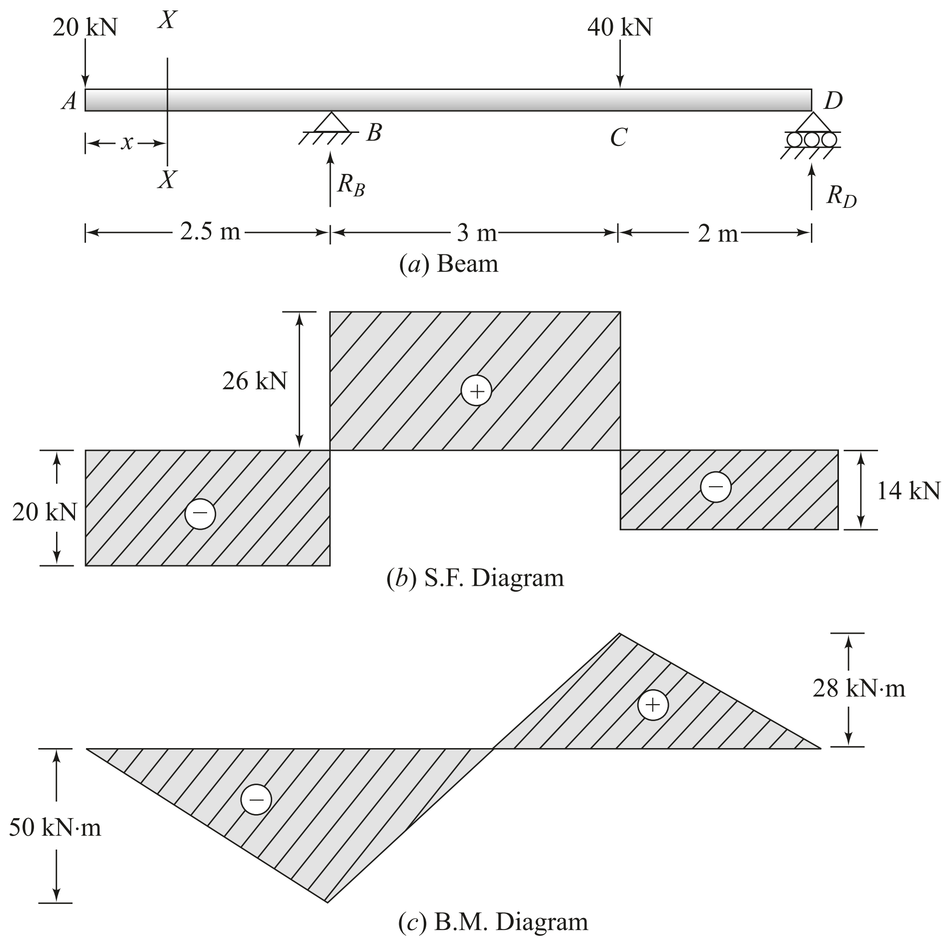


Fig. 4.35

Solution:

Reactions at *B* and *D*

Taking moments of the forces about *A*, we have

$$R_B \times 2.5 + R_D \times 7.5 = 40 \times 5.5$$

$$= 220 \qquad \ldots (1)$$

and $$R_B + R_D = 20 + 40 = 60 \text{ kN} \qquad \ldots (2)$$

From equation (2)

$$R_B = 60 - R_D$$

Substituting R_B in equation (1), we get

$$(60 - R_D) \times 2.5 + R_D \times 7.5 = 220$$

or $$R_D = 14 \text{ kN } (\uparrow)$$

On using equation (2), we get

$$R_B = 60 - 14 = 46 \text{ kN } (\uparrow)$$

Calculations for shear forces

Consider a section XX of the beam at a distance x from A. Shear force at the section is

$$V = -20 \text{ kN}$$

The shear force between A and B is constant at – 20 kN.

Shear force just to the right of B

$$= -20 + R_B = -20 + 46 = +26 \text{ kN}$$

The shear force between B and C is constant at 26 kN.

Shear force just to the right of $C = +26 - 40 = -14$ kN

The shear force between C and D is constant at – 14 kN.

The shear force diagram is shown in Fig. 4.35 (*b*).

Calculations for bending moments

Bending moment at the section is

$$M_x = -20x$$

Bending moment at A is

$$M_A = 0 \quad (\text{for } x = 0)$$

Bending moment at B is

$$M_B = -20 \times 2.5 \quad (\text{for } x = 2.5 \text{ m})$$

$$= -50 \text{ kN·m}$$

Bending moment at C is

$$M_C = -20 \times (2.5 + 3) + R_B \times 3$$

$$= +28 \text{ kN·m}$$

Bending moment D is

$$M_D = -20 \times (2.5 + 3 + 2) + 46 \times (3 + 2) - 40 \times 2$$

$$= 0$$

The bending moment diagram is shown in Fig. 4.35 (*c*).

SHORT ANSWER QUESTIONS

1. Differentiate between statically determine and indeterminate beams. Give two examples of each type.
2. What is a pinned support? How does it differ from a roller support?
3. Why is bending moment so called?
4. What are positive and negative bending moments alternatively called?
5. What type of bending moment acts on a simple beam and on a cantilever beam?
6. Which two types of loading a trapezoidal distribution consists of?
7. What is the relationship between shear force and bending moment?
8. What is point of contraflexure?

MULTIPLE CHOICE QUESTIONS

1. The shear force at any section of a beam is positive, if

(*a*) the left part is moving up and the right part moving down

(*b*) the right part is moving up and the left part moving down

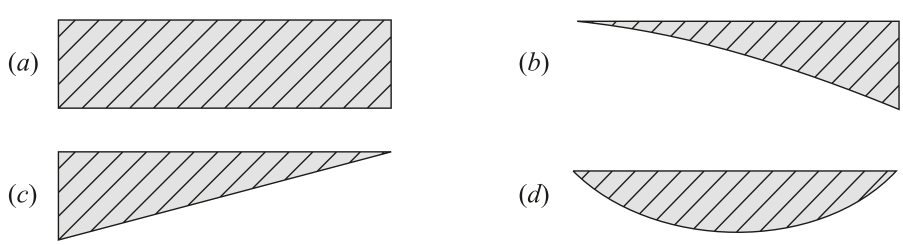
(*c*) the left part is moving up and the right part does not move

(*d*) none of these.

2. The maximum bending moment, when a point load W is acting at the free end of a cantilever beam of length l, is

(*a*) $\frac{Wl}{2}$ (*b*) $\frac{Wl}{4}$ (*c*) Wl (*d*) $\frac{Wl}{8}$.

3. The bending moment diagram for a cantilever beam carrying a point load at its free end is given as



4. The maximum bending moment for a cantilever beam carrying a *udl* of intensity *w*/unit length over its entire span is given as

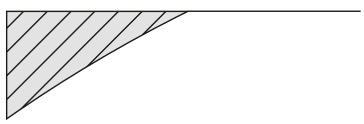
(*a*) $\frac{wl}{4}$ (*b*) $\frac{wl^2}{8}$ (*c*) $\frac{wl^2}{2}$ (*d*) $-\frac{wl^2}{2}$.

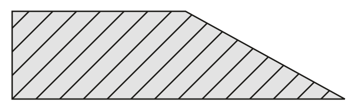
5. The variation of bending moment for a cantilever beam carrying a *udl* of intensity *w*/unit length over its entire span is shown by

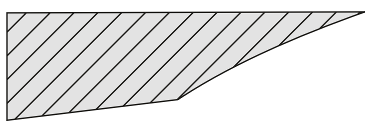
(*a*) a straight line (*b*) a second degree parabola

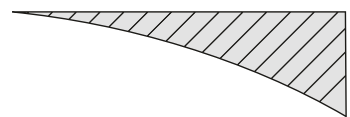
(*c*) a third degree parabola (*d*) an ellipse.

6. The bending moment diagram for a cantilever beam carrying a *udl* over its certain length from the free end is shown as

(*a*)


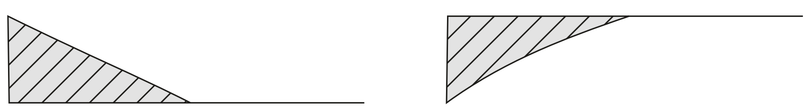
(*b*)


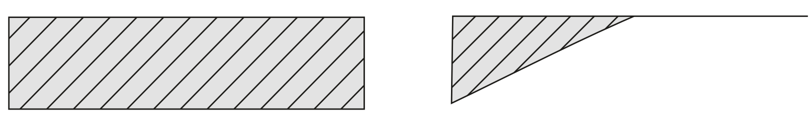
(*c*)


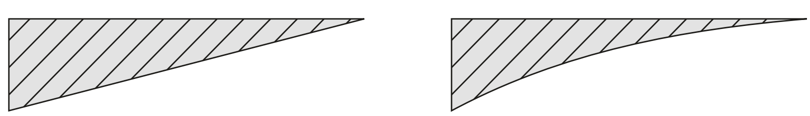
(*d*)


7. The shear force and the bending moment diagrams for a cantilever beam carrying a *udl* over its certain length from the fixed end are given as

(*a*)


(*b*)


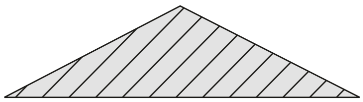
(*c*)


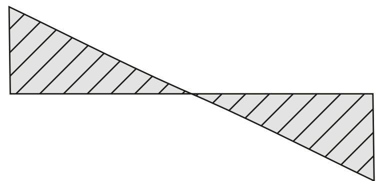
(*d*)


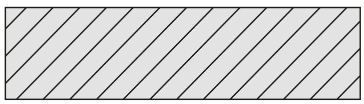
8. The maximum bending moment for a simply supported beam subjected to a point load at its centre is given as

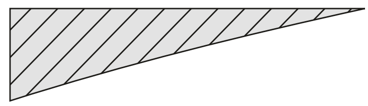
(a) $\frac{Wl}{8}$ (b) $\frac{Wl^2}{4}$ (c) $\frac{Wl}{4}$ (d) $\frac{Wl}{2}$.

9. The shear force diagram for a simply supported beam carrying a *udl* over its entire span is shown as

(a)


(b)


(c)


(d)


10. The maximum bending moment for a simply supported beam subjected to a *udl* of intensity *w*/unit length over its entire span is given as

(a) $\frac{wl^2}{6}$ (b) $\frac{wl^3}{24}$ (c) $\frac{wl^2}{8}$ (d) $\frac{wl^2}{12}$.

11. The reactions at the two supports of a simply supported beam carrying a *udl* of intensity *w*/unit length over its entire span are given as

(a) $\frac{w}{2}, \frac{w}{2}$ (b) $\frac{wl}{4}, \frac{wl}{4}$ (c) $wl, \frac{wl}{2}$ (d) $\frac{wl}{2}, \frac{wl}{2}$.

12. The load and shear force relationship is

(a) $d^2F = wdx^2$ (b) $dw = Fdx$ (c) $dF = wdx$ (d) $F = wdx$.

13. The shear force and bending moment relationship is

(a) $F = \frac{d^2M}{dx^2}$ (b) $M = \frac{d^2F}{dx^2}$ (c) $M = \frac{dF}{dx}$ (d) $F = \frac{dF}{dx}$.

14. The maximum bending moment for a simply supported beam carrying a *udl* which varies from zero at each end to *w* per unit length at its mid-point is given as

(a) $\frac{wl^2}{8}$ (b) $\frac{wl^3}{12}$ (c) $\frac{wl^2}{12}$ (d) $\frac{wl^2}{24}$.

15. The reactions at the two supports for the beam in Question No. 14 are given as

(*a*) $\frac{wl}{4}, \frac{wl}{2}$ (*b*) $\frac{wl}{4}, \frac{wl}{4}$ (*c*) $\frac{wl}{8}, \frac{wl}{8}$ (*d*) $\frac{wl}{8}, \frac{wl}{4}$.

16. The maximum bending moment for a simply supported beam carrying a *udl* which varies from zero at one end to *w* per unit length at other end is given as

(*a*) $\frac{wl^2}{24}$ (*b*) $\frac{wl^2}{8}$ (*c*) $\frac{wl^2}{9\sqrt{3}}$ (*d*) $\frac{wl^2}{8\sqrt{3}}$.

ANSWERS

1. (*a*) **2.** (*c*) **3.** (*c*) **4.** (*d*) **5.** (*b*) **6.** (*c*) **7.** (*b*) **8.** (*c*) **9.** (*b*)

10. (*c*) **11.** (*d*) **12.** (*c*) **13.** (*d*) **14.** (*c*) **15.** (*b*) **16.** (*c*).

EXERCISES

1. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.36. Also indicate the location and magnitude of maximum bending moment. Is there any contraflexure?

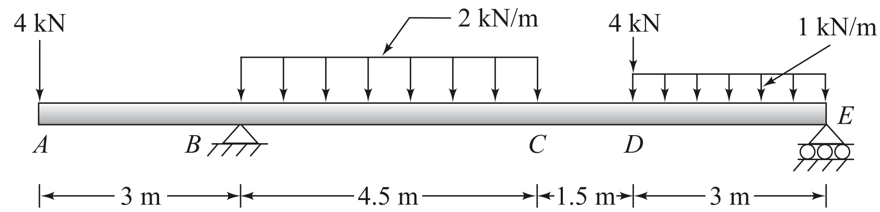


Fig. 4.36

(*Ans.* M_{max} = + 22.74 kN·m at 3 m from E; Contraflexure exists at 4.09 m from A).

2. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.37.

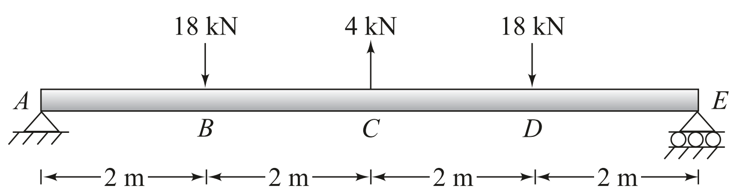


Fig. 4.37

(*Ans.* $M_C = M_{max}$ = + 28 kN·m).

3. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.38. Also determine the location and magnitude of the maximum bending moment.

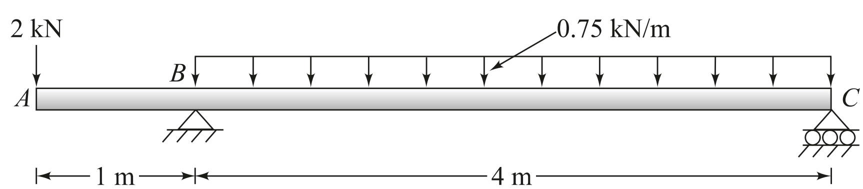


Fig. 4.38

(*Ans.* M = – 2 kN.m at B; M_{max} = + 0.667 kN·m, 1.333 m left of C).

4. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.39.

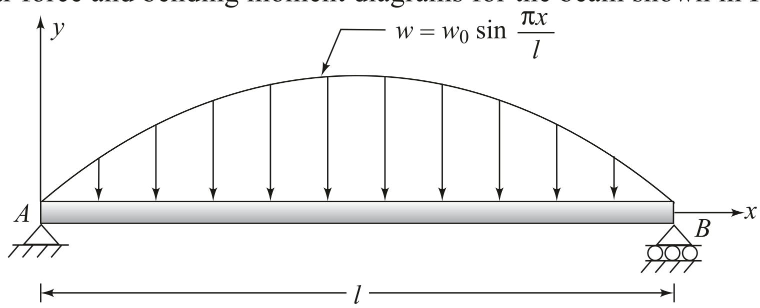


Fig. 4.39

$$\left(Ans.\ F = w_o \left(\frac{l}{\pi}\right) \cos\left(\frac{\pi x}{l}\right);\ M = w_o \left(\frac{l}{\pi}\right)^2 \sin\left(\frac{\pi x}{l}\right);\ M_{\max} = w_o \left(\frac{l}{\pi}\right)^2,\ \text{at } x = \frac{l}{2}\right).$$

5. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.40.

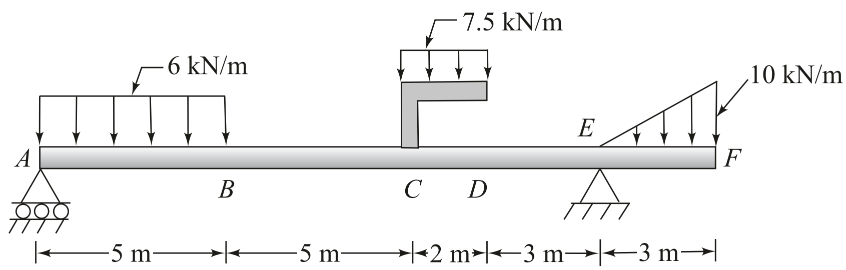


Fig. 4.40

(*Ans.* B.M. just to the left of $C = +\ 85$ kN·m; B.M. just to the right of $C = +\ 100$ kN·m).

6. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.41.

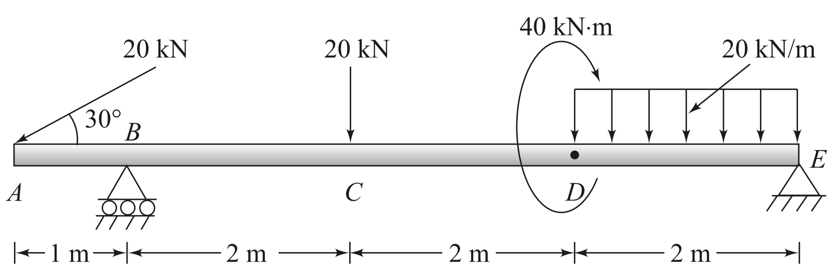


Fig. 4.41

(*Ans.* B.M. just to the right of $D = +\ 50$ kN·m; B.M. just to the left of $D = +\ 10$ kN·m).

7. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.42.

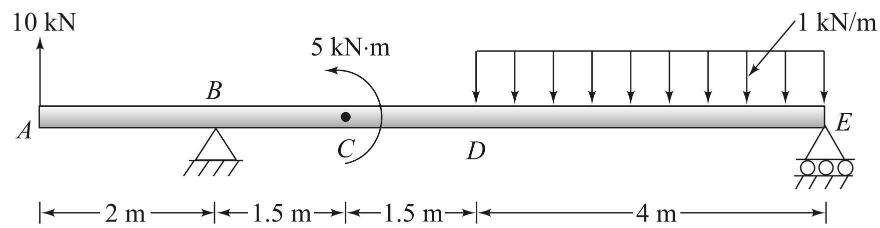


Fig. 4.42

(*Ans.* $M_B = M_{max} = +\,20$ kN·m).

8. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.43.

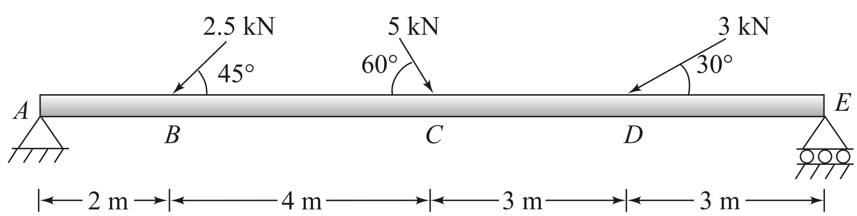


Fig. 4.43

(*Ans.* $M_{max} = 17$ kN·m).

9. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.44.

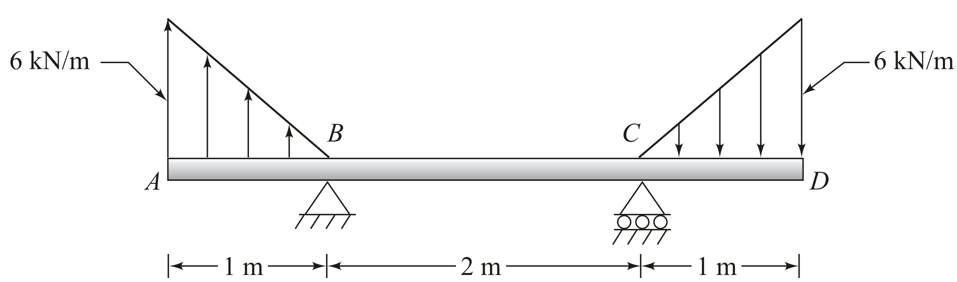


Fig. 4.44

(*Ans.* $M_{max} = \pm\, 2$ kN·m).

10. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.45.

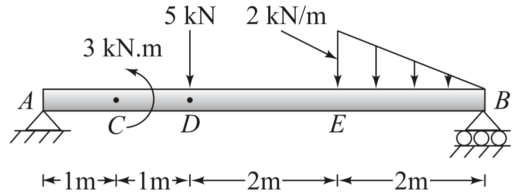


Fig. 4.45

(*Ans.* $M_C = 4.266$ kN·m; $M_D = 5.546$ kN·m; $M_E = 4.107$ kN·m).

11. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.46.

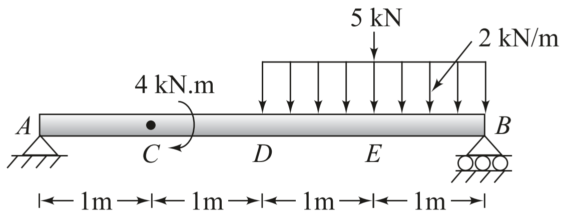


Fig. 4.46

(*Ans.* *B.M.* just to the left of $C = 1.25$ kN·m
B.M. just to the right of $C = 5.25$ kN·m
$M_D = 6.5$ kN·m, $M_E = 6.75$ kN·m).

12. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.47.

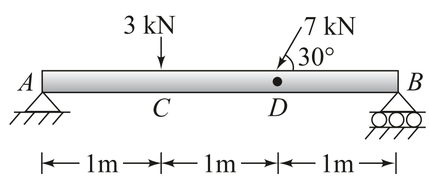


Fig. 4.47

(*Ans.* $M_C = 3.16$ kN·m; $M_D = 3.33$ kN·m).

13. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.48.

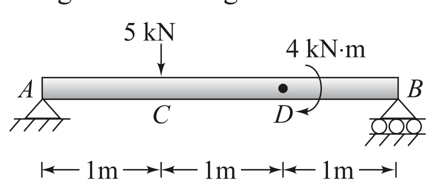


Fig. 4.48

(*Ans.* $M_C = 2$ kN·m, *B.M.* just to the right of $D = 3$ kN·m
B.M. just to the left of $D = -1$ kN·m).

14. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.49.

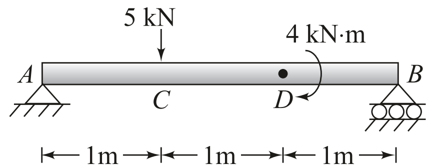


Fig. 4.49

(*Ans.* $F_B = -3$kN, $F_A = 2$ kN
For AC, shear force is constant at 2kN.
For BC, shear force is constant at – 3kN.
$M_A = M_B = 0$
M (left of D) = –1 kN.m
M (right of D) = 3 kN.m = M_{max}
$M_C = +2$kN.m).

15. Draw the shear force and bending moment diagrams for the beam shown in Fig. 4.50.

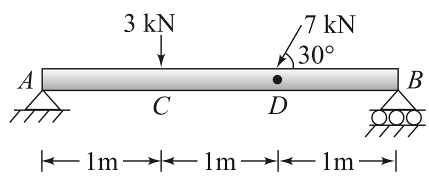


Fig. 4.50

(*Ans.* $F_B = -3.33$ kN
For BD, the shear force is constant at –3.33 kN.
F (left of D) = 0.17 kN
For CD, the shear force is constant at 0.17 kN.
F (left of C) = 3.17 kN
For AC, the shear force is constant at 3.17 kN.
$M_A = M_B = 0$).

❑❑❑

5 Stresses in Beams

D. J. Jourawski
(1821-1891)

D. J. Jourawski, born in 1821, was a Russian bridge and railway engineer. In 1844, only two years after graduating from the Institute of Engineers of Ways of Communication in St. Petersburg, he was assigned the task of designing and constructing a major bridge on the first railway line from Moscow to St. Petersburg. He developed the now widely used approximate theory for shear stresses in beams, also called the shear formula and applied his theory to various shapes of beams.

LEARNING OBJECTIVES

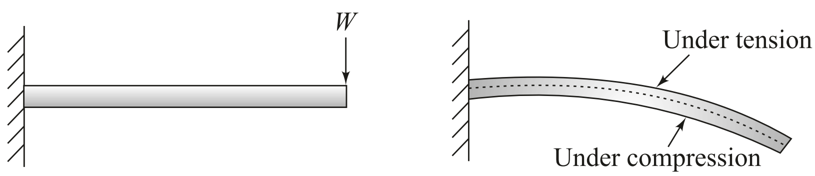
After reading this chapter, you will be able to answer some of the following questions:

- What is meant by pure bending?
- What is an elastic curve?
- What does the section modulus of a cross section indicate?
- What is a composite beam?
- Why does the neutral axis always pass through the centroid?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_5

5.1 PURE BENDING IN BEAMS

When external loads are applied on a beam, it bends in a curve due to bending moments produced. Stresses produced in the beam may be shear and bending. If only bending stresses are considered and shear force is neglected, the beam is said to be under pure bending or simple bending. Bending stresses may be tensile and compressive. If a cantilever beam is loaded as shown in Fig. 5.1(*a*), its upper layer is under tension and the bottom layer is in compression as is shown in Fig. 5.1(*b*). In between the two layers, there is a plane where no stress is acting. Such a plane is known as *neutral plane***.** The magnitude of bending stress at any depth of a beam is proportional to its distance from the neutral plane.



(*a*) Loaded beam (*b*) Stressed beam

Fig. 5.1

5.2 SIMPLE BENDING THEORY

The beam upon loading bends in the form of a curve, known as *elastic curve*. Following assumptions are considered for the simple bending equation:

- The material of the beam is isotropic and homogeneous.
- The transverse sections of the beam, which were plane before bending remain so even after bending.
- The Young's modulus of elasticity of the beam material remains same in tension as well as in compression.
- The stresses produced are within the elastic limits.
- The radius of curvature of the beam is very large as compared to its cross-sectional dimensions.
- There is no resultant push or pull on the cross-section of the beam.
- The loads are applied in the plane of bending.

Let us consider certain portion of a beam being subjected to pure bending as shown in Fig. 5.2, where GH represents the neutral plane. The beam bends in a curve of radius R, known as radius of curvature. The two transverse sections AC and BD meet at point O, known as centre of curvature, making an angle θ between them.

A layer EF of beam at a distance y from the neutral plane is considered.

$$GH = R\theta$$

and
$$EF = (R + y)\,\theta$$

The strain produced in EF is given as

$$\in \;=\; \frac{EF - GH}{GH} = \frac{(R + y)\,\theta - R\theta}{R\theta} = \frac{y}{R} \qquad \text{... (5.1)}$$

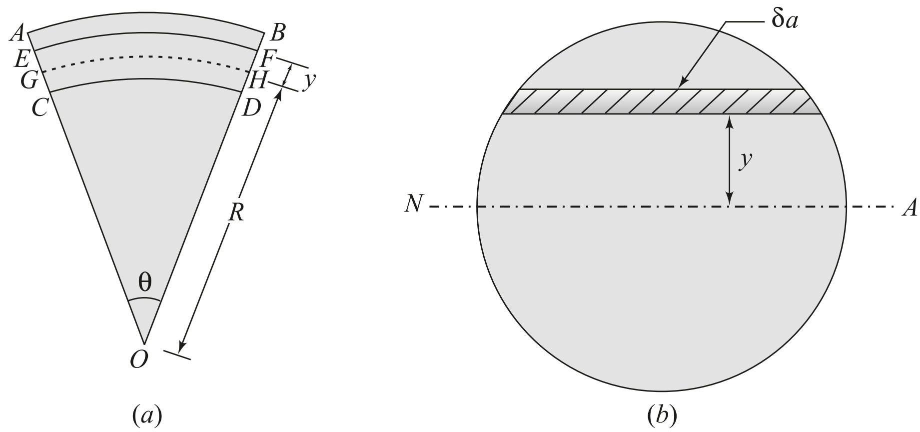


Fig. 5.2

From Hooke's law, we have

$$\frac{\text{Stress}}{\text{Strain}} = E$$

$$\frac{\sigma_b}{\in} = E$$

or $$\in = \frac{\sigma_b}{E} \qquad ...(5.2)$$

where σ_b = Bending stress

E = Modulus of elasticity of the beam material

From equations (5.1) and (5.2), we have

$$\frac{y}{R} = \frac{\sigma_b}{E}$$

or $$\frac{\sigma_b}{y} = \frac{E}{R} \qquad ...(5.3)$$

For a given loading condition, $\frac{E}{R}$ is constant. Hence, the bending stress varies directly proportional to its distance from the neutral plane. Bending stress is maximum at outermost layer of beam, where y is maximum and zero at neutral plane, where y is zero.

The transverse section of beam is shown in Fig. 5.2 (b). An elemental area δa is considered at a distance y from the neutral axis.

Force acting on the elemental area

$$= \sigma_b \times \delta a$$

Moment of this force about the neutral axis (NA)

$$= \sigma_b \times \delta a \times y$$

$$= \frac{E}{R} y^2 \, \delta a \qquad \text{(using equation (5.3))}$$

The moment of resistance is

$$\sum \frac{E}{R} y^2 \, \delta a$$

The moment of resistance is equal to bending moment M.

$$M = \sum \frac{E}{R} y^2 \, \delta a = \frac{E}{R} \sum y^2 \, \delta a$$

where $\sum y^2 \, \delta a$ = Second moment of area about the neutral axis = I

Hence, $$M = \frac{E}{R} \, . \, I$$

or $$\frac{M}{I} = \frac{E}{R} \qquad \text{... (5.4)}$$

Comparing equations (5.3) and (5.4), we have

$$\frac{\sigma_b}{y} = \frac{E}{R} = \frac{M}{I} \qquad \text{... (5.5)}$$

Equation (5.5) is known as simple *bending equation* or *flexure formula*.

5.3 POSITION OF THE NEUTRAL AXIS

Let us consider the cross-section of a beam (Fig. 5.2 (*b*)). For equilibrium of the section, net force acting on it must be zero.

Force acting on the elemental area

$$= \frac{E}{R} \, y \, \delta a$$

Total force on the beam section

$$= \int \frac{E}{R} y \, \delta a = \frac{E}{R} \int y \, \delta a$$

For equilibrium $$\frac{E}{R} \int y \, \delta a = 0$$

or $$\int y \, \delta a = 0$$

since $\dfrac{E}{R}$ is a constant quantity.

In the above equation, $\delta a \neq 0$ but $y = 0$, indicating that the distance from the neutral axis to the centroid of the cross-sectional area must be zero. Hence, the neutral axis always passes through the centre of the area *i.e.*, its centroid.

5.4 SECTION MODULUS

From bending equation, we have

$$\frac{M}{I} = \frac{\sigma_b}{y}$$

The bending stress is maximum, when y is maximum.

$$\frac{M}{\sigma_{b_{max}}} = \frac{I}{y_{max}} = S$$

where $$S = \text{Section modulus} = \frac{I}{y_{max}}$$

The unit of S is m^3, and it is a measure of strength of beam section. The bending stress can be expressed as

$$\sigma_{b_{max}} = \frac{M}{S} \qquad \text{... (5.6)}$$

5.5 COMPOSITE BEAM

A composite (flitched) beam is made of two different materials, for example, steel and concrete or wood and iron. One of the materials of the beam is called reinforcing material and is used to increase the strength of the beam. The basic assumptions of simple bending theory are applicable to composite beams.

A composite beam consisting of a wooden beam and two steel plates is shown in Fig. 5.3. The two materials are rigidly connected and hence they have the same radius of curvature and same strains are produced in both of them.

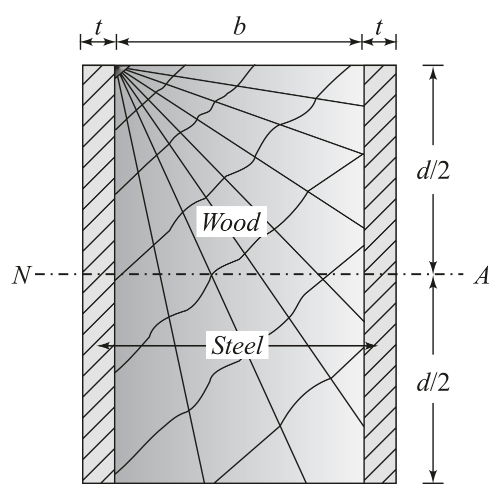


Fig. 5.3

Let σ_s = Stress produced in the steel plate

σ_w = Stress produced in wood

E_s = Modulus of elasticity of steel

E_w = Modulus of elasticity of wood

The stresses produced are considered at a certain fixed distance from the neutral axis.

Now Strain in steel = Strain in wood

$$\frac{\sigma_s}{E_s} = \frac{\sigma_w}{E_w}$$

or $$\frac{\sigma_s}{\sigma_w} = \frac{E_s}{E_w} = m \quad \text{... (5.7)}$$

where m = Modular ratio

The moment of resistance of the steel plate is given as

$$M_s = \sigma_s \frac{I_s}{y} \quad \text{... (5.8)}$$

where I_s = Moment of inertia of the steel plate about the neutral axis (*NA*)

y = distance from neural axis

The moment of resistance of wood is given as

$$M_w = \sigma_w \frac{I_w}{y} \quad \text{... (5.9)}$$

where I_w = Moment of inertia of the wooden section

The moment of resistance of the composite beam is the sum of the two moments of resistance.

$$M = M_s + M_w \quad \text{...(5.10)}$$

$$= \sigma_s \frac{I_s}{y} + \sigma_w \frac{I_w}{y} = \frac{1}{y}[\sigma_s I_s + \sigma_w I_w] \quad \text{...(5.11)}$$

$$= \frac{\sigma_w}{y}[mI_s + I_w] \quad \text{(using equation (5.7)) ... (5.12)}$$

But $$I_w = \frac{bd^3}{12}$$

$$I_s = 2.\frac{td^3}{12} = \frac{td^3}{6}$$

and $$y = \frac{d}{2}$$

Using I_s, I_w and y in equation (5.11), we have

$$M = \frac{\sigma_w d^2}{6}(b + 2mt) \quad \text{... (5.13)}$$

or $$M = \frac{\sigma_s d^2}{6}\left(\frac{b}{m} + 2t\right) \qquad \text{... (5.14)}$$

Hence, the moment of resistance of the composite beam is similar to that of a wooden beam of width ($b + 2mt$) and depth d. The resulting wooden beam is called equivalent beam. Alternatively, it is also similar to that of an equivalent steel beam of width $\left(\frac{b}{m} + 2t\right)$ and depth d.

5.6 BEAMS OF UNIFORM STRENGTH

We have seen that bending moment is maximum at the centre and zero at the two support ends of a simply supported beam. If the beam is designed in such a way that it has uniform bending stress over the entire span so that it has uniform strength throughout its length, then the resulting beam is called a beam of uniform strength. For that, beam is expected to have minimum cross-section at the two supports and gradually increasing towards the centre of the beam, thus preventing wastage of material. The change in cross-section is made effective by

(*a*) Keeping width of beam constant, but reducing its depth

(*b*) Keeping depth of beam constant, but reducing its width

(*a*) **When width is constant and depth is varying.**

Consider a simply supported beam of length l carrying a point load W at the centre (Fig. 5.4). The reactions at the two supports are found as

$$R_A = R_B = \frac{W}{2}$$

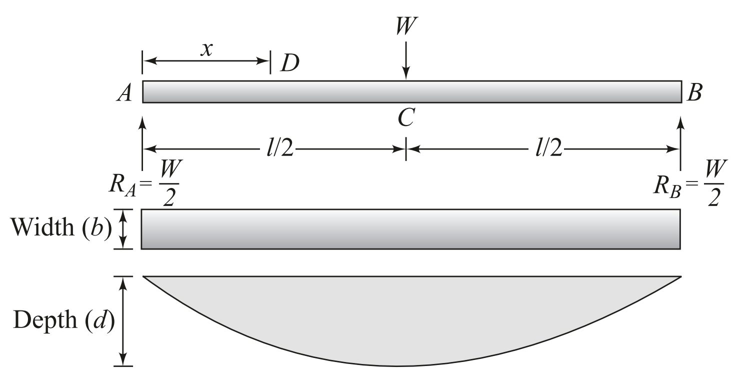


Fig. 5.4

Let D be a point at a distance x from A, where depth is required to be found.

The bending moment at D is given as

$$M_D = \frac{W}{2}x$$

The moment of resistance is $\sigma \times \frac{bd_D^2}{6}$. (using bending equation) where d_D represents the diameter of the beam at D.

Equating the two moments, we have

$$\sigma \times \frac{bd_D^2}{6} = \frac{W}{2}x$$

or $$d_D = \sqrt{\frac{3Wx}{\sigma b}} \qquad \text{... (5.15)}$$

Since W, σ and b are constants for a given beam, hence depth at any section is proportional to $\sqrt{x}$, thereby indicating that the variation of depth is parabolic between points A and B.

The depth at C (where $x = \frac{l}{2}$) is

$$d = \sqrt{\frac{3Wl}{2\sigma b}} \qquad \text{... (5.16)}$$

(*b*) **When depth is constant and width is varying.**

Consider the same loading condition. Depth and width are shown in Fig. 5.5. The bending moment at D is again $\frac{W}{2}x$.

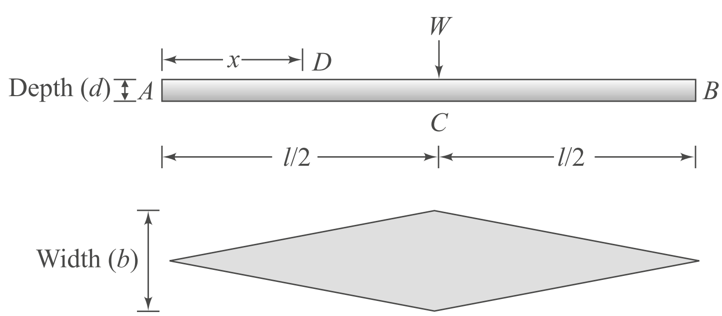


Fig. 5.5

The moment of resistance is $\sigma \times \frac{b_D d^2}{6}$.

On equating the two moments, we have

$$\sigma \times \frac{b_D d^2}{6} = \frac{W}{2} \times x$$

or $$b_D = \frac{3Wx}{\sigma d^2} \qquad \text{... (5.17)}$$

W, σ, and d are constants for the beam, hence width at any section is proportional to x.

The width at C $\left(\text{where } x = \dfrac{l}{2}\right)$ is

$$b = \frac{3Wl}{2\sigma d^2} \qquad \text{...(5.18)}$$

Example 5.1

A 5 m cantilever beam of cross-section 150 mm × 300 mm weighing 0.05 kN/m carries an upward concentrated load of 30 kN at its free end (Fig. 5.6). Determine the maximum bending stress at a section 2m from the free end.

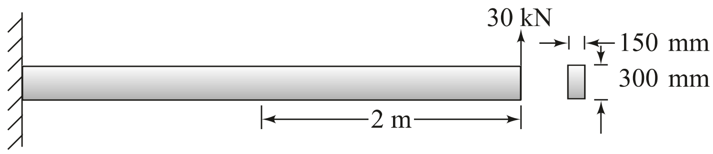


Fig. 5.6

Solution: The bending moment at the section (2 m from the free end) is given as

$$M = 30 \times 2 - \left(0.50 \times 2 \times \frac{2}{2}\right)$$

$$= 59 \text{ kN.m} = 59 \times 10^6 \text{ N.mm}$$

The moment of inertia of the beam section about the neutral axis (NA) is given as

$$I = \frac{1}{12} \times 150 \times 300^3 = 3.375 \times 10^8 \text{ mm}^4$$

Using bending equation, we have

$$\frac{\sigma_b}{y} = \frac{M}{I}$$

or

$$\sigma_b = \frac{M}{I} \cdot y$$

$$= \frac{59 \times 10^6}{3.375 \times 10^8} \cdot \left(\frac{300}{2}\right) = 26.22 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

The top layer of the beam is under compression and the bottom layer is in tension. Both layers are subjected to an equal stress of 26.22 N/mm^2.

Example 5.2

A 5m long steel beam having an I-section is simply supported at its ends (Fig. 5.7). The tensile stress in beam does not exceed 25 MPa. Determine the safe uniformly distributed load to be placed on the entire span of the beam.

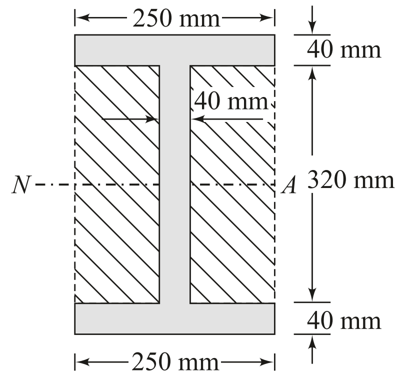


Fig. 5.7

Solution: The moment of inertia of the beam section about the neutral axis is given as

$$I = \text{M.I. of rectangle } 250 \text{ mm} \times 400 \text{ mm} - 2 \text{ (M.I. of rectangle } 105 \text{ mm} \times 320 \text{ mm)}$$

$$= \frac{1}{12} \times 250 \times 400^3 - 2 \times \frac{1}{12} \times 105 \times 320^3 \text{ mm}^4$$

$$= 7.598 \times 10^{-4} \text{ m}^4$$

If w kN/m be the uniformly distributed load to be placed on the entire beam, then the bending moment under this condition is

$$M = \frac{wl^2}{8} = \frac{w \times 25}{8} \text{ kN·m}$$

Using bending equation, we have

$$\frac{M}{I} = \frac{\sigma_b}{y}$$

$$\frac{w \times 25}{8} \times \frac{1}{7.598 \times 10^{-4}} = \frac{25 \times 10^6}{10^3 \times \left(\frac{400}{2} \times 10^{-3}\right)}$$

Solving for w, we get $w = 30.392$ kN/m **Ans.**

Example 5.3

A simple steel beam of 4 m span carries a uniform load of 6 kN/m over its entire span and a point load 2 kN at its centre. If the permissible stress does not exceed 100 MPa, find the cross-section of the beam assuming depth to be twice of breadth.

Solution: Given,

Length of the beam, $l = 4$ m

Uniform load, $w = 6$ kN/m $= 6 \times 10^3$ N/m

Point load, $W = 2$ N $= 2$ kN $= 2 \times 10^3$ N

Bending stress, $\sigma_b = 100 \times 10^6$ N/m^2

Let Width of the beam $= b$

Depth of the beam $= d$

$d = 2b$ (Given)

The maximum bending moment due to *udl* and point load is given as

$$M = \frac{wl^2}{8} + \frac{Wl}{4}$$

$$= \frac{6\times10^3\times4^2}{8} + \frac{2\times10^3\times4}{4} = 14000 \text{ N·m}$$

The moment of inertia of the beam section about the neutral axis is calculated as

$$I = \frac{1}{12}\,bd^3 = \frac{1}{12}\cdot b\cdot(2b)^3 = \frac{2}{3}\,b^4$$

$$y = \frac{d}{2} = \frac{2b}{2} = b$$

Using bending equation, we have

$$\frac{M}{I} = \frac{\sigma_b}{y}$$

$$\frac{14000}{\frac{2}{3}b^4} = \frac{100\times10^6}{b}$$

Solving for b, we get $b = 0.0594$ m $= 59.4$ mm

and $d = 2b = 118.8$ mm

Hence, the cross-section of the beam is 59.4 mm × 118.8 mm. **Ans.**

Example 5.4

A simply supported beam of cross-section 50 mm by 50 mm having a length of 800 mm is capable of carrying a point load of 3 kN at its centre. This beam is required to be replaced by a cantilever beam of the same material having cross-section 50 mm by 100 mm and length 1500 mm. Determine the maximum point load that can be placed at the free end of the cantilever.

Solution:

For simply supported beam

Load on the beam, $W_1 = 3$ kN

Length of the beam, $l_1 = 800$ mm

Width of the beam, $b = 50$ mm

Depth of the beam, $d = 50$ mm

The distance of the neutral axis from the outermost layers is

$$y_1 = \frac{50}{2} = 25 \text{ mm}$$

The moment of inertia of the beam section about the neutral axis is obtained as

$$I_1 = \frac{1}{12} \times 50 \times 50^3 = 520833.33 \text{ mm}^4$$

The bending moment is given as

$$M_1 = \frac{W_1 l_1}{4} = \frac{3 \times 800}{4} = 600 \text{ kN·mm}$$

Using bending equation, we have

$$\frac{\sigma_b}{y_1} = \frac{M_1}{I_1}$$

or
$$\sigma_b = \frac{M_1}{I_1} . y_1 = \frac{600 \times 25}{520833.33} = 0.0288 \text{ kN/mm}^2$$

For cantilever beam

Load on the beam, $W_2 = ?$

Length of the beam, $l_2 = 1500$ mm

Width of the beam, $b = 50$ mm

Depth of the beam, $d = 100$ mm

The distance of the neutral axis from the outermost layer is

$$y_2 = \frac{100}{2} = 50 \text{ mm}$$

The moment of inertia of beam section about the neutral axis is

$$I_2 = \frac{1}{12} \times 50 \times 100^3 = 4166666.7 \text{ mm}^4$$

The bending moment is found as

$$M_2 = W_2 l_2 = 1500 \ W_2 \text{ kN·mm}$$

Since both beams are made of the same material, hence equal bending stresses will be developed in both of them.

Again using bending equation, we have

$$\frac{\sigma_b}{y_2} = \frac{M_2}{I_2}$$

$$\frac{0.0288}{50} = \frac{1500\,W_2}{4166666.7}$$

or $$W_2 = 1.6 \text{ kN}$$ **Ans.**

Example 5.5

A simple beam of length 5 m carries two types of loads : a *udl* of 6 kN/m is acting over the entire span and a point load of 2 kN at a distance 2 m from the left support. Cross-section of the beam is shown in Fig. 5.8. Calculate the maximum bending stress at a distance 3.5 m from the left support.

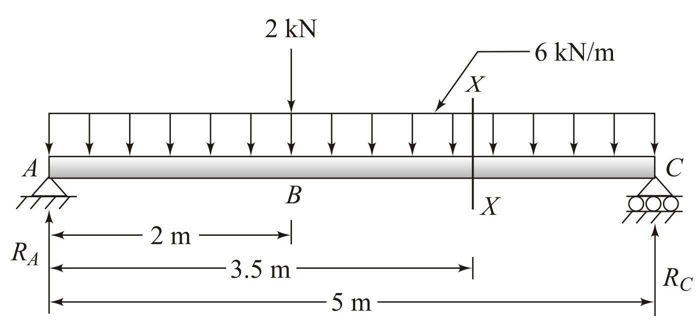
Solution: **Support reactions at *A* and *C***

Taking moments of the forces about *A*, we get

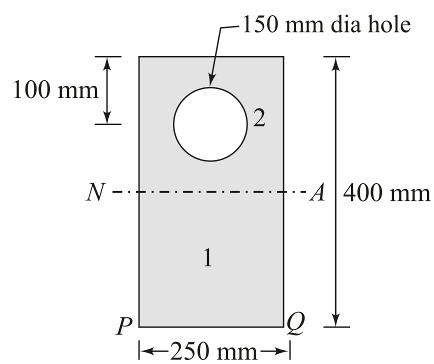
$$R_C \times 5 = 2 \times 2 + \left(6 \times 5 \times \frac{5}{2}\right)$$

or $$R_C = 15.8 \text{ kN } (\uparrow)$$

But $$R_A + R_C = 2 + (6 \times 5) = 32 \text{ kN}$$



(*a*) Loaded beam



(*b*) Beam cross-section

Fig. 5.8

or $$R_A = 32 - R_C = 32 - 15.8$$
$$= 16.2 \text{ kN } (\uparrow)$$

Bending moment at the desired section *XX*

The bending moment at the section is given as

$$M = R_A \times 3.5 - 2 \times (3.5 - 2) - 6 \times 3.5 \times \frac{3.5}{2} = 16.95 \text{ kN·m}$$

Calculation of moment of inertia

Component	*Area 'a' of the component (mm^2)*	*Distance 'y' of the centroid of component from PQ (mm)*	*ay (mm^3)*
Rectangle (1)	$400 \times 250 = 100000$	$\frac{400}{2} = 200$	20000000
Circle (2)	$\frac{\pi}{4} \times 150^2 = 17671.46(-)$	$(400 - 100) = 300$	5301437.6 (–)
Total	$\sum a = 82328.54$	—	$\sum ay = 14698562$

The distance of the centroid of the whole section or the neutral axis from PQ is given as

$$\bar{y} = \frac{\sum ay}{\sum a}$$

$$= \frac{14698562}{82328.54} = 178.5 \text{ mm}$$

Hence, $y_t = 178.5$ mm

$$y_c = (400 - 178.5) \text{ mm} = 221.5 \text{ mm}$$

Now $I_{NA} = I_{\text{rectangle}} - I_{\text{hole}}$

$$= \left[\left\{\frac{1}{12} \times 250 \times 400^3 + 250 \times 400 \times (200 - 178.5)^2\right\}\right.$$

$$\left. - \left\{\frac{\pi}{64} \times 150^4 + \frac{\pi}{4} \times 150^2 \times (221.5 - 100)^2\right\}\right] \times 10^{-12} \text{ m}^4$$

$$= 1.093 \times 10^{-3} \text{ m}^4$$

Bending stress at *XX*

Using bending equation, we have

$$\frac{\sigma_b}{y} = \frac{M}{I}$$

or $\sigma_b = \frac{M}{I} \cdot y_c$ (since $y_c > y_t$)

$$= \frac{16.95 \times 10^3}{1.093 \times 10^{-3}} \times 221.5 \times 10^{-3} \text{ N/m}^2 = 3432343.1 \text{ N/m}^2$$

$$= 3.43 \text{ MPa}$$ **Ans.**

Example 5.6

Compare the weights of two equally strong beams of circular sections made of same material, one being of solid section and the other of hollow section with inside diameter being $\frac{2}{3}$ of outside diameter.

Solution: Let diameter of the solid cicular beam $= D$

Inside diameter of the hollow circular beam $= D_i$

Ouside diameter of the hollow circular beam $= D_0$

Given, $D_i = \frac{2}{3} D_0$

The section modulus of the solid beam is given as

$$S_S = \frac{1}{y} = \frac{\pi}{64} \frac{D^4}{\left(\frac{D}{2}\right)} = \frac{\pi}{32} D^3$$

The section modulus of the hollow beam is given as

$$S_H = \frac{\pi}{64} (D_0^4 - D_i^4) / \frac{D_0}{2} = \frac{\pi D_0^3}{32}\left[1 - \left(\frac{D_i}{D_o}\right)^4\right]$$

$$= \frac{\pi D_0^3}{32}\left[1 - \left(\frac{2}{3}\right)^4\right] = \frac{65\pi D_0^3}{32 \times 81}$$

Since the beams are of equal strength, hence their section modului are equal.

$$S_S = S_H$$

$$\frac{\pi D^3}{32} = \frac{65\pi D_0^3}{32 \times 81}$$

$$\left(\frac{D}{D_0}\right)^3 = \frac{65}{81}$$

or $$\frac{D}{D_0} = 0.9292$$

Being same material for the two beams, their bending stresses are equal. In other words, ratio of their weights is equal to the ratio of their cross-sectional areas.

If W_S = Weight of the solid beam

W_H = Weight of the hollow beam

Then $$\frac{W_S}{W_H} = \frac{\pi}{4} D^2 \Big/ \frac{\pi}{4} (D_0^2 - D_i^2)$$

$$= \frac{D^2}{(D_o^2 - D_i^2)} = \frac{D^2}{D_o^2\left[1-\left(\frac{D_i}{D_o}\right)^2\right]}$$

Using ratios $\frac{D}{D_0}$ and $\frac{D_i}{D_0}$, we get

$$\frac{W_S}{W_H} = \frac{(0.9292)^2}{\left[1-\left(\frac{2}{3}\right)^2\right]} = 1.55$$

$$W_S = 1.55\ W_H$$ **Ans.**

Example 5.7

A wire of diameter d is wound round a cylinder of diameter D. Determine the bending stress produced on the cross-section of the wire. Hence or otherwise, find the minimum radius to which a 10 mm diameter circular and of high tensile steel can be bent without undergoing permanent deformation. Take yield stress = 1700 N/mm^2 and E = 200 GPa. What is the magnitude of bending moment necessary for this?

Solution: Using bending equation, we have

$$\frac{\sigma_b}{y} = \frac{E}{R}$$

$$\frac{\sigma_b}{(d/2)} = \frac{E}{(D/2)} \qquad \left(y = \frac{d}{2} \text{ and } R = \frac{D}{2}\right)$$

$$\sigma_b = E \times \frac{d}{D}$$ **Ans.**

or

$$D = \frac{E \times d}{\sigma_b}$$

$$= \frac{200 \times 10^9 \times 10 \times 10^{-3}}{1700 \times 10^6}\text{ m} \qquad \left(\begin{array}{l}\sigma_b = 1700\text{N/mm}^2 \\ d = 10\,\text{mm}\end{array}\right)$$

$$= 1.17647\text{ m} = 1176.47\text{ mm}$$

Now

$$R = \frac{D}{2} = \frac{1176.47}{2} = 588.23\text{ mm}$$ **Ans.**

Again using bending equation, we have

$$M = \frac{\sigma_b}{(d/2)} \times I = \frac{2\sigma_b}{d} \times \frac{\pi}{64} d^4$$

$$= \frac{\pi d^3}{32} \times \sigma_b = \frac{\pi \times (10 \times 10^{-3})^3 \times 1700 \times 10^6}{32} = 166.9\text{ N.m}$$ **Ans.**

Example 5.8

The shear force diagram for a rectangular cross-section beam *AD* is shown in Fig. 5.9. Width of the beam is 100 mm and depth is 250 mm. Determine the maximum bending stress in the beam.

Solution: Given,

Width of the beam, $b = 100$ mm

Depth of the beam, $d = 250$ mm

Fig. 5.9 shows that shear force at *D* is zero and it gradually increases to 5 kN (15 kN – 10 kN) at *C* indicating that there is a *udl* of intensity 5 kN/m between *C* and *D*. At *C*, the shear force increases further by 5 kN. There is no load on *BC*, hence the SFD between *B* and *C* remains constant. At *B*, the shear force further increses by 5 kN so that between *A* and *B*, the shear force remains constant at 15 kN. The loaded beam is shown in Fig. 5.10.

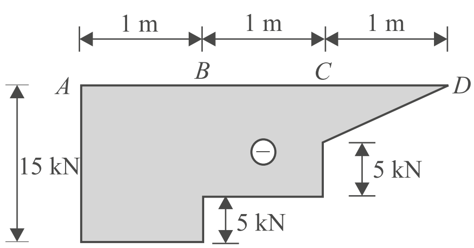


Fig. 5.9 Shear Force Diagram (SFD).

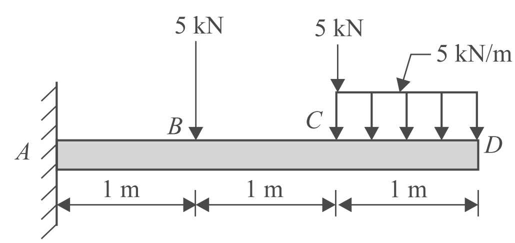


Fig. 5.10 Loaded Beam.

The bending moments at different points are given as

$$M_D = 0$$

$$M_C = -5 \times 1 \times \frac{1}{2}$$

$$= -2.5 \text{ kN.m}$$

$$M_B = \left[-5 \times 1 - 5 \times 1 \times \left(1 + \frac{1}{2}\right)\right]$$

$$= -12.5 \text{ kN.m}$$

$$M_A = \left[-5 \times 1 - 5 \times (1+1) - 5 \times 1\left(1 + 1 + \frac{1}{2}\right)\right]$$

$$= -27.5 \text{ kN.m} = M_{\text{max}}$$

The moment of inertia of the cross-section of the beam is found as

$$I = \frac{1}{12} bd^3 = \frac{1}{12} \times (100 \times 10^{-3}) \times (250 \times 10^{-3})^3 \text{ m}^4$$

$$= 1.302 \times 10^{-4} \text{ m}^4$$

The maximum bending stress is given by

$$\sigma_b = \frac{M}{I} \times \frac{d}{2} \qquad (y = d/2)$$

$$= \frac{-27.5 \times 10^3}{1.302 \times 10^{-4}} \times \frac{250 \times 10^{-3}}{2} \times \frac{1}{10^6} \text{ MPa}$$

$$= -26.4 \text{ MPa}$$

$$= 26.4 \text{ MPa (Compressive)}$$ **Ans.**

Example 5.9

A composite beam is made by placing two steel plates, 10 mm thick and 200 mm deep, one each on both sides of a wooden section 100 mm wide and 200 mm deep. Determine moment of resistance of the section of beam. Given, $\frac{E_s}{E_w}$ = 20. The stress in the wood should not exceed 7.5 N/mm^2.

Solution: The composite beam is shown in Fig. 5.11.

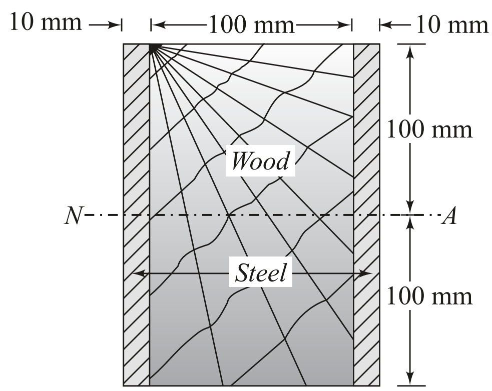


Fig. 5.11

Given, Thickness of the steel plate, t = 10 mm

Depth of the steel plate and the wooden section, d = 200 mm

Width of the wooden section, b = 100 mm

Modular ratio, $m = \frac{\sigma_s}{\sigma_w} = \frac{E_s}{E_w} = 20$

Stress in wood, $\sigma_w = 7.5 \text{ N/mm}^2$

Stress in the steel plate, $\sigma_s = m\sigma_w$

$= 20 \times 7.5 = 150 \text{ N/mm}^2$

The moment of resistance of the steel plate is given as

$$M_s = \sigma_s \cdot \frac{I_s}{y} \qquad \text{(using equation (5.8))}$$

$$= \sigma_s\left(2 \cdot \frac{td^3}{12}\right) \cdot \frac{1}{(d/2)}$$

$$= \frac{\sigma_s\, td^2}{3}$$

$$= \frac{150 \times 10 \times (200)^2}{3}$$

$$= 20000 \text{ kN·mm}$$

The moment of resistance of the wooden section is given as

$$M_w = \sigma_w \frac{I_w}{y} \qquad \text{(using equation (5.9))}$$

$$= \sigma_w \frac{bd^3}{12} \cdot \frac{1}{(d/2)}$$

$$= \frac{\sigma_w\, bd^2}{6} = \frac{7.5 \times 100 \times (200)^2}{6}$$

$$= 5000 \text{ kN·mm}$$

Hence, the moment of resistance of the composite beam is given as

$$M = M_s + M_w$$

$$= (20000 + 5000) \text{ kN.mm}$$

$$= 25000 \text{ kN·mm}$$ **Ans.**

Example 5.10

A flitched beam consists of two timber joists each 80 mm wide × 250 mm deep, is reinforced by a steel plate 25 mm wide and 180 mm deep, placed symmetrically in the groove made in the centre and fixed firmly (Fig. 5.12). Find the moment of resistance of beam. The maximum bending stress in timber is not to exceed 8.5 N/mm^2. Take $E_s = 2 \times 10^5$ N/mm^2 and $E_w = 10^4$ N/mm^2.

Solution:

For timber joist

Width, $b = 80$ mm

Depth, $d = 250$ mm

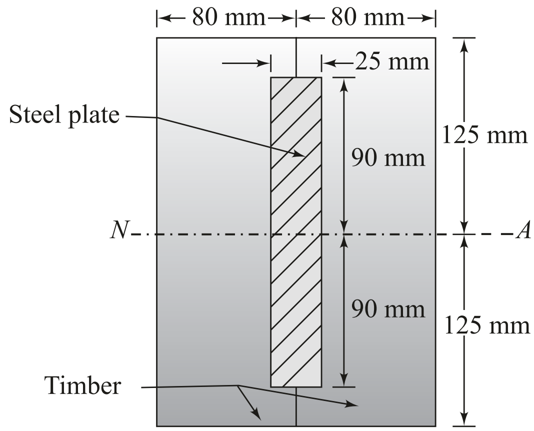


Fig. 5.12

Maximum bending stress,

$$\sigma_w = 8.5 \text{ N/mm}^2$$

For steel plate

Width, $b = 25$ mm

Depth, $d = 180$ mm

and $m = \dfrac{E_s}{E_w} = 20$

The stress in the timber at a distance 90 mm from neutral axis (*NA*) is given as

$$\sigma'_w = \frac{8.5}{125} \times 90 = 6.12 \text{ N/mm}^2$$

Strain in the timber at 90 mm from *NA* is

$$\in_w = \frac{\sigma'_w}{E_w} = \frac{6.12}{10^4} = 6.12 \times 10^{-4}$$

At 90 mm from neutral axis, strains in timber as well as in steel are same.

Maximum stress in steel is given as

$$\sigma_s = E_s \times \in_s$$

$$= 2 \times 10^5 \times 6.12 \times 10^{-4} = 122.4 \text{ N/mm}^2 \qquad (\in_s = \in_w)$$

The moment of resistance of timber is given as

$$M_w = \sigma_w \cdot \frac{I_w}{y} \qquad \text{(using equation (5.9))}$$

$$= 8.5 \times \frac{1}{125}\left(\frac{1}{12} \times 160 \times 250^3 - \frac{1}{12} \times 25 \times 180^3\right)$$

$$= 13340467 \text{ N·mm}$$

The moment of resistance of steel is given as

$$M_s = \sigma_s \cdot \frac{I_s}{y} \qquad \text{(using equation (5.8))}$$

$$= 122.4 \times \frac{1}{90}\left(\frac{1}{12} \times 25 \times 180^3\right) = 16524000 \text{ N·mm}$$

The total moment of resistance of the flitched beam is given as

$$M = M_w + M_s$$

$$= (13340467 + 16524000) \text{ N·mm}$$

$$= 29864467 \text{ N·mm}$$ **Ans.**

5.7 SHEAR STRESSES IN BEAMS

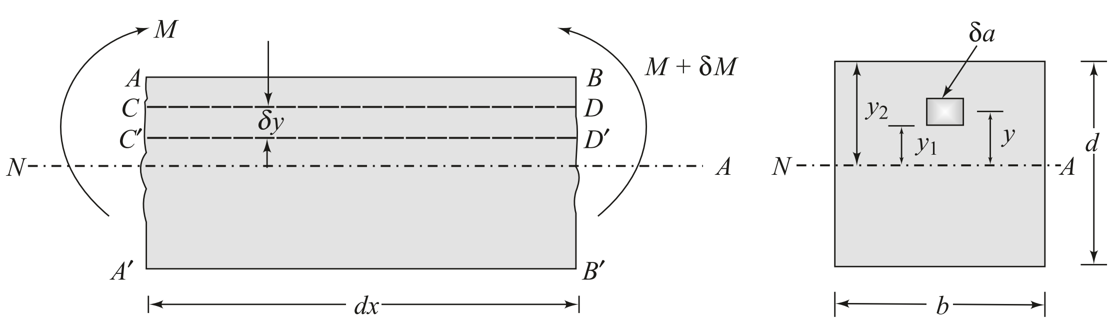
In the derivation of bending equation, the beam is assumed to be subjected to pure bending and no shear force is considered. As a result, only longitudinal stresses, called bending stresses are produced in such cases. Now if beam is loaded in the transverse direction, as is the case most common in practice, shear stresses and bending moments both are produced. Shear stresses are due to shear force acting along depth of the transverse cross-section of beam. For equilibrium of the cross-section, complementary shear stresses of equal intensity are considered on longitudinal planes parallel to the axis of beam. Calculation of shear stresses are important although shear deformations are relatively small and bending stresses are dominating in majority of the cases.

5.8 SHEAR STRESS DISTRIBUTION (GENERAL CASE)

Let us consider a small length $AB = dx$ of the beam (Fig. 5.13).

M = Bending moment at left side, that is, at AA'

$M + \delta M$ = Bending moment at right side, that is, at BB'



(*a*) Longitudinal section (*b*) Cross-section

Fig. 5.13

Consider an elementary area δa at a height y from the neutral axis.

From bending equation, we have

$$\sigma_{AA'} = \frac{M}{I} y \qquad \text{(bending stress at } AA'\text{)}$$

and

$$\sigma_{BB'} = \frac{(M + \delta M)}{I} y \qquad \text{(bending stress at } BB'\text{)}$$

The force acting on face CC' of the elemental area at A is

$$F_1 = \frac{M}{I} y\, \delta a$$

The force acting on face DD' of the elemental area at B is

$$F_2 = \frac{(M + \delta M)}{I} y\, \delta a$$

where $\delta a = b.\, \delta y$

Net unbalancing force (longitudinal) acting on the elementary area between A and B

$$= F_2 - F_1$$

$$= \frac{(M + \delta M)}{I} y\, \delta a - \frac{M}{I} y\, \delta a = \frac{\delta M}{I} y\, \delta a$$

Total unbalanced force acting on the area between A and B

$$= \int_{y_1}^{y_2} \frac{\delta M}{I} y\, \delta a$$

This force is balanced by shear force (longitudinal) acting on the area between A and B and is equal to

$$\tau.b.dx$$

where τ = Shear stress at the section

Equating two forces, we have

$$\tau\, .\, b\, .\, dx\, . = \int_{y_1}^{y_2} \frac{\delta M}{I} y\, \delta a$$

or

$$\tau = \frac{dM}{I\, .\, bdx} \int_{y_1}^{y_2} y\, \delta a = \frac{1}{Ib}\left(\frac{dM}{dx}\right) \int_{y_1}^{y_2} y\, \delta a$$

$$= \frac{1}{Ib} V \int_{y_1}^{y_2} y\, \delta a = \frac{VQ}{Ib} = \frac{V}{Ib} A\bar{y} \qquad \text{... (5.19)}$$

where V = Vertical shear force at the section

$\int_{y_1}^{y_2} y\,\delta a$ = First moment of area of the cross-section above $C'D'$ about the neutral axis

$= Q$

$= A\bar{y}$

$\bar{y}$ = Distance of $C.G.$ of area above the plane $C'D'$ where the shear stress is τ

For a given section of beam, I and b are constants. Hence, the variation of shear stress depends upon $A\bar{y}$. The shear stress is maximum for the maximum value of $A\bar{y}$ and is minimum for the minimum value of $A\bar{y}$. Accordingly, the shear stress is maximum at the neutral axis and zero at extreme faces of beam.

5.9 SHEAR STRESS DISTRIBUTION IN A RECTANGULAR CROSS-SECTION

Refer Fig. 5.14.

Let b = Width of the beam

d = Depth of the beam

τ_{max} = Maximum shear stress

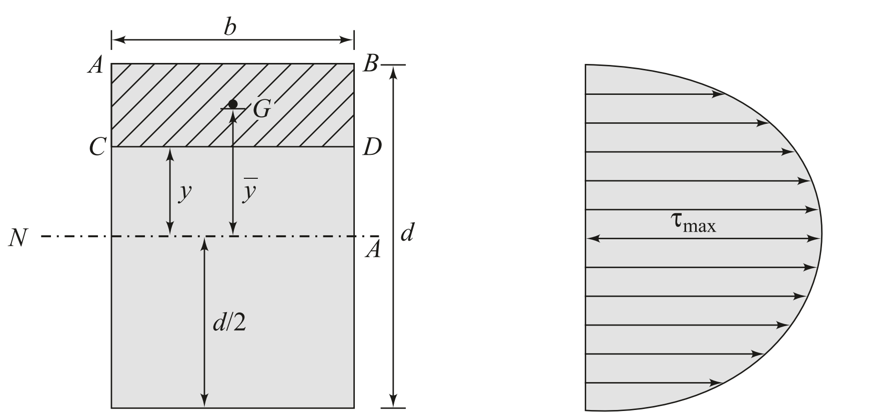
τ = Shear stress at CD

y = Distance of plane at CD from the neutral axis

A = Distance of $C.G.$ of $ABDC$

I = Moment of inertia of cross-section about the neutral axis

$= \dfrac{1}{12} bd^3$



(a) Rectangular corss-section (b) Shear stress variation (parabolic)

Fig. 5.14

$$A = \text{Area } ABDC = \left(\frac{d}{2} - y\right) b$$

$$\bar{y} = y + \frac{1}{2}\left(\frac{d}{2} - y\right) = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence, the moment of area $ABDC$ about the neutral axis is given as

$$A\bar{y} = \left(\frac{d}{2} - y\right) b \times \frac{1}{2}\left(\frac{d}{2} + y\right) = \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Using equation (5.19), we have

$$\tau = \frac{V}{Ib} A\bar{y} = \frac{V}{Ib} \times \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

$$= \frac{V}{2I}\left(\frac{d^2}{4} - y^2\right) \qquad \text{... (5.20)}$$

This is the equation of a parabola, hence the shear stresss variation is parabolic in nature (Fig. 5.14 (*b*)).

For maximum shear stress, put $y = 0$.

Equation (5.20) reduces to an equation giving maximum value of shear stress as

$$\tau_{\max} = \frac{V}{2I} \cdot \frac{d^2}{4}$$

$$= \frac{V}{2 \times \frac{1}{12} bd^3} \times \frac{d^2}{4} = \frac{3V}{2bd} \qquad \text{... (5.21)}$$

For minimum shear stress, put $y = \dfrac{d}{2}$

Equation (5.20) reduces to an equation giving minimum value of shear stress as

$$\tau_{\min} = 0 \qquad \text{... (5.22)}$$

Hence, the shear stress is zero at extreme faces of the cross-section for which $y = \dfrac{d}{2}$.

The average shear stress is given as

$$\tau_{av} = \frac{V}{bd} \qquad \text{... (5.23)}$$

Comparing this equation with equation (5.21), we find

$$\tau_{\max} = \frac{3}{2}\tau_{av} \qquad \text{... (5.24)}$$

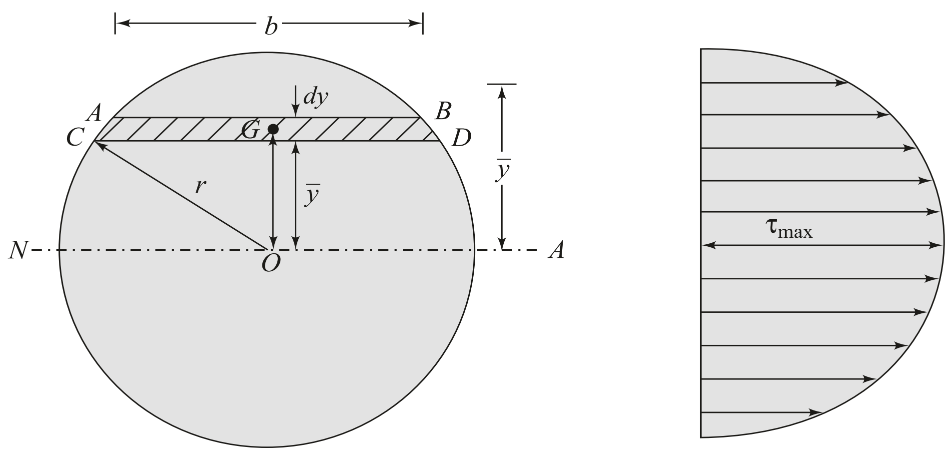
5.10 SHEAR STRESS DISTRIBUTION IN A CIRCULAR CROSS-SECTION

Refer Fig. 5.15.

Let r = Radius of the circular section

I = Moment of inertia of the beam cross-section about the neutral axis

$$= \frac{\pi}{4} r^4 = \frac{\pi}{64} d^4, \ d \text{ being diameter}$$



(*a*) Circular section (*b*) Shear stress variation

Fig. 5.15

Consider an elementary strip *ABDC* of thickness *dy* at a distance *y* from the neutral axis.

b = Width of the strip

$$= 2\sqrt{r^2 - y^2} \qquad ...(5.25)$$

Moment of the strip about the neutral axis

$$= \text{Area of the strip} \times y$$

$$= bdy \times y$$

$$= (2\sqrt{r^2 - y^2} \times dy) \times y = 2y\sqrt{r^2 - y^2}\ dy$$

The moment of the area above *CD* about the neutral axis is $A\bar{y}$, given by

$$A\bar{y} = \int_{y=y}^{y=r} 2y\sqrt{r^2 - y^2}\ dy \qquad ...(5.26)$$

From equation (5.25)

$$b = \sqrt{r^2 - y^2}$$

or $$b^2 = 4(r^2 - y^2)$$

$$= 4r^2 - 4y^2$$

Differentiating both sides w.r.t. y, we have

$$2b\frac{db}{dy} = 0 - 4 \times 2y$$

$$= -8y$$

or $$ydy = -\frac{bdb}{4} \qquad \text{... (5.27)}$$

Making necessary changes in the limits of integration, we have

when $y = y$, $b = b$

$y = r$, $b = 0$

Using equations (5.25), (5.27) and changed limits of integration, equation (5.26) changes to

$$A\bar{y} = \int_b^0 \frac{b^2\,db}{4} = -\frac{1}{4}\left(\frac{b^3}{3}\right)_b^0 = \frac{1}{12}b^3 \qquad \text{... (5.28)}$$

Using equation (5.19), we have

$$\tau = \frac{V}{Ib}A\bar{y} = \frac{V}{Ib} \times \frac{1}{12}b^3 = \frac{Vb^2}{12I}$$

$$= \frac{V}{12I}4(r^2 - y^2) = \frac{V}{3I}(r^2 - y^2) \qquad \text{... (5.29)}$$

This is an equation of parabola suggesting that shear stress variation is parabolic in nature. The shear stress is maximum, when $y = 0$ (at the neutral axis).

$$\tau_{\max} = \frac{V}{3I} \times r^2 \qquad \text{... (5.30)}$$

The shear stress is minimum, when $y = r$.

$$\tau_{\min} = 0 \qquad \text{... (5.31)}$$

On substituting I in equation (5.30), we find

$$\tau_{\max} = \frac{V}{3 \times \frac{\pi}{4}r^4} \times r^2$$

$$= \frac{4}{3}\frac{V}{\pi r^2} \qquad \text{... (5.32)}$$

The average shear stress is given as

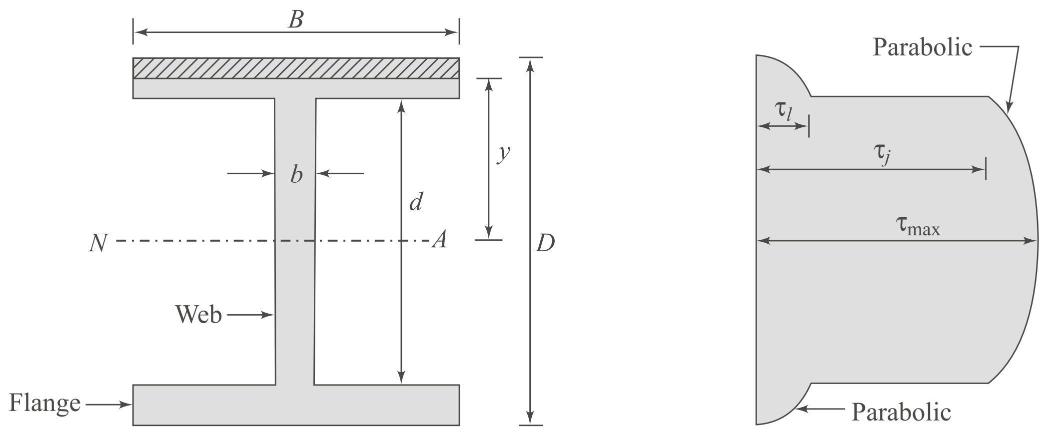
$$\tau_{av} = \frac{V}{\pi r^2} \quad \text{...(5.33)}$$

Comparing equations (5.32) and (5.33), we have

$$\tau_{max} = \frac{4}{3}\tau_{av} \quad \text{... (5.34)}$$

5.11 SHEAR STRESS DISTRIBUTION IN AN I-SECTION

Refer Fig. 5.16.



(*a*) I-section

(*b*) Shear stress variation

Fig. 5.16

Let D = Depth of I-section

d = Depth of the web

B = Width of the flange

b = Width of the web

I = Moment of inertia of I-section about the neutral axis (NA)

V = Shear force at the section

Shear stress distribution in the flange

For the hatched portion at a distance y from the neutral axis, we have

$$A\bar{y} = \left[B\left(\frac{D}{2} - y\right)\right]\left[y + \frac{1}{2}\left(\frac{D}{2} - y\right)\right]$$

where $\bar{y}$ is the distance of centroid of the hatched portion from the neutral axis.

or

$$A\bar{y} = \frac{B}{2}\left(\frac{D^2}{4} - y^2\right) \quad \text{... (5.35)}$$

Using equation (5.19) and making necessary changes, we have

$$\tau = \frac{V}{IB} \times \frac{B}{2}\left(\frac{D^2}{4} - y^2\right) = \frac{V}{2I}\left(\frac{D^2}{4} - y^2\right) \qquad \text{... (5.36)}$$

This is an equation of parabola, hence shear stress variation in flange is parabolic in nature. At upper part of the flange, where $y = \frac{D}{2}$

$$\tau_u = 0 \qquad \text{... (5.37)}$$

At lower part of the flange, where $y = \frac{d}{2}$

$$\tau_l = \frac{V}{8I}(D^2 - d^2) \qquad \text{... (5.38)}$$

Shear stress distribution in the web

Consider Fig. 5.17.

For any section XX in the web, we have

$$A\bar{y} = \left[B\left(\frac{D-d}{2}\right)\right]\left[\frac{d}{2} + \frac{1}{2}\left(\frac{D-d}{2}\right)\right] + \left[b\left(\frac{d}{2} - y_1\right)\right]\left[y_1 + \frac{1}{2}\left(\frac{d}{2} - y_1\right)\right]$$

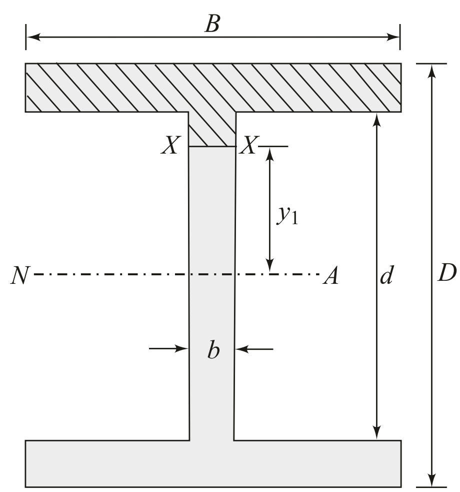


Fig. 5.17

$$= \frac{B}{8}(D^2 - d^2) + \frac{b}{2}\left(\frac{d^2}{4} - y_1^2\right) \qquad \text{...(5.39)}$$

Using equation (5.19), we have

$$\tau = \frac{V}{Ib} A\bar{y}$$

$$= \frac{V}{8Ib}\ [B\,(D^2 - d^2) + b\,(d^2 - 4y_1^2)] \qquad \text{....(5.40)}$$

The shear stress increases as y_1 decreases and is maximum when $y_1 = 0$. The variation of the shear stress is parabolic in the web section.

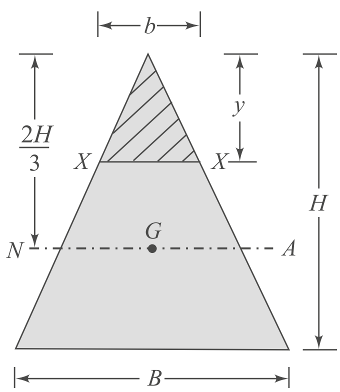
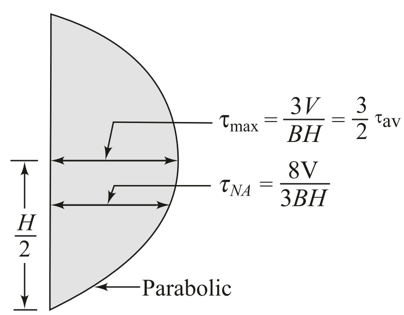
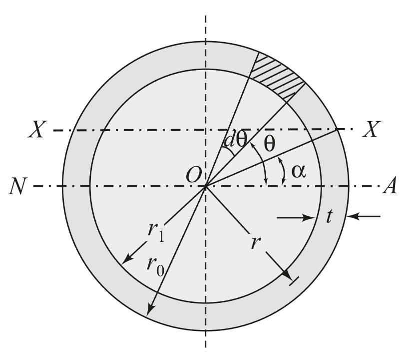
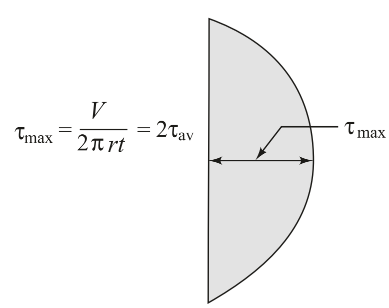
$$\tau_{max} = \frac{V}{8Ib}\ [B\,(D^2 - d^2) + bd^2 \qquad \text{... (5. 41)}$$

At the junction of web and flange (where $y_1 = \frac{d}{2}$), the shear stress is found to be

$$\tau_j = \frac{VB}{8Ib}\ (D^2 - d^2) \qquad \text{...(5.42)}$$

The shear stress distributions for some other sections are given in Table 5.1.

Table 5.1

Section	***Section figure***	***Shear stress distribution***
1. Triangular		
2. Thin circular		

Example 5.11

A *T*-section beam shown in Fig. 5.18 is subjected to a shear force of 10 kN. Draw the shear stress distribution diagram.

Solution: Distance of the neutral axis (*NA*) from the top surface of the flange is

$$\frac{500\times 20\times 10+600\times 20\times (300+20)}{500\times 20+600\times 20} = 179 \text{ mm}$$

Distance of the neutral axis from *PQ* is

$$620 - 179 \text{ mm} = 441 \text{ mm}$$

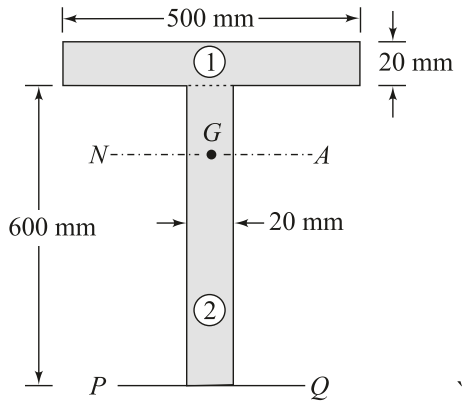


Fig. 5.18

The moment of inertia of the section about the neutral axis (*NA*) is given as

$$I = I_1 + I_2$$

$$= \left[\frac{500\times 20^3}{12}+500\times 20\times (179-10)^2\right]+\left[\frac{20\times (600)^3}{12}+20\times 600\,(441-300)^2\right]$$

(using parallel-axes theorem)

$$= 8.84 \times 10^8 \text{ mm}^4$$

Distance of the centre of gravity of the flange from the neutral axis is

$$\bar{y}_f = 179 - 10 = 169 \text{ mm}$$

Using equation (5.19), shear stress at lower part of the flange is given as

$$\tau_l = \frac{V}{Ib} A_f \bar{y}_f = \frac{10\times 10^3\times (500\times 20)\times 169}{8.84\times 10^8\times 500} = 0.038 \text{ N/mm}^2$$

Maximum shear stress

For flange and web above the neutral axis, we have

$$A\bar{y} = 500 \times 20 \times (179 - 10) + (600 - 441) \times 20 \times \frac{(600 - 441)}{2} = 1.69 \times 10^6 \text{ mm}^3$$

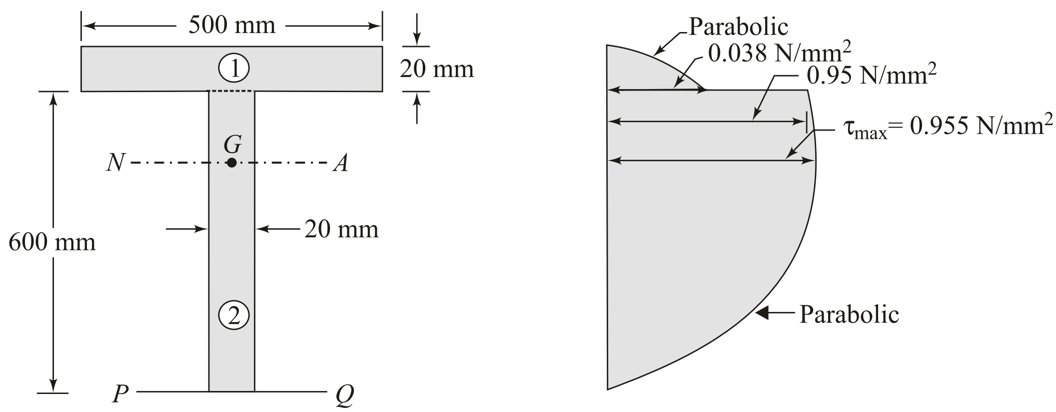
The maximum shear stress occurs at the neutral axis (NA), given by

$$\tau_{max} = \frac{VA\bar{y}}{Ib} = \frac{10\times10^3\times1.69\times10^6}{8.84\times10^8\times20} = 0.955\text{ N/mm}^2$$

The shear stress at the junction of the flange and web is

$$0.038\times\frac{500}{20} = 0.95\text{ N/mm}^2$$

Shear stress at top of the flange and bottom of the web is zero. The shear stress distribution is shown in Fig. 5.19 (*b*).



(*a*) T-section (*b*) Shear stress variation

Fig. 5.19

Example 5.12

An *I*-section beam shown in Fig. 5.20 is subjected to a bending moment of 50 kN.m at its certain section. Find the shear force at the section, if the maximum stress is limited to 100 N/mm^2.

Solution: Refer Fig. 5.20.

$\bar{y}$ = Distance of the *C.G.* of the flange from the neutral axis = 165 mm

A = Area of each flange

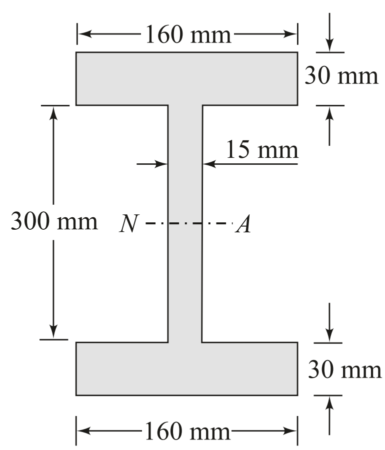


Fig. 5.20

$$= 160 \times 30 = 4800 \text{ mm}^2$$

$$I = \frac{1}{12} \times 160 \times 360^3 - 2 \times \frac{1}{12} \times 72.5 \times 300^3 = 2.9583 \times 10^8 \text{ mm}^4$$

The shear stress at the lower part of the flange is given as

$$\tau = \frac{VA\bar{y}}{Ib} = \frac{V \times 4800 \times 165}{2.9583 \times 10^8} = 1.7848 \times 10^{-4}\, V \qquad \text{... (1)}$$

The bending stress at the lower part of the flange is given as

$$\sigma = \frac{M}{I} y \qquad \text{(using bending equation)}$$

$$= \frac{50 \times 10^3 \times 10^3}{2.9583 \times 10^8} \times 150 = 25.35 \text{ N/mm}^2$$

The maximum stress (principal) is given as

$$100 = \frac{\sigma}{2} + \frac{\sqrt{\sigma^2 + 4\tau^2}}{2}$$

$$100 = 12.675 + \frac{\sqrt{642.62 + 4\tau^2}}{2}$$

Solving for τ, we get

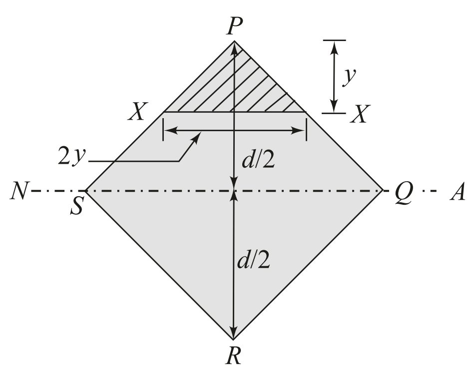
$$\tau = 86.4 \text{ N/mm}^2$$

Also $\tau = 1.7848 \times 10^{-4}\, V$ (using equation (1))

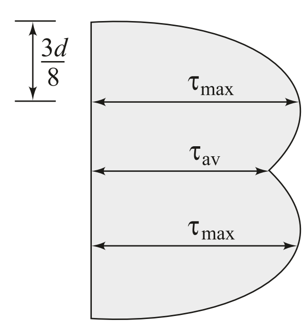
which gives $V = \frac{86.4}{1.7848 \times 10^{-4}} = 4.84 \times 10^5$ N **Ans.**

Example 5.13

A beam of square cross-section is placed on one of its diagonal horizontally (Fig. 5.21(*a*)). It is subjected to a shear force *V*. Draw the shear stress distribution diagram.



(*a*) Beam cross-section



(*b*) Shear stress variation

Fig. 5.21

Solution: Refer Fig. 5.21.

Let a = Side of the square $PQRS$

d = Diagonal of the square = $\sqrt{2}a$

Consider a section XX at a distance y from P.

The Moment of inertia of the square section about the neutral axis (NA) is given as

$$I = 2 \times \frac{d \times \left(\frac{d}{2}\right)^3}{12} = \frac{d^4}{48}$$

Cross-sectional area above XX, $A = \frac{1}{2} \times 2y \times y = y^2$

The shear stress at XX is given as

$$\tau = \frac{V \,.\, A\bar{y}}{Ib} \qquad \text{(using equation (5.19))}$$

$$= \frac{V \,.\, y^2 \,.\left(\frac{d}{2} - \frac{2y}{3}\right)}{\frac{d^4}{48} \times 2y}$$

$$= \frac{6V}{a^4}\left(\frac{a}{\sqrt{2}}y - \frac{2}{3}y\right)^2 \qquad \text{(in terms of } a\text{)} \qquad \dots (1)$$

This is the equation of a parabola, suggesting that the variation of the shear stress is parabolic in nature.

For τ to be maximum, we have

$$\frac{d\tau}{dy} = 0$$

which gives $$y = \frac{3a}{4\sqrt{2}} = \frac{3d}{8}$$

Position of the maximum shear stress from the neutral axis (NA) is

$$\frac{3d}{8} - \frac{d}{8} = \frac{a}{4\sqrt{2}}$$

Using equation (1), we have

$$\tau_{\max} = \frac{6V}{a^4}\left(\frac{a}{\sqrt{2}} \cdot \frac{3a}{4\sqrt{2}} - \frac{2}{3} \cdot \frac{9a^2}{32}\right) = \frac{9}{8} \times \frac{V}{a^2} \qquad \dots (2)$$

The average shear stress occurs at the neutral axis, given by

$$\tau_{av} = \frac{V}{a^2}$$

Now $$\frac{\tau_{\max}}{\tau_{av}} = \frac{9}{8} = 1.125$$

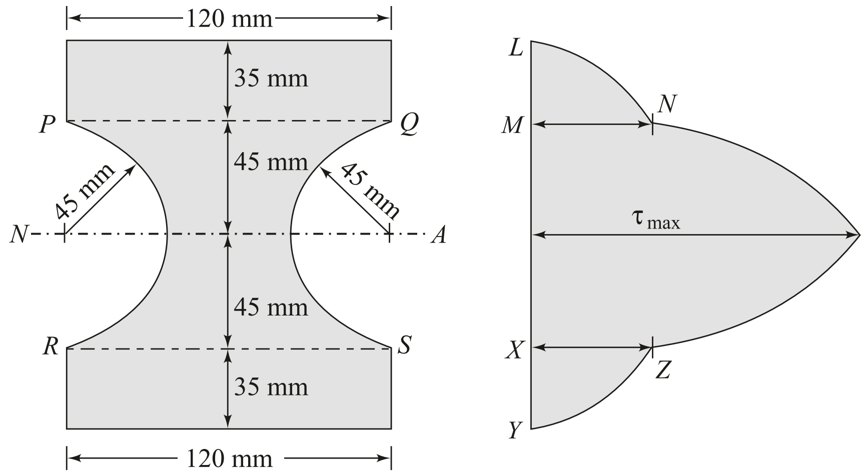
The shear stress distribution is shown in Fig. 5.21 (*b*).

Example 5.14

The cross-section (Fig. 5.22) of a steel beam is subjected to a shear force of 20 kN. Draw the shear stress distribution diagram.

Solution: The moment of inertia of cross-section about the neutral axis (NA) is given as

$$I = \frac{1}{12} \times 120 \times 160^3 - \frac{\pi}{64}(90)^4 = 3.77 \times 10^7 \text{ mm}^4$$



(a) Beam cross-section (b) Shear stress variation

Fig. 5.22

The moment of interia is calculated by assuming that given cross-section is made of a rectangular cross-section 120 mm × 160 mm minus two semi circles equivalent to a full circle of diameter (45 + 45) mm = 90 mm.

Shear stresses at L and Y (where $\bar{y} = 0$) are zero.

Shear stress at *Q* and *S*

$$\text{Area above } Q, A = (120 \times 35) \text{ mm}^2 = 4200 \text{ mm}^2$$

Distance of the centroid of this area from the neutral axis (NA) is

$$\bar{y} = 45 + \frac{35}{2} = 62.5 \text{ mm}$$

Now
$$\tau = \frac{VA\bar{y}}{Ib}$$

$$= \frac{20 \times 10^3 \times 4200 \times 62.5}{3.77 \times 10^7 \times 120} = 1.16 \text{ N/mm}^2$$

This is the shear stress at Q. By symmetry, the shear stress at S = 1.16 N/mm^2, that is, $MN = XZ$ = 1.16 N/mm^2. The variation of the shear stress between L and N, and between Y and Z is parabolic in nature.

Shear stress at the neutral axis

$$A\bar{y} = 120 \times 80 \times \frac{80}{2} - \left(\frac{\pi \times 45^2}{2} \times \frac{4 \times 45}{3\pi}\right) = 323250 \text{ mm}^3$$

Now $$\tau_{NA} = \tau_{\max} = \frac{VA\bar{y}}{Ib} = \frac{20 \times 10^3 \times 323250}{3.77 \times 10^7 \times 30} = 5.71 \text{ N/mm}^2$$

The shear stress distribution is shown in Fig. 5.22 (*b*).

Example 5.15

A T-section beam 300 mm deep and 150 mm wide, has flange and web thickness of 30 mm. It carries a *udl* of the intensity 5 kN/m over the overhang portion of the beam and a point load of 40 kN (Fig. 5.23). Draw the bending stress and shear stress distribution diagram for the beam.

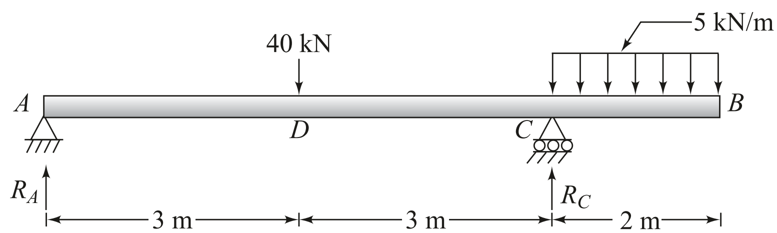


Fig. 5.23

Solution: The beam *AB* has a total span of 8 m with overhang portion *BC* of length 2 m.

Support reactions at *A* and *C*

Using $\Sigma M_A = 0$, we have

$$R_C \times 6 - 5 \times 2 \times \left(3 + 3 + \frac{2}{2}\right) - 40 \times 3 = 0$$

or $$R_C = 31.67 \text{ kN } (\uparrow)$$

Now $$R_A + R_C = 40 + 5 \times 2 = 50 \text{ kN}$$

or $$R_A = 18.33 \text{ kN } (\uparrow)$$

For any section between *C* and *D*, at a distance *x* from *A*, the bending moment is

$$M_x = R_A x - 40\,(x - 3)$$

$$= 18.33x - 40x + 120 = -21.67x + 120$$

Bending moment at *D*, where $x = 3$ m, is

$$M_D = -21.67 \times 3 + 120 = 54.99 \text{ kN·m} = M_{\max}$$

Bending moment at *C*, where $x = 6$ m, is

$$M_C = -21.67 \times 6 + 120 = -10.02 \text{ kN·m}$$

Calculations for bending stresses

Refer Fig. 5.24(*a*).

If the neutral axis is located at a distance of $\bar{y}$ from the top surface of the flange, then

$$\bar{y} = \frac{\left(150 \times 30 \times \frac{30}{2}\right) + 30 \times 270 \times \left(30 + \frac{270}{2}\right)}{(150 \times 30) + (30 \times 270)} = 111.42 \text{ mm}$$

The moment of inertia of whole section about the neutral axis (*NA*) is given as

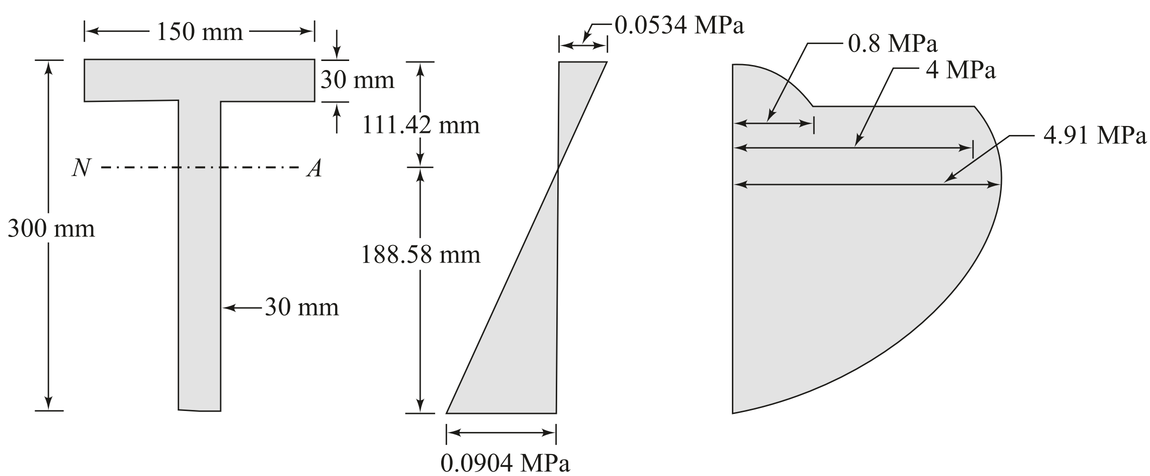
$$I = \frac{150 \times 30^3}{12} + 150 \times 30 \times \left[111.42 - \frac{30}{2}\right]^2 + \frac{30 \times 270^3}{12} + 30 \times 270 \times \left(188.58 - \frac{270}{2}\right)^2$$

$$= 1.146 \times 10^8 \text{ mm}^4$$

The bending stress at the top surface of the flange is compressive, given by

$$\sigma_C = \frac{M_{\max}}{I} \times \bar{y}$$

$$= \frac{54.99}{1.146 \times 10^8} \times 111.42 = 0.0534 \text{ MPa}$$



(*a*) T-section (*b*) Bending stress distribution (*c*) Shear stress distribution

Fig 5.24

The bending stress at the lower part of the beam is tensile, given by

$$\sigma_t = \frac{54.99}{1.146 \times 10^8} \times (300 - \bar{y})$$

$$= \frac{54.99}{1.146 \times 10^8} \times (300 - 111.42) = 9.04 \times 10^{-5} \text{ kN/mm}^2$$

$$= 0.0904 \text{ MPa}$$

Calculations for shear stresses

Since $R_C > R_A$, hence maximum shear force is equal to

$$R_C = 31.67 \text{ kN, which occurs at } C \ (= V).$$

$$(A\bar{y} \text{ at the lower part of flange}) = 150 \times 30 \times (111.42 - 15)$$

$$= 433890 \text{ mm}^3$$

($\bar{y}$ = Distance of the centroid of the flange from the neutral axis)

$$b = 150 \text{ mm}$$

Hence, the shear stress at the lower part of the flange is given as

$$\tau_l = \frac{VA\bar{y}}{Ib}$$

$$= \frac{31.67 \times 433890}{1.146 \times 10^8 \times 150} \text{ kN/mm}^2$$

$$= 8 \times 10^{-4} \text{ kN/mm}^2 = 0.8\text{MPa}$$

The shear stress in the web (at the junction of the flange and web) is given as

$$\tau_j = \frac{31.67 \times 433890}{1.146 \times 10^8 \times 30} \text{ kN/mm}^2 \qquad (b = 30 \text{ mm})$$

$$= 4 \times 10^{-3} \text{ kN/mm}^2 = 4 \text{ MPa}$$

The shear stress at the neutral axis is maximum, given by

$$\tau_{NA} = \tau_{\max}$$

$$= \frac{31.67 \times 533328.25}{1.146 \times 10^8 \times 30}$$

$$= 4.91 \times 10^{-3} \text{ kN/mm}^2 = 4.91 \text{ MPa}$$

$$\left[\ b = 30 \text{ mm and } A\bar{y} = 150 \times 30 \times (111.42 - 15) + 30 \times (111.42 - 30) \times \left(\frac{111.42 - 30}{2}\right)\right.$$

$$\left. = 533328.25 \text{ mm}^2\right]$$

The shear stress distribution diagram is shown in Fig. 5.24 (*c*) and the bending stress distribution diagram in Fig 5.24 (*b*).

SHORT ANSWER QUESTIONS

1. What is meant by pure bending?
2. What is flexural rigidity? What does it indicate?
3. What is the importance of neutral axis?
4. What is section modulus? What is its importance?
5. What is meant by a beam of uniform strength?
6. What is a composite beam? Where is it useful?
7. What are the commonly used cross-sections of a beam?
8. What are the advantages of an I-section beam?

MULTIPLE CHOICE QUESTIONS

1. The section modulus is the ratio of

(*a*) Moment of inertia and bending stress

(*b*) Moment of inertia and the distance from neutral axis

(*c*) Moment of inertia and modulus of elasticity

(*d*) Modulus of elasticity and moment of inertia.

2. The section modulus for a circular beam of diameter *d* is given as

(*a*) $\frac{\pi}{64} d^4$ (*b*) $\frac{\pi}{64} d^3$ (*c*) $\frac{\pi}{32} d^3$ (*d*) $\frac{\pi}{64} d^2$.

3. The section modulus for a rectangular beam of width *b* and depth *d* is given as

(*a*) $\frac{1}{12} bd^3$ (*b*) $\frac{1}{6} bd^3$ (*c*) $\frac{1}{12} bd^2$ (*d*) $\frac{1}{6} bd^2$.

4. The moment of inertia of a circular section of diameter *d* about the neutral axis is given as

(*a*) $\frac{\pi}{64} d^4$ (*b*) $\frac{\pi}{32} d^4$ (*c*) $\frac{\pi}{64} d^3$ (*d*) $\frac{\pi}{32} d^3$.

5. The bending stress is proportional to

(*a*) moment of inertia (*b*) modulus of elasticity

(*c*) its distance from neutral axis (*d*) radius of curvature.

6. The bending stress is maximum at

(*a*) neutral axis (*b*) top layer of beam

(*c*) bottom layer of beam (*d*) top and bottom layer of beam.

7. The bending stress is zero at

(*a*) neutral axis (*b*) top layer of beam

(*c*) bottom layer of beam (*d*) top and bottom layer of beam.

8. The strength of a beam depends upon

(*a*) modulus of elasticity (*b*) bending moment

(*c*) section modulus (*d*) radius of curvature.

9. The bending equation is valid for a beam subjected to

(*a*) only bending moment and no shear force (*b*) combined bending and shear force

(*c*) shear force only (*d*) normal force only.

10. A composite beam is made of

(*a*) more than one material (*b*) three materials

(*c*) more than one cross-section (*d*) plastic material.

11. The shear stress at neutral axis of a rectangular section is

(*a*) average shear stress (*b*) maximum shear stress

(*c*) minimum shear stress (*d*) none of these.

12. For a triangular section, the shear stress is maximum at a height of

(*a*) $\frac{H}{4}$ from the base (*b*) $\frac{2H}{3}$ from the base

(*c*) $\frac{H}{3}$ from the base (*d*) $\frac{H}{2}$ from the apex.

13. The shear stress variation across a rectangular section is

(*a*) hyperbolic (*b*) parabolic (*c*) circular (*d*) elliptical.

14. The relationship between maximum shear stress and average shear stress for a rectangular section is given as

(*a*) $\tau_{av} = 1.33\tau_{\max}$ (*b*) $\tau_{\max} = 1.33\tau_{av}$ (*c*) $\tau_{av} = 1.5\tau_{\max}$ (*d*) $\tau_{\max} = 1.5\tau_{av}$.

15. The ratio of maximum shear stress and average shear stress for a triangular section is

(*a*) 0.66 (*b*) 1.33 (*c*) 1.5 (*d*) 0.75.

16. In the equation of shear stress, $\bar{y}$ is defined as the distance between the

(*a*) two extreme faces of section

(*b*) neutral axis and the *C.G.* of the area above the neutral axis

(*c*) neutral axis and the extreme face of the section

(*d*) none of these.

17. Compared to the bending deformation, the shear deformation is

(*a*) large (*b*) small (*c*) very large (*d*) zero.

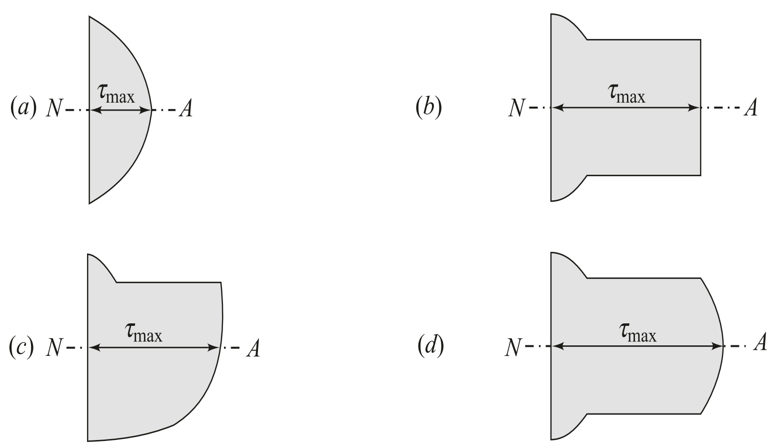
18. The shear stress varies directly proportional to

(*a*) moment of inertia about the neutral axis
(*b*) width of the beam
(*c*) distance between the neutral axis and the centroid of the area above the neutral axis
(*d*) normal stress.

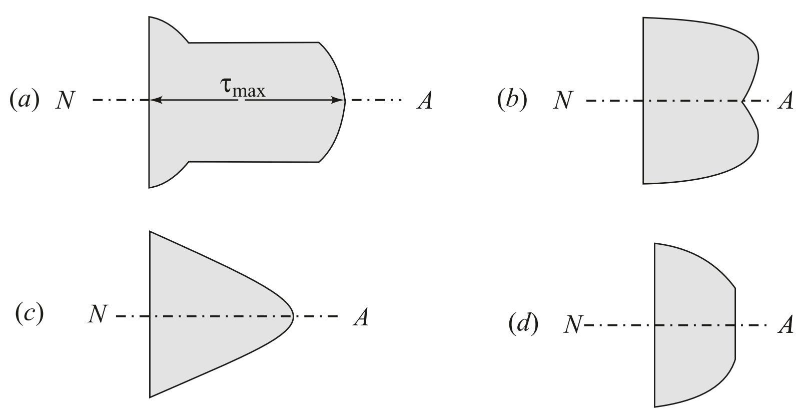
19. The shear stress is maximum, where the

(*a*) bending stress is minimum
(*b*) bending stress is maximum
(*c*) bending stress is zero
(*d*) bending stress is negative.

20. The shear stress variation for an *I*-section is represented as



21. For a beam of square cross-section and whose one of the diagonals is placed horizontally, the shear stress variation is shown as



22. For a beam of cross-section shown in Fig. 5.25, the shear stress variation is shown as

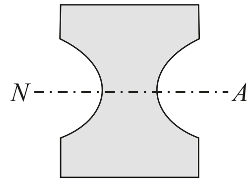
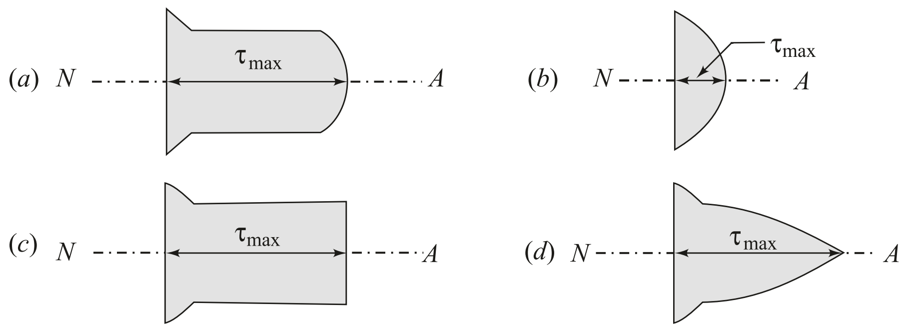


Fig. 5.25



ANSWERS

1. (*b*)	**2.** (*c*)	**3.** (*d*)	**4.** (*a*)	**5.** (*c*)	**6.** (*d*)	**7.** (*a*)	**8.** (*c*)
9. (*a*)	**10.** (*a*)	**11.** (*b*)	**12.** (*d*)	**13.** (*b*)	**14.** (*d*)	**15.** (*c*)	**16.** (*b*)
17. (*b*)	**18.** (*c*)	**19.** (*c*)	**20.** (*d*)	**21.** (*b*)	**22.** (*d*).		

EXERCISES

1. An *I*-section steel beam (Fig. 5.25) is 5 m long and simply supported at ends. Find the permissible uniform load to be placed on the beam. The maximum stress in tension does not exceed 25 N/mm^2.

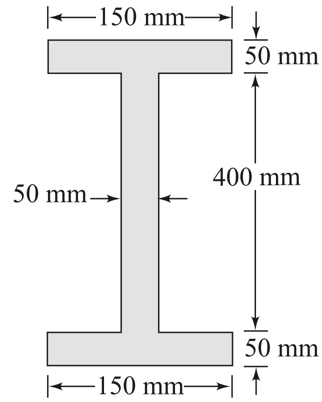


Fig. 5.25

(*Ans.* 32.93 N/mm).

2. A simply supported beam of length 4 m carries a point load of 40 kN at the centre and is supported at ends. Find the cross-section of beam assuming depth to be twice the width. The maximum bending stress in beam is not to exceed 200 N/mm^2. (*Ans.* $b = 67$ mm, $d = 134$ mm).

3. Compare the weights of two equally strong beams of circular sections made of the same material, one is solid and the other hollow with inside diameter being 2/5 of outside diameter.

(*Ans.* $W_s = 1.14\ W_H$).

4. A rectangular section is to be cut from a circular log of wood of diameter *D*. Find the dimensions of the strongest section in bending.

$$\left(\textit{Ans.} \quad b = \frac{D}{\sqrt{3}}, d = D\sqrt{\frac{2}{3}} \right).$$

5. A cast iron pipe having inside diameter 300 mm and outside diameter 350 mm is used to carry water and is supported at two ends 12 m apart. The specific weight of cast iron and water are 7×10^4 N/m^3 and 1.0×10^4 N/m^3 respectively. Determine the maximum bending stress induced in the pipe, if it is running full of water. (*Ans.* 1.447 N/mm^2).

6. A timber beam of rectangular section section 60 mm wide × 150 mm deep is reinforced by two steel plates having 12 mm thickness on both sides. The composite beam is a simply supported beam of length 2.5 m and carries a load of 10 kN at its centre. Find the depth of steel plates. The maximum stress in the timber section is not to exceed 12 N/mm^2. The modular ratio, *m* is 14. (*Ans.* 92.53 mm).

7. A flitched beam consists of a wooden section 150 mm wide × 200 mm deep. It is to be reinforced by providing a steel plate at its bottom. The steel plate is 150 mm wide and 10 mm thick. The maximum stress in wood is not to exceed 8×10^6 Pa and the modular ratio, *m* is 15. Determine the moment of resistance of beam. (*Ans.* 1.33×10^8 N·mm).

8. A flitched beam consists of two timber joists each 120 mm wide × 250 mm deep, is reinforced by a steel plate 30 mm wide and 200 mm deep, placed symmetrically in the centre and fixed to the timber joists. Find the maximum uniformly distributed load to be placed on beam of simply supported nature of length 6 m. The maximum stress in timber is not to exceed 10 N/mm^2. Given, the modular ratio, *m* is 20. (*Ans.* 14 kN/m).

9. A *T*-section beam, symmetrical about the vertical axis has the flange 100 mm × 10 mm and web 120 mm × 10 mm. What is the percentage of shear force shared by the web ? (*Ans.* 95.1%).

10. An *I*-section beam shown in Fig. 5.26 is subjected to a shear force of 50 kN. Find the magnitude and position of the maximum shear stress.

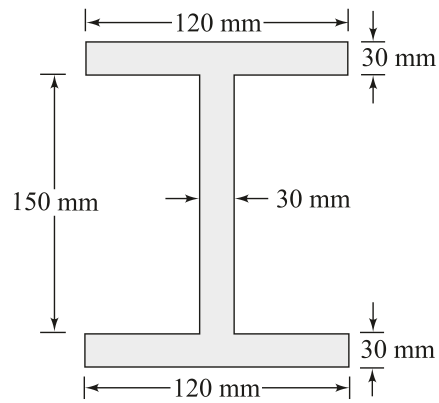


Fig. 5.26

(*Ans.* 12.2 N/mm^2 at the neutral axis, which is located at 105 mm from the base).

11. A *T*-section beam shown in Fig. 5.27 is subjected to a shear force of 40 kN. Find the magnitude and position of maximum shear stress. Also draw the shear stress distribution diagram.

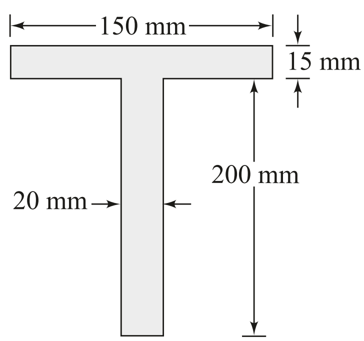
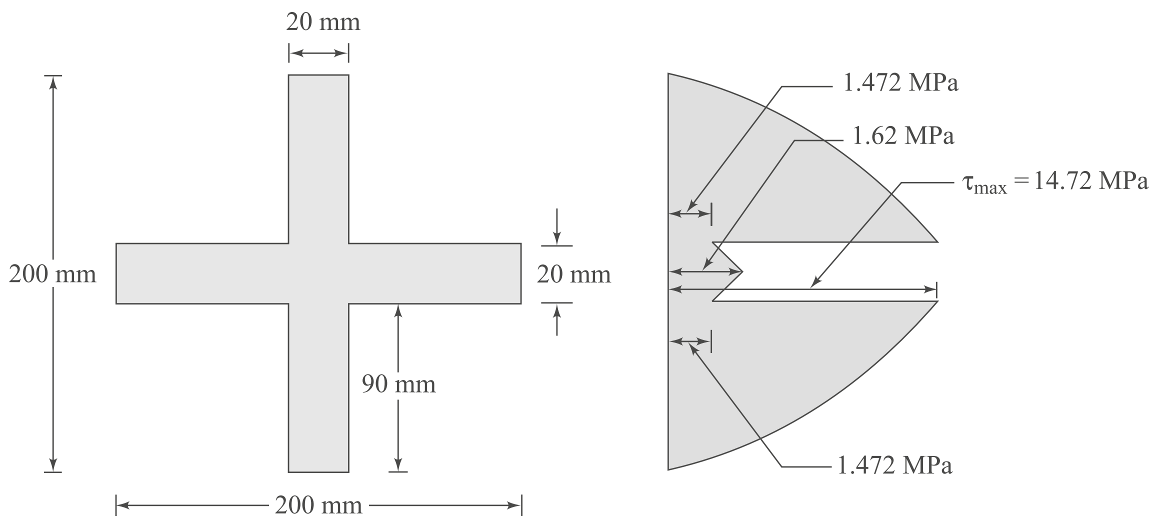


Fig. 5.27

(*Ans.* 12.81 N/mm^2 at neutral axis, which is located at 76.3 mm from the top surface of the flange).

12. Find the ratio of the maximum shear stress and mean shear stress for a hexagonal cross-section, whose one of the diagonals is placed horizontally. (*Ans.* 1.2).

13. An ornamental beam in the form of a cross-bar (Fig. 5.28 (*a*)) has a span of 4 m and carries a *udl* of 20 kN/m inclusive of its weight. Determine the maximum shear stress in the cross-section and draw the shear stress diagram.



(*a*) Cross-bar (*b*) Shear stress diagram

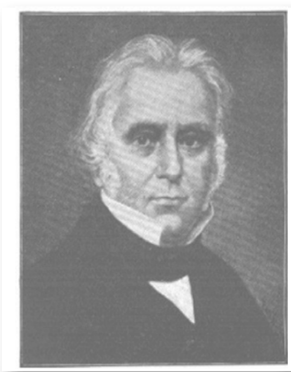
Fig. 5.28

(*Ans.* 14.72 MPa, for second part of the problem refer Fig. 5.27 (*b*)).

14. A wooden beam of rectangular section 200 mm by 300 mm is used as a simply supported beam carrying a *udl* of *w* ton/m. What is the maximum value of *w*, if the maximum shear stress developed in the beam section is limited to 5 N/mm^2 and the span length is 6 m ? (*Ans.* 6.66 N/m).

❑❑❑

6 Deflections of Beams



William Herrick Macaulay
(1853-1936)

William Herrick Macaulay, born on 16 November 1853, was a British mathematician. He developed the Macaulay's method, which is used in structural analysis to determine the deflection of beams. The method is based on the use of a singularity function, which helps to formulate a single equation of moments for all the loads acting on the beam for which the boundary conditions apply uniformly without considering any discontinuity, and is particularly useful for cases of discontinuous and/or discrete loading.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- How are slope and deflection defined?
- What is the limitation of double integration method?
- When is Macaulay's method useful?
- What is Mohr's theorem?
- What is the principle of superposition?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_6

6.1 INTRODUCTION

A loaded beam is initially straight but remains no longer straight in due course of time because of deflection of its axis. Deflection is the shifting of a point from its initial position in the transverse direction, while slope at a point is the angle made by the tangent drawn on the deflected beam at that point with the original axis of the beam.

Slope and deflection are required to be known for the proper design of a beam in order to avoid its failure. Following methods are used to find the slope and deflection of a beam:

- Double integration method
- Macaulay's method
- Moment-area method
- Conjugate beam method
- Method of superposition
- Strain energy method

6.2 DIFFERENTIAL EQUATION OF FLEXURE

Let us consider a loaded simply supported beam as shown in Fig. 6.1 (*a*). It deflects in a manner as shown in Fig. 6.1 (*b*). The axis of the beam bends in a curve known as elastic curve.
Consider two points *A* and *B* on the elastic curve.

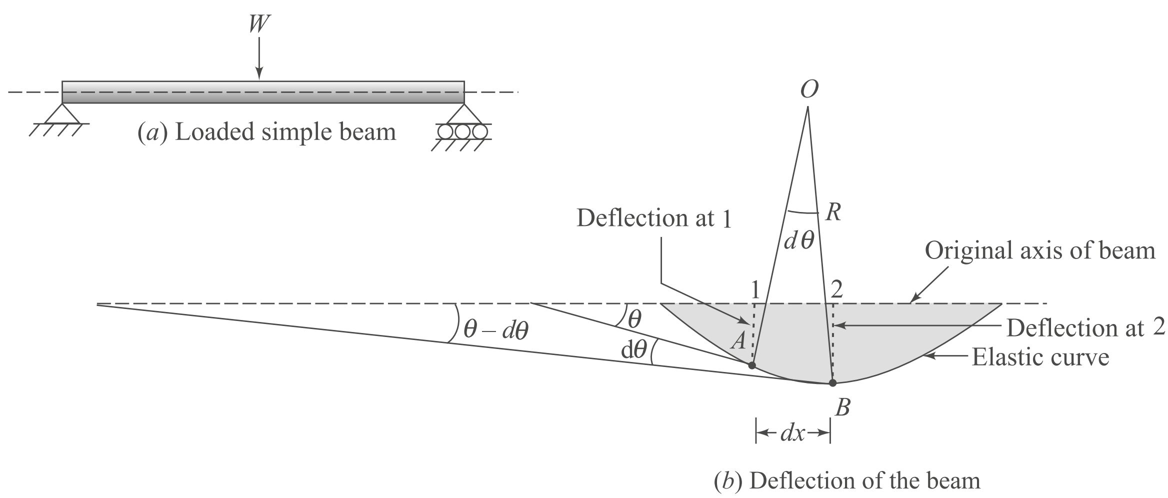


Fig. 6.1

Let θ = Angle made by the tangent drawn at point *A* with the axis of the beam

= Slope at point *A*

$d\theta$ = Angle between two normals drawn on the elastic curve at points *A* and *B*

R = Radius of curvature of the bent beam

$$dx = \text{Small segment between } A \text{ and } B$$

$$AB = Rd\theta$$

or $$\frac{1}{R} = \frac{d\theta}{AB} = \frac{d\theta}{dx} \qquad \text{(for small deflection, } AB \approx dx\text{)} \qquad \ldots (6.1)$$

Slope at point B is $(\theta - d\theta)$. It means that slope decreases with increase of dx and hence $\frac{d\theta}{dx}$ is negative.

Equation (6.1) modifies to

$$\frac{1}{R} = -\frac{d\theta}{dx} \qquad \ldots (6.2)$$

Now $$\frac{dy}{dx} = -\tan\theta \approx -\theta \qquad \text{(for small value of } \theta\text{)}$$

Differentiating w.r.t. x, we have

$$\frac{d^2y}{dx^2} = -\frac{d\theta}{dx} = \frac{1}{R} \qquad \text{(using equation (6.2))} \ldots (6.3)$$

Using bending equation, we have

$$\frac{1}{R} = \frac{M}{EI} \qquad \ldots (6.4)$$

Substituting equation (6.4) in equation (6.3), we get

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \quad \text{or} \quad EI\frac{d^2y}{dx^2} = M \qquad \ldots (6.5)$$

The product EI is known as the *flexural rigidity*, and the equation (6.5) is called *differential equation of flexure* or *differential equation of elastic curve.*

The slope is given by integrating equation (6.5) as

$$\frac{dy}{dx} = \frac{1}{EI}\int Mdx \qquad \ldots (6.6)$$

The deflection is obtained by integrating equation (6.6) as

$$y = \frac{1}{EI}\int \left(\int Mdx\right) dx \qquad \ldots (6.7)$$

6.3 SIGN CONVENTIONS

- The x and y axes are positive to the right and upward, respectively.
- The deflection y is negative downward.

- The slope $\frac{dy}{dx}$ is positive when measured counterclockwise with respect to the positive x axis, and is negative when measured clockwise.
- The bending moment M is positive when it produces compression in the upper part of the beam, and is negative when it produces tension in the upper part.

In Fig. 6.2, M and θ_B are positive, whereas y and θ_A are negative.

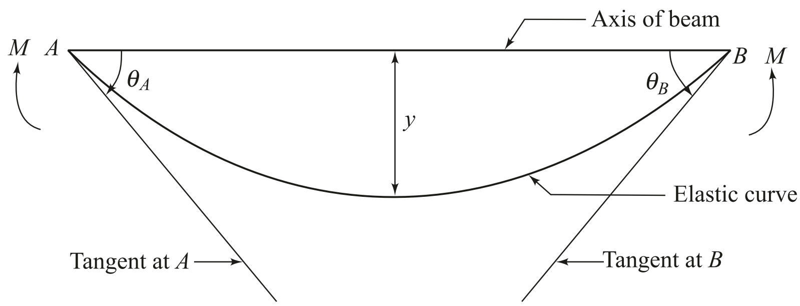


Fig. 6.2

6.4 DOUBLE INTEGRATION METHOD

This method is based on finding slope and deflection using equations (6.6) and (6.7).

6.4.1 Cantilever Beam carrying a Point Load at its Free End

Refer Fig. 6.3.

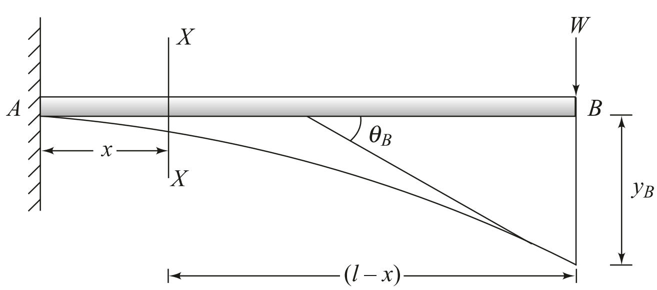


Fig. 6.3

The bending moment at the selected section XX at a distance x from the fixed end A is given as

$$M_x = -W(l - x) \qquad \text{... (6.8)}$$

Using differential equation of flexure, we have

$$\begin{aligned} EI\frac{d^2y}{dx^2} &= M \\ &= -W(l - x) \qquad \text{... (6.9)} \end{aligned}$$

On integration, we get

$$EI\frac{dy}{dx} = -Wlx + \frac{Wx^2}{2} + C_1 \qquad \text{... (6.10)}$$

Further integration gives

$$EIy = -Wl\frac{x^2}{2} + \frac{Wx^3}{6} + C_1x + C_2 \qquad \text{... (6.11)}$$

where C_1 and C_2 are the constants of integration.

The boundary conditions are:

At A, where $x = 0$ $\qquad \frac{dy}{dx} = 0$ and $y = 0$

which gives $C_1 = 0$ and $C_2 = 0$

Equations (6.10) and (6.11) on substituting C_1 and C_2 are reduced to

$$\frac{dy}{dx} = \frac{1}{EI}\left[-Wlx + \frac{Wx^2}{2}\right] \qquad \text{... (6.12)}$$

and

$$y = \frac{1}{EI}\left[-\frac{Wlx^2}{2} + \frac{Wx^3}{6}\right] \qquad \text{... (6.13)}$$

Equation (6.12) is a slope equation and equation (6.13), a deflection equation.

The slope at point B (where $x = l$) is given by

$$\theta_B = \left(\frac{dy}{dx}\right)_{x=l} = \frac{1}{EI}\left(-Wl\,.\,l + \frac{Wl^2}{2}\right)$$

$$= -\frac{Wl^2}{2El} \qquad \text{... (6.14)}$$

The deflection at B (where $x = l$) is found to be maximum, given by

$$y_B = y_{\max} = \frac{1}{EI}\left(-Wl\cdot\frac{l^2}{2} + \frac{Wl^3}{6}\right) = -\frac{Wl^3}{3El} \qquad \text{... (6.15)}$$

The negative sign shows the downward deflection.

The deflection at A (where $x = 0$) is zero.

6.4.2 Cantilever Beam carrying *udl* over its Entire Span

Refer Fig. 6.4.

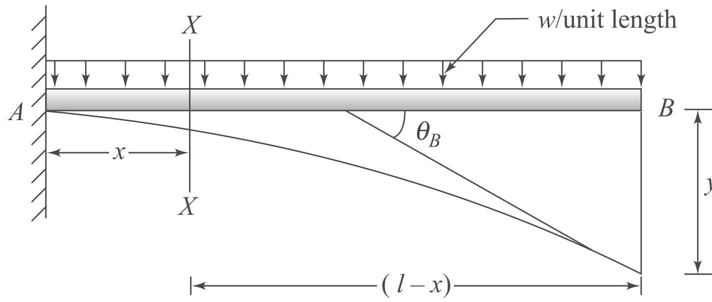


Fig. 6.4

The bending moment at the section is

$$M_x = -w(l-x)\frac{(l-x)}{2}$$

$$= -\frac{w(l-x)^2}{2} \quad \text{... (6.16)}$$

Using differential equation of flexure, we have

$$EI\frac{d^2y}{dx^2} = M$$

$$= -\frac{w(l-x)^2}{2} \quad \text{... (6.17)}$$

On integration, we have

$$\frac{dy}{dx} = -\frac{1}{EI}\left[-\frac{w}{2}\frac{(l-x)^3}{3} + C_1\right] \quad \text{... (6.18)}$$

Further integration gives

$$y = -\frac{1}{EI}\left[\frac{w}{6}\frac{(l-x)^4}{4} + C_1 x + C_2\right] \quad \text{... (6.19)}$$

where C_1 and C_2 are the constants of integration.

The boundary conditions are:

At A, where $x = 0$ $\quad \frac{dy}{dx} = 0$

and $\quad y = 0$

Equation (6.18) on substituting boundary condition gives

$$C_1 = \frac{wl^3}{6}$$

Equation (6.19) on substituting boundary condition gives

$$C_2 = -\frac{wl^4}{24}$$

Hence, equations (6.18) and (6.19) on substituting C_1 and C_2 are reduced to

$$\frac{dy}{dx} = -\frac{1}{EI}\left[-\frac{w}{6}(l-x)^3 + \frac{wl^3}{6}\right] \quad \text{... (6.20)}$$

and

$$y = -\frac{1}{EI}\left[\frac{w(l-x)^4}{24} + \frac{wl^3}{6}x - \frac{wl^4}{24}\right] \quad \text{... (6.21)}$$

Equations (6.20) and (6.21) are used to find the slope and deflection respectively.

For slope at B, put $x = l$ in equation (6.20).

$$\left(\frac{dy}{dx}\right)_{x=l} = \theta_B = -\frac{wl^3}{6EI} \quad \text{... (6.22)}$$

For deflection at B, put $x = l$ in equation (6.21).

$$y_B = -\frac{1}{EI}\left[\frac{wl^3}{6}l - \frac{wl^4}{24}\right]$$

$$= -\frac{wl^4}{8EI} \quad \text{... (6.23)}$$

The negative sign shows the downward deflection.

6.4.3 Cantilever Beam subjected to a Pure Couple at its Free End

Refer Fig. 6.5.

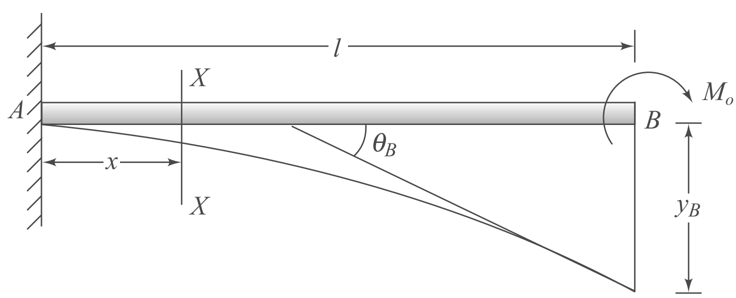


Fig. 6.5

Bending moment at the section is

$$M_x = -M_o \quad \text{... (6.24)}$$

Using differential equation of flexure, we have

$$EI\frac{d^2y}{dx^2} = M$$

$$= -M_o \quad \text{... (6.25)}$$

On integration, we have

$$EI\frac{dy}{dx} = -M_o x + C_1 \qquad \text{... (6.26)}$$

Further integration gives

$$EIy = -M_o \frac{x^2}{2} + C_1 x + C_2 \qquad \text{... (6.27)}$$

where C_1 and C_2 are the constants of integration.

The boundary conditions are:

At A, where $x = 0$ $\quad \frac{dy}{dx} = 0$ and $y = 0$

which gives $\quad C_1 = 0$ and $C_2 = 0$

Hence, equations (6.26) and (6.27) on substituting C_1 and C_2 are reduced to

$$\frac{dy}{dx} = -\frac{1}{EI}\left[M_o\, x\right] \qquad \text{... (6.28)}$$

$$y = -\frac{1}{EI}\left[M_o \frac{x^2}{2}\right] \qquad \text{... (6.29)}$$

Equations (6.28) and (6.29) are used to find the slope and deflection respectively.

For slope at B, put $x = l$ in equation (6.28).

$$\theta_B = \left(\frac{dy}{dx}\right)_{x=l} = -\frac{M_o l}{EI}. \qquad \text{... (6.30)}$$

For deflection at B, put $x = l$ in equation (6.29).

$$y_B = -\frac{M_o\, l^2}{2EI} \qquad \text{... (6.31)}$$

The negative sign shows the downward deflection.

6.4.4 Cantilever Beam carrying a Point Load anywhere on its Span

Refer Fig. 6.6.

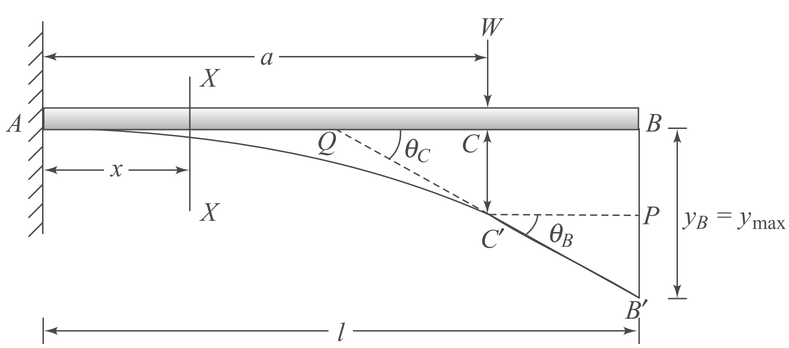


Fig. 6.6

Consider a section XX at a distance x from A in the portion AC.

Bending moment at the section is

$$M_x = -W(a-x) \qquad \text{... (6.32)}$$

Using differential equation of flexure, we have

$$EI\frac{d^2y}{dx^2} = M$$

$$= -W(a-x) \qquad \text{... (6.33)}$$

Integration of the above equation results in

$$EI\frac{dy}{dx} = \frac{W(a-x)^2}{2} + C_1 \qquad \text{... (6.34)}$$

Further integration gives

$$EIy = -\frac{W}{2}\frac{(a-x)^3}{3} + C_1 x + C_2 \qquad \text{... (6.35)}$$

where C_1 and C_2 are the constants of integration.

The boundary conditions are:

At A, where $x = 0$ $\qquad \dfrac{dy}{dx} = 0$ and $y = 0$

which gives $\qquad C_1 = -\dfrac{Wa^2}{2}$

$$C_2 = \frac{Wa^3}{6}$$

Equations (6.34) and (6.35) on substituting C_1 and C_2 are reduced to

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{W(a-x)^2}{2} - \frac{Wa^2}{2}\right] \qquad \text{... (6.36)}$$

$$y = \frac{1}{EI}\left[-\frac{W(a-x)^3}{6} - \frac{Wa^2}{2}x + \frac{Wa^3}{6}\right] \qquad \text{... (6.37)}$$

Equation (6.36) is a slope equation and equation (6.37), a deflection equation.

For slope at C, put $x = a$ in equation (6.36).

$$\theta_C = \left(\frac{dy}{dx}\right)_{x=a} = -\frac{Wa^2}{2EI} \qquad \text{... (6.38)}$$

Since there is no load on the portion BC, hence $C'B'$ remains straight indicating that slopes at the points B and C are equal.

$$\theta_B = \theta_C = -\frac{Wa^2}{2EI} \qquad \text{... (6.39)}$$

For deflection at C, put $x = a$ in equation (6.37).

$$y_C = \frac{1}{EI}\left[-\frac{Wa^3}{2}+\frac{Wa^3}{6}\right] = -\frac{Wa^3}{3EI} \qquad \text{... (6.40)}$$

In $\Delta\, B'C'P$

$$\tan\theta_B = \frac{PB'}{PC'} = \frac{PB'}{(l-a)}$$

$\tan\theta_B \approx \theta_B$ (for small value of θ_B)

$$PB' = (l-a)\,\theta_B$$

$$= -(l-a)\,\frac{Wa^2}{2EI} \qquad \text{(using equation (6.39))}$$

Hence, the deflection at B is given as

$$y_B = BP + PB' = y_C + PB'$$

or

$$y_B = y_{\max} = -\frac{Wa^3}{3EI} - (l-a)\frac{Wa^2}{2EI}$$

$$= -\left[\frac{Wa^3}{3EI} + (l-a)\frac{Wa^2}{2EI}\right] \qquad \text{... (6.41)}$$

The negative sign shows the downward deflection.

6.4.5 Cantilever Beam carrying Gradually Varying Load

Refer Fig. 6.7.

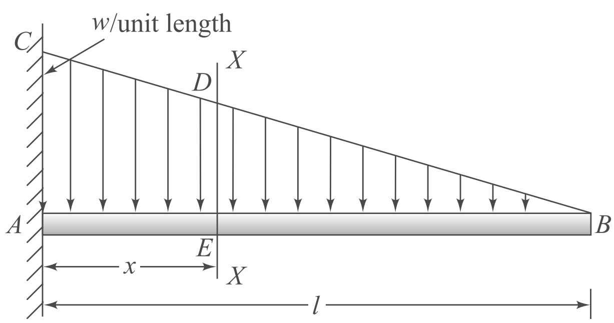


Fig. 6.7

Consider a section XX at a distance x from the fixed end A.

Comparing Δs ABC and EBD.

$$\frac{AC}{ED} = \frac{AB}{EB}$$

or

$$ED = \frac{AC}{AB}\cdot EB = \frac{w}{l}(l-x)$$

Bending moment at the section is

$$M_x = -\text{(Area of the triangle } EBD) \times \frac{EB}{3}$$

$$= -\frac{1}{2}(l-x) \times \frac{w}{l}(l-x) \times \frac{(l-x)}{3}$$

$$= -\frac{w(l-x)^3}{6l} \qquad \text{... (6.42)}$$

Using equation (6.5), we have

$$EI\frac{d^2y}{dx^2} = M = -\frac{w(l-x)^3}{6l} \qquad \text{... (6.43)}$$

On integration, we get

$$EI\frac{dy}{dx} = +\frac{w}{6l} \cdot \frac{(l-x)^4}{4} + C_1 \qquad \text{... (6.44)}$$

Further integration gives $EIy = -\frac{w}{24l} \cdot \frac{(l-x)^5}{5} + C_1 x + C_2$ ... (6.45)

where C_1 and C_2 are the constants of integration.

The boundary conditions are:

At A, where $x = 0$ $\quad \frac{dy}{dx} = 0$ and $y = 0$

which gives $\quad C_1 = -\frac{wl^3}{24}$

$$C_2 = +\frac{wl^4}{120}$$

Equations (6.44) and (6.45) on substituting C_1 and C_2 are reduced to

$$\frac{dy}{dx} = \frac{1}{EI}\left[+\frac{w}{24l}(l-x)^4 - \frac{wl^3}{24}\right] \qquad \text{... (6.46)}$$

and

$$y = \frac{1}{EI}\left[-\frac{w}{120l}(l-x)^5 - \frac{wl^3}{24}x + \frac{wl^4}{120}\right] \qquad \text{... (6.47)}$$

For slope at B, put $x = l$ in equation (6.46).

$$\theta_B = \left(\frac{dy}{dx}\right)_{x=l}$$

$$= -\frac{wl^3}{24EI} \qquad \text{... (6.48)}$$

For deflection at B, put $x = l$ in equation (6.47).

$$y_B = -\frac{wl^4}{30EI} \quad \text{... (6.49)}$$

The negative sign shows the downward deflection.

6.4.6 Simple Beam carrying a Central Point Load

Refer Fig. 6.8.

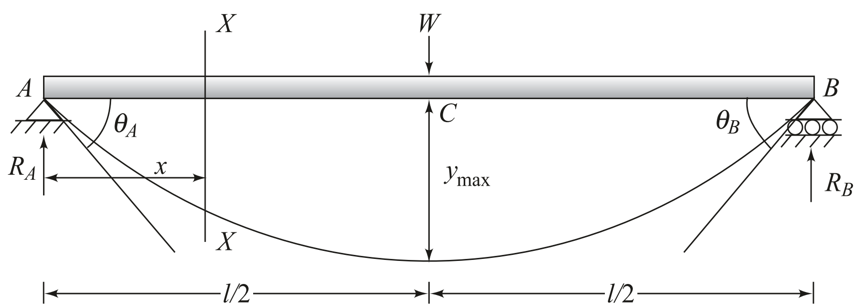


Fig. 6.8

Using $\sum M_A = 0$, we have

$$R_B \times l = W \times \frac{l}{2}$$

or
$$R_B = \frac{W}{2}(\uparrow)$$

Now
$$R_A + R_B = W$$

or
$$R_A = W - R_B = \frac{W}{2}(\uparrow)$$

Consider a section XX at a distance x from A.

Bending moment at the section is

$$M_x = +\frac{W}{2}x$$

Using equation (6.5), we have

$$EI\frac{d^2y}{dx^2} = M = \frac{W}{2}x \quad \text{... (6.50)}$$

On integration, we get

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} + C_1 \quad \text{... (6.51)}$$

Further integration gives
$$EIy = \frac{Wx^3}{12} + C_1x + C_2 \quad \text{... (6.52)}$$

where C_1 and C_2 are the constants of integration.

The boundary conditions are:

$$\frac{dy}{dx} = 0 \text{ at } C \quad (\text{for} \quad x = \frac{l}{2})$$

$$y = 0 \text{ at } A \quad (\text{for} \quad x = 0)$$

which gives $$C_1 = -\frac{Wl^2}{16} \quad \text{and} \quad C_2 = 0$$

Equations (6.51) and (6.52) on substituting C_1 and C_2 are reduced to

$$\frac{dy}{dx} = \frac{1}{EI}\left[+\frac{W}{4}x^2 - \frac{Wl^2}{16}\right] \qquad \text{... (6.53)}$$

$$y = \frac{1}{EI}\left[+\frac{W}{12}x^3 - \frac{Wl^2}{16}x\right] \qquad \text{... (6.54)}$$

Equation (6.53) can be used to find the slope at any point between A and C.

For slope at A, put $x = 0$ in equation (6.53).

$$\theta_A = \left(\frac{dy}{dx}\right)_{x=0} = -\frac{Wl^2}{16EI} \qquad \text{... (6.55)}$$

and $$\theta_B = -\theta_A = \frac{Wl^2}{16EI} \qquad \text{... (6.56)}$$

Equation (6.54) can be used to find the deflection at any point between A and C.

For deflection at C, put $x = \frac{l}{2}$ in equation (6.54).

$$y_C = y_{\max} = \frac{1}{EI}\left[+\frac{Wl^3}{96} - \frac{Wl^3}{32}\right] = -\frac{Wl^3}{48EI} \qquad \text{... (6.57)}$$

Thr negative sign shows the downward deflection.

6.4.7 Simple Beam carrying *udl* over its Entire Span

Refer Fig. 6.9.

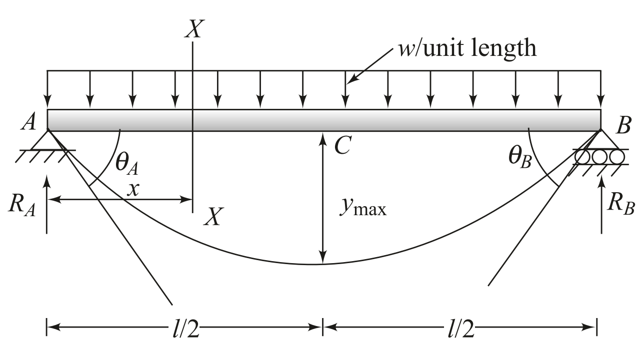


Fig. 6.9

Support reactions at *A* and *B*

Using $\Sigma M_A = 0$, we have

$$R_B \times l = w \times l \times \frac{l}{2}$$

or $$R_B = \frac{wl}{2} (\uparrow)$$

Now $$R_A + R_B = wl$$

or $$R_A = \frac{wl}{2} (\uparrow)$$

Consider a section *XX* at a distance *x* from *A* within *A* and *C*.

Bending moment at the section is

$$M_x = R_A x - wx \times \frac{x}{2}$$

$$= \frac{wlx}{2} - \frac{wx^2}{2} \qquad \text{... (6.58)}$$

Using differential equation of flexure, we have

$$EI \frac{d^2 y}{dx^2} = M$$

$$= \frac{wlx}{2} - \frac{wx^2}{2} \qquad \text{... (6.59)}$$

On integration, we get

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{wl}{4} x^2 - \frac{w}{6} x^3 + C_1\right] \qquad \text{... (6.60)}$$

Further integration gives $$y = \frac{1}{EI}\left[\frac{wl\,x^3}{12} - \frac{wx^4}{24} + C_1 x + C_2\right] \qquad \text{... (6.61)}$$

where C_1 and C_2 are the constants of integration.

The boundary conditions are:

$$\frac{dy}{dx} = 0 \text{ at } C \quad (\text{for} \quad x = \frac{l}{2})$$

$$y = 0 \text{ at } A \quad (\text{for} \quad x = 0)$$

which gives $$C_1 = -\frac{wl^3}{24} \quad \text{and} \quad C_2 = 0$$

Equations (6.60) and (6.61) on substituting C_1 and C_2 become

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{wl}{4}x^2 - \frac{w}{6}x^3 - \frac{wl^3}{24}\right] \qquad \text{... (6.62)}$$

$$y = \frac{dy}{dx}\left[\frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3}{24}x\right] \qquad \text{... (6.63)}$$

Equation (6.62) can be used to find the slope at any point between A and C.

For slope at A, put $x = 0$ in equation (6.62).

$$\theta_A = \left(\frac{dy}{dx}\right)_{x=0} = -\frac{wl^3}{24EI} \qquad \text{... (6.64)}$$

$$\theta_B = -\theta_A = \frac{wl^3}{24EI} \qquad \text{... (6.65)}$$

Equation (6.63) can be used to find the deflection at any point between A and C.

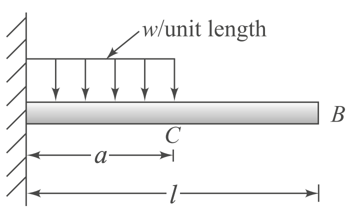
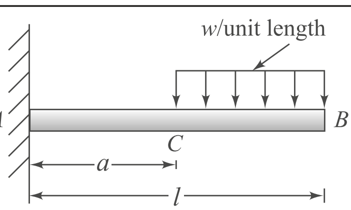
For deflection at C, put $x = \frac{l}{2}$ in equation (6.63).

$$y_C = y_{\max} = \frac{1}{EI}\left[\frac{wl^4}{96} - \frac{wl^4}{384} - \frac{wl^4}{48}\right] = -\frac{5}{384}\cdot\frac{wl^4}{EI} \qquad \text{... (6.66)}$$

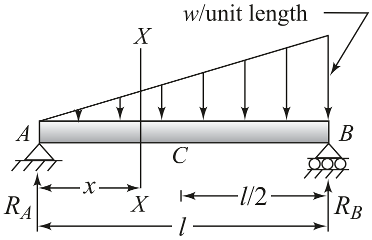
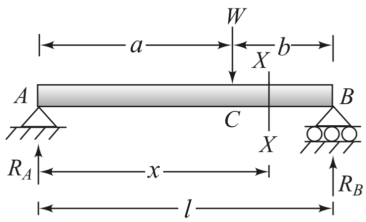
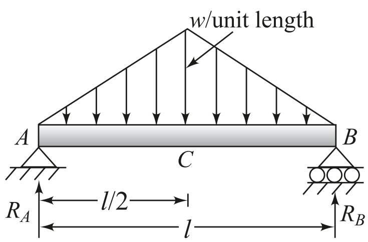
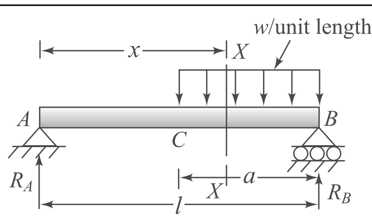
Thr negative sign shows the downward deflection.

Slope and deflection under different loading conditions are given in Table 6.1.

Table 6.1

S.N.	***Loading condition***	***Slope*** **(θ)**	***Deflection*** **(*y*)**	***Remarks***
1.		$\theta_A = 0$ $\theta_B = \theta_C$ $= -\frac{wa^3}{6EI}$	$y_A = 0$ $y_C = -\frac{wa^4}{8EI}$ $y_B = -\left[\frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l-a)\right]$	The elastic curve is straight between B and C because of no load on that portion.
2.		$\theta_A = 0$ $\theta_B = -\left[\frac{w}{6EI} \times (l^3 - a^3)\right]$	$y_A = 0$ $y_B = -\left[\frac{w}{24EI} \times (3l^4 - 4a^3l + a^4)\right]$	The elastic curve is continuous between A and B.

Contd...

S.N.	*Loading condition*	*Slope* (θ)	*Deflection* (*y*)	*Remarks*
3.		$\theta_A = -\dfrac{7wl^3}{360EI}$ $\theta_B = \dfrac{wl^3}{45EI}$	$y_A = y_B = 0$ $y_{max} = -\dfrac{13wl^4}{2000EI}$ $y_C = -\dfrac{147\,wl^4}{2340\,EI}$	• The maximum deflection occurs at a distance $x = 0.522\ l$ from A. • C is centre of the beam.
4.		$\theta_A = -\dfrac{Wb\,(l^2 - b^2)}{6EIl}$ $\theta_B = -\,\theta_B$ $= \dfrac{Wb\,(l^2 - b^2)}{6EIl}$	$y_A = y_B = 0$ $y_C = -\dfrac{Wa^2b^2}{3EIl}$ $y_{max} =$ $-\dfrac{Wb\,(l^2 - b^2)^{3/2}}{9\sqrt{3}\ EIl}$	• The maximum deflection occurs at $x = \sqrt{\dfrac{l^2 - b^2}{3}}$
5.		$\theta_A = -\dfrac{5wl^3}{192EI}$ $\theta_B = -\,\theta_A$ $= \dfrac{5wl^3}{192EI}$ $\theta_C = 0$	$y_A = y_B = 0$ $y_C = y_{max}$ $= -\dfrac{wl^4}{120EI}$	• Total load on beam $= \dfrac{wl}{2}$ • $R_A = R_B = \dfrac{wl}{4}$
6.		$\theta_A = -\dfrac{w}{6EI} \times$ $\left[\dfrac{a^2l}{2} - \dfrac{a^4}{4l} - (l-a)^3\right]$ $\theta_B = -\theta_A = \dfrac{1}{EI}\left[\dfrac{wa^2}{4}\right.$ $+ \dfrac{wl}{6}(l-a)$ $\left. + \dfrac{wa^2l}{12} - \dfrac{wa^4}{24l}\right]$	$y_A = y_B = 0$ $y_C = y_{max}$ $= -\dfrac{0.0056\,wl^4}{EI}$	The maximum deflection occurs at C, where $x = (l - a)$.

Example 6.1

A 3 m cantilever beam having rectangular cross-section is loaded with a point load of 10 kN at its free end. Find the cross-section of the beam. The maximum bending stress is not to exceed 5 N/mm^2 and the maximum deflection is restricted to 8 mm. Take $E = 2 \times 10^4$ N/mm^2.

Solution: Given,

Length of the beam, $l = 3$ m

Load on the beam, $W = 10$ kN $= 10 \times 10^3$ N

Bending stress, $\sigma_b = 5$ N/mm^2

Maximum deflection, $y_{\max} = 8$ mm

Let width of the beam $= b$

Depth of the beam $= d$

The moment of inertia of the cross-section of beam about the neutral axis is given as

$$I = \frac{1}{12} bd^3$$

The maximum negative bending moment is given as

$$M = W \times l$$

$$= 10^4 \times 3 \times 10^3 = 3 \times 10^7 \text{ N·mm}$$

The maximum bending stress is given as

$$\sigma_{\max} = \frac{M \times y}{I} \qquad \text{(using bending equation)}$$

$$5 = \frac{3 \times 10^7 \times \frac{d}{2}}{\frac{1}{12} bd^3}$$

or $$bd^2 = 3.6 \times 10^7 \qquad \dots (1)$$

The maximum negative deflection occurs at the free end of the cantilever, given by

$$y_{\max} = \frac{Wl^3}{3EI} \qquad \text{(using Equation (6.15))}$$

$$8 = \frac{10^4 \times (3 \times 1000)^3}{3 \times 2 \times 10^4 \times \frac{bd^3}{12}}$$

or $$bd^3 = 6.75 \times 10^9 \qquad \dots (2)$$

Solving equations (1) and (2), we get

$$d = 187.5 \text{ mm}$$

and $$b = 1024 \text{ mm}$$ **Ans.**

Example 6.2

A 2 m simple beam having cross-section 150 mm × 500 mm carries a point load of 20 kN at a distance of 0.5 m from the left end. Find the slope at the two ends, deflection under the load and the maximum deflection. Take $E = 2 \times 10^4$ N/mm^2.

Solution: Refer Fig. 6.10.

Given,

Length of the beam, $l = 2$ m $= 2000$ mm

Width of the beam $= 150$ mm

Depth of the beam $= 500$ mm

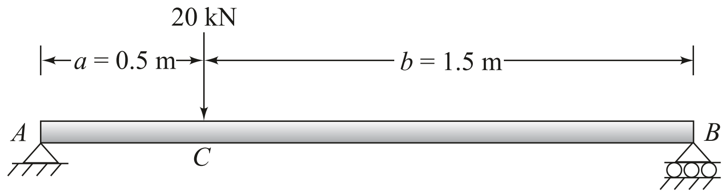


Fig. 6.10

Load on the beam, $W = 20 \text{ kN} = 2 \times 10^4$ N

Distance of load from A, $a = 0.5$ m

Distance of load from B, $b = 1.5$ m

The moment of inertia of the cross-section of beam about the neutral axis is given as

$$I = \frac{1}{12} \times 150 \times 500^3$$

$$= 1.5625 \times 10^9 \text{ mm}^4$$

The slope at the end A is given as

$$\theta_A = -\frac{Wb\,(l^2 - b^2)}{6EIl} \qquad \text{(see Table 6.1)}$$

$$= -\frac{2 \times 10^4 \times 1.5 \times 10^3 \times (2000^2 - 1500^2)}{6 \times 2 \times 10^4 \times 1.5625 \times 10^9 \times 2000}$$

$$= -1.4 \times 10^{-4} \text{ radian}$$

$$= -8.02 \times 10^{-3} \text{ degree}$$ **Ans.**

The slope at the end B is given as

$$\theta_B = \frac{Wb\,(l^2 - b^2)}{6EIl} \qquad \text{(see Table 6.1)}$$

$$= -\frac{2\times10^4\times1.5\times10^3\times(2000^2-1500^2)}{6\times2\times10^4\times1.5625\times10^9\times2000}$$

$$= -1.0\times10^{-4}\text{ radian}$$

$$= -5.7\times10^{-3}\text{ degree} \qquad \textbf{Ans.}$$

The deflection under the load is

$$y_C = -\frac{Wa^2b^2}{3EIl} \qquad \text{(see Table 6.1)}$$

$$= -\frac{2\times10^4\times(0.5\times1000)^2\times(1.5\times1000)^2}{3\times2\times10^4\times1.5625\times10^9\times2000} = -0.06\text{ mm} \qquad \textbf{Ans.}$$

The maximum deflection is

$$y_{\max} = -\frac{Wb\,(l^2-b^2)^{3/2}}{9\sqrt{3}EIl} \qquad \text{(see Table 6.1)}$$

$$= -\frac{2\times10^4\times1.5\times1000\,(2000^2-1500^2)^{3/2}}{9\sqrt{3}\times2\times10^4\times1.5625\times10^9\times2000} = -0.071\text{ mm} \qquad \textbf{Ans.}$$

The negative sign shows the downward deflection.

Example 6.3

A cantilever beam of uniform section and of length l carries two concentrated loads: W at the free end and $2W$ at a distance 'a' from the free end. Starting from the first principles, determine the deflection under the load $2W$.

If the cantilever is made from a steel tube of circular section of 100 mm external diameter and 6 mm thickness and l = 1.5 m, a = 0.6 m, determine the value of W so that the maximum bending stress is 140 MPa. Calculate the maximum deflection for the loading. Take E = 200 GPa.

Solution: Refer Fig. 6.11.

Consider a section XX in AC at a distance x from the free end B.

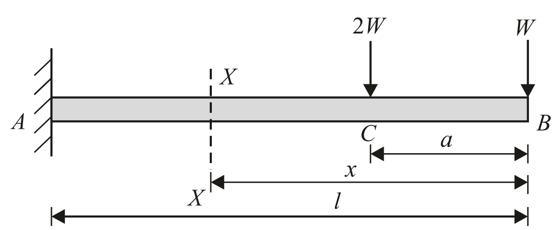


Fig. 6.11

The bending moment at the section is

$$M_x = -[Wx + 2W(x-a)]$$

Using differential equation of flexure, we have

$$EI\,\frac{d^2y}{dx^2} = M_x$$

$$= -\,[Wx + 2W(x-a)]$$

On integration, we get

$$EI\,\frac{dy}{dx} = -\frac{Wx^2}{2} - 2W\,\frac{(x-a)^2}{2} + C_1$$

$$= -\,\frac{Wx^2}{2} - W\,(x-a\,)^2 + C_1 \qquad \text{... (1)}$$

Further integration gives

$$EIy = -\,\frac{W}{2}\cdot\frac{x^3}{3} - W\cdot\,\frac{(x-a)^3}{3} + C_1x + C_2$$

$$= -\,\frac{Wx^3}{6} - \frac{W(x-a)^3}{3} + C_1x + C_2 \qquad \text{... (2)}$$

where C_1 and C_2 are the constants of integration.

The boundary conditions are:

At A, where $x = l$ $\qquad \frac{dy}{dx} = 0$

and $\qquad y = 0$

From equation (1), we have

$$0 = -\frac{Wl^2}{2} - W\,(l-a)^2 + C_1$$

$$= -\frac{Wl^2}{2} - Wl^2 - Wa^2 + 2Wla + C_1$$

or $\qquad C_1 = \frac{3}{2}\,Wl^2 + Wa^2 - 2Wla$

From equation (2), we have

$$0 = -\,\frac{W}{6}\,l^3 - \frac{W}{3}\,(l-a)^3 + \frac{3}{2}\,Wl^3 + Wa^2l - 2Wl^2.\,a + C_2$$

$$= -\,\frac{W}{6}\,l^3 - \frac{W}{3}\,l^3 + 3\cdot\frac{W}{3}\cdot l^2\,a - 3\cdot\frac{W}{3}\,la^2 + \frac{W}{3}\,a^3 + \frac{3}{2}\,Wl^3 + Wa^2l - 2Wl^2a + C_2$$

$$= Wl^3\left(-\frac{1}{6} - \frac{1}{3} + \frac{3}{2}\right) - Wl^2\,a + \frac{W}{3}\,a^3 + C_2$$

$$= Wl^3\left(\frac{-1-2+9}{6}\right) - Wl^2\,a + \frac{W}{3}\,a^3 + C_2$$

$$= Wl^3 - Wl^2 a + \frac{W}{3} a^3 + C_2$$

or $$C_2 = -Wl^3 - \frac{W}{3} a^3 + Wl^2 a$$

Equation (2) on substituting C_1 and C_2 becomes

$$EIy = -\frac{W}{6}x^3 - \frac{W}{3}(x-a)^3 + \left(\frac{3}{2}Wl^2 + Wa^2 - 2Wla\right)x - Wl^3 - \frac{W}{3}a^3 + Wl^2a \quad \text{... (3)}$$

Deflection at C (for which $x = a$) is given by

$$y = \frac{1}{EI}\left(-\frac{W}{6}a^3 + \frac{3}{2}Wl^2a + Wa^3 - 2Wla^2 - Wl^3 - \frac{W}{3}a^3 + Wl^2a\right)$$

$$= \frac{1}{EI}\left[Wa^3\left(-\frac{1}{6} + 1 - \frac{1}{3}\right) + Wl^2a\left(\frac{3}{2} + 1\right) - 2Wla^2 - Wl^3\right]$$

$$= \frac{W}{EI}\left(\frac{a^3}{2} + \frac{5}{2}l^2a - 2la^2 - l^3\right)$$ **Ans.**

The following parameters of the tube are given:

External diameter, $d_0 = 100$ mm

Thickness, $t = 6$ mm

Length, $l = 1.5$ m

Length, $a = 0.6$ m

Bending stress, $\sigma_b = 140$ MPa $= 140$ N/mm^2

Young's modulus, $E = 200$ GPa $= 200 \times 10^3$ N/mm^2

Internal diameter of the tube, $d_i = d_0 - 2t$

$= 100 - (2 \times 6) = 88$ mm

The maximum negative bending moment occurs at A, given by

$$M_{max} = Wl + 2W(l - a)$$

$$= Wl + 2Wl - 2Wa$$

$$= 3Wl - 2Wa$$

$$= 3W \times 1.5 - 2 \times 0.6\,W \quad \text{(substituting } l \text{ and } a\text{)}$$

$$= 4.5W - 1.2\,W$$

$$= 3.3W \text{ N.m } (W \text{ is assumed to be in } N)$$

The moment of inertia of the cross-section of the tube about the neutral axis is

$$I = \frac{\pi}{64}(d_o^4 - d_i^4)$$

$$= \frac{\pi}{64}(100^4 - 88^4) = 1964991 \text{ mm}^4$$

Now

$$\frac{\sigma_b}{\left(\frac{d_0}{2}\right)} = \frac{M}{I}$$

$$\frac{140}{\left(\frac{100}{2}\right)} = \frac{3.3W \times 10^3}{1964991}$$

or

$$W = 1667.3 \text{ N}$$ **Ans.**

The maximum deflection occurs at B (for $x = 0$), given by

$$y_{max} = -\frac{Wl^2(l-a)}{EI} \quad \text{(using equation (3))}$$

$$= -\frac{1667.3 \times (1.5 \times 10^3)^2}{200 \times 10^3 \times 1964991} \times (1.5 - 0.6) \times 10^3 \text{ mm}$$

$$= -8.6 \text{ mm}$$ **Ans.**

The negative sign shows the downward deflection.

6.5 MACAULAY'S METHOD

A single equation of moments satisfying all the boundary conditions at a time is difficult to formulate in case of double integration method. The Macaulay's method, an improvement over the double integration method, removes this difficulty by using a single equation of moments for all the loads acting on the beam and the constants of integration apply equally to all the sections of the beam. It is a very useful method to find the deflection of a beam involving discontinuous or discrete loading.

The following points are considered during the solution of problems using the Macaulay's method (Fig. 6.12).

- Work is always started from the extreme left end of the beam.
- The constants of integration are written just after the first term.
- This method uses step function of the form

$$f_n(x) = (x-a)^n \text{ such that } f_n(x) = 0 \text{ for } x < a \text{ and}$$

$$f_n(x) = (x-a)^n \text{ for } x > a$$

where n = 0 for any applied moment on the beam
= 1 for point load
= 2 for *udl*

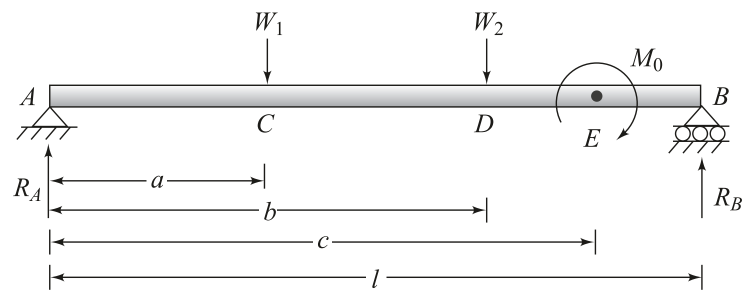


Fig. 6.12

- The negative terms inside the brackets are omitted.
- The *udl* should be extended up to the right end of the beam, if it is not so. Negative *udl* should be applied for the extended part to make a balance. Using Macaulay's method, the equation of moment is written as

$$M = R_A x - W_1 (x - a) - W_2 (x - b) + M_0 (x - c)^0 \qquad \text{... (6.66)}$$

The method is illustrated by the following examples.

Example 6.4

Two point loads of 5 kN and 15 kN are acting on a 5 m simple beam at 1 m and 2 m respectively from the left end. Find the following:

(*a*) the slopes at the two ends,

(*b*) the deflections under the applied loads,

(*c*) the position and magnitude of the maximum deflection. Take $E = 90$ GPa, $I = 18 \times 10^6$ mm^4.

Solution: Refer Fig. 6.13.

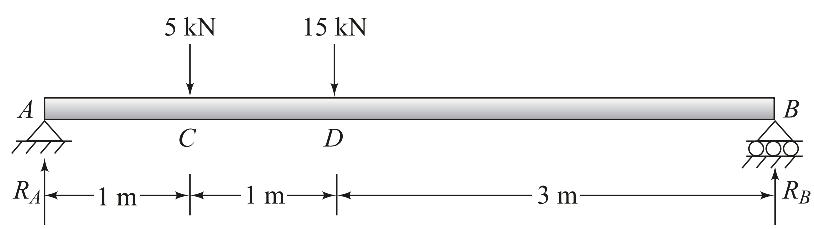


Fig. 6.13

Support reactions at *A* and *B*

Using $\Sigma M_A = 0$, we have

$$R_B \times 5 = 5 \times 1 + 15 \times 2$$

or

$$R_B = 7 \text{ kN } (\uparrow)$$

Now

$$R_A + R_B = 5 + 15 = 20 \text{ kN}$$

or $$R_A = 20 - R_B = 20 - 7$$
$$= 13 \text{ kN } (\uparrow)$$

Using Macaulay's method, the bending moment at any section in BD at a distance x from A is given as

$$M_x = R_A\, x \,|-5\,(x-1)| - 15\,(x-2)$$
$$= 13x \,|-5\,(x-1)| - 15\,(x-2) \qquad \text{(using equation (6.66))} \quad ...(1)$$

Using differential equation of flexure, we have

$$EI\,\frac{d^2y}{dx^2} = M$$
$$= 13x\,|-5(x-1)| - 15\,(x-2) \qquad ...\ (2)$$

On integrating, we get

$$\frac{dy}{dx} = \frac{1}{EI}\left[13\frac{x^2}{2} + C_1 \left| -\frac{5\,(x-1)^2}{2} \right| - 15\,\frac{(x-2)^2}{2}\right] \qquad ...\ (3)$$

Further integration gives

$$y = \frac{1}{EI}\left[\frac{13}{6}x^3 + C_1\,x + C_2 \left| -\frac{5}{6}(x-1)^3 \right| - \frac{15}{6}(x-2)^3\right] \qquad ...\ (4)$$

where C_1 and C_2 are the constants of integration.

The boundary conditions are:

At A, where $x = 0$, $y = 0$ which gives $C_2 = 0$ (neglecting negative terms)

At B, where $x = 5$ m, $y = 0$.

Using this boundary condition, we find C_1 with the help of equation (4) as

$$0 = \frac{1}{EI}\left[\frac{13}{6}\times 5^3 + C_1 \times 5 + 0 - \frac{5}{6}\times 4^3 - \frac{15}{6}\times 3^3\right]$$

Solving for C_1, we get

$$C_1 = -30$$

Substituting C_1 in equation (3), we have

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{13}{2}x^2 - 30 \left| -\frac{5}{2}(x-1)^2 \right| - \frac{15}{2}(x-2)^2\right] \qquad ...\ (5)$$

This is the equation of slope, giving slope for any section of the beam.

For slope at A, put $x = 0$ and neglect the negative terms.

$$\theta_A = \frac{1}{EI}\left[\frac{13}{2}\times 0 - 30\right] = -\frac{30}{EI}$$

Now $$EI = \frac{90 \times 10^9}{10^3} \text{ kN/m}^2 \times 18 \times 10^6 \times 10^{-12} \text{ m}^4 \qquad (1 \text{ mm}^4 = 10^{-12} \text{ m}^4)$$

$$= 1620 \text{ kN·m}^2$$

Hence, $$\theta_A = -\frac{30}{1620}$$

$$= -0.0185 \text{ radian} = -1.06 \text{ degree}$$ **Ans.**

For slope at B, put $x = 5$ m.

$$\theta_B = \frac{1}{EI}\left[\frac{13}{2} \times 5^2 - 30 - \frac{5}{2}(5-1)^2 - \frac{15}{2}(5-2)^2\right]$$

$$= 0.0154 \text{ radian} = 0.884 \text{ degree}$$ **Ans.**

Now substituting C_1 and C_2 in equation (4), we have

$$y = \frac{1}{EI}\left[\frac{13}{6}x^3 - 30x \left| -\frac{5}{6}(x-1)^3 \right| -\frac{15}{6}(x-2)^3\right] \qquad \text{... (6)}$$

This is the equation of deflection, giving deflection for any section of the beam.

For deflection at C, put $x = 1$ m in equation (6) and neglect the negative terms within the small brackets.

$$y_C = \frac{1}{EI}\left[\frac{13}{6} \times 1^3 - 30 \times 1\right]$$

$$= \frac{1}{1620}\left[\frac{13}{6} - 30\right]$$

$$= -0.0171 \text{ m}$$

$$= -17.1 \text{ mm}$$ **Ans.**

For deflection at D, put $x = 2$ m in equation (6).

$$y_D = \frac{1}{EI}\left[\frac{13}{6} \times 2^3 - 30 \times 2 - \frac{5}{6}(2-1)^3\right]$$

$$= \frac{1}{1620}\left[\frac{104}{6} - 60 - \frac{5}{6}\right] = -0.02685 \text{ m}$$

$$= -26.85 \text{ mm}$$ **Ans.**

For maximum deflection

$$\frac{dy}{dx} = 0$$

On equating equation (5) to zero, we have

$$0 = \frac{1}{EI}\left[\frac{13}{2}x^2 - 30 - \frac{5}{2}(x-1)^2 - \frac{15}{2}(x-2)^2\right]$$

or $$3.5x^2 - 35x + 62.5 = 0$$

Solving for x, we get

$$x = 2.327 \text{ m (neglecting higher value)}$$

Hence, the deflection is maximum at a distance $x = 2.327$ m from A.

The maximum deflection is obtained by putting $x = 2.327$ m in equation (6).

$$y_{\max} = \frac{1}{EI}\left[\frac{13}{6}\times(2.327)^3 - 30\times 2.327 - \frac{5}{6}(2.327-1)^3 - \frac{15}{6}(2.327-2)^3\right]$$

$$= -\frac{1}{EI}\times 44.54$$

$$= -\frac{1}{1620}\times 44.54 = -0.02749 \text{ m} = -27.49 \text{ mm}$$ **Ans.**

The negative sign shows the downward deflection.

Example 6.5

Find the slope and deflection of a loaded cantilever beam shown in Fig. 6.14. Also, find the position and magnitude of maximum deflection. Take flexural rigidity EI to be equal to 10^5 kN·m^2.

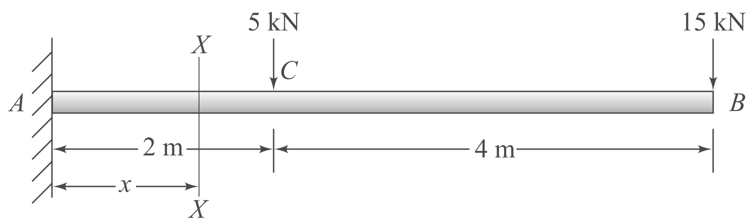


Fig. 6.14

Solution: Using Macaulay's method, the bending moment at any section at a distance x from A is given as

$$M_x = -15(6-x) \mid -5(2-x) \qquad \text{... (1)}$$

Using differential equation of flexure, we have

$$EI\frac{d^2y}{dx^2} = M$$

$$= -15(6-x) \mid -5(2-x) \qquad \text{... (2)}$$

Integrating equation (2), we get

$$EI\frac{dy}{dx} = \frac{15(6-x)^2}{2} + C_1 \left| + \frac{5(2-x)^2}{2}\right. \qquad \text{... (3)}$$

Integrating equation (3), we get

$$EIy = -\frac{15(6-x)^3}{6} + C_1 x + C_2 \left| - \frac{5(2-x)^3}{6}\right. \qquad \text{... (4)}$$

where C_1 and C_2 are the constants of integration.

Using boundary conditions

$$\frac{dy}{dx} = 0 \text{ and } y = 0 \text{ at } x = 0, \text{ we get}$$

$$C_1 = -280$$

$$C_2 = 546.67$$

Equation (3) is reduced to

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{15}{2}(6-x)^2 - 280 + \frac{5}{2}(2-x)^2\right] \quad \text{... (5)}$$

This is the slope equation and can be used to find slope for any section of the beam.

For slope at C, put $x = 2$ m in equation (5).

$$\theta_C = \left(\frac{dy}{dx}\right)_{x=2\text{m}} = \frac{1}{EI}\left[\frac{15}{2}\times 4^2 - 280\right] \quad \text{... (6)}$$

$$= -\frac{1}{EI}\times 160 = -\frac{1}{10^5}\times 160$$

$$= -0.0016 \text{ radian} = -0.091 \text{ degree}$$ **Ans.**

The maximum slope occurs at free end of the beam, given by

$$\theta_B = \left(\frac{dy}{dx}\right)_{x=6}$$

$$= -\frac{280}{EI} = -\frac{280}{10^5}$$

$$= -0.0028 \text{ radian}$$

$$= -0.160 \text{ degree}$$ **Ans.**

Equation (4) is reduced to

$$y = \frac{1}{EI}\left[-\frac{15}{6}(6-x)^3 - 280x + 546.67 - \frac{5}{6}(2-x)^3\right] \quad \text{... (6)}$$

This is the equation of deflection and can give deflection for any section of the beam.

For deflection at C, put $x = 2$ m in equation (6).

$$y_C = \frac{1}{EI}\left[-\frac{15}{6}\times 4^3 - 280\times 2 + 546.67\right]$$

$$= -\frac{173.33}{EI} = -0.001734 \text{ m} = -1.734 \text{ mm}$$ **Ans.**

The maximum deflection occurs at the free end of the beam ($x = 6$ m), given by

$$y_B = y_{\max} = \frac{1}{EI}\,[-280 \times 6 + 546.67]$$

$$= -\frac{1133.33}{10^5} = -0.01134 \text{ m} = -11.34 \text{ mm} \qquad \textbf{Ans.}$$

The negative signs associated with y_C and y_B show the downward deflections at these points.

Example 6.6

For a simple beam loaded as shown in Fig. 6.15, find slope and deflection at important points. Take flexural rigidity as *EI*.

Solution: Refer Fig. 6.15.

Support reactions at *A* and *B*

Using $\Sigma M_A = 0$, we have

$$R_B \times 12 = 25 \times 2 + 3 \times (2 + 2) \times 6 + 82 = 50 + 72 + 82 = 204$$

or $$R_B = 17 \text{ kN } (\uparrow)$$

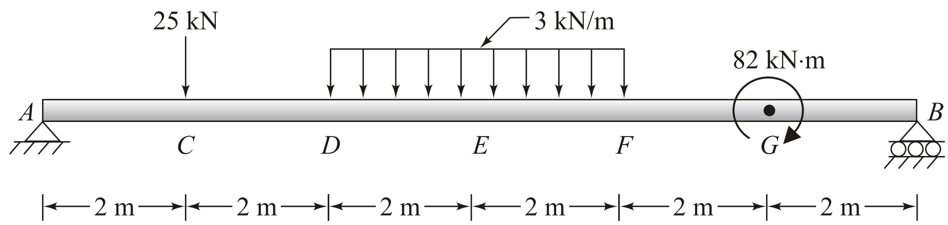


Fig. 6.15

Now $$R_A + R_B = 25 + 3 \times 4 = 37 \text{ kN}$$

or $$R_A = 37 - R_B = 37 - 17 = 20 \text{ kN } (\uparrow)$$

Using Macaulay's method, the bending moment at any section at a distance x from A is found by extending *udl* upto the point B and then applying negative *udl* of the same intensity for the compensating part to make a balance (Fig. 6.16).

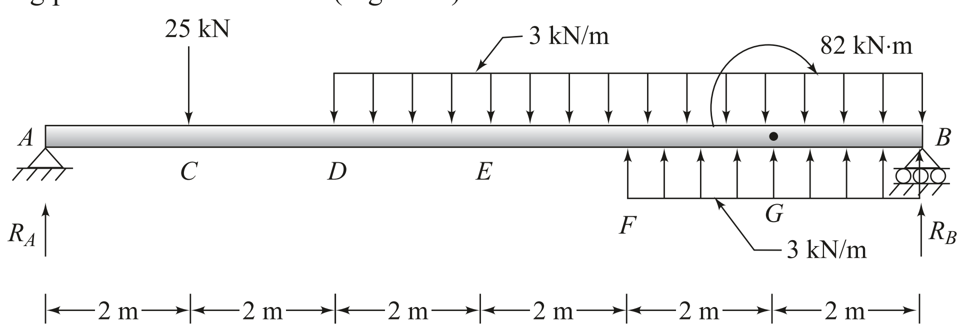


Fig. 6.16

The bending moment equation at the section is

$$M_x = R_A\, x - 25\,(x-2) - 3\,(x-4)\,\frac{(x-4)}{2}\,\Bigg| + 3\,(x-8)\,\frac{(x-8)}{2}\Bigg| + 82\,(x-10)^0$$

$$= 20x \mid - 25\,(x-2) \mid - \frac{3}{2}(x-4)^2 \Bigg| + \frac{3}{2}(x-8)^2 \Bigg| + 82(x-10)^0 \qquad \text{... (1)}$$

Using differential equation of flexure, we have

$$EI\,\frac{d^2y}{dx^2} = M = 20x \mid - 25(x-2) \mid - \frac{3}{2}(x-4)^2 \Bigg| + \frac{3}{2}(x-8)^2 + 82(x-10)^0 \qquad \text{... (2)}$$

Integrating equation (2), we get

$$EI\,\frac{dy}{dx} = 20\cdot\frac{x^2}{2} + C_1 \Bigg| - 25\cdot\frac{(x-2)^2}{2}\Bigg| - \frac{3}{2}\cdot\frac{(x-4)^3}{3}\Bigg| + \frac{3}{2}\cdot\frac{(x-8)^3}{3}\Bigg| + 82\,(x-10)$$

or

$$\frac{dy}{dx} = \frac{1}{EI}\left[10x^2 + C_1 \Big| - \frac{25}{2}(x-2)^2 \Big| - \frac{1}{2}(x-4)^3 \Big| + \frac{1}{2}(x-8)^3 \Big| + 82\,(x-10)\right] \qquad \text{...(3)}$$

Integrating equation (3), we have

$$y = \frac{1}{EI}\left[\frac{10x^3}{3} + C_1 x + C_2 \Bigg| - \frac{25}{2}\cdot\frac{(x-2)^3}{3}\Bigg| - \frac{1}{2}\cdot\frac{(x-4)^4}{4}\Bigg| + \frac{1}{2}\cdot\frac{(x-8)^4}{4}\Bigg| + \frac{82\,(x-10)^2}{2}\right]$$

$$= \left[\frac{10}{3}x^3 + C_1 x + C_2 \Big| - \frac{25}{6}(x-2)^3 \Big| - \frac{1}{8}(x-4)^4 \Big| + \frac{1}{8}(x-8)^4 \Big| + 41\,(x-10)^2\right] \qquad \text{... (4)}$$

The boundary conditions are:

At A, where $x = 0$, $y = 0$.

Equation (4) on using this boundary condition gives $C_2 = 0$. Negative terms within small brackets are omitted.

At B, where $x = 12$ m, $y = 0$.

Using this boundary condition, we find C_1 with the help of equation (4) as

$$0 = \frac{1}{EI}\left[\frac{10}{3}\times 12^3 + C_1 \times 12 + 0 - \frac{25}{6}\times 10^3 - \frac{1}{8}\times 8^4 + \frac{1}{8}\times 4^4 + 41\times 2^2\right]$$

Solving for C_1, we get

$$C_1 = -\,106.44$$

Substituting C_1 in equation (3), we have

$$\frac{dy}{dx} = \frac{1}{EI}\left[10\,x^2 - 106.44 \Big| - \frac{25}{2}(x-2)^2 \Big| - \frac{1}{2}(x-4)^3 \Big| + \frac{1}{2}(x-8)^3 \Big| + 82\,(x-10)\right] \qquad \text{... (5)}$$

This is the equation of slope, giving slope at any section of the beam.

Substituting C_1 and C_2 in equation (4), we have

$$y = \frac{1}{EI}\left[\frac{10}{3}x^3 - 106.44x\Big| - \frac{25}{6}(x-2)^3\Big| - \frac{1}{8}(x-4)^4\Big| + \frac{1}{8}(x-8)^4\Big| + 41(x-10)^2\right] \quad \text{... (6)}$$

This is the equation of deflection, giving deflection at any section of the beam.

Determination of slopes at various points using equation (5)

Slope at A, where $x = 0$, is

$$\theta_A = -\frac{106.44}{EI} \quad \text{(neglecting negative terms)}$$

Slope at C, where $x = 2$ m, is

$$\theta_C = \left(\frac{dy}{dx}\right)_{x=2} = \frac{1}{EI}[10 \times 2^2 - 106.44] \quad \text{(neglecting negative terms)}$$

$$= -\frac{66.44}{EI}$$

Slope at D, where $x = 4$ m, is

$$\theta_D = \left(\frac{dy}{dx}\right)_{x=4} = \frac{1}{EI}\left[10 \times 4^2 - 106.44 - \frac{25}{2} \times 2^2\right] = \frac{3.56}{EI} \quad \text{(neglecting negative terms)}$$

Slope at E, where $x = 6$ m, is

$$\theta_E = \left(\frac{dy}{dx}\right)_{x=6} = \frac{1}{EI}\left[10 \times 6^2 - 106.44 - \frac{25}{2} \times 4^2 - \frac{1}{2} \times 2^2\right] = \frac{49.56}{EI}$$

Slope at F, where $x = 8$ m, is

$$\theta_F = \left(\frac{dy}{dx}\right)_{x=8} = \frac{1}{EI}\left[10 \times 8^2 - 106.44 - \frac{25}{2} \times 6^2 - \frac{1}{2} \times 4^3\right] = \frac{51.56}{EI}$$

Slope at G, where $x = 10$ m, is

$$\theta_G = \left(\frac{dy}{dx}\right)_{x=10} = \frac{1}{EI}\left[10 \times 10^2 - 106.44 - \frac{25}{2} \times 8^2 - \frac{1}{2} \times 6^3 + \frac{1}{2} \times 2^3\right] = -\frac{10.44}{EI}$$

Slope at B, where $x = 12$ m, is

$$\theta_B = \left(\frac{dy}{dx}\right)_{x=12} = \frac{1}{EI}\left[10 \times 12^2 - 106.44 - \frac{25}{2} \times 10^2 - \frac{1}{2} \times 8^3 + \frac{1}{2} \times 4^3 + 82 \times 2\right] = \frac{23.56}{EI}$$

Determination of deflections at various points using equation (6)

Deflection at A, where $x = 0$, is

$$y_C = 0 \text{ (for check)}$$

Deflection at C, where $x = 2$ m, is

$$y_C = \frac{1}{EI}\left[\frac{10}{3}\times 2^3 - 106.44\times 2\right] = -\frac{79.77}{EI} \qquad \text{(neglecting negative terms)}$$

Deflection at D, where $x = 4$ m, is

$$y_D = \frac{1}{EI}\left[\frac{10}{3}\times 4^3 - 106.44\times 4 - \frac{25}{6}\times 2^3\right] = -\frac{245.76}{EI} \qquad \text{(neglecting negative terms)}$$

Deflection at E, where $x = 6$ m, is

$$y_E = \frac{1}{EI}\left[\frac{10}{3}\times 6^3 - 106.44\times 6 - \frac{25}{6}\times 4^3 - \frac{1}{8}\times 2^4\right] = -\frac{187.39}{EI}$$

Deflection at F, where $x = 8$ m, is

$$y_F = \frac{1}{EI}\left[\frac{10}{3}\times 8^3 - 106.44\times 8 - \frac{25}{6}\times 6^3 - \frac{1}{8}\times 4^4\right] = -\frac{76.85}{EI}$$

Deflection at G, where $x = 10$ m, is

$$y_G = \frac{1}{EI}\left[\frac{10}{3}\times 10^3 - 106.44\times 10 - \frac{25}{6}\times 8^3 - \frac{1}{8}\times 6^4 + \frac{1}{8}\times 2^4\right] = -\frac{24.4}{EI}$$

Deflection at B, where $x = 12$ m, is

$$y_B = 0 \text{ (for check)}$$

6.6 MOMENT-AREA METHOD

The moment-area method was introduced by a famous German civil engineer Otto Mohr (1835–1918). It is a semigraphical method in which the integration of the bending moment is carried out indirectly using the geometric properties of the area under the bending moment diagram. It is based on the application of two theorems known as Mohr's theorems and is equally valid for determinate or indeterminate beams. The method takes care of slope and deflection by pure bending and no shear consideration is made. The two theorems are stated below.

First Moment-Area Theorem (Mohr's First Theorem): The change in slope between any two points on a beam is equal to the net area of the bending moment diagram between these two points, divided by the flexural rigidity (EI) of the beam.

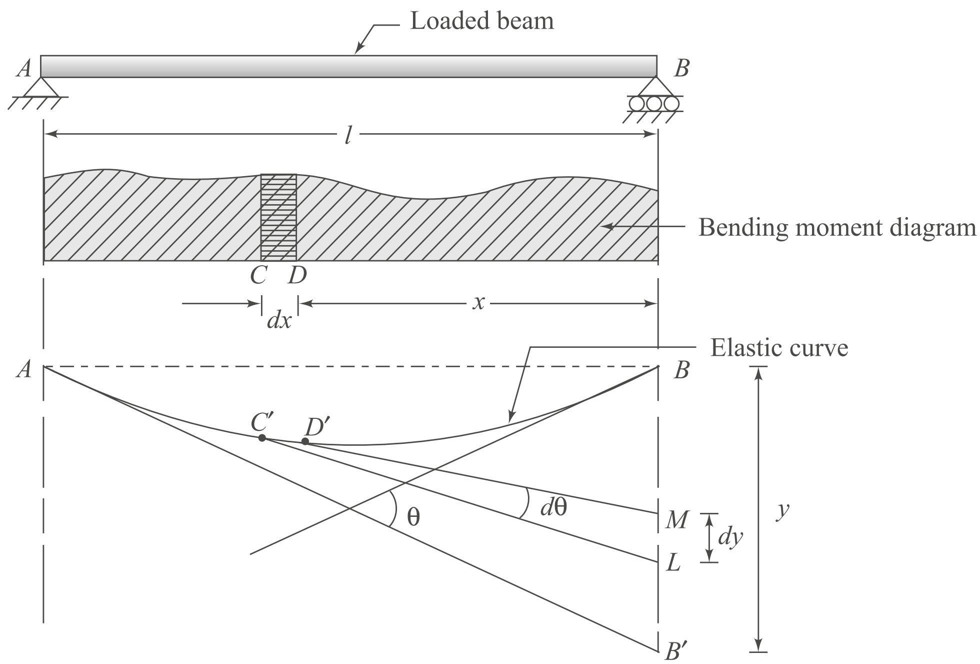


Fig. 6.17

Consider a beam AB of certain length l and loaded arbitrarily (Fig. 6.17). Take an element CD of length dx at a distance x from end B of the beam, where bending moment is M.

Let $d\theta$ = Change of slope for the element CD between C and D

θ = Angle between two tangents drawn at A and B

The differential equation of flexure is

$$EI\frac{d^2y}{dx^2} = M$$

or

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

Intergrating the above equation between the limits A and B, we have

$$\left(\frac{dy}{dx}\right)_B^A = -\int_B^A \frac{M}{EI}\,dx$$

$$\left(\frac{dy}{dx}\right)_A - \left(\frac{dy}{dx}\right)_B = -\int_B^A \frac{Mdx}{EI}$$

or

$$\theta_A - \theta_B = -\int_B^A \frac{Mdx}{EI}$$

or $$\theta_B - \theta_A = \int_B^A \frac{Mdx}{EI}$$

or $$\theta = \int_0^l \frac{Mdx}{EI} \qquad \text{... (6.67)}$$

where $$\theta_A = \left(\frac{dy}{dx}\right)_A = \text{Slope at } A$$

$$\theta_B = \left(\frac{dy}{dx}\right)_B = \text{Slope at } B$$

Equation (6.67) is the mathematical expression of the Mohr's first theorem.

If the flexural rigidity (EI) of the beam is constant, then equation (6.67) can be written as

$$\theta_B - \theta_A = \frac{1}{EI}\int_B^A M\,dx \qquad \text{... (6.68)}$$

Hence, the change in slope between A and B

$$= \frac{\text{Net area of the bending moment diagram between } A \text{ and } B}{EI}$$

Second Moment-Area Theorem (Mohr's Second Theorem): The vertical displacement of a point on a straight beam, measured from the tangent drawn at another point on the beam, is equal to the moment of area of bending moment diagram between these two points about the point, where this deflection occurs, divided by the flexural rigidity of beam.

The theorem helps to find deflection of a point with respect to another point of fixed position for which $x = 0$ and slope θ is also zero and this point is considered as reference point. That is why, it is very much useful for cantilever beams, fixed beams and symmetrically loaded simply supported beams for which position of zero slope is well defined.

From equation (6.67), we have

$$d\theta = \text{Angle between the tangents drawn at } C \text{ and } D$$

$$= \frac{Mdx}{EI} \qquad \text{...(6.69)}$$

From Fig. 6.17, we get

$$LM = x\,d\theta$$

$$= x \cdot \frac{Mdx}{EI} \qquad \text{(on substituting } d\theta\text{)}$$

For total length of the beam, we have

$$\int_B^A LM = \int_B^A \frac{Mxdx}{EI}$$

or $$BB' = \int_0^l \frac{Mxdx}{EI} \quad \text{... (6.70)}$$

Hence, $$y = \frac{1}{EI}\int_0^l x \cdot Mdx \quad \text{... (6.71)}$$

Hence, deflection of B w.r.t. A

$$= \frac{\text{Moment of the area of the bending moment diagram between } A \text{ and } B}{EI}$$

Equation (6.70) is the mathematical expression of the Mohr's second theorem.

6.6.1 Cantilever Beam carrying a Point Load at its Free End

Refer Fig. 6.18. A point load W is acting at free end B of the cantilever beam of length l.

To find the slope

The slope at fixed end of the cantilever beam is zero.

$$\theta_A = 0$$

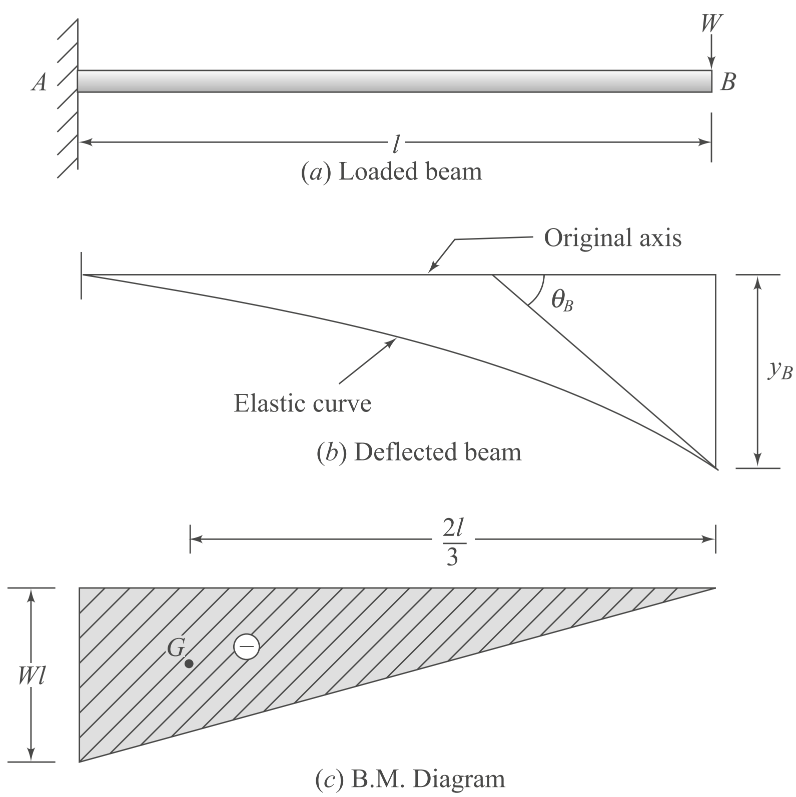


Fig. 6.18

Slope at free end B is

$$\theta_B = \text{Change in slope of the cantilever beam between } A \text{ and } B \text{ (from } B \text{ to } A)$$

Using Mohr's first theorem, we have

$$\theta_B = \frac{\text{Area of the bending moment diagram between } A \text{ and } B}{EI}$$

$$= \frac{1}{EI}\left[\frac{1}{2}\times Wl \times l\right] = \frac{Wl^2}{2EI} = \text{Maximum slope} \qquad \text{... (6.72)}$$

To find the deflection

Using Mohr's second theorem, the deflection at B is given as

$$y_B = \frac{\text{Moment of the area of } BMD \text{ between } A \text{ and } B \text{ about } B}{EI}$$

$$= \frac{1}{EI}\left(\frac{Wl^2}{2}\times\frac{2l}{3}\right) = \frac{Wl^3}{3EI} = \text{Maximum deflection} \qquad \text{... (6.73)}$$

6.6.2 Cantilever Beam carrying a *udl* over its Entire Span

Refer Fig. 6.19. A cantilever beam of length l is carrying a *udl* of intensity w/m length over the entire span.

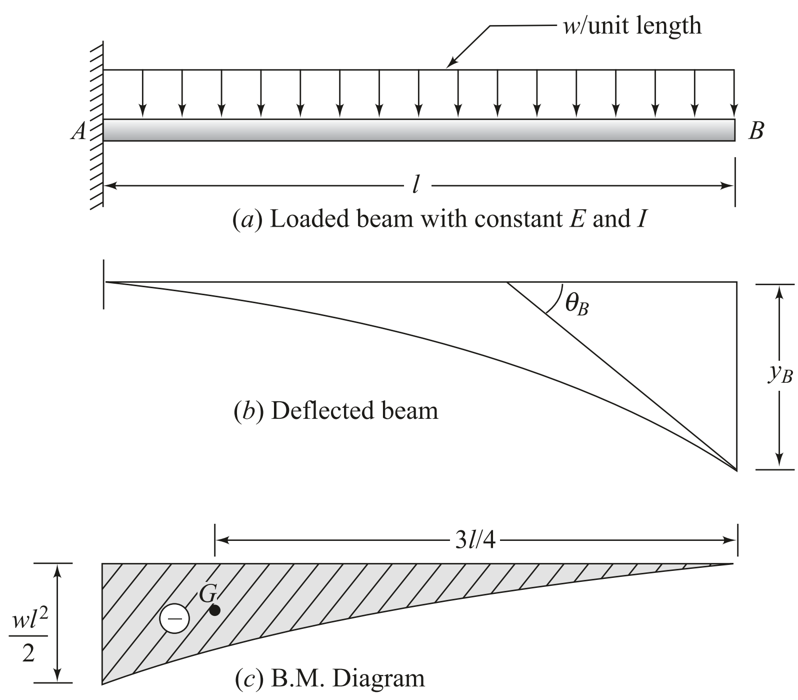


Fig. 6.19

To find the slope

Slope at A $\theta_A = 0$

and θ_B = Slope at B

$$= \frac{\text{Area of } BMD \text{ between } A \text{ and } B}{EI} \quad \text{(using Mohr's first theorem)}$$

$$= \frac{1}{EI}\left(\frac{1}{3}\times\frac{wl^2}{2}\times l\right) = \frac{wl^2}{6EI} = \text{Maximum slope} \qquad \text{... (6.74)}$$

To find the deflection

Using Mohr's second theorem, the deflection at B is given as

$$y_B = y_{\max} = \frac{1}{EI}(\text{Moment of the area of } BMD \text{ between } A \text{ and } B \text{ about } B)$$

$$= \frac{1}{EI}\left(\frac{wl^3}{6}\times\frac{3l}{4}\right) = \frac{wl^4}{8EI} \qquad \text{... (6.75)}$$

6.6.3 Simple Beam carrying a Central Point Load

Refer Fig. 6.20. A point load W is acting at centre C of the beam of length l.

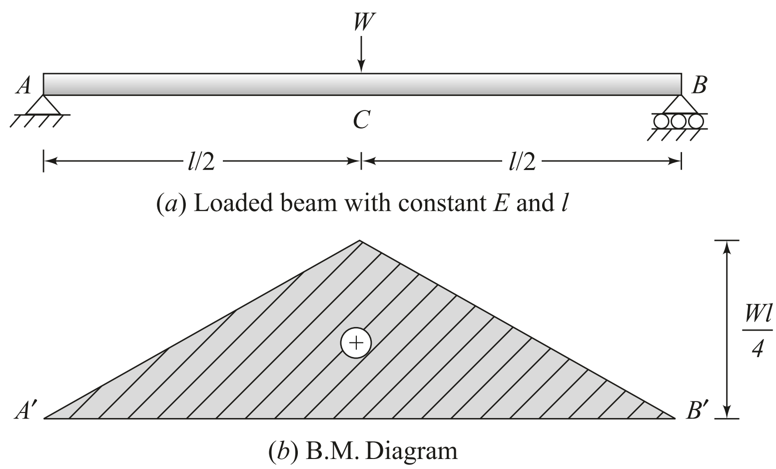


Fig. 6.20

To find the slope

Since the load is placed at the centre of the beam, hence the deflection on both sides of it is symmetrical and the slope at that point is zero.

Slope at A is $\theta_A = \frac{\text{Area of } BMD \text{ between } A \text{ and } C}{EI}$

$$= \frac{\frac{1}{2}\times\frac{Wl}{4}\times\frac{l}{2}}{EI} = \frac{Wl^2}{16EI} \qquad \text{... (6.76)}$$

Slope at B is

$$\theta_B = \frac{\text{Area of } BMD \text{ between } B \text{ and } C}{EI}$$

$$= \frac{Wl^2}{16EI} \qquad \text{... (6.77)}$$

To find the deflection

The deflections at A and B are zero.

Deflection at C is

$$y_C = \frac{\text{Moment of the area of } BMD \text{ between } A \text{ and } C \text{ about } A}{EI}$$

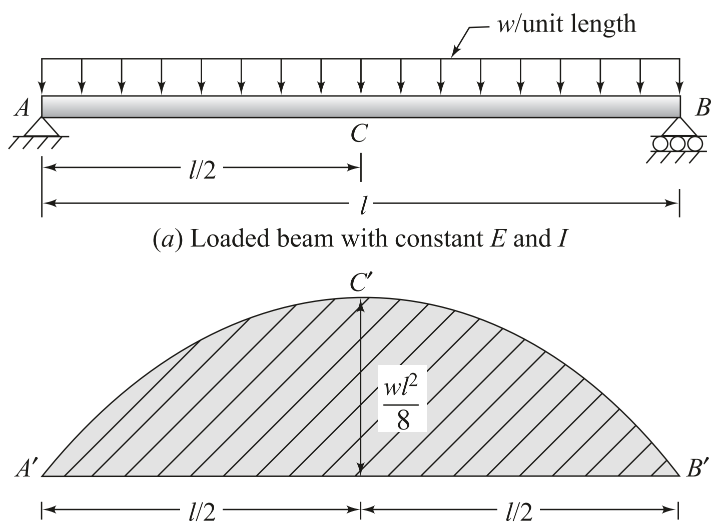
$$= \frac{Wl^2}{16\,EI} \times \left(\frac{2}{3} \times \frac{l}{2}\right)$$

$$= \frac{Wl^3}{48EI} \qquad \text{... (6.78)}$$

Since the centroid of the area of BMD between A and C lies at a distance of $\left(\frac{2}{3} \times \frac{l}{2}\right)$ from A'.

6.6.4 Simple Beam carrying a *udl* over its Entire Span

Refer Fig. 6.21. A *udl* of intensity *w/m* length is acting over the entire span *l*.



(*a*) Loaded beam with constant E and I

(*b*) B.M. Diagram

Fig. 6.21

To find the slope

Since intensity of *udl* acting over the entire span of the beam is uniform, hence the deflection at the midspan is maximum and slope at that point is zero.

Slope at A is

$$\theta_A = \frac{\text{Area of } BMD \text{ between } A \text{ and } C}{EI}$$

$$= \frac{1}{EI} \times \frac{1}{2}\left(\frac{4}{3} \times \frac{l}{2} \times \frac{wl^2}{8}\right)$$

$$= \frac{wl^3}{24EI} \qquad \text{... (6.79)}$$

Slope at B is $$\theta_B = \frac{\text{Area of } BMD \text{ between } B \text{ and } C}{EI} = \frac{wl^3}{24EI} \qquad \text{... (6.80)}$$

To find the deflection

The deflections at A and B are zero.

Deflection at C is

$$y_C = \frac{\text{Moment of the area of } BMD \text{ between } A \text{ and } C \text{ about } A}{EI}$$

$$= \frac{wl^3}{24EI} \times \left(\frac{l}{2} - \frac{3}{8} \times \frac{l}{2}\right) = \frac{5wl^4}{384EI} \qquad \text{... (6.81)}$$

Since distance of the centroid of parabolic section from C' is

$$\bar{x} = \frac{3}{8} \times \frac{l}{2} = \frac{3}{16} l, \text{ and its distance from } A' \text{ is } \left(\frac{l}{2} - \frac{3}{16} l\right) = \frac{5}{16} l.$$

Example 6.7

A 6 m simple beam is carrying a *udl* of 25 kN/m over its entire span and two point loads of 15 kN at 1.5 m from both ends. Find the slope and deflection at important locations of beam, using moment-area method. Take $E = 200$ GPa and $I = 3.32 \times 10^{-4}$ m^4.

Solution: Refer Fig. 6.22.

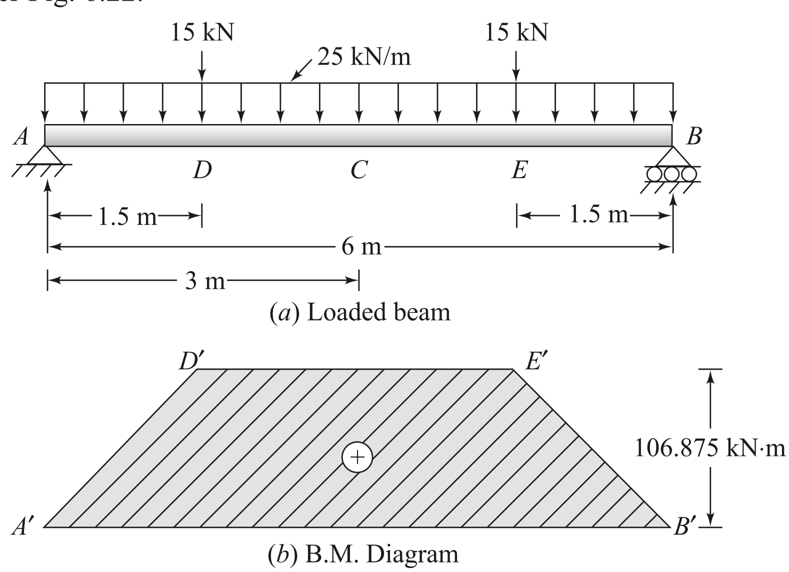


Fig. 6.22

Reactions at *A* and *B*

Using $\Sigma M_A = 0$, we have

$$R_B \times 6 = 15 \times 1.5 + 15 \times 4.5 + 25 \times 6 \times \frac{6}{2}$$

or $$R_B = 90 \text{ kN } (\uparrow)$$

and $$R_A + R_B = 15 + 15 + 25 \times 6 = 180 \text{ kN}$$

or $$R_A = 180 - 90 = 90 \text{ kN } (\uparrow)$$

Bending moment diagram (*BMD*)

The bending moments at *A* and *B* are zero.

Bending moment at $E = R_B \times 1.5 - 25 \times 1.5 \times \frac{1.5}{2} = 106.875$ kN·m

Similarly, the bending moment at $D = 106.875$ kN·m

To find the slope

The slope at the centre of the beam (point *C*) is zero because the tangent to the elastic curve at that point is horizontal.

Slope at *A* is $$\theta_A = \frac{\text{Area of } BMD \text{ between } A \text{ and } D}{EI} \qquad \text{(using Mohr's first theorem)}$$

$$= \frac{1}{EI}\left(\frac{1}{2} \times 1.5 \times 106.875\right) = \frac{\frac{1}{2} \times 1.5 \times 106.875}{200 \times 10^6 \times 3.32 \times 10^{-4}}$$

$$= 0.0012 \text{ radian}$$ **Ans.**

Similarly, the slope at *B* is

$$\theta_B = 0.0012 \text{ radian}$$ **Ans.**

To find the deflection

The deflections at *A* and *B* are zero.

Deflection at *D* is $$y_D = \frac{\text{Moment of the area of } BMD \text{ between } A \text{ and } D \text{ about } A}{EI}$$

$$= \frac{\left(\frac{1}{2} \times 1.5 \times 106.875\right) \times \frac{2}{3} \times 1.5}{200 \times 10^6 \times 3.32 \times 10^{-4}} \text{ m} = 1.2 \times 10^{-3} \text{ m}$$

$$= 1.2 \text{ mm}$$ **Ans.**

Similarly, the deflection at *E* is

$$y_E = 1.2 \text{ mm}$$ **Ans.**

Deflection at C (midspan) is

$$y_C = \frac{\text{Moment of area of } BMD \text{ between } A \text{ and } C \text{ about } A}{EI}$$

$$= \frac{\left[\frac{1}{2} \times (1.5+3) \times 106.875\right] \times \left[3 - \frac{1.5^2 + 1.5 \times 3 + 3^2}{3\,(1.5+3)}\right]}{200 \times 10^6 \times 3.32 \times 10^{-4}} \text{ m}$$

$= 6.64 \times 10^{-3}$ m $= 6.64$ mm **Ans.**

Example 6.8

A 3 m propped cantilever beam carries a *udl* of 2 kN/m over its entire span and is supported at its free end. Find the prop reaction when the free end does not yield. Take $EI = 10^4$ kN·m^2.

Solution: Refer Fig. 6.23.

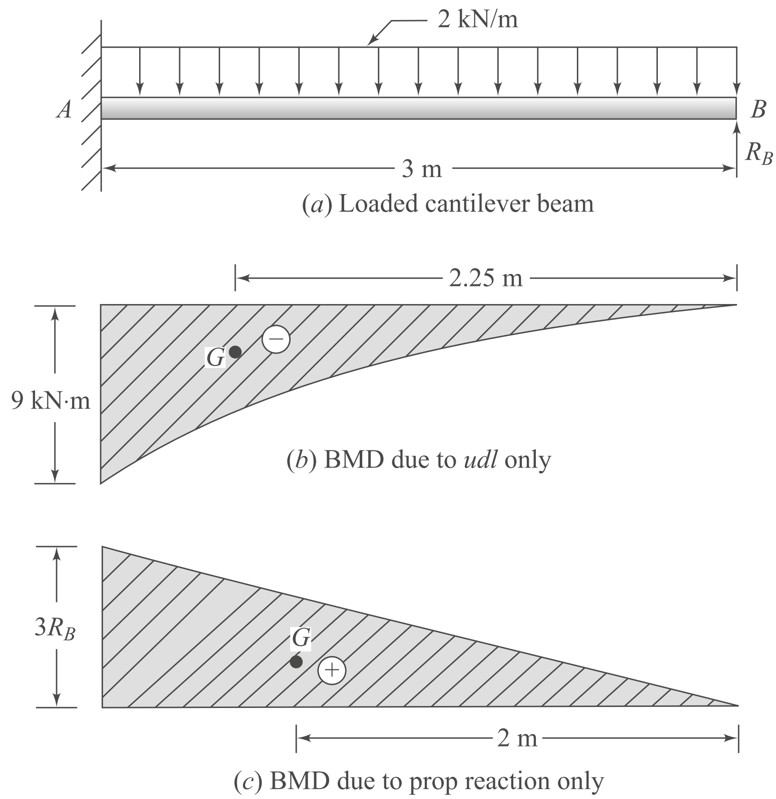


Fig. 6.23

Let R_B be the prop reaction acting at the point B. Fig. 6.23 (b) gives the BMD when only *udl* is supposed to be acting on the cantilever and there is no prop reaction. Fig. 6.23 (c) gives the BMD when only prop reaction acts, and there is no *udl* on the cantilever.

The deflection at B, when no prop reaction acts, is given as

$$y_B = \frac{\text{Moment of the area of } BMD \text{ between } A \text{ and } B \text{ about } B}{EI} \quad \text{(using Mohr's first theorem)}$$

$$= \frac{1}{EI}\left(\frac{1}{3} \times \frac{wl^2}{2} \times l\right) \times \frac{3l}{4}$$

$$= \frac{1}{10^4}\left(\frac{1}{3} \times \frac{2 \times 3^2}{2} \times 3\right) \times \frac{3}{4} \times 3 = 2.025 \times 10^{-3} \text{ m} = 2.025 \text{ mm}$$

This deflection is directed downward.

In case, when no *udl* were acting on the cantilever beam, then the cantilever would have deflected upward by an amount given by

$$y_{B'} = \frac{\text{Moment of the area of } BMD \text{ between } A \text{ and } B \text{ about } B}{EI}$$

$$= \frac{1}{EI}\left(\frac{1}{2} \times R_B \times l \times l\right) \times \frac{2}{3} \times l$$

Because height of the triangle is $R_B\, l$ and its $C.G.$ is located at a distance of $\frac{2}{3}\, l$ from the apex.

or $$y_{B'} = \frac{1}{10^4}\left(\frac{1}{2} \times R_B \times 3 \times 3\right) \times \frac{2}{3} \times 3 = 9 \times 10^{-4}\, R_B$$

Since end B does not yield, it means that both deflections y_B and $y_B{'}$ are the same.

$$2.025 \times 10^{-3} = 9 \times 10^{-4}\, R_B$$

or $$R_B = \frac{2.025 \times 10^{-3}}{9 \times 10^{-4}} = 2.25 \text{ kN}(\uparrow)$$ **Ans.**

Example 6.9

A cantilever beam of length $2l$ carries two point loads as shown in Fig. 6.24. The flexural rigidity for the part AB is $2EI$ and for the part BC *is* EI. Find the slope and deflection at free end of the beam.

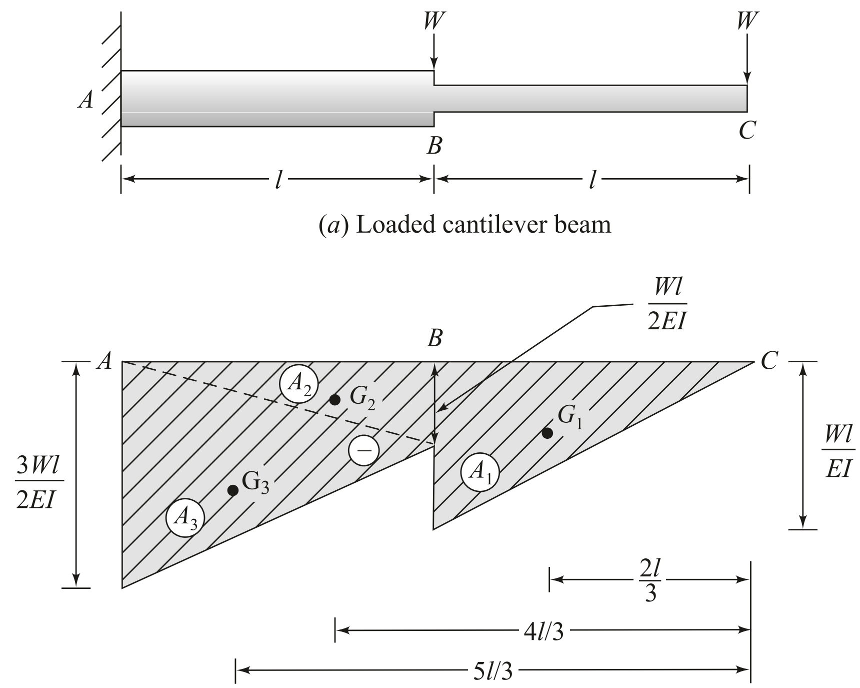
Solution: Refer Fig. 6.24.

The (M/EI) diagram is divided into three parts (triangles A_1, A_2 and A_3). Their centre of gravities G_1, G_2 and G_3 are located at $\frac{2l}{3}$, $\frac{4l}{3}$ and $\frac{5l}{3}$ respectively from the point C. The (M/EI) areas are found as

$$A_1 = \frac{1}{2} \times \frac{Wl}{EI} \times l = \frac{Wl^2}{2EI}$$

$$A_2 = \frac{1}{2} \times \frac{Wl}{2EI} \times l = \frac{Wl^2}{4EI}$$

and $$A_3 = \frac{1}{2} \times \frac{3Wl}{2EI} \times l = \frac{3Wl^2}{4EI}$$



(*a*) Loaded cantilever beam

(*b*) (*M/EI*) Diagram

Fig. 6.24

The slope and deflection at the fixed end of the beam are zero.

$$\theta_A = 0 \text{ and } y_A = 0$$

Using first moment-area theorem, the slope at C is given as

$$\theta_C = \text{Area of the } (M/EI) \text{ diagram between } A \text{ and } C$$

$$= A_1 + A_2 + A_3$$

$$= \frac{Wl^2}{2EI} + \frac{Wl^2}{4EI} + \frac{3Wl^2}{4EI} = \frac{3Wl^2}{2EI} \qquad \textbf{Ans.}$$

Using second moment-area theorem, the deflection at C is given as

$$y_c = \text{Moment of the } (M/EI) \text{ diagram between } A \text{ and } C \text{ about } C$$

$$= A_1 \times \frac{2l}{3} + A_2 \times \frac{4l}{3} + A_3 \times \frac{5l}{3}$$

$$= \frac{Wl^2}{2EI} \times \frac{2}{3}l + \frac{Wl^2}{4EI} \times \frac{4l}{3} + \frac{3Wl^2}{4EI} \times \frac{5l}{3} = \frac{23Wl^3}{12EI} \qquad \textbf{Ans.}$$

Example 6.10

In Example 6.9, if $W = 5$ kN, $l = 2$ m and $EI = 1 \times 10^4$ kN·m^2, find the slope and deflection at free end of the beam.

Solution: The slope at the free end is given as

$$\theta_C = \frac{3Wl^2}{2EI}$$

$$= \frac{3 \times 5 \times 2^2}{2 \times 10^4} = 0.003 \text{ radian} = 0.171° \qquad \textbf{Ans.}$$

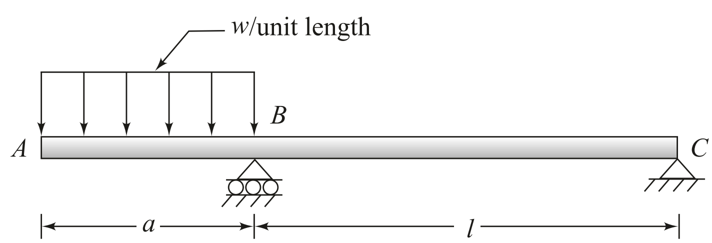
The deflection at the free end is given as

$$y_C = \frac{23Wl^3}{12EI} = \frac{23 \times 5 \times 2^3}{12 \times 10^4} = 0.0076 \text{ m} = 7.6 \text{ mm} \qquad \textbf{Ans.}$$

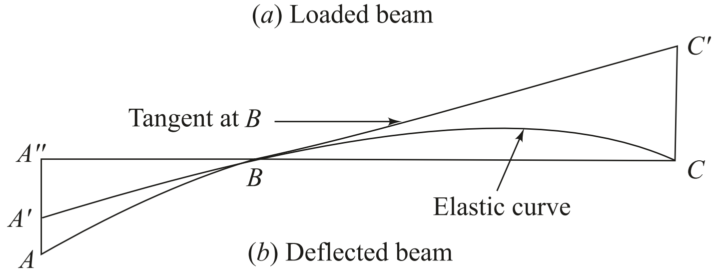
Example 6.11

For the beam shown in Fig. 6.25, find the deflection at point A.

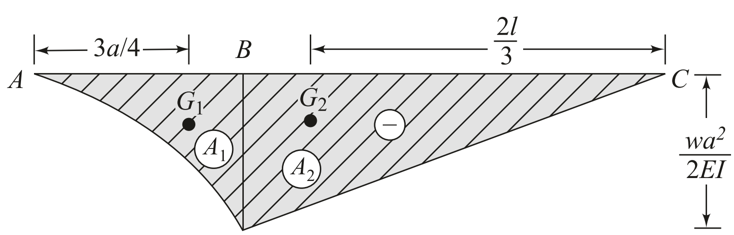
Solution: Refer Fig. 6.25. The (M/EI) diagram is shown in Fig. 6.25 (c). It consists of a parabolic segment and a triangle.



(a) Loaded beam



(b) Deflected beam



(c) (M/EI) Diagram

Fig. 6.25

The (M/EI) area for the parabolic segment is found as

$$A_1 = \frac{1}{3} \times \left(\frac{wa^2}{2EI}\right) \times a = \frac{wa^3}{6EI}$$

The (M/EI) area for the triangle is found as

$$A_2 = \frac{1}{2} \times \frac{wa^2}{2EI} \times l$$

$$= \frac{wa^2 l}{4EI}$$

The reference tangent is drawn at B.

The tangential deflection of C w.r.t. B, using second moment-area theorem is given as

$$y_{C/B} = \text{Moment of the area of } (M/EI) \text{ diagram between } B \text{ and } C \text{ about } C$$

$$= \frac{wa^2 l}{4EI} \times \frac{2l}{3} = \frac{wa^2 l^2}{6EI}$$

Comparing Δs $A'BA''$ and BCC', we have

$$A'A'' = \frac{y_{C/B} \times a}{l} = \frac{wa^2 l^2}{6EI} \times \frac{a}{l} = \frac{wa^3 l}{6EI}$$

The tangential deflection of A w.r.t. B is given as

$$y_{A/B} = \text{Moment of the area of } (M/EI) \text{ diagram between } A \text{ and } B \text{ about } A$$

$$= \frac{wa^3}{6EI} \times \frac{3}{4} a = \frac{wa^4}{8EI}$$

Hence, the deflection at A is given as

$$y_A = y_{A/B} + A'A''$$

$$= \frac{wa^4}{8EI} + \frac{wa^3 l}{6EI} = \frac{wa^3}{2EI}\left[\frac{a}{4} + \frac{l}{3}\right]$$

Ans.

6.7 CONJUGATE BEAM METHOD

The conjugate beam method is regarded as the modification of moment area method. The moment area method is convenient for the beams of constant flexural rigidity but this method is particularly useful for the beams of variable flexural rigidity. The method starts with the formation of a conjugate beam, which is an imaginary beam of length equal to the length of the actual beam but the load on acting it is not the actual load, rather it is the elastic weight $\frac{M}{EI}$ corresponding to the actual load, acting at the point of actual load.

The conjugate beam method is based on two theorems stated below.

Conjugate beam theorem *I*

The slope at any section of a loaded beam is equal to the shear force at the corresponding section of the conjugate beam.

Conjugate beam theorem *II*

The deflection at any section of a loaded beam is equal to the bending moment at the corresponding section of the conjugate beam.

6.7.1 Simple Beam carrying a Point Load at its Centre

Refer Fig. 6.26.

Figure 6.26 (*c*) shows a loaded conjugate beam and the loading over it is triangular. The (*M/EI*) diagram for the actual beam is the elastic weight (*M/EI*) for the conjugate beam.

Let $R_{A'}$ = Reaction at A' of the conjugate beam

$R_{B'}$ = Reaction at B' of the conjugate beam

$$R_{A'} = R_{B'} = \frac{\text{Total load on the conjugate beam}}{2}$$

$$= \frac{\frac{1}{2} \times l \times \frac{Wl}{4EI}}{2} = \frac{Wl^2}{16EI}$$

According to the first theorem, the shear force at any section of the conjugate beam is equal to the slope at that section for the actual beam.

θ_A = Slope at A of the actual beam

= Shear force at A of the conjugate beam $= \dfrac{Wl^2}{16EI}$

Similarly θ_B = Slope at B of the actual beam

= Shear force at B of the conjugate beam $= \dfrac{Wl^2}{16EI}$

Hence,
$$\theta_A = \theta_B = \frac{Wl^2}{16EI} \qquad ...(6.82)$$

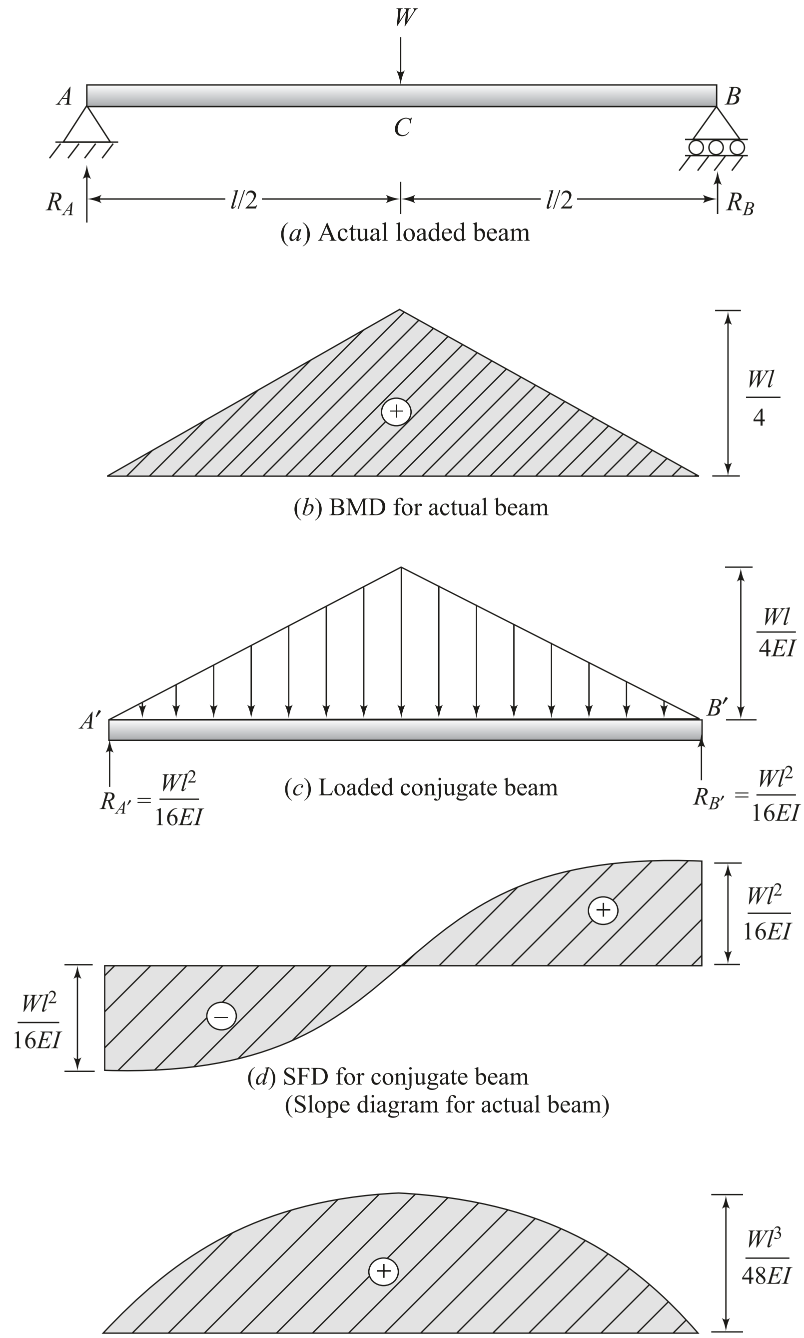
According to the second theorem, the bending moment at any section of the conjugate beam is equal to the deflection at that section for the actual beam.

y_C = Deflection at C of the actual beam

= Bending moment at C of the conjugate beam

$$= R_{A'} \times \frac{l}{2} - \frac{1}{2} \times \frac{l}{2} \times \frac{Wl}{4EI} \times \left(\frac{1}{3} \times \frac{l}{2}\right)$$

$$= \frac{Wl^2}{16EI} \times \frac{l}{2} - \frac{Wl^3}{96EI} = \frac{Wl^3}{48EI} \qquad ... (6.83)$$



(*a*) Actual loaded beam

(*b*) BMD for actual beam

(*c*) Loaded conjugate beam

(*d*) SFD for conjugate beam (Slope diagram for actual beam)

(*e*) BMD for conjugate beam (Deflection diagram for actual beam)

Fig. 6.26

6.7. 2 Simple Beam carrying a Point Load not at the Centre

Refer Fig. 6.27.

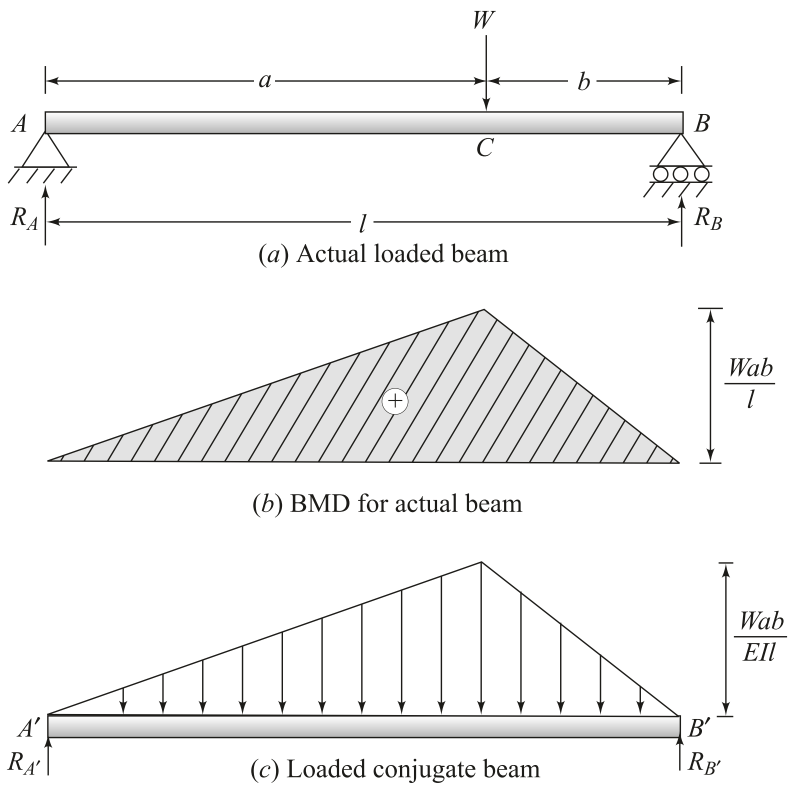


Fig 6.27

Support reactions for actual beam *AB*

$$R_A = \frac{Wb}{l} \text{ and } R_B = \frac{Wa}{l}$$

The bending moment diagram for actual beam is shown in Fig. 6.27 (*b*).

The maximum bending moment at *C* is $\frac{Wab}{l}$ ·

The conjugate beam with equivalent load is shown in Fig. 6.27 (*c*).

Support reactions for conjugate beam *A′ B′*

Using $\Sigma M_{A'} = 0$, we have

$$R_{B'} \times l = \left(\frac{1}{2} \times a \times \frac{Wab}{EIl}\right) \times \frac{2}{3} a + \left(\frac{1}{2} \times b \times \frac{Wab}{EIl}\right) \times \left(a + \frac{b}{3}\right)$$

$$= \frac{Wa^3 b}{3EIl} + \frac{Wab^2}{6EIl}(3a + b) = \frac{Wab}{6EIl}[2a^2 + 3ab + b^2]$$

or
$$R_{B'} = \frac{Wab}{6EIl^2}[2a^2 + 3ab + b^2]$$
$$= \frac{Wab}{6EIl}(l + a) \qquad (a + b = l)$$

and
$$R_{A'} + R_{B'} = \frac{1}{2} \times l \times \frac{Wab}{EIl}$$

On solving, we get
$$R_{A'} = \frac{Wab}{6EIl}(l + b)$$

Using first theorem, we have

θ_A = Slope at A of the actual beam

= Shear force at A of the conjugate beam
$$= \frac{Wab}{6EIl}(l + b) \qquad \text{... (6.84)}$$

and θ_B = Slope at B of the actual beam

= Shear force at B of the conjugate beam
$$= \frac{Wab}{6EIl}(l + a) \qquad \text{... (6.85)}$$

Using second theorem, we have

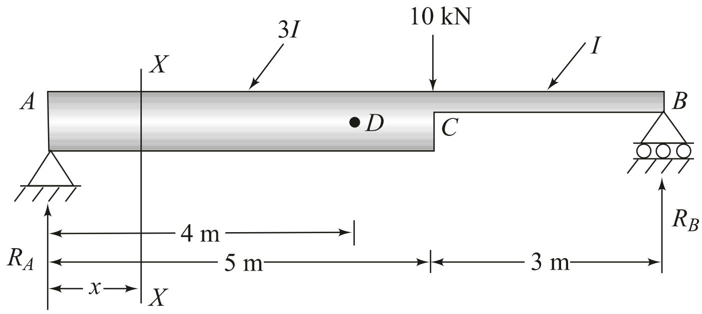
y_C = Deflection at C of the actual beam

= Bending moment at C of the conjugate beam
$$= R_{B'} \times b - \frac{1}{2} \times b \times \frac{Wab}{EIl} \times \frac{b}{3}$$
$$= \frac{Wab}{6EIl}(l + a) \times b - \frac{Wab^3}{6EIl} = \frac{Wa^2b^2}{3EIl} \qquad \text{... (6.86)}$$

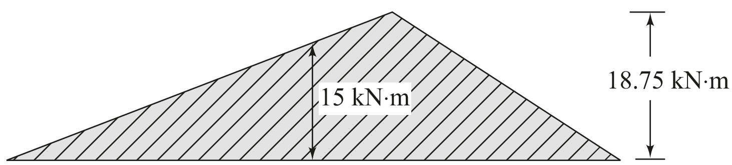
Example 6.12

For the beam shown in Fig. 6.28, find the slope at the support points and deflection under the given load. Also, find the deflection at the centre of the beam. Take E = 200 GPa and $I = 4 \times 10^{-5}$ m^4.

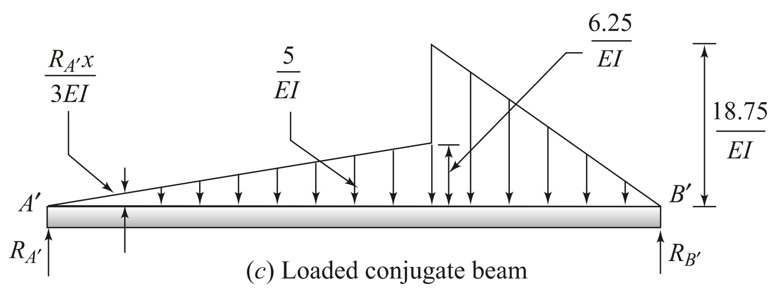
Solution: Refer Fig. 6.28.



(*a*) Actual loaded beam



(*b*) BMD for actual beam



(*c*) Loaded conjugate beam

Fig. 6.28

Support reactions for the actual beam

Using $\sum M_A = 0$, we have

$$R_B \times 8 = 10 \times 5$$

or $$R_B = 6.25 \text{ kN } (\uparrow)$$

Now $$R_A + R_B = 10 \text{ kN}$$

and $$R_A = 10 - 6.25 \text{ kN} = 3.75 \text{ kN } (\uparrow)$$

***BMD* for the actual beam**

Bending moment at C is

$$M_C = R_B \times 3 = 6.25 \times 3$$
$$= 18.75 \text{ kN·m}$$

Bending moment at D is

$$M_D = R_A \times 4 = 3.75 \times 4 = 15 \text{ kN·m}$$

The *BMD* for actual beam is shown in Fig. 6.28 (*b*).

Conjugate beam (Fig. 6.28 (*c*))

$$\text{Load intensity just to the left of } C = 18.75 \times \frac{1}{E\,(3I)} = \frac{6.25}{EI}$$

$$\text{Load intensity just to the right } C = 18.75 \times \frac{1}{EI} = \frac{18.75}{EI}$$

$$\text{Load intensity at } D = R_A \times 4 \times \frac{1}{E\,(3I)} = \frac{3.75 \times 4}{3EI} = \frac{5}{EI}$$

Support reactions at A' and B'

Using $\Sigma M_{B'} = 0$, we have

$$R_{A'} \times 8 = \left[\left(\frac{1}{2} \times 5 \times \frac{6.25}{EI}\right) \times \left(3 + \frac{5}{3}\right) + \left(\frac{1}{2} \times 3 \times \frac{18.75}{EI}\right) \times \left(\frac{2}{3} \times 3\right)\right]$$

$$= \frac{72.91}{EI} + \frac{56.25}{EI} = \frac{129.16}{EI}$$

or $$R_{A'} = \frac{16.14}{EI}$$

and $$R_{A'} + R_{B'} = \frac{1}{2} \times 5 \times \frac{6.25}{EI} + \frac{1}{2} \times 3 \times \frac{18.75}{EI} = \frac{43.75}{EI}$$

or $$R_{B'} = \frac{43.75}{EI} - \frac{16.14}{EI} = \frac{27.61}{EI}$$

(*a*) Slopes at *A* and *B*

Slope at *A* is

$$\theta_A = \text{Shear force at } A' = R_{A'}$$

$$= \frac{16.14}{EI} = \frac{16.14}{200 \times 10^6 \times 4 \times 10^{-5}}$$

$$= 0.002 \text{ radian} = 0.115°$$ **Ans.**

Slope at *B* is

$$\theta_B = \text{Shear force at } B' = R_{B'}$$

$$= \frac{27.61}{EI} = \frac{27.61}{200 \times 10^6 \times 4 \times 10^{-5}}$$

$$= 0.0034 \text{ radian} = 0.197°$$ **Ans.**

(*b*) Deflection under the given load

Deflection at C is

$$y_C = \text{Bending moment at } C \text{ of the conjugate beam}$$

$$= R_{A'} \times 5 - \frac{1}{2} \times 5 \times \frac{6.25}{EI} \times \frac{5}{3}$$

$$= \frac{16.14}{EI} \times 5 - \frac{25 \times 6.25}{6EI}$$

$$= \frac{16.14 \times 5}{200 \times 10^6 \times 4 \times 10^{-5}} - \frac{25 \times 6.25}{6 \times 200 \times 10^6 \times 4 \times 10^{-5}}$$

$$= 6.83 \times 10^{-3} \text{ m} = 6.83 \text{ mm}$$ **Ans.**

(*c*) Deflection at the midpoint *D*

Deflection at D is

$$y_D = \text{Bending moment at } D \text{ of the conjugate beam}$$

$$= R_{A'} \times 4 - \frac{1}{2} \times 4 \times \frac{5}{EI} \times \frac{4}{3} = \frac{16.14}{EI} \times 4 - \frac{40}{3EI}$$

$$= \frac{16.14 \times 4}{200 \times 10^6 \times 4 \times 10^{-5}} - \frac{40}{3 \times 200 \times 10^6 \times 4 \times 10^{-5}}$$

$$= 6.4 \times 10^{-3} \text{ m} = 6.4 \text{ mm}$$ **Ans.**

Example 6.13

For the beam shown in Fig. 6.29, find the deflection at the centre of the beam. Take $E = 200$ GPa and $I = 300 \times 10^{-5}$ m^4.

Solution: Refer Fig. 6.29.

Support reactions for the actual beam

Using $\Sigma M_A = 0$, we have

$$R_B \times 6 = 250 \times 3$$

or $$R_B = 125 \text{ kN } (\uparrow)$$

Now $$R_A + R_B = 250 \text{ kN}$$

and $$R_A = 250 - 125$$

$$= 125 \text{ kN } (\uparrow)$$

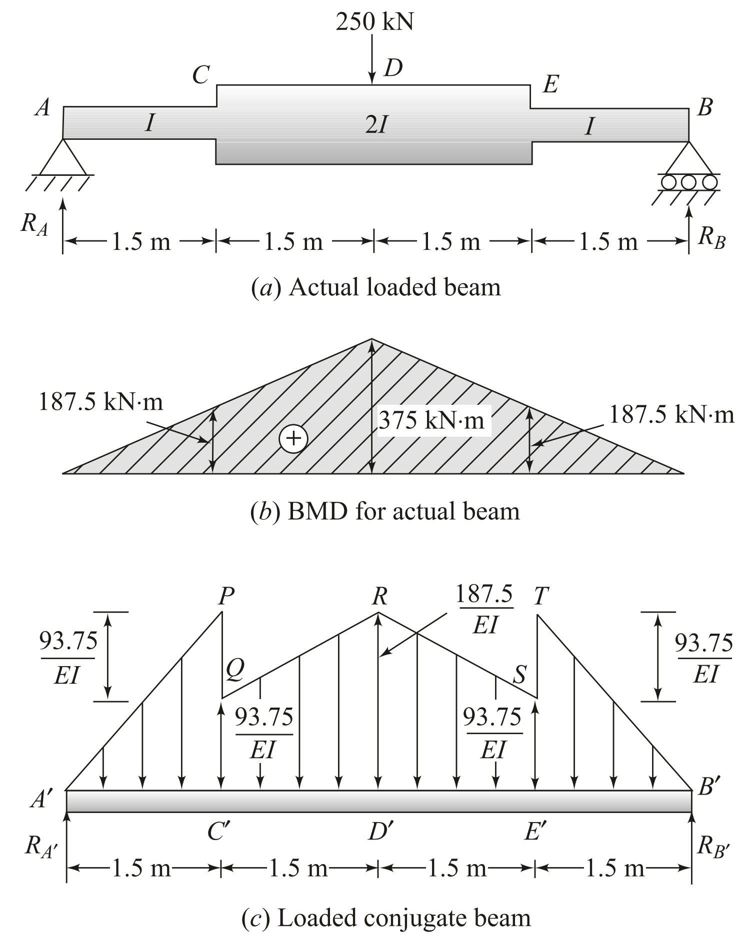


Fig. 6.29

***BMD* for the actual beam**

Bending moment at D is $M_D = 125 \times 3 = 375$ kN·m

Bending moments at C and E are $M_C = M_E = 125 \times 1.5 = 187.5$ kN·m

The *BMD* is shown in Fig. 6.29 (*b*).

Conjugate beam (Fig. 6.29 (*c*))

Load at D' $= \dfrac{375}{E(2I)} = \dfrac{375}{2EI} = \dfrac{187.5}{EI}$

Load just to the left of C' $= \dfrac{187.5}{EI} = PC'$

$=$ Load just to the right of $E' = TE'$

Load just to the right of C' $= \dfrac{187.5}{2EI} = \dfrac{93.75}{EI}$

$= $ Load just to the left of $E' = SE'$

Hence, $$PQ = QC' = \frac{93.75}{EI}$$

and $$TS = SE' = \frac{93.75}{EI}$$

Support reactions at A' and B'

Since load on the conjugate beam is symmetrical, hence the support reactions are equal, each being equal to one-half of the total load on the conjugate beam.

$$R_{A'} = R_{B'} = \text{Area of } \Delta\, A'\, PC' + \text{Area of trapezium } C'QRD'$$

$$= \left[\frac{1}{2} \times 1.5 \times \frac{187.5}{EI}\right] + \frac{1}{2}\left(\frac{93.75}{EI} + \frac{187.5}{EI}\right) \times 1.5 = \frac{351 \cdot 5625}{EI}$$

Deflection at the midpoint D of the actual beam

Deflection at D is

y_D = Bending moment at D' of the conjugate beam

$$= R_{A'} \times 3 - \text{area } A'\, PC' \times \left(1.5 + \frac{1.5}{3}\right) - \text{area } C'QRD' \times \frac{C'D'}{3} \times \left(\frac{2QC' + RD'}{QC' + RD'}\right)$$

$$= \frac{351.5625}{EI} \times 3 - \frac{140.625}{EI} \times 2 - \frac{210.9375}{EI} \times \frac{1.5}{3} \times \left(\frac{2 \times \frac{93.75}{EI} + \frac{187.5}{EI}}{\frac{93.75}{EI} + \frac{187.5}{EI}}\right)$$

$$= \frac{1054.6875}{EI} - \frac{281.25}{EI} - \frac{140.625}{EI} = \frac{632.8125}{EI}$$

$$= \frac{632.8125}{\left(\frac{200 \times 10^9}{10^3}\right) \times 300 \times 10^{-5}} = 1.054 \times 10^{-3} \text{ m} = 1.054 \text{ mm.}$$ **Ans.**

Example 6.14

For the beam shown in Fig. 6.30, find the slope at the support points and the deflection at the centre. Take

$$E = 200 \text{ GPa}$$

$$I = 250 \times 10^{-4} \text{ m}^4$$

Solution: Refer Fig. 6.30.

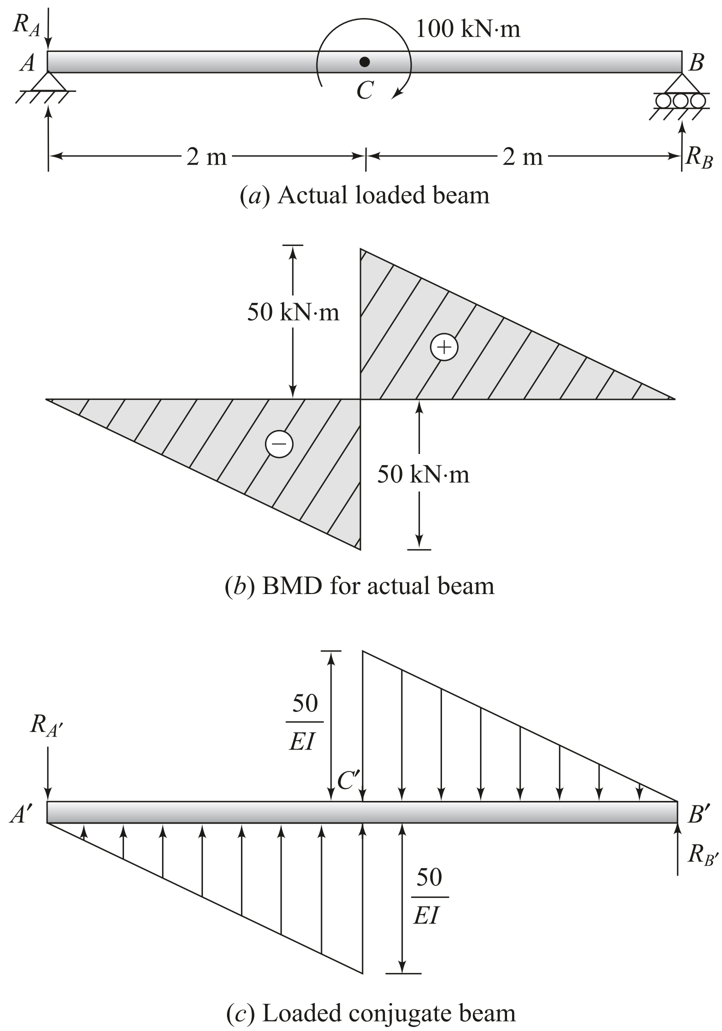


Fig. 6.30

Support reactions for the actual beam

Using $\sum M_A = 0$, we have

$$R_B \times 4 = 100$$

or $$R_B = \frac{100}{4} = 25 \text{ kN } (\uparrow)$$

Now $$R_A + R_B = 0$$

or $$R_A = -R_B = -25 \text{ kN}$$

$$= 25 \text{ kN } (\downarrow)$$

***BMD* for the actual beam**

The bending moments at A and B are zero.

$$M_A = M_B = 0$$

Bending moment just to the right of C

$$= R_B \times 2 = 25 \times 2 = 50 \text{ kN·m}$$

Bending moment just to the left of $C = R_B \times 2 - 100$

$$= 50 - 100 = -50 \text{ kN·m}$$

Conjugate beam (Fig. 6.30 (*c*))

Load just to the right of $C' = \dfrac{50}{EI}$

Load just to the left of $C' = \dfrac{50}{EI}$

Support reactions at *A′* and *B′*

Using $\sum M_{A'} = 0$, we have

$$R_{B'} \times 4 + \frac{1}{2} \times 2 \times \frac{50}{EI} \times \frac{2}{3} \times 2 = \frac{1}{2} \times 2 \times \frac{50}{EI} \times \left(2 + \frac{2}{3}\right)$$

$$R_{B'} \times 4 + \frac{200}{3EI} = \frac{400}{3EI}$$

or $$R_{B'} = \frac{400 - 200}{3EI \times 4} = \frac{50}{3EI} (\uparrow)$$

and $$R_{A'} = \frac{50}{3EI} (\downarrow)$$

Slope at the support points *A* and *B* of the actual beam

Slope at A = Shear force at A' of the conjugate beam

$$= R_{A'}$$

$$= \frac{50}{3EI}$$

$$= \frac{50}{3 \times \left(\dfrac{200 \times 10^9}{10^3}\right) \times 250 \times 10^{-4}} = 3.33 \times 10^{-6} \text{ radian}$$ **Ans.**

Similarly Slope at B = Shear force at B' of the conjugate beam

$$= 3.33 \times 10^{-6} \text{ radian}$$ **Ans.**

Deflection at *C*

Deflection at C is

$$y_C = \text{Bending moment at } C' \text{ of the conjugate beam}$$

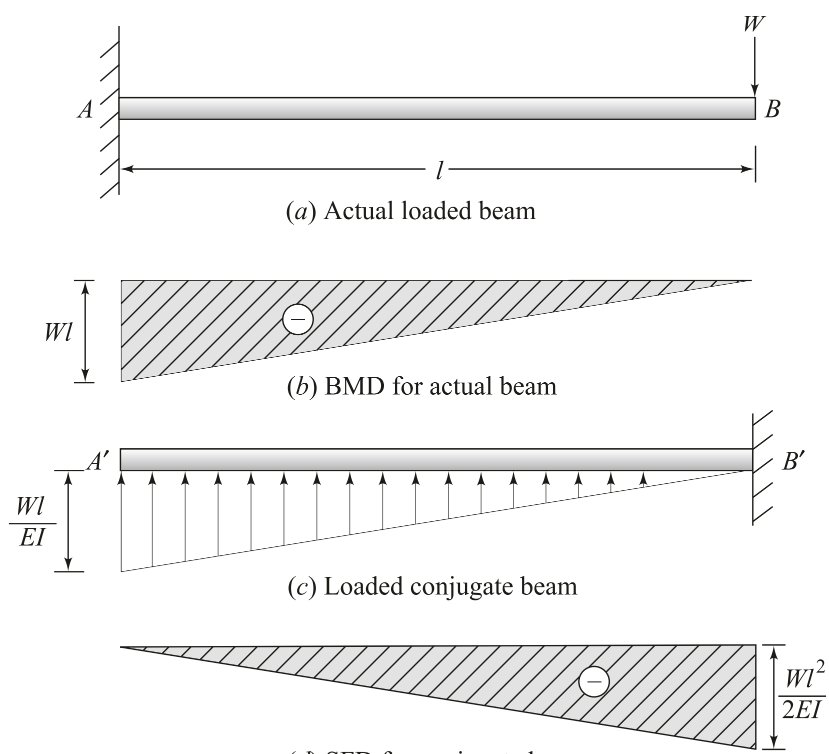
$$= R_{A'} \times 2 - \frac{1}{2} \times 2 \times \frac{50}{EI} \times \frac{2}{3}$$

$$= \frac{50}{3EI} \times 2 - \frac{100}{3EI} = 0$$ **Ans.**

Example 6.15

Find the slope and deflection at free end of a cantilever beam of length l carrying a point load W at its free end.

Solution: Refer Fig. 6.31.



(*a*) Actual loaded beam

(*b*) BMD for actual beam

(*c*) Loaded conjugate beam

(*d*) SFD for conjugate beam

Fig. 6.31

The free and fixed ends of the cantilever beam are converted to fixed and free ends respectively for the conjugate beam.

Slope at *B* of the actual beam

Slope at B is

$$\theta_B = \text{Shear force at } B' \text{ of the conjugate beam}$$

$$= \frac{1}{2} \times l \times \frac{Wl}{EI}$$

$$= \frac{Wl^2}{2EI}$$ **Ans.**

Deflection at B of the actual beam

Deflection at B is

$$y_B = \text{Bending moment at } B' \text{ of the conjugate beam}$$

$$= \frac{1}{2} \times l \times \frac{Wl}{EI} \times \frac{2}{3} \times l$$

$$= \frac{Wl^3}{3EI}$$ **Ans.**

Example 6.16

Find the slope and deflection at the free end of a cantilever beam of length l carrying a point load at a certain distance l_1 from the fixed end.

Solution: Refer Fig. 6.32.

Slope at C is

$$\theta_C = \text{Shear force at } C' \text{ of the conjugate beam}$$

$$= \frac{1}{2} \times l_1 \times \frac{Wl_1}{EI}$$

$$= \frac{Wl_1^2}{2EI}$$ **Ans.**

Deflection at C is

$$y_c = \text{Bending moment at } C' \text{ of the conjugate beam}$$

$$= \frac{1}{2} \times l_1 \times \frac{Wl_1}{EI} \times \frac{2}{3} l_1$$

$$= \frac{Wl_1^3}{3EI}$$ **Ans.**

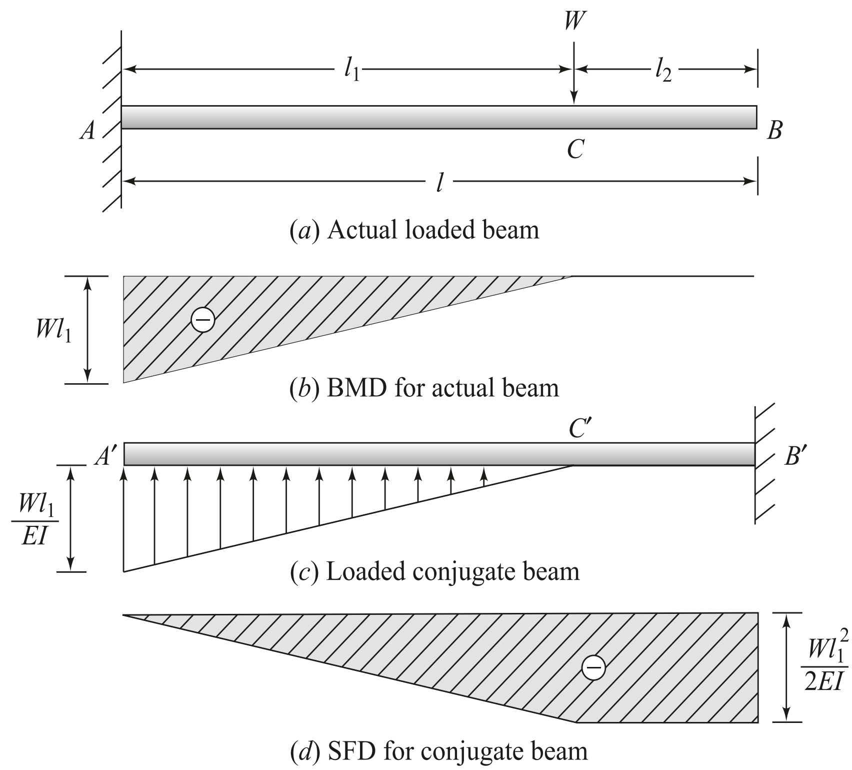


Fig. 6.32

Example 6.17

For a cantilever beam loaded as shown in Fig. 6.33, find the deflection at its free end. Take $E = 200$ GPa and $I = 1 \times 10^{-4}$ m^4.

Solution: Refer Fig. 6.33.

Bending moment at C is

$$M_C = 10 \times 2 = 20 \text{ kN·m}$$

Bending moment at A is

$$M_A = 10 \times 4 + 20 \times 2 = 80 \text{ kN·m}$$

The deflection at B is obtained as

$$y_B = \text{Bending moment at } B' \text{ of the conjugate beam}$$

$$= \left[\frac{1}{2} \times 2 \times \frac{20}{EI} \times \frac{2}{3} \times 2\right] + \left[\frac{20}{EI} \times 2 \times \left(2 + \frac{2}{2}\right)\right] + \left[\frac{1}{2} \times 2 \times \frac{60}{EI} \times \left(2 + \frac{2}{3} \times 2\right)\right]$$

$$= \frac{1}{EI} \times \frac{1040}{3}$$

$$= 0.01733 \text{ m} = 17.33 \text{ mm}$$ **Ans.**

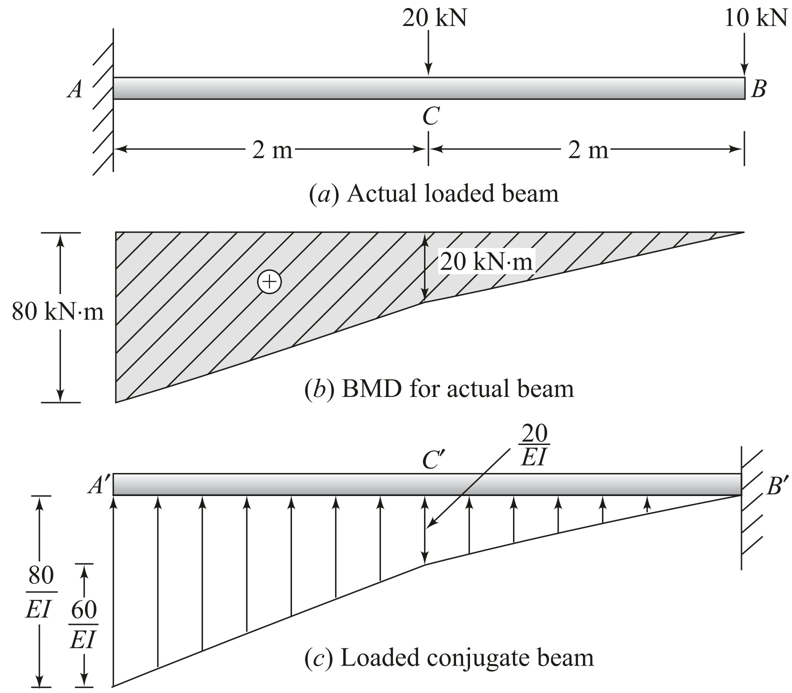


Fig. 6.33

Example 6.18

Find the slope and deflection at free end of a 3m cantilever beam carrying a *udl* of intensity 2 kN/m over its entire span. Take $E = 200$ GPa and $I = 2 \times 10^{-5}$ m^4.

Solution: Refer Fig. 6.34.

Bending moment at A is

$$M_A = \frac{2 \times 3^2}{2} = 9 \text{ kN·m}$$

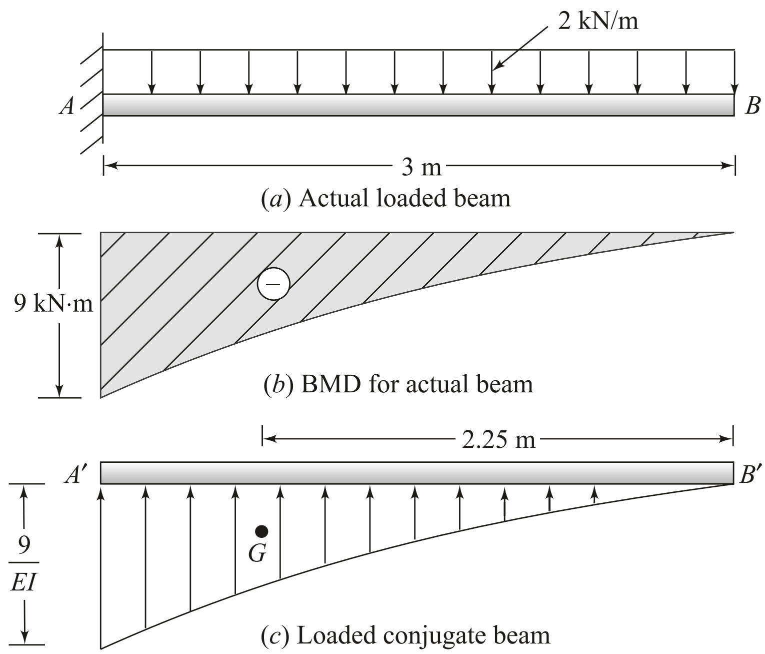
The variation of bending moment between A and B is parabolic.
The centroid of the loaded conjugate beam acts at a distance of

$$\frac{3}{4} \times 3 = 2.25 \text{ m from } B'.$$

The slope at B is found as θ_B = Shear force at B' of the conjugate beam

$$= \frac{1}{3} \times 3 \times \frac{9}{EI} = \frac{9}{EI}$$

$$= \frac{9}{\left(\dfrac{200 \times 10^9}{10^3}\right) \times 2 \times 10^{-5}} = 0.00225 \text{ radian} = 0.129°$$ **Ans.**



(*a*) Actual loaded beam

(*b*) BMD for actual beam

(*c*) Loaded conjugate beam

Fig. 6.34

The deflection at B is obtained as

$$y_B = \text{Bending moment at } B' \text{ of the conjugate beam}$$

$$= \frac{1}{3} \times 3 \times \frac{9}{EI} \times 2.25 = \frac{20.25}{EI}$$

$$= \frac{20.25}{\left(\dfrac{200 \times 10^9}{10^3}\right) \times 2 \times 10^{-5}}$$

$$= 5.06 \times 10^{-3} \text{ m} = 5.06 \text{ mm}$$ **Ans.**

6.8 METHOD OF SUPERPOSITION

This method is mainly suitable for linearly elastic deformation problems where load-deformation relationship is linear and the deformations produced are very small as compared to the transverse dimensions of the beam. The method can be used to study the deformation behaviour of a structure under a given set of loads. According to this method, the total deflection of a beam at any section is equal to the algebraic sum of the deflections produced separately at that section due to each load.

Consider a cantilever beam carrying three point loads at certain distances from the fixed end (Fig. 6.35).

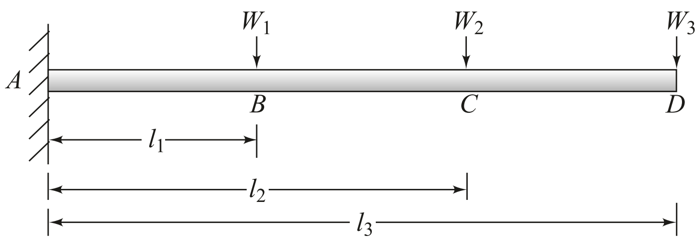


Fig. 6.35

Let y_1 = Deflection produced at B due to load W_1 only

y_2 = Deflection produced at B due to load W_2 only

y_3 = Deflection produced at B due to load W_3 only

The total deflection produced at D, according to method of superposition is

$$y = y_1 + y_2 + y_3 \quad \text{... (6.87)}$$

Example 6.19

Find the slope and deflection at the free end of a cantilever beam of length 6 m as shown in Fig. 6.36, using method of superposition.

Solution: Refer Fig. 6.36.

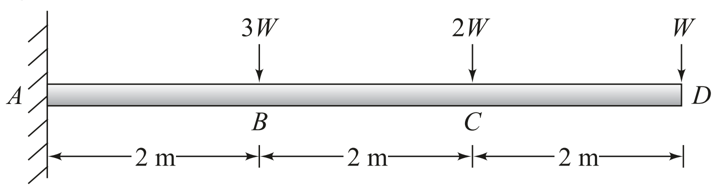


Fig. 6.36

Let θ_B and y_B = Slope and deflection at free end D due to $3W$ only (acting at B)

θ_C and y_C = Slope and deflection at D due to $2W$ only (acting at C)

θ_D and y_D = Slope and deflection at D due to W only (acting at D)

The slopes and deflections at various points of the beam are given as

$$\theta_B = \frac{3W \times 2^2}{2EI} = \frac{6W}{EI}$$

$$y_B = \frac{3W \times 2^3}{3EI} + \theta_B \times (2+2) = \frac{8W}{3EI} + \frac{6W}{EI} \times 4 = \frac{80W}{3EI}$$

$$\theta_C = \frac{2W \times (2+2)^2}{2EI} = \frac{16W}{EI}$$

$$y_C = \frac{2W \times (2+2)^3}{3EI} + \theta_c \times 2 = \frac{128W}{3EI} + \frac{16W}{EI} \times 2 = \frac{224W}{3EI}$$

$$\theta_D = \frac{W \times (2+2+2)^2}{2EI} = \frac{18W}{EI}$$

$$y_D = \frac{W \times (2+2+2)^3}{3EI} + \theta_D \times 0 = \frac{216W}{3EI}$$

Hence, the slope at the free end is

$$\theta_B + \theta_C + \theta_D = \frac{6W}{EI} + \frac{16W}{EI} + \frac{18W}{EI} = \frac{40W}{EI}$$ **Ans.**

And the deflection at the free end is

$$y_B + y_C + y_D = \frac{80W}{3EI} + \frac{224W}{3EI} + \frac{216W}{3EI} = \frac{520W}{3EI}$$ **Ans.**

SHORT ANSWER QUESTIONS

1. How is deflection of a beam defined?
2. Why is elastic curve so called?
3. Which method uses singularity function to find the deflections of beams?
4. What is Castigliano's theorem? What is its use?
5. Which method uses Mohr's theorem to find the deflections of beams?
6. What is conjugate beam? How is it useful in finding the deflections of beams?
7. How is the method of superposition useful in finding the deflections of beams?

MULTIPLE CHOICE QUESTIONS

1. The deflected neutral surface of a beam after bending is called

(*a*) deflected surface (*b*) bent surface (*c*) elastic curve (*d*) plastic curve.

2. The differential equation of flexure is

(*a*) $EI\dfrac{d^2y}{dx^2} = M$ (*b*) $EI\dfrac{dy}{dx} = M$ (*c*) $EI\dfrac{d^3y}{dx^3} = M$ (*d*) $EM\dfrac{dy}{dx} = I$.

3. The deflection produced by bending is

(*a*) equal to the deflection produced by shear

(*b*) less than the deflection produced by shear

(*c*) greater than the deflection produced by shear

(*d*) unpredictable.

4. The flexural rigidity is the product of

(*a*) modulus of elasticity and mass moment of inertia

(*b*) modulus of rigidity and area moment of inertia

(*c*) modulus of rigidity and mass moment of inertia

(*d*) modulus of elasticity and area moment of inertia.

5. The slope and deflection at the fixed end of a cantilever beam are

(*a*) zero, maximum (*b*) zero, zero

(*c*) maximum, minimum (*d*) maximum, zero.

6. The slope and deflection at the centre of a simple beam carrying a central point load are

(*a*) zero, zero (*b*) zero, maximum

(*c*) maximum, zero (*d*) minimum, maximum.

7. Which method uses Mohr's theorem for finding the slope and deflection of a beam?

(*a*) Macaulay's method (*b*) Integration method

(*c*) Moment area method (*d*) Conjugate beam method.

8. The slope at any section of a beam is equal to which parameter of the conjugate beam ?

(*a*) bending moment (*b*) slope

(*c*) deflection (*d*) shear force.

9. The deflection at any section of a beam is equal to which parameter of the conjugate beam ?

(*a*) shear force (*b*) slope (*c*) deflection (*d*) bending moment.

10. The conjugate beam method is the most suitable method for finding the

(*a*) slope and deflection of a uniform sectional beam

(*b*) slope and deflection of a non-uniform sectional beam

(*c*) slope of a uniform sectional beam

(*d*) deflection of a uniform sectional beam.

11. According to moment-area method, the change in slope between any two sections of a beam is equal to the

(*a*) moment of area of (M/EI) diagram between two sections

(*b*) area of the bending moment diagram between two sections

(*c*) area of (M/EI) diagram between two sections

(*d*) area of the shear force diagram between two sections.

12. According to moment area method, deflection at any section of a beam w.r.t. a reference point is equal to the

(*a*) area of (M/EI) diagram between section and reference point

(*b*) moment of area of (M/EI) diagram between section and reference point

(*c*) area of the bending moment diagram between section and reference point

(*d*) area of the shear force diagram between section and reference point.

13. If the diameter of a circular sectional beam is doubled, its deflection is reduced by

(*a*) 16 times (*b*) 4 times (*c*) 8 times (*d*) 32 times.

ANSWERS

1. (*c*) **2.** (*a*) **3.** (*c*) **4.** (*d*) **5.** (*b*) **6.** (*b*) **7.** (*c*) **8.** (*d*) **9.** (*d*)
10. (*b*) **11.** (*c*) **12.** (*b*) **13.** (*a*).

EXERCISES

1. A cantilever beam of length 3 m and cross-section 150 mm wide × 300 mm deep is loaded with a point load of 30 kN at its free end. In addition to this, it also carries a *udl* of intensity 20 kN per metre length over its span. Find the following:

(*a*) the maximum slope and deflection

(*b*) the slope and deflection at 2 m from the fixed end

Take $G = 210$ GPa.

(*Ans.* (*a*) 0.182°, 6.67 mm
(*b*) 0.167°, 3.57 mm).

2. A simply supported beam carries triangularly distributed symmetrical load as shown in Fig. 6.37. Find the maximum deflection.

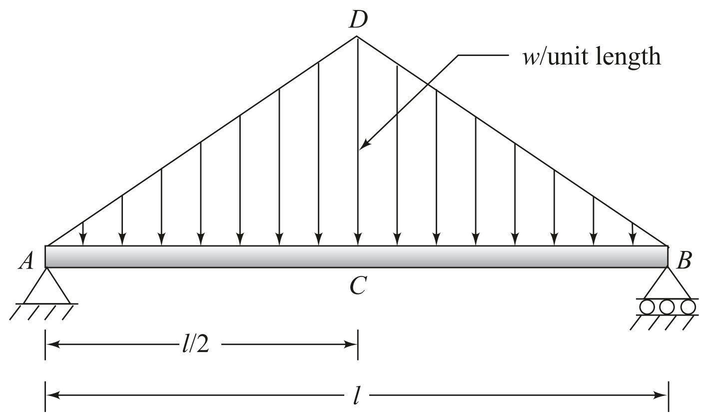


Fig. 6.37

$$\left(Ans. \quad \frac{wl^4}{120EI} \text{ at } C\right).$$

3. Using moment area method, find the slope and deflection at the free end of the cantilever beam shown in Fig. 6.38.

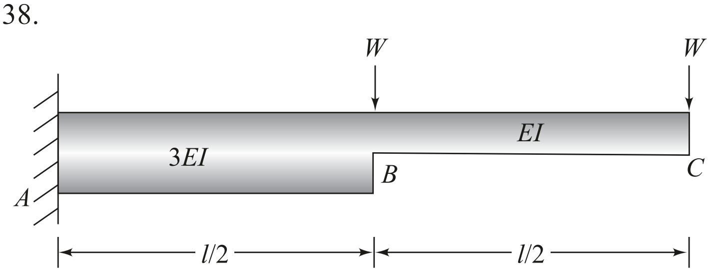


Fig. 6.38

$$\left(\begin{aligned} Ans. \quad \theta_c &= \frac{7}{24}\frac{Wl^2}{EI} \\ y_c &= \frac{25Wl^3}{72\,EI} \end{aligned}\right).$$

4. Using moment area method, find the slope and deflection at the free end of the cantilever beam shown in Fig. 6.39.

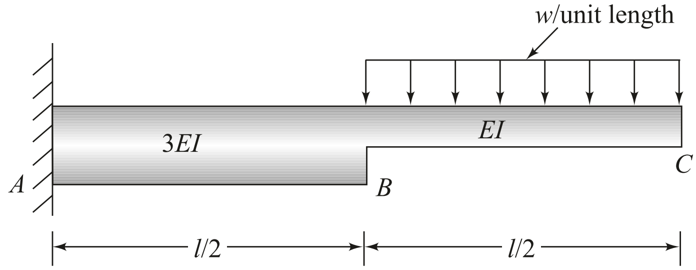


Fig. 6.39

$\left(Ans.\quad \dfrac{wl}{6}\right)$.

5. Using moment area method or otherwise, find the deflection at point A, for the beam shown in Fig. 6.40. Take $EI = 1 \times 10^3$ kN·m^2.

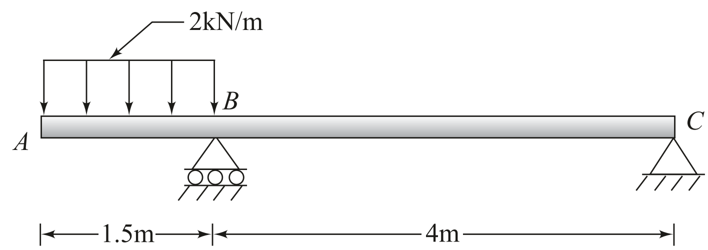


Fig. 6.40

(*Ans.* 5.76 mm).

6. Using moment area method or otherwise, find the deflection at point D, for the beam shown in Fig. 6.41. Take $EI = 1 \times 10^3$ kN·m^2.

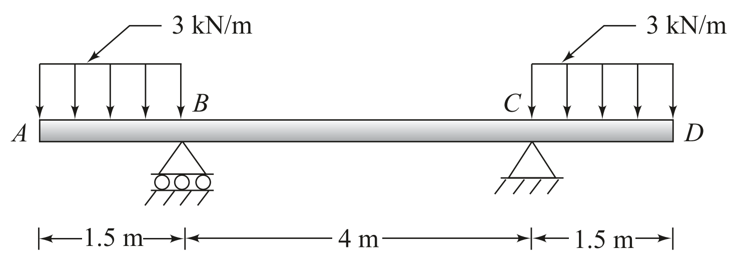


Fig. 6.41

(*Ans.* 12 mm).

7. For a beam ABC shown in Fig. 6.42, find the value of P in terms of w, if deflection at the end C has to be zero. Use conjugate beam method.

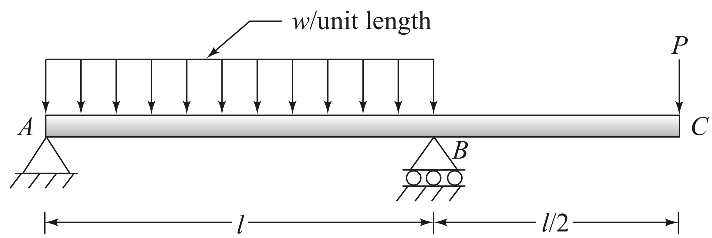


Fig. 6.42

$$\left(Ans.\quad \frac{wl}{6}\right).$$

8. Using conjugate beam method, find out slope at A and deflection at C for the beam shown in Fig. 6.43. Take $I_{AC} = 2I_{BC}$

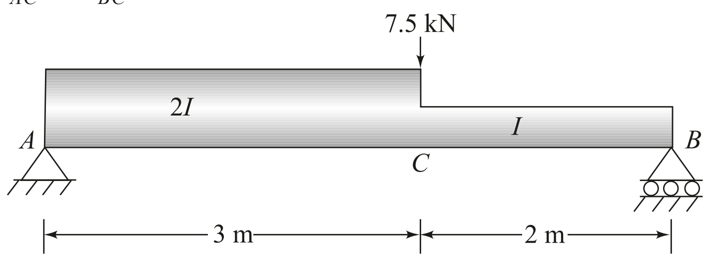


Fig. 6.43

$$\left(Ans.\quad \frac{129}{20EI}, \frac{63}{5EI}\right).$$

9. Find the maximum deflection of a simple beam having flexural rigidity EI and loaded with a distributed load which varies from zero at both ends to w/unit length at the midspan.

$$\left(Ans.\quad \frac{wl^4}{120EI} \text{ at midspan}\right).$$

10. Find the deflection at the free end of a cantilever beam having flexural rigidity EI and loaded with a triangular load as shown in Fig. 6.44.

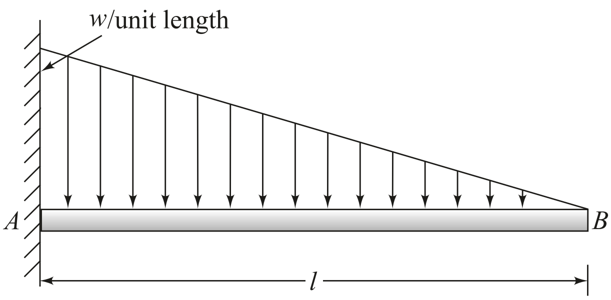


Fig. 6.44

$$\left(Ans.\quad \frac{wl^4}{30EI}\right).$$

11. Find the central deflection of a simple beam having flexural rigidity EI due to uniform load w as shown in Fig. 6.45.

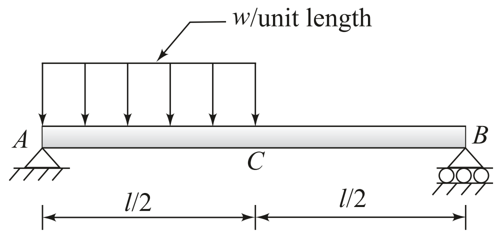


Fig. 6.45

$$\left(Ans.\quad \frac{5}{768}\cdot\frac{wl^4}{EI}\right).$$

12. Find the central deflection of a simple beam having flexural rigidity EI due to two point loads, both equal to W as shown in Fig. 6.46.

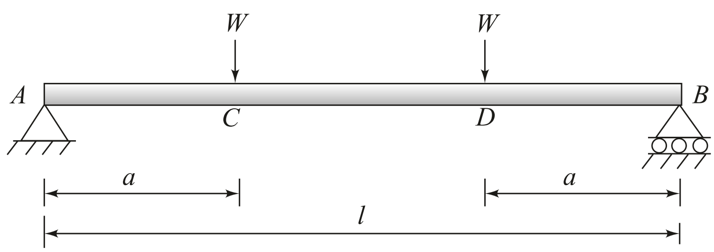


Fig. 6.46

$$\left(Ans.\quad \frac{Wa}{24EI}(3l^2 - 4a^2)\right).$$

13. Find the deflection of a cantilever beam having flexural rigidity EI due to uniform load w acting over the middle half of the beam as shown in Fig. 6.47.

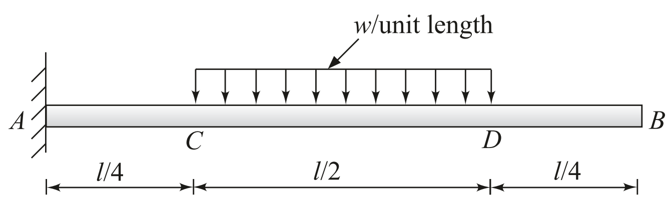


Fig. 6.47

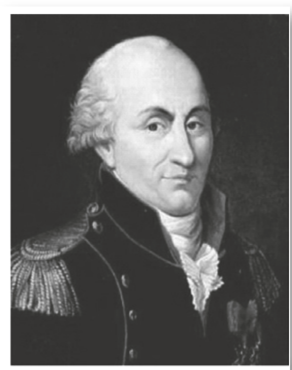
$$\left(Ans.\quad \frac{7}{64}\cdot\frac{wl^4}{EI}\right).$$

14. Find the central deflection of a simple beam of length 7 m, which carries a uniform load that varies from 15 kN/m at one end to 60 KN/m at the other end. The cross-section of the beam is 450 mm deep and the maximum bending stress is 100 MPa. Take E = 210 GPa.

(*Ans.* 21.9 mm).

❑❑❑

7 Torsion of Circular Members



Charles-Augustin
de Coulomb
(1736-1806)

Charles-Augustin de Coulomb, born on 14 June 1736, was a French physicist and engineer. He is best known for his Coulomb's law used in electrostatics for finding the electrostatic force of attraction or repulsion. The SI unit of electric charge, the Coulomb (C), was named in his honour. He derived the torsion formula in about 1775. Coulomb leaves a legacy as a pioneer in the field of geotechnical engineering for his contribution to the design of retaining walls. His name is one of the 72 names inscribed on the Eiffel Tower.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- What is Torsion?
- How much power is transmitted by a rotating shaft?
- What does torsional rigidity signify?
- What is polar modulus?
- Why is polar moment of inertia used in the torsion formula?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_7

7.1 INTRODUCTION

The members in torsion are subjected to twisting action about their longitudinal axes. It has wide engineering applications, most common among them is the transmission shaft which is used to transmit power from one point to another *e.g.,* from the prime mover (steam turbine, gas turbine etc.) to a machine or from a motor to a machine tool, or from the engine to rear axle of an automobile. Solid as well as hollow shafts can be used for the transmission of power.

The torsion formula was derived by a French physicist Charles A. Coulomb in about 1775. His study was confined to circular members. Torsions of non-circular members first introduced by Louis Navier were further improved by St.Venant and Prandtl.

7.2 TORSION EQUATION

The following *assumptions* are made for circular members under torsion:

- The shaft is straight and has uniform cross-section throughout its length.
- The plane sections remain plane and do not get distorted after being subjected to torsion.
- Stresses induced in the shaft are within elastic limit.
- The twist along the shaft is uniform.
- The shaft material is homogeneous and isotropic.

Consider a solid circular shaft which is rigidly connected at one end. If a torque T is applied to the other end, the shaft will twist, with its free end rotating through an angle θ, called the angle of twist (Fig. 7.1). The angle θ varies in proportion to T for a certain range of values of T. Also, it is proportional to the length of the shaft.

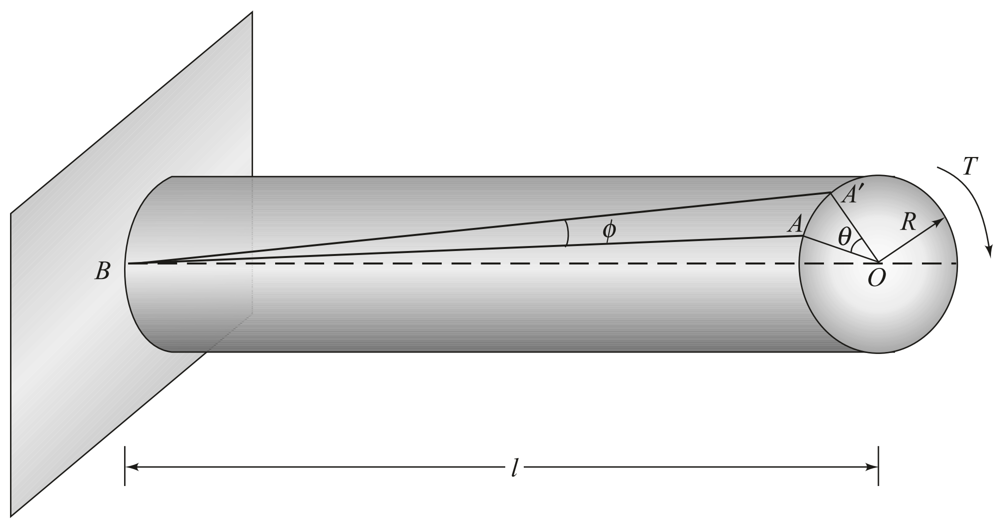


Fig. 7.1 A solid circular shaft subjected to torsion.

Let l = Length of the shaft

R = Radius of the shaft

G = Modulus of rigidity or shear modulus of the shaft material

ϕ = Shear strain

τ = Shear stress

With the application of torque, point A at the radial distance shifts to A' making angle θ at the centre.

$$\angle ABA' = \phi$$
$$\angle AOA' = \theta$$

ϕ and θ both are expressed in radians.

In $\Delta\, ABA'$
$$\tan\phi = \frac{AA'}{AB} = \frac{AA'}{l}$$

or
$$\phi = \frac{AA'}{l} \qquad (\tan\phi \simeq \phi \text{ for small angle})$$

$$AA' = \phi\, l \qquad \text{... (7.1)}$$

Again
$$\theta = \frac{AA'}{OA} = \frac{AA'}{R}$$

$$AA' = \theta R \qquad \text{...(7.2)}$$

From equations (7.1) and (7.2), we have

$$\phi l = \theta R$$

$$\phi = \frac{\theta R}{l} = \phi_{max} \qquad \text{... (7.3)}$$

It shows that the shear strain ϕ at a given point of a shaft under torsion is proportional to the angle of twist θ and is maximum at the surface of the shaft.

Similarly for any point at a radial distance r, we can write

$$\phi = \frac{\theta r}{l} \qquad \text{... (7.4)}$$

It simply means that the shear strain varies linearly with the distance from the axis of the shaft.

From definition of shear modulus

$$G = \frac{\tau}{\phi} \qquad \text{... (7.5)}$$

or
$$\phi = \frac{\tau}{G} = \frac{\theta R}{l} \qquad \text{(using equation (7.3))} \quad \text{... (7.6)}$$

or
$$\frac{\tau_{max}}{R} = \frac{G\theta}{l} = \frac{\tau}{r} \qquad \text{(using equation (7.4))} \quad \text{... (7.7)}$$

Equation (7.7) shows that the shear stress varies linearly with the radial distance r, and is maximum at the surface of the shaft, but the relation holds good, if Hooke's law is obeyed (Fig. 7.2).

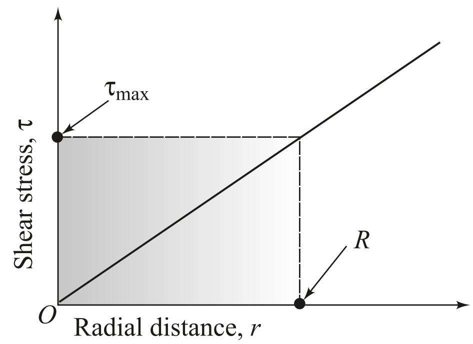
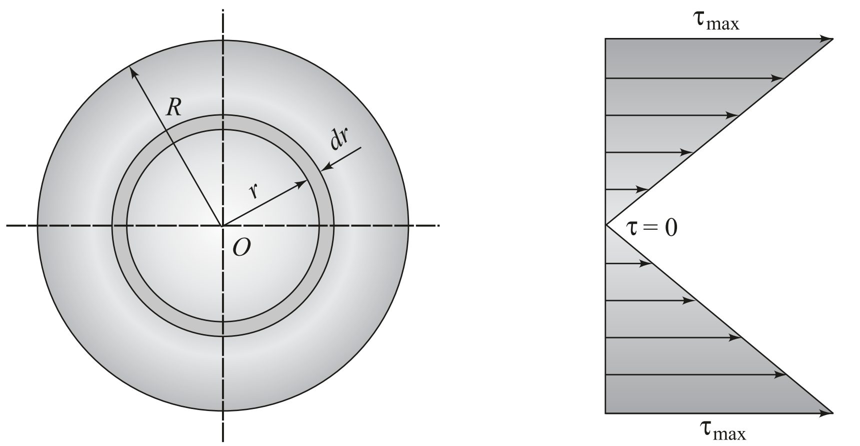


Fig. 7.2

Now consider an elemental ring at a radial distance r from the axis of the shaft (Fig. 7.3). The variation of shear stress is also shown in the same figure.



(*a*) Cross-section of the shaft (*b*) Variation of shear stress

Fig. 7.3

Shear force acting on the ring = $(2\pi\, r\, dr)\tau$

Torque acting on the ring $= (2\pi\, r\, dr \times \tau) \times r = 2\pi\, r^2 dr\tau$ ... (7.8)

From equation (7.7), we have

$$\tau = \frac{t_{max}}{R} \cdot r$$

Using τ in equation (7.8), we get

Torque acting on the ring $= 2\pi\, \frac{\tau_{max}}{R}\, r^3\, dr$

Hence, total torque acting on the entire cross-section is

$$T = \int_0^R 2\pi\, \frac{\tau_{max}}{R}\, r^3\, dr = \frac{2\pi\, \tau_{max}}{R} \left(\frac{r^4}{4} \right)_0^R$$

$$= \frac{2\pi\, \tau_{\max}}{R} \cdot \frac{R^4}{4} = \frac{\tau_{\max}}{R} \cdot \left(\frac{\pi R^4}{2}\right) = \frac{\tau_{\max}}{R} \cdot J$$

where $J = \dfrac{\pi}{2} R^4$ = Polar moment of inertia of the cross-section

$= I_{XX} + I_{YY}$

or
$$\frac{T}{J} = \frac{\tau_{\max}}{R} \qquad \text{... (7.9)}$$

From equations (7.7) and (7.9), we have

$$\frac{T}{J} = \frac{\tau_{\max}}{R} = \frac{G\theta}{l} \qquad \text{... (7.10)}$$

This is known as *torsion equation* and is also valid for a hollow circular shaft with suitable changes in diameter.

7.3 TORSIONAL RIGIDITY

We have, from torsion formula, we have

$$\frac{T}{J} = \frac{G\theta}{l}$$

or
$$JG = \text{Torsional rigidity} = \frac{Tl}{\theta} \qquad \text{... (7.11)}$$

Hence, the product of polar moment of inertia and modulus of rigidity is known as torsional rigidity. It is a measure of strength of shaft against torsion, and is defined as the torque acting on a shaft of unit length producing unit twist.

7.4 POLAR MODULUS

From torsion formula, we have

$$\frac{T}{J} = \frac{\tau_{\max}}{R}$$

or
$$\frac{J}{R} = Z_p = \text{Polar modulus} = \frac{T}{\tau_{\max}} \qquad \text{... (7.12)}$$

For a solid circular shaft of diameter d, the polar modulus is given as

$$\underset{\text{(Solid shaft)}}{Z_p} = \frac{\frac{\pi}{32} d^4}{\left(\frac{d}{2}\right)} = \frac{\pi}{16} d^3 \qquad \text{... (7.13)}$$

For a hollow circular shaft of external diameter d_0 and internal diameter d_i, the polar modulus is given as

$$\underset{\text{(Hollow shaft)}}{Z_p} = \frac{\frac{\pi}{32} (d_0^4 - d_i^4)}{\left(\frac{d_0}{2}\right)} = \frac{\pi}{16} \cdot \frac{(d_0^4 - d_i^4)}{d_0} \qquad \text{... (7.14)}$$

7.5 POWER TRANSMITTED BY A SHAFT

The power generated by a prime mover has to be finally transmitted to some other devices by means of rotating shafts of solid or hollow circular sections.

Let the shaft rotating at N rpm be subjected to a torque T.

Angle turned by the shaft in one rotation = 2π radian

Angle turned by the shaft in N rotations or angle turned per minute = $2\pi N$ radians

Power transmitted by the shaft is given as

$$P = \text{Angle turned per minute} \times \text{torque} = 2\pi NT \qquad \text{... (7.15)}$$

- If we consider angle turned per second and T in N·m, then equation (7.15) can be rewritten as

$$P = \frac{2\pi NT}{60} \text{ (Watt)}$$

$$= \frac{2\pi NT}{60 \times 1000} = \frac{\pi NT}{30000} \text{ (kW)} \qquad \text{... (7.16)}$$

- If torque is expressed in kgf·m and number of revolutions are considered on second basis, then equation (7.15) can be expressed as

$$P = \frac{2\pi NT}{60 \times 75} = \frac{2\pi NT}{4500} \text{ (hp)} \qquad (1 \text{ hp} = 75 \text{ kgf.m/s}) \qquad \text{... (7.17)}$$

Example 7.1

Fig. 7.4 shows the attachment of four pulleys to two different types of shafts. Pulleys B and C are connected by a hollow shaft with inside and outside diameters of 50 mm and 80 mm respectively. Other pulleys are connected by solid shafts of equal diameters. The torque acting on each pulley is shown in the figure. Find the following parameters:

(a) the maximum and minimum shear stress induced in the hollow shaft.

(b) the diameter of the solid shaft, if the maximum shear stress is not to exceed 50 MPa.

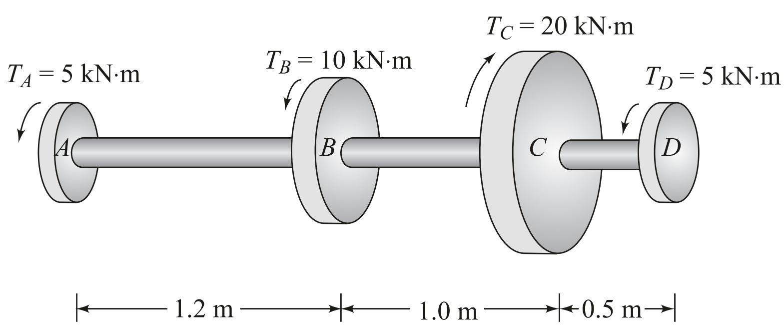


Fig. 7.4

Solution: Given,

Inside diameter of the hollow shaft, d_i = 50 mm

Outside diameter of the hollow shaft, d_o = 80 mm (*a*) The net torque acting on the hollow shaft *BC* is given as

$$T_{BC} = (5 + 10) = 15 \text{ kN·m (Clockwise)}$$

The polar moment of inertia of the section is found as

$$J = \frac{\pi}{32}(d_o^4 - d_i^4)$$

$$= \frac{\pi}{32}\left[\left(\frac{80}{1000}\right)^4 - \left(\frac{50}{1000}\right)^4\right]\text{m}^4 = 3.4 \times 10^{-6} \text{ m}^4$$

Using torsion formula, we have

$$\frac{T_{BC}}{J} = \frac{\tau_{\max}}{\left(\frac{d_o}{2}\right)}$$

Hence, the maximum shear stress induced in the shaft *BC* is given as

$$\tau_{\max} = \frac{T_{BC}}{J} \times \frac{d_o}{2}$$

$$= \frac{15}{3.4 \times 10^{-6}} \times \left(\frac{80}{2} \times \frac{1}{1000}\right)$$

$$= 1.76 \times 10^5 \text{ kN/m}^2$$ **Ans.**

The minimum shear stress induced in the shaft *BC* is given as

$$\tau_{\min} = \frac{T_{BC}}{J} \times \left(\frac{d_i}{2}\right)$$

$$= \frac{15}{3.4 \times 10^{-6}} \times \left(\frac{50}{2} \times \frac{1}{1000}\right) = 1.1 \times 10^5 \text{ kN/m}^2$$ **Ans.**

(*b*) Shafts *AB* and *CD* are solid shafts.

Let d = Diameter of the two shafts

Again from torsion formula

$$\frac{T}{J} = \frac{\tau}{(d/2)}$$

Both solid and hollow shafts are subjected to the same torque of 5 kN·m.

$$\frac{5}{\frac{\pi}{32}d^4} = \frac{50 \times 10^6}{10^3} \times \frac{1}{(d/2)}$$

Solving for *d*, we get

$$d = 0.08 \text{ m} = 80 \text{ mm}$$ **Ans.**

Example 7.2

Three pulleys are connected by two shafts as shown in Fig. 7.5. Assuming that both shafts are solid, find the maximum shear stress induced in the shaft *AB* and *BC*.

Solution: Given,

Diameter of the shaft *AB*, $d_1 = 25$ mm

Diameter of the shaft *BC*, $d_2 = 35$ mm

Length of the shaft *AB* = 1.0 m

Length of the shaft *BC* = 1.5 m

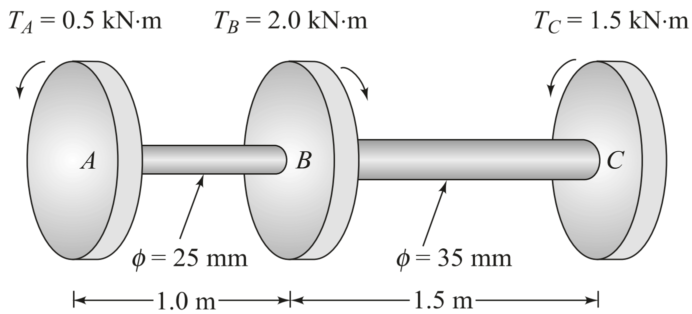


Fig. 7.5

The torque exerted on the shaft *AB* is

$$T_{AB} = 0.5 \text{ kN·m (clockwise)}$$

Using torsion formula, we have

$$\frac{T_{AB}}{J} = \frac{\tau_{max}}{\left(\frac{d_1}{2}\right)}$$

or

$$\frac{T_{AB}}{\frac{\pi}{32} \times d_1^4} = \frac{\tau_{max}}{\left(\frac{d_1}{2}\right)}$$

Hence, the maximum shear stress in the shaft *AB* is given as

$$\tau_{max} = \frac{T_{AB} \times 32}{\pi d_1^3 \times 2}$$

$$= \frac{0.5 \times 32}{\pi \times \left(\frac{25}{1000}\right)^3 \times 2} = 162.97 \text{ MPa} \qquad \textbf{Ans.}$$

The torque exerted on the shaft *BC* is $T_{BC} = 1.5$ kN.m (clockwise).

The maximum shear stress in the shaft BC is given as

$$\tau_{\max} = \frac{T_{BC} \times 32}{\pi d_2^3 \times 2}$$

$$= \frac{1.5 \times 32}{\pi \times \left(\frac{35}{1000}\right)^3 \times 2} = 178.18 \text{ MPa} \qquad \textbf{Ans.}$$

Example 7.3

Four pulleys are connected by solid shafts as the shown in Fig. 7.6. Torque acting on each pulley is shown in the figure. Which shaft experiences the maximum shear stress? Also, find the magnitude of that stress.

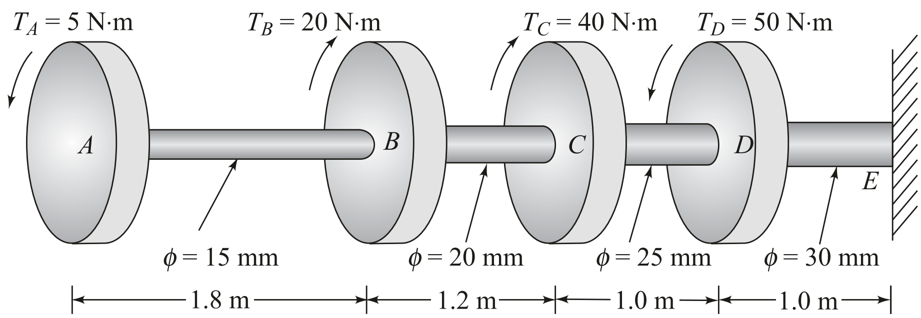


Fig. 7.6

Solution: Refer Fig. 7.6.

Given,

Shaft parameters	*Shaft AB*	*Shaft BC*	*Shaft CD*	*Shaft DE*
Diameter	15 mm	20 mm	25 mm	30 mm
Length	1.8 m	1.2 m	1.0 m	1.0 m
Torque	5 N·m (clockwise)	(20 – 5) = 15 N·m (anticlockwise)	(20 + 40 – 5) = 55N·m (anticlockwise)	(20 + 40 – 5 – 50) = 5N·m (anticlockwise)

Using torsion formula, τ is given as

$$\tau = \frac{T}{J} \times \frac{d}{2}$$

Hence, the shear stress induced in the shaft AB is

$$\tau_{AB} = \frac{5}{\frac{\pi}{32} \times \left(\frac{15}{1000}\right)^4} \times \left(\frac{15}{2 \times 1000}\right)$$

$$= 7.54 \text{ MPa}$$

Similarly, the shear stress induced in the shaft *BC* is

$$\tau_{BC} = \frac{15}{\frac{\pi}{32}\times\left(\frac{20}{1000}\right)^4}\times\left(\frac{20}{2\times1000}\right) = 9.55 \text{ MPa}$$

Shear stress in the shaft *CD* is

$$\tau_{CD} = \frac{55}{\frac{\pi}{32}\times\left(\frac{25}{1000}\right)^4}\times\left(\frac{25}{2\times1000}\right) = 17.92 \text{ MPa}$$

And the shear stress in the shaft *DE* is

$$\tau_{DE} = \frac{5}{\frac{\pi}{32}\times\left(\frac{30}{1000}\right)^4}\times\left(\frac{30}{2\times1000}\right) = 0.94 \text{ MPa}$$

Since τ_{CD} is the biggest among all the values, hence the maximum shear stress is induced in the shaft *CD* and its magnitude is 17.92 MPa. **Ans.**

Example 7.4

An electric motor exerts a torque of 2.5 kN·m at *D* as shown in Fig. 7.7. Find the maximum shear stress induced in the shafts *AB*, *BC* and *CD*, assuming that they are all solid.

Solution: The details of each shaft are given below.

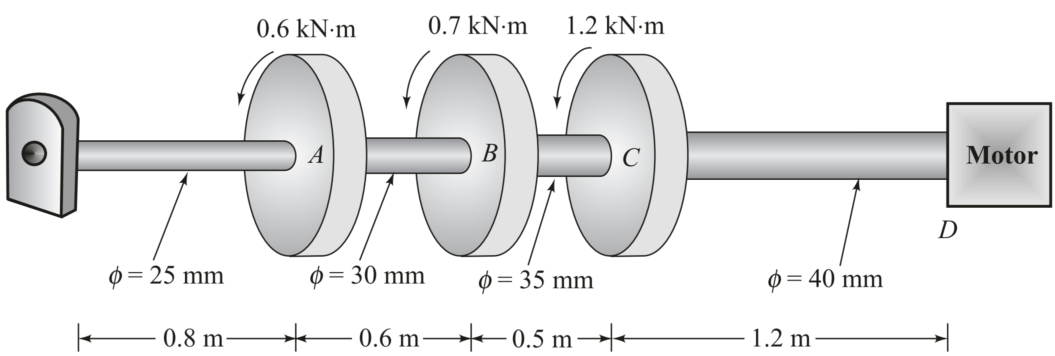


Fig. 7.7

Shaft parameters	***Shaft AB***	***Shaft BC***	***Shaft CD***
Diameter	30 mm	35 mm	40 mm
Length	0.6 m	0.5 m	1.2 m
Torque	0.6 kN·m (clockwise)	(0.6 + 0.7) = 1.3 kN·m (clockwise)	(0.6 + 0.7 + 1.2) = 2.5 kN·m (clockwise)

For shaft *AB*

The maximum shear stress induced in the shaft is

$$\tau = \frac{T}{J} \times \frac{d}{2}$$

$$= \frac{0.6}{\frac{\pi}{32} \times \left(\frac{30}{1000}\right)^4} \times \left(\frac{30}{2} \times \frac{1}{1000}\right) \qquad \left(\text{as } J = \frac{\pi}{32} d^4\right)$$

$$= 113.17 \text{ MPa}$$ **Ans.**

For shaft *BC*

The maximum shear stress induced in the shaft is

$$\tau = \frac{1.3}{\frac{\pi}{32} \times \left(\frac{35}{1000}\right)^4} \times \left(\frac{35}{2} \times \frac{1}{1000}\right)$$

$$= 154.42 \text{ MPa}$$ **Ans.**

For shaft *CD*

The maximum shear stress induced in the shaft is

$$\tau = \frac{2.5}{\frac{\pi}{32} \times \left(\frac{40}{1000}\right)^4} \times \left(\frac{40}{2} \times \frac{1}{1000}\right) = 198.94 \text{ MPa}$$ **Ans.**

Example 7.5

An electric motor running at 600 rpm exerts a torque of 3 kN·m at *D* as shown in Fig. 7.8. Find angle of twist between (*a*) *B* and *C* and (*b*) between *B* and *D*. Take G = 80 GPa.

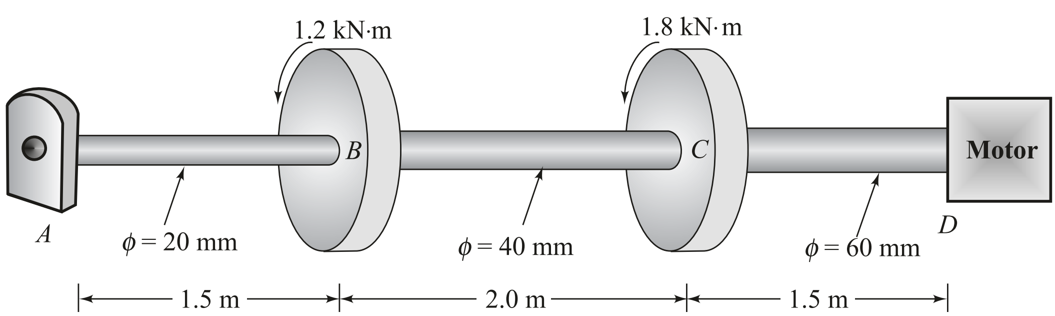


Fig. 7.8

Solution: Refer Fig. 7.8.

The details of each shaft are given below.

Shaft parameters	*Shaft BC*	*Shaft CD*
Diameter	40 mm	60 mm
Length	2.0 m	1.5 m
Torque	1.2 kN·m (clockwise)	1.2 + 1.8 = 3 kN·m (clockwise)

$$G = 80 \text{ GPa} = \frac{80 \times 10^9}{10^3} = 80 \times 10^6 \text{ kPa}$$

Using torsion formula, θ is given as

$$\theta = \frac{Tl}{JG}$$

Hence, the angle of twist produced in the shaft BC is given as

$$\theta_{BC} = \frac{1.2 \times 2}{\frac{\pi}{32} \times \left(\frac{40}{1000}\right)^4 \times 80 \times 10^6}$$

$= 0.1193$ radian $= 6.84^{\circ}$ (anticlockwise) **Ans.**

The angle of twist produced in the shaft CD is given as

$$\theta_{CD} = \frac{3 \times 1.5}{\frac{\pi}{32} \times \left(\frac{60}{1000}\right)^4 \times 80 \times 10^6}$$

$= 0.0442$ radian $= 2.53^{\circ}$ (anticlockwise) **Ans.**

Hence, the angle of twist between B and D is

$$\theta_{BD} = \theta_{BC} + \theta_{CD}$$

$= (6.84 + 2.53)^{\circ} = 9.37^{\circ}$ (anticlockwise) **Ans.**

Example 7.6

Find angle of twist between A and B and between A and C for the arrangement of pulleys and shafts shown in Fig. 7.9. Take $G = 80$ GPa.

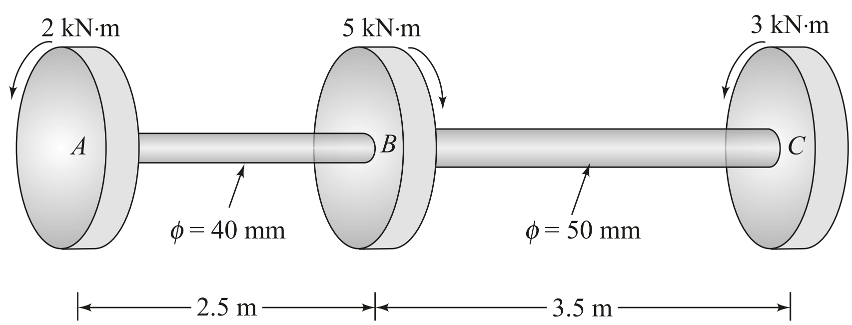


Fig. 7.9

Solution: Refer Fig. 7.9.

The details of each shaft are given below.

Shaft parameters	*Shaft AB*	*Shaft BC*
Diameter	40 mm	50 mm
Length	2.5 m	3.5 m
Torque	2 kN·m (clockwise)	3 kN·m (clockwise)

$$G = 80 \text{ GPa} = \frac{80 \times 10^9}{10^3} = 80 \times 10^6 \text{ kPa}$$

Using torsion formula, the angle of twist for the shaft AB is given as

$$\theta_{AB} = \frac{Tl}{JG}$$

$$= \frac{2 \times 2.5}{\frac{\pi}{32} \times \left(\frac{40}{1000}\right)^4 \times 80 \times 10^6}$$

$$= 0.248 \text{ radian} = 14.25^\circ \text{ (clockwise)}$$ **Ans.**

The angle of twist for the shaft BC is given as

$$\theta_{BC} = \frac{3 \times 3.5}{\frac{\pi}{32} \times \left(\frac{50}{1000}\right)^4 \times 80 \times 10^6}$$

$$= 0.214 \text{ radian} = 12.25^\circ \text{ (clockwise)}$$

Hence, the angle of twist between A and C is given as

$$\theta_{AC} = \theta_{AB} + \theta_{BC} = 14.25^\circ + 12.25^\circ = 26.5^\circ \quad \text{(clockwise)}$$ **Ans.**

Example 7.7

Determine angle of twist between A and C and between A and E for the problem given in Example 7.3. Take $G = 220$ GPa.

Solution: The angles of twist for different shafts are given in the following table.

Shaft	*Angle of twist* (θ)
AB	$\theta_{AB} = \dfrac{5 \times 1.8}{\frac{\pi}{32} \times \left(\frac{15}{1000}\right)^4 \times 220 \times 10^9} \times \dfrac{180}{\pi} \left(\text{using } \theta = \dfrac{Tl}{JG}\right)$ $= 0.471^\circ$ (anticlockwise)
BC	$\theta_{BC} = \dfrac{15 \times 1.2}{\frac{\pi}{32} \times \left(\frac{20}{1000}\right)^4 \times 220 \times 10^9} \times \dfrac{180}{\pi} = 0.298^\circ$ (clockwise)

Contd...

CD	$\theta_{CD} = \dfrac{55 \times 1.0}{\dfrac{\pi}{32} \times \left(\dfrac{25}{1000}\right)^4 \times 220 \times 10^9} \times \dfrac{180}{\pi} = 0.373^\circ$ (clockwise)
DE	$\theta_{DE} = \dfrac{5 \times 1.0}{\dfrac{\pi}{32} \times \left(\dfrac{30}{1000}\right)^4 \times 220 \times 10^9} \times \dfrac{180}{\pi} = 0.016^\circ$ (clockwise)

The angle of twist between *A* and *C* is given as

$$\theta_{AC} = \theta_{AB} - \theta_{BC}$$
$$= (0.471 - 0.298)^\circ = 0.173^\circ \text{ (anticlockwise)}$$ **Ans.**

The angle of twist between *A* and *E* is given as

$$\theta_{AE} = \theta_{AB} - (\theta_{BC} + \theta_{CD} + \theta_{DE})$$
$$= 0.471^\circ - (0.298^\circ + 0.373^\circ + 0.016^\circ)$$
$$= -0.216^\circ = 0.216^\circ \text{ (clockwise)}$$ **Ans.**

Example 7.8

A stepped steel shaft is shown in Fig. 7.10. A torque of 100 N·m is acting at *C* and another torque of 200 N·m is acting at a distance 2 m from *A*. Determine angular displacement of the free end, if the maximum shear stress in the shaft is limited to 50 MPa. Take *G* = 100 GPa.

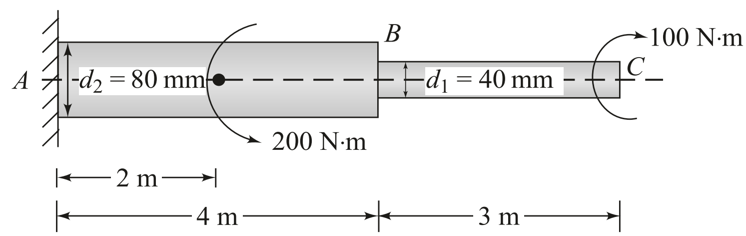


Fig. 7.10

Solution:

For shaft *BC*

Length, $l_1 = 3$ m

Diameter, $d_1 = 40$ mm

Torque, $T_1 = 100$ N·m (clockwise)

Using torsion equation, we have

$$\frac{T}{J} = \frac{G\theta}{l}$$

or $$\theta_{BC} = \frac{T_1 l_1}{J_1 G}$$

$$= \frac{100 \times 3}{\frac{\pi}{32} \times \left(\frac{40}{1000}\right)^4 \times 100 \times 10^9} \times \frac{180}{\pi} \text{ degree} = 0.684^\circ \text{ (clockwise)}$$

For shaft *AB*

Length, $l_2 = 4$ m

Diameter, $d_2 = 80$ mm

Torque, $T_2 = (200 - 100) = 100$ N·m (anticlockwise)

Now $\theta_{AB} = \dfrac{T_2 \times 2}{J_2\, G}$ (Since torque is acting at a distance of 2 m from *A*)

$$= \frac{100 \times 2}{\frac{\pi}{32} \times \left(\frac{80}{1000}\right)^4 \times 100 \times 10^9} \times \frac{180}{\pi} \text{ degree}$$

$$= 0.028^\circ \text{ (anticlockwise)}$$

Hence, the angular displacement of the free end or angle of twist between *A* and *C* is given as

$$\theta_{AC} = (0.684 - 0.028)^\circ$$

$$= 0.656^\circ \text{ (clockwise)}$$ **Ans.**

Example 7.9

A stepped steel shaft shown in Fig. 7.11, is subjected to a torque of 100 N·m (anticlockwise) at *C* and another torque of 200 N·m (clockwise) at *B*. Determine angle of twist at the free end. Shear stress in the shaft is not to exceed 60 MPa and modulus of rigidity is 84 GPa.

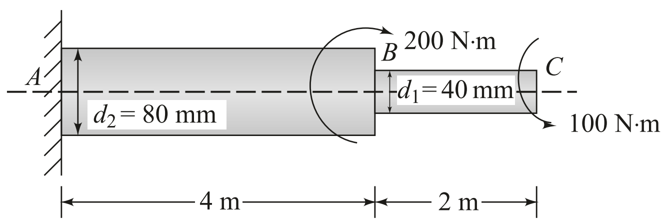


Fig. 7.11

Solution:

For shaft *BC*

Length, $l_1 = 2$ m

Diameter, $d_1 = 40$ mm

Torque, $T_1 = 100$ N·m (Anticlockwise)

Using torsion equation, the angle of twist in the shaft *BC* is given as

$$\theta_{BC} = \frac{T_1\, l_1}{J_1\, G}$$

$$= \frac{100 \times 2}{\frac{\pi}{32} \times \left(\frac{40}{1000}\right)^4 \times 84 \times 10^9} \times \frac{180}{\pi} = 0.542^\circ \text{ (anticlockwise)}$$

For shaft *AB*

Length, $l_2 = 4$ m

Diameter, $d_2 = 80$ mm

Torque, $T_2 = 200 - 100 = 100$ N·m (clockwise)

The angle of twist in the shaft *AB* is given as

$$\theta_{AB} = \frac{T_2\, l_2}{J_2\, G}$$

$$= \frac{100 \times 4}{\frac{\pi}{32} \times \left(\frac{80}{1000}\right)^4 \times 84 \times 10^9} \times \frac{180}{\pi} = 0.067^\circ \text{ (clockwise)}$$

Hence, the net angle of twist at the free end of the shaft

$$= \theta_{BC} - \theta_{AB} = (0.542 - 0.067)^\circ = 0.475^\circ \text{ (anticlockwise)} \qquad \textbf{Ans.}$$

Example 7.10

A composite shaft made of steel and aluminium is fixed at *A* and *C*. A torque of 100 N·m is applied at point *B* (Fig. 7.12). Determine the following:

(*a*) the resisting torques induced at the supports

(*b*) the maximum shear stress induced in each shaft.

Take $G_s = 80$ GPa and $G_{Al} = 30$ GPa.

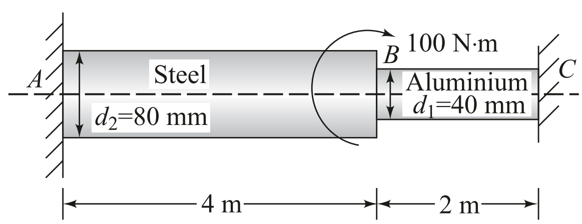


Fig. 7.12

Solution:

For shaft *AB*

Length, $l_S = 4$ m

Diameter, $d_S = 80$ mm

For shaft *BC*

Length, $l_{Al} = 2$ m

Diameter, $d_{Al} = 40$ mm

Let T_A and T_C be the torques induced (anticlockwise) at ends A and C respectively.

Given, $$T_A + T_C = 100 \text{ N·m} \quad \text{... (1)}$$

Since the ends of the shaft are fixed, hence they can't rotate.

$$\theta_{C/A} = \text{Angle of twist of end } C \text{ w.r.t. } A$$
$$= 0$$

$$\frac{T_C \times l_{Al}}{J_{Al}\, G_{Al}} + \frac{T_C \times l_S}{J_S\, G_S} - \frac{100 \times l_S}{J_S\, G_S} = 0 \quad \text{... (2)}$$

$$T_C \left[\frac{2}{\frac{\pi}{32} \times \left(\frac{40}{1000}\right)^4 \times 30 \times 10^9} + \frac{4}{\frac{\pi}{32} \times \left(\frac{80}{1000}\right)^4 \times 80 \times 10^9} \right] = \frac{100 \times 4}{\frac{\pi}{32} \times \left(\frac{80}{1000}\right)^4 \times 80 \times 10^9}$$

On solving, we get $T_C = 4.47$ N·m **Ans.**

From equation (1), we have

$$T_A = 100 - T_C$$
$$= 100 - 4.47 = 95.53 \text{ N·m} \qquad \textbf{Ans.}$$

The maximum shear stress induced in the aluminium part is given as

$$\tau_{Al} = \frac{T_C \times \left(\frac{d_{Al}}{2}\right)}{J_{Al}}$$

$$= \frac{4.47 \times \left(\frac{40}{2000}\right)}{\frac{\pi}{32} \times \left(\frac{40}{1000}\right)^4} = 3.55 \times 10^5 \text{ Pa} \qquad \textbf{Ans.}$$

The maximum shear stress induced in the steel part is given as

$$\tau_s = \frac{T_A \times \left(\frac{d_s}{2}\right)}{J_s}$$

$$= \frac{95.53 \times \left(\frac{80}{2000}\right)}{\frac{\pi}{32} \times \left(\frac{80}{1000}\right)^4} = 9.50 \times 10^5 \text{ Pa} \qquad \textbf{Ans.}$$

Example 7.11

A 3 m long hollow steel shaft of outside diameter 75 mm is transmitting a power of 150 kW at 1000 rpm. Find thickness of the shaft, if the maximum shear stress in shaft is limited to 40 MPa. Take G = 80 GPa.

Solution: Given,

Length of the shaft,	$l = 3$ m
Outside diameter of the shaft,	$d_0 = 75$ mm
Power to be transmitted,	$P = 150$ kW
Revolutions per minute,	$N = 1000$
Maximum shear stress in the shaft,	$\tau_{max} = 40$ MPa

Using power equation, T is given as

$$T = \frac{P \times 60 \times 1000}{2\pi N}$$

$$= \frac{150 \times 60 \times 1000}{2\pi \times 1000} = 1432.4 \text{ N·m}$$

Now
$$\frac{T}{J} = \frac{\tau_{max}}{\left(\frac{d_0}{2}\right)}$$

$$\frac{T}{\frac{\pi}{32}(d_o^4 - d_i^4)} = \frac{\tau_{max}}{\left(\frac{d_0}{2}\right)}$$

or
$$\frac{1432.4}{\frac{\pi}{32}\left[\left(\frac{75}{1000}\right)^4 - d_i^4\right]} = \frac{40 \times 10^6}{\left(\frac{75}{2 \times 1000}\right)} = 1.06 \times 10^9$$

Solving for d_i, we get

$$d_i = 0.0651 \text{ m} = 65.1 \text{ mm}$$

Hence, the thickness of the hollow shaft is

$$t = \frac{d_o - d_i}{2} = \frac{75 - 65.1}{2} = 4.95 \text{ mm}$$ **Ans.**

Example 7.12

A 3 m long solid shaft transmits 15 kW at 1200 rpm. Find the required diameter of the shaft, assuming that maximum shear stress in the shaft is limited to 25 MPa and angle of twist is not to exceed 5°. Take G = 80 GPa.

Solution: Given,

Length of the shaft, $l = 3$ m

Power being transmitted, $P = 15$ kW

Revolutions per minute, $N = 1200$

Maximum shear stress, $\tau_{max} = 25 \text{ MPa} = 25 \times 10^6$ Pa

Angle of twist, $\theta = 5^\circ = \dfrac{\pi}{180} \times 5 = 0.087$ radian

Modulus of rigidity, $G = 80 \text{ GPa} = 80 \times 10^9$ Pa

Let d be diameter of the shaft. Using power equation, T is given as

$$T = \frac{P \times 60 \times 1000}{2\pi N} = \frac{15 \times 60 \times 1000}{2\pi \times 1200} = 119.36 \text{ N·m}$$

Using torsion formula, we have

$$\frac{T}{J} = \frac{G\theta}{l}$$

$$\frac{T}{\frac{\pi}{32} d^4} = \frac{G\theta}{l}$$

$$\frac{119.36}{\frac{\pi}{32} d^4} = \frac{80 \times 10^9 \times 0.087}{3}$$

Solving for d, we get $d = 26.9$ mm

Again
$$\frac{T}{J} = \frac{\tau_{max}}{\left(\frac{d}{2}\right)}$$

$$\frac{T}{\frac{\pi}{32} d^4} = \frac{2 \times \tau_{max}}{d}$$

$$\frac{T \times 32}{\pi d^3} = 2 \times \tau_{max}$$

$$\frac{119.36 \times 32}{\pi d^3} = 2 \times 25 \times 10^6$$

Solving for d, we get

$$d = 29 \text{ mm}$$

Of the two calculated values of d, we accept the largest one. Why? Hence diameter of the shaft is 29 mm. **Ans.**

Example 7.13

A solid shaft of length 3.5 m and diameter 25 mm rotates at a frequency of 40 Hz. Find the maximum power to be transmitted by the shaft, assuming that maximum shear stress in the shaft is limited to 40 MPa and angle of twist does not exceed 6°. Take G = 80 GPa.

Solution: Given,

Length of the shaft, $l = 3.5$ m

Diameter of the shaft, $d = 25$ mm

Revolutions per minute, $N = 40 \times 60 = 2400$

Maximum shear stress, $\tau_{max} = 40 \text{ MPa} = 40 \times 10^6 \text{ Pa}$

Angle of twist, $\theta = 6^o = \frac{\pi}{180} \times 6 = 0.104$ radian

Modulus of rigidity, $G = 80 \text{ GPa} = 80 \times 10^9 \text{ Pa}$

From torsion formula, we have

$$\frac{T}{J} = \frac{G\theta}{l}$$

$$T = \frac{JG\theta}{l} = \frac{\frac{\pi}{32} d^4 \times G \times \theta}{l}$$

$$= \frac{\frac{\pi}{32} \times \left(\frac{25}{1000}\right)^4 \times 80 \times 10^9 \times 0.104}{3.5} = 91.16 \text{ N·m}$$

Again

$$\frac{T}{J} = \frac{\tau_{max}}{(d/2)}$$

or

$$T = \frac{J \times 2 \times \tau_{max}}{d}$$

$$= \frac{\frac{\pi}{32} d^4 \times 2 \times \tau_{max}}{d} = \frac{\pi}{32} d^3 \times 2 \times \tau_{max}$$

$$= \frac{\pi}{32} \times \left(\frac{25}{1000}\right)^3 \times 2 \times 40 \times 10^6 = 122.71 \text{ N·m}$$

Of the two calculated values of T, we take the lowest one *i.e.*, the accepted torque is 91.16 N·m.

Now

$$P = \frac{2\pi NT}{60 \times 1000}$$

$$= \frac{2\pi \times 2400 \times 91.16}{60 \times 1000} = 22.91 \text{ kW}$$ **Ans.**

Example 7.14

A 3 m long solid steel shaft transmits a power of 20 kW at 2000 rpm. Find the smallest permissible diameter of the shaft, if the maximum shear stress is not to exceed 50 MPa. Also find the corresponding angle of twist. Take G = 80 GPa.

Solution: Given,

Length of the shaft, $l = 3$ m

Power to be transmitted, $P = 20$ kW

Revolutions per minute, $N = 2000$

Maximum shear stress, $\tau_{max} = 50$ MPa

Modulus of rigidity, $G = 80 \text{ GPa} = 80 \times 10^9$ Pa

Let T be the torque applied on the shaft. Using power equation, T is given as

$$T = \frac{P \times 60 \times 1000}{2\pi N}$$

$$= \frac{20 \times 60 \times 1000}{2\pi \times 2000} = 95.5 \text{ N·m}$$

Using torsion formula, we have

$$\frac{T}{J} = \frac{\tau_{max}}{(d/2)} \quad \text{(where, } d \text{ is diameter of the shaft)}$$

or

$$\frac{T}{\frac{\pi}{32} d^4} = \frac{\tau_{max}}{\frac{d}{2}}$$

or

$$\frac{95.5}{\frac{\pi}{32} d^4} = \frac{50 \times 10^6}{(d/2)}$$

Solving for d, we get

$$d = 21.34 \text{ mm}$$ **Ans.**

Again

$$\frac{T}{J} = \frac{G\theta}{l}$$

$$\frac{T}{\frac{\pi}{32} d^4} = \frac{G\theta}{l}$$

$$\frac{95.5}{\frac{\pi}{32}\left(\frac{21.34}{1000}\right)^4} = \frac{80 \times 10^9 \times \theta}{3}$$

Solving for θ, we get

$$\theta = 0.175 \text{ radian} = 10^{\circ}$$ **Ans.**

Example 7.15

A solid circular shaft transmits a power of 100 kW at 1500 rpm. For a limiting shear stress of 70 MPa, find diameter of the shaft. The solid shaft is replaced by a hollow shaft with inside diameter being equal to 0.75 times the outside diameter. What percent of saving in weight can be achieved by the replacement, if both shafts are of equal length, made of the same material, being subjected to equal maximum shear stress and are transmitting equal powers at same speed?

Solution: Given,

Power to be transmitted, $P = 100$ kW

Revolutions per minute, $N = 1500$

Maximum shear stress, $\tau_{max} = 70 \text{ MPa} = 70 \times 10^6$ Pa.

Let T be the torque applied on the shaft. Using power equation, T is given as

$$T = \frac{P \times 60 \times 1000}{2\pi N}$$

$$= \frac{100 \times 60 \times 1000}{2\pi \times 1500} = 636.62 \text{ N·m} \qquad \text{... (1)}$$

From torsion formula, we have

$$\frac{T}{J} = \frac{\tau_{max}}{(d/2)}$$

$$\frac{T}{\frac{\pi}{32} d^4} = \frac{2 \times \tau_{max}}{d}$$

$$\frac{636.62}{\frac{\pi}{32} d^3} = 2 \times 70 \times 10^6$$

Solving for d, we get

$$d = 35.91 \text{ mm}$$ **Ans.**

Let d_o = Outside diameter of the hollow shaft

d_i = Inside diameter of the hollow shaft

$d_i = 0.75 d_o$ (Given)

Since two shafts are transmitting the same power at equal speed, hence they are being subjected to same torque.

For hollow shaft

$$T_H = \frac{J \times \tau_{max} \times 2}{d_o}$$

$$= \frac{\frac{\pi}{32}(d_0^4 - d_i^4) \times \tau_{max} \times 2}{d_0}$$

$$= \frac{\frac{\pi}{32} d_0^4 \left[1 - \left(\frac{d_i}{d_0}\right)^4\right] \times \tau_{max} \times 2}{d_0}$$

$$= \frac{\pi}{32} d_0^3 \left[1 - (0.75)^4\right] \times 70 \times 10^6 \times 2 = 9395632.3\, d_0^3$$

Now $\quad 9395632.3\, d_0^3 = 636.62$ (using equation (1))

Solving for d_o, we get

$$d_0 = 40.76 \text{ mm}$$

and $\quad d_i = 0.75\, d_0 = 30.57$ mm

If density of the material for the two shafts is ρ, then

Weight of hollow shaft, W_H = Density × Area × length × g

$$= \rho \times \frac{\pi}{4}(d_0^2 - d_i^2) \times l \times g$$

Weight of solid shaft, $\quad W_S = \rho \times \frac{\pi}{4} d^2 \times l \times g$

The percentage saving in the weight is

$$\frac{W_S - W_H}{W_S} \times 100$$

$$= \frac{d^2 - (d_0^2 - d_i^2)}{d^2} \times 100$$

$$= \frac{(35.91)^2 - (40.76^2 - 30.57^2)}{(35.91)^2} \times 100 = 43.63\%$$ **Ans.**

Example 7.16

A hollow shaft of diameter ratio 0.6 running at 150 rpm is required to drive a screw propeller fitted to a vessel, whose speed is 10 m/s for an expenditure of 12000 shaft horse power. The efficiency of the propeller is 70%. Determine the shaft diameter, if the maximum shearing stress in the shaft is 80 N/mm^2.

Solution: Given

Revolution per minute, N = 150 rpm

Speed of the vessel, V = 10 m/s

Power consumption, P = 12000 HP

Efficiency of the propeller, η = 70% = 0.70

Maximum shear stress, τ_{max} = 80 N/mm^2

The power equation is

$$P = \frac{2\pi NT}{4500} \times \frac{1}{\eta}$$

$$12000 = \frac{2\pi \times 150 \times T}{4500} \times \frac{1}{0.70}$$

or
$$T = \frac{12000 \times 4500 \times 0.70}{2\pi \times 150} \text{ kgf.m}$$
$$= 40107 \text{ kgf.m}$$
$$= 40107 \times 9.81 \text{ N.m} = 393449.7 \text{ N.m}$$

Let d_o = Outside diameter of the shaft

d_i = Inside diameter of the shaft

$$\frac{d_i}{d_o} = 0.6 \quad \text{(Given)}$$

The polar moment of inertia of the shaft is found as

$$J = \frac{\pi}{32}(d_0^4 - d_i^4)$$
$$= \frac{\pi}{32} d_0^4 \left\{1 - \left(\frac{d_i}{d_0}\right)^4\right\}$$
$$= \frac{\pi}{32} d_0^4 \{1 - (0.6)^4\} = 0.0854\, d_0^4$$

The maximum shear stress is given as

$$\tau_{max} = \frac{T}{J} \times \frac{d_0}{2}$$

or
$$80 = \frac{393449.7 \times 1000}{0.0854 d_0^4} \times \frac{d_0}{2}$$

Solving for d_0, we get d_0 = 306.5 mm **Ans.**

$$d_i = 0.6\, d_0$$
$$= 183.9 \text{ mm}$$ **Ans.**

Example 7.17

Two shafts are made of same material and are of equal lengths. One of them is solid and another one is hollow. The ratio of inside and outside diameters for the hollow shaft is 0.65. They are subjected to the same torque and same maximum shear stress. Compare the weights of the two shafts.

Solution: Let d = Diameter of the solid shaft

d_o = Outside diameter of the hollow shaft

d_i = Inside diameter of the hollow shaft

$$\frac{d_i}{d_o} = 0.65 \quad \text{(Given)}$$

From torsion formula, we have

$$\frac{T}{J} = \frac{\tau_{max}}{R}$$

Since the torque and maximum shear stress for the two shafts are same, hence

$$\frac{T}{\tau_{max}} = \frac{J}{R} = \text{Constant}$$

In terms of respective diameters, we have

$$\frac{J_S}{\left(\frac{d}{2}\right)} = \frac{J_H}{\left(\frac{d_o}{2}\right)}$$

$$\frac{\frac{\pi}{32} d^4}{\left(\frac{d}{2}\right)} = \frac{\frac{\pi}{32}(d_o^4 - d_i^4)}{\left(\frac{d_o}{2}\right)}$$

$$d^3 = d_o^3\left[1 - \left(\frac{d_i}{d_o}\right)^4\right]$$

$$\left(\frac{d}{d_o}\right)^3 = 1 - (0.65)^4 = 0.821$$

or $$\frac{d}{d_o} = 0.936 \qquad \text{... (1)}$$

For the same material and same length, weight is proportional to the cross-sectional area of the respective shaft.

$$\frac{W_S}{W_H} = \frac{\frac{\pi}{4} d^2}{\frac{\pi}{4}(d_o^2 - d_i^2)}$$

$$= \frac{d^2}{d_o^2\left[1 - \left(\frac{d_i}{d_o}\right)^2\right]}$$

$$= \frac{(d/d_o)^2}{1 - \left(\frac{d_i}{d_o}\right)^2} = \frac{(0.936)^2}{1 - (0.65)^2} \qquad \text{(using equation (1))}$$

$$= 1.517$$ **Ans.**

Example 7.18

A solid circular shaft 200 mm in diameter has the same cross-sectional area as a hollow circular shaft of the same material with inside diameter of 150 mm. For the same maximum shear stress, determine the ratio of torque transmitted by the hollow shaft to that by the solid shaft. Also, compare the angle of twist in the above shaft for equal length and same maximum shear stress. From the above results, what will be your conclusions regarding strength and stiffness of the two shafts?

Solution: Given,

Diameter of the solid shaft, $d = 200$ mm

Inside diameter the hollow shaft, $D_i = 150$ mm

Let D_0 = Outside diameter of the hollow shaft

l = Length of the two shafts

G = Modulus of rigidity for the two shafts

T_S = Torque transmitted by the solid shaft

T_H = Torque transmitted by the hollow shaft

θ_S = Angle of twist produced in the solid shaft

θ_H = Angle of twist produced in the hollow shaft

τ_{max} = Maximum shear stress for the two shafts

J_S = Polar moment of inertia for the solid shaft

J_H = Polar moment of inertia for the hollow shaft

The cross-sectional areas of the solid and hollow shafts are equal.

$$\frac{\pi}{4}d^2 = \frac{\pi}{4}(D_0^2 - D_i^2)$$

$$d^2 = D_0^2 - D_i^2$$

$$200^2 = D_0^2 - 150^2$$

Solving for D_0, we get $D_0 = 250$ mm

The torque transmitted by the solid shaft is given as

$$T_s = \frac{\tau_{max}}{(d/2)} \cdot J_s \qquad \left(\text{using } \frac{T}{J} = \frac{\tau_{max}}{(d/2)}\right)$$

$$= \frac{\pi d^3}{16}\tau_{max} \qquad \text{... (1)}$$

The torque transmitted by the hollow shaft is given as

$$T_H = \frac{\tau_{max}}{\left(\frac{D_0}{2}\right)} \cdot J_H$$

$$= \frac{\pi(D_0^4 - D_i^4)}{16 D_0}\tau_{max} \qquad \text{... (2)}$$

Dividing equation (2) by equation (1), we have

$$\frac{T_H}{T_S} = \frac{D_0^4 - D_i^4}{D_0 \cdot d^3}$$

$$= \frac{250^4 - 150^4}{250 \times 200^3} = 1.7 \qquad \textbf{Ans.}$$

Now $$\theta_S = \frac{\tau_{\max}}{(d/2)} \times \frac{l}{G}$$

and $$\theta_H = \frac{\tau_{\max}}{(D_0/2)} \times \frac{l}{G}$$

Hence, $$\frac{\theta_H}{\theta_S} = \frac{d}{D_0} = \frac{200}{250} = 0.8 \qquad \textbf{Ans.}$$

Conclusions:

(*a*) More torque transmission means more stronger the shaft is. Hence, the hollow shaft is 1.7 times stronger than the solid shaft.

(*b*) Lesser angle of twist produced in the shaft means more stiffer the shaft is. Since $\theta_H < \theta_S$, hence the hollow shaft is stiffer than the solid shaft.

Example 7.19

A solid steel shaft of diameter 40 mm is placed inside an aluminium tube and both of them are connected to a fixed support at one end. The other end of the two members are connected by a rigid plate as shown in Fig. 7.13. Find maximum torque to be applied to the plate, if maximum shear stresses in the steel shaft and the aluminium tube are limited to 110 MPa and 65 MPa respectively. Take G_S = 80 GPa and G_{Al} = 28 GPa.

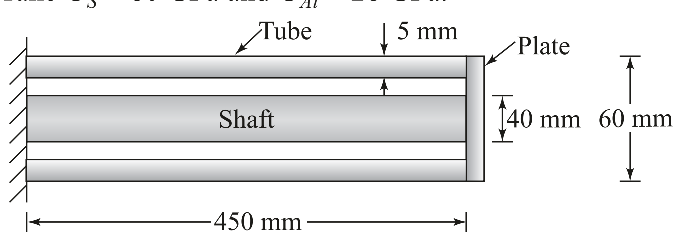


Fig. 7.13

Solution: Given,

For steel shaft	*For aluminium tube*
Diameter, d = 40 mm Shear modulus, G_S = 80 GPa = 80×10^9 Pa	Outside diameter, d_0 = *60 mm* Thickness, t = 5 mm Inside diameter, $d_i = d_o - 2t$ $= 60 - 2 \times 5 =$ 50 mm

Contd...

Maximum shear stress, $\tau_{\max_s} = 110$ MPa $= 110 \times 10^6$ Pa	Shear modulus, $G_{Al} = 28$ GPa $= 28 \times 10^9$ Pa Maximum shear stress, $\tau_{\max_{Al}} = 65$ MPa $= 65 \times 10^6$ Pa
Length, $l_s = 450$ mm	Length, $l_{Al} = 450$ mm

The torque applied is distributed to the shaft and tube.

Let T_1 = Torque exerted by the tube on the plate

T_2 = Torque exerted by the shaft on the plate

T = Net torque to be applied on the plate

θ_1 = Angle of twist produced in the tube

θ_2 = Angle of twist produced in the shaft

J_1 = Polar moment of inertia of the tube

J_2 = Polar moment of inertia of the shaft

$$J_1 = \frac{\pi}{32}(d_o^4 - d_i^4)$$

$$= \frac{\pi}{32}(60^4 - 50^4) \times 10^{-12} \text{ m}^4 = 6.58 \times 10^{-7} \text{ m}^4$$

$$J_2 = \frac{\pi}{32} d^4 = \frac{\pi}{32} \times 40^4 \times 10^{-12} \text{ m}^4 = 2.51 \times 10^{-7} \text{ m}^4$$

Now $T = T_1 + T_2$ ... (1)

and $\theta_1 = \theta_2$ ... (2)

From torsion formula, we have

$$\theta = \frac{Tl}{JG}$$

$$\frac{T_1\, l_{Al}}{J_1\, G_{Al}} = \frac{T_2\, l_S}{J_2\, G_S}$$

$$\frac{T_1 \times \left(\frac{450}{1000}\right)}{6.58 \times 10^{-7} \times 28 \times 10^9} = \frac{T_2 \times \left(\frac{450}{1000}\right)}{2.51 \times 10^{-7} \times 80 \times 10^9}$$

or $T_1 = 0.917\, T_2$ ... (3)

Check for shear stresses

Using torsion formula, the torque

$$T_1 = \frac{2 \times \tau_{\max_{Al}} \times J_1}{d_o} = \frac{2 \times 65 \times 10^6 \times 6.58 \times 10^{-7}}{60 \times 10^{-3}} = 1427.3 \text{ N·m}$$

The corresponding value of T_2, by using equation (3), is given as

$$T_2 = \frac{T_1}{0.917} = \frac{1427.3}{0.917} = 1556.48 \text{ N·m}$$

The maximum shear stress induced in the shaft, by using T_2, is given as

$$\tau_{\max_s} = \frac{T_2 \times d}{2J_2}$$

$$= \frac{1556.48 \times 40 \times 10^{-3}}{2 \times 2.51 \times 10^{-7}} \text{ N/m}^2 = 124 \text{ MPa}$$

Since 124 MPa > 110 MPa *i.e.*, maximum shear stress induced in the shaft is more than its allowable value, which is wrong. Hence, we accept $\tau_{\max_s}$ = 110 MPa.

We find T_2 corresponding to $\tau_{\max_s}$ = 110 MPa

$$T_2 = \frac{2 \times \tau_{\max_s} \times J_2}{d}$$

$$= \frac{2 \times 110 \times 10^6 \times 2.51 \times 10^{-7}}{40 \times 10^{-3}} = 1380.5 \text{ N·m}$$

and $\quad T_1 = 0.917T_2 = 0.917 \times 1380.5 = 1265.9$ N·m (using equation (3))

Hence, the net torque to be applied on the plate is

$$T = T_1 + T_2$$

$$= (1265.9 + 1380.5) = 2646.4 \text{ N·m}$$ **Ans.**

Example 7.20

A horizontal shaft 12 m in length is fixed at its ends (Fig. 7.14). When viewed from the left end, axial couples of 50 kN.m clockwise and 75 kN.m counter clockwise act at 5 m and 9 m from the left end respectively. Determine the end fixing couples and the position, where the shaft suffers no angular twist.

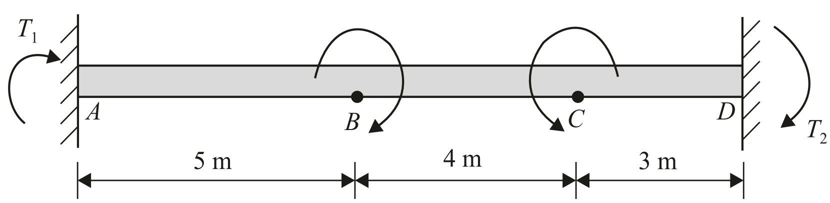


Fig. 7.14

Solution: Refer Fig. 7.14.

Given, Length of the shaft, l = 12 m

Couple at B, T_B = 50 kN.m (Clockwise)

Couple at C, T_C = 75 kN.m (Anticlockwise)

Let T_1 and T_2 be the reactive torques at A and D respectively.

Hence, $\quad T_1 + T_2 = 75 - 50 = 25$ kN.m ... (1)

Using torsion formula, we have $\theta = \dfrac{Tl}{JG}$

Angle of twist at C (from left end) is

$$\theta_1 = \frac{5T_1}{JG} + \frac{(T_1 + 50) \times 4}{JG}$$

Angle of twist at C (from right end) is

$$\theta_2 = \frac{3T_2}{JG}$$

Now $\theta_1 = \theta_2$

$$\frac{5T_1}{JG} + \frac{(T_1 + 50) \times 4}{JG} = \frac{3T_2}{JG}$$

$$5T_1 + 4T_1 + 200 = 3T_2$$

or $9T_1 - 3T_2 + 200 = 0$ ... (2)

and $T_1 + T_2 - 25 = 0$ (from equation (1))

Solving equations (1) and (2), we get

$$T_2 = 35.42 \text{ kN.m}$$

and $T_1 = -10.42$ kN.m

The negative sign associated with T_1 is indicative of its anticlockwise nature.

Let at a distance x (in BC) from A, angular twist is zero, which means

$$-\frac{10.42 \times 5}{JG} + \frac{(50 - 10.42) \times (x - 5)}{JG} = 0$$

$$-52.1 + 39.58x - 197.9 = 0$$

or $x = 6.32$ m

Hence, the shaft suffers no angular twist at 6.32 m from A. **Ans.**

Example 7.21

For the shaft loaded as shown in Fig. 7.15, calculate the maximum shear stress induced and the angle of twist for cross-section at A. The modulus of rigidity is G.

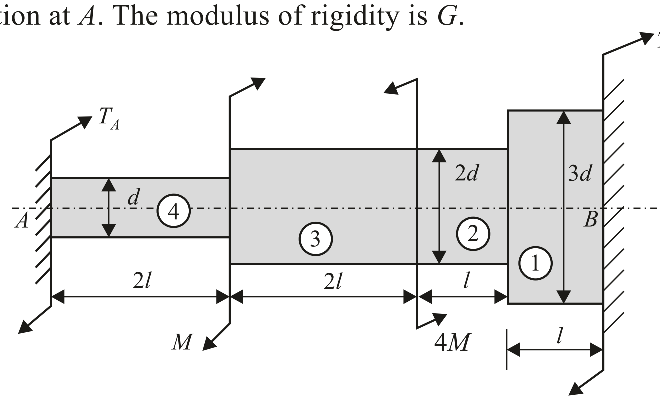


Fig. 7.15

Solution: The algebraic sum of the torques acting on the shaft is zero. T_A and T_B are the reactive torques at A and B respectively.

$$T_A + M - 4M + T_B = 0$$

or

$$T_A + T_B = 4M - M = 3M \qquad \text{... (1)}$$

Using principle of superimposition, angular twists produced independently by M, $4M$ and T_B are considered. Then these twists are added algebraically to get the total twist of 'free end' B w.r.t. fixed end A. This should be zero as the end B is fixed.

$$\frac{T_B \times l}{J_1 G} + \frac{T_B \times l}{J_2 G} + \frac{T_B \times 2l}{J_3 G} + \frac{T_B \times 2l}{J_4 G} + \frac{M \times 2l}{J_4 G} - \frac{4M \times 2l}{J_3 G} - \frac{4M \times 2l}{J_4 G} = 0$$

or

$$\frac{l}{G}\left[\frac{T_B}{J_1} + \frac{T_B}{J_2} + \frac{2T_B}{J_3} + \frac{2T_B}{J_4} + \frac{2M}{J_4} - \frac{8M}{J_3} - \frac{8M}{J_4}\right] = 0$$

$$\frac{T_B}{J_1} + \frac{T_B}{J_2} + \frac{2T_B}{J_3} + \frac{2T_B}{J_4} + \frac{2M}{J_4} - \frac{8M}{J_3} - \frac{8M}{J_4} = 0$$

$$T_B\left(\frac{1}{J_1} + \frac{1}{J_2} + \frac{2}{J_3} + \frac{2}{J_4}\right) + M\left(\frac{2}{J_4} - \frac{8}{J_3} - \frac{8}{J_4}\right) = 0$$

$$T_B\left[\frac{1}{\frac{\pi}{32}(3d)^4} + \frac{1}{\frac{\pi}{32}(2d)^4} + \frac{2}{\frac{\pi}{32}(2d)^4} + \frac{2}{\frac{\pi}{32}d^4}\right] + M\left[\frac{2}{\frac{\pi}{32}d^4} - \frac{8}{\frac{\pi}{32}(2d)^4} - \frac{8}{\frac{\pi}{32}d^4}\right] = 0$$

$$T_B\left(\frac{1}{81} + \frac{1}{16} + \frac{2}{16} + 2\right) + M\left(2 - \frac{8}{16} - 8\right) = 0$$

$$2.1998\, T_B - 6.5\, M = 0$$

Hence,

$$T_B = 2.954\, M$$

and

$$T_A = 3M - T_B \qquad \text{(using equation (1))}$$

$$= 0.046\, M$$

Since $T_B > T_A$, hence maximum shear stress is induced on account of T_B.

The maximum shear stress is

$$\tau_{max} = \frac{T_B}{J_4} \times \frac{d}{2}$$

$$= \frac{T_B}{\frac{\pi}{32}d^4} \times \frac{d}{2}$$

$$= \frac{2.954\, M \times 32}{2\pi d^3} = \frac{15.04\, M}{d^3} \qquad \textbf{Ans.}$$

The angle of twist is
$$\theta_A = \frac{T_B \times 2l}{J_4 G} = \frac{T_B \times 2l}{\frac{\pi}{32} d^4 \times G}$$
$$= \frac{32 \times 2T_B \times l}{\pi d^4 G}$$
$$= \frac{64 \times 2.954\, M \times l}{\pi d^4 G} \quad \text{(on substituting } T_B\text{)}$$
$$= 60.18 \frac{Ml}{Gd^4}$$
Ans.

Example 7.22

The compound shaft shown in Fig. 7.16 is built-in at the two ends. It is subjected to a twisting moment T at the middle. What is the ratio of the reaction torques T_1 and T_2 at the ends?

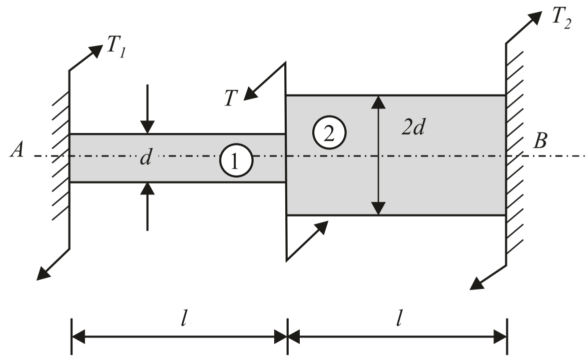


Fig. 7.16

Solution: Refer Fig. 7.16.

$$T_1 + T_2 = T \text{ (Given)} \quad \text{... (1)}$$

We will use principle of superimposition for solving this problem. We consider the angular twists produced by T and T_2 independently, and then add them together algebraically to get the total twist of 'free end' B w.r.t. fixed end A. This should be zero, as the end B is fixed, and hence can't rotate.

$$\frac{T_2 l}{J_2 G} + \frac{T_2 l}{J_1 G} - \frac{Tl}{J_1 G} = 0$$
$$\frac{l}{G}\left[\frac{T_2}{J_2} + \frac{T_2}{J_1} - \frac{T}{J_1}\right] = 0$$
$$\frac{T_2}{J_2} + \frac{T_2}{J_1} - \frac{T_1 + T_2}{J_1} = 0 \quad \text{(using equation (1))}$$
$$\frac{T_2}{J_2} + \frac{T_2}{J_1} - \frac{T_1}{J_1} - \frac{T_2}{J_1} = 0$$

$$\frac{T_2}{J_2} - \frac{T_1}{J_1} = 0$$

or
$$\frac{T_1}{T_2} = \frac{J_1}{J_2} = \frac{\frac{\pi}{32}d^4}{\frac{\pi}{32}(2d)^4} = \frac{1}{16} \qquad \textbf{Ans.}$$

7.6 EFFECT OF STRESS CONCENTRATION

In case of stepped shaft, where cross-section of the shaft abruptly changes at any point along its length, stress concentrations are found to occur near the discontinuity (Fig. 7.17).

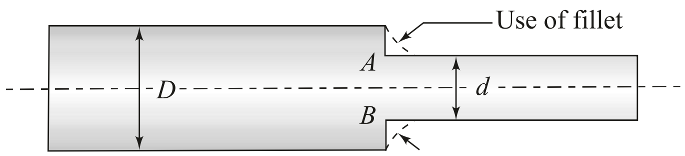


Fig. 7.17 A stepped shaft using a fillet.

In the figure, A and B are the locations of high stress concentration. These are the weakest points in the shaft. A factor, known as stress-concentration factor (K), is introduced in the torsion analysis of such shafts (Fig. 7.18). Stress concentrations are reduced by the use of fillets, where change in cross-section takes place gradually. The stress concentration factor depends upon two factors.

(a) The ratio of the diameters of the shaft at two ends (D/d), and

(b) The ratio of the radius of the fillet to the smallest diameter of the shaft $\left(\frac{r}{d}\right)$.

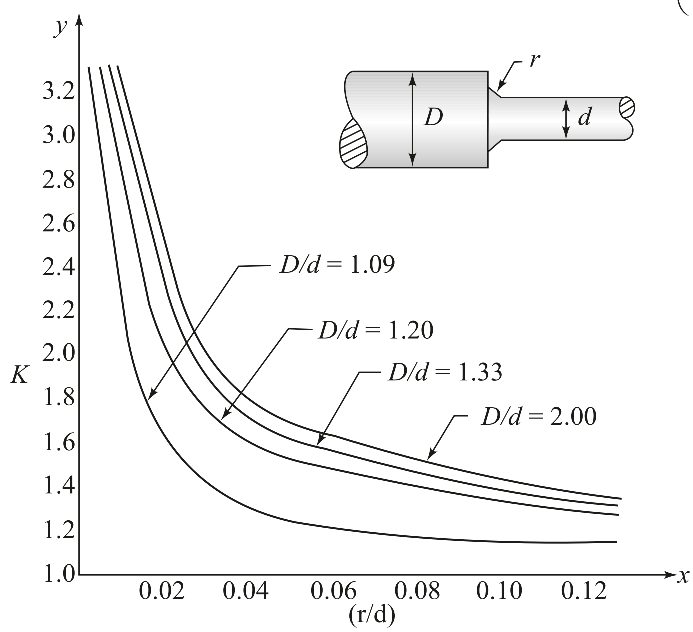


Fig. 7.18 Stress concentration factor and use of fillet.

The maximum shear stress induced at the fillet locations may be expressed as

$$\tau_{max} = \frac{K.\,T.\,d}{2J} \quad \text{... (7.18)}$$

where K = Stress-concentration factor

d = Diameter at smallest end of the shaft

J = Polar moment of inertia based on smallest diameter of the shaft

T = Torque

Equation (7.18) is valid only if Hooke's law is obeyed, that is, the maximum shear stress does not exceed the proportional limit of the shaft material.

Example 7.23

A stepped shaft shown in Fig. 7.19 is subjected to a torque of 500 N·m. Find the maximum shear stress in the shaft when the fillet radius is 2.5 mm.

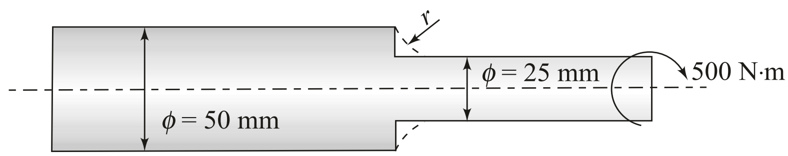


Fig. 7.19

Solution: Refer Fig. 7.16.

Given,

Diameter at the bigger end, D = 50 mm

Diameter at the smallest end, d = 25 mm

Radius of the fillet, r = 2.5 mm

Torque to be applied, T = 500 N·m

The ratio $\frac{r}{d} = \frac{2.5}{25} = 0.10$

The ratio $\frac{D}{d} = \frac{50}{25} = 2.0$

Corresponding to $\frac{r}{d} = 0.10$ and $\frac{D}{d} = 2.0$, the value of stress concentration factor, K from Fig. 7.15 is 1.45. Hence, the maximum shear stress induced in the shaft is given as

$$\tau_{max} = \frac{K.\,T.\,d}{2J} \quad \text{(using equation (7.18))}$$

$$= \frac{1.45 \times 500 \times (25/1000)}{2 \times \frac{\pi}{32} \times \left(\frac{25}{1000}\right)^4} = 236.31 \text{ MPa} \quad \textbf{Ans.}$$

Example 7.24

A stepped shaft shown in Fig. 7.20 transmits 100 kW of power at 850 rpm. Find the smallest permissible radius of the fillet, assuming that maximum shear stress in the shaft is limited to 60 MPa.

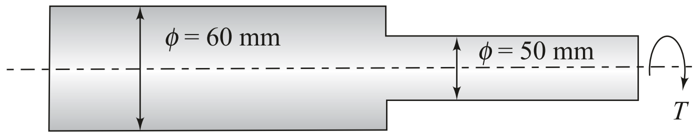


Fig. 7.20

Solution: Given,

Diameter at the bigger end, $D = 60$ mm

Diameter at the smaller end, $d = 50$ mm

Revolutions per minute, $N = 850$

Power to be transmitted, $P = 100$ kW

Maximum shear stress, $\tau_{max} = 60 \text{ MPa} = 60 \times 10^6$ Pa

Let T be the torque applied on the shaft.

The equation of power is

$$P = \frac{2\pi NT}{60 \times 1000}$$

or

$$T = \frac{P \times 60 \times 1000}{2\pi N} = \frac{100 \times 60 \times 1000}{2\pi \times 850} = 1123.44 \text{ N·m}$$

The maximum shear stress in the shaft is

$$\tau_{max} = \frac{K\,.\,T\,.\,d}{2J}$$

$$60 \times 10^6 = \frac{K \times 1123.44 \times \left(\frac{50}{1000}\right)}{2 \times \frac{\pi}{32} \times \left(\frac{50}{1000}\right)^4}$$

Solving for K, we get

$$K = 1.31$$

The ratio $$\frac{D}{d} = \frac{60}{50} = 1.2$$

Corresponding to $K = 1.31$ and $\frac{D}{d} = 1.2$, ratio $\frac{r}{d}$ is found to be 0.11. (using Fig. 7.15)

$$\frac{r}{d} = 0.11$$

or $$r = 0.11 \times d = 0.11 \times 50 = 5.5 \text{ mm}$$

Hence, the minimum radius of the fillet to be used is 5.5 mm. **Ans.**

7.7 TORSION OF A TAPERED SHAFT

Consider a tapered shaft of diameter d_1 at the bigger end and diameter d_2 at smaller end (Fig. 7.21).

Let τ_1 = Shear stress at the bigger end

τ_2 = Shear stress at the smaller end

G = Shear modulus of the shaft material

T = Torque applied at the two ends

l = Length of the shaft

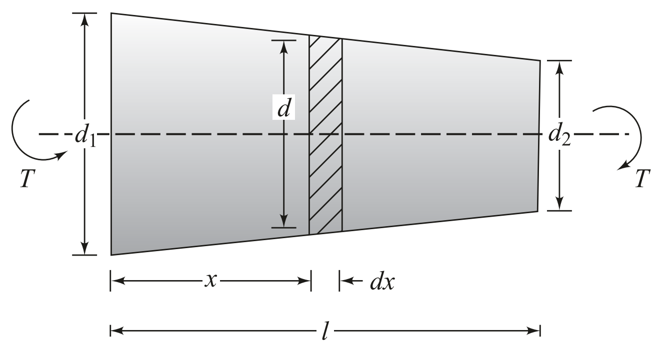


Fig. 7.21

Consider a small length dx of diameter d at a distance x from bigger end.

The entire section of shaft is subjected to same torque T.

$$\frac{\pi}{16} d_1^3 \tau_1 = \frac{\pi}{16} d_2^3 \tau_2 = \frac{\pi}{16} d^3 \tau$$

where τ = Shear stress at x

The above equation, on simplification, becomes

$$d_1^3 \tau_1 = d_2^3 \tau_2 = d^3 \tau \quad \text{... (7.19)}$$

The angle of twist produced in the length dx is

$$d\theta = \frac{Tdx}{JG} \quad \left(\text{using } \frac{T}{J} = \frac{G\theta}{l}\right)$$

$$= \frac{Tdx}{\frac{\pi}{32} d^4 G} = \frac{32T}{\pi d^4 G} dx \quad \text{... (7.20)}$$

The diameter d can be expressed as

$$d = d_1 - \frac{(d_1 - d_2)x}{l}$$

$$= d_1 - Kx, \quad \text{where } K = \left(\frac{d_1 - d_2}{l}\right)$$

On substituting d, equation (7.20) becomes

$$d\theta = \frac{32T\,dx}{\pi[d_1 - Kx]^4\,G}$$

The total angle of twist for the entire shaft of length l is

$$\theta = \frac{32T}{\pi G}\int_0^l \frac{dx}{(d_1 - Kx)^4} = \frac{32T}{\pi G}\int_0^l (d_1 - Kx)^{-4}\,dx$$

$$= \frac{32T}{\pi G}\left[\frac{(d_1 - Kx)^{-3}}{3K}\right]_0^l = \frac{32T}{3\pi GK}\left[(d_1 - Kx)^{-3}\right]_0^l$$

$$= \frac{32T}{3\pi GK}\left[(d_1 - Kl)^{-3} - d_1^{-3}\right]$$

$$= \frac{32Tl}{3\pi G(d_1 - d_2)}\left[\left(d_1 - \frac{d_1 - d_2)}{l}\cdot l\right)^{-3} - d_1^{-3}\right]$$

$$= \frac{32Tl}{3\pi G(d_1 - d_2)}\left[d_2^{-3} - d_1^{-3}\right] = \frac{32Tl}{3\pi G(d_1 - d_2)}\left[\frac{1}{d_2^3} - \frac{1}{d_1^3}\right]$$

$$= \frac{32Tl}{3\pi G(d_1 - d_2)}\left[\frac{d_1^3 - d_2^3}{d_1^3\,d_2^3}\right]$$

$$= \frac{32Tl}{3\pi G(d_1 - d_2)}\left[\frac{(d_1 - d_2)(d_1^2 + d_1\,d_2 + d_2^2)}{d_1^3\,d_2^3}\right]$$

$$= \frac{32Tl}{3\pi G}\left[\frac{d_1^2 + d_1\,d_2 + d_2^2}{d_1^3\,d_2^3}\right] \qquad \text{... (7.21)}$$

Maximum shear stress is induced at smaller end of the shaft because of smallest diameter at that point, and its value is given as

$$\tau_{\max} = \frac{16T}{\pi d_2^3} \qquad \text{... (7.22)}$$

7.8 TORSION OF A THIN CIRCULAR TUBE

Consider a circular tube of outside diameter d_0 and inside diameter d_i, being subjected to a torque T (Fig. 7.22).

Thickness of the tube is

$$t = \frac{d_0 - d_i}{2}$$

The polar moment of inertia of the cross-section is found as

$$J = \frac{\pi}{32}(d_0^4 - d_i^4) = \frac{\pi}{32}\left[d_0^4 - (d_0 - 2t)^4\right]$$

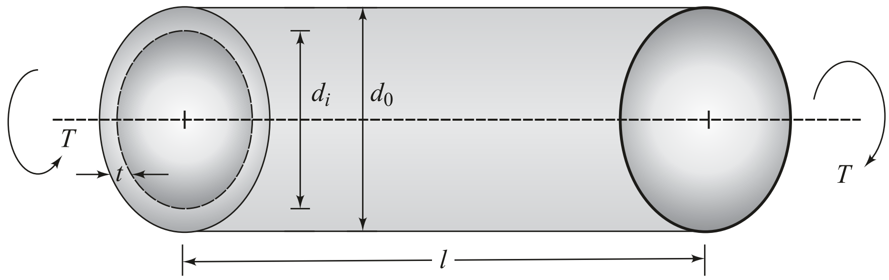


Fig. 7.22

$$= \frac{\pi}{32}\left[d_0^4 - d_0^4\left(1-\frac{2t}{d_0}\right)^4\right] = \frac{\pi d_0^4}{32}\left[1-\left(1-\frac{2t}{d_0}\right)^4\right]$$

$$= \frac{\pi d_0^4}{32}\left[1-\left(1-\frac{8t}{d_0}+\ldots\right)\right]$$

$$= \frac{\pi d_0^4}{32}\times\frac{8t}{d_0} \qquad \text{(ignoring higher powers of } t\text{)}$$

$$= \frac{\pi d_0^3\, t}{4}$$

The torque T is given as

$$T = \frac{2\times\tau_{max}\times J}{d_0} \qquad \text{(using torsion formula)}$$

$$= \frac{2\times\tau_{max}\times\pi d_0^3\, t}{4\times d_0} \qquad \text{(on substituting } J\text{)}$$

$$= \frac{\pi}{2} d_0^2\; t\,\tau_{max} \qquad \ldots (7.23)$$

The angle of twist produced in the tube is given as

$$\theta = \frac{Tl}{JG} = \frac{Tl}{\left(\frac{\pi d_0^3\, t}{4}\right)\times G} \qquad \text{(on substituting } J\text{)}$$

$$= \frac{4Tl}{\pi d_0^3\, t G} \qquad \ldots (7.24)$$

7.9 STRAIN ENERGY DUE TO TORSION

The strain energy or the workdone due to torsion is given by

$$U = \frac{1}{2}\times T\times\theta \qquad \ldots (7.25)$$

where torque is assumed to be applied gradually and is increasing from zero to its maximum value.

From torsion formula, for a solid circular shaft, we have

$$\frac{T}{J} = \frac{G\theta}{l} = \frac{\tau_{\max}}{(d/2)}$$

Hence,
$$T_s = \frac{J \times \tau_{\max}}{(d/2)}$$

$$= \frac{\pi}{32} d^4 \times \frac{\tau_{\max} \times 2}{d} = \frac{\pi d^3}{16} \tau_{\max}$$

and
$$\theta_s = \frac{\tau_{\max} \times l}{G \times (d/2)} = \frac{2 \times \tau_{\max} \times l}{Gd}$$

Substituting T_s and θ_s in equation (7.25), we have for a solid shaft

$$U_s = \frac{1}{2} \times \frac{\pi d^3}{16} \tau_{\max} \times \frac{2 \times \tau_{\max} \times l}{Gd}$$

$$= \frac{\tau_{\max}^2}{4G}\left(\frac{\pi}{4} d^2 \times l\right) = \frac{\tau_{\max}^2}{4G} \times V_S \qquad \text{... (7.26)}$$

where
$$V_s = \frac{\pi}{4} d^2 \times l = \text{Volume of the solid shaft}$$

For a hollow shaft of inside diameter d_i and outside diameter d_o, the values of T and θ are:

$$T_H = \frac{\pi d_0^3}{16}\left[1 - \left(\frac{d_i}{d_o}\right)^4\right] \tau_{\max}$$

and
$$\theta_H = \frac{2 \times \tau_{\max} \times l}{G d_0}$$

Substituting T and θ in equation (7.25), we get

$$U_H = \frac{1}{2} \times \frac{\pi d_0^3}{16}\left[1 - \left(\frac{d_i}{d_o}\right)^4\right] \tau_{\max} \times \frac{2 \times \tau_{\max} \times l}{G d_o}$$

$$= \frac{\pi d_0^2}{16}\left[1 - \left(\frac{d_i}{d_o}\right)^4\right] \times \frac{\tau_{\max}^2 \times l}{G}$$

$$= \frac{\pi d_0^2}{16} \times \frac{(d_o^2 - d_i^2)(d_o^2 + d_i^2)}{d_o^4} \times \frac{\tau_{\max}^2 \times l}{G}$$

$$= \left[\frac{\pi}{4}(d_o^2 - d_i^2) \times l\right] \times \frac{\tau_{\max}^2}{2G} \times \frac{(d_o^2 + d_i^2)}{2 d_o^2} = V \times \frac{\tau_{\max}^2}{2G} \times 1$$

where
$$V_H = \frac{\pi}{4}(d_o^2 - d_i^2) \times l = \text{Volume of the hollow shaft}$$

and $$\frac{d_o^2 + d_i^2}{2d_o^2} = 1, \text{ when } d_o \simeq d_i$$

Hence, $$U_H = \frac{\tau_{max}^2}{2G} \times V_H \qquad \text{... (7.27)}$$

Comparing equation (7.27) with equation (7.26), we find that a hollow shaft can store approximately twice strain energy as compared to a solid shaft, and it is a very important point in favour of a hollow shaft.

Example 7.25

A shaft circular in section (Fig. 7.23) and of length l is subjected to a variable torque given by Kx^2/l^2, where x is the distance measured from one end of the shaft and K is a constant. Find the angle of twist for the shaft by use of Castigliano's theorem. Torsional rigidity of the shaft is G.

Solution: The torque is $$T = \frac{Kx^2}{l^2}.$$

Consider a small length dx of the shaft at a distance x from the right end.

The angle of twist, using the Castigliano's theorem, is given by

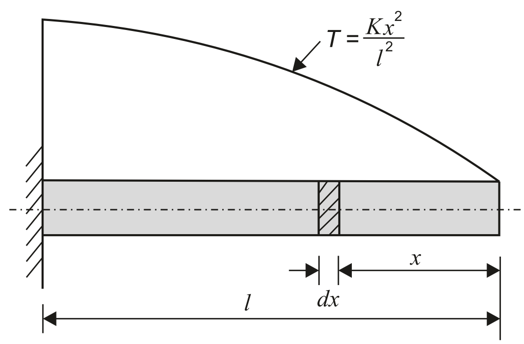


Fig. 7.23

$$\theta = \int_0^l \frac{Tdx}{JG} = \frac{1}{JG}\int_0^l Tdx$$

$$= \frac{1}{JG}\int_0^l \frac{Kx^2}{l^2}dx \qquad \text{(on substituting } T\text{)}$$

$$= \frac{K}{JGl^2}\int_0^l x^2 dx$$

$$= \frac{K}{JGl^2}\left(\frac{x^3}{3}\right)_0^l = \frac{Kl}{3JG} \qquad \textbf{Ans.}$$

SHORT ANSWER QUESTIONS

1. What effect is observed when a shaft is twisted?
2. How does the shear stress vary with radius of a shaft?
3. What are the values of shear stress at the neutral axis and on the surface of a shaft ?
4. What is polar modulus? What is its SI unit?
5. What is torsional rigidity? What is its significance?
6. What is polar moment of inertia? How does it differ from second moment of area?
7. What is the relationship between polar moment of inertia and second moment of area for a solid circular shaft?
8. How does the power transmission vary with the rpm of a shaft?
9. What does the horse power mean?
10. How are kilowatt and horse power related?

MULTIPLE CHOICE QUESTIONS

1. The shear stress produced in a circular shaft due to pure torsion is
   (*a*) proportional to radius of the shaft
   (*b*) inversely proportional to diameter of the shaft
   (*c*) inversely proportional to radius of the shaft
   (*d*) inversely proportional to the square of radius of the shaft.

2. The torsional rigidity is defined as the
   (*a*) product of torque and length
   (*b*) product of polar moment of inertia and modulus of rigidity
   (*c*) sum of polar moment of inertia and modulus of rigidity
   (*d*) product of polar moment of inertia and angle of twist.

3. The torsional stiffness is defined as the
   (*a*) ratio of torque to angle of twist
   (*b*) product of torque and angle of twist
   (*c*) sum of modulus of rigidity and angle of twist
   (*d*) ratio of torque to polar moment of inertia.

4. The torsion equation is

(*a*) $\dfrac{T}{I} = \dfrac{G\theta}{L} = \dfrac{\tau}{r}$ (*b*) $\dfrac{T}{J} = \dfrac{G\theta}{L} = \dfrac{\tau}{r}$

(*c*) $\dfrac{T}{J} = \dfrac{G\theta}{L} = \dfrac{\tau}{D}$ (*d*) $\dfrac{J}{T} = \dfrac{G\theta}{L} = \dfrac{\tau}{r}$.

where the symbols have their usual meanings.

5. A solid steel shaft of length 2 m is running at 150 rpm, transmitting a power of 50 kW. The angle of twist produced in the shaft is 1° and the permissible shear stress is 50 MPa. What is the diameter of the shaft, if $G = 8 \times 10^4$ N/mm^2 ?

(*a*) 68.7 mm (*b*) 82.55 mm

(*c*) 67.8 mm (*d*) 65.7 mm.

6. The ratio of the torques transmitted by a hollow and a solid shaft, both made of same material, length and weight is

(*a*) $\dfrac{n^2+1}{n\sqrt{n^2+1}}$ (*b*) $\dfrac{n^2+1}{n\sqrt{n^2-1}}$ (*c*) $\dfrac{n^2+1}{n\sqrt{n+1}}$ (*d*) $\dfrac{n^2-1}{n\sqrt{n+1}}$

where $n = \dfrac{d_o}{d_i}$.

7. The ratio of the weights of two shafts in Question No. 6 is

(*a*) $\left(1-\dfrac{1}{n^2}\right) \Big/ \left(1-\dfrac{1}{n^4}\right)^{2/3}$ (*b*) $\left(1-\dfrac{1}{n^2}\right) \Big/ \left(1+\dfrac{1}{n^4}\right)^{2/3}$

(*c*) $\left(1+\dfrac{1}{n^2}\right) \Big/ \left(1-\dfrac{1}{n^4}\right)^{2/3}$ (*d*) $\left(1+\dfrac{1}{n^2}\right) \Big/ \left(1+\dfrac{1}{n^4}\right)^{2/3}$.

8. The power transmitted by a shaft is given as

(*a*) $\dfrac{\pi NT}{30}$ (W) (*b*) $\dfrac{\pi NT}{4500}$ (hp) (*c*) $\dfrac{2\pi NT}{75}$ (hp) (*d*) $\dfrac{\pi NT}{30}$ (kW).

where the symbols have their usual meanings.

9. The formula used for finding angle of twist produced in a tapered shaft of length L being subjected to a torque T is

(*a*) $\dfrac{TL}{3\pi G}\left[\dfrac{r_1 + r_2 + 2r_1 r_2}{(r_1 r_2)^3}\right]$ (*b*) $\dfrac{TL}{3\pi G}\left[\dfrac{r_1 + r_2 + 2r_1 r_2}{(r_1 r_2)^3}\right]$

(*c*) $\dfrac{2TL}{3\pi G}\left[\dfrac{r_1^2 + r_2^2 + r_1 r_2}{(r_1 r_2)^3}\right]$ (*d*) $\dfrac{3TL}{2\pi G}\left[\dfrac{r_1^2 + r_2^2 + r_1 r_2}{(r_1 r_2)^3}\right]$.

where r_1 and r_2 are the radii at smaller and bigger end respectively.

10. For a stepped shaft where two or more shafts are connected end-to-end, the angle of twist produced and the torques transmitted are given respectively as

(*a*) $\theta_1 - \theta_2, T_1 + T_2$ (*b*) $\theta_1 + \theta_2, T_1 = T_2$

(*c*) $\frac{\theta_1 + \theta_2}{2}, T_1 = \frac{T_2}{2}$ (*d*) $\frac{\theta_1 - \theta_2}{2}, T_1 = T_2$.

11. For a composite shaft made of two concentric shafts one solid and other hollow, the torque T is given as

(*a*) $T_1 - T_2$ (*b*) $\frac{T_1 + T_2}{2}$ (*c*) $T_1 + T_2$ (*d*) $\frac{T_1}{2} + T_2$.

12. For the composite shaft in Question No. 11, the angle of twist produced is equal to

(*a*) $\theta = \theta_1 + \theta_2$ (*b*) $\theta = \theta_1 - \theta_2$ (*b*) $\theta = \frac{\theta_1 - \theta_2}{2}$ (*d*) $\theta = \theta_1 = \theta_2$.

13. For a shaft being subjected to a torque T, the variation of the shear stress w.r.t. its radius is

(*a*) linear (*b*) parabolic (*c*) hyperbolic (*d*) none of these.

14. The shear stress for a shaft being subjected to a torque T is minimum at

(*a*) half of radius from the axis (*b*) axis of the shaft

(*c*) equal radial distances from the axis (*d*) its both ends.

15. The polar modulus of a shaft of diameter d is given as

(*a*) $\frac{\pi d^3}{32}$ (*b*) $\frac{\pi d^4}{64}$ (*c*) $\frac{\pi d^3}{16}$ (*d*) $\frac{\pi d^4}{32}$.

16. The shear strain in a circular shaft varies

(*a*) linearly with the distance from axis of the shaft

(*b*) inversely proportional to the distance from axis of the shaft

(*c*) inversely proportional to the square of the distance from axis of the shaft

(*d*) linearly with the square of the distance from the shaft.

17. The shear strain is maximum

(*a*) at a distance equal to one-third from axis of the shaft

(*b*) at the centre of shaft

(*c*) on the surface of shaft

(*d*) none of these.

ANSWERS

1. (*a*) **2.** (*b*) **3.** (*a*) **4.** (*b*) **5.** (*b*) **6.** (*b*) **7.** (*a*) **8.** (*a*) **9.** (*c*)

10. (*b*) **11.** (*c*) **12.** (*d*) **13.** (*a*) **14.** (*b*) **15.** (*c*) **16.** (*a*) **17.** (*c*)

EXERCISES

1. Find the lowest speed at which 250 kW could be transmitted through a shaft of diameter 63 mm. The maximum shear stress is limited to 50 MPa. If length of the shaft is 6 m, find the angle of twist. Take G = 80 GPa.

(*Ans.* 972.5 rpm, 6.82°).

2. A solid circular shaft transmits a power of 300 kW at 100 rpm. If the maximum shear stress is limited to 80 Pa, find diameter of the shaft. What percent of saving in weight can be achieved, if solid shaft is replaced by a hollow shaft with inside diameter being equal to 0.6 times the outside diameter. Both shafts are of equal length, made of same material and are being subjected to the same maximum shear stress.

(*Ans.* 122.15 mm, 29.72%).

3. A hollow shaft of inside diameter 80 mm and outside diameter 120 mm is twisted by 1.5°. Find the required torque, if length of the shaft is 2 m. Also, find shear stress at the inner and outer surfaces of the shaft. Take G = 80 GPa.

(*Ans.* 17 kN·m, 41.6 MPa, 62.85 MPa).

4. (*a*) A hollow circular shaft of inner radius 30 mm and outer radius 50 mm and length 1 m is subjected to a twisting moment so that the angular twist in the shaft is 0.57°. Find the maximum shear angle in the shaft.

(*b*) A hollow shaft of outer radius 100 mm and inner radius 40 mm is subjected to a twisting moment. The maximum shear stress developed in the shaft is 50 MPa. Find the shear stress at the inner radius of the shaft.

(*Ans.* (*a*) 0.028°; (*b*) 20 MPa).

5. A hollow steel shaft of outside diameter 90 mm is to transmit a power of 250 kW at 1200 rpm. Find its inside diameter, if the maximum shear stress in the shaft is limited to 50 MPa.

(*Ans.* 82.96 mm).

6. Two pulleys are attached to different shafts as shown in Fig. 7.24. The motor attached exerts a torque of 3.0 N·m at C. Find the maximum shear stress induced in the shafts AB and BC, assuming them to be solid.

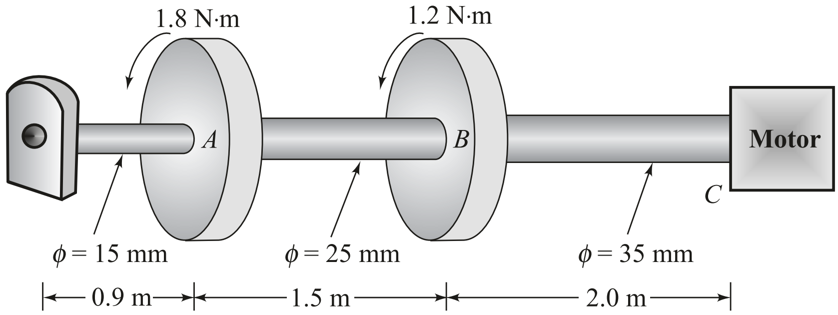


Fig. 7.24

(*Ans.* 0.586 MPa, 0.356 MPa).

7. Part *AB* of the shaft shown in Fig. 7.25 has 25 mm diameter and is subjected to a maximum shear stress of 75 MPa. Part *BC* has 60 mm diameter and is subjected to a maximum shear stress of 50 MPa. Neglecting the effect of stress concentration, find the torque T_1 applied at *A* and the reaction torque exerted by the support.

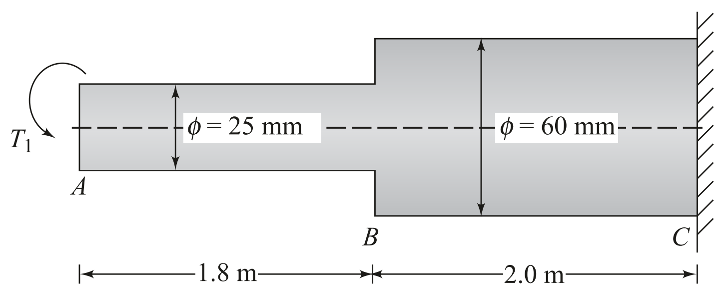


Fig. 7.25

(*Ans.* 230 N·m, 2120.5 N·m (anticlockwise)).

8. Two shafts, one solid and another hollow, are made of same material and are of equal length. They are subjected to same torque. The ratio of inside and outside diameters for the hollow shaft is 0.75. If both shafts are subjected to the same maximum shear stress, compare their weights.

$$\left(Ans. \quad \frac{W_S}{W_H} = 1.77 \right).$$

9. Find the maximum torque that can be applied safely to a shaft of diameter 300 mm. The permissible angle of twist is 1.5° in a length of 7.5 m and the maximum shear is limited to 42 MPa. Take G = 84.4 GPa.

(*Ans.* 222.6 kN·m).

10. A horizontal shaft of length 12 m is fixed at its ends. When viewed from the left end, axial couples of 50 kN·m clockwise and 75 kN·m anticlockwise act at 5 m and 9 m from the left end respectively. Determine the end fixing couples and the position, where the shaft suffers no angular twist.

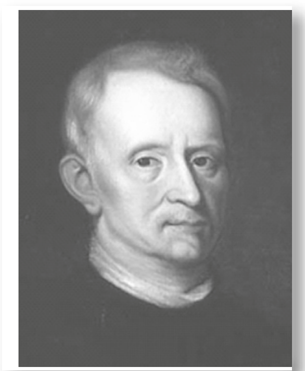
(*Ans.* 10.42 kN·m, Clockwise (left end);
35.42 kN·m, Anticlockwise (right end);
6.32 m from the left end).

11. A solid circular shaft is subjected to an axial torque *T* and a bending moment *M*. If $M = kT$, find the ratio of the maximum principal stress to the maximum shear stress in terms of *k*. Find the power transmitted by a 50 mm diameter shaft at a speed of 300 rpm, when $k = 0.4$ and the maximum shear stress is 75 MPa.

$$\left(\textit{Ans.} \quad 1 + \frac{k}{\sqrt{(1+k^2)}},\ 57.6 \text{ kW} \right).$$

❑❑❑

8 Springs



Robert Hooke
(1635-1703)

Robert Hooke, born on 18 July 1635, was an English inventor, microscopist, physicist, surveyor, astronomer, biologist and polymath. He founded the law of elasticity, widely known as Hooke's Law, in 1660, which forms the basis of design for many engineering structures and components. He originated the word 'Cell' in biology, which describes the tiniest component of a living system. He is the inventor of the iris diaphragm in cameras, the universal joint used in the motor vehicles and the balance wheel in a watch. He discovered the red spot of Jupiter and was the first to report about the rotation of the planet. He devised the world's first compound microscope that used diaphragm.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- How does the spring deflection vary with the load applied?
- Why is leaf spring also called carriage spring?
- What is Wahl correction factor?
- Why is helical spring also called torsion spring?
- How does the spring combination in series and parallel differ?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_8

8.1 INTRODUCTION

A spring is a device used to absorb or store energy and release it when required. The deformation produced in the spring is not of permanent nature and vanishes on removal of load due to its elastic nature. It finds application in mechanical clocks, shock absorbers, reciprocating mechanisms, automobile elements etc.

8.2 SPRING TERMINOLOGY

- *Proof Load* : It is the maximum load a spring can be subjected without undergoing permanent deformation.
- *Proof stress* : It is the stress corresponding to the proof load.
- *Resilience* : It is the strain energy stored in the spring when loaded within elastic limit. Once the load is removed, the energy is given up or released.
- *Proof resilience* : It is the maximum strain energy stored in the spring when loaded within elastic limit.
- *Modulus of Resilience* : It is the maximum strain energy stored per unit volume.
- *Stiffness* : It is the load required to produce unit deflection in the spring.

8.3 CLASSIFICATION OF SPRINGS

Various types of springs are employed for different applications. Important types are discussed below.

Helical springs are in the form of a helix made of a rod of circular, square or rectangular cross-section, but the circular section is most frequently used. Square or rectangular sections are used in heavy duty springs. The main disadvantages of non-circular wire section springs include non-uniform stress distribution and less energy absorbing capacity of the spring.

Helical springs are also known as torsion springs, because they are subjected to torsion. Helical springs may be in the cylindrical or conical form.

Cylindrical helical springs are subjected to tension or compression. A conical helical spring is used where there is a space problem or a single spring with a variable stiffness is desired. They are used for heavy loads and deflections. Helical springs are used in shock absorbers, reciprocating mechanisms, front axle suspension of vehicles, etc. They are classified into two groups depending upon angle of helix.

In the *close coiled helical spring*, angle of helix is very small and can be approximated to zero so that plane of the coil is at right angle to the axis of the spring.

In the *open coiled helical spring,* plane of the coil of the spring makes certain angle with the horizontal.

Depending upon the nature of applications, helical springs can be of compression and tension type.

Laminated springs or *leaf springs* are made of a number of thin curved plates of uniform thickness but different lengths which are placed over each other and clamped together at the centre. They may be of cantilever, semi-elliptical or elliptical type. The most common type of leaf spring is semi-elliptical leaf spring. For very heavy loads to be supported, it can be elliptical leaf spring. Laminated springs are also called carriage springs and find application in the suspension system of front or rear axles of cars, trucks, trains etc. Sometimes they are also known as bending springs because being subjected to bending only.

Flat springs are made of a flat strip and may be in the form of a cantilever or a simple beam. The end of the flat cantilever spring can be guided along a definite path as it deflects. Thus, the spring may act as a structural member as well as an energy absorbing device.

Spiral spring consists of a flat strip wound in the form of a spiral and is loaded in torsion. The major stresses produced in the spring are tensile and compressive due to bending by the applied load.

Various types of springs are shown in Fig. 8.1.

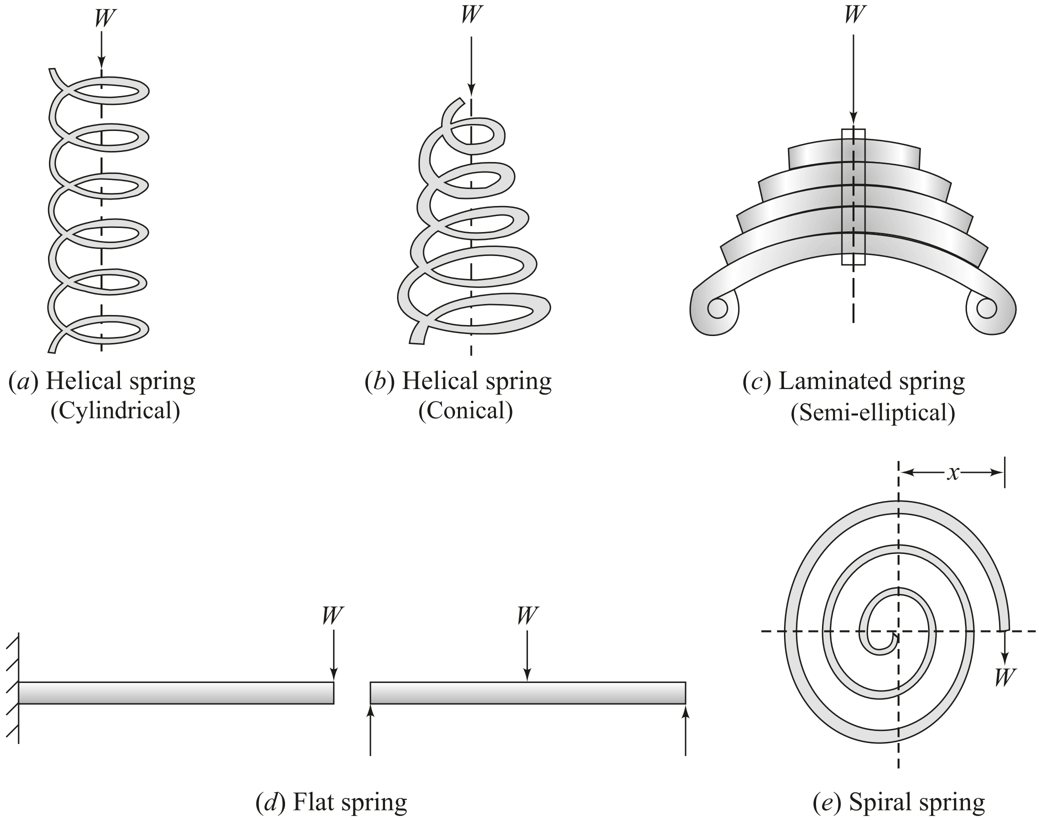


Fig. 8.1 Types of springs.

Sometimes springs are used to support a body that has vibrational problems. Sofa springs are the examples of conical helical springs. Demerits of the spring include difficulty in its manufacture and buckling during the use.

8.4 LOAD-DEFLECTION CURVE

If we plot load-deflection diagram for a spring, a straight line is obtained suggesting that deflection is directly proportional to the load applied. For an example, if the load is doubled, then deflection will also be doubled (Fig. 8.2). The straight relationship is valid only if the spring is stressed within elastic limit. It is equally applicable in case when a torque or moment is acting in place of a load and linear deflection is replaced by angular deflection.

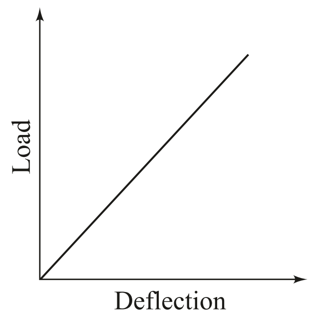


Fig. 8.2 Load Versus deflection curve for a spring.

8.5 LEAF SPRING

A leaf spring, also called laminated spring, is shown in Fig. 8.3. The long leaf or plate fastened to the supports is called the main leaf or master leaf. Its ends are bent to form an eye. If heavy loading is to be applied, one or more full length leaves are provided below the master leaf. The bundle of leaves having same thickness but different length are clamped together at the centre. All the leaves are bent to same curvature. The spring rests on the axle of the vehicle and is pin jointed to the chasis at the eyes.

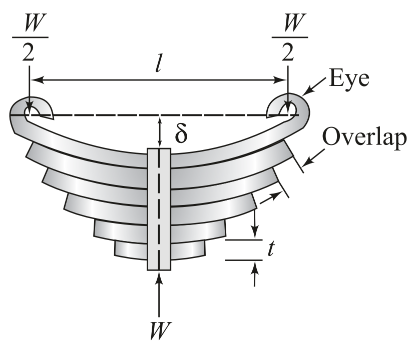


Fig. 8.3 A leaf spring.

Consider a leaf spring hinged at both ends and carrying a load W at the centre. The load W is distributed equally at the two eyes.

Let	Span of the spring (biggest leaf)	$= l$
	Width of each plate	$= b$
	Thickness of each plate	$= t$
	Number of plates in the spring	$= n$
	Bending stress in the plate	$= \sigma_b$
	Deflection of the master plate	$= \delta$
	Radius of curvature of each plate	$= R$

The bending moment is maximum at the centre and goes on decreasing towards the ends (eyes); therefore maximum resistance of moment is required at the centre and less towards the ends. That is why leafs of gradually reducing lengths are used in the spring. The law of variation of bending moment is linear.

Using bending equation, the moment of resistance is found to be

$$M_r = \frac{n\sigma_b bt^2}{6} \quad \text{... (8.1)}$$

The maximum bending moment due to load applied on the spring is given as

$$M = \frac{Wl}{4} \quad \text{... (8.2)}$$

Equation (8.2) is obtained on comparing the spring with a simply supported beam loaded with a point load at the centre.

From the two equations (8.1) and (8.2), we have

$$\frac{Wl}{4} = \frac{n\sigma_b \cdot bt^2}{6}$$

or

$$\sigma_b = \frac{3Wl}{2nbt^2} \quad \text{... (8.3)}$$

The radius of curvature can be obtained by considering Fig. 8.4.

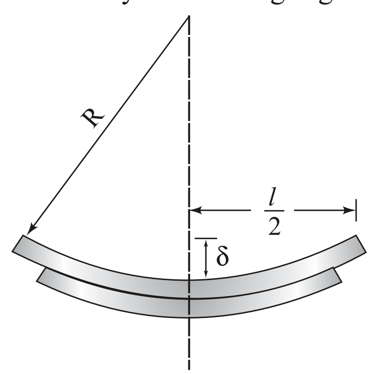


Fig. 8.4

$$\left(\frac{l}{2}\right)^2 + (R - \delta)^2 = R^2$$

or

$$\delta = \frac{l^2}{8R} \quad \text{... (8.4)}$$

From bending equation

$$\frac{\sigma_b}{(t/2)} = \frac{E}{R}$$

or

$$R = \frac{Et}{2\sigma_b} \quad \text{... (8.5)}$$

Substituting R in equation (8.4), we get

$$\delta = \frac{\sigma_b l^2}{4Et} \quad \text{... (8.6)}$$

where E = Modulus of elasticity of the spring material

Another equation for deflection can be obtained by substituting σ_b in equation (8.6) as

$$\delta = \frac{3Wl^3}{8n\,Ebt^3} \quad \text{... (8.7)}$$

The strain energy stored in the spring is due to the work done by the load acting on it, given by

$$U = \frac{1}{2} \cdot W \cdot \delta \quad \text{... (8.8)}$$

Substituting δ in equation (8.8), we get strain energy as

$$U = \frac{1}{2} \times W \times \frac{3Wl^3}{8n\,Ebt^3}$$

or

$$U = \frac{3W^2l^3}{16n\,Ebt^3} \quad \text{... (8.9)}$$

The strain energy can also be expressed in terms of bending stress and volume of spring, given by

$$U = \frac{\sigma_b^2}{6E} \times \text{Volume of the spring} \quad \text{... (8.10)}$$

where Volume of spring $= \frac{n}{2} lbt$

Example 8.1

A steel carriage spring is 800 mm long and carries a central load of 6 kN. The plates are 70 mm wide and 5 mm thick. Determine the number of plates in the spring to sustain a maximum bending stress of 200 N/mm^2. What will be the deflection in the spring? To what radius should each plate be curved so that it becomes straight under the given load? Take E = 200 kN/mm^2.

Solution: Given,

Length of the biggest plate, l = 800 mm
Central load on the spring, W = 6 kN = 6000 N
Width of the plate, b = 70 mm
Thickness of the plate, t = 5 mm
Bending stress induced in the spring, σ_b = 200 N/mm^2
Modulus of elasticity of spring material, E = 200 kN/mm^2 = 2 × 10^5 N/mm^2

Let

Number of plates in the spring = n
Deflection of the spring = δ
Radius of curvature of each plate = R

Using equation (8.3) for bending stress, we have

$$\sigma_b = \frac{3Wl}{2nbt^2}$$

$$200 = \frac{3 \times 6000 \times 800}{2 \times n \times 70 \times 5^2}$$

or $$n = 20.57$$

Hence, the number of plates can be taken to be 21.

Ans.

Now, using equation (8.7) for deflection, we have

$$\delta = \frac{3Wl^3}{8nEbt^3}$$

or $$\delta = \frac{3 \times 6000 \times (800)^3}{8 \times 21 \times 2 \times 10^5 \times 70 \times 5^3} = 31.34 \text{ mm}$$ **Ans.**

For the radius of curvature, use equation (8.4).

$$\delta = \frac{l^2}{8R}$$

or $$R = \frac{l^2}{8\delta} = \frac{(800)^2}{8 \times 31.34} = 2.552 \text{ m}$$ **Ans.**

Example 8.2

A laminated steel spring of length 900 mm carries a central proof load of 8 kN. The maximum deflection at the centre is 60 mm and the bending stress should not exceed 0.5 kN/mm^2. Determine the thickness, width, number of plates, and the radius to which the plates should be bent. Assume the plate width to be ten times its thickness. Take E = 200 kN/mm^2.

Solution: Given,

Length of the biggest plate, l = 900 mm

Load on the spring, W = 8 kN = 8000 N

Deflection of the spring, δ = 60 mm

Bending stress induced in the spring, σ_b = 0.5 kN/mm^2 = 500 N/mm^2

Modulus of elasticity, E = 200 kN/mm^2 = 2 × 10^5 N/mm^2

Let Width of the plate = b

Thickness of the plate = t

Number of plates in the spring = n

Given, $$b = 10t \qquad \text{... (1)}$$

Using equation (8.7) for deflection, we have

$$\delta = \frac{3Wl^2}{8nEbt^3}$$

$$60 = \frac{3 \times 8000 \times (900)^3}{8n \times 2 \times 10^5 \times 10t \times t^3}$$

or $$nt^4 = 18225 \qquad \text{...(2)}$$

Using equation (8.3) for bending stress, we have

$$\sigma_b = \frac{3Wl}{2nbt^2}$$

$$500 = \frac{3 \times 8000 \times 900}{2n \times 10t \times t^2}$$

or $$nt^3 = 2160 \qquad ...(3)$$

Dividing equation (2) by equation (3), we get

$$t = \frac{18225}{2160} = 8.43 \text{ mm}$$

Hence, the thickness of the plate can be chosen to be 8 mm. **Ans.**

From equation (1)

$$b = 10t = 10 \times 8 = 80 \text{ mm} \qquad \textbf{Ans.}$$

From equation (3)

$$n = \frac{2160}{8^3} = 4.21$$

Hence, the number of plates can be 5. **Ans.**

The actual deflection under the proof load is given as

$$\delta_p = \frac{3 \times 8000 \times (900)^3}{8 \times 5 \times 2 \times 10^5 \times 80 \times (8)^3} = 53.4 \text{ mm} \qquad \text{(using equation (8.7))}$$

The radius of curvature is found by using equation (8.4).

$$R = \frac{l^2}{8\delta_p}$$

$$= \frac{(900)^2}{8 \times 53.4} = 1.896 \text{ m}$$

Hence, the radius of curvature of each plate = 1.896 m. **Ans.**

Example 8.3

A steel carriage spring of length 1.5 m having plate width 150 mm and thickness 10 mm is subjected to a bending stress of 200 N/mm^2. The spring during its straightening absorbs 150 joule of energy. Find the number of plates and their radius of curvature. Take E = 200kN/mm^2.

Solution: Given,

Length of the biggest plate, l = 1.5 m = 1500 mm

Width of the plate, b = 150 mm

Thickness of the plate, t = 10 mm

Modulus of Elasticity of spring material, E = 200 kN/mm^2 = 2 × 10^5 N/mm^2

Bending stress induced in the spring, $\sigma_b = 200$ N/mm^2

Strain energy stored in the spring, $U = 150$ joules

Let, Number of plates, $= n$

Radius of curvature of each plate, $= R$

The strain energy stored in the spring is given by the equation (8.10) as

$$U = \frac{\sigma_b^2}{6E} \times \frac{n}{2} lbt$$

$$150 \times 10^3 = \frac{(200)^2 \times n \times 1500 \times 150 \times 10}{6 \times 2 \times 10^5 \times 2}$$

or $n = 4$ **Ans.**

To find the radius of curvature, use equation (8.5).

$$R = \frac{Et}{2\sigma_b} = \frac{2 \times 10^5 \times 10}{2 \times 200} = 5 \text{ m}$$ **Ans.**

Example 8.4

A steel carriage spring of length 1 m has 15 leaves, each 150 mm wide and 10 mm thick. It is subjected to a maximum bending stress of 300 N/mm^2. Find the curvature of each leaf.

A load of 200 N is dropped on the spring without exceeding the given bending stress. Neglecting the loss of energy at impact, find the maximum height through which the load falls on the spring. Take $E = 200$ kN/mm^2.

Solution: Given,

Length of the biggest leaf, $l = 1$ m $= 1000$ mm

Number of leaves in the spring, $n = 15$

Width of the leaf, $b = 150$ mm

Thickness of the leaf, $t = 10$ mm

Bending stress induced in the spring, $\sigma_b = 300$ N/mm^2

Load to be dropped on the spring, $W = 200$ N

Modulus of elasticity, $E = 200$ kN/mm^2

$= 2 \times 10^5$ N/mm^2

Let

Radius of curvature of each plate $= R$

Height of load $= h$

The radius of curvature R can be found by using equation (8.5).

$$R = \frac{Et}{2\sigma_b} = \frac{2 \times 10^5 \times 10}{2 \times 300} = 3.34 \text{ m}$$ **Ans.**

The deflection in the spring is given by equation (8.4).

$$\delta = \frac{(1000)^2}{8 \times 3.34 \times 10^3} = 37.5 \text{ mm}$$

The potential energy at a height h is given as

$$P.E = W(\delta + h) = 200\,(37.5 + h) \qquad \dots (1)$$

The potential energy is stored in the spring as strain energy given by equation (8.10).

$$U = \frac{\sigma_b^2}{6E} \times \frac{n}{2} lbt$$

$$= \frac{(300)^2}{6 \times 2 \times 10^5} \times \frac{15}{2} \times 1000 \times 150 \times 10 = 843.75 \text{ joule} \qquad \dots (2)$$

Equating equations (1) and (2), we have

$$200\,(37.5 + h) = 843.75 \times 10^3$$

or $$h = 4.18 \text{ m}$$ **Ans.**

8.6 QUARTER-ELLIPTIC LEAF SPRING

The analysis of a quarter-elliptic leaf spring (also called cantilever laminated spring) can be made by comparing the spring with a cantilever beam loaded with a point load at its free end (Fig. 8.5). In this case, maximum bending moment is given as

$$M = Wl \qquad \dots (8.11)$$

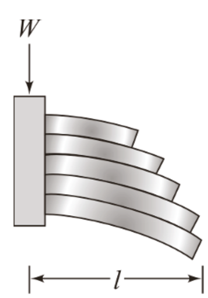


Fig. 8.5 A quarter-elliptic leaf spring.

Using equations (8.1) and (8.11), we get the equation for maximum bending stress.

$$\sigma_{b_{\max}} = \frac{6Wl}{nbt^2} \qquad \dots (8.12)$$

where the symbols have their usual meanings.

Deflection in the spring is given as

$$\delta = \frac{6Wl^3}{nE\,bt^3} \qquad \text{... (8.13)}$$

Radius of curvature of each leaf is found to be

$$R = \frac{l^2}{2\delta} = \frac{Et}{2\sigma_{b_{\max}}} \qquad \text{... (8.14)}$$

Example 8.5

A quarter-elliptic leaf spring has 10 leaves with cross-section of each leaf being 100 mm × 10 mm. The spring is 500 mm long. The bending stress is not to exceed 300 N/mm^2. Determine the following parameters of the spring.

(*a*) the maximum load which can be applied on the spring

(*b*) the deflection produced and

(*c*) the radius of curvature of leaves

Take E = 200 kN/mm^2.

Solution: Given,

Number of leaves in the spring, $n = 10$

Width of the plate, $b = 100$ mm

Thickness of the plate, $t = 10$ mm

Lenth of the biggest plate, $l = 500$ mm

Maximum bending stress induced in the spring, $\sigma_{b_{\max}} = 300$ N/mm^2

Modulus of elasticity, E = 200kN/mm^2 = 2 × 10^5 N/mm^2

(*a*) Use equation (8.12) to get the load W acting on the spring.

$$\sigma_{b_{\max}} = \frac{6Wl}{nbt^2}$$

$$300 = \frac{6W \times 500}{10 \times 100 \times 10^2}$$

or $W = 10$ kN **Ans.**

(*b*) The deflection can be obtained by using equation (8.13).

$$\delta = \frac{6Wl^3}{nE\,bt^3} = \frac{6 \times 10 \times 10^3 \times (500)^3}{10 \times 2 \times 10^5 \times 100 \times 10^3} = 37.5 \text{ mm}$$ **Ans.**

(*c*) Use equation (8.14) for the radius of curvature.

$$R = \frac{l^2}{2\delta} = \frac{500^2}{2 \times 37.5} = 3.34 \text{ m}$$ **Ans.**

Example 8.6

A quarter-elliptic leaf spring of length 700 mm is subjected to a load of 3 kN, producing a deflection of 100 mm. The cross-section of the spring is 70 mm × 10 mm. Determine the following parameters of the spring:

(*a*) the number of leaves in the spring

(*b*) the maximum bending stress induced in the spring and

(*c*) the height through which if the given load is dropped on the spring, is to produce a maximum bending stress of 1 kN/mm^2

Take E = 200 kN/mm^2.

Solution: Given,

Length of the biggest plate,	l = 700 mm
Load on the spring,	W = 3 kN = 3 × 10^3 N
Deflection produced in the spring,	δ = 100 mm
Modulus of elasticity,	E = 200 kN/mm^2 = 2 × 10^5 N/mm^2
Width of the plate,	b = 70 mm
Thickness of the plate,	t = 10 mm

(*a*) Use equation (8.13) for number of leaves in the spring.

$$\delta = \frac{6Wl^3}{nE\,bt^3}$$

$$100 = \frac{6\times3\times10^3\times(700)^3}{10\times2\times10^5\times70\times10^3}$$

or n = 4.41

Hence, the number of leaves can be chosen to be 5. **Ans.**

(*b*) The maximum bending stress induced in the spring can be obtained by using equation (8.12).

$$\sigma_{b_{\max}} = \frac{6Wl}{nbt^2} = \frac{6\times3\times10^3\times700}{5\times70\times10^2} = 360\text{ N/mm}^2$$ **Ans.**

(*c*) If W_e is the equivalent gradually applied load to produce the given bending stress. Equation (8.12) is again used to find W_e.

$$1000 = \frac{6W_e\times700}{5\times70\times10^2}$$

or W_e = 8.34 kN

This is the equivalent impact load corresponding to the given load of 3 kN.

The deflection produced by the impact load is given as

$$\delta_e = \frac{6\times8.34\times10^3\times(700)^3}{5\times2\times10^5\times70\times10^3} = 245\text{ mm.}$$

Using principle's of energy, we have

$$\text{Loss of potential energy} = \text{Gain of strain energy}$$

$$W(h + \delta_e) = \frac{1}{2} \times W_e \times \delta_e$$

where, h is the height through which the load falls.

$$3 \times 10^3 (h + 245) = \frac{1}{2} \times 8.34 \times 10^3 \times 245$$

or $h = 95.2$ mm **Ans.**

8.7 SPIRAL SPRING

A spiral spring consists of a flat strip of rectangular section wound into a spiral (Fig. 8.1). It is used to produce torsion on the axle to which one of its end (inner one) is attached. If a force W, acting at a radius x, is applied to the axis of the spring, a twisting moment Wx is set up.

Let

Bending stress induced in the spring $= \sigma_b$

Width of spring strip $= b$

Length of spring strip $= l$

Thickness of spring strip $= t$

The maximum bending stress in the spring is found to be

$$\sigma_{b_{\max}} = \frac{12Wx}{bt^2} \quad \text{(using bending equation)} \quad ...(8.15)$$

$$= \frac{12M}{bt^2} \quad ...(8.16)$$

where $M = Wx$ = Winding torque or turning moment on the spindle

The angular deflection of the spring is given as

$$\theta \text{ (radian)} = \frac{Wxl}{EI} = \frac{Ml}{EI} \quad ...(8.17)$$

where E = Modulus of elasticity of the spring material

I = Moment of inertia of the spring cross-section

The strain energy stored in the spring is given as

$$U = \frac{1}{2} \times M \times \theta$$

$$= \frac{1}{2} \times Wx \times \frac{Wxl}{EI} \quad \text{...(on substituting } \theta\text{)}$$

$$= \frac{W^2 x^2 l}{2EI} \quad ...(8.18)$$

The number of turns given to the spindle is given as

$$n = \frac{\theta}{2\pi} \qquad \text{...(8.19)}$$

Example 8.7

A flat spiral spring with rectangular section 10 mm × 0.5 mm is 5 metre long. One end of it is attached to a small spindle and other to a fixed point. The bending stress is not to exceed 0.5 kN/mm^2. Find the following parameters of the spring:

(*a*) the maximum turning moment applied to the spindle

(*b*) the number of turns and

(*c*) the strain energy stored in the spring

The modulus of elasticity of the spring material can be taken to be 200 kN/mm^2.

Solution: Given,

Width of the spring section, $b = 10$ mm

Thickness of the spring section, $t = 0.5$ mm

Length of the spring, $l = 5 \text{ m} = 5 \times 10^3$ mm

Bending stress induced in the spring, $\sigma_{b_{max}} = 0.5 \text{ kN/mm}^2 = 500 \text{ N/mm}^2$

Modulus of elasticity of spring material, $E = 200 \text{ kN/mm}^2 = 2 \times 10^5 \text{ N/mm}^2$

(*a*) The maximum turning moment applied to the spindle is given by equation (8.16).

$$\sigma_{b_{max}} = \frac{12M}{bt^2}$$

$$500 = \frac{12M}{10 \times (0.5)^2}$$

or $M = 104.16$ N.mm **Ans.**

(*b*) The angular deflection θ of the spring is given by using equation (8.17).

$$\theta = \frac{Ml}{EI}$$

$$= \frac{104.16 \times 5 \times 10^3}{2 \times 10^5 \times \left[\frac{1}{12} \times 10 \times (0.5)^3\right]} = 25 \text{ radian}$$

For number of turns, use equation (8.19).

$$n = \frac{\theta}{2\pi} = \frac{25}{2\pi} = 3.97$$

Hence, the number of turns can be chosen to be 4. **Ans.**

(*c*) Strain energy stored in the spring can be found by using equation (8.18).

$$U = \frac{1}{2} \times M \times \theta$$

$$= \frac{1}{2} \times 104.16 \times 25$$

$$= 1.3 \text{ joules}$$ **Ans.**

Example 8.8

A flat spiral spring having cross-section 20 mm × 0.5 mm is subjected to a maximum bending stress of 1 kN/mm^2. The spring can store an energy of 10 Joule. With E = 200 kN/mm^2, determine the following parameters of the spring:

(*a*) the maximum torque required

(*b*) the number of turns given to the spindle and

(*c*) the length of the spring.

Solution: Given,

Width of the spring section, $b = 20$ mm

Thickness of the spring section, $t = 0.5$ mm

Maximum bending stress, $\sigma_{b_{max}} = 1$ kN/mm^2 = 1000 N/mm^2

Energy stored in the spring, $U = 10$ Joule

(*a*) The maximum torque required can be obtained by using equation (8.16).

$$\sigma_{b_{max}} = \frac{12M}{bt^2}$$

$$1000 = \frac{12M}{20 \times (0.5)^2}$$

or $$M = 416.6 \text{ N·mm}$$

Hence, the required torque is 416.6 N·mm. **Ans.**

(*b*) The strain energy is given as

$$U = \frac{1}{2} \times M \times \theta$$

$$10 \times 10^3 = \frac{1}{2} \times 416.6 \times \theta$$

or $$\theta = 48 \text{ radian}$$

The number of turns is given by equation (8.19).

$$n = \frac{\theta}{2\pi} = \frac{48}{2\pi} = 7.64$$ **Ans.**

(*c*) The length of the spring can be found by using equation (8.17).

$$\theta = \frac{Ml}{EI}$$

$$48 = \frac{416.6 \times l}{2 \times 10^5 \times \left[\frac{1}{12} \times 20 \times (0.5)^3\right]}$$

or $$l = 4.8\text{m}$$ **Ans.**

8.8 HELICAL SPRING

8.8.1 Close Coiled Helical Spring subjected to an Axial Load

Fig. 8.6 shows a close coiled helical spring subjected to an axial load W.

Let R = Mean radius of the coil

d = Diameter of the spring wire

n = Number of turns or coils in the spring

θ = Angle of twist produced in the spring wire

δ = Deflection produced by the applied load W

τ = Shear stress induced in the wire of spring

G = Modulus of rigidity of spring material

Each section of the wire of spring is subjected to torsional shear stress and hence such springs are also called torsion springs. Bending effect, on account of its negligible value is neglected. Effect of direct stress is also neglected due to similar reasons.

Total length of the spring wire, $l = 2\pi Rn$

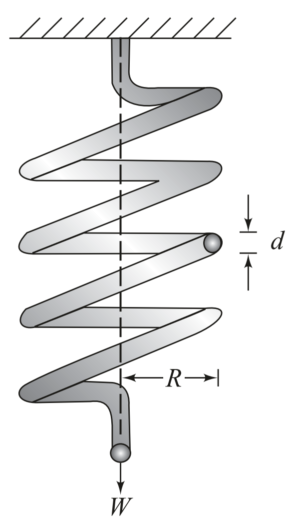


Fig. 8.6 A close coiled helical spring subjected to an axial load.

Using torsion formula, we have

$$\frac{T}{J} = \frac{G\theta}{l} \qquad \text{... (8.20)}$$

where $T = WR$ = Torque applied on the spring

J = Polar moment of inertia of the spring wire

$$= \frac{\pi}{32} d^4$$

Substituting T, J and l in equation (8.20), we have

$$\theta = \frac{64WR^2 n}{Gd^4} \qquad \text{...(8.21)}$$

This is the required expression for the angle of twist expressed in radian.

Shear stress induced in the spring is found by using torsion formula.

$$\frac{T}{J} = \frac{\tau}{r}$$

where r is radius of the spring wire.

On substituting T and J, the above equation gives shear stress in the outermost fibre of the spring wire as

$$\tau = \frac{16WR}{\pi d^3} \qquad \text{...(8.22)}$$

Equation (8.22) gives maximum shear stress induced in the spring wire. While deriving this equation, the effect of the curvature of the wire is neglected. This is true only when the spring is subjected to static loads, because yielding of the spring material will relieve the stresses. Also, the effect of direct stress has been neglected.

Curvature of the spring wire in the calculation of maximum shear stress was introduced by A.M. Wahl, who is famous for his book 'Mechanical Springs', considered as the bible of spring design. Accordingly, the equation for the shear stress is modified as

$$\tau_{max} = \frac{8WD}{\pi d^3} \times K_W \qquad \text{... (8.23)}$$

where K_W = Wahl correction factor

$$= \frac{8C-1}{4C-4} + \frac{0.615}{C}$$

C = Spring index = $\frac{D}{d}$ for circular spring wires

The strain energy stored in the spring is given as

$$U = \frac{1}{2} T\theta$$

$$= \frac{1}{2} WR \times \frac{64WR^2 n}{Gd^4} = \frac{32W^2 R^3 n}{Gd^4} \quad \text{... (8.24)}$$

The strain energy in terms of shear stress can be expressed as

$$U = \frac{\tau^2}{4G} \times \text{Volume of the spring} \quad \text{... (8.25)}$$

The workdone on the spring is equal to the strain energy stored in it.

$$U = \frac{1}{2} W\delta = \frac{1}{2} T\theta \quad \text{... (8.26)}$$

or
$$\delta = \frac{2U}{W}$$

$$= \frac{2}{W}\left(\frac{32W^2 R^3 n}{Gd^4}\right) \quad \text{(on substituting } U\text{)}$$

$$= \frac{64WR^3 n}{Gd^4} \quad \text{... (8.27)}$$

This is the required expression for deflection produced in the spring under the axial load W.

Axial stiffness of the spring, K is defined as

$$K = \frac{W}{\delta} = \frac{Gd^4}{64R^3 n} \quad \text{... (8.28)}$$

8.8.2 Close Coiled Helical Spring subjected to an Axial Twist

Let the spring shown in Fig. 8.7 is subjected to an axial couple at its free end. The effect of this may be to open or close the spring, resulting in change in number of turns/coils of the spring. The number of turns in the spring increases if the torque resulting from the applied couple closes the spring and number of turns decreases if the torque opens the spring. During the analysis of such spring, effect of torsional shear stress is neglected and only bending stress is considered.

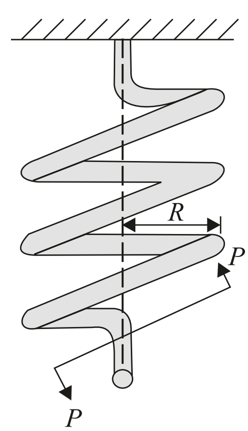


Fig. 8.7 A close coiled helical spring subjected to an axial twist.

The length of the spring is obtained as

$$l = 2\pi R_1 n = 2\pi R_2 n' \qquad ...(8.29)$$

where R_1 and R_2 = Initial and final mean radius of the coil respectively

n and n' = Initial and final number of coils in the spring respectively

Length of the spring remains constant, but radius of coil and number of coils change when axial couple is applied.

Using bending equation, the bending moment M is given as

$$M = EI\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \qquad ...(8.30)$$

Using equation (8.29), equation (8.30) transforms to

$$M = \frac{2\pi EI}{l}\ [n - n'] \qquad ...\ (8.31)$$

The angle of twist (in radian) at the free end of spring is given as

$$\phi = 2\pi\ (n - n') \qquad ...\ (8.32)$$

Using equation (8.31), we find ϕ to be

$$\phi = \frac{Ml}{EI} \qquad ...\ (8.33)$$

If the spring is made of circular wire of diameter d, then

$$I = \frac{\pi}{64} d^4$$

Equation (8.33) on using equations (8.29) and (8.31) becomes

$$\phi = \frac{128MRn}{Ed^4} \qquad (R_1 \approx R_2 = R) \qquad ...\ (8.34)$$

The angular stiffness of the spring is given as

$$K_1 = \frac{M}{\phi} = \frac{EI}{l}$$

$$= \frac{Ed^4}{128Rn} \qquad ...\ (8.35)$$

The bending stress induced in the spring is calculated by using bending equation.

$$\frac{M}{I} = \frac{\sigma_b}{y}$$

where σ_b = Maximum bending stress induced in the spring

$$y = \frac{d}{2}$$

Hence,

$$\sigma_b = \frac{32M}{\pi d^3} \qquad ...\ (8.36)$$

The strain energy stored in the spring is given as

$$U = \frac{1}{2}M\phi$$

$$= \frac{64M^2 Rn}{Ed^4} \quad \text{... (8.37)}$$

The strain energy can also be expressed in terms of bending stress, given as

$$U = \frac{\sigma_b^2}{8E} \times \text{Volume of the spring} \quad \text{... (8.38)}$$

8.8.3 Open Coiled Helical Spring subjected to an Axial Load

The figure for an open coiled helical spring under the action of an axial load W is similar to as shown in Fig. 8.6. The load produces two types of moments, one has twisting action and another has bending effect. The two moments are :

$$\left.\begin{aligned} \text{Twisting moment,} \quad T &= WR\cos\alpha \\ \text{Bending moment,} \quad M &= WR\sin\alpha \end{aligned}\right\} \quad \text{... (8.39)}$$

where α = Angle of helix

The equivalent twisting moment is obtained as

$$T_e = \sqrt{T^2 + M^2}$$

$$= \sqrt{W^2R^2\cos^2\alpha + W^2R^2\sin^2\alpha} = WR \quad \text{... (8.40)}$$

The equivalent bending moment is obtained as

$$M_e = \frac{1}{2}[M + \sqrt{T^2 + M^2}]$$

$$= \frac{1}{2}[WR\sin\alpha + \sqrt{W^2R^2\cos^2\alpha + W^2R^2\sin^2\alpha}]$$

$$= \frac{1}{2}[WR\sin\alpha + WR]$$

$$= \frac{WR}{2}[1 + \sin\alpha] \quad \text{... (8.41)}$$

The maximum direct stress induced in the spring wire is given as

$$\sigma_b = \frac{32M_e}{\pi d^3} \quad \text{(using bending equation)} \quad \text{... (8.42)}$$

The maximum shear stress induced in the spring wire is given as

$$\tau = \frac{16T_e}{\pi d^3} \quad \text{(using torsion equation)} \quad \text{... (8.43)}$$

The total length of wire in the spring is given as

$$l = 2\pi nR\sec\alpha \quad \text{... (8.44)}$$

The strain energy stored in the spring is due to torsion and bending effect both.

The strain energy due to torsion is given as

$$U_t = \frac{1}{2}T\theta = \frac{T^2 l}{2GJ} \qquad \text{(using torsion equation)} \qquad \text{... (8.45)}$$

where θ = Angle of twist produced in the spring wire

$$= \frac{64WR^2 n \sin\alpha}{d^4}\left[\frac{1}{G} - \frac{2}{E}\right] \qquad \text{... (8.46)}$$

The strain energy due to bending moment is found as

$$U_b = \frac{1}{2}M\theta$$

$$= \frac{M^2 l}{2EI} \qquad \left(\text{using } \frac{M}{I} = \frac{E}{R} \text{ and } \frac{l}{R} = \theta\right) \text{... (8.47)}$$

The total strain energy stored in the spring is given as

$$U_T = U_t + U_b$$

$$= \frac{l}{2}\left[\frac{T^2}{GJ} + \frac{M^2}{EI}\right] \qquad \text{... (8.48)}$$

$$= \frac{W^2 R^2 l}{2}\left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right] \qquad \text{... (8.49)}$$

Workdone on the spring is also found as

$$\frac{1}{2}W\delta = U_T \qquad \text{... (8.50)}$$

Comparing equations (8.49) and (8.50), we find the value of deflection δ as

$$\delta = WR^2 l\left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right] \qquad \text{... (8.51)}$$

For a spring made of circular wire of diameter d, we have

$$I = \frac{\pi d^4}{64} \text{ and } J = \frac{\pi d^4}{32}$$

Now equation (8.51) changes to

$$\delta = \frac{64WR^3 n \sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right] \qquad \text{... (8.52)}$$

On comparing this equation with close-coiled helical spring, where α is zero, equation (8.52) reduces to equation (8.27).

8.8.4 Open Coiled Helical Spring subjected to an Axial Twist

Let an open coiled helical spring be subjected to a torque M which produces both twisting and bending.

$$\left.\begin{aligned}\text{Twisting moment, } T' &= M\sin\alpha\\ \text{Bending moment, } M' &= M\cos\alpha\end{aligned}\right\} \qquad \text{... (8.53)}$$

The equivalent twisting moment is found as

$$T_e = \sqrt{T'^2 + M'^2}$$

$$= \sqrt{M^2\sin^2\alpha + M^2\cos^2\alpha} = M \qquad \text{... (8.54)}$$

The equivalent bending moment is found as

$$M_e = \frac{1}{2}[M' + \sqrt{T'^2 + M'^2}]$$

$$= \frac{1}{2}[M\cos\alpha + \sqrt{M^2\sin^2\alpha + M^2\cos^2\alpha}]$$

$$= \frac{M}{2}[1 + \cos\alpha] \qquad \text{... (8.55)}$$

The maximum direct stress induced in the spring is given as

$$\sigma_b = \frac{32M_e}{\pi d^3} \qquad \text{(using bending equation)} \quad \text{... (8.56)}$$

The maximum shear stress induced in the spring is given as

$$\tau = \frac{16T_e}{\pi d^3} \qquad \text{(using torsion equation)} \quad \text{... (8.57)}$$

The total strain energy stored in the spring is given as

$$U_T = \frac{1}{2}T'\theta' + \frac{1}{2}M'\phi' \qquad \text{... (8.58)}$$

where θ' and ϕ' are the angles of twist due to T' and M' respectively, and are expressed as

$$\theta' = \frac{T'l}{GJ} \text{ and } \phi' = \frac{M'l}{EI} \qquad \text{... (8.59)}$$

Also $$U_T = \frac{1}{2}M\phi \qquad \text{... (8.60)}$$

where ϕ is the net angle of twist because of the combined effects.

Comparing equations (8.58) and (8.60) and using equations (8.53) and (8.59), we have

$$\phi = \frac{64MRn\sec\alpha}{d^4}\left[\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right] \qquad \text{... (8.61)}$$

(on using $l = 2\pi Rn\sec\alpha$)

On comparing this equation with close coiled helical spring, where $\alpha = 0$ and subjected to an axial twist M, we find that equation (8.61) is similar to equation (8.34).

The deflection produced in the spring with circular wire is found to be

$$\delta = \frac{64MR^2 n \sin\alpha}{d^4}\left[\frac{1}{G} - \frac{2}{E}\right] \quad ...(8.62)$$

Example 8.9

A close coiled helical spring of circular section having a mean coil diameter of 60 mm is subjected to an axial load of 80 N applied at the end of spring producing a shear stress of 100 N/mm^2 and a deflection of 50 mm. Find the diameter, the number of coils, the length of the spring wire, and the strain energy stored in the spring. Take $G = 80$ kN/mm^2.

Solution: Given,

Mean diameter of the coil, $D = 60$ mm
Axial load on the spring, $W = 80$ N
Shear stress induced in the spring, $\tau = 100$ N/mm^2
Deflection produced in the spring, $\delta = 50$ mm
Modulus of rigidity of spring material, $G = 80$ kN/mm$^2 = 8 \times 10^4$ N/mm^2

The diameter of the spring wire can be found by using equation (8.22).

$$\tau = \frac{16WR}{\pi d^3}$$

or

$$d = \left(\frac{16WR}{\pi\tau}\right)^{1/3}$$

$$= \left(\frac{16\times 80\times\frac{60}{2}}{\pi\times 100}\right)^{1/3} = 4.96 \text{ mm}$$

Hence, the diameter of the spring wire is 4.96 mm. **Ans.**

Using equation (8.27), number of coils in the spring can be found.

$$\delta = \frac{64WR^3 n}{Gd^4}$$

$$50 = \frac{16\times 80\times\left(\frac{60}{2}\right)^3\times n}{8\times 10^4\times (4.96)^4}$$

or $n = 17.5$ **Ans.**

The length of the spring wire is given by

$$l = 2\pi Rn$$

$$= 2\pi\times\frac{60}{2}\times 17.5 = 3.29 \text{ m}$$ **Ans.**

The strain energy stored in the spring can be found using equation (8.24).

$$U = \frac{32W^2R^3n}{Gd^4}$$

$$= \frac{32\times 80^2 \times \left(\frac{60}{2}\right)^3 \times 17.5}{8\times 10^4 \times (4.96)^4} = 1.99 \text{ joule} \qquad \textbf{Ans.}$$

Example 8.10

A close coiled helical spring made of steel wire of diameter 6 mm has 15 coils. The spring has mean coil diameter of 100 mm and is subjected to an axial load of W producing a maximum shear stress of 100 N/mm^2. Find the load W, the deflection produced as a result of the load applied and the work done to produce the deflection. Take G = 80 kN/mm^2.

Solution: Given,

Spring wire diameter, d = 6 mm

Number of coils, n = 15

Mean coil diameter, D = 100 mm

Shear stress induced in the spring, τ = 100 N/mm^2

Modulus of rigidity of the spring material, G = 80 kN/mm^2 = 8 × 10^4 N/mm^2

Load W can be calculated by using equation (8.22).

$$\tau = \frac{16WR}{\pi d^3}$$

$$100 = \frac{16\times W\times\left(\frac{100}{2}\right)}{\pi\times 6^3}$$

or W = 84.8 N **Ans.**

The deflection produced in the spring can be found by using equation (8.27).

$$\delta = \frac{64WR^3n}{Gd^4}$$

$$= \frac{64\times 84.8\times\left(\frac{100}{2}\right)^3\times 15}{8\times 10^4\times 6^4} = 98.1 \text{ mm} \qquad \textbf{Ans.}$$

The workdone on the spring is given as

$$\frac{1}{2}W\delta = \frac{1}{2}\times 84.8\times 98.1 = 4.16 \text{ Joule} \qquad \textbf{Ans.}$$

Example 8.11

A close coiled helical spring made of steel wire of diameter 7 mm has an axial stiffness of 6 N per mm and an angular stiffness of 100 N·mm per degree angle of twist. Find the mean radius of the coil, the number of turns in the spring and its length. Take $E = 200$ kN/mm^2 and $G = 80$ kN/mm^2.

Solution: Given,

Spring wire diameter, $d = 7$ mm

Axial stiffness of the spring, $K = 6$ N/mm

Angular stiffness of the spring, $K_1 = 100$ N·mm/degree

The axial stiffness of the spring is given by using equation (8.28).

$$K = \frac{W}{\delta} = \frac{Gd^4}{64R^3n}$$

$$6 = \frac{8\times10^4\times7^4}{64\times R^3n}$$

or $$R^3n = 500208.33 \qquad \text{...(1)}$$

The angular stiffness is given by equation (8.35).

$$K = \frac{M}{\phi} = \frac{Ed^4}{128Rn}\times\frac{\pi}{180}$$

$$100 = \frac{2\times10^5\times7^4}{128\times Rn}\times\frac{\pi}{180}$$

or $$Rn = 654.77 \qquad \text{...(2)}$$

Using equations (1) and (2), we get

or $$R = 27.64 \text{ mm} \qquad \textbf{Ans.}$$

From equation (2)

$$n = \frac{654.77}{R} = 23.7 \qquad \textbf{Ans.}$$

The length of the spring wire is given as

$$l = 2\pi Rn = 2\pi\times27.64\times23.7 = 4.11 \text{ m} \qquad \textbf{Ans.}$$

Example 8.12

A close coiled helical spring made of 15 mm steel wire is subjected to an axial load of 200 N. The spring has 15 coils and its mean coil radius is 100 mm. Find the deflection, the maximum shear stress and the strain energy stored in the spring per unit voulme.

Now the axial load is replaced by an axial torque of 11 N·m. Find the axial twist, the maximum bending stress and the strain energy stored per unit volume of the spring.

Take $G = 80$ kN/mm^2 and $E = 200$ kN/mm^2.

Solution: Given,

Diameter of the spring wire, $d = 15$ mm
Axial load on the spring, $W = 200$ N
Number of coils in the spring, $n = 15$
Mean radius of the coil, $R = 100$ mm
Axial torque on the spring, $M = 11$ N·m
Modulus of rigidity, $G = 80 \text{ kN/mm}^2 = 8 \times 10^4 \text{ N/mm}^2$
Modulus of elasticity, $E = 200 \text{ kN/mm}^2 = 2 \times 10^5 \text{ N/mm}^2$

The deflection produced in the spring is given by equation (8.27).

$$\delta = \frac{64WR^3n}{Gd^4}$$

$$= \frac{64 \times 200 \times 100^3 \times 15}{8 \times 10^4 \times 15^4} = 47.4 \text{ mm}$$ **Ans.**

Use equation (8.22) to find the maximum shear stress induced in the spring.

$$\tau = \frac{16WR^3}{\pi d^3} = \frac{16 \times 200 \times 100}{\pi \times 15^3} = 30.18 \text{ N/mm}^2$$ **Ans.**

Use equation (8.25) for the strain energy stored in the spring.

$$U = \frac{\tau^2}{4G} \times \text{Volume of the spring } (V)$$

or $$\frac{U}{V} = \frac{\tau^2}{4G} = \frac{(30.18)^2}{4 \times 8 \times 10^4} = 2.84 \times 10^{-3} \text{ N·mm/mm}^3$$ **Ans.**

Use equation (8.34) for the axial twist in the spring.

$$\phi = \frac{128MRn}{Ed^4}$$

$$= \frac{128 \times 11 \times 10^3 \times 100 \times 15}{2 \times 10^5 \times 15^4} = 0.208 \text{ radian} = 11.95^\circ$$ **Ans.**

For the bending stress in the spring, use equation (8.36).

$$\sigma_b = \frac{32M}{\pi d^3} = \frac{32 \times 11 \times 10^3}{\pi \times 15^3} = 33.19 \text{ N/mm}^2$$ **Ans.**

Use equation (8.38) for the strain energy stored in the spring.

$$U = \frac{\sigma_b^2}{8E} \times \text{Volume of the spring } (V)$$

or $$\frac{U}{V} = \frac{(33.19)^2}{8 \times 2 \times 10^5} = 6.88 \times 10^{-4} \text{ N·mm/mm}^3$$ **Ans.**

Example 8.13

A close coiled helical spring made of steel wire is subjected to an axial load of 50 N. The maximum shear stress induced in the spring is limited to 100 N/mm^2. The stiffness of the spring is 0.8 N/mm of compression and its solid length is 50 mm. Determine the following parameters:

(*a*) the diameter of the spring wire

(*b*) the mean radius of the coil and

(*c*) the number of coils in the spring

Take G = 80 kN/mm^2.

Solution: Given,

Axial load on the spring, $W = 50$ N

Maximum shear stress in the spring, $\tau = 100$ N/mm^2

Stiffness of the spring, $K = 0.8$ N/mm

Modulus of rigidity of spring material, $G = 80$ kN/mm^2 = 8×10^4 N/mm^2

Let d = Diameter of the spring wire

n = Number of coils in the spring

Given, solid length of the spring, $nd = 50$ mm

(*a*) Use equation (8.28) for the stiffness of the spring.

$$K = \frac{Gd^4}{64R^3n}$$

or
$$0.8 = \frac{8\times10^4\times d^4}{64R^3n} \qquad \text{... (1)}$$

Use equation (8.22) for the shear stress in the spring.

$$\tau = \frac{16WR}{nd^3}$$

$$100 = \frac{16\times50\times R}{nd^3}$$

or
$$R = 0.392d^3 \qquad \text{... (2)}$$

Since
$$nd = 50$$

or
$$n = \frac{50}{d}$$

Using n and R in equation (1), we have

$$0.8 = \frac{8\times10^4\times d^4}{64\times(0.392d^3)^3\times\left(\frac{50}{d}\right)}$$

Solving for d, we get

$$d = 4.76 \text{ mm}$$ **Ans.**

(*b*) Using equation (2), we have

$$R = 0.392d^3$$
$$= 0.392 \times (4.76)^3$$
$$= 42.4 \text{ mm} \qquad \textbf{Ans.}$$

(*c*) The number of coils in the spring is given as

$$n = \frac{50}{d} = \frac{50}{4.76} = 10.5 \qquad \textbf{Ans.}$$

Example 8.14

A close coiled helical spring made of steel wire of 4 mm diameter has mean coil radius of 30 mm. The number of turns in the spring is 5 and its pitch, when no load is acting on it, is 20 mm. Find the axial load to be applied gradually on the spring so that the pitch reduces to minimum *i.e*, coils touch each other. An impact load of 10 N is allowed to fall on the spring through a certain height so that again the pitch is minimum. Find the height. Take G = 80 kN/mm^2.

Solution: Given,

Diameter of the spring wire, d = 4 mm

Mean coil radius of the spring, R = 30 mm

Number of turns in the spring, n = 5

Pitch while unloaded = 20 mm

Impact load on the spring, = 10 N

Modulus of rigidity of spring material, G = 80 kN/mm^2 = 8 × 10^4 N/mm^2

The deflection in the spring (during no load condition) for one turn (Fig. 8.8) is given as

$$\delta = \text{Pitch} - \text{Spring wire diameter}$$
$$= 20 - 4 = 16 \text{ mm}$$

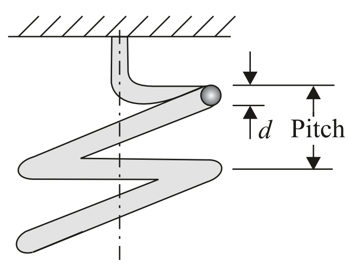


Fig. 8.8

Using equation (8.27), the deflection in the spring is given as

$$\delta = \frac{64WR^3n}{Gd^4}$$

$$16 = \frac{64W \times 30^3 \times 5}{8 \times 10^4 \times 4^4}$$

$$W = 37.9 \text{ N}$$ **Ans.**

The total deflection in the spring is given as

$$\delta_t = \text{Deflection in one turn} \times \text{number of turns}$$
$$= 16 \times 5 = 80 \text{ mm}$$

The strain energy stored in the spring is given as

$$U = \frac{1}{2}W\delta_t = \frac{1}{2} \times 37.9 \times 80$$
$$= 1516 \text{ N·mm} = 1.51 \text{ joule} \quad \text{...(1)}$$

Loss in potential energy, when load of 10 N is released from a height h, is

$$= 10\ (h + 80) \quad \text{...(2)}$$

Equating equations (1) and (2), we have

$$10\ (h + 80) = 1516$$

or $$h = 71.6 \text{ mm}$$

Hence, the height of the load is 71.6 mm. **Ans.**

Example 8.15

A close coiled helical spring of free length 200 mm and mean coil radius 40 mm is subjected to a maximum stress of 100 N/mm^2. Find the diameter of the spring wire and the number of turns in the spring, assuming that the spring can store a maximum strain energy of 30 joule, when the pitch is reduced to a minimum. Take G = 80 kN/mm^2.

Solution: Given,

Mean coil radius of the spring, R = 40 mm

Maximum shear stress induced in the spring, τ = 100 N/mm^2

Free length of the spring, = 200 mm

Strain energy stored in the spring, U = 30 joule

Modulus of rigidity of spring material, G = 80 kN/mm^2 = 8 × 10^4 N/mm^2

Free length of the spring is its total length in uncompressed state. The solid length is its total length in compressed state. They are related to each other as

$$\text{Solid length + Deflection} = \text{Free length}$$
$$nd + \delta = \text{Free length}$$

or $$nd = 200 - \delta \quad \text{... (1)}$$

The shear stress is given by equation (8.22).

$$\tau = \frac{16WR}{\pi d^3}$$

$$100 = \frac{16W \times 40}{\pi d^3}$$

or $$W = 0.49d^3 \qquad \text{... (2)}$$

The strain energy stored in the spring is given by equation (8.25).

$$U = \frac{\tau^2}{4G} \times \text{Volume of the spring } (V)$$

$$30 \times 10^3 = \frac{(100)^2}{4 \times 8 \times 10^4} \times V$$

or $$V = 96 \times 10^4 \text{ mm}^3 \qquad \text{... (3)}$$

But volume of the spring is given as

$$V = \text{Length of the spring} \times \text{Cross-sectional area of the spring wire}$$

$$= 2\pi Rn \times \frac{\pi}{4} d^2$$

$$96 \times 10^4 = 2\pi \times 40 \times n \times \frac{\pi}{4} d^2$$

$$nd^2 = 4863.4$$

or $$n = \frac{4863.4}{d^2} \qquad \text{... (4)}$$

The strain energy is also given as

$$U = \frac{1}{2} W\delta$$

$$30 \times 10^3 = \frac{1}{2} \times 0.49d^3 \times \delta \qquad \text{(on substituting } W \text{ from equation (2))}$$

$$\delta = \frac{122448.98}{d^3}$$

On substituting n and δ in equation (1), we have

$$\frac{4863.4}{d^2} \times d = 200 - \frac{122448.98}{d^3}$$

Solving for d, we get $d = 25.3$ mm **Ans.**

From equation (4)

$$n = \frac{4863.4}{(25.3)^2} = 7.6 \qquad \textbf{Ans.}$$

Example 8.16

A body of mass 25 kg moving with a velocity of 3 m/sec is to be stopped by using a close coiled helical spring of 6 mm wire diameter and having a mean coil radius of 30 mm. The maximum shear stress induced in the spring is limited to 500 N/mm^2. Find the number of turns in the spring and the deflection produced. Take G = 80 kN/mm^2.

Solution: Given,

Diameter of the spring wire, $d = 6$ mm

Mean radius of the coil, $R = 30$ mm

Maximum shear stress induced in the spring,

$$\tau = 500 \text{ N/mm}^2$$

The strain energy stored in the spring is given as

$$U = \frac{\tau^2}{4G} \times \text{Volume of the spring } (V)$$

The kinetic energy of the mass is given as

$$K.E. = \frac{1}{2} mV^2 = \frac{1}{2} \times 25 \times 3^2 = 112.5 \text{ jolues}$$

Equating the two energies, we have

$$112.5 \times 10^3 = \frac{(500)^2}{4 \times 8 \times 10^4} \times V$$

or $$V = 1.44 \times 10^5 \text{ mm}^3$$

Also $$V = 2\pi R n \times \frac{\pi}{4} d^2$$

$$1.44 \times 10^5 = 2\pi \times 30 \times n \times \frac{\pi}{4} \times 6^2$$

or $$n = 27$$ **Ans.**

To determine the deflection produced in the spring, an equivalent load is calculated which when applied gradually on the spring may cause the given shear stress.

Using equation (8.22), we have

$$\tau = \frac{16WR}{\pi d^3}$$

$$500 = \frac{16W \times 30}{\pi \times 6^3}$$

or $$W = 706.8 \text{ N}$$

The strain energy stored in the spring is equal to kinetic energy of the moving mass, given as

$$U = \frac{1}{2} \times W \times \delta \times = 112.5 \times 10^3$$

or $$\delta = \frac{2 \times 112.5 \times 10^3}{706.8} = 318.3 \text{ mm}$$ **Ans.**

Example 8.17

An open coiled helical spring of 8 mm wire diameter has 10 turns and mean coil radius of 60 mm. It is subjected to an axial load of 150 N. If the pitch of the spring is 70 mm, find the following:

(*a*) the maximum direct and shear stress induced in the spring wire

(*b*) the angle of twist and

(*c*) the deflection of the free end

Take $E = 200$ kN/mm^2 and $G = 80$ kN/mm^2.

Solution: Given,

Diameter of the spring wire,	$d = 8$ mm
Number of turns in the spring,	$n = 10$
Mean coil radius of the spring,	$R = 60$ mm
Axial load on the spring,	$W = 150$ N
Pitch of the spring,	$p = 70$ mm

The angle of helix α is found using Fig. 8.9.

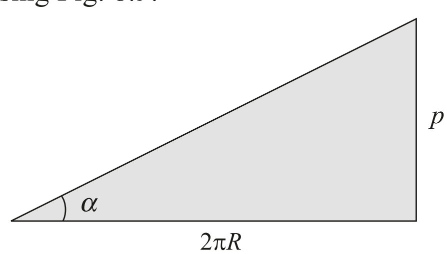


Fig. 8.9

$$\tan\alpha = \frac{p}{2\pi R}$$

$$= \frac{70}{2\pi \times 60} = 0.185$$

or $$\alpha = 10.51^{\circ}$$

Twisting moment, $T = WR\cos\alpha$

Bending moment, $M = WR\sin\alpha$

The equivalent twisting moment is given by equation (8.40).

$$T_e = WR$$
$$= 150 \times 60 = 9000 \text{ N·mm}$$

The equivalent bending moment is given by equation (8.41).

$$M_e = \frac{WR}{2}[1+\sin\alpha]$$
$$= \frac{150\times 60}{2}[1+\sin 10.51^\circ] = 5320.8 \text{ N·mm}$$

(*a*) The maximum direct stress in the spring wire is given by equation (8.42).

$$\sigma_b = \frac{32 M_e}{\pi d^3}$$
$$= \frac{32\times 5320.8}{\pi\times 8^3}$$
$$= 105.8 \text{ N/mm}^2$$ **Ans.**

The maximum shear stress in the spring wire is given by equation (8.43).

$$\tau = \frac{16 T_e}{\pi d^3} = \frac{16\times 9000}{\pi\times 8^3} = 89.5 \text{ N/mm}^2$$ **Ans.**

(*b*) The angle of twist of spring wire is given by equation (8.46).

$$\phi = \frac{64 W R^2 n \sin\alpha}{d^4}\left[\frac{1}{G}-\frac{2}{E}\right]$$
$$= \frac{64\times 150\times 60^2\times 10\times \sin 10.51^\circ}{8^4}\left[\frac{1}{8\times 10^4}-\frac{2}{2\times 10^5}\right]$$
$$= 0.038 \text{ radian} = 2.20^\circ$$ **Ans.**

(*c*) The deflection of the spring is given by equation (8.52).

$$\delta = \frac{64 W R^3 n \sec\alpha}{d^4}\left[\frac{\cos^2\alpha}{G}+\frac{2\sin^2\alpha}{E}\right]$$
$$= \frac{64\times 150\times 60^3\times 10\times \sec 10.51^\circ}{8^4}\left[\frac{\cos^2 10.51^\circ}{8\times 10^4}+\frac{2\sin^2 10.51^\circ}{2\times 10^5}\right]$$
$$= 63.9 \text{ mm}$$ **Ans.**

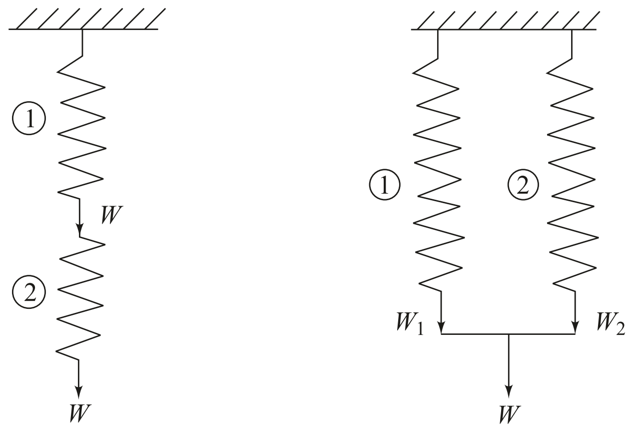
8.9 COMBINATION OF SPRINGS

Springs may be combined together in two ways.

- Series combination
- Parallel combination

8.9.1 Series Combination

The combination of springs in series is shown in Fig. 8.10 (*a*). Each spring carries the same load.



(*a*) Series combination (*b*) Parallel combination

Fig. 8.10

Let W = Load applied on the spring

K_1 and K_2 = Stiffnesses of springs (1) and (2) respectively

K = Stiffness of the combined spring

δ_1 and δ_2 = Extensions produced in two springs respectively

The total deflection in the combined spring is the sum of the deflections produced individually in the two springs.

$$\delta = \delta_1 + \delta_2 \qquad \text{... (8.63)}$$

The deflection produced in spring (1) is

$$\delta_1 = \frac{W}{K_1} \qquad \text{... (8.64)}$$

The deflection produced in spring (2) is

$$\delta_2 = \frac{W}{K_2} \qquad \text{... (8.65)}$$

Adding the two equations (8.64) and (8.65), we have

$$\delta_1 + \delta_2 = W\left[\frac{1}{K_1} + \frac{1}{K_2}\right]$$

or

$$\delta = W\left[\frac{1}{K_1} + \frac{1}{K_2}\right] \qquad \text{... (8.66)}$$

If the two springs are replaced by a single spring of stiffness K producing a deflection δ, then

$$\delta = \frac{W}{K} \qquad \text{... (8.67)}$$

Comparing equations (8.66) and (8.67), we get

$$\frac{W}{K} = W\left[\frac{1}{K_1} + \frac{1}{K_2}\right]$$

or
$$\frac{1}{K} = \frac{1}{K_1} + \frac{1}{K_2} \qquad \text{... (8.68)}$$

8.9.2 Parallel Combination

Two springs connected in parallel are shown in Fig. 8.10 (*b*). Here, each spring carries different load and the load applied on the combined spring is the sum of the loads shared by the two springs. Deflection produced in each spring is the same and so is the deflection produced in the combined spring.

Let W_1 = Load shared by the spring (1)

W_2 = Load shared by the spring (2)

Then
$$W = W_1 + W_2 \qquad \text{... (8.69)}$$

$$\delta K = \delta K_1 + \delta K_2$$

or
$$K = K_1 + K_2 \qquad \text{... (8.70)}$$

Example 8.18

Two close coiled helical springs are connected in series. Both springs have 15 turns and the same mean coil diameter of 50 mm. Find the diameter of spring wire of one of the two springs if the diameter of other spring wire is 5 mm. Also, find the load to be applied on the equivalent spring and the its corresponding deflection. The stiffness of the equivalent spring is 0.5 N/mm and the maximum shear stress is limited to 150 N/mm^2. Given G = 80 kN/mm^2.

Solution: Given,

Number of turns in both springs, $n_1 = n_2 = 15$

Mean coil radius of both springs, $R_1 = R_2 = \frac{50}{2} = 25$ mm

Diameter of wire in the first spring, $d_1 = 5$ mm

Stiffness of equivalent spring, $K = 0.5$ N/mm

Maximum shear stress induced in the equivalent spring, $\tau = 150$ N/mm^2

Let Diameter of wire in the second spring $= d_2$

Load on equivalent spring $= W$

When connected in series, both springs will carry the same load W, which will be acting on the equivalent spring, but will have different deflections.

The deflection in the first spring δ_1 is given as

$$\delta_1 = \frac{64WR_1^3 n_1}{Gd_1^4} \qquad \text{(using equation (8.27))}$$

$$= \frac{64 \times W \times 25^3 \times 15}{8 \times 10^4 \times 5^4} = 0.3\ W$$

The deflection in the second spring δ_2 is given as

$$\delta_2 = \frac{64WR_2^3 n_2}{Gd_2^4}$$

$$= \frac{64 \times W \times (25^3) \times 15}{8 \times 10^4 \times d_2^4} = \frac{187.5W}{d_2^4}$$

Both deflections are added to get the total deflection in the equivalent spring.

$$\delta = \delta_1 + \delta_2$$

$$= 0.3W + \frac{187.5W}{d_2^4} \quad \text{... (1)}$$

But $$\delta = \frac{W}{K} = \frac{W}{0.5} \quad \text{... (2)}$$

Comparing equations (1) and (2), we have

$$\frac{W}{0.5} = W\left[0.3 + \frac{187.5}{d_2^4}\right]$$

$$\left(\frac{1}{0.5} - 0.3\right) = \frac{187.5}{d_2^4}$$

or $$d_2 = 3.24 \text{ mm}$$ **Ans.**

Using equation (8.22), we have

$$\tau = \frac{16WR}{\pi d_2^3}$$

$$150 = \frac{16W \times 25}{\pi \times (3.24)^3}$$

or $$W = 40 \text{ N}$$ **Ans.**

While using equation (8.22), spring with smaller wire diameter is used to avoid its failure. The mean coil diameter of the equivalent spring is also the same as for the two springs.

Using equation (2), we get deflection in the equivalent spring as

$$\delta = \frac{W}{0.5} = \frac{40}{0.5} = 80 \text{ mm}$$ **Ans.**

Example 8.19

Two close coiled helical springs are connected in parallel to take a load of 1500 N. Both springs have 16 coils and are made of equal wire diameter of 15 mm. Their mean coil diameters are 60 mm and 80 mm respectively. Find the loads shared by the two springs and the maximum shear stresses induced in them. Also, find the deflection in the equivalent spring. Take $G = 80$ kN/mm^2 for both springs.

Solution: Given,

Load on the equivalent spring, $W = 1500$ N

Number of coils in two springs, $n_1 = n_2 = 16$

Wire diameter of the two springs, $d_1 = d_2 = 15$ mm

Mean coil radius of the first spring, $R_1 = \frac{60}{2} = 30$ mm

Mean coil radius of the second spring, $R_2 = \frac{80}{2} = 40$ mm

When connected in parallel, both springs produce the same deflection but share different loads. The total load acting on the equivalent spring is the sum total of the loads shared by both springs. Let W_1 and W_2 are the respective loads on the two springs. Then,

$$W = W_1 + W_2$$

or

$$1500 = W_1 + W_2 \quad \text{... (1)}$$

Deflections in the two springs are given as

$$\delta = \delta_1 = \delta_2$$

$$= \frac{64W_1R_1^3n_1}{Gd_1^4} = \frac{64W_2R_2^3n_2}{Gd_2^4} \quad \text{... (2)}$$

or

$$W_1 \times 30^3 = W_2 \times 40^3 \quad (n_1 = n_2 = 16 \text{ and } d_1 = d_2 = 15 \text{ mm})$$

$$W_1 = \frac{64}{27}W_2$$

Substituting W_1 in equation (1), we have

$$1500 = \frac{64}{27}W_2 + W_2$$

or

$$W_2 = 445 \text{ N}$$ **Ans.**

From equation (1) we find W_1.

$$1500 = W_1 + 445$$

or

$$W_1 = 1055 \text{ N}$$ **Ans.**

The shear stress in the first spring is given by using equation (8.22).

$$\tau_1 = \frac{16W_1R_1}{\pi d_1^3} = \frac{16 \times 1055 \times 40}{\pi \times 15^3} = 47.76 \text{ N/mm}^2$$ **Ans.**

The shear stress in the second spring is given as

$$\tau_2 = \frac{16W_2R_2}{\pi d_2^3} = \frac{16\times 455\times 40}{\pi\times 15^3} = 26.86 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

From equation (2), we have

$$\delta = \frac{64W_1R_1^3n_1}{Gd_1^4} = \frac{64\times 1055\times 30^3\times 16}{8\times 10^4\times 15^4} = 7.2 \text{ mm}$$

Hence, the deflection in the equivalent spring is 7.2 mm. **Ans.**

Example 8.20

A composite spring made of two close coiled helical springs is subjected to an axial load of 100 N. One of the two springs is placed inside the other and both springs are concentric. The inner spring has 10 coils and is 15 mm shorter than the outer spring of 4 mm wire diameter with number of coils 12 and mean coil diameter 50 mm. The radial clearance between the two springs is 1 mm. Find the stiffness and the wire diameter of the inner spring, if the axial load applied on the composite spring produces a deflection of 30 mm in the outer spring. Take G = 80 kN/mm^2 for both springs.

Solution: Given,

Axial load on the composite spring, $W = 100$ N
Number of coils in the inner spring, $n_1 = 10$
Number of coils in the outer spring, $n_2 = 12$
Mean coil diameter of the outer spring, $D_2 = 50$ mm
Wire diameter of the outer spring, $d_2 = 4$ mm
Deflection produced in the outer spring, $\delta_2 = 30$ mm

The axial load applied on the composite spring is shared by the two springs. Let W_1 and W_2 be the load shared by the inner and outer spring respectively, then

$$W = W_1 + W_2$$

or $$W_1 + W_2 = 100 \text{ N (Given)} \qquad \dots (1)$$

Use equation (8.27) to get the load (W_2) shared by the outer spring in order to deflect it by 30 mm.

$$\delta_2 = \frac{64W_2R_2^3n_2}{Gd_2^4}$$

or $$W_2 = \frac{\delta_2\times G\times d_2^3}{64R_2^3n_2} = 51.2 \text{ N}$$

and $$W_1 = (100 - 51.2) \text{ N} = 48.8 \text{ N} \qquad \text{(using equation (1))}$$

The deflection produced by the load W_1 in the inner spring is given as

$$\delta_1 = (30 - 15) = 15 \text{ mm}$$

The stiffness of the inner spring is given as

$$K_1 = \frac{W_1}{\delta_1} = \frac{48.8}{15} = 3.25 \text{ N/mm} \qquad \textbf{Ans.}$$

The mean coil diameter of the inner spring is given as

$$D_1 = D_2 - d_2 - 2 \times 1 - d_1 = 50 - 4 - 2 \times 1 - d_1 = 44 - d_1$$

Again using equation (8.27) for the inner spring, we have

$$\delta_1 = \frac{64 W_1 R_1^3 n_1}{G d_1^4}$$

$$\frac{W_1}{\delta_1} = K_1 = \frac{G d_1^4}{64 R_1^3 n_1} = \frac{8 \times 10^4 \, d_1^4}{64 \times \left(\dfrac{44 - d_1}{2}\right)^3 \times 10}$$

$$= \frac{8 \times 10^4 \times 8 \times d_1^4}{640(44 - d_1)^3}$$

or $$3.25 = \frac{10^3 \, d_1^4}{(44 - d_1)^3} \qquad \text{(using } K_1\text{)}$$

Solving for d_1, we get

$$d_1 = 3.8 \text{ mm} \qquad \textbf{Ans.}$$

SHORT ANSWER QUESTIONS

1. What is the purpose of using a spring?
2. What is axial stiffness of a spring?
3. Why is leaf spring also called carriage spring?
4. Where is a spiral spring used?
5. Why is helical spring also called torsion spring?
6. How is close coiled helical spring different from open coiled helical spring?
7. What is the purpose of combining springs?
8. How is axial twisting different from axial loading?
9. What are the construction features of a leaf spring?
10. How is series combination of springs different from parallel combination?

MULTIPLE CHOICE QUESTIONS

1. The deformation produced in the spring is said to be

(*a*) semi-elastic (*b*) plastic (*c*) elastic (*d*) visco-elastic.

2. The stiffness of the spring is defined as a ratio of

(*a*) load and angle of twist (*b*) load and deflection
(*c*) load and strain energy (*d*) load and strain.

3. In a close-coiled helical spring, the

(*a*) plane of the coil and axis of the spring are closely attached
(*b*) angle of helix is large
(*c*) plane of the coil is normal to the axis of the spring
(*d*) deflection is small.

4. A conical helical spring is used, where

(*a*) space is a problem
(*b*) more stiffness is required
(*c*) more load is to be taken
(*d*) less deflection is required.

5. The load-deflection curve of a spring is a straight line, when spring is stressed

(*a*) upto yield point
(*b*) upto ultimate point
(*c*) upto failure point
(*d*) within the elastic limit.

6. The strain energy stored in a leaf spring is given as

(*a*) $\dfrac{3W^2l^3}{16nEbt^3}$ (*b*) $\dfrac{5Wl^3}{16nEbt^3}$

(*c*) $\dfrac{3W^2l}{16nEbt^3}$ (*d*) $\dfrac{3W^2l^3}{16nEbt^2}$.

where the symbols have their usual meanings.

7. The maximum bending stress developed in the wire of spiral spring is

(*a*) $\dfrac{12M}{bt}$ (*b*) $\dfrac{6M}{bt^2}$ (*c*) $\dfrac{12M}{bt^2}$ (*d*) $\dfrac{12M}{b^2t}$.

where the symbols have their usual meanings.

8. The deflection produced in a close coiled helical spring when subjected to an axial load is

(*a*) $\dfrac{64W^3Rn}{Gd^4}$ (*b*) $\dfrac{64WR^3n}{Gd^4}$

(*c*) $\dfrac{64WR^3n}{G^4d}$ (*d*) $\dfrac{64WR^2n^2}{G^4d}$.

where the symbols have their usual meanings.

9. Wahl's correction factor is introduced to

(*a*) increase the number of coils in the spring
(*b*) take care of extra load on the spring
(*c*) take care of curvature of spring wire
(*d*) take care of extra stiffness in the spring.

10. The spring index is defined as a ratio of

(*a*) load and deflection
(*b*) mean coil diameter and spring wire diameter
(*c*) load and angle of twist
(*d*) mean coil diameter and length of spring wire.

11. For two springs being connected in series, which of the following statements is correct?

(*a*) The deflection produced in the equivalent spring is the sum of the deflections produced in the individual spring.
(*b*) The total weight is the sum of the weights acting separately on the two springs.
(*c*) The equivalent stiffness is the sum of the individual stiffness.
(*d*) The equivalent stiffness is the product of the individual stiffness.

12. The equivalent stiffness, when two springs are connected in series, is given by the expression

(*a*) $\dfrac{1}{K} = \dfrac{K_1K_2}{K_1 + K_2}$ (*b*) $K^2 = \dfrac{K_1 + K_2}{K_1K_2}$

(*c*) $K = \dfrac{K_1K_2}{K_1 + K_2}$ (*d*) $K = K_1 + K_2$.

13. For two springs being connected in parallel, which of the following statements is correct?

(*a*) The equivalent stiffness is the sum of the individual stiffness.
(*b*) The equivalent load is the sum of the individual load.
(*c*) The equivalent deflection is the sum of the individual deflection.
(*d*) The equivalent deflection is the product of the individual deflection.

14. For parallel combination of two springs, the expression for the equivalent stiffness is

(*a*) $\dfrac{1}{K} = \dfrac{K_1 K_2}{K_1 + K_2}$ (*b*) $\dfrac{1}{K} = \dfrac{K_1 + K_2}{K_1 K_2}$

(*c*) $K = K_1 + K_2$ (*d*) $K = \sqrt{K_1 K_2}$.

ANSWERS

1. (*c*) **2.** (*b*) **3.** (*c*) **4.** (*a*) **5.** (*d*) **6.** (*a*) **7.** (*c*) **8.** (*b*) **9.** (*c*)
10. (*b*) **11.** (*a*) **12.** (*c*) **13.** (*a*) and (*b*) **14.** (*c*)

EXERCISES

1. A quarter-elliptic laminated spring of length 700 mm is loaded with a load of 3 kN producing a deflection of 100 mm. The cross-section of the spring wire is 70 mm wide by 7 mm thick. Find the following:

(*a*) the number of leaves in the spring

(*b*) the maximum stress induced in the spring and

(*c*) the height through which the load is released, giving a maximum stress of 1 kN/mm^2

Take E = 200 kN/mm^2.

(*Ans.* (*a*) 13 (*b*) 282.57 N/mm^2 (*c*) 269.3 mm).

2. A close coiled helical spring of mean coil radius equal to 6 times the wire diameter is subjected to an axial load of 120 N. It can absorb 5 joules of energy producing a deflection of 50 mm. The maximum stress in the spring is not to exceed 80 N/mm^2. Find the mean coil diameter, the wire diameter, the number of turns in the spring and the length of the spring. Take G = 80 kN/mm^2.

(*Ans.* 55 mm, 4.58 mm, 18.42, 3.18 m).

3. A close coiled helical spring of mean coil diameter 100 mm is subjected to a torque of 6.5 N·m producing angular twist of 60^o. The deflection of the spring under the action of an axial load of 250 N is 120 mm. Find Poisson's ratio of the spring wire. (*Ans.* 0.191).

4. A close coiled helical spring of 10 mm wire diameter has 15 turns with mean coil diameter of 120 mm. It is subjected to an axial load of 500 N. Find the shear stress in the spring wire (*a*) neglecting correction factor and (*b*) considering correction factor.

(*Ans.* (*a*) 152.78 N/mm^2 (*b*) 171.03 N/mm^2).

5. An open coiled helical spring of 8 mm wire diameter has mean coil radius of 120 mm and 10 turns. The angle of helix is 20^o. It is subjected to an axial torque of 3 N·m. Find the following:

(*a*) the maximum direct and shear stress induced in the spring wire and

(*b*) the angle of twist. Take G = 80 kN/mm^2 and E = 200 kN/mm^2.

(*Ans.* (*a*) 57.88 N/mm^2, 29.84 N/mm^2 (*b*) 17.65^o).

6. What is the error in the calculation of stiffness if open coiled helical spring is mistaken to be close coiled helical spring. Given, angle of helix = 30^o and E/G = 4.0. Assume all the parameters of the two springs to be same. (*Ans.* 1% (negative)).

7. A close coiled helical spring is of 80 mm mean coil diameter. The spring extends by 37.75 mm when loaded axially by a weight of 500 N. There is an angular rotation of 45° when the spring is subjected to an axial couple of magnitude 20 N·m. Determine Possion's ratio of the material of the spring. (*Ans.* 0.2016).

8. A truck weighing 25 kN and moving at 2.5 m/s has to be brought to rest by a buffer. Find how many springs each of 25 coils will be required to store energy of motion during compression of 0.2 m. The spring is made of 25 mm diameter steel rod coiled to a mean diameter of 0.2 m. Take G = 100 GPa. (*Ans*. 16.3).

9. A close coiled helical spring is made of 10 mm diameter steel wire, the coil consisting of 10 complete turns with a mean diameter of 120 mm. The spring carries an axial pull of 200 N. Determine the shear stress induced in the spring neglecting the effect of stress concentration. Determine also the deflection of the spring, its stiffness and the strain energy stored by it, if modulus of rigidity of spring material is 80 GPa.

(*Ans*. 61.11 MPa, 34.56 mm, 5787 N/m, 3.456 joules).

10. Find the maximum shear stress and the deflection produced in a helical spring of the following specifications, if it has to absorb the energy of 1 kN.m.

Mean diameter of the spring = 100 mm

Diameter of the spring steel wire = 20 mm

Number of coils = 30

Modulus of rigidity of steel = 85 GPa.

(*Ans*. 338.87 MPa, 187.87 mm).

11. A close coiled helical spring made of 6 mm wire diameter and mean coil diameter 100 mm extends by 45 mm under an axial load of 50 N. The same spring when firmly fixed at one end rotates through 90º under a torque of 5.7 N.m. Calculate the value of Poisson's ratio for the material. (*Ans*. 0.3).

12. Find the load required to produce an extension of 8 mm in an open coiled helical spring made of 6 mm wire diameter, having 10 coils of mean diameter 76 mm with a helix angle of 20º. Also, calculate the bending and shear stresses produced in the surface of the spring wire. What would be the angular twist at the free end of the spring, when subjected to an axial torque of 1.5 N.m? Take E = 210 GPa and G = 70 GPa.

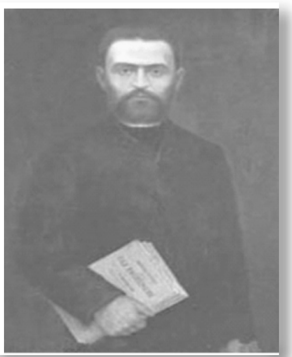
(*Ans*. 20 N, 12.3 MPa, 16.8 MPa, 17.3°).

13. A compound spring is made of two close coiled helical springs connected in series, where each spring has 12 coils at a mean diameter of 25 mm. Find the diameter of the wire in one of the springs, if the diameter of wire in the other spring is 2.5 mm and the stiffness of the compound spring is 700 N/m. Estimate the greatest load that can be carried by the composite spring and the corresponding extension for a maximum shearing stress of 180 MPa.

(*Ans*. 63.2 mm, 44.2 N).

❑❑❑

9 Strain Energy



Carlo Alberto Castigliano
(1847-1884)

Carlo Alberto Castigliano, born on 9 November 1847, was an Italian mathematician and physicist. He is well known for his Castigliano's theorem, which is used for determining the displacements in a linear-elastic system using the partial derivatives of strain energy.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- Why is strain energy so named?
- How does suddenly applied load differ from gradually applied load?
- What is shear strain energy?
- What is volumetric strain energy?
- What is Castigliano's theorem?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_9

9.1 INTRODUCTION

When an external load is applied on a body, it tends to deform the body. During the process, internal forces are induced in the body to counter the effect of external load. If the body regains its shape and size, on removed of load, it is said to be within elastic limit. This limit is defined by Hooke's law which states that within this limit, stress is directly proportional to strain. The internally induced force acting on unit area of the body is called stress. Energy is stored in the body during deformation process and this energy is called strain energy. The workdone to produce the deformation is equal to the strain energy stored in the body.

9.2 STRAIN ENERGY DUE TO DIRECT LOADS

The load acting on a body may be of the following types:

(*a*) Gradually applied load (*b*) Suddenly applied load (*c*) Impact load

9.2.1 Strain Energy due to Gradually Applied Load

Consider a bar of length l placed vertically and one end of it is attached at the ceiling (Fig. 9.1). This is the most general case.

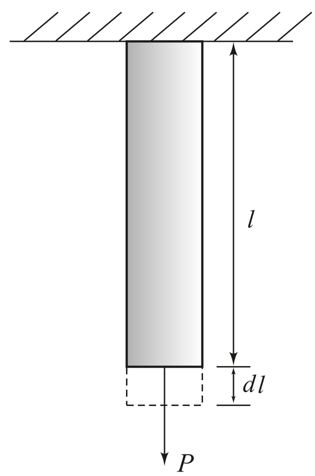


Fig. 9.1

Let P = Gradually applied load

l = Length of the bar

A = Cross-sectional area of the bar

δl = Deflection produced in the bar

σ = Axial or direct stress induced in the bar. It may be tensile or compressive, depending upon if the bar under consideration is under tension or compression load.

E = Modulus of elasticity of the bar material

The work done by the load is given as

$$W = \frac{1}{2} \cdot P \cdot \delta l \qquad \text{... (9.1)}$$

By definition $\quad \sigma = \dfrac{P}{A}$

or $\quad P = \sigma A$

From Hooke's law $\quad \delta l = \dfrac{Pl}{AE} = \dfrac{\sigma l}{E}$

On substituting δl, equation (9.1) becomes

$$W = \frac{1}{2}\frac{P^2 l}{AE} \qquad \text{(in terms of } P\text{)} \qquad \text{... (9.2)}$$

$$= \frac{1}{2}\frac{\sigma^2 Al}{E} \qquad \text{(in terms of } \sigma\text{)} \qquad \text{... (9.3)}$$

$$= \frac{1}{2}\frac{\sigma^2 V}{E} \qquad \text{(in terms of } \sigma \text{ and } V\text{)} \qquad \text{...(9.4)}$$

where $\quad V$ = Volume of the bar = Al

The strain energy stored in the bar is equal to the workdone by the load.

$$U = \frac{\sigma^2}{2E} \times V \qquad \text{...(9.5)}$$

$$= \frac{1}{2} \times \sigma \times \frac{\sigma}{E} \times V = \frac{1}{2} \times \text{Stress} \times \text{Strain} \times \text{Volume} \qquad \text{...(9.6)}$$

The strain energy stored per unit volume is known as strain energy density, given as

$$\frac{U}{V} = \text{Strain energy density} = \frac{\sigma^2}{2E} \qquad \text{...(9.7)}$$

The load versus deflection curve is shown in Fig. 9.2. δ_e stands for the elastic deflection and P_e for the elastic load. The area of the curve gives the workdone by the load to produce the deflection.

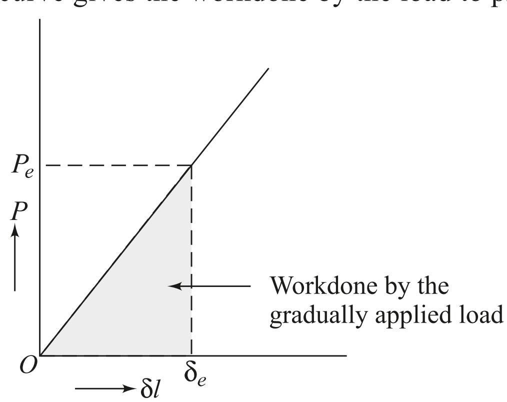


Fig. 9.2

If the bar is not of uniform cross-sectional area as shown in Fig. 9.3, then strain energy stored in the bar is not represented by equation (9.6).

The strain energy under this condition is given by the Equation (9.8) as

$$U = \frac{1}{2}\frac{P^2}{E}\int_0^l \frac{dx}{A} \quad \text{...(9.8)}$$

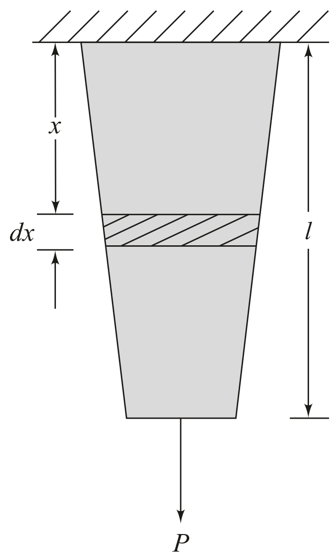


Fig. 9.3

9.2.2 Strain Energy due to Suddenly Applied Load

When the load is applied suddenly, workdone by the load is given as

$$W = P \cdot \delta l \quad \text{... (9.9)}$$

The applied load has the same value during the deformation of the bar, hence average load is P and not $\frac{P}{2}$. The workdone by the load is shown in Fig. 9.4.

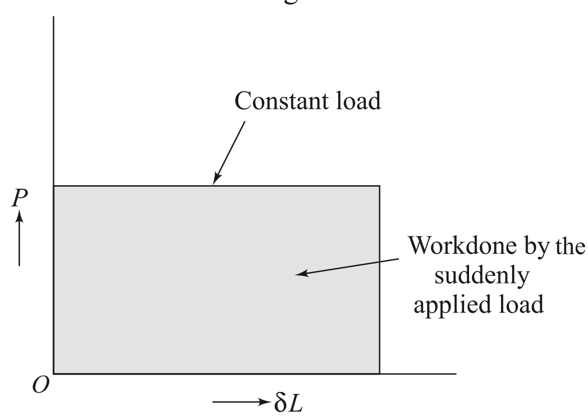


Fig. 9.4

The strain energy stored in the bar is given by equation (9.5).

$$U = \frac{\sigma^2}{2E} \times V$$

Equating equations (9.5) and (9.9), we have

$$P \cdot \delta l = \frac{\sigma^2}{2E} \times V$$

$$P\left(\frac{\sigma \cdot l}{E}\right) = \frac{\sigma^2}{2E} \times Al \qquad \left(\delta l = \frac{\sigma l}{E}\right)$$

$$P = \frac{\sigma}{2} \cdot A$$

or

$$\sigma = \frac{2P}{A} \qquad \text{... (9.10)}$$

Hence, the stress induced by the suddenly applied load is two times the stress produced by the same load but applied gradually.

9.2.3 Strain Energy due to Impact or Shock Load

Consider the situation, when a load P is allowed to fall freely on the collar attached to the lower end of a bar.

Let, h = Height through which the load falls on the collar

The workdone by the falling load is given as

$$W = P\,(h + \delta l) \qquad \text{... (9.11)}$$

It is important to note here that the load does not change and the same load acts throughout the process (Fig. 9.5).

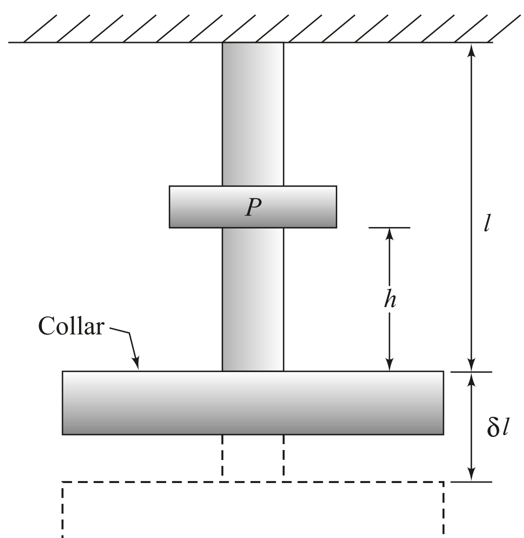


Fig. 9.5

Equating workdone to the strain energy stored in the bar, we have

$$P(h+\delta l) = \frac{\sigma^2}{2E} \times V \quad \text{... (9.12)}$$

$$P\left(h+\frac{\sigma l}{E}\right) = \frac{\sigma^2}{2E} \times Al \qquad \left(\delta l = \frac{\sigma l}{E} \text{ and } V = Al\right)$$

or
$$\left(\frac{Al}{2E}\right)\sigma^2 - \left(\frac{Pl}{E}\right)\sigma - Ph = 0 \quad \text{...(9.13)}$$

This is a quadratic equation in terms of σ. Its two values are given as

$$\sigma = \frac{P}{A}\left[1 \pm \sqrt{\left(1+\frac{2AEh}{Pl}\right)}\right] \quad \text{...(9.14)}$$

The effect of the load is to elongate the bar, hence it produces tensile stress in the bar. Therefore, possibility of negative stress does not arise. Equation (9.14) accordingly reduces to

$$\sigma = \frac{P}{A}\left[1 + \sqrt{\left(1+\frac{2AEh}{Pl}\right)}\right] \quad \text{...(9.15)}$$

when $h = 0$, then equation (9.15) becomes

$$\sigma = \frac{2P}{A} \quad \text{...(9.16)}$$

The equation is same as equation (9.10). Hence, in case of zero height, stress induced in the bar is equal to stress produced by the same load but applied suddenly.

Now suppose, if the deflection δl is very small as compared to height h, equation (9.12) reduces to

$$Ph = \frac{\sigma^2}{2E} \times Al$$

or
$$\sigma = \sqrt{\frac{2EPh}{Al}} \quad \text{...(9.17)}$$

9.3 STRAIN ENERGY DUE TO SHEAR

Consider a cube $ABCD$ of side l being subjected to a shear force P which is applied tangentially to the side CD. As a result, side CD changes to $C'D'$ (Fig. 9.6).

Let ϕ = Angle turned by the side AD as a result of P being applied

τ = Shear stress induced on the side CD

A = Area of each side

$= l^2$

G = Modulus of rigidity of the cube material

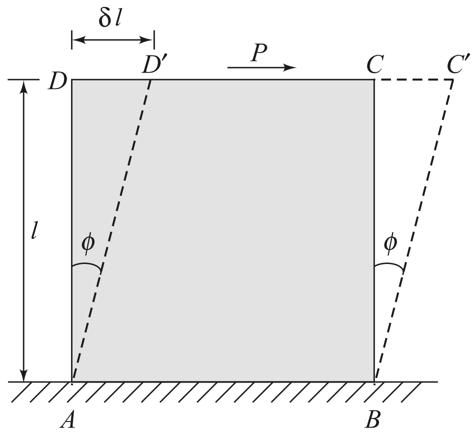


Fig. 9.6

The shear force P is given as

$$P = \tau \cdot A = \tau l^2 \qquad ...(9.18)$$

In Δ ADD'

$$\tan\phi = \frac{DD'}{AD}$$

When ϕ is very small

$$\tan\phi \simeq \phi = \frac{\delta l}{l} = \text{Shear strain}$$

or

$$\delta l = \phi l \qquad ...(9.19)$$

The workdone by the shear force is given as

$$W = \frac{1}{2}P\delta l = \frac{1}{2}\times \tau l^2 \times \phi l \qquad \text{(on substituting } P \text{ and } \delta l\text{)}$$

$$\left(\phi = \frac{\tau}{G}\right)$$

$$= \frac{1}{2}\times\tau\times\phi\times l^3 = \frac{1}{2}\times\tau\times\frac{\tau}{G}\times l^3$$

$$= \frac{\tau^2 l^3}{2G} \qquad ...(9.20)$$

Hence, the strain energy stored in the cube is given as

$$U = \frac{\tau^2}{2G}\times V \qquad ...(9.21)$$

where $V = l^3$ = Volume of the cube.

9.4 STRAIN ENERGY DUE TO PURE BENDING

Consider a small length dx of the beam at a distance x from A. Let the bending moment acting on the length be M. (Fig. 9.7).

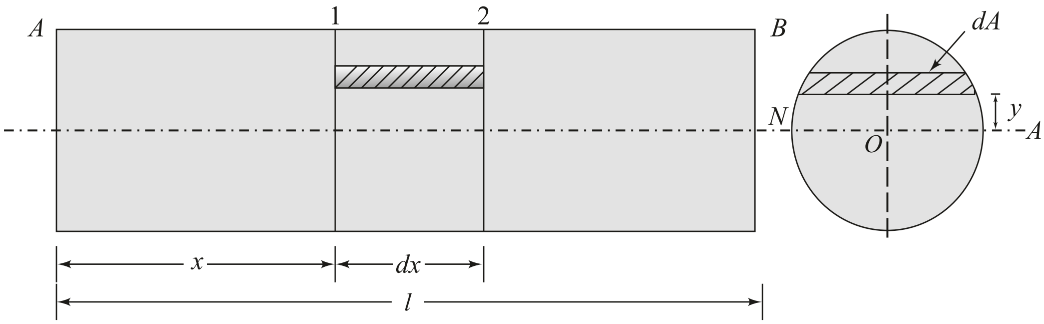


Fig. 9.7

Bending stress acting on the elementary area (dA) at a distance y from the neutral axis (NA) is given by using bending equation as

$$\sigma = \frac{M}{I} \cdot y \qquad \text{... (9.22)}$$

where I = Moment of inertia of the cross-section about the neutral axis (NA)

The strain energy stored in the elementary area is

$$= \frac{\sigma^2}{2E} \times \text{Volume of the elementary area}$$

$$= \frac{\sigma^2}{2E}\, dx \cdot dA$$

$$= \frac{M^2}{I^2} \frac{y^2}{2E}\, dx \;\; dA \qquad \text{(on substituting } \sigma \text{ from equation (9.22))}$$

The strain energy stored in the beam between 1 and 2 is given as

$$dU = \frac{M^2}{2EI^2} dx \int y^2 dA$$

$$= \frac{M^2}{2EI^2} dx \cdot I = \frac{M^2}{2EI} dx \qquad \left(I = \int y^2 dA\right)$$

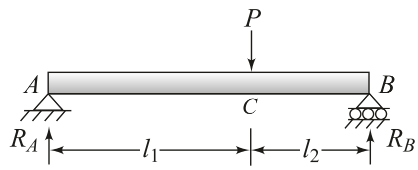
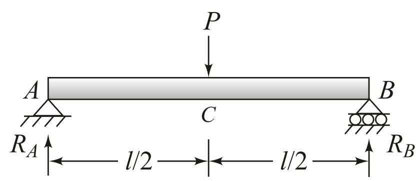
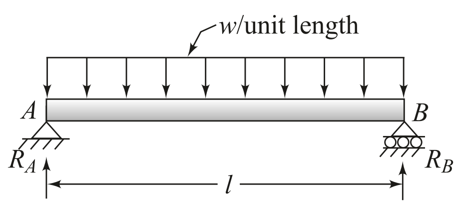
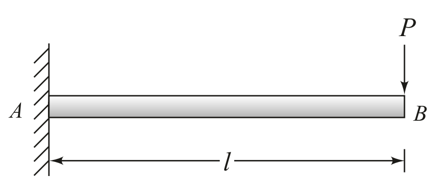
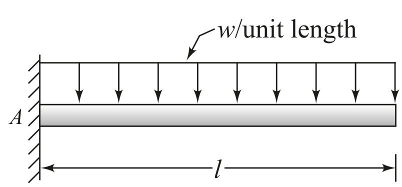
The strain energy stored in the entire beam is obtained as

$$U = \int_0^l dU$$

$$= \int_0^l \frac{M^2}{2EI} dx \qquad \text{...(9.23)}$$

Strain energy stored in different beams due to bending under different loading conditions are given in Table 9.1.

Table 9.1 Strain energy due to Bending

Loaded beam	*Loading figure*	*Strain energy stored*
1. A simple beam with a point load not at its centre		$\frac{P^2 l_1^2 l_2^2}{6EIL}$ $(l_1 > l_2)$
2. A simple beam with a point load at its centre		$\frac{P^2 l^3}{96EI}$
3. A simple beam with *udl* over the entire span		$\frac{P^2 l^3}{240EI}$ where, $P = wl$ = Total load on the beam
4. A cantilever beam with a point load at free end		$\frac{P^2 l^3}{6EI}$
5. A cantilever beam with *udl* over the entire span		$\frac{P^2 l^3}{40EI}$ where, $P = wl$ = Total load on the beam

9.5 STRAIN ENERGY DUE TO PRINCIPAL STRESSES

Consider a cube of side l being subjected to three mutually perpendicular principal stresses σ_1, σ_2 and σ_3 along x, y and z axes respectively. (Fig. 9.8).

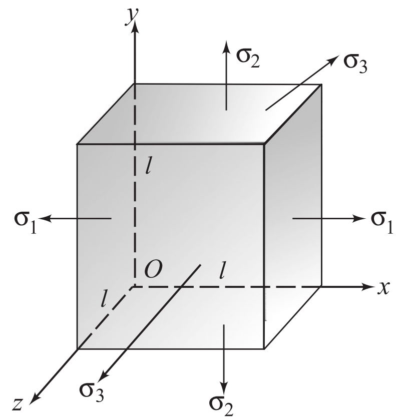


Fig. 9.8

If $\in_1$, $\in_2$ and $\in_3$ are the strains produced in x, y and z directions respectively due to σ_1, σ_2 and σ_3, then

$$\left.\begin{aligned} \in_1 &= \frac{\sigma_1}{E} - v\frac{\sigma_2}{E} - v\frac{\sigma_3}{E} \\ \in_2 &= \frac{\sigma_2}{E} - v\frac{\sigma_3}{E} - v\frac{\sigma_1}{E} \\ \in_3 &= \frac{\sigma_3}{E} - v\frac{\sigma_1}{E} - v\frac{\sigma_2}{E} \end{aligned}\right\} \qquad ...(9.24)$$

where v = Poisson's ratio

E = Modulus of elasticity

The strain energy stored in the cube is given as

$$U = \frac{1}{2} \times \text{Stress} \times \text{Strain} \times \text{Volume}$$

$$= \frac{1}{2}\sigma_1 \in_1 V + \frac{1}{2}\sigma_2 \in_2 V + \frac{1}{2}\sigma_3 \in_3 V$$

$$= \frac{V}{2}(\sigma_1 \in_1 + \sigma_2 \in_2 + \sigma_3 \in_3) \qquad ...(9.25)$$

where V = Volume of the cube $= l^3$

Substituting $\in_1$, $\in_2$ and $\in_3$ in equation (9.25), we have

$$U = \frac{V}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \qquad ...(9.26)$$

Strain energy stored per unit volume of the cube is given as

$$\frac{U}{V} = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \quad ...(9.27)$$

Equation (9.27) does not contain any shear stress term because principal planes do not have shear stresses.

For a uniaxial stress system

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

and $\sigma_1 = \sigma$

Equation (9.26) reduces to

$$U = \frac{\sigma^2}{2E} \times V$$

For a biaxial stress system

$$\sigma_3 = 0$$

Equation (9.26) reduces to

$$U = \frac{V}{2E}\left[\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2\right] \quad ...(9.28)$$

9.6 STRAIN ENERGY DUE TO VOLUMETRIC STRAIN

Refer Fig. 9.9.

Let $\in_V$ = Volumetric strain produced in the cube

K = Bulk modulus of elasticity

$\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ = Stress of equal intensity applied on all sides of the cube

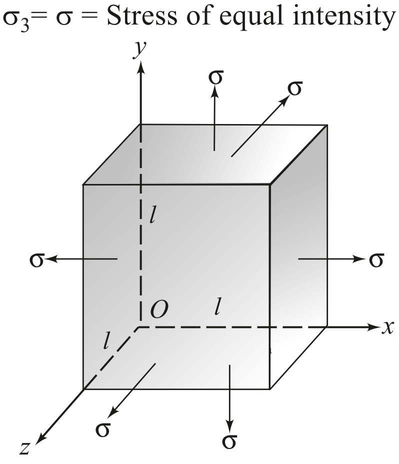


Fig. 9.9

The volumetric strain is produced as a result of changes in all sides of the cube due to σ_1, σ_2 and σ_3.

By definition

$$K = \frac{\sigma}{\in_v}$$

But $$\in_V = \in_1 + \in_2 + \in_3 = 3\in = \frac{3\sigma}{E}(1 - 2v)$$
(using equation (9.24))...(9.29)

Strain produced in all the sides of the cube is same, since equal stress intensity is acting on them. The workdone in any of the three directions is given as

$$W = \frac{1}{2} \times \text{Load} \times \text{Deflection}$$

$$= \frac{1}{2} \times \sigma \cdot l^2 \times \in l = \frac{1}{2}\sigma \times \in \times l^3 \qquad \left(\in = \frac{\delta l}{l}\right)$$

The total workdone is given as

$$W_t = 3W = \frac{3}{2} \times \sigma \times \in \times l^3$$

$$= \frac{\sigma}{2} \times 3\in \times V$$

$$= \frac{\sigma}{2} \times \frac{3\sigma}{E}(1 - 2v) \times V \qquad \text{(using equation (9.29))}$$

$$= \frac{3}{2}\frac{\sigma^2}{E}(1 - 2v) \times V \qquad ...(9.30)$$

But $$E = 3K(1 - 2v)$$

or $$\frac{E}{3(1-2v)} = K \qquad ...(9.31)$$

Substituting K in equation (9.30), we get

$$W_t = \frac{1}{2} \times \sigma^2 \times \frac{1}{K} \times V = \frac{\sigma^2}{2K} \times V$$

Hence, the strain energy stored in the cuble is given as

$$U = W_t = \frac{\sigma^2}{2K} \times \text{Volume} \qquad ...(9.32)$$

9.7 SHEAR STRAIN ENERGY DUE TO PRINCIPAL STRESSES

Let a rectangular body be subjected to three mutually perpendicular principal stresses (Fig. 9.10).

Strain energy stored in the body is given as

$$\frac{U}{V} = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$
(using equation (9.27))

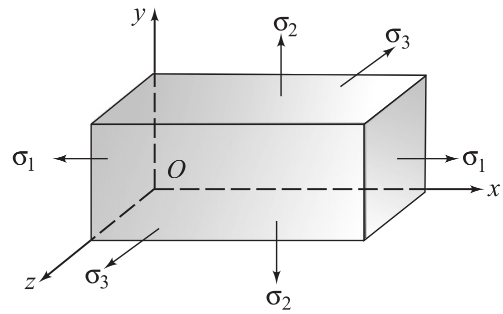


Fig. 9.10

The average stress is given as

$$\sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \qquad ...(9.33)$$

If $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, then the strain energy equation reduces to

$$\frac{U}{V} = \frac{3\sigma_{av}^2}{2E}(1 - 2v) \qquad ... (9.34)$$

Equation (9.34) is same as equation (9.32) and it gives the strain energy due to change in volume of the body.

The volumetric strain energy due to σ_{av} is given as

$$\frac{U}{V} = \frac{3\sigma_{av}^2}{2E}(1 - 2v)$$

$$= \frac{3}{2E}\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)^2 (1 - 2v) \qquad \text{(using equation (9.33))}$$

$$= \frac{1}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2 (1 - 2v)$$

$$= \left(\frac{1 - 2v}{6E}\right)\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \qquad ... (9.35)$$

On per unit volume basis, we have

Shear strain energy = Total strain energy – Volumetric strain energy ... (9.36)

Using equations (9.27) and (9.35), the shear strain energy per unit volume is

$$U_s = \left(\frac{1 + v}{6E}\right)\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \qquad ... (9.37)$$

But $\qquad E = 2G(1 + v)$

On substituting E in equation (9.37), equation (9.37) becomes

$$U_s = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \quad \text{... (9.38)}$$

This is the required expression for the shear strain energy on per unit volume basis.

If $\sigma_2 = \sigma_3 = 0$

and $\sigma_1 = \sigma$

Hence, the shear strain energy per unit volume is

$$U_s = \frac{\sigma^2}{6G} \quad \text{... (9.39)}$$

9.8 CASTIGLIANO'S THEOREM

The Castigliano's theorem is named after the Italian engineer Alberto Castigliano (1847-1884), who first proposed it. It is based on the work-energy principle, and is mainly used to determine the deflection of a structure from the strain energy of the structure under different loading conditions. There are two statements of this theorem as discussed below.

First theorem

The partial derivative of the strain energy with respect to any displacement produced within elastic limit, as a result of the application of external forces on a given member, gives forces in the direction of displacements. Mathematically,

$$P_k = \frac{\partial U}{\partial \delta_k} \quad \text{... (9.40)}$$

where U = Strain energy

δ_k = Displacement in the direction of k

P_k = Force in the direction of k

Second theorem

The partial derivative of the strain energy with respect to a force produced, within elastic limit, gives the displacement in the direction of force. Mathematically,

$$\delta_k = \frac{\partial U}{\partial P_k} \quad \text{... (9.41)}$$

Example 9.1

A solid steel rod of 5 m length and 10 mm diameter is subjected to an axial load of 5 kN. Find the stresses induced in the rod if the load is applied (*a*) gradually (*b*) suddenly and (*c*) with impact after falling through a height of 150 mm. Also, find the strain energy stored in the rod under the given conditions. Take $E = 200$ kN/mm^2.

Solution: Given,

Length of the steel rod, l = 5 m = 5 × 1000 mm

Diameter of the rod, d = 10 mm

Axial load, P = 5 kN = 5 × 10^3 N

Height of the impact load, h = 150 mm

(*a*) When load is gradually applied, the stress induced in the rod is given as

$$\sigma = \frac{P}{A}$$

where A = Cross-sectional area of the rod

$$= \frac{\pi}{4}d^2 = \frac{\pi}{4}\times 10^2 = 78.54 \text{ mm}^2$$

Hence, the stress in the rod is

$$\sigma = \frac{5\times 10^3}{78.54} = 63.66 \text{ N/mm}^2$$ **Ans.**

The strain energy stored in the rod is given by Equation (9.3).

$$U = \frac{1}{2}\frac{\sigma^2 Al}{E}$$

$$= \frac{1}{2}\times\frac{(63.66)^2\times 78.54\times 5\times 1000}{2\times 10^5}\text{ N·mm}$$

$$= 3978.88 \text{ N·mm} = 3.978 \text{ joule}$$ **Ans.**

(*b*) When load is suddenly applied, the stress induced in the rod is given as

$$\sigma = 2\frac{P}{A}$$

$$= \frac{2\times 5\times 10^3}{78.54} = 127.32 \text{ N/mm}^2$$ **Ans.**

The strain energy stored in the rod is

$$U = \frac{\sigma^2}{2E}\times V$$

$$= \frac{(127.32)^2\times 78.54\times 5\times 1000}{2\times 2\times 10^5\times 10^3} \qquad (V = Al)$$

$$= 15.91 \text{ joules}$$ **Ans.**

(*c*) During the impact loading, the stress induced is given by using equation (9.15).

$$\sigma = \frac{P}{A}\left[1+\sqrt{1+\frac{2AEh}{Pl}}\right]$$

$$= \frac{5\times 10^3}{78.54}\left[1+\sqrt{1+\frac{2\times 78.54\times 2\times 10^5\times 150}{5\times 10^3\times 5\times 10^3}}\right]$$

$$= 940 \text{ N/mm}^2$$ **Ans.**

The strain energy stored in the rod is given as

$$U = \frac{\sigma^2}{2E}\times V = \frac{(940)^2\times 78.54\times 5\times 10^3}{2\times 2\times 10^5\times 10^3}$$

$$= 867.47 \text{ joules}$$ **Ans.**

Example 9.2

A cube of side 100 mm fixed at the bottom is subjected to a shear force of 50 kN on its top face. Find strain energy stored in the cube and modulus of resilience. Take $G = 80$ kN/mm^2.

Solution: Given,

Side of the cube, $l = 100$ mm

Shear force applied, $P = 50 \text{ kN} = 5 \times 10^4$ N

Volume of the cube, $V = l^3$

$= (100 \text{ mm})^3 = 10^6 \text{ mm}^3$

The shear stress produced, $\tau = \dfrac{P}{A} = \dfrac{5\times 10^4}{100\times 100} = 5 \text{ N/mm}^2$

The strain energy stored in the cube is given by equation (9.21).

$$U = \frac{\tau^2}{2G}\times V$$

$$= \frac{5^2}{2\times 8\times 10^4}\times 10^6 \text{ N·mm} = 156.25 \text{ N·mm}$$

$$= 0.156 \text{ joule}$$ **Ans.**

The modulus of resilience is the strain energy stored per unit volume of the cube, given by

$$\frac{U}{V} = \frac{156.25}{10^6} \text{ N·mm/mm}^3$$

$$= 1.5625 \times 10^{-4} \text{ N.mm/mm}^3$$ **Ans.**

Example 9.3

A rod of diameter 10 mm and length 1.5 m hangs vertically from the ceiling of a roof. A collar is attached at its lower end on which a load of 250 N falls from a height of 200 mm. Find the strain energy absorbed and the instantaneous deflection of the rod. Take $E = 200$ kN/mm^2.

Solution: Given,

Diameter of the rod, $d = 10$ mm

Length of the rod, $l = 1.5 \text{ m} = 1.5 \times 1000 = 1500$ mm

Impact load to be applied, $P = 250$ N

Height through which the load falls, $h = 200$ mm

The cross-sectional area of the rod A given by

$$A = \frac{\pi}{4}d^2$$

$$= \frac{\pi}{4} \times 10^2 = 78.54 \text{ mm}^2$$

The stress induced in the rod is given by equation (9.15).

$$\sigma = \frac{P}{A}\left[1 + \sqrt{1 + \sqrt{\frac{2AEh}{Pl}}}\right]$$

$$= \frac{250}{78.54} \times \left[1 + \sqrt{1 + \frac{2 \times 78.54 \times 2 \times 10^5 \times 200}{250 \times 1500}}\right]$$

$$= 415.22 \text{ N/mm}^2$$ **Ans.**

The strain energy stored in the rod is given by equation (9.5).

$$U = \frac{\sigma^2}{2E} \times V = \frac{\sigma^2}{2E} \times Al$$

$$= \frac{(415.22)^2 \times 78.54 \times 1500}{2 \times 2 \times 10^5 \times 10^3}$$

$$= 50.77 \text{ joules}$$ **Ans.**

The instantaneous deflection of the rod is given as

$$\delta L = \frac{\sigma l}{E}$$

$$= \frac{415.22 \times 1500}{2 \times 10^5} = 3.11 \text{ mm}$$ **Ans.**

Example 9.4

A rectangular block is subjected to three mutually perpendicular tensile stresses of magnitude 60, 70 and 80 N/mm^2. Calculate strain energy and shear strain energy. The Poisson's ratio is 0.3. Take E = 200 kN/mm^2.

Solution: Given,

Tensile stresses, $\sigma_1 = 60$ N/mm^2

$\sigma_2 = 70$ N/mm^2

$\sigma_3 = 80$ N/mm^2

Poisson's ratio, $v = 0.3$

The strain energy stored per unit volume of the block is given by equation (9.27).

$$\frac{U}{V} = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_2^3 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

$$= \frac{1}{2\times 2\times 10^5}[60^2 + 70^2 + 80^2 - 2\times 0.3(60\times 70 + 70\times 80 + 80\times 60)]$$

$$= 0.0153 \text{ N·mm/mm}^3$$ **Ans.**

The relationship between E and G is given as

$$E = 2G\,(1 + \nu)$$

or $$G = \frac{E}{2(1+v)} = \frac{2\times 10^5}{2(1+0.3)} = 7.7 \times 10^4 \text{ N/mm}^2$$

The shear strain energy per unit volume is given by equation (9.38).

$$\frac{U}{V} = \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$= \frac{1}{12\times 7.7\times 10^4}[(60-70)^2 + (70-80)^2 + (80-60)^2]$$

$$= 6.5 \times 10^{-4} \text{ N·mm/mm}^3$$ **Ans.**

Example 9.5

Find the weight which falls through a height of 5 m on a collar attached to the lower end of a vertical rod of diameter 40 mm and length 3 m. The deflection produced in the rod is 5 mm.

Take E = 200 GPa.

Solution: Given,

Height of the weight, $h = 5$ m

Diameter of the vertical rod, $d = 40$ mm

Length of the rod, $l = 3$ m

Deflection produced in the rod, $\delta l = 5$ mm

The cross-sectional area of the rod is A, given by

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times\left(\frac{40}{1000}\right)^2$$

$$= 1.256 \times 10^{-3}\ \text{m}^2$$

The instantaneous stress produced in the rod is given as

$$\sigma = \frac{E\delta l}{l}$$

$$= \frac{200\times10^9\times5\times10^{-3}}{3} = 3.34 \times 10^8\ \text{N/m}^2$$

Using equation (9.15), we have

$$\sigma = \frac{P}{A}\left[1+\sqrt{\left(1+\frac{2AEh}{Pl}\right)}\right]$$

$$3.34 \times 10^8 = \frac{P}{1.256\times10^{-3}}\left[1+\sqrt{\left(1+\frac{2\times1.256\times10^{-3}\times200\times10^9\times5}{P\times3}\right)}\right]$$

Solving for P, we get $P = 210$ N **Ans.**

Example 9.6

A cantilever beam of length l carries a *udl* of intensity w per unit length over its entire span and a point load P at its free end. Find the maximum deflection of the beam using Castigliano's theorem.

Solution: Refer Fig. 9.11.

The bending moment due to load P at a distance x from the free end is given as

$$M_1 = -Px$$

The bending moment due to *udl* at x is given as

$$M_2 = -\frac{wx^2}{2}$$

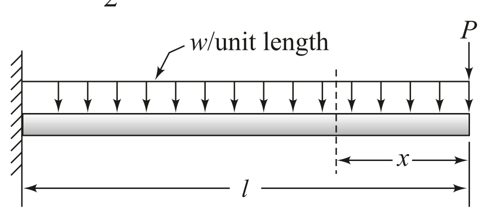


Fig. 9.11

The combined bending moment at x is given as

$$M = -\left(Px+\frac{wx^2}{2}\right)$$

The strain energy stored in the beam is given as

$$U = \int_0^l \frac{M^2 dx}{2EI}$$

The deflection in the direction of load applied using second Castigliano's theorem is obtained as

$$\delta = \frac{\partial U}{\partial P}$$

$$= \frac{\partial\left[\int_0^l \frac{M^2 dx}{2EI}\right]}{\partial P} = \int_0^l \frac{2M}{2EI} \cdot \frac{\partial M}{\partial P} \cdot dx$$

But
$$\frac{\partial M}{\partial P} = -x$$

Hence,
$$\delta = \int_0^l \frac{M}{EI}(-x)dx$$

$$= \frac{1}{EI}\int_0^l \left\{-\left(Px + \frac{wx^2}{2}\right)\right\}(-x)dx \qquad \text{(on substituting } M\text{)}$$

$$= \frac{1}{EI}\int_0^l \left(Px^2 + \frac{wx^3}{2}\right)dx$$

$$= \frac{1}{EI}\left[\frac{Px^3}{3} + \frac{wx^4}{8}\right]_0^l = \frac{1}{EI}\left[\frac{Pl^3}{3} + \frac{wl^4}{8}\right] \qquad \textbf{Ans.}$$

when $P = 0$, then

$$\delta = \frac{wl^4}{8EI}$$

SHORT ANSWER QUESTIONS

1. Why is strain energy so called?
2. What is strain energy density?
3. What is the strain energy stored in a shaft when the shaft is subjected to a torque T and produces an angle of twist θ?
4. What is Castigliano's theorem? Where is it used?
5. What is the basic principle of Castigliano's theorem?

MULTIPLE CHOICE QUESTIONS

1. Energy stored in a material during its deformation is known as

(*a*) elastic energy (*b*) plastic energy (*c*) strain energy (*d*) potential energy.

2. Strain energy stored per unit volume in a bar subjected to a gradually applied load P on its cross-section A is given as

(*a*) $\dfrac{P^2}{2A^2E}$ (*b*) $\dfrac{P^2}{2AE}$ (*c*) $\dfrac{P}{2A^2E}$ (*d*) $\dfrac{2P}{A^2E}$.

3. Workdone by a gradually applied load P on a body producing a deflection δl is given as

(*a*) $P\,.\,\delta l$ (*b*) $\dfrac{P\cdot\delta l}{2}$ (*c*) $\dfrac{P^2\delta l}{2}$ (*d*) $\dfrac{P\delta l^2}{2}$

4. The maximum strain energy stored in a body at the elastic limit is called

(*a*) resilience (*b*) modulus of resilience
(*c*) proof resilience (*d*) potential energy.

5. The workdone by a suddenly applied load P producing a deflection δl is given as

(*a*) $\dfrac{P\cdot\delta l}{2}$ (*b*) $P\cdot\delta l$ (*c*) $\dfrac{P^2\cdot\delta l}{2}$ (*d*) $P\cdot\delta l^2$.

6. The strain energy stored in a shaft of length l subjected to a torque T with its polar moment of inertia J is

(*a*) $\dfrac{T^2l}{2JG}$ (*b*) $\dfrac{T^2l}{JG}$ (*c*) $\dfrac{2T^2l}{JG}$ (*d*) $\dfrac{Tl}{2JG}$.

7. The strain energy stored in a simply supported beam of span l loaded with a central point load P and moment of inertia I is

(*a*) $\dfrac{Pl^3}{48EI}$ (*b*) $\dfrac{P^2l^2}{96EI}$ (*c*) $\dfrac{P^2l^3}{96EI}$ (*d*) $\dfrac{P^2l^3}{24EI}$.

8. The strain energy stored in a cantilever beam of span l carrying a point load P at its free end with moment of inertia I is

(*a*) $\dfrac{P^2l^3}{48EI}$ (*b*) $\dfrac{P^2l^3}{6EI}$ (*c*) $\dfrac{P^2l^3}{24EI}$ (*d*) $\dfrac{P^2l^3}{96EI}$.

9. The strain energy stored per unit volume in a body subjected to two mutually perpendicular principal stresses σ_1 and σ_2 with Poisson's ratio v is

(*a*) $\dfrac{1}{2E}(\sigma_1^2+\sigma_2^2-2v\sigma_1\sigma_2)$ (*b*) $\dfrac{1}{E}(\sigma_1+\sigma_2+\sigma_1\sigma_2)$

(*c*) $\dfrac{1}{E}(\sigma_1^2+\sigma_2^2-2v\sigma_1\sigma_2)$ (*d*) $\dfrac{1}{E}(\sigma_1+\sigma_2-\sigma_1\sigma_2)$.

10. The strain energy stored per unit volume in a cube subjected to a stress intensity σ on its all sides with bulk modulus K is

(*a*) $\dfrac{\sigma}{2K}$ (*b*) $\dfrac{\sigma}{2K^2}$ (*c*) $\dfrac{\sigma^2}{2K}$ (*d*) $\dfrac{\sigma^2}{K^2}$.

11. The strain energy stored per unit volume in a cube subjected to three mutually perpendicular principal stresses σ_1, σ_2 and σ_3 with $\in_1$, $\in_2$ and $\in_3$ being the strains produced in the respective directions of the stresses is

(*a*) $\sigma_1\in_1 + \sigma_2\in_2 + \sigma_3\in_3$ (*b*) $\dfrac{1}{2}(\sigma_1\in_1 + \sigma_2\in_2 + \sigma_3\in_3)$

(*c*) $\dfrac{1}{2}(\sigma_1^2\in_1 + \sigma_2^2\in_2 + \sigma_3^3\in_3)$ (*d*) $\dfrac{1}{2}(\sigma_1\in_1^2 + \sigma_2\in_2^2 + \sigma_3\in_3^2)$.

12. The stress produced by a suddenly applied load is how many times the stress produced by the gradually applied load?

(*a*) four times (*b*) three times (*c*) two times (*d*) eight times.

13. The stress produced in a bar of length l and cross-section A when a load P is dropped on it from a height h producing negligible deflection is given as

(*a*) $\sqrt{\dfrac{2EPh}{Al}}$ (*b*) $\sqrt{\dfrac{EPh}{Al}}$ (*c*) $\sqrt{\dfrac{EPh}{2Al}}$ (*d*) $\sqrt{\dfrac{EPh}{3Al}}$.

14. The strain energy stored per unit volume in a body subjected to a shear stress τ is given as

(*a*) $\dfrac{2\tau^2}{G}$ (*b*) $\dfrac{\tau^2}{2G}$ (*c*) $\dfrac{\tau}{2G^2}$ (*d*) $\dfrac{\tau^2}{G}$.

15. Two bodies of equal cross-sections and equal lengths absorb equal strain energy when loaded differently. The first body is carrying an axial load P and the other one is acting as a cantilever beam with a point load P at its free end. The stresses produced in the two bodies are related as

(*a*) $\sigma_1 = \dfrac{\sigma_2}{3}$ (*b*) $\sigma_1 = \sigma_2$ (*c*) $\sigma_1 = \dfrac{\sigma_2}{2}$ (*d*) $\sigma_1 = \dfrac{\sigma_2^2}{2}$.

16. For a straight bar of length l being subjected to a tensile load P, the strain energy is given as

(*a*) $\dfrac{Pl^2}{2AE}$ (*b*) $\dfrac{P^2l}{2AE}$ (*c*) $\dfrac{Pl^2}{AE^2}$ (*d*) $\dfrac{Pl}{A^2E}$.

17. For a bar of length l being subjected to a bending moment M, the strain energy is given as

(*a*) $\dfrac{M^2l}{2EI}$ (*b*) $\dfrac{Ml^2}{2EI}$ (*c*) $\dfrac{Ml^2}{EI}$ (*d*) $\dfrac{Ml^2}{EI}$.

ANSWERS

1. (*c*) **2.** (*a*) **3.** (*b*) **4.** (*c*) **5.** (*b*) **6.** (*a*) **7.** (*c*) **8.** (*b*) **9.** (*a*)
10. (*c*) **11.** (*b*) **12.** (*c*) **13.** (*a*) **14.** (*b*) **15.** (*a*) **16.** (*b*) **17.** (*a*).

EXERCISES

1. A vertical bar of length 2 m is fixed at its lower end. It is striked by a horizontally moving body of mass 5 kg at its upper end with a certain velocity V producing a maximum stress of 500 MPa in the bar. Find V. Take E = 200 GPa. (*Ans.* 3.61 m/s).

2. A bar of length 2 m is subjected to an axial load of 25 kN. The diameter of the bar for one-half of its length is 30 mm and for the other half 60 mm. Calculate the strain energy stored in the bar. Take E = 200 GPa. (*Ans.* 2.76 joules).

3. Show that the strain energy stored in a beam of length l having rectangular cross-section supported at the ends and loaded with a central point load P is given as $\frac{P^2 l^3}{96EI}$, where the symbols have their usual meanings.

4. A load of 20 kN falls through a height of 1 m on a rectangular beam of length 1 m with cross-section 20 mm wide × 30 mm deep supported at the ends. Find the instantaneous stress developed and the strain energy stored in the beam. Take E = 200 GPa. (*Ans.* 12747.18 N/mm^2, 27 kJ).

5. A load of 2 kN is allowed to fall freely from a height of 1 m at the centre of a circular simply supported beam of length 3 m and diameter 80 mm. Find the deflection of the beam. Take E = 200 GPa. (*Ans.* 77.65 mm).

6. Using Castigliano's theorem, find the deflection at the centre of a simply supported beam of length l carrying a triangular load which varies from zero at one end to w/unit length at another end.

$$\left(Ans.\ \frac{1}{156.25}\frac{wl^4}{EI}\right).$$

7. A hollow shaft having the outside and the inside diameter ratio of 2 is subjected to a maximum shear stress of τ. Show that the strain energy stored in the shaft on per unit volume basis is given by $\frac{5\tau^2}{16G}$, where G is the shear modulus of the shaft material.

8. A 1m long bar is subjected to an axial pull which induces a maximum stress of 150 MPa. The area of cross-section of the bar is 2×10^{-4} m^2 over a length of 0.95 m and for the central 0.05 m length the sectional area is equal to 1×10^{-4} m^2. Assuming that E for the bar material is 200 GPa, calculate the strain energy stored in the bar. (*Ans.* 2.953 joules).

9. A 500 mm × 180 mm rolled steel beam is simply supported over a span of 6m. A load of 20 kN is dropped on to the middle of the beam from a height of 12.5 mm. Find the maximum instantaneons deflection and the maximum stress induced. The second moment of area of the section equals to 45218×10^{-8} m^4 and the Young's modulus is 200 GPa. (*Ans.* 6.026 mm, 36.715 MPa).

10. Using the Castigliano's theorem, calculate the vertical deflection at the middle of a simply supported beam which carries a uniformly distributed load of intensity w over the full span l. The flexural rigidity EI of the beam is constant and only strain energy of bending is to be considered.

$$\left(Ans.\ \frac{5wl^4}{384EI}\right).$$

11. A steel tube having outside and inside diameter of 100 mm and 60 mm respectively is bent into the form of a quadrant of 2 m radius as shown in Fig. 9.12. One end is rigidly attached to a horizontal base plate to which a tangent to that end is perpendicular, and the free end supports a load of 1 kN. Determine the vertical and horizontal deflections of the free end under this load using the Castigliano's theorem. Take $E = 200$ GPa.

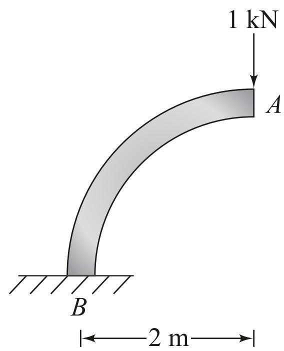


Fig. 9.12

(*Ans.* 7.353 mm, – 4.681 mm).

12. A shaft circular in section (Fig. 9.13) and of length l is subjected to a variable torque given by $\frac{kx^2}{i^2}$, where x is the distance measured from one end of the shaft and k is a constant. Find the angle of twist for the shaft using Castigliano's theorem. Torsional rigidity of the shaft is JG.

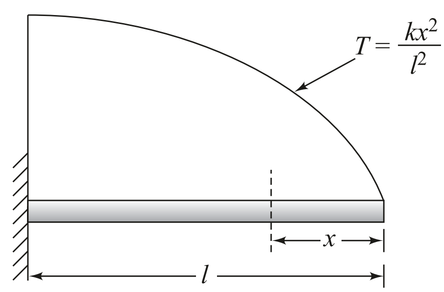


Fig. 9.13

$\left(Ans. \ \frac{kl}{3JG}\right)$.

❑❑❑

10

Theory of Elastic Failure

Henri Edouard Tresca
(1814-1885)

Henri Edouard Tresca, born on 12 October 1814, was a French mechanical engineer. He discovered the Tresca yield criterion, also called maximum shear stress theory, which is one of the two main failure criteria used today for ductile materials along with von Mises yield criterion. He became an honorary member of the American Society of Mechanical Engineers (ASME) in 1882. He is called the 'father of plasticity' for his contribution in the field of non-elastic deformation.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- Why is the yield strength most important criteria of a material's failure?
- Which is the simplest theory of a material's failure?
- Why is the Tresca's yield criterion the most popular theory of a material's failure?
- What is a principal plain?
- What is the difference between strain energy and shear strain energy?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_10

10.1 INTRODUCTION

When a material is stressed, there is a possibility for it to fail. Sometimes it fails without giving any indication and sometimes the failure is delayed. The analysis of failure of materials helps to know the actual reasons of their failure. There are five theories relating to a material's failure, but none of them give the exact reason of failure because of the complexity of compound stress system under multiaxial loading. Yield point is the stage in the material, beyond which permanent deformation occurs, and it is the most important failure criteria. The different theories of failure, also called failure criteria, are discussed below.

10.2 MAXIMUM NORMAL STRESS THEORY

This theory is also known as maximum principal stress theory or Rankine's theory, named after a Scottish civil engineer William John Macquorn Rankine (1820–1872). It is the oldest, as well as the simplest, of all the theories of failure, and applies well to brittle materials in all ranges of stresses, provided a tensile principal stress exists. According to this theory, a material fails, when the maximum principal stress developed due to external load reaches the ultimate strength of the material under uniaxial tension condition.

Consider a rectangular body being subjected to two normal stresses σ_x and σ_y along x and y directions respectively (Fig. 10.1). Let σ_u be the ultimate stress in simple tension.

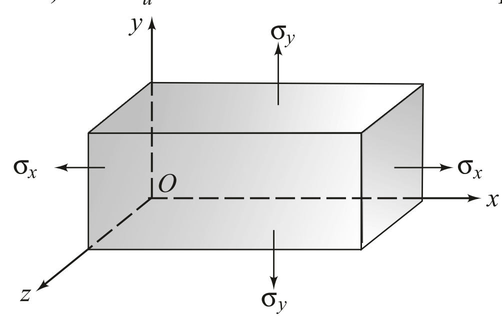


Fig. 10.1

The principal stresses are given as

$$\sigma_{1,\,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}{2} \qquad \text{...(10.1)}$$

The maximum principal stress is given as

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}{2} \qquad \text{...(10.2)}$$

The minimum principal stress is given as

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}{2} \qquad \text{...(10.3)}$$

According to this theory, $\sigma_1 = \sigma_u$

$$\frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2} = \sigma_u \qquad ...(10.4)$$

The normalised form of the Rankine's theory for a plane stress condition is

$$\frac{\sigma_1}{\sigma_u} = \pm 1 \text{ or } \frac{\sigma_2}{\sigma_u} = \pm 1 \qquad ...(10.5)$$

and is graphically represented as shown in Fig. 10.2, where σ_1 and σ_2 are the principal stresses and σ_u is the ultimate strength in the uniaxial tension test. The boundary of the square defines the failure criterion. The material is safer for the principal stresses lying within the square, but fails when the stresses lie on or outside the square.

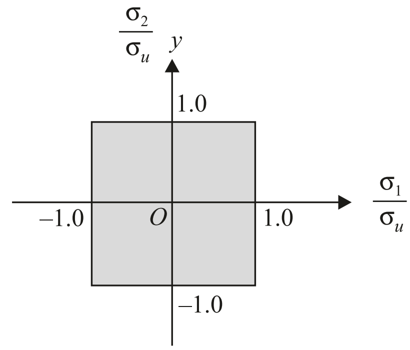


Fig. 10.2 Plotting of the maximum normal stress theory.

10.3 MAXIMUM NORMAL STRAIN THEORY

This theory is also called maximum principal strain theory or Saint Venant's criterion in honour of a French mathematician Saint Venant. The theory holds reasonably well for cast iron (a brittle material), but is not generally used these days as other theories give better result. According to this theory, a material fails, when the maximum principal strain reaches the strain at the yield point in the simple tension test.

Consider a complex stress system in which a body is subjected to three mutually perpendicular stresses σ_x, σ_y and σ_z (Fig. 10.3). Let the major principal strain occurs in the direction of σ_x and σ_{yp} is the yield point stress.

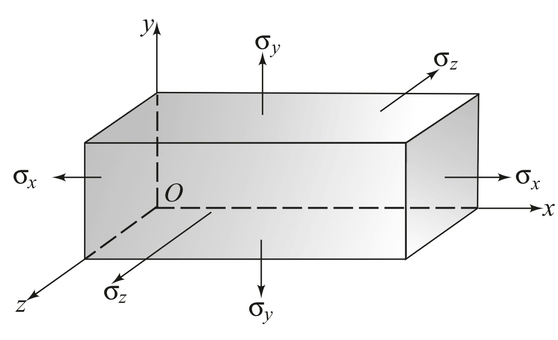


Fig. 10.3

Principal strain in the direction of σ_x is given as

$$\frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E} \qquad ...(10.6)$$

where

E = Modulus of elasticity

v = Poisson's ratio

According to this theory,

$$\frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E} = \frac{\sigma_{yp}}{E}$$

or

$$\sigma_x - v\,(\sigma_y + \sigma_z) = \sigma_{yp} \qquad ...(10.7)$$

10.4 MAXIMUM TOTAL STRAIN ENERGY THEORY

This theory is also called Haigh's criterion. According to this theory, for a body being subjected to a complex stress system, failure occurs, when the total strain energy per unit volume reaches the total strain energy at the yield point in the simple tension test. The theory gives fairly good results for ductile materials, but is not used these days as other improved theories are available.

Consider a body which is subjected to three mutually perpendicular stresses (Fig.10.3).

The strain energy per unit volume under the loading condition is given as

$$\frac{U}{V} = \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2v\,(\sigma_x\,\sigma_y + \sigma_y\,\sigma_z + \sigma_z\,\sigma_x)\right] \qquad ...(10.8)$$

The strain energy per unit volume at the yield point, when σ_{yp} is the yield stress, is given as

$$\frac{U_{yp}}{V} = \frac{\sigma_{yp}^2}{2E} \qquad ...(10.9)$$

According to this theory,

$$\frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2v\,(\sigma_x\,\sigma_y + \sigma_y\,\sigma_z + \sigma_z\,\sigma_x)\right] = \frac{\sigma_{yp}^2}{2E}$$

or

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2v\,(\sigma_x\,\sigma_y + \sigma_y\,\sigma_z + \sigma_z\,\sigma_x) = \sigma_{yp}^2 \qquad ...(10.10)$$

10.5 MAXIMUM SHEAR STRESS THEORY

This theory is also known as Tresca's yield criterion, named after a French engineer Henri Edouard Tresca (1814–1885). This theory gives better result when applied to ductile materials, and values obtained are very near to experimental values, hence oftenly used for ductile materials in machine design and is one of the widely used laws of plasticity. According to this theory, a material fails, when the maximum shear stress developed in the material equals to the maximum shear stress at the yield point in the uniaxial tension test.

Consider a body being subjected to two mutually perpendicular normal stresses σ_x and σ_y and a shear stress τ (Fig. 10.4).

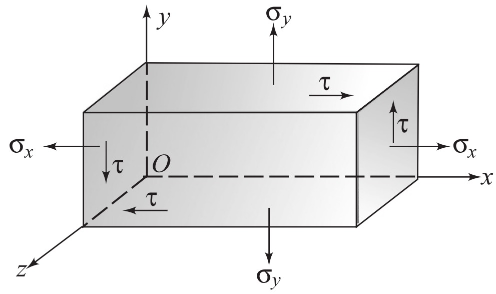


Fig. 10.4

The maximum shear stress is given as

$$\tau_{max} = \pm \frac{\sigma_1 - \sigma_2}{2} \qquad ...(10.11)$$

$$= \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

where σ_1 = Major principal stress

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}{2}$$

σ_2 = Minor principal stress

$$= \frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}{2}$$

The maximum shear stress, when the body is subjected to a stress σ_{yp}, is given as

$$\tau_{max_{yp}} = \frac{\sigma_{yp}}{2} \qquad ...(10.12)$$

According to this theory,

$$\tau_{max} = \tau_{max_{yp}}$$

$$\pm \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_{yp}}{2}$$

$$\pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2} = \frac{\sigma_{yp}}{2}$$

or

$$\pm \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2} = \sigma_{yp} \qquad ...(10.13)$$

The normalised form of the Tresca's yield criterion for a plane stress condition is

$$\frac{\sigma_1}{\sigma_{yp}} = \pm 1 \text{ or } \frac{\sigma_2}{\sigma_{yp}} = \pm 1 \text{ or } \frac{\sigma_1}{\sigma_{yp}} - \frac{\sigma_2}{\sigma_{yp}} = \pm 1 \qquad ...(10.14)$$

and its graphical representation forms a hexagon, called Tresca's hexagon as shown in Fig. 10.5, where σ_1 and σ_2 are the principal stresses and σ_{yp} is the yield point stress in the uniaxial tension test.

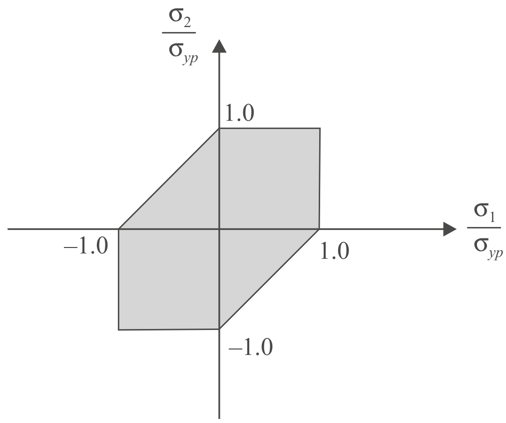


Fig. 10.5 Plotting of the maximum shear stress theory.

The boundary of the hexagon defines the failune criterion. The material is safer for the shear stresses within the hexagon, but fails when the stresses lie on or outside the hexagon.

10.6 MAXIMUM DISTORTION ENERGY THEORY

This theory is also known as von Mises yield criterion, named after a German-American mathematician Richard von Mises (1883–1953) or octahedral shear theory or maximum shear strain energy theory, and is the most popular theory for predicting yielding in ductile materials. According to this theory, a material begins yielding, when the maximum shear strain energy (also called distortion energy) per unit volume equals to the shear strain energy per unit volume at the yield point in the uniaxial tension test.

For a complex stress system shown in Fig. 10.3, the energy of distortion is given as

$$\frac{U_s}{V} = \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \quad ...(10.15)$$

where σ_1, σ_2 and σ_3 are three mutually perpendicular principal stresses.

The shear strain energy at the yield point, when the body is subjected to a stress σ_{yp}, is given as

$$\frac{U_{yp}}{V} = \frac{\sigma_{yp}^2}{6G} \quad ...(10.16)$$

According to this theory,

$$\frac{U_s}{V} = \frac{U_{yp}}{V}$$

$$\frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = \frac{\sigma_{yp}^2}{6G}$$

or $$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_{yp}^2 \quad ...(10.17)$$

The normalised form of the maximum distortion energy theory for a plane stress condition is

$$\left(\frac{\sigma_1}{\sigma_{yp}}\right)^2 + \left(\frac{\sigma_2}{\sigma_{yp}}\right)^2 - \left(\frac{\sigma_1}{\sigma_{yp}}\right)\left(\frac{\sigma_2}{\sigma_{yp}}\right) = 1 \quad ...(10.18)$$

and its graphical representation forms an ellipse as shown in Fig. 10.6. The bounding of the ellipse defines the failure criteria. The material is safer for the principal stresses lying within the ellipse, but fails if the stresses lie on or outside the ellipse. The corresponding Tresca's hexagon is shown by dashed lines.

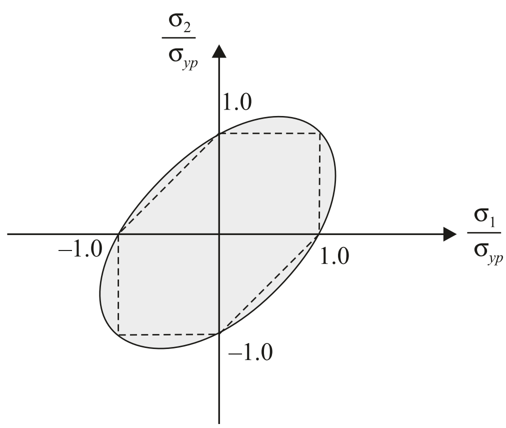


Fig. 10.6 Plotting of the maximum distortion energy theory.

Example 10.1

Two direct stresses are acting at two mutually perpendicular planes in a material. Both of them are tensile, and are 150 N/mm^2 and 80 N/mm^2 respectively. Find the shear stress acting on the planes to consider the material's failure according to maximum principal stress theory, maximum shear stress theory and shear strain energy theory. Take yield stress to be equal to 300 N/mm^2.

Solution: Given,

Direct stress in x-direction, σ_x= 150 N/mm^2 (Tensile)

Direct stress in y-direction, σ_y= 80 N/mm^2 (Tensile)

Yield stress, σ_{yp} = 300 N/mm^2

The major principal stress σ_1 is given as

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}{2}$$

$$= \frac{150+80}{2} + \frac{\sqrt{(150-80)^2 + 4\tau^2}}{2}$$

$$= 115 + \frac{\sqrt{4900 + 4\tau^2}}{2} \qquad \text{...(1)}$$

The minor principal stress σ_2 is given as

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}{2}$$

$$= 115 - \frac{\sqrt{4900 + 4\tau^2}}{2} \qquad \text{...(2)}$$

According to the maximum principal stress theory, we have

$$\sigma_1 = \sigma_{yp} = 300 \text{ N/mm}^2$$

$$115 + \frac{\sqrt{4900 + 4\tau^2}}{2} = 300$$

Solving for τ, we get

$$\tau = 181.66 \text{ N/mm}^2$$ **Ans.**

The maximum shear stress τ_{max} is given as

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

Using equations (1) and (2), we get

$$\tau_{max} = \sqrt{4900 + 4\tau^2}$$

According to the maximum shear stress theory, we have

$$\tau_{max} = \sigma_{yp} = 300 \text{ N/mm}^2$$

or $$\sqrt{4900 + 4\tau^2} = 300$$

Solving for τ, we get

$$\tau = 145.86 \text{ N/mm}^2$$ **Ans.**

The shear strain energy per unit volume, for a biaxial stress system, is given as

$$\frac{U_s}{V} = \frac{1}{12G} [(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2] \qquad (\sigma_3 = 0)$$

$$= \frac{1}{6G} [\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2]$$

According to the shear strain energy theory, we have

$$\frac{1}{6G}\,[\sigma_1^2 + \sigma_2^2 - \sigma_1\,\sigma_2] = \frac{\sigma_{yp}^2}{6G}$$

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\,\sigma_2 = \sigma_{yp}^2 = (300)^2$$

or

$$\left[115 + \frac{\sqrt{4900 + 4\tau^2}}{2}\right]^2 + \left[115 - \frac{\sqrt{4900 + 4\tau^2}}{2}\right]^2$$

$$- \left(115 + \frac{\sqrt{4900 + 4\tau^2}}{2}\right)\left(115 - \frac{\sqrt{4900 + 4\tau^2}}{2}\right) = 9 \times 10^4$$

Solving for τ, we get

$$\tau = 156.09 \text{ N/mm}^2$$ **Ans.**

Example 10.2

A circular shaft of diameter 50 mm is subjected to a maximum bending moment of 3×10^5 N·mm and a twisting moment of 5×10^5 N·mm. Find the factor of safety of the shaft using maximum shear stress theory. The yield stress of the shaft is not to exceed 90 N/mm^2.

Solution: Given,

Diameter of the shaft, $d = 50$ mm

Bending moment, $M = 3 \times 10^5$ N·mm

Twisting moment, $T = 5 \times 10^5$ N·mm

Yield stress of the shaft, $\sigma_{yp} = 90$ N/mm^2

The major principal stress is given as

$$\sigma_1 = \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

$$= \frac{16}{\pi \times (50)^3}\left[3 \times 10^5 + \sqrt{(3 \times 10^5)^2 + (5 \times 10^5)^2}\right] = 36 \text{ N/mm}^2$$

The minor principal stress is given as

$$\sigma_2 = \frac{16}{\pi d^3}\left[M - \sqrt{M^2 + T^2}\right]$$

$$= \frac{16}{\pi \times (50)^3}\left[3 \times 10^5 - \sqrt{(3 \times 10^5)^2 + (5 \times 10^5)^2}\right]$$

$$= -\,11.53 \text{ N/mm}^2$$

The negative sign signifies the compressive nature of σ_2.

The maximum shear stress is given as

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{36 + 11.53}{2} = 23.76 \text{ N/mm}^2$$

The working simple stress corresponding to τ_{max} is given as

$$\sigma_w = 2 \times \tau_{max}$$

$$= 47.52 \text{ N/mm}^2$$

Hence, the factor of safety is

$$n = \frac{\sigma_{yp}}{\sigma_w} = \frac{90}{47.52} = 1.9$$ **Ans.**

Example 10.3

For a complex stress system, three principal stresses are $2\sigma'$, $1.5\sigma'$ and $-1\sigma'$. The stress in simple tension at the elastic limit is 200 N/mm^2. Find the value of σ' according to (*a*) the maximum principal stress theory (*b*) the maximum principal strain theory (*c*) the total strain energy theory (*d*) the maximum shear stress theory and (*e*) the distortion energy theory. Take Poisson's ratio to be 0.25.

Solution: Given,

Major principal stress, $\sigma_1 = 2\sigma'$

Minor principal stress, $\sigma_2 = -1\sigma'$

Third principal stress, $\sigma_3 = 1.5\sigma'$

Stress at the elastic limit, $\sigma_e = 200$ N/mm^2

Poisson's ratio, $v = 0.25$

The minor principal stress is compressive because of negative sign associated with it.

(*a*) Using the maximum principal stress theory, we have

Major principal stress σ_1 = Stress at the elastic limit σ_e

$$\sigma_1 = 200$$

$$2\sigma' = 200$$

or $\sigma' = 100$ N/mm^2 **Ans.**

(*b*) Using the maximum principal strain theory, we have

$$\frac{\sigma_1}{E} - v\frac{\sigma_2}{E} - v\frac{\sigma_3}{E} = \frac{\sigma_e}{E}$$

(On replacing σ_x, σ_y and σ_z by σ_1, σ_2 and σ_3 respectively in equation (10.6).)

or $\sigma_1 - v\sigma_2 - v\sigma_3 = \sigma$

or $\quad 2\sigma' - 0.25 \times (-1\sigma') - 0.25\,(1.5\sigma') = 200$

On solving, we get

$$\sigma' = 106.66 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

(*c*) Using the total strain energy theory, we have

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\upsilon\,(\sigma_1\,\sigma_2 + \sigma_2\,\sigma_3 + \sigma_3\sigma_1) = \sigma_e^2$$

(On replacing σ_x, σ_y and σ_z by σ_1, σ_2 and σ_3 respectively in equation (10.9).)

$$(2\sigma')^2 + (-1\sigma')^2 + (1.5\sigma')^2 - 2 \times 0.25\,[(2\sigma' \times (-1\sigma') + (-1\sigma') \times (1.5\sigma') + (1.5\sigma') \times 2\sigma')] = (200)^2$$

On solving, we get

$$\sigma' = 73 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

(*d*) The maximum shear stress is given as

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{2\sigma' - (-1\sigma')}{2} = 1.5\sigma'$$

According to the maximum shear stress theory, we have

$$\tau_{max} = \frac{\sigma_e}{2} = \frac{200}{2} = 100$$

$$1.5\sigma' = 100$$

or
$$\sigma' = 66.67 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

(*e*) Using the distortion energy theory, we have

$$\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = \sigma_e^2$$

or
$$\frac{1}{2}[(2\sigma' + 1\sigma')^2 + (-1\sigma' - 1.5\sigma')^2 + (1.5\sigma' - 2\sigma')^2] = (200)^2$$

On solving, we get

$$\sigma' = 71.8 \text{ N/mm}^2 \qquad \textbf{Ans.}$$

Example 10.4

A circular shaft of 80 mm diameter is subjected to combined bending and twisting moments, the bending moment being four times the twisting moment. Find the allowable twisting moment according to (*a*) the maximum principal stress theory (*b*) the maximum shear stress theory and (*c*) the distortion energy theory. Given, the stress at the elastic limit is 4 N/mm^2 and the factor of safety is 3.

Solution: Given,

Diameter of the shaft, $d = 80$ mm

Elastic limit stress, $\sigma_e = 4$ N/mm^2

Factor of safety, $n = 3$

Let T = Twisting moment

M = Bending moment

$M = 4T$ (Given)

The major principal stress is given as

$$\sigma_1 = \frac{16}{\pi d^3}(M + \sqrt{M^2 + T^2})$$

$$= \frac{16}{\pi \times (80^3)}(4T + \sqrt{(4)^2 + T^2})$$

$$= \frac{16T}{\pi \times (80)^3}(4 + \sqrt{17}) = 8 \times 10^{-5} T$$

The minor principal stress is given as

$$\sigma_2 = \frac{16T}{\pi \times (80)^3}(4 - \sqrt{17}) = -1.22 \times 10^{-6} T$$

The working stress at the elastic limit is given as

$$\sigma_w = \frac{\sigma_e}{n} = \frac{4}{3} = 1.33 \text{ N/mm}^2$$

(*a*) Using the maximum principal stress theory, we have

$$\sigma_1 = \sigma_w$$

$$8 \times 10^{-5}\, T = 1.33$$

On solving, we get

$$T = 1.66 \times 10^4 \text{ N·mm}$$ **Ans.**

(*b*) The maximum shear stress is given as

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{8 \times 10^{-5}\, T + 1.22 \times 10^{-6}\, T}{2} = 4.06 \times 10^{-5}\, T$$

Using the maximum shear stress theory, we have

$$\tau_{max} = \frac{\sigma_w}{2}$$

$$4.06 \times 10^{-5}\, T = \frac{1.33}{2}$$

On solving, we get

$$T = 1.63 \times 10^4 \text{ N·mm}$$ **Ans.**

(*c*) For a biaxial stress system, the distortion energy theory reduces to

$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = \sigma_w^2$$

$$(8\times10^{-5}\ T)^2 + (-1.22\times10^{-6}\ T)^2 - 8\times10^{-5}\ T\times(-1.22\times10^{-6}\ T) = (1.33)^2$$

Solving for T, we get

$$T = 1.65 \times 10^4 \text{ N·mm}$$ **Ans.**

Example 10.5

A circular shaft of diameter 100 mm is subjected to a bending moment and a twisting moment. Find the relationship between the two moments assuming that they are causing failure alone according to (*a*) the maximum principal stress theory (*b*) the maximum principal strain theory (*c*) the total strain energy theory (*d*) the maximum shear stress theory and (*e*) the distortion energy theory. Take Poisson's ratio to be 0.25.

Solution: Given,

Diameter of the shaft, $d = 100$ mm

Poisson's ratio, $v = 0.25$

Let M = Bending moment

T = Twisting moment

σ_e = Stress at the elastic limit

σ_1 = Major principal stress

σ_2 = Minor principal stress

The bending stress, due to bending moment M acting alone, is given as

$$\sigma_b = \frac{32M}{\pi d^3}$$

The maximum shear stress, due to twisting moment, that is, torque T acting alone, is given as

$$\tau = \frac{16T}{\pi d^3}$$

Principal stresses are

$$\sigma_{1,2} = \pm\frac{16T}{\pi d^3}$$

(*a*) Using the principal stress theory, we have

$$\sigma_1 = \sigma_e$$

$$\frac{16T}{\pi d^3} = \frac{32M}{\pi d^3}$$ (here $\sigma_e = \sigma_b$ = bending stress)

or $$T = 2M$$ **Ans.**

(*b*) For a biaxial stress system, the maximum principal strain theory is expressed as

$$\sigma_1 - v\sigma_2 = \sigma_e$$

(On replacing σ_x and σ_y by σ_1 and σ_2 respectively in equation (10.6).)

$$\frac{16T}{\pi d^3} - 0.25\left(-\frac{16T}{\pi d^3}\right) = \frac{32M}{\pi d^3} \qquad \text{(Here } \sigma_e = \sigma_b = \text{Bending stress)}$$

On simplification, we get

$$T = 1.6\, M$$ **Ans.**

(*c*) The total strain energy theory, for a biaxial stress system, is expressed as

$$\sigma_1^2 + \sigma_2^2 - 2v\,\sigma_1\,\sigma_2 = \sigma_e^2$$

(Using equation (10.10) and replacing σ_x and σ_y by σ_1 and σ_2 respectively.)

$$\left(\frac{16T}{\pi d^3}\right)^2 + \left(-\frac{16T}{\pi d^3}\right)^2 - 2 \times 0.25 \times \frac{16T}{\pi d^3} \times \left(-\frac{16T}{\pi d^3}\right) = \left(\frac{32M}{\pi d^3}\right)^2$$

On simplification, we get

$$T = 1.264\, M$$ **Ans.**

(*d*) Using the maximum shear stress theory, we have

$$\tau = \frac{\sigma_e}{2} = \frac{\sigma_b}{2}$$

$$\frac{16T}{\pi d^3} = \frac{1}{2} \times \frac{32M}{\pi d^3}$$

or $$T = M$$ **Ans.**

(*e*) For a biaxial stress system, the distortion energy theory is expressed as

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\,\sigma_2 = \sigma_e^2$$

$$\left(\frac{16T}{\pi d^3}\right)^2 + \left(-\frac{16T}{\pi d^3}\right)^2 - \left(\frac{16T}{\pi d^3}\right)\left(-\frac{16T}{\pi d^3}\right) = \left(\frac{32M}{\pi d^3}\right)^2$$

On simplification, we get

$$T = 1.155\, M$$ **Ans.**

Example 10.6

Find the maximum principal stress developed in a cylindrical shaft 8 cm in diameter and subjected to a bending moment of 2.5 kN.m and a twisting moment of 4.2 kN.m. If the yield stress of the shaft material is 300 MN/m^2, determine the factor of safety according to the maximum shearing stress theory of failure.

Solution: Given,

Diameter of the shaft, $d = 8$ cm $= 0.08$ m

Bending moment, $M = 2.5$ kN.m

Twisting moment, $T = 4.2$ kN.m

Yield stress, $\sigma_{yp} = 300$ MN/m^2

The bending stress is given as

$$\sigma_b = \frac{32M}{\pi d^3} = \frac{32 \times 25 \times 10^3}{\pi \times (0.08)^3 \times 10^6} \text{ MN/m}^2 = 49.73 \text{ MN/m}^2$$

The shear stress is given as

$$\tau = \frac{16T}{\pi d^3}$$

$$= \frac{16 \times 4.2 \times 10^3}{\pi \times (0.08)^3 \times 10^6} \text{ MN/m}^2 = 41.78 \text{ MN/m}^2$$

The maximum principal stress is given as

$$\sigma_1 = \frac{\sigma_b}{2} + \frac{1}{2}\sqrt{\sigma_b^2 + 4\tau^2} \qquad (\because \sigma_x = \sigma_b \text{ and } \sigma_y = 0)$$

$$= \frac{49.73}{2} + \frac{1}{2}\sqrt{(49.73)^2 + 4 \times (41.78)^2}$$

$$= 73.48 \text{ MN/m}^2$$ **Ans.**

The maximum shear stress is given as

$$\tau_{max} = \frac{1}{2}\sqrt{\sigma_b^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{(49.73)^2 + 4 \times (41.78)^2} = 48.62 \text{ MN/m}^2$$

According to the maximum shear stress theory, failure occurs when τ_{max} equals to one-half of the yield stress. Hence, the factor of safety is

$$n = \frac{\sigma_{yp}}{2 \times \tau_{max}} = \frac{300}{2 \times 48.62} = 3.08$$ **Ans.**

Example 10.7

A thin walled circular tube of wall thickness t and mean radius r is subjected to an axial load P and a torque T in a combined tension-torsion experiment.

(*a*) Determine the state of stress exising in the tube in terms of P and T.

(*b*) Using the von Mises failure criterion, show that the failure takes place, when

$$\sqrt{\sigma^2 + 3\tau^2} = \sigma_{yp}$$

where σ_{yp} is the yield stress in uniaxial tension, σ and τ are respectively the axial and torsional shearing stresses in the tube.

Solution: The von Mises theory, for a two-dimensional stress system, is given as

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_{yp}^2 \qquad \text{... (1)}$$

where σ_1 and σ_2 are the two principal stresses and σ_{yp} is the yield stress in uniaxial tension.

Refer Fig. 10.7 for the state of stress in the tube. A small element on the surface of the tube with stresses acting on it is shown in the figure.

The direct stress due to axial load P is given as

$$\sigma = \frac{P}{A} = \frac{P}{\pi r^2} \qquad \text{... (2)}$$

The shear stress due to torque T is

$$\tau = \frac{T}{J} \times r$$

Substituting $J = \frac{\pi}{2} r^4$ in the shear stress equation, we get

$$\tau = \frac{2T}{\pi r^3} \qquad \text{... (3)}$$

The principal stresses w.r.t. two axes system are given as

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

and
$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

But
$$\sigma_y = 0,\ \sigma_x = \sigma \text{ and } \tau_{xy} = \tau$$

Hence,
$$\sigma_1 = \frac{\sigma}{2} + \frac{\sqrt{\sigma^2 + 4\tau^2}}{2}$$

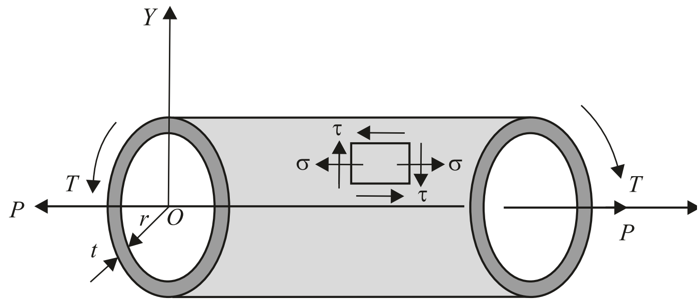


Fig. 10.7

and $$\sigma_2 = \frac{\sigma}{2} - \frac{\sqrt{\sigma^2 + 4\tau^2}}{2}$$

Substituting σ_1 and σ_2 in equation (1), we have

$$\frac{\sigma^2}{4} + \frac{\sigma^2 + 4\tau^2}{4} + 2 \cdot \frac{\sigma}{2} \cdot \frac{\sqrt{\sigma^2 + 4\tau^2}}{4} + \frac{\sigma^2}{4} + \frac{\sigma^2 + 4\tau^2}{4} - 2 \cdot \frac{\sigma}{2} \cdot \frac{\sqrt{\sigma^2 + 4\tau^2}}{2} - \left(\frac{\sigma^2}{4} - \frac{\sigma^2 + 4\tau^2}{4}\right) = \sigma_{yp}^2$$

$$\frac{\sigma^2}{4} + 3 \cdot \frac{\sigma^2 + 4\tau^2}{4} = \sigma_{yp}^2$$

$$\frac{\sigma^2}{4} + \frac{3}{4}\sigma^2 + 3 \cdot \frac{4\tau^2}{4} = \sigma_{yp}^2$$

$$\sigma^2 + 3\tau^2 = \sigma_{yp}^2$$

Hence, $$\sqrt{\sigma^2 + 3\tau^2} = \sigma_{yp}$$ **Proved.**

Example 10.8

Three exactly similar specimens of mild steel tube are 37.5 mm in external diameter and 31.25 mm in internal diameter. One of these tubes was tested in tension and the limit of proportionality was achieved at a load of 70 kN. The second was tested in torsion whereas the third was tested in torsion with a superimposed bending moment of 350 N.m. If the failure criteria is maximum shear stress, estimate the torque at which the two specimens would fail.

Solution: Given,

Internal diameter of the tube, $d_i = 31.25$ mm

External diameter of the tube, $d_0 = 37.5$ mm

Bending moment, $M = 350$ N.m

The area of the tube is

$$A = \frac{\pi}{4}(d_0^2 - d_1^2)$$

$$= \frac{\pi}{4}(37.5^2 - 31.25^2) = 337.4 \text{ mm}^2$$

The polar moment of inertia of the cross-section of the tube is obtained as

$$J = \frac{\pi}{32}(d_o^4 - d_i^4)$$

$$= \frac{\pi}{32}(37.5^4 - 31.25^4) = 100517.7 \text{ mm}^4$$

The stress at the limit of proportionality is given as

$$\sigma_e = \frac{70 \times 10^3}{337.47} \text{ N/mm}^2$$

$$= 207.42 \text{ N/mm}^2$$

According to the maximum shear stress theory, the maximum shear stress is given by

$$\tau_{\max} = \frac{\sigma_e}{2}$$

$$= \frac{207.42}{2} = 103.71 \text{ N/mm}^2$$

The equivalent torque T_e is given by

$$T_e = \sqrt{M^2 + T^2}$$

Now $$\sqrt{M^2 + T^2} = \frac{\tau_{\max} \times J}{\left(\frac{d_0}{2}\right)}$$ (using torsion equation)

$$= \frac{103.71 \times 100517.7 \times 2}{37.5} = 555983.5 \text{ N.mm}$$

or $$\sqrt{(350 \times 10^3)^2 + T^2} = 555983.5$$

Solving for T, we get

$$T = 432 \text{ N.m}$$ **Ans.**

Example 10.9

A mild steel shaft of 50 mm diameter is subjected to a bending moment of 2 kN.m and a torque T. If the yield point of steel in tension is 200 MPa, find the maximum value of the theory torque without causing yielding of the shaft material according to (*a*) the maximum principal stress theory, (*b*) the maximum shear stress theory and (*c*) the maximum distortion strain energy theory.

Solution: Given,

Diameter of the shaft, $d = 50$ mm

Bending moment, $M = 2$ kN.m $= 2 \times 10^6$ N.mm

Yield point stress, $\sigma_{yp} = 200$ MPa $= 200$ N/mm^2

The bending stress is given by

$$\sigma_b = \frac{M}{I} \times y \qquad \text{(using bending equation)}$$

$$= 2 \times 10^6 \times \frac{1}{\frac{\pi}{64} \times 50^4} \times \frac{50}{2} \text{ N/mm}^2 = 162.975 \text{ N/mm}^2$$

The shear stress is given by

$$\tau = \frac{T}{J} \times \frac{d}{2} \qquad \text{(using torsion equation)}$$

$$= T \times \frac{1}{\frac{\pi}{32} \times 50^4} \times \frac{50}{2} \text{ N/mm}^2 = 4.07 \times 10^{-5} T \text{ N/mm}^2$$

where T is the torque in N.mm.

The maximum principal stress is given by

$$\sigma_1 = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{162.975}{2} + \frac{\sqrt{(162.975)^2 + 4 \times (4.07 \times 10^{-5})^2 T^2}}{2}$$

$$= 81.488 + \frac{\sqrt{26560.85 + 6.626 \times 10^{-9} T^2}}{2}$$

The minimum principal stress is given by

$$\sigma_2 = 81.488 - \frac{\sqrt{26560.85 + 6.626 \times 10^{-9} T^2}}{2}$$

The maximum shear stress is

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{\sqrt{26560.85 + 6.626 \times 10^{-9} T^2}}{2}$$

(*a*) According to the maximum principal stress theory, we have

$$\sigma_1 = \sigma_{yp}$$

or $$81.488 + \frac{\sqrt{26560.85 + 6.626 \times 10^{-9} T^2}}{2} = 200$$

Solving for T, we get

$$T = 2114284.1 \text{ N.mm} = 2114.3 \text{ N.m}$$ **Ans.**

(*b*) According to the maximum shear stress theory, we have

$$\tau_{max} = \frac{\sigma_{yp}}{2}$$

or $$\frac{\sqrt{26560.85 \times 6.626 \times 10^{-9} T^2}}{2} = \frac{200}{2}$$

Solving for T, we get

$$T = 1424164.6 \text{ N.mm} = 1424.16 \text{ N.m}$$ **Ans.**

(*c*) According to the distortion energy theory, we have

$$(\sigma_1 - \sigma_2)^2 + \sigma_1^2 + \sigma_2^2 = 2\sigma_{yp}^2 \qquad (\because \sigma_3 = 0)$$

or $$26560.85 + 6.626 \times 10^{-9} T^2 + \left(81.488 + \frac{\sqrt{26560.85 + 6.626 \times 10^{-9} T^2}}{2}\right)^2$$

$$+ \left(81.488 - \frac{\sqrt{26560.85 + 6.626 \times 10^{-9} T^2}}{2}\right)^2 = 2 \times (200)^2$$

Solving for T, we get

$$T = 1644478.6 \text{ N.mm} = 1644.48 \text{ N.m}$$ **Ans.**

Example 10.10

A cube of 5 mm side is loaded as shown in Fig. 10.5. (*a*) Determine the principal stresses σ_1, σ_2 and σ_3. (*b*) Will the cube yield, if the yield strength of the material is 70 MPa? Use von Mises theory.

Solution: Refer Fig. 10.8.

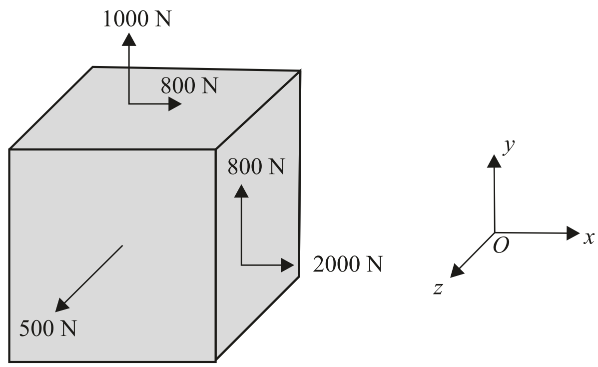


Fig. 10.8

Side of the cube, $a = 5$ mm

Yield strength, $\sigma_{yp} = 70$ MPa $= 70$ N/mm^2

Area of each face of the cube $= a^2 = 25$ mm^2

(*a*) Normal stress in *x*-direction is

$$\sigma_x = \frac{2000}{25} = 80 \text{ N/mm}^2$$

Normal stress in *y*-direction is

$$\sigma_y = \frac{1000}{25} = 40 \text{ N/mm}^2$$

Normal stress in *z*-direction is

$$\sigma_z = \frac{500}{25} = 20 \text{ N/mm}^2$$

Shear stress in *xy* plane is

$$\tau_{xy} = \frac{800}{25} = 32 \text{ N/mm}^2$$

Since no shear stress is acting on the plane of 500 N force, hence it is one of the principal planes.

Hence, $\sigma_3 = \sigma_z = 20$ N/mm^2 **Ans.**

The other two principal stresses are given as

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}{2}$$

$$= \frac{80 + 40}{2} \pm \frac{\sqrt{(80 - 40)^2 + 4 \times 32^2}}{2} = 60 \pm 37.73 \text{ N/mm}^2$$

Hence, $\sigma_1 = 60 + 37.73 \text{ N/mm}^2$

$= 97.73 \text{ N/mm}^2$ **Ans.**

and $\sigma_2 = 60 - 37.73 \text{ N/mm}^2$

$= 22.27 \text{ N/mm}^2$ **Ans.**

(*b*) Now $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$

$= (97.73 - 22.27)^2 + (22.27 - 20)^2 + (20 - 97.73)^2$

$= 11741.32 \text{ N/mm}^2$

and $2\sigma_{yp}^2 = 2 \times 70^2 = 9800 \text{ N/mm}^2$

Since $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 > 2\sigma_{yp}^2$, hence according to the von Mises theory, the yielding will occur. **Ans.**

SHORT ANSWER QUESTIONS

1. How is a failure criterion useful in the analysis of a material's failure?
2. Which theory predicts a brittle failure more accurately?
3. Which theory predicts a ductile failure more accurately?
4. What is yield point? What does it signify?
5. Why is a factor of safety used in engineering design?
6. How are the principal stress and the principal strain useful in predicting a material's failure?
7. How does direct strain energy differ from shear strain energy?
8. Which failure criterion is also called Rankine's theory? For which type of materials this theory is applied?
9. Which failure theory is also called Tresca's yield criterion? For which type of materials this theory is applied?
10. Why is the maximum distortion energy theory also called the octahedral shear theory?

MULTIPLE CHOICE QUESTIONS

1. The most important point in the consideration of a material's failure is
(*a*) ultimate point (*b*) yield point
(*c*) failure point (*d*) elastic limit.

2. The maximum principal stress theory is also known as
(*a*) Haigh's theory (*b*) St.Venant's theory
(*c*) Rankine's theory (*d*) von Mises theory.

3. Factor of safety is defined as the ratio of
(*a*) ultimate stress to working stress (*b*) working stress to ultimate stress
(*c*) yield stress to ultimate stress (*d*) lateral strain to longitudinal strain.

4. The principal stresses are given as

(*a*) $\dfrac{\sigma_x + \sigma_y}{2} \pm \dfrac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}{2}$ (*b*) $\dfrac{\sigma_x - \sigma_y}{2} \pm \dfrac{\sqrt{(\sigma_x + \sigma_y)^2 - 4\tau^2}}{2}$

(*c*) $\dfrac{\sigma_x + \sigma_y}{2} \pm \dfrac{\sqrt{(\sigma_x - \sigma_y)^2 - 4\tau^2}}{2}$ (*d*) $\sigma_x + \sigma_y \pm \dfrac{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}{2}$.

5. Hooke's law is valid for
(*a*) brittle materials (*b*) ductile materials
(*c*) isotropic materials (*d*) isotropic and homogenous materials.

6. For a body being subjected to three mutually perpendicular stresses σ_x, σ_y and σ_z, principal strain in the *x*-direction is

(*a*) $\dfrac{\sigma_x}{E} - \upsilon\dfrac{\sigma_y}{E} + \upsilon\dfrac{\sigma_z}{E}$ (*b*) $\dfrac{\sigma_x}{E} + \upsilon\dfrac{\sigma_y}{E} + \upsilon\dfrac{\sigma_z}{E}$

(*c*) $\dfrac{\sigma_x}{E} - \upsilon\dfrac{\sigma_y}{E} - \upsilon\dfrac{\sigma_z}{E}$ (*d*) $\dfrac{\sigma_x}{E} - \dfrac{\upsilon}{2}(\sigma_y + \sigma_z)$.

7. The maximum shear stress theory gives better results for
(*a*) brittle materials (*b*) ductile materials
(*c*) brittle and ductile materials both (*d*) nonmetallic materials.

8. For a biaxial stress system, the strain energy per unit volume is

(*a*) $\dfrac{1}{2E}(\sigma_x^2 - \sigma_y^2 + 2\upsilon\,\sigma_x\,\sigma_y)$ (*b*) $\dfrac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\upsilon\,\sigma_x\,\sigma_y)$

(*c*) $\dfrac{1}{2G}(\sigma_x^2 + \sigma_y^2 - 2\upsilon\,\sigma_x\,\sigma_y)$ (*d*) $\dfrac{1}{2G}(\sigma_x^2 - \sigma_y^2 + 2\upsilon\,\sigma_x\,\sigma_y)$.

9. According to the Tresca's theory, the failure occurs, when the

(*a*) major principal stress exceeds the elastic limit stress

(*b*) maximum principal strain exceeds the elastic limit strain

(*c*) maximum shear stress exceeds the maximum shear stress at the elastic limit

(*d*) distortion energy per unit volume exceeds the distortion energy per unit volume at the elastic limit.

10. The shear strain energy per unit volume, for a biaxial stress system, in which σ_1 and σ_2 are the major and minor principal stresses respectively, is given as

(*a*) $\frac{1}{12G}[\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2]$ (*b*) $\frac{1}{6G}[\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2]$

(*c*) $\frac{1}{6G}[\sigma_1^2 - \sigma_2^2 + \sigma_1 \sigma_2]$ (*d*) $\frac{1}{6G}[\sigma_1^2 - \sigma_2^2 + \sigma_1 \sigma_2]$.

11. The von Mises equivalent stress, for a biaxial stress system, is defined as

(*a*) $\sqrt{\sigma_1^2 - \sigma_2^2 + \sigma_1 \sigma_2}$ (*b*) $\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2}$

(*c*) $\sqrt{\sigma_1 + \sigma_2 - \sigma_1 \sigma_2}$ (*d*) $\sqrt{\sigma_1 - \sigma_2 + \sigma_1 \sigma_2}$.

where σ_1 and σ_2 are the major and minor principal stresses respectively.

12. The maximum shear stress is given as

(*a*) $\frac{\sigma_1 + \sigma_2}{2}$ (*b*) $\frac{\sigma_1 - \sigma_2}{2}$

(*c*) $\frac{\sigma_1^2 + \sigma_2^2}{2}$ (*d*) $\frac{(\sigma_1 + \sigma_2)^2}{2}$.

where σ_1 and σ_2 are the major and minor principal stresses respectively.

13. Which one of the groups consisting of two theories is said to be modern because of their closeness to experimental values ?

(*a*) Rankine's and St. Venant's theory (*b*) Tresca's and Rankine's theory

(*c*) Tresca's and von Mises theory (*d*) Rankine's and von Mises theory.

14. The shear strain energy per unit volume at the elastic limit for a body being subjected to a stress σ is

(*a*) $\frac{\sigma^2}{6E}$ (*b*) $\frac{\sigma^2}{6G}$

(*c*) $\sigma^2\left(\frac{1}{6E} + \frac{1}{6G}\right)$ (*d*) $\frac{\sigma}{6G}$.

15. The maximum shear stress at the elastic limit for a body being subjected to a stress σ is

(*a*) 2σ (*b*) $\frac{\sigma}{3}$

(*c*) $\frac{\sigma}{2}$ (*d*) $\frac{\sigma}{5}$.

ANSWERS

1. (*b*) **2.** (*c*) **3.** (*a*) **4.** (*a*) **5.** (*d*) **6.** (*c*) **7.** (*b*) **8.** (*b*) **9.** (*c*)
10. (*b*) **11.** (*b*) **12.** (*b*) **13.** (*c*) **14.** (*b*) **15.** (*c*)

EXERCISES

1. A cylindrical shell made of mild-steel plate yields at 200 N/mm^2 in uniaxial tension. The diameter of the shell is 2 m and its thickness is 20 mm. Find the pressure at which the failure occurs according to (*a*) the maximum shear stress theory and (*b*) the distortion energy theory.

(*Ans.* (*a*) 4 N/mm^2 (*b*) 4.62 N/mm^2).

2. For a biaxial stress system, σ_x = 80 N/mm^2 and σ_y = 40 N/mm^2. Find the equivalent stress at the elastic limit assuming that the failure occurs due to the maximum principal strain theory. Take Poisson's ratio to be 0.25. (*Ans.* 7×10^3 N/mm^2).

3. A pressure vessel of inside radius 300 mm is subjected to a pressure of 2.0 N/mm^2. Find the thickness of the vessel according to total strain energy theory. The factor of safety is 3 and the yielding occurs in simple tension at 240 N/mm^2. Take Poisson's ratio to be 0.3.

(*Ans.* 7.31 mm).

4. A cylindrical tube of outside diameter 120 mm and thickness 3 mm is subjected to a torque of 2×10^4 N·m. The stress at the elastic limit in simple tension is 1200 N/mm^2. Calculate the factor of safety according to (*a*) the maximum shear stress theory and (*b*) the distortion energy theory.

(*Ans.* (*a*) 1.89 (*b*) 2.18).

5. A shaft of diameter 50 mm is subjected to a torque of 300 N·m and an axial thrust. For a factor of safety of 3, find the maximum value of the thrust according to (*a*) the maximum shear stress theory and (*b*) the distortion energy theory. The failure occurs at a stress of 100 N/mm^2 at the elastic limit. (*Ans.* (*a*) 8.56×10^4 N (*b*) 8.9×10^4 N).

6. A thick spherical pressure vessel of inner radius 150 mm is subjected to an internal pressure of 80 MPa. Calculate its wall thickness based upon (*a*) the maximum principal stress theory and (*b*) the total strain energy theory. Poisson's ratio = 0.30 and yield strength = 300 MPa.

(*Ans.* 20.227 mm, 28.87 mm).

7. A component in an aircraft flap actuator can be adequately modelled as a cylindrical bar subjected to an axial force of 8 kN, a bending moment of 55 N·m and torsional moment of 30 N·m. A 20 mm diameter solid bar of 7075-T6 aluminium having σ_u = 591 MPa, σ_{yt} = 542 MPa and σ_{ys} = 271 MPa is recommended for its use. Determine the factor of safety according to (*a*) the maximum principal stress theory and (*b*) the maximum shear stress theory. (*Ans.* 7.35 , 5.269).

8. A thin cylindrical pressure vessel with closed ends having inside diameter 500 mm is required to withstand an internal pressure of 4 MPa. Find the thickness of the vessel taking a factor of safety of 4 and assuming the yield stress of 360 MPa, according to the following failure criteria:

(*a*) the maximum shear stress theory.

(*b*) the shear strain energy theory. (*Ans.* 11.1 mm, 9.62 mm).

❑❑❑

11

Buckling of Columns

Leonhard Euler
(1707-1783)

Leonhard Euler, born on 15 April 1707, was a great Swiss mathematician and physicist. He is a seminal figure in the history of mathematics and one of the most prolific mathematicians who worked in almost all areas of mathematics such as geometry, infinitesimal calculus, trigonometry, algebra and number theory. He introduced several notations used in mathematics such as the letter e for the base of the natural logarithm, which approximately equals to 2.71828 and is also known as Euler number, the Greek letter $\sum$ for summation, and the letter i to denote imaginary unit. He introduced the concept of a function and used $f(x)$ to denote function f applied to a quantity x. The Euler constant γ (gamma) is named after him. He is also known for his work in mechanics, fluid mechanics, optics and astronomy. He was elected a foreign member of the Royal Swedish Academy of Sciences in 1755.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- What is meant by buckling of a column?
- What does slenderness ratio signify?
- What is crippling stress?
- What is the equivalent length of a column?
- What are the limitations of the Euler's formula?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_11

11.1 INTRODUCTION

Columns are long, slender structural members designed to support axial compressive loads. Vertical pillars used in the construction of a building are suitable examples of columns. The loaded slender column may deflect laterally and fail by bending (buckling) rather than failing by direct compression. Buckling is one of the major causes of failures in structures. So design of a column is essential to ensure that load carried by it is within the safe permissible limit and can sustain it without buckling under the specified load.

11.2 IMPORTANT TERMINOLOGY

- *Buckling* or *crippling or critical load* is the maximum limiting load at which the column either buckles or tends to buckle. Buckling of column means its lateral displacement during the maximum loading condition and it depends upon the length and cross-sectional area of the column.

- *Slenderness ratio* is the ratio of effective length of the column to its least radius of gyration. The columns are usually classified according to their slenderness ratio. The crippling stress depends on this ratio. The smaller the slenderness ratio, the higher is the crippling stress.

- *Safe load* is the actual load to be placed on the column without producing any buckling, and is given as

$$\text{Safe load} = \frac{\text{Buckling load}}{\text{Factor of safety } (n)}$$

11.3 CLASSIFICATION OF COLUMNS

- *Short columns* have length less than 8 times their least lateral dimension (diameter) or their slenderness ratio is less than 30. Short columns do not buckle but fail by yielding and hence can be subjected to maximum permissible compressive stress.
- *Long columns* have very large lengths as compared to their lateral dimensions and their slenderness ratio is greater than 120. Such columns fail mainly by buckling and the effect of direct compressive stress is almost negligible.
- *Medium columns* have slenderness ratio in between 30 and 120 and fail due to the combined effect of buckling and direct compressive stress.

11.4 EULER'S THEORY

The Euler's theory is applicable for long columns where buckling is the major cause for their failure and the effect of direct compressive stress is ignored on account of its negligible value. Here the buckling load depends on the modulus of elasticity of the column material, but not on its yield strength.

The following *assumptions* are made while using the Euler's theory.

- The column is initially perfectly straight and has uniform cross-section throughout its length.
- The material of the column is homogeneous and isotropic.
- The column fails due to buckling (bending) only and the effect of direct compression is neglected.
- The column has very large length as compared to its lateral dimensions.
- Hooke's law is obeyed by the material of the column *i.e.*, the load acting on the column are within elastic limit.
- The column is subjected to perfectly axial load.
- The weight of the column is not considered.

The following *end conditions* are used for the long columns in the Euler's formula.

- Both ends of the column are hinged or pinned.
- Both ends are fixed.
- One end is fixed and other end hinged.
- One end is fixed and other end free.

The *equivalent length* or the *effective length* of the column is defined as the distance between the adjacent points of inflexion on the elastic curves. The point of inflexion is positioned near the end of the column. The effective column length according to different end conditions are given in Table 11.1.

Table 11.1 Effective length of Columns

End conditions	***Effective length (l_e)***
• Both ends hinged/pinned	$l_e = l$ (l is the actual length of the column)
• Both ends fixed	$l_e = l/2$
• One end fixed and other end hinged	$l_e = \dfrac{l}{\sqrt{2}}$
• One end fixed and other end free	$l_e = 2l$

11.4.1 Euler's Formula (when Both Ends of the Column are Hinged or Pinned)

Consider a long column AB of length l loaded with an axial compressive load P and hinged (fixed in position but not in direction) at A and B (Fig. 11.1).

Let P be the critical load which is responsible for buckling of the column and y be the lateral deflection at a distance x from the end B.

Moment at the distance x is

$$M = -Py$$

The differential equation of flexure is

$$EI\frac{d^2y}{dx^2} = M$$

$$EI\frac{d^2y}{dx^2} = -Py$$

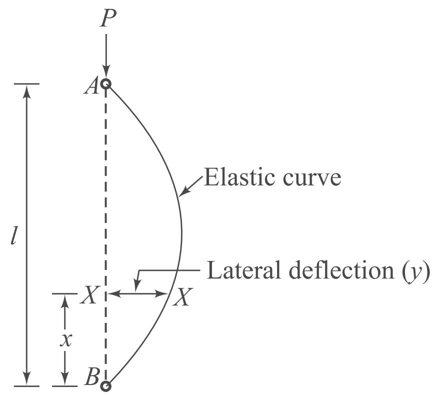


Fig. 11.1

$$EI\frac{d^2y}{dx^2} + Py = 0$$

or $$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = 0 \qquad ...(11.1)$$

Solving the above equation, we get

$$y = C_1\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2\sin\left(\sqrt{\frac{P}{EI}}\,x\right) \qquad ...(11.2)$$

where C_1 and C_2 are the constants of integration.

The boundary condition is:

At B, where $x = 0$, $y = 0$

On substituting boundary condition in equation (11.12), we have

$$0 = C_1\cos\left(\sqrt{\frac{P}{EI}}\times 0\right) + C_2\sin\left(\sqrt{\frac{P}{EI}}\times 0\right)$$

$$= C_1\times 1 + C_2\times 0 = C_1$$

which gives $$C_1 = 0$$

Also at A, where $x = l$, $y = 0$

On substituting boundary condition in equation (11.12), we get

$$0 = C_2\sin\left(\sqrt{\frac{P}{EI}}\,l\right)$$

For LHS to be zero, either C_2 or $\sin\left(\sqrt{\frac{P}{EI}}\,l\right)$ are required to be zero. If C_2 is zero, no deflection occurs in the column, and hence C_2 can not be zero.

$$\sin\left(\sqrt{\frac{P}{EI}}\,l\right) = 0$$

which means $$\sqrt{\frac{P}{EI}}\,l = 0, \pi, 2\pi, 3\pi, \ldots$$

Zero value is not admissible and values other than π carry no practical significance.

$$\sqrt{\frac{P}{EI}}\,l = \pi$$

Squaring both sides, we have

$$\frac{P}{EI}l^2 = \pi^2$$

or $$P = \frac{\pi^2 EI}{l^2} = P_{cr} \qquad \text{... (11.3)}$$

This is the required expression for the load (critical), which buckles the given column with both of its ends hinged.

The lateral deflection is maximum at $x = \frac{l}{2}$.

11.4.2 Euler's Formula (when Both Ends of the Column are Fixed)

Consider a column AB of length l fixed (both in position and direction) at both ends and is subjected to a crippling load P (Fig. 11.2). Since the ends are fixed, hence they have zero slope and zero deflection. But there are restraint moments say M at each end.

To derive the required formula, a section XX is considered at a distance x from B, where the lateral deflection due to applied load is y. The net bending moment at x is $M - Py$.

Hence, $$EI\frac{d^2y}{dx^2} = M - Py$$

$$\frac{d^2y}{dx^2} = \frac{M}{EI} - \frac{P}{EI}y$$

or $$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = \frac{M}{EI} \qquad \text{...(11.4)}$$

The general solution of the above equation is

$$y = C_1\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2\sin\left(\sqrt{\frac{P}{EI}}\,x\right) + \frac{M}{P} \qquad \text{...(11.5)}$$

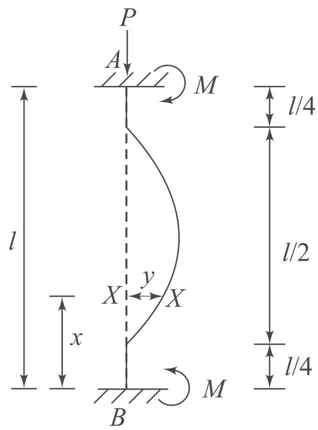


Fig. 11.2

where C_1 and C_2 are the constants of integration.

Differentiating equation (11.5) w.r.t. x, we get

$$\frac{dy}{dx} = -C_1\sqrt{\frac{P}{EI}}\sin\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}\,x\right) \quad ...(11.6)$$

The boundary condition is:

At B, where $x = 0$, $\frac{dy}{dx} = 0$

From equation (11.6), we have

$$0 = 0 + C_2\sqrt{\frac{P}{EI}}$$

or $$C_2\sqrt{\frac{P}{EI}} = 0$$

But $$\sqrt{\frac{P}{EI}} \neq 0, \text{ as } P \neq 0$$

or $$C_2 = 0$$

Also at B, where $x = 0$, $y = 0$.

From equation (11.5), we have

$$0 = C_1 + \frac{M}{P}$$

or $$C_1 = -\frac{M}{P}$$

Equation (11.5) on substituting C_1 and C_2 becomes

$$y = -\frac{M}{P}\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + \frac{M}{P} \quad ...(11.7)$$

And at A, where $x = l$, $y = 0$

$$0 = -\frac{M}{P}\cos\left(\sqrt{\frac{P}{EI}}\,l\right) + \frac{M}{P}$$

$$\cos\left(\sqrt{\frac{P}{EI}}\,l\right) = 1$$

or
$$\sqrt{\frac{P}{EI}}\,l = 0, 2\pi, 4\pi, 6\pi, \ldots$$

Taking the least significant value, we have

$$\sqrt{\frac{P}{EI}}\,l = 2\pi$$

Squaring both sides, we have

$$\frac{P}{EI}l^2 = 4\pi^2$$

or
$$P = \frac{4\pi^2 EI}{l^2} = P_{cr} \qquad \ldots(11.8)$$

It is the required expression for the critical load which buckles the column. This equation is same as for a column of length $\frac{l}{2}$ with both ends hinged. At the same time, on comparing equation (11.8) with equation (11.3), we find that the buckling load for a column with both ends fixed is four times as compared to a column with both ends hinged.

11.4.3 Euler's Formula (when One End of the Column is Fixed and Other End Hinged)

Consider a column AB of length l which is hinged at A but fixed at B (Fig. 11.3). Let M be restraint moment at B. To counter this, another moment is considered by applying a horizontal force F at point A.

The bending moment at a distance x from B, where the lateral deflection is y, is $F(l - x) - Py$.

Hence,
$$EI\frac{d^2y}{dx^2} = F(l - x) - Py$$

or
$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{F(l - x)}{EI} \qquad \ldots(11.9)$$

The general solution of the above equation is

$$y = C_1\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2\sin\left(\sqrt{\frac{P}{EI}}\,x\right) + \frac{F}{P}(l - x) \qquad \ldots(11.10)$$

where C_1 and C_2 are the constants of integration.

The boundary condition is:

At B, where $x = 0$, $y = 0$.

From equation (11.10), we have

$$0 = C_1 + \frac{Fl}{P}$$

or
$$C_1 = -\frac{Fl}{P}$$

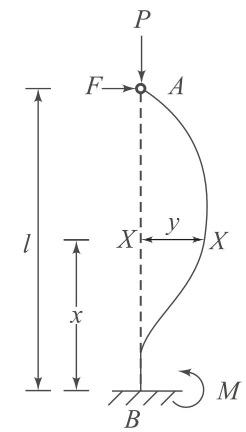


Fig. 11.3

Differentiating equation (11.10) w.r.t. x, we get

$$\frac{dy}{dx} = -C_1\sqrt{\frac{P}{EI}}\sin\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2\sqrt{\frac{P}{EI}}\;\cos\left(\sqrt{\frac{P}{EI}}\,x\right) - \frac{F}{P} \qquad ...(11.11)$$

Also at B, where $x = 0$, $\dfrac{dy}{dx} = 0$.

From equation (11.11), we have

$$0 = C_2\sqrt{\frac{P}{EI}} - \frac{F}{P}$$

or
$$C_2 = \frac{F}{P}\sqrt{\frac{EI}{P}}$$

And at A, where $x = l$, $y = 0$

Equation (11.10) on substituting the above boundary condition, and C_1 and C_2 reduces to

$$0 = -\frac{Fl}{P}\cos\left(\sqrt{\frac{P}{EI}}\,l\right) + \frac{F}{P}\sqrt{\frac{EI}{P}}\,\sin\left(\sqrt{\frac{P}{EI}}\,l\right)$$

$$\frac{Fl}{P}\cos\left(\sqrt{\frac{P}{EI}}\,l\right) = \frac{F}{P}\sqrt{\frac{EI}{P}}\,\sin\left(\sqrt{\frac{P}{EI}}\,l\right)$$

$$l \cos\left(\sqrt{\frac{P}{EI}}\, l\right) = \sqrt{\frac{EI}{P}} \sin\left(\sqrt{\frac{P}{EI}}\, l\right)$$

or $$\tan\left(\sqrt{\frac{P}{EI}}\, l\right) = \sqrt{\frac{P}{EI}}\, l \qquad ...(11.12)$$

There are two possible values of $\left(\sqrt{\frac{P}{EI}}\, l\right)$ to hold the above relationship good. These values are 0 and 4.493 radian.

But $$\sqrt{\frac{P}{EI}}\, l \neq 0, \text{ as } P \neq 0$$

or $$\sqrt{\frac{P}{EI}}\, l = 4.493$$

Squaring both sides, we have

$$\frac{P}{EI} l^2 = 20.187 = 2\pi^2 \text{ (approx.)}$$

or $$P = \frac{2\pi^2 EI}{l^2} = P_{cr} \qquad ...(11.13)$$

On comparing this equation with equation (11.3), we find that buckling load is two times the load required when both ends of the column are hinged or pinned. In other words, a column of effective length $\frac{l}{\sqrt{2}}$ with both ends hinged will have the same crippling load.

11.4.4 Euler's Formula (when One End of the Column is Fixed and Other End Free)

Consider a column AB of length l which is fixed at B but free (both in position and in direction) at A (Fig. 11.4).

The bending moment at a distance x from B, where the lateral deflection is y, is $P\,(d - y)$, d being the deflection at the free end.

Hence, $$EI \frac{d^2 y}{dx^2} = P\,(d - y) = Pd - Py$$

or $$\frac{d^2 y}{dx^2} + \frac{Py}{EI} = \frac{Pd}{EI} \qquad ...(11.14)$$

The general solution of the above equation is

$$y = C_1 \cos\left(\sqrt{\frac{P}{EI}}\, x\right) + C_2 \sin\left(\sqrt{\frac{P}{EI}}\, x\right) + d \qquad ...(11.15)$$

where C_1 and C_2 are the constants of integration.

The boundary condition is:

At B, where $x = 0$, $y = 0$

From equation (11.15), we have

$$0 = C_1 + d$$

or $$C_1 = -d$$

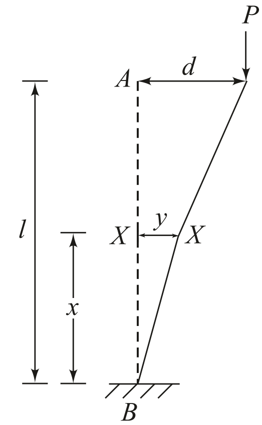


Fig. 11.4

Differentiating equation (11.15), w.r.t. x, we get

$$\frac{dy}{dx} = -C_1\sqrt{\frac{P}{EI}}\sin\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}\,x\right) \quad ...(11.16)$$

Also at B, where $x = 0$, $\frac{dy}{dx} = 0$

From equation (11.16), we have

$$0 = d\sqrt{\frac{P}{EI}} \times 0 + C_2\sqrt{\frac{P}{EI}}$$

$$C_2\sqrt{\frac{P}{EI}} = 0$$

But $\sqrt{\frac{P}{EI}} \neq 0$, as $P \neq 0$ or $C_2 = 0$

Equation (11.15) on substituting C_1 and C_2 reduces to

$$y = -d\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + d \quad ...(11.17)$$

And at A, where $x = l$, $y = d$

Equation (11.17) on substituting the above boundary condition becomes

$$d = -d\cos\left(\sqrt{\frac{P}{EI}}\,l\right) + d$$

$$d\cos\left(\sqrt{\frac{P}{EI}}\,l\right) = 0$$

But $$d \neq 0$$

$$\cos\left(\sqrt{\frac{P}{EI}}\,l\right) = 0$$

or $$\sqrt{\frac{P}{EI}}\,l = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Taking the least significant value, we have

$$\sqrt{\frac{P}{EI}}\,l = \frac{\pi}{2}$$

Squaring both sides, we get

$$\frac{P}{EI}l^2 = \frac{\pi^2}{4} \quad \text{or} \quad P = \frac{\pi^2 EI}{4l^2} = P_{cr} \qquad \text{...(11.18)}$$

Comparing this equation with equation (11.3), we find that crippling load in this case is one-fourth of the load, when a column with both ends hinged, is used. In other words, a column of effective length ($2l$) with both ends hinged will have the same crippling load.

11.4.5 Crippling Stress

The value of the stress corresponding to the critical or crippling load is called crippling stress or critical stress and is denoted by σ_{cr}.

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{l^2 A} \qquad \text{(using equation (11.3))}$$

$$= \frac{\pi^2 EAr^2}{l^2 A} \qquad (I = Ar^2, \text{ where } r = \text{Radius of gyration})$$

$$= \frac{\pi^2 E}{(l/r)^2} \qquad \text{...(11.19)}$$

Where (l/r) is called the *slenderness ratio* of the column. If the slenderness ratio is based on the effective length of the column, it is known as effective slenderness ratio, (l_e/r). The equation (11.19) shows that crippling stress varies inversely proportional to the square of the slenderness ratio, meaning thereby, higher the slenderness ratio, smaller is the crippling stress. Hence minimum value of the radius of gyration, r should be used in the calculation of slenderness ratio and the crippling stress in a column. A crippling stress versus slenderness ratio plot for structural steel is shown in Fig. 11.5.

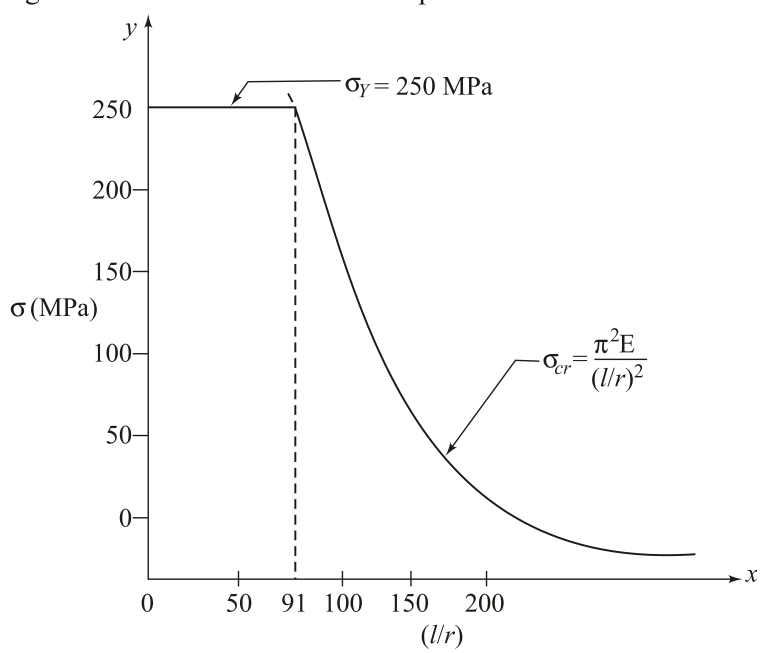


Fig. 11.5 σ_{cr} versus (l/r) plot for structural steel.

If the crippling stress σ_{cr} is higher than the yield strength σ_Y, then it is of no interest, because the column will yield in compression and cease to be elastic before it has a chance to buckle.

11.4.6 Limitations of Euler's Formula

From the Fig. 11.5, it is clear that for higher value of crippling stress, slenderness ratio is low. At the same time, the crippling stress of the column cannot be more than its yield strength. Hence, it puts a limitation on the use of Euler's formula and the formula is not very useful if the slenderness ratio is less than a particular value.

Using the condition that the crippling stress should be less than the yield strength, for structural steel, we have

$$\sigma_{cr} \leq \sigma_Y$$

$$\frac{\pi^2 E}{(l/r)^2} \leq 250 \times 10^6 \qquad \text{(using equation (11.19))}$$

$$\frac{\pi^2 \times 210 \times 10^9}{(l/r)^2} \leq 250 \times 10^6 \qquad \text{(using } E = 210 \text{ GPa)}$$

$$\frac{\pi^2 \times 210 \times 10^9}{250 \times 10^6} \leq (l/r)^2$$

$$8290.46 \leq (l/r)^2$$

$$(l/r) \geq 91 \qquad \text{...(11.20)}$$

Hence, the Euler's formula for a structural steel column with its both ends hinged is valid only when the slenderness ratio (l/r) is greater than or equal to 91.

Example 11.1

A pin-ended square cross-section column of length 3 m is subjected to a compressive stress of 10 MPa. Using a factor of safety of 2.5, find the cross-section, if the column is to safely support (*a*) a 100 kN load and (*b*) a 200 kN load. Take $E = 15$ GPa.

Solution: Given,

Length of the column, $l = 3$ m

Compressive stress on the column, $\sigma = 10$ MPa

Factor of safety = 2.5

(*a*) The crippling load is given as

$$P = \text{Factor of safety} \times \text{Safe load}$$
$$= 2.5 \times 100$$
$$= 250 \text{ kN}$$

Using equation (11.3), we have

$$P = \frac{\pi^2 EI}{l^2}$$

or $$I = \frac{Pl^2}{\pi^2 E}$$

$$= \frac{250 \times 10^3 \times 3^2}{\pi^2 \times 15 \times 10^9} \text{ m}^4$$

$$= 1.52 \times 10^{-5} \text{ m}^4$$

I for a square cross-section of side a is given as

$$I = \frac{a^4}{12}$$

$$= 1.52 \times 10^{-5}$$

or $$a = 0.1162 \text{ m}$$

$$= 116.2 \text{ mm}$$

Hence, the acceptable cross-section is 117 mm × 117 mm. **Ans.**

Check

The normal stress produced in the column is given as

$$\sigma_d = \frac{100}{(0.117)^2} \times \frac{1}{10^3} \text{ MPa}$$

$$= 7.3 \text{ MPa}$$

Since $\sigma' < 10$ MPa, hence the calculated cross-section is within safe limit and is acceptable.

(*b*) Crippling load is

$$P = 2.5 \times 200$$

$$= 500 \text{ kN}$$

The moment of inertia I is given as

$$I = \frac{500 \times 10^3 \times 3^2}{\pi^2 \times 15 \times 10^9} \text{ m}^4$$

$$= 3.04 \times 10^{-5} \text{ m}^4$$

$$= \frac{a^4}{12}$$

or $$a = 0.1382 \text{ m}$$

$$= 138.2 \text{ mm}$$

Check

The normal stress produced in the column is given as

$$\sigma_d = \frac{200}{(0.1382)^2} \times \frac{1}{10^3} \text{ MPa} = 10.47 \text{ MPa}$$

Since $\sigma' > 10$ MPa, the permissible compressive stress in the column, hence the above cross-section is not acceptable. In that case, cross-section is found on the basis of given permissible compressive stress in the column.

Area of the cross-section is

$$A = \frac{200}{10} \times \frac{1}{10^3} \text{ m}^2$$

$$= 0.02 \text{ m}^2$$

or $$a = \sqrt{A} = 0.1414 \text{ m}$$

$$= 141.4 \text{ mm}$$

Hence, the acceptable cross-section is 142 mm × 142 mm. **Ans.**

Example 11.2

A structural steel column is in the form of a tube of thickness 15 mm and external diameter 250 mm and is 3 m long. Its one end is fixed while the other end is free. Find the maximum axial load to be applied on the column. How this load varies with other end conditions? Take E = 200 GPa.

Solution: Given,

Thickness of the column, $t = 15$ mm

External diameter, $d_0 = 250$ mm

Length of the column, $l = 3$ m

Let d_i be the internal diameter of the column (Fig. 11.6).

Now $$d_i + 2t = d_0$$

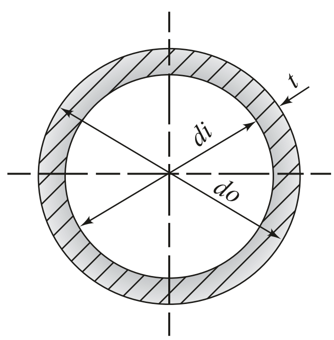


Fig. 11.6

or $$d_i = d_0 - 2t$$

$$= 250 - 2 \times 15 = 220 \text{ mm}$$

The moment of inertia of the cross-section of the column is given as

$$I = \frac{\pi}{64}\left[\left(\frac{250}{1000}\right)^4 - \left(\frac{220}{1000}\right)^4\right] \text{m}^4 = 7.67 \times 10^{-5} \text{ m}^4$$

The crippling load, when one end of the column is fixed while the other end free, is given as

$$P = \frac{\pi^2 EI}{4l^2} \qquad \text{(using equation (11.18))}$$

$$= \frac{\pi^2 \times 200 \times 10^9 \times 7.67 \times 10^{-5}}{4 \times 3^2} \times \frac{1}{10^3} \text{ kN}$$

$$= 4205.54 \text{ kN}$$ **Ans.**

The crippling loads with other end conditions are given as:

End conditions	***Crippling load***
Both ends hinged	$P = \frac{\pi^2 EI}{l^2}$ $= \frac{\pi^2 \times 200 \times 10^9 \times 7.67 \times 10^{-5}}{3^2} \times \frac{1}{10^3}$ kN $= 16822.2$ kN **Ans.**
Both ends fixed	$P = \frac{4\pi^2 EI}{l^2}$ $= 4 \times \left(\frac{\pi^2 EI}{l^2}\right)$ $= 4 \times 16822.2$ kN $= 67288.8$ kN **Ans.**
One end fixed and other end hinged	$P = \frac{2\pi^2 EI}{l^2}$ $= 2 \times \left(\frac{\pi^2 EI}{l^2}\right)$ $= 2 \times 16822.2 = 33644.4$ kN **Ans.**

Example 11.3

A steel column of diameter 60 mm and hinged at both ends is subjected to an axial load. Calculate the minimum length of the column for which Euler's formula is valid. Take $E = 200$ GPa and limit of proportionality = 220 MPa.

Solution: Given,

Diameter of the column, $d = 60 \text{ mm} = \frac{60}{1000} \text{ m} = 0.06 \text{ m}$

The moment of inertia of the cross-section of the column is given as

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64} \times (0.06)^4 \text{ m}^4$$

$$= 6.36 \times 10^{-7} \text{ m}^4$$

The Euler's formula under the given end condition is given as

$$P = \frac{\pi^2 EI}{l^2} \qquad \text{(using equation (11.3))}$$

$$\sigma \times \frac{\pi}{4} d^2 = \frac{\pi^2 EI}{l^2}$$

or

$$l = \sqrt{\frac{4\pi\, EI}{\sigma\, d^2}}$$

$$= \sqrt{\frac{4\pi \times 200 \times 10^9 \times 6.36 \times 10^{-7}}{220 \times 10^6 \times (0.06)^2}} \text{ m} = 1.42 \text{ m.} \qquad \textbf{Ans.}$$

Example 11.4

A *T*-section column of length 2 m with flange dimension 100 mm × 20 mm and web dimension 120 mm × 20 mm is subjected to an axial load. Find the crippling load using Euler's formula, assuming that the column is hinged at both ends. Take E = 200 GPa.

Solution: Refer Fig. 11.7.

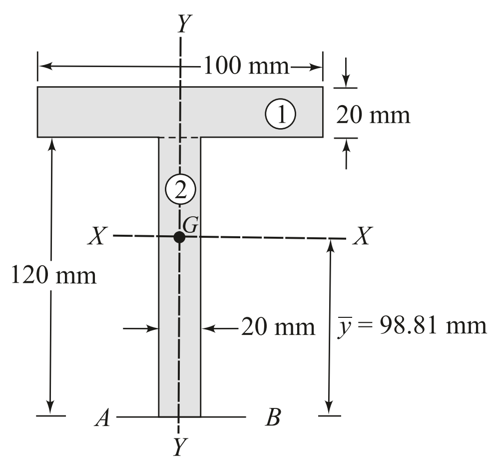


Fig. 11.7

Given, Length of the column, l = 2 m

Young's modulus, E = 200 GPa = 200×10^9 Pa

The centroid G of the section lies on YY-axis, because it is the axis of symmetry.

Calculation of moment of inertia *I*

Area of (1), $a_1 = 100 \times 20 = 2000 \text{ mm}^2$

Area of (2), $a_2 = 120 \times 20 = 2400 \text{ mm}^2$

Distance of the centroid of (1) from AB is

$$y_1 = \left(120 + \frac{20}{2}\right) \text{mm} = 130 \text{ mm}$$

The distance of the centroid of (2) from AB is

$$y_2 = \frac{120}{2} \text{ mm} = 60 \text{ mm}$$

The distance of the centroid of the whole section from AB is

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$$= \frac{2000 \times 130 + 2400 \times 60}{2000 + 2400} = 98.81 \text{ mm}$$

The moment of inertia of the whole section about XX is given as

$$I_{XX} = \left[\frac{1}{12} \times 100 \times 20^3 + 100 \times 20 \times (130 - 98.81)^2\right]$$

$$+ \left[\frac{1}{12} \times 20 \times 120^3 + 120 \times 20 \times (98.81 - 60)^2\right] \text{mm}^4$$

$$= 8507217.5 \text{ mm}^4 = 8.507 \times 10^{-6} \text{ m}^4$$

The moment of inertia of the whole section about YY is given as

$$I_{YY} = \left(\frac{1}{12} \times 20 \times 100^3 + \frac{1}{12} \times 120 \times 20^3\right) \text{mm}^4$$

$$= 1746666.7 \text{ mm}^4$$

$$= 1.746 \times 10^{-6} \text{ m}^4$$

Now $$I_{min} = I_{YY} = 1.746 \times 10^{-6} \text{ m}^4$$

The Euler's formula under the given end condition is given as

$$P = \frac{\pi^2 E I_{min}}{l^2} \quad \text{(using equation (11.3))}$$

$$= \frac{\pi^2 \times 200 \times 10^9 \times 1.746 \times 10^{-6}}{2^2} \times \frac{1}{10^3} \text{ kN}$$

$$= 861.6 \text{ kN}$$

Hence, the crippling load is 861.6 kN. **Ans.**

11.5 EMPIRICAL FORMULAE

For long columns, where slenderness ratio is large, failure is closely predicted by Euler's formula, and the value of crippling stress σ_{cr} is observed to depend on the modulus of elasticity E of the material used, but not on its yield strength σ_{yp}. Failure of very short columns and compression blocks occurs essentially as a result of yield, and $\sigma_{cr} = \sigma_{yp}$. Medium columns fail due to the combined effect of σ_Y and E. In this range, column failure is an extremely complex phenomenon, and test data have been used extensively to guide the development of specifications and design formulae.

Empirical relations express the allowable stress or critical stress in terms of effective slenderness ratio. Since a single formula is usually not adequate for all values of $\left(\frac{l_e}{r}\right)$, different formulae, each with a definite range of applicability, have been developed for various materials. Some typical empirical formulae are:

- Rankine-Gordon Formula
- Johnston's Parabolic Formula
- Straight Line Formula

11.5.1 Rankine-Gordon Formula

It is also known as Rankine's formula, named after a Scottish civil engineer William John Macquorn Rankine (1820 -1872) who gave the final formula in this regard. The formula is equally valid for short columns as well as long columns.

The Rankine's formula is

$$\frac{1}{P_r} = \frac{1}{P_c} + \frac{1}{P} \quad \text{...(11.21)}$$

where P_r = Crippling load according to the Rankine's formula

P_c = Direct crushing load to cause the failure

= $\sigma_c . A$ (σ_c is the maximum allowable compressive stress for the column material, which is oftenly the yield stress σ_Y and A is the cross-sectional area of the column)

P = Crippling load according to the Euler's formula

$$= \frac{\pi^2 EI}{l_e^2} \quad \text{(using equation (11.3))}$$

Hence,

$$\frac{1}{P_r} = \frac{1}{\sigma_c\ A} + \frac{1}{\left(\frac{\pi^2\ EI}{l_e^2}\right)}$$

$$= \frac{1}{\sigma_c\ A} + \frac{1}{\left(\frac{\pi^2\ EAr^2}{l_e^2}\right)} \quad (I = Ar^2)$$

$$= \frac{1}{\sigma_c\ A} + \frac{1}{\pi^2\ EA}\left(\frac{l_e}{r}\right)^2$$

$$= \frac{1}{A}\left[\frac{1}{\sigma_c} + \frac{1}{\pi^2\ E}\left(\frac{l_e}{r}\right)^2\right]$$

or
$$\frac{A}{P_r} = \frac{1}{\sigma_c} + \frac{1}{\pi^2 E}\left(\frac{l_e}{r}\right)^2$$

$$= \frac{1+\left(\frac{\sigma_c}{\pi^2 E}\right)\left(\frac{l_e}{r}\right)^2}{\sigma_c} = \frac{1+K_1\left(\frac{l_e}{r}\right)^2}{\sigma_c}$$

where
$$K_1 = \frac{\sigma_c}{\pi^2 E} = \text{Constant for a particular column material}$$

or
$$\frac{P_r}{A} = \sigma_r = \frac{\sigma_C}{1+K_1\left(\frac{l_e}{r}\right)^2} \qquad \text{...(11.22)}$$

where σ_r = Rankine's critical stress for the column material

or
$$P_r = \frac{\sigma_c A}{1+K_1\left(\frac{l_e}{r}\right)^2} \qquad \text{...(11.23)}$$

Values of σ_c and K_1 for pinned ends columns made of different materials are given in Table 11.2. The stresses are the ultimate (crushing) stresses, which need to be divided by a factor of safety.

Table 11.2 Values of σ_c and K_1

Column material	σ_c ***(MPa)***	$K_1 = \sigma_c/\pi^2 E$
Mild steel	320	1/7500
Cast iron	550	1/1600
Wrought iron	250	1/9000
Timber	40	1/3000

11.5.2 Johnston's Parabolic Formula

This formula was developed by an American civil engineer B.G. Johnston.

According to this formula, the Johnston's crippling load P_j is given as

$$P_j = A\left[\sigma_c - K_2\left(\frac{l_e}{r}\right)^2\right] \qquad \text{...(11.24)}$$

$$\frac{P_j}{A} = \sigma_j = \text{Johnson's critical stress}$$

$$= \sigma_c - K_2\left(\frac{l_e}{r}\right)^2 \qquad \text{...(11.25)}$$

where K_2 = Constant for a particular column material

$$= \frac{\sigma_c^2}{4\pi^2 E} = \frac{\sigma_c^2}{64E} \text{ for pinned ends}$$

11.5.3 Straight Line Formula

According to the straight line formula, the crippling load is given as

$$P = A\left[\sigma_c - K_3\left(\frac{l_e}{r}\right)\right] \quad ...(11.26)$$

where K_3 = Constant depending upon column material

The crippling stress is

$$\sigma_{st} = \frac{P}{A} = \left[\sigma_c - K_3\left(\frac{l_e}{r}\right)\right] \quad ...(11.27)$$

The values of K_2 and K_3 are given in Table 11.3.

Table 11.3 Values of K_2 and K_3

Column material	σ_c ***(MPa)***	K_2	K_3
Mild steel	320	0.000057	0.0053
Cast iron	550	0.00016	0.008
Wrought iron	250	0.000039	0.0053

The above stated three empirical formulae used to approximate test data are shown in Fig. 11.8.

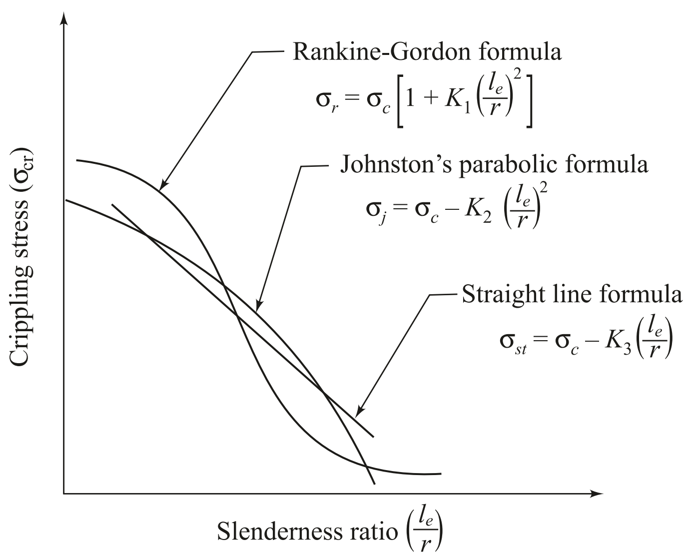


Fig. 11.8

11.6 IS CODE FORMULA (IS: 800-1962)

This is another empirical formula, which is an Indian standard.

(*a*) For slenderness ratio $\left(\frac{l_e}{r}\right)$ varying between 0 and 160

The allowable average axial compressive stress, σ_c for the above range of slenderness ratio is given as

$$\sigma_c = \frac{(\sigma_{yp}/n)}{1+0.20\sec\left[\left(\frac{l_e}{r}\right)\times\sqrt{\frac{n\times\sigma_c'}{4E}}\right]} \quad \text{...(11.28)}$$

where σ_c' = Stress obtained from secant formula

σ_{yp} = Guaranteed minimum yield stress

= 260 N/mm^2 for mild steel

n = Factor of safety = 1.68

(*b*) For slenderness ratio ≥ 160

For this range of slenderness ratio, σ_c is given as

$$\sigma_c = \sigma_c'\left[1.2-\frac{1}{800}\left(\frac{l_e}{r}\right)\right] \quad \text{...(11.29)}$$

11.7 SECANT FORMULA (FOR ECCENTRIC LOADING)

In earlier cases we considered buckling of columns due to centric axial compressive loading. But the load applied to a column is never perfectly centric, rather it is eccentric *i.e.*, the load is acting at a certain distance away from the axis.

Here the given eccentric load is replaced by a centric load P and a couple of moment $M_A = Pe$ (Fig. 11.9).

Consider a section at a distance x from the top end A, where the lateral deflection is y.

Bending moment at the section $= M_A + Py = Pe + Py$

Using differential equation of flexure, we have

$$EI\frac{d^2y}{dx^2} = -P(y+e) \quad \text{...(11.30)}$$

$$\frac{d^2y}{dx^2} = -\frac{P}{EI}(y+e)$$

Substituting $\frac{P}{EI} = K^2$, we have

$$\frac{d^2y}{dx^2} + K^2 y = -K^2 e \qquad \text{...(11.31)}$$

The general solution of equation (11.31) is

$$y = C_1 \sin Kx + C_2 \cos Kx - e \qquad \text{...(11.32)}$$

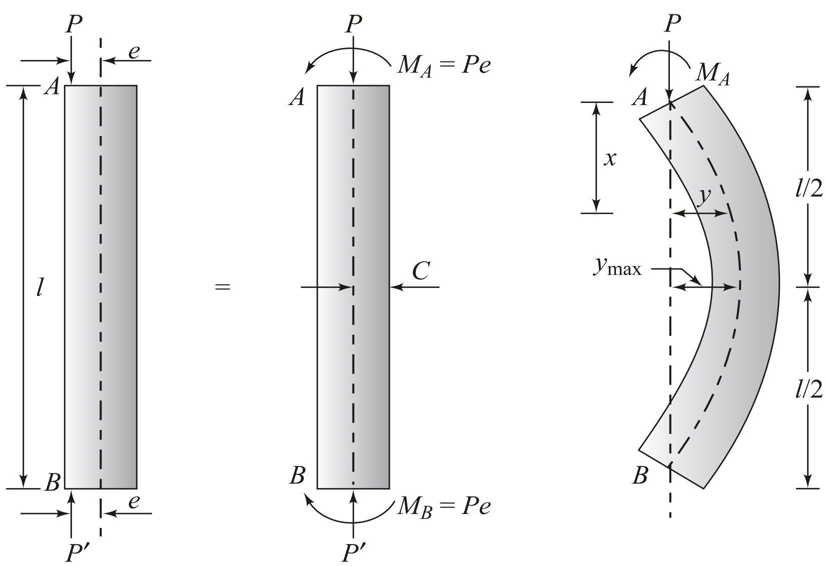


Fig. 11.9

where C_1 and C_2 are the constants of integration.

The boundary conditions are:

At A, where $x = 0$, $\quad y = 0$

$$0 = 0 + C_2 - e$$

or $\quad C_2 = e$

And at B, where $x = l$, $\quad y = 0$

$$0 = C_1 \sin Kl + e \cos Kl - e$$

$$C_1 \sin Kl = e\,(1 - \cos Kl)$$

$$C_1\, 2 \sin\frac{Kl}{2} \cos\frac{Kl}{2} = e\, 2 \sin^2\frac{Kl}{2}$$

$$C_1 \cos\frac{Kl}{2} = e \sin\frac{Kl}{2}$$

or $\quad C_1 = e \tan\frac{Kl}{2}$

Hence, equation (11.32) on substituting C_1 and C_2 becomes

$$y = e \tan\frac{Kl}{2} \sin Kx + e \cos Kx - e$$

$$= e\left[\tan\frac{Kl}{2}\sin Kx + \cos Kx - 1\right] \quad ...(11.33)$$

Deflection is maximum, when $x = \dfrac{l}{2}$

$$y_{\max} = e\left[\tan\frac{Kl}{2}\sin\left(\frac{Kl}{2}\right) + \cos\left(\frac{Kl}{2}\right) - 1\right]$$

$$= e\left[\frac{\sin^2\left(\frac{Kl}{2}\right) + \cos^2\left(\frac{Kl}{2}\right)}{\cos\left(\frac{Kl}{2}\right)} - 1\right] = e\left[\sec\left(\frac{Kl}{2}\right) - 1\right] \quad ...(11.34)$$

Restoring $K = \sqrt{\dfrac{P}{EI}}$, we have

$$y_{\max} = e\left[\sec\left(\sqrt{\frac{P}{EI}}\,\frac{l}{2}\right) - 1\right] \quad ...(11.35)$$

From the above equation it is observed that $y_{\max}$ becomes infinite, when

$$\sqrt{\frac{P}{EI}}\,\frac{l}{2} = \frac{\pi}{2} \quad ...(11.36)$$

But it does not happen actually and the deflection is kept to a practical value by not allowing P to reach a critical value which satisfies equation (11.36).

Hence, from equation (11.36), we have

$$P = \frac{\pi^2 EI}{l^2} = P_{cr} \quad ...(11.37)$$

This is same as equation (11.3) for a column under a centric load.

From equation (11.37), we have

$$EI = \frac{P_{cr}\, l^2}{\pi^2}$$

Substituting EI in equation (11.35), we get

$$y_{\max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \quad ...(11.38)$$

The maximum stress occurs, where the bending moment is maximum *i.e.*, at mid-point $\left(x = \dfrac{l}{2}\right)$, and is the sum of the normal stresses due to axial load (direct stress) and due to bending couple at that section (bending stress).

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max} \times C}{I} \quad ...(11.39)$$

where C is the distance of the outermost fibre of the column from its neutral axis.

$$M_{max} = Py_{max} + M_A$$

$$= P\,(y_{max} + e) \quad (M_A = Pe)$$

and $$I = Ar^2$$

Substituting M_{max} and I in equation (11.39), we have

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{(y_{max} + e)\,C}{r^2}\right] \quad ...(11.40)$$

Substituting y_{max} from equation (11.35), we get

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{eC}{r^2}\sec\left(\sqrt{\frac{P}{EI}}\,\frac{l}{2}\right)\right] \quad ...(11.41)$$

The above equation gives the maximum stress induced in a column with its both ends hinged and an eccentricity e. The formula can be used for other end conditions with suitable effective length.

Using $I = Ar^2$ and taking effective length condition, equation (11.41) modifies to

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{eC}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\,\frac{l_e}{r}\right)\right] \quad ...(11.42)$$

This is known as secant formula, due to *sec* term present in the above equation.

It is valid for all types of columns and for any value of slenderness ratio $\left(\frac{l_e}{r}\right)$.

Another equation for σ_{max} containing critical load is

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{eC}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right] \quad ...(11.43)$$

Equation (11.43) is obtained by substituting equation (11.38) in equation (11.40).

Example 11.5

A compressive member of length 2.5 m has outside and inside diameters of 40 mm and 25 mm respectively. Determine the crippling load using Rankine's formula assuming both ends fixed. The crushing stress of the material is 500 N/mm^2 and Rankine's constant is $\frac{1}{650}$.

Solution: Given,

Length of the member, $l = 2.5$ m

Outside diameter, $d_o = 40$ mm

Inside diameter, $d_i = 25$ mm

Constant, $K_1 = \dfrac{1}{650}$

$\sigma_c = 500$ N/mm^2

The cross-sectional area, $A = \dfrac{\pi}{4}(40^2 - 25^2)$

$= 243.75\pi$ mm^2

The moment of inertia of the cross-section is given as

$$I = \frac{\pi}{64}(40^4 - 25^4) = 33896.48\pi \text{ mm}^4$$

Now $I = Ar^2$

or $r = \sqrt{\dfrac{I}{A}} = \sqrt{\dfrac{33896.48\pi}{243.75\pi}}$

$= 11.79$ mm

The effective length of the member is $l_e = \dfrac{l}{2} = \dfrac{2.5}{2} = 1.25$ m

According to the Rankine's formula, the crippling load is

$$P_r = \frac{\sigma_c A}{1 + K_1\left(\dfrac{l_e}{r}\right)^2}$$

$$= \frac{500 \times 243.75\pi}{1 + \dfrac{1}{650} \times \left(\dfrac{1.25 \times 1000}{11.79}\right)^2}$$

$= 20930.1$ N

$= 20.93$ kN **Ans.**

Example 11.6

A solid circular rod of diameter 60 mm and length 2 m is used as a column. One of the ends of the column is fixed while the other end free. Taking a factor of safety of 3, find the safe load using (*a*) Rankine-Gordon formula and (*b*) Euler's formula. Take $E = 120$ GPa.

Rankine's constant $= \dfrac{1}{600}$ and the yield stress $= 300$ N/mm^2.

Solution: Given,

Diameter of the column, $d = 60$ mm

Length of the column, $l = 2$ m

Factor of safety, $n = 3$

Rankine's constant, $K_1 = \dfrac{1}{600}$

Crushing stress, $\sigma_c = 300\ \text{N/mm}^2$

Equivalent length of the column,

$$l_e = 2l = 2 \times 2\ \text{m} = 4\ \text{m}$$

Cross-sectional area of the column is

$$A = \frac{\pi}{4} \times 60^2 = 2827.43\ \text{mm}^2$$

Moment of inertia, $I = \dfrac{\pi}{64} \times 60^4 = 636172.51\ \text{mm}^4$

Now $I = Ar^2$, r being radius of gyration

or $r = \sqrt{\dfrac{I}{A}} = \sqrt{\dfrac{636172.51}{2827.43}} = 15\ \text{mm}$

According to the Rankine's formula, the crippling load is

$$P_r = \frac{\sigma_c A}{1 + K_1 \left(\dfrac{l_e}{r}\right)^2}$$

$$= \frac{300 \times 2827.43}{1 + \dfrac{1}{600} \times \left(\dfrac{4 \times 1000}{15}\right)^2}$$

$$= 7097\ \text{N} = 7.097\ \text{kN}$$

Rankine's safe load $= \dfrac{7.097}{3} = 2.36\ \text{kN}$ **Ans.**

According to the Euler's formula, the crippling load is

$$P = \frac{\pi^2 EI}{l_e^2}$$

$$= \frac{\pi^2 \times 120 \times 10^9 \times 636172.51 \times 10^{-12}}{4^2}\ \text{N}$$

$$= 47090\ \text{N} = 47.09\ \text{kN}$$

Euler's safe load $= \dfrac{47.09}{3} = 15.7\ \text{kN}$ **Ans.**

Example 11.7

Find the greatest length of a column of cross-section 50 mm by 50 mm with its one end fixed and the other end hinged to carry a working load of 50 kN. Assume a factor of safety of 3. Take $K_1 = \dfrac{1}{1600}$ and the yield stress = 400 N/mm^2.

Solution: Given,

Factor of safety, $n = 3$

Working load, $P = 50$ kN

Buckling load, $P_r = P \times n = 50 \times 3 = 150$ kN

Cross-sectional area, $A = 50 \times 50 = 2500 \text{ mm}^2$

Moment of inertia, $I = \dfrac{50^4}{12} = 520833.33 \text{ mm}^4$

Now $I = Ar^2$, r being radius of gyration.

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{520833.33}{2500}} = 14.43 \text{ mm}$$

According to the Rankine's formula, the buckling load is

$$P_r = \frac{\sigma_c A}{1 + K_1\left(\dfrac{l_e}{r}\right)^2}$$

$$150 \times 10^3 = \frac{400 \times 2500}{1 + \dfrac{1}{1600} \times \left(\dfrac{l_e}{14.43}\right)^2}$$

On solving, we get $l_e = 1374 \text{ mm} = 1.374 \text{ m}$

Under the given end condition

Effective length, $l_e = \dfrac{\text{Actual length}}{\sqrt{2}}$

Actual length of the column $= \sqrt{2}\, l_e = 1.94$ m **Ans.**

Example 11.8

A tubular column of length 3 m has one end fixed and other free (Fig. 11.10). Its cross-sectional area is 2500 mm^2 and the moment of inertia 3.8×10^6 mm^4. (*a*) Find the allowable centric load for the column and the corresponding normal stress using Euler's formula assuming a factor of safety of 2. (*b*) If the allowable load obtained in (*a*) is applied eccentrically at a distance of 20 mm from the axis of the column, determine the lateral deflection of the top of the column and the maximum normal stress in the column. Take $E = 210$ GPa and $C = 60$ mm.

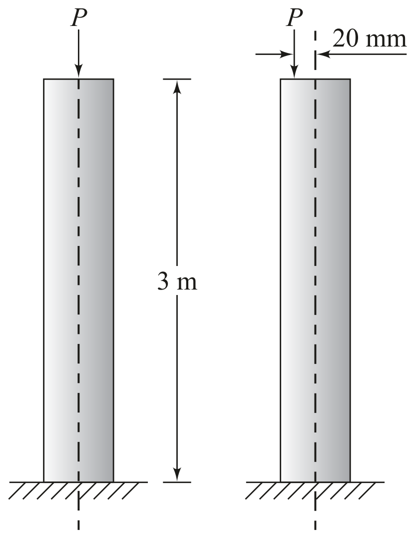


Fig. 11.10

Solution: Given,

Length of the column, $l = 3$ m

Cross-sectional area, $A = 2500$ mm^2

Moment of inertia, $I = 3.8 \times 10^6$ mm^4

Factor of safety, $n = 2$

Eccentricity, $e = 20$ mm

Distance of the outermost fibre from the neutral axis,

$$C = 60 \text{ mm}$$

Modulus of elasticity, $E = 210 \text{ GPa} = 120 \times 10^9$ Pa

Under the given end condition, the effective length is

$$l_e = 2l = 2 \times 3 = 6 \text{ m}$$

The radius of gyration r is given as

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{3.8 \times 10^6}{2500}} = 39 \text{ mm}$$

Using the Euler's formula, the buckling load for the column is

$$P_{cr} = \frac{\pi^2 EI}{l_e^2}$$

$$= \frac{\pi^2 \times 210 \times 10^9 \times 3.8 \times 10^6 \times 10^{-12}}{6^2} \times \frac{1}{10^3} \text{ kN}$$

$$= 218.8 \text{ kN}$$

Allowable centric load $= \dfrac{P_{cr}}{n}$

$= \dfrac{218.8}{2}$ kN = 109.4 kN **Ans.**

Allowable normal stress $= \dfrac{109.4 \times 10^3}{2500}$ N/mm^2

= 43.76 N/mm^2 = 43.76 MPa **Ans.**

(*b*) Using Secant formula, the maximum lateral deflection of the top of the column is given as

$$y_{\max} = e\left[\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} - 1\right] \qquad \text{(using equation (11.38))}$$

$$= 20\left[\sec\left(\frac{\pi}{2} \times \sqrt{\frac{109.4}{218.8}}\right) - 1\right] \text{mm}$$

= 20 (2.252 – 1) mm = 25 mm **Ans.**

Since the lateral deflection occurs under the allowable load, hence it is the maximum deflection.

The maximum normal stress in the column is given as

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{eC}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right] \qquad \text{(using equation (11.43))}$$

$$= \frac{109.4 \times 10^3}{2500}\left[1 + \frac{20 \times 60}{39^2}\sec\left(\frac{\pi}{2} \times \sqrt{\frac{109.4}{218.8}}\right)\right]$$

= 121.51 N/mm^2 = 121.51 MPa **Ans.**

Example 11.9

A circular rod of diameter 35 mm and length 750 mm is subjected to an axial compressive load of 90 kN at an eccentric distance of *e* from the axis of the rod. Assuming the lateral deflection at the mid-point of the rod to be 0.75 mm, find (*a*) the distance *e*, and (*b*) the maximum stress in the rod. Take E = 200 GPa.

Solution: Given,

Diameter of the rod, d = 35 mm

Length of the rod, l = 750 mm

Axial load on the rod, P = 90 kN

(*a*) The rod is supposed to be hinged at both ends.

Effective length $l_e = l$ = 750 mm

Since the lateral deflection occurs at the midpoint of the rod, hence it is the maximum deflection.

$$y_{max} = 0.75 \text{ mm}$$

Using equation (11.35), we have

$$y_{max} = e\left[\sec\left(\sqrt{\frac{P}{EI}}\frac{l}{2}\right) - 1\right]$$

In the above equation, the effective length is used.

Now $$\sqrt{\frac{P}{EI}} \times \frac{l}{2} = \sqrt{\frac{90 \times 10^3}{200 \times 10^9 \times \frac{\pi}{64} \times \left(\frac{35}{1000}\right)^4}} \times \frac{750}{2 \times 1000} = 0.926 \text{ radian} = 53.1^\circ$$

Hence, $$y_{max} = e[\sec 53.1^\circ - 1]$$

or $$e = \frac{y_{max}}{(\sec 53.1^\circ - 1)} = \frac{0.75}{(1.665 - 1)}$$

$$= 1.13 \text{ mm}$$ **Ans.**

(*b*) Moment of inertia of the section is

$$I = \frac{\pi}{64} d^4$$

$$= \frac{\pi}{64} \times 35^4 \text{ mm}^4$$

$$= 73661.75 \text{ mm}^4$$

Cross-sectional area, $$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times 35^2 = 962.11 \text{ mm}^2$$

Radius of gyration, $$r = \sqrt{\frac{I}{A}}$$

$$= \sqrt{\frac{73661.75}{962.11}} = 8.75 \text{ mm}$$

The maximum stress in the rod is given by equation (11.41).

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{eC}{r^2} \sec\left(\sqrt{\frac{P}{EI}}\frac{l}{2}\right)\right]$$

$$= \frac{90 \times 10^3}{962.11}\left[1 + 1.13 \times \left(\frac{35}{2}\right) \times \frac{1}{(8.75)^2} \times \sec 53.1^\circ\right] \qquad \left(C = \frac{35}{2}\right)$$

$$= 133.78 \text{ N/mm}^2$$

$$= 133.78 \text{ MPa}$$ **Ans.**

Example 11.10

A square rod of length 0.5 m and cross-section 25 mm by 25 mm is acting as a column whose one end is fixed and the other end free. A load of 20 kN is acting at an eccentric distance of 5 mm from the axis of the rod. Determine (*a*) the lateral deflection of the top of the column and (*b*) the maximum stress in the rod. Take E = 100 GPa.

Solution: Given,

Length of the column, $l = 0.5$ m

Load on the column, $P = 20$ kN

Eccentricity, $e = 5$ mm

Cross-sectional area of the column is

$$A = 25 \text{ mm} \times 25 \text{ mm} = 625 \text{ mm}^2$$

Moment of inertia, $I = \dfrac{25^4}{12} = 32552 \text{ mm}^4$

Radius of gyration, $r = \sqrt{\dfrac{I}{A}} = \sqrt{\dfrac{32552}{625}} = 7.21$ mm

Effective length of the column, $l_e = 2l = 2 \times 0.5$ m = 1 m

Now $$\sqrt{\frac{P}{EI}}\frac{l}{2} = \sqrt{\frac{20 \times 10^3}{100 \times 10^9 \times 32552 \times 10^{-12}}} \times \frac{1}{2}$$

(here effective length is used)

$$= 1.239 \text{ radian} = 71^\circ$$

(*a*) The lateral deflection of the top of the column is given as

$$y_{\max} = e\left[\sec\left(\sqrt{\frac{P}{EI}}\frac{l}{2}\right) - 1\right]$$

$$= 5\,[\sec 71^\circ - 1]$$

$$= 10.36 \text{ mm}$$ **Ans.**

(*b*) The maximum stress in the rod is given as

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{eC}{r^2}\sec\left(\sqrt{\frac{P}{EI}}\frac{l}{2}\right)\right]$$

$$= \frac{20 \times 10^3}{625}\left[1 + 5 \times \left(\frac{25}{2}\right) \times \frac{1}{(7.21)^2} \times \sec 71^\circ\right] \qquad \left(C = \frac{25}{2}\right)$$

$$= 150.2 \text{ N/mm}^2$$

$$= 150.2 \text{ MPa}$$ **Ans.**

SHORT ANSWER QUESTIONS

1. What is the difference between column and beam?
2. What is slenderness ratio? State its values for short and long columns.
3. What is equivalent length of a column? What is its value for a pin-ended long column?
4. What is meant by end conditions for a column?
5. What is crippling stress? What is its significance?
6. For which type of column, the Euler's formula is applicable?
7. What is the suitable material for a column?
8. How does the Johnston parabolic formula differ from the Rankine-Gordon formula?
9. Why is it difficult to predict the failure of an intermediate column?
10. What is Secant formula? Why is it so named? For which type of loading it is used?
11. What is meant by eccentric loading on a column?
12. What is the Indian standard for a column's failure?

MULTIPLE CHOICE QUESTIONS

1. A structural member subjected to an axial compressive load is called a

(*a*) strut (*b*) beam (*c*) shaft (*d*) lever.

2. The Euler's formula is valid for

(*a*) short column (*b*) medium column

(*c*) long column (*d*) short and long columns both.

3. The Rankine-Gordon formula is valid for

(*a*) short column (*b*) medium column

(*c*) long column (*d*) short and long columns both.

4. The Euler's critical load for a column with both ends hinged is

(*a*) $\dfrac{4\pi^2 EI}{l^2}$ (*b*) $\dfrac{2\pi^2 EI}{l^2}$

(*c*) $\dfrac{\pi^2 EI}{4l^2}$ (*d*) $\dfrac{\pi^2 EI}{l^2}$.

5. The effective length of a column with both ends fixed is equal to

(*a*) the actual length (*b*) one-half of the actual length

(*c*) two times the actual length (*d*) $\dfrac{1}{\sqrt{2}}$ times the actual length.

6. The ratio of equivalent length of a column to its least radius of gyration is known as

(*a*) factor of safety (*b*) Poisson's ratio

(*c*) slenderness ratio (*d*) moment of inertia.

7. The slenderness ratio is less than 30 for

(*a*) long columns (*b*) short columns

(*c*) medium columns (*d*) short and medium columns both.

8. A short column fails mainly due to

(*a*) buckling

(*b*) compressive stress

(*c*) combined effect of buckling and compressive stress

(*d*) tensile stress.

9. The radius of gyration of a circular section of diameter 50 mm is

(*a*) 25 mm (*b*) 50 mm (*c*) 12.5 mm (*d*) 20 mm.

10. The crippling stress varies

(*a*) directly proportional to the slenderness ratio

(*b*) inversely proportional to the slenderness ratio

(*c*) inversely proportional to the cubic power of the slenderness ratio

(*d*) inversely proportional to the square of the slenderness ratio.

11. For Euler's formula to be valid, the crippling stress of the column is

(*a*) more than its yield strength

(*b*) less than its yield strength

(*c*) equal to its yield strength

(*d*) equal to its ultimate strength.

12. For a long column, the slenderness ratio is greater than

(*a*) 30 (*b*) 90

(*c*) 120 (*d*) 200.

ANSWERS

1. (*a*) **2.** (*c*) **3.** (*b*) **4.** (*d*) **5.** (*b*) **6.** (*c*) **7.** (*b*) **8.** (*b*) **9.** (*c*)

10. (*d*) **11.** (*b*) **12.** (*c*).

EXERCISES

1. The design of a hollow steel rod requires a cross-sectional area of 1000 mm^2. Knowing that the effective length of the rod is 3 m, determine the critical load, when the ratio of outer diameter to inner diameter is 2. Take E = 200 GPa. (*Ans.* 29 kN)

2. The cross-sections of the two brass rods used as compression members, each of 2 m effective length are shown in Fig. 11.11. Find the following:

(*a*) the wall thickness of the hollow square rod for which the rods have the same cross-sectional area

(*b*) using E = 100 GPa, determine the critical load of each rod.

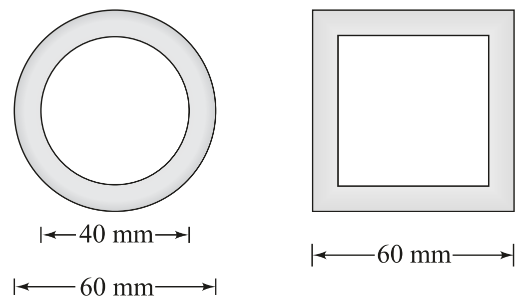


Fig. 11.11

(*Ans.* (*a*) 7.47 mm; (*b*) 126 kN, 181.7 kN)

3. Find the dimension d so that the two columns shown in Fig. 11.12 has the same weight. Also, find the crippling loads for the two columns.

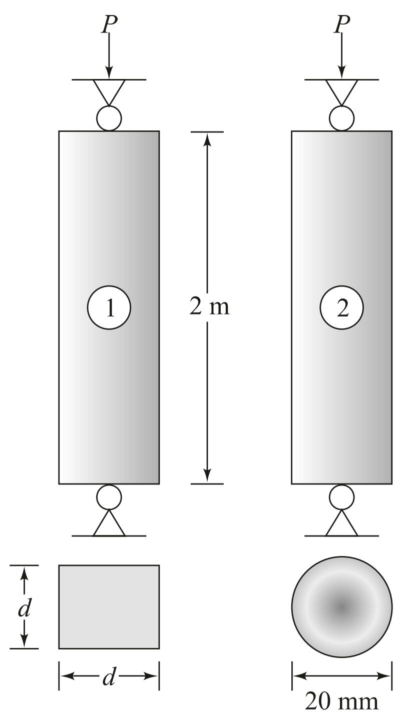


Fig. 11.12

Take $E_1 = 210$ GPa

$E_2 = 80$ GPa

$\rho_1 = 6 \times 10^3$ kg/m^3

$\rho_2 = 3 \times 10^3$ kg/m^3

(*Ans.* 12.53 mm, 1.06 kN, 1.55 kN).

4. A circular rod shown in Fig. 11.13 is used as a compressive member. Find (*a*) the deflection of the top of the column and (*b*) the maximum stress in the rod. Take $E = 150$ GPa.

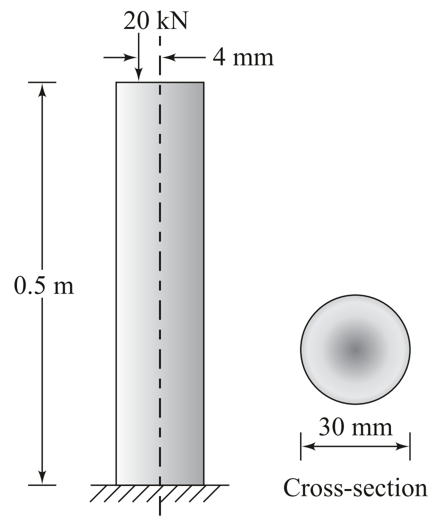


Fig. 11.13

(*Ans.* (*a*) 2.56 mm, 77.82 MPa).

5. A square rod of length 0.65 m and cross-section 20 mm × 20 mm is acting as a column whose both ends are hinged. A load of 15 kN is acting at an eccentric distance *e* and the lateral deflection of the top of the column is 3 mm. Find (*a*) the distance *e* and (*b*) the maximum stress in the rod. Take $E = 210$ GPa.

(*Ans.* 8.12 mm, 162.75 MPa).

6. A circular rod of length 3 m is acting as a column and is hinged at both ends (Fig. 11.14). It is subjected to an axial load *P* at an eccentricity of 15 mm. Find the load *P* which will buckle the column. Also, find *P* at which the maximum allowable stress reaches 120 MPa.

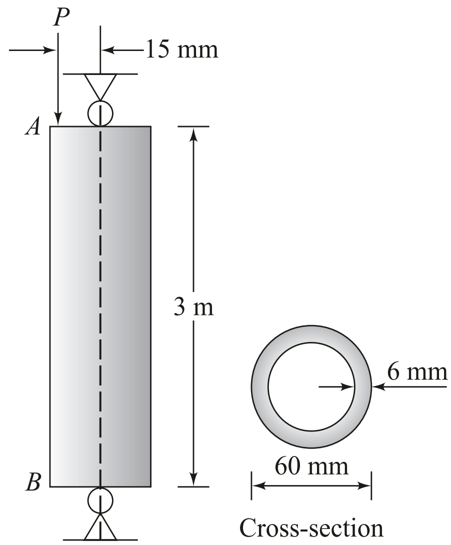


Fig. 11.14

(*Ans.* 85.3 kN, 36.4 kN).

7. Find the shortest length of a hinged-hinged steel column having a rectangular cross-section 60 mm × 100 mm, for which the elastic Euler's formula applies. Take yield strength and modulus of elasticity for steel as 250 MPa and 200 GPa respectively. (*Ans.* 1.54 m).

8. A round steel rod of 15 mm diameter and 2 m long is subjected to a gradually increasing axial compressive load. Using Euler's formula, find the buckling load. Also find the maximum lateral deflection corresponding to buckling condition. Both ends of the column can be taken as hinged. Assume Young's modulus for steel as equal to 200 GPa and the yield stress as 240 N/mm^2.

(*Ans.* 1226.3 N, 80.52 mm).

9. A vertical column 6 m high is fixed at base and a clockwise moment of 1.4 kN·m is applied at the top of the column. A horizontal force *P* is applied to the colum at a height of 3 m above the base so as to give a counter clockwise moment. Determine the value of *P* so that the horizontal deflections at the top of the colum and at the point of application of *P* shall be equal (*i*) when the defections are on the same side (*ii*) when the deflections are on the opposite sides of the vertical line through the foot of the column. (*Ans.* 1.4 kN, 1.0 kN).

10. A tubular steel strut is 6.5 cm external diameter and 5 cm internal diameter. It is 2.5 m long and has hinged ends. The load is parallel to the axis but is eccentric. Find the maximum eccentricity for a crippling load of 0.75 of the Euler value. The yield stress is 320 MPa and E = 210 GPa.

(*Ans.* 5.57 mm).

11. A straight length of steel bar, 1.5 m long and 20 mm × 5 mm section is compressed longitudinally until it buckles. Assuming that the Euler's formula apply to this case, estimate the maximum central deflection before the steel passes the yield point of 320 MPa. Take E = 210 GPa.

(*Ans.* 138 mm).

❑❑❑

12 Pressure Vessels

Gabriel Lame
(1795-1870)

Gabriel Lame, born on 22 July 1795, was a noted French mathematician who made original contributions to the fields of mathematics, physics, thermodynamics and applied mechanics. His greatest contribution to mathematics was the introduction of curvilinear coordinates, and their use in pure and applied mathematics. Lame used his new coordinate system to transform Laplace's equation into ellipsoidal coordinates to a form where the variables were separable, and solve the resulting form of the generalized Laplace's equation. Gauss considered Lame the foremost French mathematician of his time who has been immortalized in French society along with Laplace and Pascal. Lame is most famous for his study of classes of ellipse-like curves, known as Lame curves. His significant contribution to engineering was to accurately define the stresses and capabilities of a press fit joint, as found in a dowel pin in a housing. The Lame's constants used in mechanics of solids, although not defined by Lame, were named after his death, to honour his contributions to the field of mechanics. In the field of civil engineering, he is credited with planning roads, highways and bridges that were built in Petersburg in Russia in 1820s. In 1854, he was elected a foreign member of the Royal Swedish Academy of Sciences. He is one of the greatest French scientists whose name is inscribed on the Eiffel Tower.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- How are thin and thick pressure vessels classified?
- What is hoop stress? How is it related to longitudinal stress?
- What is the significance of Lame's constant?
- What is the purpose of a compound cylinder?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_12

12.1 INTRODUCTION

Pressure vessels are used to store liquids and gases under pressure. They can be thin-walled or thick-walled depending upon their diameter-thickness (d/t) ratio. If this ratio is equal to or greater than 20, then the pressure vessels are categorised as thin-walled, whereas thick-walled pressure vessels have this ratio less than 20. Most commonly used pressure vessels are cylindrical and spherical pressure vessels. Typical examples of pressure vessels include pipes, tanks, boilers and reservoirs.

12.2 STRESSES IN A THIN CYLINDRICAL SHELL

The distribution of stress in thin-walled vessels is constant across the thickness, as their inside and outside radii are nearly equal. Consider a thin cylindrical vessel of inside radius r and wall thickness t containing a fluid of gauge pressure p, as shown in Fig. 12.1 (a).

The shell is subjected to two types of stresses, namely longitudinal stress (σ_l) acting along the axis of the shell and hoop or circumferential stress (σ_h) acting along its circumference.

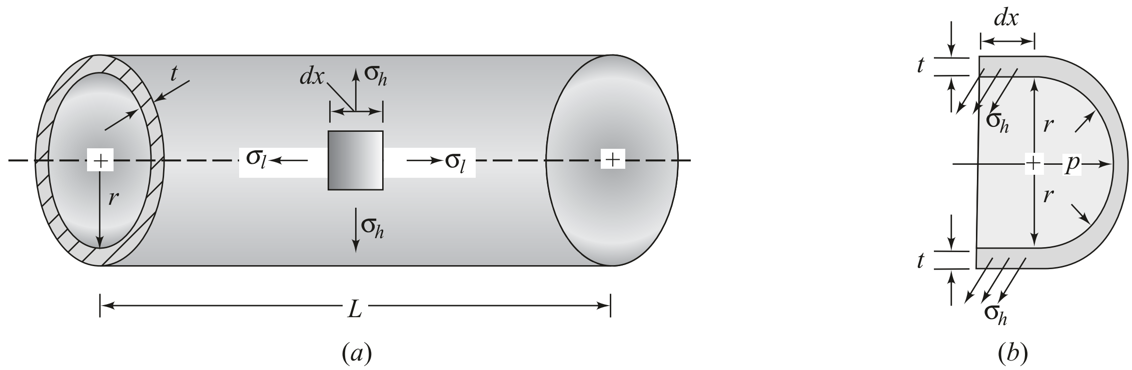


Fig. 12.1

Now consider a small element of length dx of the wall of the vessel (Fig. 12.1(b)). Because of the axis symmetry of the vessel, no shear stress is acting on the element. Hence, σ_h and σ_l are treated as principal stresses, where σ_h is the major principal stress and σ_l, the minor principal stress.

Considering the equilibrium of the element in the transverse direction, that is, in the direction normal to its length, we have

$$\sigma_h\,(2t\,.\,dx) - p\,(2rdx) = 0$$

$$\sigma_h\,t = pr$$

or

$$\sigma_h = \frac{pr}{t} \qquad \text{... (12.1)}$$

$$= \frac{pd}{2t} \qquad \text{...(12.2)}$$

where d is the inside diameter of the shell.

This is the required expression for hoop stress or circumferential stress, which is tensile in nature.

Now considering the equilibrium of the element in the longitudinal direction, that is, along the axis of the vessel, we have

$$\sigma_l\,(2\pi rt) - p\,(\pi r^2) = 0$$

Since the area of the fluid section is πr^2 and that of wall section $2\pi rt$.

or
$$\sigma_l = \frac{pr}{2t} \quad \text{...(12.3)}$$

$$= \frac{pd}{4t} \quad \text{...(12.4)}$$

This is the required expression for the longitudinal stress.

Comparing equations (12.1) and (12.3), we get

$$\sigma_l = \frac{\sigma_h}{2}$$

or
$$\sigma_h = 2\sigma_l \quad \text{...(12.5)}$$

Hence, the hoop stress is twice of the longitudinal stress and both stresses act at right angle to each other.

The two stresses are principal stresses because their planes contain no shear stress. Under that condition, the maximum shear stress is

$$\tau_{\max} = \frac{\sigma_h - \sigma_l}{2} \quad \text{...(12.6)}$$

$$= \frac{\frac{pr}{t} - \frac{pr}{2t}}{2} = \frac{pr}{4t} \quad \text{...(12.7)}$$

$$= \frac{pd}{8t} \quad \text{...(12.8)}$$

There are two types of joints in thin cylindrical shells, one is circumferential joint and another is longitudinal joint.

If η_h = Efficiency of the circumferential joint

η_l = Efficiency of the longitudinal joint

Then
$$\sigma_h = \frac{pd}{2t\eta_l} = \frac{pr}{t\eta_l} \quad \text{...(12.9)}$$

and
$$\sigma_l = \frac{pd}{4t\eta_h} = \frac{pr}{2t\eta_h} \quad \text{...(12.10)}$$

The pressure p used in different equations is the gauge pressure. The following relation exists between gauge pressure and absolute pressure.

Gauge pressure + Atmospheric pressure = Absolute pressure

The design of a thin cylindrical shell is based on hoop stress σ_h, because it is the maximum stress produced in the shell. If σ_w is the allowable stress, then

$$\sigma_h \leq \sigma_w$$

$$\frac{pr}{t} \leq \sigma_w$$

or

$$t \geq \frac{pr}{\sigma_w} \quad \text{...(12.11)}$$

12.3 VOLUMETRIC STRAIN FOR A THIN CYLINDRICAL SHELL

Due to change in volume of a thin cylindrical shell as a result of the two stresses, the volumetric strain is produced in it.

Let l = Length of the shell

r = Inside radius of the shell

t = Thickness of the shell

v = Poisson's ratio

p = Fluid pressure inside the shell

E = Modulus of elasticity of the shell material

The circumferential or hoop strain produced is given as

$$\epsilon_h = \frac{\sigma_h}{E} - v\frac{\sigma_l}{E} \quad \text{...(12.12)}$$

$$= \frac{pd}{2tE} - v\frac{pd}{4tE}$$

$$= \frac{pd}{2tE}\left(1 - \frac{v}{2}\right) = \frac{pd}{4tE} \quad \text{...(12.13)}$$

The change in diameter is connected with the hoop strain, and is given as

$$d(d) = \epsilon_h \times d \quad \text{...(12.14)}$$

The longitudinal strain is given as

$$\epsilon_l = \frac{\sigma_l}{E} - v\frac{\sigma_h}{E} \quad \text{...(12.15)}$$

$$= \frac{pd}{4tE} - v\frac{pd}{2tE} = \frac{pd}{2tE}\left(\frac{1}{2} - v\right) = \frac{pd}{4tE}\,(1 - 2v) \quad \text{...(12.16)}$$

The change in length is connected with the longitudinal strain, and is given as

$$dl = \epsilon_l \times l \quad \text{...(12.17)}$$

The change in volume of the shell occurs due to change in its length and radius. The initial volume of the shell is

$$V = \frac{\pi}{4} d^2 \times l \qquad \text{...(12.18)}$$

The changed volume is

$$V + dV = \frac{\pi}{4} [d + d]^2 \times (l + dl)$$

where dV = Change in volume

Hence, the change in volume dV is

$$(V + dV) - V = \frac{\pi}{2} ld \cdot d(d) + \frac{\pi}{4} d^2 \cdot dl \qquad \text{(neglecting smaller quantities)}$$

$$\text{Volumetric strain} = \frac{\text{Change in volume}}{\text{Initial volume}}$$

or

$$\in_V = \frac{\frac{\pi}{2} ld \cdot d(d) + \frac{\pi}{4} d^2 \cdot dl}{\frac{\pi}{4} d^2 l} = 2 \frac{d(d)}{d} + \frac{dl}{l} \qquad \text{...(12.19)}$$

Now the hoop strain is defined as

$$\in_h = \frac{\text{Change in circumference}}{\text{Initial circumference}}$$

$$= \frac{\text{Changed circumference} - \text{Initial circumference}}{\text{Initial circumference}}$$

$$= \frac{\pi[d + d(d)] - \pi d}{\pi d} = \frac{d(d)}{d} \qquad \text{...(12.20)}$$

Using equation (12.20) in equation (12.19), we have

$$\in_V = 2\in_h + \in_l \qquad \text{...(12.21)}$$

where $\in_l = \frac{dl}{l}$

Now using equations (12.13) and (12.16) in equation (12.21), we get

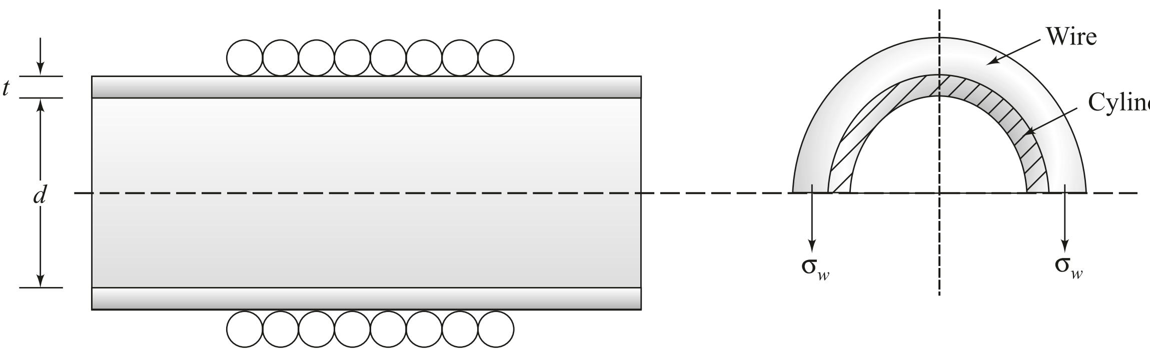
$$\in_V = 2 \times \frac{pd}{4tE}(2 - v) + \frac{pd}{4tE}(1 - 2v)$$

$$= \frac{pd}{4tE}[4 - 2v + 1 - 2v] = \frac{pd}{4tE}(5 - 4v) \qquad \text{...(12.22)}$$

This is the required expression for the volumetric strain for a thin cylindrical shell.

12.4 WIRE WOUND THIN CYLINDERS

To strengthen a thin cylinder against its bursting because of high fluid pressure, steel wires are wound around it (Fig. 12.2). These wires are under tension because of their stretching but they compress the cylinder, producing a negative *i.e.*, compressive circumferential stress which helps to reduce the chance of bursting. The resultant stress in the cylinder is the difference of the circumferential stress produced due to fluid pressure and compressive stress because of wire wound. But the circumferential strains produced in the cylinder and wire are the same.



(*a*) A wire wound cylinder (*b*) Half cross-section

Fig. 12.2

Example 12.1

Water is running through a cast iron pipe of diameter 750 mm at 10 MPa. Calculate the thickness of the pipe, if the maximum permissible stress is 120 MPa.

Solution: Given,

Inside diameter of the pipe, $d = 750$ mm

Pressure of water, $p = 10 \text{ MPa} = 10 \times 10^6$ Pa

Maximum stress, $\sigma_h = 120 \text{ MPa} = 120 \times 10^6$ Pa

Since hoop stress is the maximum stress induced in the pipe, hence using equation (12.2), we have

$$\sigma_h = \frac{pd}{2t}$$

or

$$t = \frac{pd}{2\sigma_h}$$

$$= 10 \times 10^6 \times \left(\frac{750}{1000}\right) \times \frac{1}{2 \times 120 \times 10^6} \text{ m}$$

$$= 0.03125 \text{ m}$$

$$= 31.25 \text{ mm}$$ **Ans.**

Example 12.2

A steel storage tank of wall thickness 5 mm, diameter 8 m and height 25 m is filled with water to a certain height *h*. Assuming a factor of safety of 3.5 and ultimate strength of steel in tension to be 390 MPa, find the value of *h*.

Solution: Given,

Thickness of the tank, $t = 5$ mm

Diameter of the tank, $d = 8$ m

Height of the tank, $h = 25$ m

Ultimate strength of steel, $\sigma_u = 390 \text{ MPa} = 390 \times 10^6$ Pa

Factor of safety, $n = 3.5$

Using equation (12.2), we have

$$\sigma_h = \frac{pd}{2t} = \sigma_u$$

or

$$390 \times 10^6 = \frac{p \times 8}{2 \times (5 \times 10^{-3})}$$

Solving for p, we get

$$p = 487500 \text{ Pa}$$

This is the maximum pressure.

The safe pressure is

$$p_s = \frac{p}{n} = \frac{487500}{3.5} = 139285.71 \text{ Pa}$$

Now

$$p_s = \rho g h$$

$$139285.71 = 1000 \times 9.81 \times h \qquad (\rho \text{ for water} = 1000 \text{ kg/m}^3)$$

On solving, we get $h = 14.2$ m **Ans.**

Example 12.3

A thin cylinder of inside diameter 450 mm is made of 5 mm thick plate. The efficiencies of the longitudinal and circumferential joints are 65% and 35% respectively. Find the largest allowable gauge pressure, if the tensile stress of the plate is limited to 90 MPa.

Solution: Given,

Inside diameter of the cylinder, $d = 450$ mm

Thickness of the cylinder, $t = 5$ mm

Efficiency of the circumferential joint, $\eta_h = 35\%$

Efficiency of the longitudinal joint, $\eta_l = 65\%$

The circumferential stress is given by

$$\sigma_h = \frac{pd}{2t\eta_l} \qquad \text{(using equation (12.9))}$$

$$90 = \frac{p \times 450}{2 \times 5 \times 0.65}$$

or

$$p = 1.3 \text{ MPa}$$

The longitudinal stress is given as

$$\sigma_l = \frac{pd}{4t\eta_h} \qquad \text{(using equation (12.10))}$$

$$90 = \frac{p \times 450}{4 \times 5 \times 0.35}$$

or $$p = 1.4 \text{ MPa}$$

Hence, the largest allowable gauge pressure is 1.3 MPa (lowest of the two values). **Ans.**

Note: If we take p = 1.4 MPa, then the value of σ_h will exceed the tensile stress in the plate, as shown below.

$$\sigma_h = \frac{1.4 \times 450}{2 \times 5 \times 0.65} = 96.92 \text{ MPa, which is greater than 90 MPa.}$$

Example 12.4

A thin cylinder of length 2.5 m and inside diameter 500 mm is made of 5 mm thick steel plate and is closed at both ends. It is subjected to an internal fluid pressure of 1.5 MPa. Find the following parameters:

(*a*) the change in length

(*b*) the final length

(*c*) the change in diameter

(*d*) the final diameter

(*e*) the change in volume

(*f*) the final volume

(*g*) the maximum shear stress induced in the cylinder

Take E = 210 GPa and v = 0.25.

Solution: Given,

Length of the cylinder, l = 2.5 m

Inside diameter of the cylinder, d = 500 mm = 500×10^{-3} m

Thickness of the plate, t = 5 mm = 5×10^{-3} m

Fluid pressure, p = 1.5 MPa = 1.5×10^{6} Pa

Modulus of elasticity, E = 210 GPa = 210×10^{9} Pa

Poisson's ratio, v = 0.25

The hoop stress is given as

$$\sigma_h = \frac{pd}{2t}$$

$$= \frac{1.5 \times 10^{6} \times 500 \times 10^{-3}}{2 \times 5 \times 10^{-3}} \text{ Pa} = 7.5 \times 10^{7} \text{ Pa}$$

The longitudinal stress is given as

$$\sigma_l = \frac{\sigma_h}{2}$$

$$= \frac{7.5\times10^7}{2} = 3.75 \times 10^7 \text{ Pa}$$

The hoop strain is given as

$$\in_h = \frac{\sigma_h}{E} - v\frac{\sigma}{E}$$

$$= \frac{7.5\times10^7}{210\times10^9} - \frac{0.25\times3.75\times10^7}{210\times10^9} = 3.125 \times 10^{-4}$$

The longitudinal strain is given as

$$\in_l = \frac{\sigma_l}{E} - v\frac{\sigma_h}{E}$$

$$= \frac{3.75\times10^7}{210\times10^9} - \frac{0.25\times7.5\times10^7}{210\times10^9} = 8.92 \times 10^{-5}$$

(*a*) Change in length of the cylinder is

$$dl = \in_l \times l$$

$$= 8.92 \times 10^{-5} \times 2.5 \times 10^3 \text{ mm} = 0.223 \text{ mm}$$ **Ans.**

Due to fluid pressure length of the cylinder increases, hence above change is increase in length.

(*b*) Final length of the cylinder

$$l_f = l + dl$$

$$= 2.5 \times 10^3 + 0.223 = 2500.223 \text{ mm}$$ **Ans.**

(*c*) Change in diameter of the cylinder

$$d(d) = \in_h \times d$$

$$= 3.125 \times 10^{-4} \times 500 \text{ mm} = 0.156 \text{ mm}$$ **Ans.**

The diameter of the cylinder increases on account of fluid pressure, hence the change in diameter is basically increase in diameter.

(*d*) Final diameter of the cylinder

$$d_f = d + d\,(d)$$

$$= (500 + 0.156) \text{ mm}$$

$$= 500.156 \text{ mm}$$ **Ans.**

(*e*) The volumetric strain is given as

$$\in_V = 2\in_h + \in_l \qquad \text{(using equation (12.21))}$$

$$= 2 \times 3.125 \times 10^{-4} + 8.92 \times 10^{-5} = 7.142 \times 10^{-4}$$

Volume of the cylinder is

$$V = \frac{\pi}{4} d^2 \times l$$

$$= \frac{\pi}{4}\ 500^2 \times 2.5 \times 10^3 \text{ mm}^3 = 4.908 \times 10^8 \text{ mm}^3$$

Hence, the change in volume of the cylinder is

$$dV = \in_V \times V$$

$$= 7.142 \times 10^{-4} \times 4.908 \times 10^8 \text{ mm}^3 = 350529.36 \text{ mm}^3 \qquad \textbf{Ans.}$$

The change in volume is increase in volume of the cylinder.

(*f*) Final volume of the cylinder is

$$V_f = V + dV$$

$$= 4.908 \times 10^8 + 350529.36 \text{ mm}^3 = 4.911 \times 10^8 \text{ mm}^3 \qquad \textbf{Ans.}$$

(*g*) The maximum shear stress induced in the cylinder is given as

$$\tau_{\max} = \frac{\sigma_h - \sigma_l}{2}$$

$$= \frac{7.5\times10^7 - 3.75\times10^7}{2} \times \frac{1}{10^6} \text{ MPa} = 18.75 \text{ MPa} \qquad \textbf{Ans.}$$

Example 12.5

If the efficiencies of the longitudinal and circumferential joints in Example 12.4 are 70% and 50% respectively, find all the enlisted parameters.

Solution: Given,

Efficiency of the circumferential joint, $\eta_h = 50$ %

Efficiency of the longitudinal joint, $\eta_l = 70$ %

The circumferential stress is gives as

$$\sigma_h = \frac{pd}{2t\eta_l}$$

$$= \frac{1.5\times500\times10^{-3}}{2\times5\times10^{-3}\times0.7} = 107.14 \text{ MPa}$$

The longitudinal stress is given as

$$\sigma_l = \frac{pd}{4t\eta_h}$$

$$= \frac{1.5\times500\times10^{-3}}{4\times5\times10^{-3}\times0.5} = 75 \text{ MPa}$$

Circumferential strain is

$$\in_h = \frac{\sigma_h}{E} - v\frac{\sigma_l}{E}$$

$$= \frac{107.14\times10^6}{210\times10^9} - \frac{0.25\times75\times10^6}{210\times10^9} = 4.2 \times 10^{-4}$$

Longitudinal strain is

$$\in_l = \frac{\sigma_l}{E} - v\frac{\sigma_h}{E}$$

$$= \frac{75\times10^6}{210\times10^9} - \frac{0.25\times107.14\times10^6}{210\times10^9} = 2.3 \times 10^{-4}$$

(*a*) Change in length, $dl = \in_l \times l$

$= 2.3 \times 10^{-4} \times 2.5 \times 10^3$ mm

$= 0.575$ mm (Increase) **Ans.**

(*b*) Final length, $l_f = l + dl$

$= (2.5 \times 10^3 + 0.575)$ mm

$= 2500.575$ mm **Ans.**

(*c*) Change in diameter, $d\,(d) = \in_h \times d$

$= 4.2 \times 10^{-4} \times 500$ mm

$= 0.21$ mm (Increase)

(*d*) Final diameter, $d_f = d + d\,(d)$

$= (500 + 0.21)$ mm

$= 500.21$ mm **Ans.**

(*e*) For change in volume

Volumetric strain, $\in_V = 2\in_h + \in_l$

$= 2 \times 4.2 \times 10^{-4} + 2.3 \times 10^{-4}$

$= 1.07 \times 10^{-3}$

Volume of the cylinder, $V = 4.908 \times 10^8$ mm^3

Hence, the change in volume, $dV = \in_V \times V$

$= 1.07 \times 10^{-3} \times 4.908 \times 10^8$ mm^3

$= 525156$ mm^3 **Ans.**

(*f*) Final volume, $V_f = V + dV$

$= (4.908 \times 10^8 + 525156)$ mm^3

$= 4.913 \times 10^8$ mm^3 **Ans.**

(*g*) Maximum shear stress, $\tau_{max} = \dfrac{\sigma_h - \sigma_l}{2}$

$$= \frac{107.14 - 75}{2} = 16.07 \text{ MPa}$$ **Ans.**

Example 12.6

A compressed air-tank of outside diameter 450 mm is of 4 m length and its wall thickness is 3 mm (Fig. 12.3). It is welded along a helix forming an angle of 20° with the transverse plane and is subjected to a pressure of 3.5 MPa. Find the following:

(*a*) the normal stress perpendicular to the weld

(*b*) the shear stress parallel to the weld.

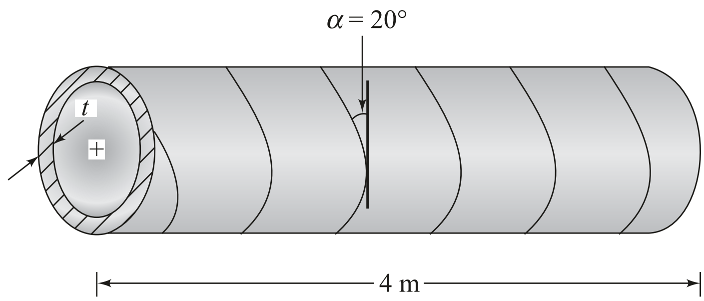


Fig. 12.3

Solution: Given,

Length of the cylinder, $l = 4$ m

Wall thickness, $t = 3$ mm

Outside diameter, $d_0 = 450$ mm

Angle of helix, $\alpha = 20°$

Air pressure, $p = 3.5 \text{ MPa} = 3.5 \times 10^6$ Pa

Inside diameter of the tank is

$$d = d_0 - 2t$$
$$= (450 - 2 \times 3) \text{ mm} = 444 \text{ mm}$$

The circumferential stress is given as

$$\sigma_h = \frac{pd}{2t}$$
$$= \frac{3.5 \times 444 \times 10^{-3}}{2 \times 3 \times 10^{-3}} = 259 \text{ MPa}$$

The longitudinal stress is given as

$$\sigma_l = \frac{\sigma_h}{2}$$
$$= \frac{259}{2} = 129.5 \text{ MPa}$$

The average stress is given as

$$\sigma_{av} = \frac{\sigma_h + \sigma_l}{2}$$

$$= \frac{259 + 129.5}{2} = 194.25 \text{ MPa}$$

σ_h and σ_l are the principal stresses. We draw the Mohr's circle taking these two stresses (Fig. 12.4).

Scale : 2 cm on x-axis $= 64.75$ MPa

Radius of Mohr's circle, $R = 64.75$ MPa

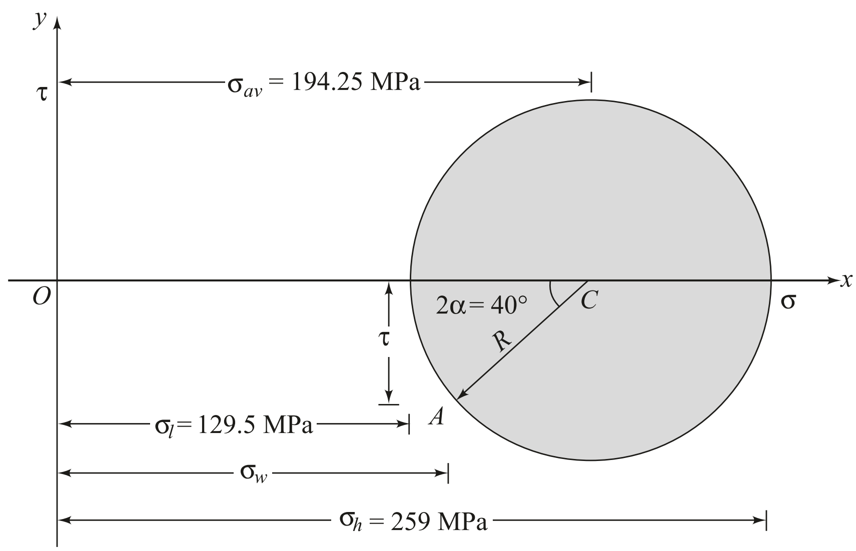


Fig. 12.4

(a) The normal stress perpendicular to the weld is given as

$$\sigma_W = \sigma_{av} - R\cos 2\alpha$$

$$= 194.25 - 64.75 \cos (2 \times 20^\circ) = 144.65 \text{ MPa}$$ **Ans.**

(b) The shear stress parallel to the weld is given as

$$\tau_W = R \sin 2\alpha$$

$$= 64.75 \sin (2 \times 20^\circ) = 41.62 \text{ MPa}$$ **Ans.**

Example 12.7

A thin cylinder of 300 mm inside diameter contains water at 1.5 MPa. Find the thickness of the cylinder, if the circumferential and longitudinal stresses are limited to 30 MPa and 20 MPa respectively.

Solution: Given,

Diameter of the cylinder, $d = 300$ mm

Circumferential stress, $\sigma_h = 30 \text{ MPa} = 30 \times 10^6$ Pa

Longitudinal stress, $\sigma_l = 20 \text{ MPa} = 20 \times 10^6$ Pa

Pressure of water, $p = 1.5 \text{ MPa} = 1.5 \times 10^6$ Pa

The circumferential stress is given as

$$\sigma_h = \frac{pd}{2t}$$

$$30 \times 10^6 = \frac{1.5 \times 10^6 \times 300 \times 10^{-3}}{2 \times t}$$

or $$t = 7.5 \text{ mm}$$

The longitudinal stress is given as

$$\sigma_l = \frac{pd}{4t}$$

$$20 \times 10^6 = \frac{1.5 \times 10^6 \times 300 \times 10^{-3}}{4 \times t}$$

or $$t = 5.62 \text{ mm}$$

Hence, the thickness of the cylinder is 7.5 mm (highest of the two values). **Ans.**

Example 12.8

A cast iron thin cylinder of inside diameter 250 mm and thickness 10 mm is closely wound by a single layer of steel wire of diameter 5 mm under a tension of 60 MPa. Find the stresses induced in the cylinder and the steel wire, if water under a pressure of 4 MPa is admitted in the cylinder.

Take $E_{CI} = 10^5$ MPa, $E_S = 2 \times 10^5$ MPa and $v = 0.25$.

Solution: Given,

Inside diameter of the cylinder, $d = 250$ mm

Thickness of the cylinder, $t = 10$ mm

Diameter of the steel wire, $d_w = 5$ mm

Tensile stress in the steel wire, $\sigma_w = 60 \text{ MPa} = 60 \times 10^6$ Pa

Pressure of water in the cylinder, $p = 4 \text{ MPa} = 4 \times 10^6$ Pa

When there is no water in the cylinder

The compressive force induced in the cylinder is equal to the tensile force in the steel wire. Consider 1 m length of the cylinder.

The compressive force developed in the cylinder is

$$2 \times \sigma_h \times t \times 1$$

where σ_h = Compressive circumferential stress in the cylinder

The tensile force exerted by wire per unit length is

$$2 \times \frac{\pi}{4} d_w^2 \times \sigma_w \times n$$

where n = Number of turns of wire per unit length

$$= \frac{1}{d_w}$$

Equating the two forces for equilibrium and substituting the value of n, we have

$$2\sigma_h \times t = 2 \times \frac{\pi}{4} \times d_w^2 \times \sigma_w \times \frac{1}{d_w}$$

On simplification, we get

$$\sigma_h = \frac{\pi d_w}{4t} \cdot \sigma_w$$

$$= \frac{\pi \times 5 \times 10^{-3}}{4 \times 10 \times 10^{-3}} = 23.56 \text{ MPa}$$

This is the value of compressive stress induced in the cylinder due to wire wound.

When water is admitted in the cylinder

Due to fluid pressure inside the cylinder, further hoop stress is developed and the tensile stress in the wire is also increased. The resultant hoop stress is the sum of the initial compressive stress due to wire wound and further tensile stress due to internal pressure. As a result, the cylinder carries more pressure.

Now suppose the wire wound cylinder is subjected to a fluid pressure p.

Let σ'_l = Londigudinal stress developed in the cylinder

σ'_h = Circumferential stress developed in the cylinder

σ'_w = Stress developed in the wire

$$\text{Longitudinal bursting force} = p \times \frac{\pi}{4} d^2 = \sigma'_l \times \pi dt \qquad \text{(for equilibrium)}$$

or

$$\sigma'_l = \frac{pd}{4t} = \frac{4 \times 10^6 \times 250 \times 10^{-3}}{4 \times 10 \times 10^{-3}} = 25 \times 10^6 \text{ Pa}$$

Diametral bursting force per unit length is

$$p \times d \times 1 = pd \qquad \text{(here } l = 1 \text{ m)}$$

For equilibrium

Bursting force per unit length = Total resisting force per unit length

$$p \times d \times 1 = \sigma'_h \times 2t \times 1 + \sigma'_w \times 2 \times \frac{\pi}{4} \times d_w^2 \times n$$

or

$$pd = \sigma'_h \times 2t + \frac{\pi d_w}{2} \times \sigma'_w \qquad \left(n = \frac{1}{d_w}\right)$$

On substituting different values, we have

$$4 \times 10^6 \times 250 \times 10^{-3} = \sigma'_h + 2 \times 10 \times 10^{-3} + \frac{\pi \times 5 \times 10^{-3}}{2} \times \sigma'_w$$

$$10^6 = 0.02\sigma'_h + 7.85 \times 10^{-3}\,\sigma'_w \qquad \text{...(1)}$$

Now for compatibility, we have

Circumferential strain in the cylinder = Circumferential strain in the wire

$$\frac{\sigma'_h}{E_{CI}} - v\frac{\sigma'_l}{E_{CI}} = \frac{\sigma'_w}{E_s}$$

On substituting different values, we have

$$\frac{\sigma'_h \times 10^6}{10^5 \times 10^6} - \frac{0.25 \times 25 \times 10^6}{10^5 \times 10^6} = \frac{\sigma'_w}{2 \times 10^5 \times 10^6}$$

or $\quad 10\sigma'_h - 0.5 \times 10^{-5}\sigma'_w = 62.5$ ...(2)

Solving equations (1) and (2), we have

$$\sigma'_w = 127.38 \text{ MPa}$$

and $\quad \sigma'_h = 69.94$ MPa

The resultant stress (hoop) in the cylinder is

$$= \sigma'_h - \sigma_h$$

$= (69.94 – 23.56)$ MPa $= 46.38$ MPa (Tensile) **Ans.**

The resultant stress in the wire is

$$= \sigma_w + \sigma'_w$$

$= (60 + 127.38)$ MPa $= 187.38$ MPa (Tensile) **Ans.**

12.5 STRESSES IN A THIN SPHERICAL SHELL

Refer Fig. 12.5.

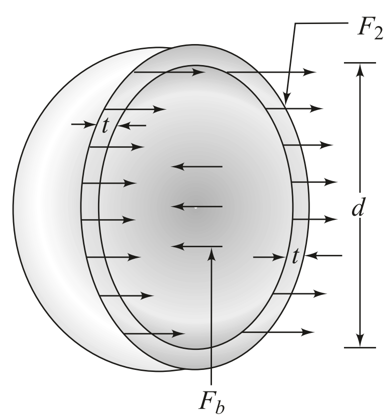


Fig. 12.5

Let $\quad d$ = Inside diameter of the shell

t = Thickness of the shell

p = Fluid pressure inside the shell

The bursting force is

$$F_b = p \times A = p \times \frac{\pi}{4} d^2 \qquad (A = \text{Area of the spherical shell})$$

The resisting force induced in the shell is due to longitudinal stress, given by

$$F_2 = \sigma_l A = \sigma_l \times \pi dt$$

Equating the two forces, we have

$$F_b = F_2$$

$$p \times \frac{\pi}{4} d^2 = \sigma_l \times \pi dt$$

or

$$\sigma_l = \frac{pd}{4t} \qquad \text{...(12.23)}$$

Because of the symmetry of the spherical shell about the two axes, we have

$$\sigma_h = \sigma_l = \sigma = \frac{pd}{4t} \qquad \text{...(12.24)}$$

Hence, the circumferential and longitudinal stresses are equal, and they are the principal stresses. The maximum shear stress in the wall of the shell is not zero, as expected, but is equal to $\frac{\sigma_h}{2} = \frac{pd}{8t}$ which is quite clear, if Mohr's circle is drawn (Fig. 12.6).

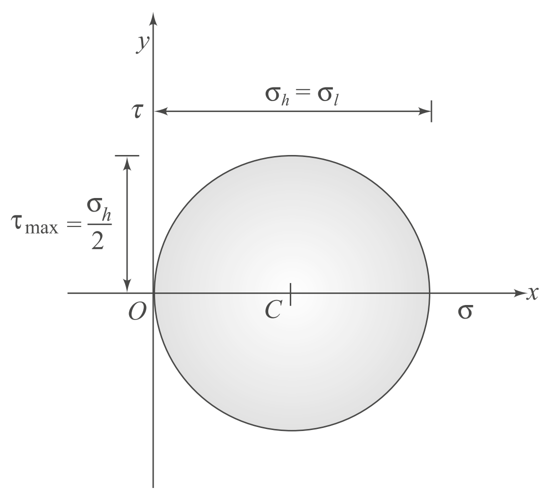


Fig. 12.6

Equation (12.24) is valid for a seamless shell *i.e.*, a shell containing no joints. In case of any joint with joint efficiency η, we have

$$\sigma_h = \sigma_l = \frac{pd}{4t\eta} \qquad \text{...(12.25)}$$

12.6 VOLUMETRIC STRAIN FOR A THIN SPHERICAL SHELL

Let d = Diameter of the shell

t = Thickness of the shell

v = Poisson's ratio

p = Fluid pressure in the shell

E = Modulus of elasticity of the shell material

The initial volume of the spherical shell is given as

$$V = \frac{pd}{4t\eta} \quad \text{...(12.26)}$$

The increased volume of the shell is

$$V + dV = \frac{\pi}{6}[d + d(d)]^3$$

where dV = Increase in volume of the shell

$d(d)$ = Change in diameter of the shell

The volumetric strain is

$$\in_V = \frac{\text{Change in volume}}{\text{Initial volume}}$$

$$= \frac{(V + dV) - V}{V} = \frac{dV}{V}$$

$$= \frac{\frac{\pi}{6}[\{d + d(d)\}^3 - d^3]}{\frac{\pi}{6}d^3}$$

$$= 3\frac{d(d)}{d} \quad \text{(neglecting smaller values)}$$

$$= 3\in_h \quad \text{...(12.27)}$$

where $\in_h = \dfrac{d(d)}{d}$ = Hoop strain

The hoop strain is also expressed as

$$\in_h = \frac{\sigma}{E} - v\frac{\sigma}{E} \quad \text{(using equation (12.12) and replacing } \sigma_h = \sigma_l = \sigma)$$

$$= \frac{pd}{4tE}(1 - v) \quad \text{(using equation (12.24)) ...(12.28)}$$

From equations (12.27) and (12.28), we have

$$\in_V = \frac{dV}{V} = \frac{3pd}{4tE}(1 - v) \quad \text{...(12.29)}$$

Hence, the change (increase) in volume of the shell is

$$dV = \frac{3pd}{4tE}(1 - v) \times V$$

$$= \frac{\pi p d^4}{8tE} \qquad \text{(on substituting } V\text{)} \qquad \dots (12.30)$$

Example 12.9

A thin spherical shell of diameter 500 mm is subjected to an internal pressure of 2 MPa. Find thickness of the shell, if the maximum stress in the shell is limited to 60 MPa. Consider the joint efficiency to be 80%.

Solution: Given,

Diameter of the shell, $d = 500$ mm

Pressure in the shell, $p = 2$ MPa $= 2 \times 10^6$ Pa

Maximum stress induced in the shell, $\sigma_h = 60$ MPa $= 60 \times 10^6$ Pa

Joint efficiency, $\eta = 80\% = 0.8$

From equation (12.25), we have

$$\sigma_h = \frac{pd}{4t\eta}$$

$$60 \times 10^6 = \frac{2 \times 10^6 \times 500 \times 10^{-3}}{4 \times t \times 0.8}$$

Solving for t, we get

$$t = 5.2 \times 10^{-3} \text{ m} = 5.2 \text{ mm}$$ **Ans.**

Example 12.10

A thin spherical shell of diameter 1 m and thickness 5 mm is filled with water at 18∞C and atmospheric pressure. What pressure will be developed if 2×10^5 mm^3 of more water is pumped into the shell ? Also, find the stress developed in the shell material. Take $E = 200$ GPa and $v = 0.25$.

Solution: Given,

Diameter of the shell, $d = 1$ m

Thickness of the shell, $t = 5$ mm

Modulus of elasticity, $E = 200$ GPa $= 200 \times 10^9$ N/m^2

Poisson's ratio, $v = 0.25$

The increased volume of the shell can accommodate the extra amount of water pumped into it.

Using equation (12.30), we have

$$dV = \frac{\pi p d^4}{8tE}(1 - v)$$

$$2 \times 10^5 \times 10^{-9} = \frac{\pi \times p \times 1^4 \times (1 - 0.25)}{8 \times 5 \times 10^{-3} \times 200 \times 10^9}$$

Solving for p, we get

$$p = 6.79 \times 10^5 \text{ N/m}^2 = 0.679 \text{ MPa} \qquad \textbf{Ans.}$$

Using equation (12.24), we have

$$\sigma_h = \sigma_l = \frac{pd}{4t}$$

$$= \frac{6.79 \times 10^5 \times 1}{4 \times 5 \times 10^{-3}} \text{ N/m}^2$$

$$= 3.395 \times 10^7 \text{N/m}^2 = 33.95 \text{ MPa} \qquad \textbf{Ans.}$$

Example 12.11

A thin spherical shell of diameter 1.5 m has wall thickness of 10 mm. Find change in diameter and change in volume of the shell, if water at a pressure of 2 MPa is admitted into it. Take $E = 200$ GPa and $v = 0.3$.

Solution: Given,

Diameter of the shell, $d = 1.5$ m

Wall thickness, $t = 10$ mm

Pressure of water, $p = 2 \text{ MPa} = 2 \times 10^6$ Pa

Modulus of elasticity, $E = 200 \text{ GPa} = 200 \times 10^9$ GPa

Poisson's ratio, $v = 0.3$

The hoop strain in any direction is given as

$$\in_h = \frac{pd}{4tE}(1 - v) \qquad \text{(using equation (12.28))}$$

$$= \frac{2 \times 10^6 \times 1.5 \times (1 - 0.3)}{4 \times 10 \times 10^{-3} \times 200 \times 10^9} = 0.0002625$$

Also $\in_h = \dfrac{d(d)}{d}$

where $d(d)$ = Change in diameter (Increase)

or $d(d) = \in_h \times d$

$$= 0.0002625 \times 1.5 \times 1000 \text{ mm} = 0.393 \text{ mm} \qquad \textbf{Ans.}$$

Volumetric strain is

$$\in_V = 3\in_h \qquad \text{(using equation (12.27))}$$

$$= 3 \times 0.0002625 = 0.0007875$$

Also $\in_V = \dfrac{dV}{V}$

where V = Volume of the shell

$$= \frac{\pi}{6}d^3 = \frac{\pi}{6} \times (1.5)^3 = 1.767 \text{ m}^3$$

Hence, $dV = \in_V \times V$

$= 0.0007875 \times 1.767 \times 10^9 \text{ mm}^3$

$= 1391512.5 \text{ mm}^3$ **Ans.**

Example 12.12

A bronze spherical shell of wall thickness 10 mm is subjected to a fluid pressure of 1.5 MPa. The maximum stress in the shell material is limited to 60 MPa. Find the diameter of the shell, if the joint efficiency of the shell is 70%.

Solution: Given,

Wall thickness of the shell, $t = 10$ mm

Fluid pressure $p = 1.5 \text{ MPa} = 1.5 \times 10^6$ Pa

Maximum stress in the shell, $\sigma_h = \sigma_l = 60 \text{ MPa} = 60 \times 10^6$ Pa

Joint efficiency, $\eta = 70\% = 0.7$

Using equation (12.25), we have

$$\sigma_h = \sigma_l = \frac{pd}{4t\eta}$$

or
$$60 \times 10^6 = \frac{1.5 \times 10^6 \times d}{4 \times 10 \times 10^{-3} \times 0.7}$$

Solving for d, we get

$d = 1.12$ m **Ans.**

12.7 CYLINDRICAL SHELL WITH HEMISPHERICAL ENDS

Refer Fig. 12.7.

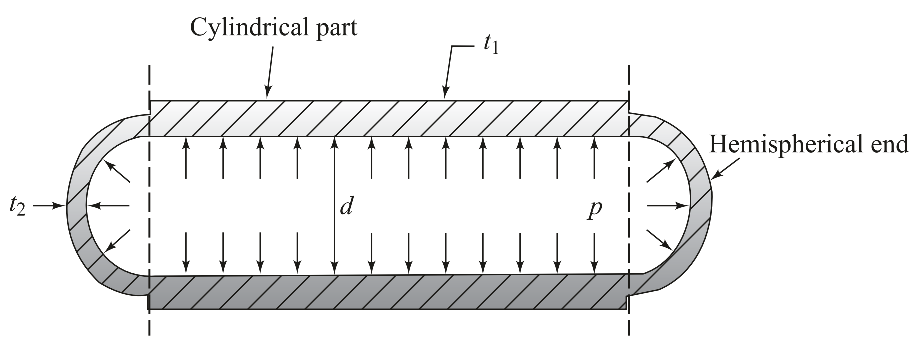


Fig. 12.7

Let
p = Fluid pressure
t_1 = Thickness of the cylindrical part
t_2 = Thickness of the hemispherical ends
d = Inside diameter of the cylindrical part
v = Poisson's ratio for the shell material

For cylinder part

Hoop stress, $$\sigma_h = \frac{pd}{2t_1} \quad \text{...(12.31)}$$

Longitudinal stress, $$\sigma_l = \frac{pd}{4t_1} \quad \text{...(12.32)}$$

Hoop strain, $$\in_h = \frac{pd}{4t_1E}(2-v) \quad \text{...(12.33)}$$

For hemispherical ends

Hoop stress, $$\sigma_h = \frac{pd}{4t_2} \quad \text{...(12.34)}$$

Hoop strain,

$$\in_h = \frac{pd}{4t_2E}(1-v) \quad \text{...(12.35)}$$

For no distortion condition at the junction of the two parts, the two hoop strains are equal.

$$\frac{pd}{4t_1E}(2-v) = \frac{pd}{4t_2E}(1-v)$$

or $$\frac{t_1}{t_2} = \frac{2-v}{1-v} \quad \text{...(12.36)}$$

But the right hand side ratio is always greater than one for the same shell material, hence t_1 is always greater than t_2.

For equal hoop stress to be produced in two parts, we have

$$\frac{pd}{2t_1} = \frac{pd}{4t_2}$$

or $$\frac{t_1}{t_2} = 2 \quad \text{...(12.37)}$$

Example 12.13

A thin cylindrical shell of diameter 600 mm and wall thickness 10 mm has hemispherical ends. Find the thickness of the hemispherical ends considering (*a*) no distortion occurs at the junction, and (*b*) equal hoop stresses are produced in two parts. Take E = 200 GPa and v = 0.25.

Solution: Given,

Diameter of the cylindrical part, d = 600 mm

Thickness of the cylindrical part, t_1 = 10 mm

Modulus of elasticity, $E = 200 \text{ GPa} = 200 \times 10^9$ Pa

Poisson's ratio, v = 0.25

Let t_2 = Thickness of the hemispherical ends (*a*) For no distortion condition at the junction of two parts, we have

$$\frac{t_1}{t_2} = \frac{2-v}{1-v} \qquad \text{(using equation (12.36))}$$

or $$\frac{10}{t_2} = \frac{2-0.25}{1-0.25}$$

Solving for t_2, we get

$$t_2 = 4.28 \text{ mm}$$ **Ans.**

(*b*) For equal hoop stress being produced in two parts, we have

$$\frac{t_1}{t_2} = 2 \qquad \text{(using equation (12.37))}$$

or $$t_2 = \frac{t_1}{2} = \frac{10}{2} = 5 \text{ mm}$$ **Ans.**

12.8 STRESSES IN THICK CYLINDERS (LAME'S THEORY)

The diameter-thickness ratio for thin shells (cylinders or spheres) is large *i.e.*, thickness is very small as compared to diameter, and hence variation of hoop stress along the thickness is negligible; in other words, it is constant over the cross-section of the thin shells. But in case of thick shells, thickness has significant value w.r.t. diameter and the variation of hoop stress is not uniform. But the longitudinal strain is constant at all the points irrespective of the radius of the cylinder.

The follwing assumptions are made in the analysis of thick cylinders:

- The material of the cylinder is homogeneous and isotropic.
- Hooke's law is obeyed, that is, the material is stressed within the elastic limit.
- Modulus of elasticity in tension and compression are same.
- Plane sections perpendicular to the axis of the cylinder remain plane even after the application of internal fluid pressure.

Consider a thick cylinder (Fig. 12.8) with the following parameters:

l = Length of the cylinder

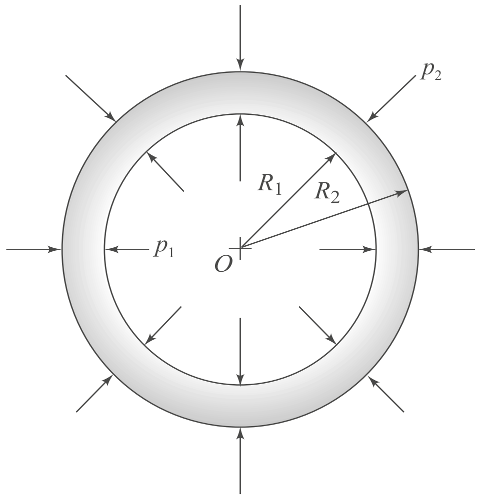
R_1 = Inside radius of the cylinder

R_2 = Outside radius of the cylinder

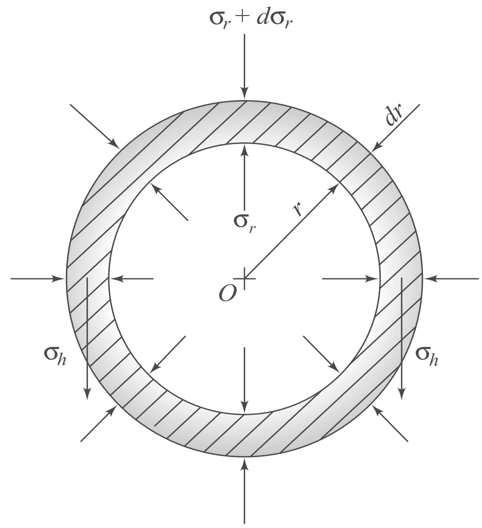
p_1 = Inner radial pressure

p_2 = Outer radial pressure

Now consider an annular ring of radius r and thickness dr in the transverse section of the cylinder.



(a) Cross-section of thick cylinder subjected to pressures p_1 and p_2



(b) An annular ring in the cross-section

Fig. 12.8

Let σ_r = Inside radial stress (pressure) on the ring

$(\sigma_r + d\sigma_r)$ = Outside radial stress (pressure) on the ring

σ_h = Circumferential or hoop stress on the ring

σ_l = Uniform longitudinal stress

Stressed σ_r and σ_h are produced due to radial pressure p_1 acting at the radius R_1 of the cylinder. On the inner surface of the cylinder *i.e.*, at radius R_1, radial stress σ_r is equal to the fluid pressure, and on the outer surface only atmospheric pressure acts.

Circumferential and longitudinal stresses are positive, when they are tensile in nature and are negative when compressive. Compressive radial stress is assumed positive and tensile radial stress negative.

The longitudinal strain is given as

$$\in_l = \frac{\sigma_l}{E} + v\frac{\sigma_r}{E} - v\frac{\sigma_h}{E}$$

$$= \frac{1}{E}[\sigma_l - v(\sigma_h - \sigma_r)] \qquad \text{... (12.38)}$$

Since σ_l, v and E are constants, hence

$$\sigma_h - \sigma_r = \text{Constant}$$

If $\sigma_h = 2A$

where A is another constant.

Then $\sigma_h = 2A + \sigma_r$ ... (12.39)

The longitudinal stress is expressed as

$$\sigma_l = \frac{p_1 \times \pi R_1^2}{\pi(R_2^2 - R_1^2)} = \frac{p_1 R_1^2}{R_2^2 - R_1^2} \qquad ...(12.40)$$

Now let us consider the equilibrium of the annular ring. Various stresses acting on it are shown in Fig. 12.8 (*b*). For equilibrium of the ring, the algebraic sum of the forces in any direction must be zero. The analysis of ring is similar to a thin cylinder.

The bursting force is

$$\begin{aligned} \sigma_r\, 2rl - & (\sigma_r + d\sigma_r)\, 2\, (r + dr)\, l \\ & = -2\sigma_r\, dr\, l - 2d\sigma_r\, r\, l \qquad \text{(neglecting smaller terms)} \end{aligned}$$

The resisting force is

$$2\sigma_h l dr$$

For equilibrium of the ring

$$\begin{aligned} \text{Bursting force} &= \text{Resisting force} \\ -2\sigma_r\, drl - 2d\sigma_r\, rl &= 2\sigma_h\, l\, dr \\ -\sigma_r\, dr - d\sigma_r\, r &= \sigma_h\, dr \end{aligned}$$

or

$$\sigma_h = -\sigma_r - r\,\frac{d\sigma_r}{dr} \qquad (12.41)$$

Substituting equation (12.41) in Equation (12.39), we have

$$\begin{aligned} -\sigma_r - r\,\frac{d\sigma_r}{dr} &= 2A + \sigma_r \\ -r\,\frac{d\sigma_r}{dr} &= 2A + 2\sigma_r = 2\,(A + \sigma_r) \\ 2\,\frac{dr}{r} &= -\frac{d\sigma_r}{A + \sigma_r} \end{aligned}$$

Integrating both sides, we have

$$\begin{aligned} 2\int \frac{dr}{r} &= \int \frac{d\sigma_r}{A + \sigma_r} \\ 2\log_e r &= -\log_e (A + \sigma_r) + \log_e B \end{aligned}$$

where $\log_e B$ is the constant of integration.

$$\begin{aligned} \log_e r^2 &= -\log_e (A + \sigma_r) + \log_e B \\ \log_e (A + \sigma_r) &= \log_e B - \log_e r^2 = \log_e \left(\frac{B}{r^2}\right) \\ A + \sigma_r &= \frac{B}{r^2} \end{aligned}$$

or

$$\sigma_r = \frac{B}{r^2} - A \qquad ...(12.42)$$

Substituting equation (12.42) in equation (12.39), we get

$$\sigma_h = \frac{B}{r^2} + A \quad \text{...(12.43)}$$

The equations (12.42) and (12.43) are known as **Lame's equations**, and are used to find out hoop and radial stresses. The constants A and B are called Lame's constants to be determined by using suitable boundary conditions.

12.8.1 General Case (when Internal and External Pressures both are acting)

At $r = R_1, \sigma_r = p_1$

when $r = R_2, \sigma_r = p_2$

Using these boundary conditions, we have

$$p_1 = \frac{B}{R_1^2} - A \quad \text{(using equation (12.42))}$$

and

$$p_2 = \frac{B}{R_2^2} - A$$

Now

$$(p_1 - p_2) = \frac{B}{R_1^2} - \frac{B}{R_2^2} = B\left[\frac{R_2^2 - R_1^2}{R_1^2 R_2^2}\right]$$

or

$$B = \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}(p_1 - p_2)$$

and

$$A = \frac{p_1 R_1^2 - p_2 R_2^2}{R_2^2 - R_1^2}$$

Substituting values of A and B in equations (12.42) and (12.43), we have

$$\sigma_r = \frac{1}{r^2}\frac{R_1^2 R_2^2}{R_2^2 - R_1^2}(p_1 - p_2) - \frac{p_1 R_1^2 - p_2 R_2^2}{R_2^2 - R_1^2}$$

$$= \frac{R_1^2 R_2^2}{r^2}\frac{(p_1 - p_2)}{R_2^2 - R_1^2} - \frac{p_1 R_1^2 - p_2 R_2^2}{R_2^2 - R_1^2} \quad \text{...(12.44)}$$

and

$$\sigma_h = \frac{R_1^2 R_2^2}{r^2}\frac{(p_1 - p_2)}{R_2^2 - R_1^2} + \frac{p_1 R_1^2 - p_2 R_2^2}{R_2^2 - R_1^2} \quad \text{...(12.45)}$$

12.8.2 When only Internal Pressure is acting

At $r = R_1, \sigma_r = p_1$

when $r = R_2$, $\sigma_r = p_2 = 0$, and outer surface of the cylinder is subjected to atmospheric pressure only.

Using equation (12.42), we have

$$p_1 = \frac{B}{R_1^2} - A \text{ and } 0 = \frac{B}{R_2^2} - A$$

Solving these two equations, we get

$$B = p_1 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}$$

and $$A = p_1 \frac{R_1^2}{R_2^2 - R_1^2}$$

Substituting values of *A* and *B* in equations (12.42) and (12.43), we have

$$\sigma_r = \frac{1}{r^2} p_1 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} - p_1 \frac{R_1^2}{R_2^2 - R_1^2}$$

$$= \frac{p_1 R_1^2}{R_2^2 - R_1^2}\left[\frac{R_2^2}{r^2} - 1\right] \qquad \text{...(12.46)}$$

and $$\sigma_h = \frac{p_1 R_1^2}{R_2^2 - R_1^2}\left[\frac{R_2^2}{r^2} + 1\right] \qquad \text{...(12.47)}$$

It has been observed that σ_r and σ_h are maximum at inner surface of the cylinder *i,e.*, at $r = R$.

or $$\sigma_{r_{\max}} = p_1 \qquad \text{...(12.48)}$$

It is compressive in nature and is taken to be positive.

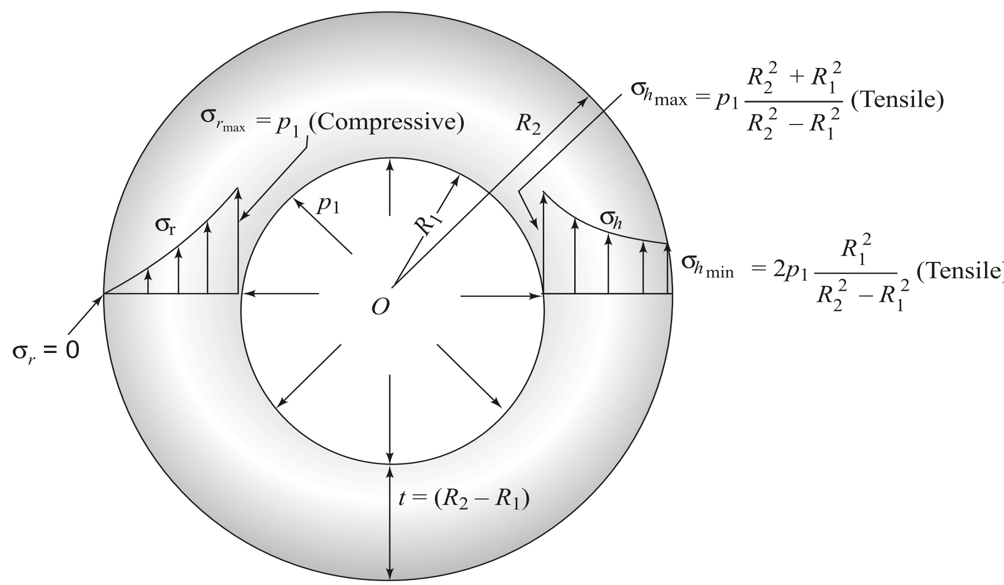


Fig. 12.9

and $$\sigma_{h_{\max}} = p_1 \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} \qquad \text{...(12.49)}$$

It is tensile in nature and is taken to be positive.

The values of σ_r and σ_h at outer surface of the cylinder *i.e.*, at $r = R_2$ are given as

$$\sigma_r = 0 \qquad ...(12.50)$$

and
$$\sigma_h = 2p_1 \ 2p_1 \frac{R_1^2}{R_2^2 - R_1^2} \text{ (Tensile)} \qquad ...(12.51)$$

The variation (parabolic) of σ_h and σ_r across thickness of the cylinder is shown in Fig. 12.9.

12.8.3 When only External Pressure is acting

The boundary conditions are:

$$\text{At } r = R_1, \sigma_r = 0$$
$$\text{when } r = R_2, \sigma_r = p_2$$

Using these boundary conditions in equation (12.42), we have

$$0 = \frac{B}{R_1^2} - A$$

and
$$p_2 = \frac{B}{R_2^2} - A$$

Solving these equations, we get

$$B = -p_2 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}$$

and
$$A = -p_2 \frac{R_2^2}{R_2^2 - R_1^2}$$

Substituting values of A and B in equations (12.42) and (12.43), we have

$$\sigma_r = \left(-p_2 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\right) + p_2 \frac{R_2^2}{R_2^2 - R_1^2}$$

$$= -\frac{p_2 R_2^2}{R_2^2 - R_1^2}\left(\frac{R_1^2}{r^2} - 1\right) \qquad ...(12.52)$$

and
$$\sigma_h = -\frac{p_2 R_2^2}{R_2^2 - R_1^2}\left(\frac{R_1^2}{r^2} + 1\right) \qquad ...(12.53)$$

The hoop stress σ_h is maximum at the inner surface of the cylinder, that is, at $r = R_1$, given as

$$\sigma_{h_{\max}} = -\ 2p_2 \frac{R_2^2}{R_2^2 - R_1^2} \qquad ...(12.54)$$

Negative sing signifies its compressive nature. The value of σ_h at outer surface of the cylinder, that is, at $r = R_2$ is given by

$$\sigma_h = -p_2 \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} \qquad ...(12.55)$$

The radial stress σ_r is maximum at the outer surface of the cylinder *i.e.*, at $r = R_2$, given by

$$\sigma_{r_{\max}} = p_2 \text{ (Compressive)} \qquad ...(12.56)$$

The minimum radial stress occurs at the inner surface of the cylinder *i.e.*, at $r = R_1$, given by

$$\sigma_{r_{\min}} = 0 \qquad ...(12.57)$$

The distribution (parabolic) of σ_h and σ_r across thickness of the cylinder is shown in Fig. 12.10.

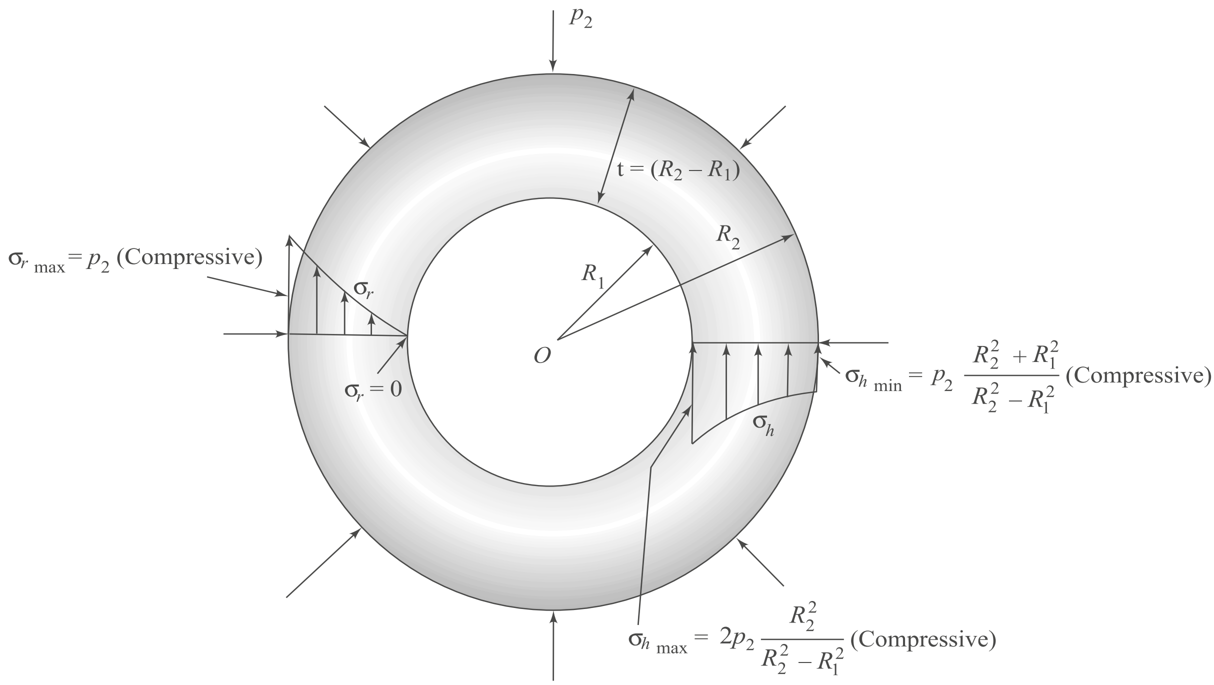


Fig. 12.10

12.8.4 When a Solid Circular Shaft is subjected to External Pressure

Shaft is considered as a thick cylinder with no inner hole. In this case, $r_1 = 0$ and $p_1 = 0$. To avoid the infinite values of stresses σ_r and σ_h, constant B has to be zero. Hence, from equations (12.42) and (12.43), we have

$$\sigma_r = -A \qquad ...(12.58)$$

$$\sigma_h = A \qquad ...(12.59)$$

12.9 LONGITUDINAL STRESS

The longitudinal stress is zero, when the cylinder is open at both ends as in the case of a gun barrel. The same situation can be visualized in a piston-cylinder arrangement where equal pressure acts on both sides of the piston. However, when the cylinder is closed at its both ends and is subjected to pressures p_1 and p_2, being internal and external respectively, then longitudinal stress σ_l is induced, which is given by considering horizontal equilibrium of the forces acting on the cylinder as

$$p_1 \pi R_1^2 - p_2 \pi R_2^2 = \sigma_l \pi (R_2^2 - R_1^2)$$

or $$\sigma_l = \frac{p_1 R_1^2 - p_2 R_2^2}{R_2^2 - R_1^2} \qquad \text{...(12.60)}$$

It is constant across the cross-section of the cylinder and is equal to A.

If no torque acts on the cylinder, the three stresses namely σ_r, σ_h and σ_l become principal stresses with σ_h has the maximum value and σ_r has the least value. σ_h (tensile) is greater than σ_l (tensile) and σ_r is compressive in nature. The maximum shear stress is given as

$$\tau_{max} = \frac{\sigma_h - (-\sigma_r)}{2} = \frac{\sigma_h + \sigma_r}{2} \qquad \text{...(12.61)}$$

Substituting values of σ_h and σ_r from equations (12.42) and (12.43), we have

$$\tau_{max} = \frac{\left(\frac{B}{r^2} + A + \frac{B}{r^2} - A\right)}{2} = \frac{B}{r^2} \qquad \text{...(12.62)}$$

τ_{max} occurs at $r = R_1$ *i.e.*, at inner surface of the cylinder and is referred to as *absolute maximum shear stress*, given by

$$\tau_{max_{absolute}} \leq \frac{B}{R_1^2} \qquad \text{...(12.63)}$$

It acts on a plane inclined at 45° to the longitudinal axis.

Example 12.14

A thick cylinder of inside diameter 200 mm and outside diameter 300 mm is subjected to internal and external pressures of 50 MPa and 15 MPa respectively. Find the hoop stress at internal and external surfaces of the cylinder, and the hoop and radial stresses at mean radius. Also, sketch the variation of the radial and hoop stresses across the thickness of the cylinder.

Solution: Given,

Inside radius of the cylinder, $R_1 = \frac{200}{2} = 100$ mm

Outside radius of the cylinder, $R_2 = \frac{300}{2} = 150$ mm

Internal pressure, $p_1 = 50$ MPa

External pressure, $p_2 = 15$ MPa

The Lame's equations under the given conditions are:

$$p_1 = \frac{B}{R_1^2} - A$$

or $$50 = \frac{B}{(100 \times 10^{-3})^2} - A \qquad \text{... (1)}$$

and $$p_2 = \frac{B}{R_2^2} - A$$

or $$15 = \frac{B}{(150 \times 10^{-3})^2} - A \qquad \text{...(2)}$$

Solving equations (1) and (2), we get

$$B = 0.63$$

and $$A = 13$$

The hoop stress at the internal surface of the cylinder is given by

$$\sigma_{h_{r=R_2}} = \frac{B}{R_1^2} + A \qquad \text{(using equation (12.43))}$$

$$= \frac{0.63}{(100 \times 10^{-3})^2} + 13 = 76 \text{ MPa} \qquad \textbf{Ans.}$$

The hoop stress at the external surface of the cylinder is given as

$$\sigma_{h_{r=150\text{ mm}}} = \frac{0.63}{(150 \times 10^{-3})^2} + 13 = 41 \text{ MPa} \qquad \textbf{Ans.}$$

The hoop stress at the mean radius *i.e.*, at $r = \dfrac{100+150}{2} = 125$ mm is given by

$$\sigma_{h_{r=125\text{ mm}}} = \frac{0.63}{(125 \times 10^{-3})^2} + 13 = 53.32 \text{ MPa} \qquad \textbf{Ans.}$$

The radial stress at the mean radius is given as

$$\sigma_{r_{r=125\text{ mm}}} = \frac{0.63}{(125 \times 10^{-3})^2} - 13 = 27.32 \text{ MPa} \qquad \textbf{Ans.}$$

The stress variation diagram is shown in Fig. 12.11.

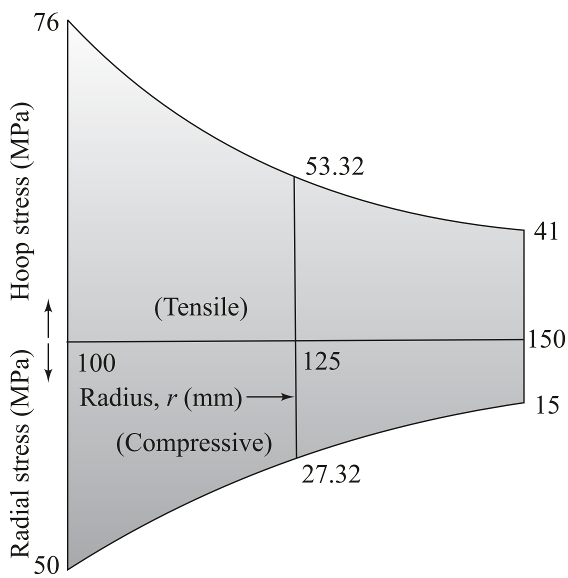


Fig. 12.11

Example 12.15

A thick cylinder of inside diameter 300 mm and outside diameter 450 mm is subjected to both internal and external pressures. Find the external pressure, if the internal pressure is 15 MPa, assuming that the tensile hoop stress at internal surface of the cylinder is 35 MPa.

Solution: Given,

Inside radius of the cylinder, $R_1 = \frac{300}{2} = 150$ mm

Outside radius of the cylinder, $R_2 = \frac{450}{2} = 225$ mm

Internal pressure, $p_1 = 15$ MPa

Hoop stress at R_1, $\sigma_h = 35$ MPa

The hoop stress is given as

$$\sigma_h = \frac{B}{r^2} + A \qquad \text{(using equation (12.43))}$$

or

$$35 = \frac{B}{(150 \times 10^{-3})^2} + A \qquad ...(1)$$

The radial stress is given as

$$\sigma_r = \frac{B}{r^2} - A \qquad \text{(using equation (12.42))}$$

or

$$15 = \frac{B}{(150 \times 10^{-3})^2} - A \qquad ...(2) \text{ (at } r = R_1, \sigma_r = p_1)$$

Solving equations (1) and (2), we get

$$B = 0.5625$$

and

$$A = 10$$

The external pressure is given as

$$p_2 = \frac{B}{R_2^2} - A \qquad (\text{at } r = R_2, \sigma_r = p_2)$$

$$= \frac{0.5625}{(225 \times 10^{-3})^2} - 10 = 1.11 \text{ MPa}$$ **Ans.**

Example 12.16

A thick cylinder of inside diameter 250 mm is required to withstand an internal pressure of 15 MPa. Find the thickness of the cylinder, assuming that the maximum permissible tensile stress is not to exceed 75 MPa.

Solution: Given,

Inside radius of the cylinder, $R_1 = \frac{250}{2} = 125$ mm

Internal pressure, $p_1 = 15$ MPa

Hoop stress, $\sigma_h \le 75$ MPa

Using Lame's equations, we have

$$\sigma_r = \frac{B}{R_1^2} - A$$

and $$\sigma_h = \frac{B}{R_1^2} + A$$

The above equations are obtained by putting $r = R_1$ in equations (12.42) and (12.43).

Here $$15 = \frac{B}{(125\times10^{-3})^2} - A$$

and $$75 = \frac{B}{(125\times10^{-3})^2} + A$$

Solving these equations, we get

$$B = 0.70$$

and $$A = 30$$

At the outer surface of the cylinder, the radial stress σ_r is zero.

Using $$\sigma_r = \frac{B}{r^2} - A$$

or $$0 = \frac{0.70}{R_2^2} - 30 \qquad (r = R_2)$$

On solving, we get

$$R_2 = 0.1527 \text{ m} = 152.7 \text{ mm}$$

Hence, the thickness of the cylinder is given as

$$t = R_2 - R_1$$
$$= 152.7 - 125 = 27.7 \text{ mm}$$ **Ans.**

Example 12.17

A thick steel cylinder of inside diameter 150 mm and outside diameter 300 mm is subjected to internal and external pressures of 60 MPa and 20 MPa respectively. Find the maximum direct and shear stresses in the cylinder.

Solution: Given,

Inside radius of the cylinder, $R_1 = \frac{150}{2} = 75$ mm

Outside radius of the cylinder, $R_2 = \frac{300}{2} = 150$ mm

Internal pressure, $p_1 = 60$ MPa

External pressure, $p_2 = 20$ MPa

Using Lame's equation, we have

$$\sigma_r = \frac{B}{r^2} - A$$

$$60 = \frac{B}{(75 \times 10^{-3})^2} - A$$

and
$$20 = \frac{B}{(150 \times 10^{-3})^2} - A$$

Solving these equations, we get

$$B = 0.3$$

and
$$A = -6.67$$

The maximum hoop stress (direct stress) occurs at internal surface of the cylinder, that is, at $r = R_1$.

$$\sigma_{h_{max}} = \frac{B}{R_1^2} + A$$

$$= \frac{0.3}{(75 \times 10^{-3})^2} - 6.67 = 46.67 \text{ MPa}$$ **Ans.**

The maximum shear stress is given as

$$\tau_{max} = \frac{\sigma_h + \sigma_r}{2}$$ (using equation (12.61))

$$= \frac{46.67 + 60}{2}$$ ($\sigma_h = \sigma_{h_{max}}$ and $\sigma_r = \sigma_{r_{max}} = p_1$)

$$= 53.33 \text{ MPa}$$ **Ans.**

Example 12.18

A thick cylinder of inside diameter 100 mm can withstand an internal pressure of 70 MPa. Find the thickness of the cylinder, if the maximum tensile and maximim shear stresses are limited to 100 MPa and 80 MPa respectively.

Solution: Given,

Inside radius of the cylinder, $R_1 = \frac{100}{2} = 50$ mm

Internal pressure, $p_1 = 70$ MPa

Maximum tensile (hoop) stress, $\sigma_{h_{max}} = 100$ MPa

Maximum shear stress, $\tau_{max} = 80$ MPa

From Lame's equation, we have

$$\sigma_r = \frac{B}{r^2} - A$$

or
$$70 = \frac{B}{(50 \times 10^{-3})^2} - A$$ ($\sigma_r = p_1$ and $r = R_1$)

Let R_2 be outside radius of the cylinder.

$$0 = \frac{B}{R_2^2} - A \qquad (\sigma_r = 0 \text{ at } r = R_2)$$

Solving these two equations, we get

$$B = \frac{0.175 R_2^2}{R_2^2 - 0.0025}$$

and

$$A = \frac{0.175}{R_2^2 - 0.0025}$$

The hoop stress is given as

$$\sigma_h = \frac{B}{r^2} + A \qquad \text{... (1)}$$

Substituting values of A and B in equation (1), we have

$$\sigma_h = \frac{1}{r^2}\frac{0.175 R_2^2}{R_2^2 - 0.0025} + \frac{0.175}{R_2^2 - 0.0025}$$

$$= \frac{0.175}{R_2^2 - 0.0025}\left[\frac{R_2^2}{r^2} + 1\right]$$

σ_h is maximum at $r = R_1$

Hence,

$$\sigma_h = \frac{0.175}{R_2^2 - 0.0025}\left[\frac{R_2^2}{(50\times 10^{-3})^2} + 1\right]$$

$$= 70 \times \frac{0.175}{R_2^2 - 0.0025}\left[\frac{R_2^2}{(50\times 10^{-3})^2} + 1\right]$$

But $\sigma_{h_{max}} = 100$ MPa (Given)

or

$$\frac{100}{70} = \frac{R_2^2 + 0.0025}{R_2^2 - 0.0025}$$

On solving, we get $R_2 = 119$ mm

On the basis of maximum shear stress

$$\tau_{max} = \frac{\sigma_h + \sigma_r}{2}$$

$$= \frac{1}{2}\times\left(70\frac{R_2^2 + 0.0025}{R_2^2 - 0.0025} + 70\right)$$

$$= \frac{70}{2}\times\left(\frac{R_2^2 + 0.0025}{R_2^2 - 0.0025} + 1\right)$$

But $$\tau_{max} = 80 \text{ MPa}$$

$$80 = \frac{70}{2} \times \left(\frac{R_2^2 + 0.0025}{R_2^2 - 0.0025} + 1 \right)$$

Solving for R_2, we get

$$R_2 = 141.5 \text{ mm}$$

Select the bigger of the two values of R_2. Hence, $R_2 = 141.5$ mm.

The thickness of the cylinder is given as

$$t = R_2 - R_1$$
$$= 141.5 - 50 = 91.5 \text{ mm}$$ **Ans.**

12.10 STRAINS IN THICK CYLINDERS

The hoop strain is connected with the change in diameter of the cylinder and is given as

$$\in_h = \frac{\text{Change in circumference}}{\text{Original circumference}}$$

$$= \frac{\pi[d + d(d)] - \pi d}{\pi d} = \frac{d(d)}{d}$$

Also $$\in_h = \frac{\sigma_h}{E} + v\frac{\sigma_r}{E} - v\frac{\sigma_l}{E}$$

$$= \frac{1}{E}\,[\sigma_h + v\,(\sigma_r - \sigma_l)] \qquad \text{...(12.64)}$$

Assuming that the cylinder is closed at both of its ends and is subjected to internal and external pressures both, the equation (12.64) changes to

$$\in_h = \left[\left\{\frac{R_1^2 R_2^2}{r^2}\frac{(p_1 - p_2)}{R_2^2 - R_1^2} + \frac{p_1 R_1^2 - p_2 R_2^2}{R_2^2 - R_1^2}\right\} + v\left\{\frac{R_1^2 R_2^2}{r^2}\frac{(p_1 - p_2)}{R_2^2 - R_1^2} - \frac{p_1 R_1^2 - p_2 R_2^2}{R_2^2 - R_1^2} - \frac{p_1 R_1^2 - p_2 R_2^2}{R_2^2 - R_1^2}\right\}\right]$$

(using equations (12.44), (12.45) and (12.60))

$$= \frac{(1+v)}{E}\left\{\frac{R_1^2 R_2^2}{r^2}\frac{(p_1 - p_2)}{R_2^2 - R_1^2}\right\} + \frac{(1+2v)}{E}\left\{\frac{p_1 R_1^2 - p_2 R_2^2}{R_2^2 - R_1^2}\right\} \qquad \text{...(12.65)}$$

The radial strain is given as

$$\in_r = \frac{1}{E}[\sigma_r + v(\sigma_h + \sigma_l)] \qquad \text{...(12.66)}$$

On substituting σ_r, σ_h and σ_l, we have

$$\in_r = \frac{1}{E}\left[\left\{\frac{R_1^2R_2^2}{r^2}\frac{(p_1-p_2)}{R_2^2-R_1^2}-\frac{p_1R_1^2-p_2R_2^2}{R_2^2-R_1^2}\right\}+v\left\{\frac{R_1^2R_2^2}{r^2}\frac{(p_1-p_2)}{R_2^2-R_1^2}+\frac{p_1R_1^2-p_2R_2^2}{R_2^2-R_1^2}+\frac{p_1R_1^2-p_2R_2^2}{R_2^2-R_1^2}\right\}\right]$$

$$= \frac{(1+v)}{E}\left\{\frac{R_1^2R_2^2}{r^2}\frac{(p_1-p_2)}{R_2^2-R_1^2}\right\}-\frac{(1-2v)}{E}\left\{\frac{p_1R_1^2-p_2R_2^2}{R_2^2-R_1^2}\right\} \qquad \text{...(12.67)}$$

The volumetric strain is defined as

$$\in_V = \frac{\text{Change in volume}}{\text{Original volume}} = \frac{dV}{V}$$

Volume of cylinder, $V = \pi r^2 l = f(r, l)$

where l = Length of the cylinder

r = Radius of the cylinder

$$dV = \frac{dV}{dr}dr + \frac{\partial V}{\partial l}\,dl$$

$$= 2\pi rl\,dr + \pi r^2\,dl$$

$$= \pi r^2 l\left(2\frac{dr}{r}+\frac{dl}{l}\right) = V\left(2\frac{dr}{r}+\frac{dl}{l}\right)$$

or

$$\frac{dV}{V} = 2\frac{dr}{r}+\frac{dl}{l}$$

$$\in_V = 2\left\{\frac{2dr}{2r}\right\}+\frac{dl}{l}$$

$$\in_V = 2\frac{d(d)}{d}+\frac{dl}{l} \qquad (d = 2r \text{ and } 2dr = 2d\,(d))$$

$$= 2\in_h + \in_l \qquad \text{...(12.68)}$$

The longitudinal strain $\in_l$ is also expressed as

$$\in_l = \frac{1}{E}[\sigma_l + v(\sigma_r - \sigma_h)] \qquad \text{...(12.69)}$$

Example 12.19

Find the ratio of thickness to internal diameter for a tube subjected to internal pressure, when the ratio of pressure to maximum hoop stress is 0.4. Also find the change in thickness of the tube, when its internal diameter is 250 mm and the internal pressure is 100 MPa.

Take $E = 2 \times 10^5$ MPa and $v = 0.3$.

Solution: Let

$$t = \text{Thickness of the tube}$$

$$p_1 = \text{Internal pressure}$$

$$= 100 \text{ MPa (Given)}$$

$$R_1 = \text{Inside radius of the tube}$$

$$= 125 \text{ mm (Given)}$$

$$R_2 = \text{Outside radius of the tube}$$

$$\sigma_h = \text{Hoop stress}$$

$$\frac{p_1}{\sigma_{h_{max}}} = 0.4 \text{ (Given)}$$

$$\sigma_{h_{max}} = \frac{p_1}{0.4} = \frac{100}{0.4} = 250 \text{ MPa} \qquad \text{(at inside radius)}$$

Using equation (12.49), we have

$$\sigma_{h_{max}} = p_1 \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}$$

$$\frac{\sigma_{h_{max}}}{p_1} = \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}$$

$$2.5 = \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}$$

or $$R_2 = 1.52R_1 = 1.52 \times 125 = 190 \text{ mm}$$

Now $$R_2 - R_1 = t$$

$$(1.52 - 1)\, R_1 = t$$

$$\frac{t}{R_1} = 0.52$$

or $$\frac{t}{2R_1} = \frac{t}{D_1} = 0.26$$ **Ans.**

The longitudinal stress σ_l is given as

$$\sigma_l = \frac{p_1 R_1^2}{R_2^2 - R_1^2} \qquad \text{(using equation (12.60) and putting } p_2 = 0\text{)}$$

$$= 2 \times 100 \times \frac{100 \times (125 \times 10^{-3})^2}{(190 \times 10^{-3})^2 - (125 \times 10^{-3})^2} = 76.3 \text{ MPa (Tensile)}$$

The hoop strain at the inner surface is given as

$$\in_h = \frac{1}{E}[\sigma_h + v(\sigma_r - \sigma_l)] \qquad \text{(using equation (12.64))}$$

$$= \frac{1}{2\times10^5}[250 + 0.3\,(100 - 76.3)] \qquad (\sigma_h = \sigma_{h\max} \text{ and } \sigma_r = p_1)$$

$$= 0.001285$$

Increase in inside radius $= 0.001285 \times R_1$

$$= 0.001285 \times 125 = 0.1606 \text{ mm}$$

Now the hoop stress at the outside radius is

$$\sigma_h = 2p_1 \frac{R_1^2}{R_2^2 - R_1^2} \qquad \text{(using equation (12.51))}$$

$$= 2 \times 100 \times \frac{(125\times10^{-3})^2}{(190\times10^{-3})^2-(125\times10^{-3})^2} = 152.6 \text{ MPa (Tensile)}$$

The hoop strain at the outer surface is given as

$$= \frac{1}{E}(\sigma_h - v\sigma_l) \qquad \text{(using equation (12.64) and putting } \sigma_r = p_2 = 0)$$

$$= \frac{1}{2\times10^5}\;(152.6 - 0.3 \times 76.3) = 0.000648$$

Increase in outside radius $= 0.000648 \times R_2$

$$= 0.000648 \times 190 = 0.12312 \text{ mm}$$

Hence, the change in thickness is given as

$$(0.1606 - 0.12312) \text{ mm}$$

$$= 0.03748 \text{ mm (Decrease)}$$ **Ans.**

Example 12.20

A tube of inside diameter 5 mm and outside diameter 10 mm is 300 mm long and is closed at one end. A fluid under a pressure of 20 MPa is forced into it through the other end, when it is already full of fluid. Find the volume of excess fluid forced into the tube.
Take $E = 5 \times 10^4$ MPa, $v = 0.3$ and $K_{\text{fluid}} = 1000$ MPa.

Solution: Given,

Inside radius of the tube, $R_1 = \frac{5}{2} = 2.5$ mm

Outside radius of the tube, $R_2 = \frac{10}{2} = 5.0$ mm

Length of the tube, $l = 300$ mm

Internal fluid pressure, $p_1 = 20$ MPa

The hoop stress is given as

$$\sigma_h = \sigma_{h\max} = p_1 \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} \qquad \text{(using equation (12.49))}$$

$$= 20\frac{5^2 + 2.5^2}{5^2 + 2.5^2} = 33.34 \text{ MPa}$$

Hoop strain is

$$\in_h = \frac{1}{E}(\sigma_h + vp_1)$$

(using equation (12.64) and putting $\sigma_r = p_1$ and $\sigma_l = 0$)

$$= \frac{1}{5\times10^4}(33.34 + 0.3 \times 20) = 0.0007868$$

Longitudinal strain is

$$\in_l = \frac{1}{E}(-v\sigma_h + vp_1)$$

$$= \frac{1}{5\times10^4}(-0.3 \times 33.34 + 0.3 \times 20) = -0.00008$$

Volumetric strain of the tube is

$$\in_{v_{\text{tube}}} = 2\in_h + \in_l \qquad \text{(using equation (12.68))}$$

$$= 2 \times 0.0007868 - 0.00008 = 0.001493$$

Volumetric strain of water is

$$\in_{v_{\text{water}}} = \frac{p_1}{K}$$

$$= \frac{20}{1000} = 0.02$$

Hence, the total volumetric strain is given as

$$\in_{v_{\text{total}}} = \in_{v_{\text{tube}}} + \in_{v_{\text{water}}}$$

$$= 0.001493 + 0.02$$

$$= 0.021493$$

The volume of fluid in the tube is obtained as

$$V = \pi R_1^2 \times l$$

$$= \pi \times (2.5)^2 \times 300 = 5890.4862 \text{ mm}^3$$

Hence, the volume of the excess fluid forced into the tube is given as

$$dV = V \times \in_{v_{\text{total}}}$$

$$= 5890.4862 \times 0.021493 = 126.6 \text{ mm}^3$$

Ans.

12.11 COMPOUND CYLINDERS

In case a thick cylinder is subjected to internal fluid pressure, hoop stress is found to be maximum at inside surface of the cylinder and minimum at its outside surface. Hence, the capacity of the cylinder to withstand the internal pressure is dependent on maximum hoop stress, which should not exceed the permissible tensile stress for its material. The purpose of the compound cylinder is to reduce the stresses on its inside radius and hence to increase the pressure bearing capacity. It is achieved by placing one cylinder over another and both cylinders are shrink fitted. Initially, the inside diameter of the outer cylinder is less than the outside diameter of the inner cylinder. When heated, due to expansion, the inner diameter of the outer cylinder expands and fits well on the outside diameter of the inner cylinder. Due to internal fluid pressure, hoop stresses are produced. The hoop stress in the inner cylinder is tensile while due to shrink fitting, the hoop stress is compressive. The resultant hoop stress at the inner radius is thus the algebraic sum of the two hoop stresses and is less than the hoop stress produced due to internal fluid pressure alone as in case of a single thick cylinder.

12.11.1 Stress due to Shrinkage

Consider a compound cylinder (Fig. 12.12).

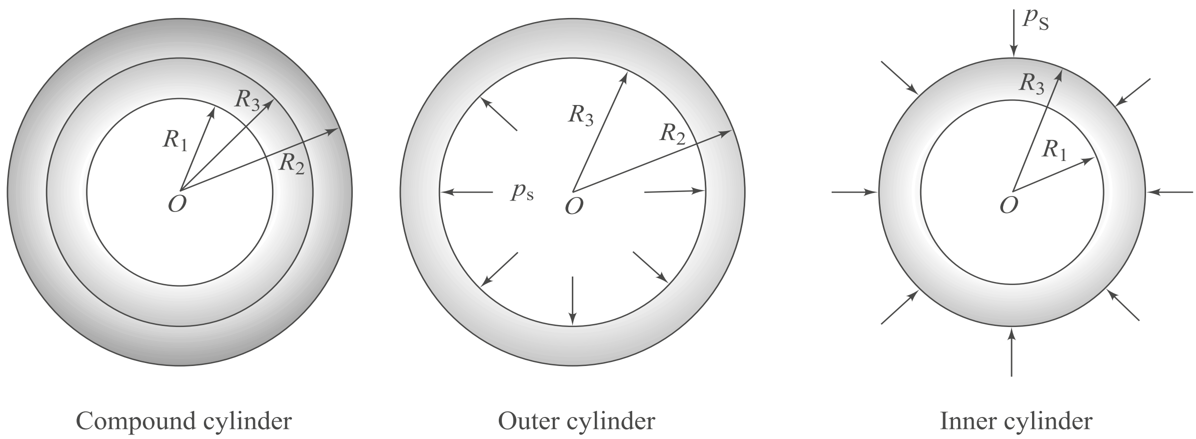


Fig. 12.12

Let p_s be the radial pressure acting at the junction of two cylinders *i.e.*, on the outer surface of the inner cylinder and on the inner surface of the outer cylinder, due to shrinkage.

Using Lame's equations, we have

Hoop stress, $$\sigma_h = \frac{B}{r^2} + A$$

Radial stress, $$\sigma_r = \frac{B}{r^2} - A$$

For outer cylinder

The boundary conditions are:

At $r = R_2,\ \sigma_r = 0$

and at $r = R_3,\ \sigma_r = p_s$

The Lame's equations, on using boundary conditions, become

$$0 = \frac{B}{R_2^2} - A$$

and $$p_s = \frac{B}{R_3^2} - A$$

Solving these equations, we get

$$B = p_s \frac{R_2^2\ R_3^2}{R_2^2 - R_3^2}$$

and
$$A = p_s \frac{R_3^2}{R_2^2 - R_3^2}$$

Substituting values of A and B in the Lame's equations, we have

$$\sigma_h = \frac{R_2^2\ R_3^2}{r^2} \frac{p_s}{R_2^2 - R_3^2} + p_s \frac{R_3^2}{R_2^2 - R_3^2}$$

$$= \frac{p_s R_3^2}{R_2^2 - R_3^2}\left[\frac{R_2^2}{r^2} + 1\right] \quad ...(12.70)$$

and
$$\sigma_r = \frac{p_s R_3^2}{R_2^2 - R_3^2}\left[\frac{R_2^2}{r^2} - 1\right] \quad ...(12.71)$$

Hoop stress at $r = R_3$ is

$$\sigma_{h_{r=R_3}} = \frac{p_s R_3^2}{R_2^2 - R_3^2}\left[\frac{R_2^2}{R_3^2} + 1\right]$$

$$= p_s \frac{R_2^2 + R_3^2}{R_2^2 - R_3^2} \quad ...(12.72)$$

Hoop stress at $r = R_2$ is

$$\sigma_{h_{r=R_2}} = \frac{R_3^2}{R_2^2 - R_3^2} 2p_s \quad ...(12.73)$$

For inner cylinder

The boundary conditions are:

At $\quad r = R_1, \sigma_r = 0$

and at $\quad r = R_3, \sigma_r = p_s$

The Lame's equations, on using boundary conditions, become

$$0 = \frac{B}{R_1^2} - A$$

and
$$p_s = \frac{B}{R_3^2} - A$$

Solving these equations, we get

$$B = -p_s \frac{R_1^2\ R_3^2}{R_3^2 - R_1^2}$$

and
$$A = -p_s \frac{R_3^2}{R_3^2 - R_1^2}$$

Substituting values of A and B in the Lame's equations, we have

$$\sigma_h = \frac{R_1^2 R_3^2}{r^2} \frac{p_s}{R_3^2 - R_1^2} - p_s \frac{R_3^2}{R_3^2 - R_1^2}$$

$$= -p_s \frac{R_3^2}{R_3^2 - R_1^2}\left(\frac{R_1^2}{r^2} + 1\right) \quad \text{...(12.74)}$$

and
$$\sigma_r = p_s \frac{R_3^2}{R_3^2 - R_1^2}\left(1 - \frac{R_1^2}{r^2}\right) \quad \text{...(12.75)}$$

Hoop stress at $r = R_1$ is

$$\sigma_{h_{r=R_1}} = -\, 2p_s \frac{R_3^2}{R_3^2 - R_1^2} \quad \text{...(12.76)}$$

Hoop stress at $r = R_3$ is

$$\sigma_{h_{r=R_3}} = -\, p_s \frac{R_3^2 + R_1^2}{R_3^2 - R_1^2} \quad \text{...(12.77)}$$

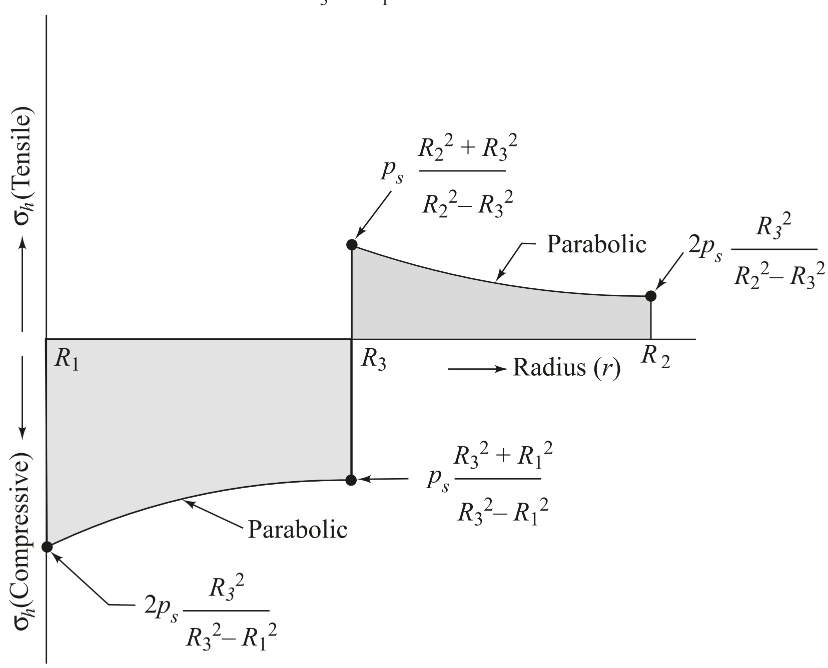


Fig. 12.13 Hoop stress distribution due to shrinkage in a compound cylinder.

The hoop stress distribution due to shrinkage is shown in Fig. 12.13. The variation is parabolic in nature.

12.11.2 Stresses due to Fluid Pressure

The inner and outer cylinders are combined together and are considered as a single thick cylinder. Let it be subjected to an internal fluid pressure p_i.

The boundary conditions are:

At $\quad r = R_1, \sigma_r = p_i$

and at $\quad r = R_2, \sigma_r = 0$

The Lame's equations, on substituting boundary conditions, become

$$p_i = \frac{B}{R_1^2} - A$$

$$0 = \frac{B}{R_2^2} - A$$

Solving these equations, we get

$$B = p_i \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}$$

and
$$A = p_i \frac{R_1^2}{R_2^2 - R_1^2}$$

Substituting values of A and B in the Lame's equations, we have

$$\sigma_h = \frac{R_1^2 R_2^2}{r^2} \frac{p_i}{R_2^2 - R_1^2} + p_i \frac{R_1^2}{R_2^2 - R_1^2}$$

$$= \frac{p_i R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{r^2} + 1\right) \quad \text{...(12.78)}$$

and
$$\sigma_r = \frac{p_i R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{r^2} - 1\right) \quad \text{...(12.79)}$$

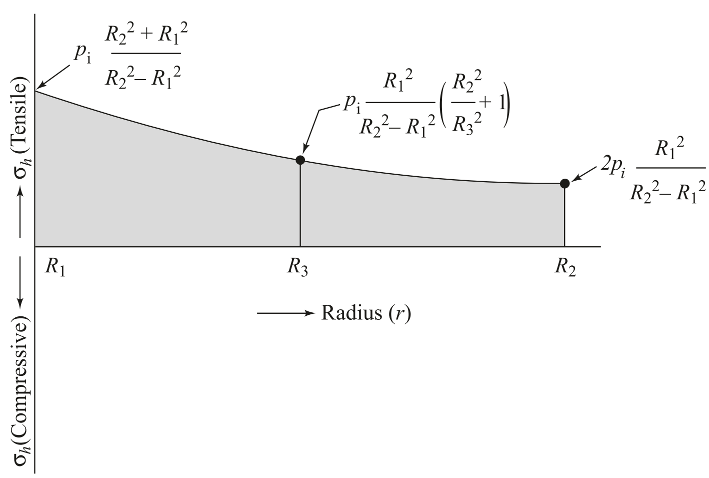


Fig. 12.14 Hoop stress distribution due to fluid pressure in a compound cylinder.

Hoop stress at $r = R_1$ is
$$\sigma_{h_{r=R_1}} = p_i \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} \qquad ...(12.80)$$

Hoop stress at $r = R_3$ is
$$\sigma_{h_{r=R_3}} = \frac{p_i R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{R_3^2} + 1\right) \qquad ...(12.81)$$

Hoop stress at $r = R_2$ is
$$\sigma_{h_{r=R_2}} = 2p_i \frac{R_1^2}{R_2^2 - R_1^2} \qquad ...(12.82)$$

The hoop stress distribution due to fluid pressure is shown in Fig. 12.14.

12.11.3 Resultant Stresses

The resultant hoop stress is the algebraic sum of the hoop stresses due to shrinkage and internal fluid pressure.

For outer cylinder

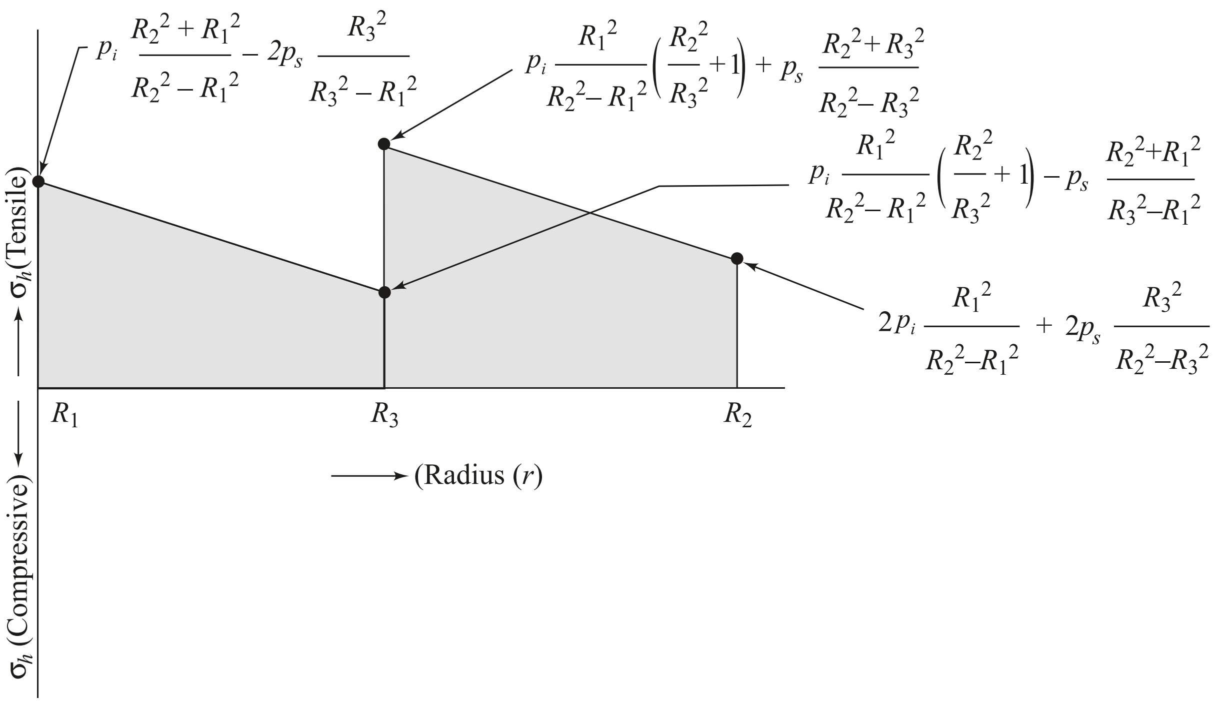
Hoop stress at $r = R_3$ is
$$\sigma_{h_{r=R_3}\text{resultant}} = \sigma_{h_{r=R_3}} \text{ due to } p_i + \sigma_{h_{r=R_3}} \text{ due to } p_s$$
$$= \frac{p_i R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{R_3^2} + 1\right) + p_s \frac{R_2^2 + R_3^2}{R_2^2 - R_3^2} \qquad ...(12.83)$$

Hoop stress at $r = R_2$ is
$$\sigma_{h_{r=R_2}\text{resultant}} = \sigma_{h_{r=R_2}} \text{ due to } p_i + \sigma_{h_{r=R_2}} \text{ due to } p_s$$
$$= 2p_i \frac{R_1^2}{R_2^2 - R_1^2} + 2p_s \frac{R_3^2}{R_2^2 - R_3^2} \qquad ...(12.84)$$

For inner cylinder

Hoop stress at $r = R_1$, is
$$\sigma_{h_{r=R_1}\text{resultant}} = \sigma_{h_{r=R_1}} \text{ due to } p_i + \sigma_{h_{r=R_1}} \text{ due to } p_s$$
$$= p_i \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} + \left(-2p_s \frac{R_3^2}{R_3^2 - R_1^2}\right)$$
$$= p_i \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2} + \left(-2p_s \frac{R_3^2}{R_3^2 - R_1^2}\right) \qquad ...(12.85)$$

Hoop stress at $r = R_3$ is
$$\sigma_{h_{r=R_3}\text{resultant}} = \sigma_{h_{r=R_3}} \text{ due to } p_i + \sigma_{h_{r=R_3}} \text{ due to } p_s$$
$$= \frac{p_i R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{R_3^2} + 1\right) + \left(-p_s \frac{R_3^2 + R_1^2}{R_3^2 - R_1^2}\right)$$
$$= \frac{p_i R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{R_3^2} + 1\right) - p_s \frac{R_3^2 + R_1^2}{R_3^2 - R_1^2} \qquad ...(12.86)$$



Fir. 12.15 Resultant hoop stress distribution in a compound cylinder.

The resultant hoop stress distribution is shown in Fig. 12.15.

12.11.4 Shrinkage Allowance

To make a strong fit at the common surface of the two cylinders in a compound cylinder, inside diameter of the outer cylinder is kept less than outside diameter of the inner cylinder. During heating or stressed condition, outer cylinder expands and fits well on inner cylinder on cooling. Shrinkage allowance is defined as the difference of inside diameter of outer cylinder and outsie diameter of inner cylinder.

Refer Fig. 12.16.

Let R_3 = Common radius of the two cylinders of the compound cylinder.

dR'_2 = Difference between the inside radius of outer cylinder and R_3. It is positive difference

dR'_3 = Difference between the outside radius of inner cylinder and R_3. It is negative difference

dR_3 = Difference in the radii of the two cylinders at the common surface before shrinkage

= Shrinkage allowance

= $dR'_2 + dR'_3$

E_1 = Modulus of elasticity for the inner cylinder

E_2 = Modulus of elasticity for the outer cylinder

v_1 = Poisson's ratio for the inner cylinder

v_2 = Poisson's ratio for the outer cylinder

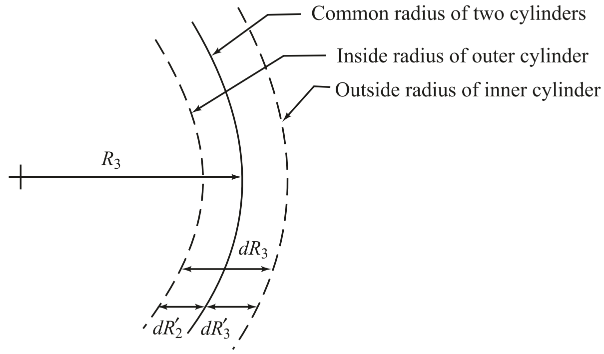


Fig. 12.16 Shrinkage allowance.

Hoop strain at R_3 in inner cylinder is given by

$$\frac{dR'_3}{R_3} = \frac{1}{E_1}[\sigma_{h_{r=R_3}} - v_1\,\sigma_{r_{r=R_3}}]$$

$$= \frac{1}{E_1}\left[-p_s\,\frac{R_3^2 + R_1^2}{R_3^2 - R_1^2} + v_1\,p_s\right] \qquad \text{(using equation (12.77) and } \sigma_{r_{r=R_3}} = -p_s)$$

$$= -\frac{p_s}{E_1}\left[\frac{R_3^2 + R_1^2}{R_3^2 - R_1^2} - v_1\right] \qquad ...(12.87)$$

Negative sign signifies its compressive nature. The hoop strain at R_3 in outer cylinder is given by

$$\frac{dR'_2}{R_3} = \frac{1}{E_2}[\sigma_{h_{r=R_3}} - v_2\,\sigma_{r_{r=R_3}}]$$

$$= \frac{1}{E_2}\left[p_s\,\frac{R_2^2 + R_3^2}{R_2^2 - R_3^2} + v_2\,p_s\right] \qquad \text{(using equation (12.72) and } \sigma_{r_{r=R_3}} = -p_s)$$

$$= \frac{p_s}{E_2}\left[\frac{R_2^2 + R_3^2}{R_2^2 - R_3^2} + v_2\right]\text{(Tensile)} \qquad ...(12.88)$$

Now $$\frac{dR'_2}{R_3} + \frac{dR'_3}{R_3} = \frac{p_s}{E_2}\left[\frac{R_2^2 + R_3^2}{R_2^2 - R_3^2} + v_2\right] + \frac{p_s}{E_1}\left[\frac{R_3^2 + R_1^2}{R_3^2 - R_1^2} - v_1\right]$$

It gives the algebraic sum of $\dfrac{dR'_2}{R_3}$ and $\dfrac{dR'_3}{R_3}$.

or $$dR'_2 + dR'_3 = \frac{p_s R_3}{E_2}\left[\frac{R_2^2 + R_3^2}{R_2^2 - R_3^2} + v_2\right] + \frac{p_s R_3}{E_1}\left[\frac{R_3^2 + R_1^2}{R_3^2 - R_1^2} - v_1\right]$$

or $$dR_3 = p_s R_3\left[\left\{\frac{1}{E_2}\left(\frac{R_2^2 + R_3^2}{R_2^2 - R_3^2} + v_2\right)\right\} + \left\{\frac{1}{E_1}\left(\frac{R_3^2 + R_1^2}{R_3^2 - R_1^2} - v_1\right)\right\}\right] \qquad ...(12.89)$$

When the two cylinders are made of the same material *i.e.*, $E_1 = E_2 = E$ and $v_1 = v_2 = v$. Substituting these values in equation (12.89), the expression for shrinkage allowance becomes

$$dR_3 = \frac{p_s R_3}{E}\left(\frac{R_2^2 + R_3^2}{R_2^2 - R_3^2} + \frac{R_3^2 + R_1^2}{R_3^2 - R_1^2}\right) \quad ...(12.90)$$

Example 12.21

A steel hoop of inside diameter 120 mm and outside diameter 180 mm is shrunk on a hollow steel cylinder of inside diameter 75 mm. The pressure of shrinkage is 30 MPa. For an internal fluid pressure of 80 MPa, find the following parameters:

(*a*) the maximum hoop stress induced in the cylinder,

(*b*) the maximum hoop stress in the hoop

(*c*) the radial pressure between the cylinder and the hoop.

Solution: Refer Fig. 12.17. Given,

Inside radius of the steel cylinder, $R_1 = \frac{70}{2} = 35$ mm

Inside radius of the steel hoop, $R_3 = \frac{120}{2} = 60$ mm

Outside radius of the steel hoop, $R_2 = \frac{180}{2} = 90$ mm

Shrinkage pressure, $p_s = 30$ MPa

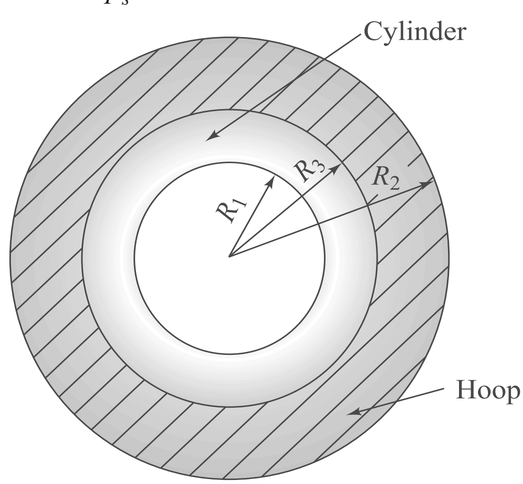


Fig. 12.17

Internal fluid pressure, $p_i = 80$ MPa

Stresses due to shrinkage

For hoop

From Lame's equation, we have

$$\sigma_r = \frac{B}{r^2} - A \quad ...(1) \text{ (using equation (12.42))}$$

The boundary conditions are:

At $r = R_3 = 60$ mm, $\sigma_r = p_s = 30$ MPa

and at $r = R_2 = 90$ mm, $\sigma_r = 0$

On using boundary conditions, we have

$$30 = \frac{B}{(60 \times 10^{-3})^2} - A$$

and

$$0 = \frac{B}{(90 \times 10^{-3})^2} - A$$

Solving these equations, we get

$$B = 0.1944$$

and

$$A = 24$$

Hence, hoop stress at $r = R_3 = 60$ mm in the hoop is

$$\sigma_h = \frac{B}{r^2} + A$$

$$= \frac{0.1944}{(60 \times 10^{-3})^2} + 24 = 78 \text{ MPa (Tensile)} \qquad ...(2)$$

For cylinder

The boundary conditions are:

At $r = R_3 = 60$ mm, $\sigma_r = p_s = 30$ MPa

and at $r = R_1 = 35$ mm, $\sigma_r = 0$

From equation (1), we have

$$30 = \frac{B}{(60 \times 10^{-3})^2} - A$$

and

$$0 = \frac{B}{(35 \times 10^{-3})^2} - A$$

Solving these equations, we get

$$B = -0.0557$$

and

$$A = -45.47$$

Hence, hoop stress at $r = R_1 = 35$ mm in the cylinder is

$$\sigma_h = \frac{B}{r^2} + A$$

$$= -\frac{0.0557}{(35 \times 10^{-3})^2} - 45.47$$

$$= -91 \text{ MPa} = 91 \text{ MPa} \quad \text{(Compressive)} \qquad ...(3)$$

Stresses due to fluid pressure

The boundary conditions are:

At $r = R_1 = 35$ mm, $\sigma_r = p_i = 80$ MPa

and at $r = R_2 = 90$ mm, $\sigma_r = 0$

Using equation (1), we have

$$80 = \frac{B}{(35\times 10^{-3})^2} - A$$

and
$$0 = \frac{B}{(90\times 10^{-3})^2} - A$$

Solving these equations, we get

$$B = 0.115$$

and
$$A = 14.2$$

Hoop stress at $r = R_1 = 35$ mm is

$$\sigma_h = \frac{B}{r^2} + A$$

$$= -\frac{0.115}{(35\times 10^{-3})^2} + 14.2 = 108 \text{ MPa (Tensile)} \qquad ...(4)$$

Hoop stress at $r = R_3 = 60$ mm is

$$\sigma_h = -\frac{0.115}{(60\times 10^{-3})^2} + 14.2 = 46.14 \text{ MPa (Tensile)} \qquad ...(5)$$

Radial stress at $r = R_3 = 60$ mm is

$$\sigma_r = \frac{B}{r^2} - A$$

$$= \frac{0.115}{(60\times 10^{-3})^2} - 14.2 = 17.74 \text{ MPa} \qquad ...(6)$$

(*a*) Maximum hoop stress induced in the cylinder is

(3) + (4)

= (– 91 + 108) MPa

= 17 MPa (Tensile) **Ans.**

(*b*) Maximum hoop stress in the hoop is

(2) + (5)

= (78 + 46.14) MPa

= 124.14 MPa **Ans.**

(*c*) Radial pressure between the cylinder and hoop is

$$p_s + (6)$$
$$= (30 + 17.74)\text{ MPa} = 47.74\text{ MPa}$$ **Ans.**

Example 12.22

A steel cylinder of outside diameter 200 mm is shrunk fitted on another steel cylinder of inside diameter 110 mm and outside diameter 170 mm. If the maximum tensile stress induced in the outer cylinder is 90 MPa, find the following parameters:

(*a*) the radial stress between the cylinders,
(*b*) the hoop stresses at the inside and outside radius of both cylinders,
(*c*) the shrinkage allowance at the common surface
(*d*) draw the stress distribution diagram
Take $E = 2 \times 10^5$ MPa and $v = 0.3$.

Solution: Refer Fig. 12.18. Given,

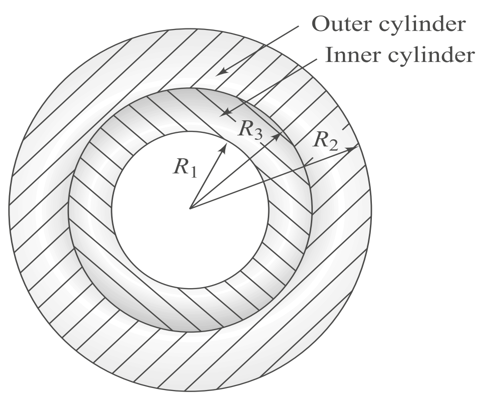


Fig. 12.18

Outside radius of the outer cylinder, $R_2 = \dfrac{200}{2} = 100$ mm

Outside radius of the inner cylinder, $R_3 = \dfrac{170}{2} = 85$ mm

Inside radius of the inner cylinder, $R_1 = \dfrac{110}{2} = 55$ mm

Maximum hoop stress at $r = R_3$, $\sigma_{h_{max}} = 90$ MPa

Outer cylinder

From Lame's equation, we have

$$\sigma_h = \frac{B}{r^2} + A$$

and $$\sigma_r = \frac{B}{r^2} - A$$

The boundary conditions are:

At $\quad r = R_3 = 85$ mm, $\sigma_h = \sigma_{h_{max}} = 90$ MPa

and at $\quad r = R_2 = 100$ mm, $\sigma_r = 0$

On using boundary conditions, we have

$$90 = \frac{B}{(85 \times 10^{-3})^2} + A$$

and
$$0 = \frac{B}{(100 \times 10^{-3})^2} - A$$

Solving these equations, we get

$$B = 0.377$$

and
$$A = 37.7$$

(*a*) The radial stress between the two cylinders is given by

$$\sigma_{r_{r=R_3}} = \frac{B}{R_3^2} - A$$

$$= \frac{0.377}{(85 \times 10^{-3})^2} - 37.7 = 14.48 \text{ MPa} \qquad \textbf{Ans.}$$

The positive sign signifies its compressive nature.

(*b*) The hoop stress at the $r = R_2 = 100$ mm is given by

$$\sigma_{h_{r=R_2}} = \frac{B}{R_2^2} + A$$

$$= \frac{0.377}{(100 \times 10^{-3})^2} + 37.7$$

$$= 75.4 \text{ MPa (Tensile)} \qquad \textbf{Ans.}$$

The hoop strain at the inner surface of the outer cylinder is given by

$$\in_h = \frac{1}{E}\left(\sigma_{h_{max}} + \nu\sigma_{r_{r=R_3}}\right) \qquad \text{(using equation (12.64) and putting } \sigma_l = 0)$$

$$= \frac{1}{2 \times 10^5}(90 + 0.3 \times 14.48) = 0.000471$$

Increase in inside diameter of the outer cylinder is

$$= 0.000471 \times 170 \text{ mm} = 0.08 \text{ mm} \qquad \text{...(1)}$$

Inner cylinder

The boundary conditions are:

At $\quad r = R_3 = 85$ mm, $\sigma_r = 14.48$ MPa

and at $\quad r = R_1 = 55$ mm, $\sigma_r = 0$

On using boundary conditions, the Lame's equations are:

$$14.48 = \frac{B}{(85 \times 10^{-3})^2} - A$$

and
$$0 = \frac{B}{(55 \times 10^{-3})^2} - A$$

Solving these equations, we get

$$B = -0.0753$$

and
$$A = -24.9$$

Hence, the hoop stress at $r = R_1 = 55$ mm is given by

$$\sigma_{h_{r=R_1}} = \frac{B}{R_1^2} + A$$

$$= \frac{-0.0753}{(55 \times 10^{-3})^2} - 24.9$$

$$= -49.8 \text{ MPa}$$ **Ans.**

The negative sign signifies its compressive nature. The hoop stress at $r = R_3 = 85$ mm is given by

$$\sigma_{h_{r=R_3}} = \frac{B}{R_3^2} + A$$

$$= \frac{-0.0753}{(85 \times 10^{-3})^2} - 24.9$$

$$= -35.32 \text{ MPa}$$ **Ans.**

The negative sign signifies its compressive nature.

The hoop strain at the outer surface of the inner cylinder is given by

$$\in_h = \frac{1}{E}(\sigma_{h_{r=R_3}} + v\sigma_{r_{r=R_3}})$$

$$= \frac{1}{2 \times 10^5}(-35.32 + 0.3 \times 14.48)$$

$$= -0.000154$$

The negative sign indicates that diameter is decreased. The decrease in outside diameter of the inner cylinder is

$$0.000154 \times 170 \text{ mm} = 0.026 \text{ mm} \qquad ...(2)$$

The shrinkage allowance at the common surface is

$$\frac{(1) + (2)}{2}$$

$$= \frac{0.08 + 0.026}{2} = 0.053 \text{ mm}$$ **Ans.**

The stress distribution across cross-section of the cylinder is shown in Fig. 12.19.

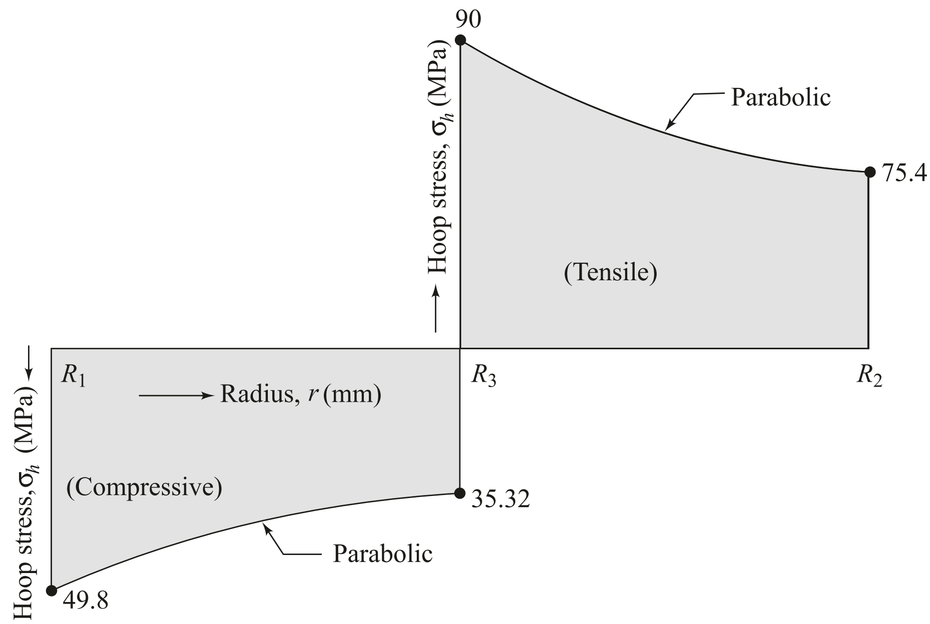


Fig. 12.19 Hoop stress distribution diagram.

Example 12.23

A bronze cylinder of outside diameter 130 mm and inside diameter 89.97 mm is forced on to a steel cylinder of outside diameter 90 mm and inside diameter 50 mm. Find the maximum resulting stresses in both cylinders. Take $E_b = 10^5$ MPa, $E_s = 2 \times 10^5$ MPa and $v_b = v_s = 0.3$.

Solution: Refer Fig. 12.20. Given,

Outside radius of the outer (bronze) cylinder, $R_2 = \dfrac{130}{2} = 65$ mm

Inside radius of the inner (steel) cylinder, $R_1 = \dfrac{50}{2} = 25$ mm

Outside radius of the inner cylinder, $R_3 = \dfrac{90}{2} = 45$ mm

Let p_s be the radial pressure at the junction of the two cylinders.

Outer cylinder

The boundary conditions are:

At $r = R_3 = 45$ mm, $\sigma_r = p_s$

and at $r = R_2 = 65$ mm, $\sigma_r = 0$

Using these boundary conditions, the Lame's equations become

$$p_s = \frac{B}{(45 \times 10^{-3})^2} - A$$

and $$0 = \frac{B}{(65\times10^{-3})^2} - A$$

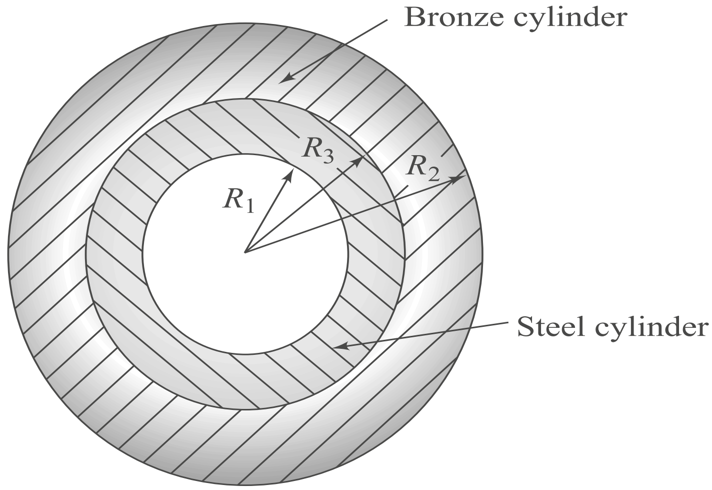


Fig. 12.20

Solving these equations, we get

$$B = 0.00388\, p_s$$

and $$A = 0.918\, p_s$$

The hoop stress at $r = R_3 = 45$ mm is given as

$$\sigma_{h_{r=R_3}} = \frac{B}{(45\times10^{-3})^2} + A$$

$$= \frac{0.00388 p_s}{(45\times10^{-3})^2} + 0.918\, p_s = 2.834 p_s \text{ (Tensile)}$$

Hoop strain at $r = R_3$ is

$$\in_h = \frac{1}{E_b}[\sigma_{h_{r=R_3}} + v_b\, \sigma_{r_{r=R_3}}]$$

$$= \frac{1}{E_b}[\sigma_{h_{r=R_3}} + v_b\, p_s] \qquad (\text{at } r = R_3,\ \sigma_r = p_s)$$

$$= \frac{1}{10^5}[2.834 p_s + 0.3\, p_s] = 3.134\times10^{-5}\, p_s$$

Increase in inside diameter of the bronze cylinder is

$$= \in_h \times 90$$

$$= 3.134\times10^{-5}\, p_s \times 90$$

$$= 2.82\times10^{-3}\, p_s \text{ mm} \qquad \ldots (1)$$

Inner cylinder

The boundary conditions are:

At $r = R_3 = 45$ mm, $\sigma_r = p_s$

and at $r = R_1 = 25$ mm, $\sigma_r = 0$

The Lame's equations, on substituting these boundary conditions, become

$$p_s = \frac{B}{(45 \times 10^{-3})^2} - A$$

and

$$0 = \frac{B}{(25 \times 10^{-3})^2} - A$$

Solving these equations, we get

$$B = -0.0009\, p_s$$

and

$$A = -1.44\, p_s$$

The hoop stress at $r = R_1 = 25$ mm is maximum, given by

$$\sigma_{h_{max}} = \frac{B}{R_1^2} + A$$

$$= \frac{-0.0009 p_s}{(25 \times 10^{-3})^2} - 1.44\, p_s = -2.88\, p_s$$

The negative sign signifies its compressive nature.

The hoop stress at $r = R_3 = 45$ mm is

$$\sigma_{h_{r=R_3}} = \frac{-0.0009 p_s}{(45 \times 10^{-3})^2} - 1.44\, p_s = -1.88\, p_s$$

The hoop strain at $r = R_3$ is given as

$$\in_h = \frac{1}{E_s}[\sigma_{h_{r=R_3}} + v_s\, \sigma_{r_{r=R_3}}]$$

$$= \frac{1}{E_s}[\sigma_{h_{r=R_3}} + v_s\, p_s] \qquad (\sigma_r = p_s)$$

$$= \frac{1}{2 \times 10^5}\, [-1.88 p_s + 0.3 p_s] = -7.9 \times 10^{-6}\, p_s$$

The negative sign signifies decrease in diameter. The decrease in outside diameter of steel cylinder is

$$\in_h \times 90$$

$$= 7.9 \times 10^{-6}\, p_s \times 90 = 7.11 \times 10^{-4}\, p_s \text{ mm} \qquad ...(2)$$

The difference in diameters is given as

$$(1) + (2)$$

$$= (2.82 \times 10^{-3}\, p_s + 7.11 \times 10^{-4}\, p_s) \text{ mm}$$

$$= 0.03 \text{ mm (Given)}$$

Hence, $p_s = 8.5$ MPa

The maximum resulting (hoop) stress occurs at $r = R_3 = 45$ mm in the bronze cylinder, given by

$$2.834\, p_s$$
$$= 2.834 \times 8.5$$
$$= 24 \text{ MPa (Tensile)} \qquad \textbf{Ans.}$$

The maximum hoop stress occurs at $r = R_1 = 25$ mm in the steel cylinder, given by

$$-2.88\, p_s$$
$$= -2.88 \times 8.5$$
$$= -24.5 \text{ MPa} \qquad \textbf{Ans.}$$

The negative sign signifies its compressive nature.

Example 12.24

A steel rod of diameter 50 mm is forced into a bronze casing of outside diameter 90 mm, producing a tensile hoop stress of 30 MPa at the outside diameter of the casing. Find the following parameters:

(*a*) the radial pressure between the rod and casing

(*b*) the shrinkage allowance

(*c*) the rise in temperature which would just eliminate the force fit

Take $E_s = 2 \times 10^5$ MPa, $v_s = 0.25$, $\alpha_s = 1.2 \times 10^{-5}$/°C

$E_b = 1 \times 10^5$ MPa, $v_b = 0.3$, $\alpha_b = 1.9 \times 10^{-5}$/°C.

Solution: Refer Fig. 12.21. Given,

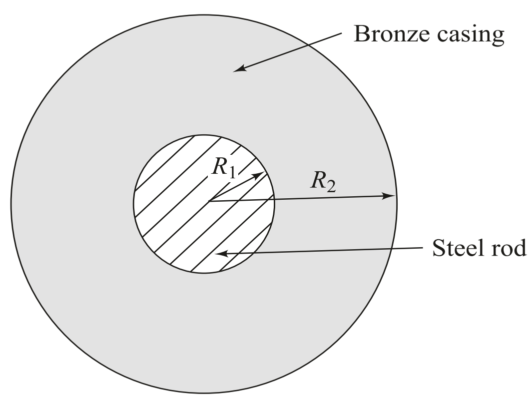


Fig. 12.21

Radius of the steel rod, $R_1 = \dfrac{50}{2} = 25$ mm

Inside radius of the casing, $R_1 = \dfrac{90}{2}$ 25 mm

Outside radius of the casing, $R_2 = \dfrac{B}{r^2} = 45$ mm

Bronze casing

Using Lame's equations, we have

$$\sigma_h = \frac{B}{r^2} + A$$

and $$\sigma_r = \frac{B}{r^2} - A$$

The boundary conditions are:

At $r = 45$ mm, $\sigma_h = 30$ MPa

and at $r = 45$ mm, $\sigma_r = 0$

Using these boundary conditions in the Lame's equations, we have

$$30 = \frac{B}{(45 \times 10^{-3})^2} + A$$

and $$0 = \frac{B}{(45 \times 10^{-3})^2} - A$$

Solving these equations, we get

$$B = 0.03$$

and $$A = 15$$

(*a*) The radial pressure between rod and casing is given as

$$25_{r_{r = 25\,\text{mm}}} = \frac{B}{(25 \times 10^{-3})^2} - A$$

$$= \frac{0.03}{(25 \times 10^{-3})^2} - 15$$

$= 33$ MPa (Compressive) **Ans.**

The hoop stress at $r = 25$ mm is given as

$$25_{r_{r = 25\,\text{mm}}} = \frac{B}{(25 \times 10^{-3})^2} + A$$

$$= \frac{0.03}{(25 \times 10^{-3})^2} + 15 = 63 \text{ MPa}$$

The hoop strain at the inside diameter of the casing is given as

$$\in_h = \frac{1}{E_b}[\sigma_{h_{r = 25\,\text{mm}}} + v_b \,.\, \sigma_{r_{r = 25\,\text{mm}}}]$$

$$= \frac{1}{1 \times 10^5}[63 + 0.3 \times 33] = 7.29 \times 10^{-4}$$

The increase in inside diameter of the casing is

$$\in_h \times 50 = 7.29 \times 10^{-4} \times 50 = 0.03645 \text{ mm} \qquad ...(1)$$

Steel rod

The hoop stress at r = 25 mm is given as

$$\sigma_h = -\sigma_{r_{r=25}} = -33 \text{ MPa}$$

Hoop strain is

$$\in_h = \frac{1}{E_s}[\sigma_h + v_s\, \sigma_{r_{r=25\text{mm}}}]$$

$$= \frac{1}{2 \times 10^5}[-33 + 0.25 \times 33] = -1.24 \times 10^{-4}$$

The negative sign signifies decrease in diameter of the rod. The decrease in diameter is

$$\in_h \times 50$$

$$= 1.24 \times 10^{-4} \times 50 = 0.0062 \text{ mm} \qquad ...(2)$$

(*b*) The difference in diameter at the common surface is given by the sum of (1) and (2).

$$0.03645 + 0.0062 = 0.04265 \text{ mm}$$

Hence, shrinkage allowance $= \dfrac{0.04265}{2} = 0.021325$ mm **Ans.**

(*c*) Let ΔT be the rise in temperature.

$$50 \times (\alpha_b - \alpha_s) \times \Delta T = 0.04265$$

$$50 \times (1.9 \times 10^{-5} - 1.2 \times 10^{-5}) \times \Delta T = 0.04265$$

Solving, we get

$$\Delta T = 122°\text{C}$$ **Ans.**

12.12 STRESSES IN A THICK SPHERICAL SHELL

Consider a thick spherical shell of inside radius R_1 and outside radius R_2 being subjected to an internal fluid pressure p (Fig. 12.22).

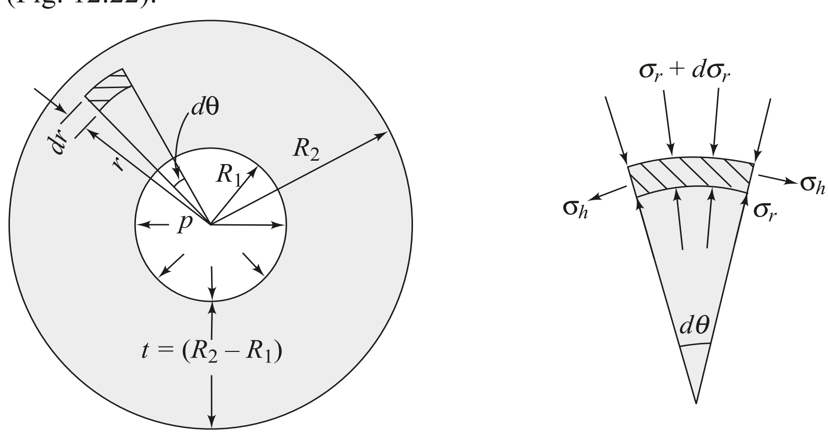


Fig. 12.22 A thick spherical shell.

To make the analysis of the spherical shell, an elementary shell of radius r and thickness dr is considered. It subtends an angle $d\theta$ at the centre of the spherical shell.

Let due to fluid pressure, the radius of the elementary shell r changes to $(r + u)$ and dr changes to $(dr + du)$.

The hoop strain in the elementary shell is given by

$$\in_h = \frac{2\pi(r+u) - 2\pi r}{2\pi r} = \frac{u}{r}$$

The radial strain in the elementary shell is given by

$$\in_r = \frac{(dr + du) - dr}{dr}$$

$$= \frac{du}{dr} = \frac{d}{dr}(r \in_h) \qquad \text{(on substituting } u\text{)}$$

$$= \in_h + r\frac{d\in_h}{dr} \qquad ...(1)$$

Let σ_h = Hoop stress at radius r

σ_r = Radial stress at radius r

$\sigma_r + d\sigma_r$ = Radial stress at radius $(r + dr)$

Bursting force acting on the elementary shell is

$$\pi r^2\sigma_r - \pi(r + dr)^2(\sigma_r + d\sigma_r)$$
$$= -\pi r(2\sigma_r\, dr + r\,.\,d\sigma_r) \qquad \text{(neglecting smaller terms)}$$

Resisting force is

$$\sigma_h\,.\,2\pi r\,.\,dr$$

For equilibrium

$$\text{Bursting force} = \text{Resisting force}$$

$$-\pi r(2\sigma_r\, dr + r.d\sigma_r) = \sigma_h\,.\,2\pi r.dr$$

$$-2\sigma_r\, dr - r\,.\,d\sigma_r = 2\sigma_h dr$$

or

$$\sigma_h = -\sigma_r - \frac{r}{2}.\frac{d\sigma_r}{dr}$$

On differentiation w.r.t. r, we have

$$\frac{d\sigma_h}{dr} = -\frac{d\sigma_r}{dr} - \frac{1}{2}\left(\frac{d\sigma_r}{dr} + r\,.\,\frac{d^2\sigma_r}{dr^2}\right)$$

There are three principal stresses at any point at a distance r from the centre of the spherical shell.

These are:

- Radial stress, σ_r (compressive)
- Hoop stress, σ_h (tensile)
- Hoop stress σ_h (tensile) acting at right angle.

Radial strain at the point is

$$\in_r = -\frac{\sigma_r}{E} - v\frac{\sigma_h}{E} - v\frac{\sigma_h}{E} = -\frac{1}{E}(\sigma_r + 2v\sigma_h) \quad ...(12.91)$$

The negative sign signifies its compressive nature. The hoop strain at the point is

$$\in_h = \frac{\sigma_r}{E} - v\frac{\sigma_h}{E} - v\frac{\sigma_h}{E}$$

$$= \frac{1}{E}[(1-v)\sigma_h + v\sigma_r] \text{ (Tensile)} \quad ...(12.92)$$

Substituting $\in_r$ and $\in_h$ in equation (1), we have

$$-\frac{1}{E}(\sigma_r + 2v\sigma_h) = \frac{1}{E}[(1-v)\sigma_h + \sigma_r] + r\frac{d}{dr}\left[(1-v)\frac{\sigma_h}{E} + v\frac{\sigma_r}{E}\right]$$

On simplification, we get

$$(1+v)\,\sigma_r + (1+v)\,\sigma_h + r\,(1-v)\,\frac{d\sigma_h}{dr} + rv\frac{d\sigma_r}{dr} = 0$$

Now, substituting σ_h and $\dfrac{d\sigma_h}{dr}$ in the above equation and simplifying, we have

$$r\frac{d^2\sigma_r}{dr^2} + 4\frac{d\sigma_r}{dr} = 0$$

Substituting $\dfrac{d\sigma_r}{dr} = R$ in the above equation results in

$$r\frac{dR}{dr} + 4R = 0$$

or

$$\frac{dR}{R} + 4\frac{dr}{r} = 0$$

On integration, we have

$$\log_e R = -4\log_e r + \log_e C_1$$

where C_1 is the constant of integration.

$$\log_e R = -\log_e r^4 + \log_e C_1$$

$$\log_e R + \log_e r^4 = \log_e C_1$$

$$\log_e (R.\, r^4) = \log_e C_1$$

$$R\,.\,r^4 = C_1$$

$$R = \frac{C_1}{r^4}$$

or

$$\frac{d\sigma_r}{dr} = \frac{C_1}{r^4}$$

On integration, we have

$$\sigma_r = -\frac{C_1}{3r^3} + C_2$$

where C_2 is another constant of integration.

Substituting σ_r and $\frac{d\sigma_r}{dr}$ in equation of σ_h, we have

$$\sigma_h = \frac{C_1}{3r^3} - C_2 - \frac{r}{2}\frac{C_1}{r^4}$$

$$= \frac{C_1}{3r^3} - C_2 - \frac{C_1}{2r^3}$$

$$= -\frac{C_1}{6r^3} - C_2$$

Using $C_1 = -6B$ and $C_2 = -A$, we get

$$\sigma_r = \frac{2B}{r^3} - A \quad ...(12.93)$$

and

$$\sigma_h = \frac{B}{r^3} + A \quad ...(12.94)$$

The hoop stress is maximum at $r = R_1$ and the radial stress zero at $r = R_2$.

Above equations are known as Lame's equations for thick spherical shells. Constants A and B are determined by using suitable boundary conditions.

Example 12.25

A thick spherical shell of inside diameter 200 mm is subjected to an internal fluid pressure of 80 MPa. Find the thickness of the shell, if the maximum permissible tensile stress in the shell is 180 MPa.

Solution: Given,

Inside radius of the spherical shell, $R_1 = \frac{200}{2} = 100$ mm

Internal fluid pressure, $p = 80$ MPa

Maximum tensile (hoop) stress, $\sigma_{h_{max}} = 180$ MPa

Using Lame's equations for the thick spherical shell, we have

$$\sigma_h = \frac{B}{r^3} + A$$

$$\sigma_r = \frac{2B}{r^3} - A$$

The boundary conditions are:

At $r = R_1 = 100$ mm, $\sigma_r = 80$ MPa

and at $r = R_1 = 100$ mm, $\sigma_h = \sigma_{h_{max}} = 180$ MPa

On using boundary conditions, the Lame's equations become

$$180 = \frac{B}{(100 \times 10^{-3})^3} + A$$

and
$$80 = \frac{2B}{(100 \times 10^{-3})^3} - A$$

Solving these equations, we get

$$B = 0.0866$$

and
$$A = 93.4$$

Now at
$$r = R_2,\ \sigma_r = 0$$

$$0 = \frac{2 \times 0.0866}{R_2^3} - 93.4$$

On solving, we get
$$R_2 = 123 \text{ mm}$$

Outside diameter of the shell = 2 × 123 = 246 mm

Hence, the thickness of the shell is given as

$$t = R_2 - R_1$$
$$= (123 - 100) \text{ mm}$$
$$= 23 \text{ mm}$$
Ans.

SHORT ANSWER QUESTIONS

1. What are pressure vessels? Name a few pressure vessels.
2. How do thin-walled pressure vessels differ from thick-walled pressure vessels?
3. What is hoop stress? Why is it so named?
4. What is the relationship between hoop stress and longitudinal stress for a thin cylindrical shell?
5. Why are steel wires wounded on a thin cylinder?
6. What is Lame's theory? For which type of pressure vessels it is used?
7. Why are cylinders compounded?
8. Why is shrinkage allowance provided in a compound cylinder? How is it defined?
9. What are the important assumptions for the Lame's theory?

MULTIPLE CHOICE QUESTIONS

1. For a thin-walled shell, the diameter-thickness ratio is

(*a*) less than 20 (*b*) greater than 20 (*c*) equal to 20 (*d*) equal to 10.

2. For a thick-walled shell, the diameter-thickness ratio is

(*a*) less than 20 (*b*) greater than 20 (*c*) equal to 20 (*d*) equal to 10.

3. Thin and thick walled pressure vessels are subjected to two types of stresses. They are

(*a*) bending stress and shear stress (*b*) tensile stress and compressive stress

(*c*) hoop stress and longitudinal stress (*d*) hoop stress and bending stress.

4. The hoop stress is also known as

(*a*) longitudinal stress (*b*) circumferential stress

(*c*) bending stress (*d*) compressive stress.

5. For a thin cylindrical shell of diameter d and thickness t, being subjected to a fluid pressure p, hoop stress is given by

(*a*) $\frac{pd}{3t}$ (*b*) $\frac{pd}{8t}$ (*c*) $\frac{pd}{2t}$ (*d*) $\frac{pd}{4t}$.

6. In the above question, the longitudinal stress is given by

(*a*) $\frac{pd}{3t}$ (*b*) $\frac{pd}{8t}$ (*c*) $\frac{pd}{2t}$ (*d*) $\frac{pd}{4t}$.

7. For a thin cylindrical shell, longitudinal stress is equal to

(*a*) hoop stress (*b*) two times the hoop stress

(*c*) three times the hoop stress (*d*) one-half of hoop stress.

8. The hoop stress is considered as

(*a*) compressive stress (*b*) bending stress

(*c*) minor principal stress (*d*) major principal stress.

9. The hoop stress and the longitudinal stress act at the following angle to each other

(*a*) 45° (*b*) 60° (*c*) 90° (*d*) 180°.

10. The difference of the hoop stress and the longitudinal stress, for a thin cylindrical shell of diameter d and thickness t, being subjected to a pressure p, is

(*a*) $\frac{pd}{4t}$ (*b*) $\frac{pd}{16t}$ (*c*) $\frac{pd}{8t}$ (*d*) $\frac{pd}{3t}$.

11. The maximum shear stress in Question No. 10 is

(*a*) $\frac{pd}{4t}$ (*b*) $\frac{pd}{16t}$ (*c*) $\frac{pd}{8t}$ (*d*) $\frac{pd}{3t}$.

12. Which is the correct relationship?

(*a*) Gauge pressure – Atmospheric pressure = Absolute pressure

(*b*) Gauge pressure + Absolute pressure = Atmospheric pressure

(*c*) Gauge pressure + Atmospheric pressure = Absolute pressure

(*d*) Gauge pressure + Vaccum pressure = Absolute pressure.

13. The hoop strain for a thin cylindrical shell of diameter (*d*), thickness (*t*), Poisson's ratio (*v*) and being subjected to a pressure (*p*) is

(*a*) $\frac{pd}{4tE}(1-v)$ (*b*) $\frac{pd}{4tE}(1-2v)$ (*c*) $\frac{pd}{4tE}(2-v)$ (*d*) $\frac{pd}{4tE}(1+v)$.

14. In Question No. 13, the longitudinal strain is

(*a*) $\frac{pd}{4tE}(1-v)$ (*b*) $\frac{pd}{4tE}(1-2v)$ (*c*) $\frac{pd}{4tE}(2-v)$ (*d*) $\frac{pd}{4tE}(1+v)$.

15. For the Question No. 13, the volumetric strain is

(*a*) $\frac{pd}{4tE}(5-3v)$ (*b*) $\frac{pd}{3tE}(5-4v)$ (*c*) $\frac{pd}{4tE}(5-4v)$ (*d*) $\frac{pd}{4tE}(5+4v)$.

16. For a thin spherical shell, the

(*a*) hoop stress is two times the longitudinal stress

(*b*) longitudinal stress is two times the hoop stress

(*c*) hoop stress is equal to one-half of the longitudinal stress

(*d*) hoop and longitudinal stresses are equal.

17. The expression for volumetric strain for a thin spherical shell of diameter (*d*), thickness (*t*) and Poisson's ratio (*v*) and being subjected to a pressure (*p*) is

(*a*) $\frac{pd}{4tE}(1+v)$ (*b*) $\frac{pd}{4tE}(5-4v)$ (*c*) $\frac{3pd}{4tE}(1-v)$ (*d*) $\frac{3pd}{4tE}(1+v)$.

18. In Question No. 17, the hoop stress is

(*a*) $\frac{pd}{2t}$ (*b*) $\frac{pd}{3t}$ (*c*) $\frac{pd}{8t}$ (*d*) $\frac{pd}{4t}$.

19. For a thin cylindrical shell with hemispherical ends, the ratio of thicknesses of the cylindrical part and hemispherical ends being subjected to equal hoop stress, is

(*a*) 0.5 (*b*) 1.5 (*c*) 2.0 (*d*) 2.5.

20. In Question No. 19, if no distortion occurs at the junction of the two parts, the ratio of thicknesses for a Poisson's ratio *v* is given as

(*a*) $\frac{1-v}{2-v}$ (*b*) $\frac{2-v}{1-v}$ (*c*) $\frac{2v-1}{1-v}$ (*d*) $\frac{1+v}{2-v}$.

21. The unit of the constant *A* in the Lame's equation is

(*a*) newton (*b*) newton-metre (*c*) newton-sec (*d*) pascal.

22. The unit of the constant B in the Lame's equation is

(*a*) newton (*b*) newton-metre

(*c*) newton-sec (*d*) pascal.

23. Lame's equations are used to find the

(*a*) bending and hoop stresses (*b*) hoop and radial stresses

(*c*) hoop and longitudinal stresses (*d*) axial and bending stresses.

24. Lame's equations are applicable in case of

(*a*) thin-walled pressure vessels

(*b*) thick-walled pressure vessels

(*c*) both thin-and thick-walled pressure vessels

(*d*) members of elliptical section.

25. In case of only internal pressure acting in the cylinder, the maximum radial stress is equal to

(*a*) one-half of internal pressure (*b*) two-third of internal pressure

(*c*) internal pressure (*d*) one-third of internal pressure.

26. In Question No. 25, maximum radial stress occurs at the

(*a*) internal surface of the cylinder (*b*) external surface of the cylinder

(*c*) centre of the cylinder (*d*) none of these.

27. The minimum radial stress occurs at the following location of a thick cylinder, when subjected to internal pressure only

(*a*) internal surface of the cylinder (*b*) external surface of the cylinder

(*c*) centre of the cylinder (*d*) none of these.

28. The hoop stress variation in a thick cylinder is shown by a/an

(*a*) straight line (*b*) parabola

(*c*) ellipse (*d*) hyperbola.

29. The maximum shear stress in case of a thick cylinder is

(*a*) $\frac{\sigma_h + \sigma_r}{2}$ (*b*) $\frac{\sigma_h - \sigma_r}{2}$ (*c*) $\frac{\sigma_h + \sigma_l}{2}$ (*d*) $\frac{\sigma_h - \sigma_l}{2}$.

where σ_h = Hoop stress

σ_r = Radial stress

σ_l = Longitudinal stress.

30. The relationship among volumetric strain hoop strain and longitudinal strain is

(*a*) Hoop strain = 2 × Volumetric strain + Longitudinal strain

(*b*) Volumetric strain = 2 × Hoop strain + Longitudinal strain

(*c*) Longitudinal strain = 2 × Hoop strain + Volumetric strain

(*d*) Longitudinal strain = 3 × Hoop strain + Volumetric strain.

31. In a thick cylinder, the radial stress at outer surface is usually

(*a*) more than zero (*b*) less than zero (*c*) zero (*d*) not defined.

32. In a thick cylinder, the radial stress at inner surface is

(*a*) independent of fluid pressure (*b*) more than fluid pressure

(*c*) less than fluid pressure (*d*) equal to fluid pressure.

33. The cylinders are compounded to

1. increase the strength of the cylinder
2. increase the pressure bearing capacity of the cylinder
3. make the distribution of hoop stress uniform

Of these:

(*a*) 1 alone is true (*b*) 2 alone is true (*c*) 2 and 3 are true (*d*) 1, 2 and 3 are true.

ANSWERS

1. (*b*)	**2.** (*a*)	**3.** (*c*)	**4.** (*b*)	**5.** (*c*)	**6.** (*d*)	**7.** (*d*)	**8.** (*d*)	**9.** (*c*)
10. (*a*)	**11.** (*c*)	**12.** (*c*)	**13.** (*c*)	**14.** (*b*)	**15.** (*c*)	**16.** (*d*)	**17.** (*c*)	**18.** (*d*)
19. (*c*)	**20.** (*b*)	**21.** (*d*)	**22.** (*a*)	**23.** (*b*)	**24.** (*b*)	**25.** (*c*)	**26.** (*a*)	**27.** (*b*)
28. (*b*)	**29.** (*a*)	**30.** (*b*)	**31.** (*c*)	**32.** (*d*)	**33.** (*b*).			

EXERCISES

1. A boiler is made of 10 mm thick plate having allowable tensile stress of 80 MPa. It is subjected to a fluid pressure of 1.5 MPa. Considering longitudinal and circumferential joint efficiencies to be 80% and 50% respectively, find the maximum diameter of the shell.

(*Ans.* 533.34 mm).

2. A thin cylinder of inside diameter 300 mm is made of 6 mm thick plate. The efficiencies of the longitudinal and circumferential joints are 75% and 50% respectively. Find the largest allowable gauge pressure, if the tensile stress of the plate is not to exceed 100 MPa.

(*Ans.* 2.0 MPa).

3. A thin cylinder of inside diameter 250 mm is made of steel plate of thickness 12 mm and having yield strength of 390 MPa. Find the maximum fluid pressure, assuming a factor of safety of 2.5 on the maximum shear stress. (*Ans.* 15 MPa).

4. A thin cylindrical shell of diameter 1.5 m and length 3.5 m is subjected to a fluid pressure of 2 MPa. Find the increase in volume of the shell, if the maximum tensile stress is limited to 60 MPa. Take E = 200 GPa and v = 0.3. (*Ans.* 2.44×10^5 mm^3).

5. A thin spherical vessel of diameter 250 mm and thickness 10 mm is filled with water. Find the extra amount of water to be pumped until the pressure reaches 3 MPa. Take E = 200 GPa and v = 0.3. (*Ans.* 1610.7 mm^3).

6. A spherical shell of diameter 2 m and wall thickness 12 mm is to accommodate 1000 mm^3 of extra water. Find the corresponding pressure. Take E = 200 GPa and v = 0.3.

(*Ans* 545.67 Pa).

7. A thin cylindrical shell of diameter 200 mm and wall thickness 6 mm has spherical ends. Determine the thickness of the hemispherical ends, assuming that no distortion occurs at the junction of the two parts. Take E = 200 GPa and v = 0.3. (*Ans.* 2.47 mm).

8. The cylinder of a hydraulic press of inside diameter 200 mm is subjected to an internal pressure of 25 MPa. Find the thickness of the cylinder, assuming that the maximum permissible tensile stress is limited to 80 MPa. Also, draw the diagram showing the variation of radial and hoop stresses across the thickness of the cylinder. (*Ans.* 38.2 mm).

9. A thick cylinder of inside diameter 400 mm and outside diameter 500 mm is subjected to internal and external pressures both. If the internal pressure is 25 MPa and the hoop stress at the inside of the cylinder is 45 MPa (Tensile), find the intensity of external pressure.

(*Ans.* 12.4 MPa (Compressive)).

10. The internal and external diameters of a thick hollow cylinder are 80 mm and 120 mm respectively. It is subjected to an external and internal pressures of 40 MPa and 120 MPa respectively. Calculate the following parameters:

(*a*) the hoop stress at external and internal surfaces,

(*b*) the radial and hoop stresses at mean radius

(*c*) plot the stresses across the cross-section of the cylinder.

(*Ans.* (*a*) 88 MPa and 168 MPa (*b*) 68.16 MPa and 116.16 MPa).

11. A compound cylinder is made by placing an outer tube of outside diameter 240 mm into an inner tube of inside diameter 120 mm. Find diameter at the junction of the two tubes, if the maximum hoop stress in the inner tube is 2/3 times the maximum hoop stress in the outer tube.

(*Ans.* 195 mm).

12. A compound cylinder consists of an outer tube of inside and outside diameters of 160 mm and 180 mm respectively. It is shrunk fitted on to an inner tube of inside diameter 80 mm and outside diameter 160.1 mm. The cylinder is subjected to an internal fluid pressure of 300 MPa. Find the maximum hoop stress developed in each tube. Take $E = 2 \times 10^5$ MPa.

(*Ans.* Outer tube: 271.9 MPa (Tensile); Inner tube: 415.3 MPa (Tensile)).

13. A steel hoop of 200 mm outer and 130 mm inner diameters, is shrunk on a hollow steel cylinder of 80 mm inner diameter, the pressure of shrinkage being 20 MPa. For an internal fluid pressure of 70 MPa, find the following parameters:

(*a*) the greatest circumferential stress induced in the cylinder,

(*b*) the radial pressure between the cylinder and the hoop

(*c*) the greatest circumferential stress in the hoop.

(*Ans.* (*a*) 32.3 MPa (Tensile) (*b*) 38.2 MPa (Compressive) (*c*) 94.2 MPa (Tensile)).

14. A high tensile steel tyre, 2 cm thick, is shrunk on a cast-iron rim having 48 cm outside diameter and 6 cm thick. Find the inside diameter of the steel tyre to the nearest thousandth of a cm, if after shrinking on, the tyre exerts a radial pressure of 50 MPa on the cast-iron rim.
Take $E_s = 210$ GPa, $E_{CI} = 100$ GPa and $v_s = v_{CI} = 0.25$. (*Ans.* 47.774 cm).

15. Calculate the thickness of a thick spherial shell of inside diameter 150 mm, being subjected to an internal fluid pressure of 25 MPa, if the maximum permissible tensile stress in the shell is 80 MPa. (*Ans.* 11.9 mm).

16. What should be the ratio of a thin cylindrical shell to the thickness of its hemispherical end for a pressure vessel subjected to internal fluid pressure so that the junction section remains free from unequal deformation?

(*Ans.* $\dfrac{2-v}{1-v}$, v being Possion's ratio).

17. A seamless spherical shell, 0.90 m in diameter and 10 mm thick, is being filled with a fluid under pressure until the volume increases by 150 cm^3. Calculate the pressure exerted by the fluid on the shell, taking modulus of elasticity for the material of the shell as 200 GPa and Poisson's ratio as 0.3. (*Ans.* 1.66 MPa).

18. A 1 mm diameter steel wire is wound around a copper tube with external and internal diameter of 140 mm and 120 mm respectively, to increase the strength of the tube against internal pressure. What initial tension must be given to the wire so that maximum allowable stresses for the tube and wire material namely 90 and 200 MPa respectively are reached simultaneously. Assume Possion's ratio for copper and steel to be 0.3 and wire winding as a thin cylinder. Modulus of elasticity for copper and steel may be assumed as 10^6 and 2×10^5 MPa respectively. The tube is open at the ends. (*Ans.* 61.35 MPa).

19. A steel tube is 18 mm internal diameter and 3 mm thick. One end is closed and the other end screwed into a pressure vessel. The projected length is 300 mm. Neglecting any constraints due to the ends, calculate the safe internal pressure for the tube if allowable stress is not to exceed 150 MPa. Calculate the increase in its internal volume under this pressure. Assume Young's modulus for the tube material as 200 GPa and Poisson's ratio 0.3.

(*Ans.* 50 MPa, 108.78 mm^3).

20. A cast iron pipe of internal diameter 200 mm and thickness 50 mm carries water under a pressure of 5 MPa. Calculate the radial and tangential stresses at the inner, middle and outer surfaces.

(*Ans.* –5 MPa, –1.76 MPa, 0; 13 MPa, 9.76 MPa, 8 MPa).

21. Find the ratio of thickness to internal diameter of a thick tube subjected to internal pressure, when the pressure is 5/8 of the value of the maximum permissible circumferential stress. Find the increase in internal diameter of such a tube of 100 mm internal diameter, when the internal pressure is 100 MPa. Take E = 200 GPa and v = 0.286. (*Ans.* 0.5408, 0.0943 mm).

22. The pressure within the cylinder of a hydraulic press is 9 MPa. The inside diameter of the cylinder is 25 mm. Determine the thickness of the cylinder wall, if the permissible tensile stress is 18 MPa. (*Ans.* 9.15 mm).

23. A thick cylindrical shell of internal diameter 150 mm has to withstand an internal fluid pressure of 50 MPa. Determine its thickness so that the maximum stress in the section does not exceed 150 MPa. (*Ans.* 31.05 mm).

24. A 1 m long thin cylinder having an internal diameter of 230 mm and wall thickness 5 mm is filled with a liquid, which exerts a pressure p to produce a change in internal volume by 12×10^{-6} m^3.

Find the values of the hoop and longitudinal stresses. How these stresses change, if joint efficiencies of 45% (hoop) and 85% (longitudinal) are considered? Also, find the necessary change in pressure p to produce a further increase in internal volume of 15%. Take E = 200 GPa and v = 0.25. (*Ans.* 28.8 MPa, 14.4 MPa, 33.9 MPa, 32 MPa, 1.86 MPa).

❑❑❑

13 Plane Trusses

James Clerk Maxwell
(1831-1879)

James Clerk Maxwell, born on 13 June 1831, was a famous Scottish mathematician and physicist. He is widely acknowledged as the nineteenth century scientist who greatly influenced physics of the twentieth century. His contributions are ranked with Newton's laws of motion and Einstein's theory of relativity as the most fundamental contributions to physics. He formulated important physical and mathematical theories; one which is widely known is Maxwell's electromagnetic theory that describes electricity, magnetism and optics. He developed the kinetic theory of gases with Clausius and is credited for Maxwell-Boltzmann distribution, a statistical means to describe the behaviour of the gases in motion. He is also known for his Maxwell diagram used in strength of materials as a graphical means to analyse the rigidity of members in a truss, which is widely used in bridges and power and communication towers. He also proved that viscosity of a fluid varied directly with temperature.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- Why is a member of a truss called a two-force member?
- What is a rigid truss?
- How does the method of sections differ from the method of joints?
- Why is a member of a truss considered weightless?
- What is the difference between a plane truss and a space truss?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_13

13.1 INTRODUCTION

A structure consists of a truss or a frame. Trusses lend strength to structures. A truss consists of straight members (bars) which are joined together either by welding or riveting. But for calculation purposes, the joints are supposed to be hinged or pin-jointed. It is commonly used to support roofs of long span in buildings, bridges or a railway platform. The members of the truss which are steel members like bars, angles, channel etc. are subjected to tension or compression forces. Load is applied at the joints of truss, which generates forces in the members.

A truss consisting of members which lie in a plane and are loaded in the same plane is called *plane truss*. Hence, a plane truss is a two-dimensional structure. It is commonly visible on the sides of bridges or on the roofs of the workshops. A *space truss* is made of non-coplanar members, and its examples include shear legs, a TV tower, mobile phone tower or transmission line tower.

Frames and machines are structures containing multiforce members, that is, the members are acted upon by three or more forces, unlike a truss where each member is a two-force member. Frames are designed to support loads and are usually stationary and fully constrained structures. Machines or mechanisms are designed to transmit and modify forces and may or may not be stationary but contain moving parts having motion relative to each other.

13.2 TYPES OF TRUSSES

Trusses can be broadly classified into two groups:

- Rigid truss
- Non-rigid truss

A rigid or perfect truss is one which does not collapse and remains in equilibrium under the action of external forces acting at the joints.

For a truss to be rigid, the following condition must exist.

$$m = 2j - 3 \quad \text{...(13.1)}$$

where m = Number of members

j = Number of joints

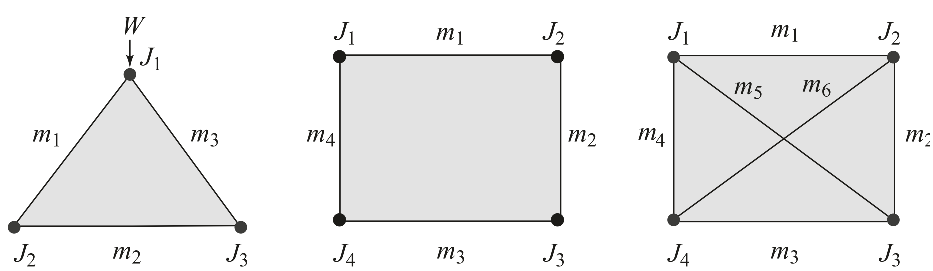
Non-rigid or imperfect truss does not obey equation (13.1). It has less number of members than the requirement or has surplus member. Accordingly it is said to be an under-rigid or over-rigid truss.

Rigid and non-rigid trusses are shown in Fig. 13.1.

For Fig. 13.1 (a)

$$j = 3$$

and $$m = 2j - 3 = 2 \times 3 - 3 = 3$$



(a) Rigid truss (b) Under-rigid truss (c) Over-rigid truss

Fig. 13.1

Hence, the truss is rigid.

For Fig. 13.1 (b)

$$j = 4$$

and $$m = 2 \times 4 - 3 = 5 \text{ (Ideal)}$$

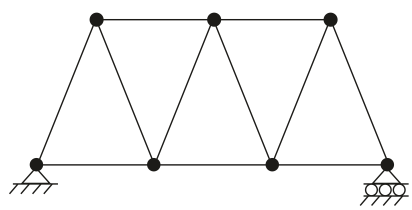
But the truss has only four members, hence it is an under-rigid truss.

For Fig. 13.1 (c)

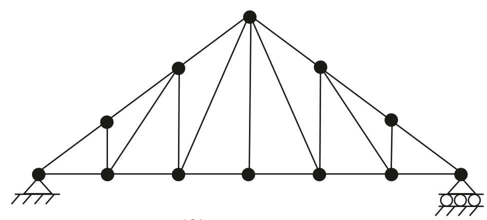
$$j = 4$$

and $$m = 2 \times 4 - 3 = 5 \text{ (Ideal)}$$

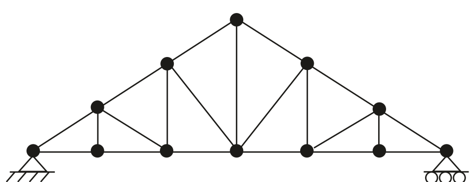
But the truss has six members making it an over-rigid truss (redundant). Some of the standard types of trusses are shown in Fig. 13.2.



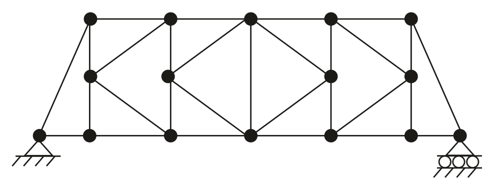
(*a*) Warren truss



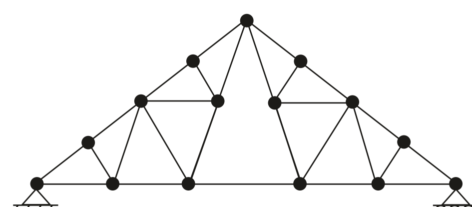
(*b*) Pratt truss



(*c*) Howe truss



(*d*) K-truss



(e) Fink truss

Fig. 13.2 Common types of trusses.

Of the two supports provided in a truss, one is a hinged support and the other a roller support. There are two reactions for the hinged support, vertical and horizontal. For roller support, there is only one normal reaction. Roller support can accomodate small changes in length of the members.

13.3 FORCES IN THE TRUSS

Each member of a truss is a two-force member, meaning thereby each member is acted upon by only two forces, one at each end. These forces have the same magnitude, same line of action-along the axis of the beam but have opposite sense. The force is only applied at the joints. The weights of the members are also assumed to be applied at the joints, half of the weight of each member being applied to each of the two joints the member connects.

The members in tension or compression are shown in Fig. 13.3. In Fig. 13.3 (*a*), the forces tend to pull the member apart, and the member is in tension, while in Fig. 13.3 (*b*), the forces tend to compress the member, and the member is in compression. The reaction (R) of the members are shown in the same figure. Hence, the member in tension will pull the joint and the member in compression will push the joint. Tension forces are assigned positive value and compression forces negative.

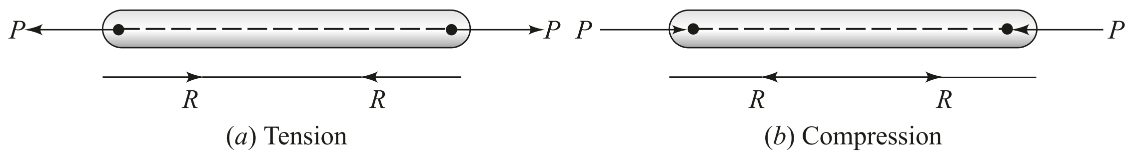


Fig. 13.3 Forces in a truss member.

13.4 ANALYSIS OF TRUSSES

The usual assumptions made in the analysis of trusses are:

- The weights of the members are negligible in comparison to the applied loads.
- The joints behave as smooth pins.
- All the loads are applied at the joints only.
- The weight of every member is divided equally on the connecting joints, for the purpose of considering their weights in the analysis.

Under these assumptions, each member of the truss is an axially loaded bar.

The internal forces in the members of the truss are obtained by the method of joints or the method of sections.

13.4.1 Analysis of Trusses by Method of Joints

This method is based on the equilibrium of forces at the joints. During the analysis, a joint is separated from the entire truss and its equilibrium is considered with the help of its free-body diagram. Since the entire truss is in equilibrium, each joint must also be in equilibrium. The equations of equilibrium, $\Sigma F_x = 0$ and $\Sigma F_y = 0$ are applied for every joint. Hence, only two equations are available for each joint.

The reactions at the supports are determined by considering the entire truss as a free body and using the equations of equilibrium of a rigid body. The method is illustrated through the following examples.

Example 13.1

In Fig 13.4, a loaded warren truss is shown. Each triangle is an equilateral one having 5 m side. Find the forces in all the members of the truss.

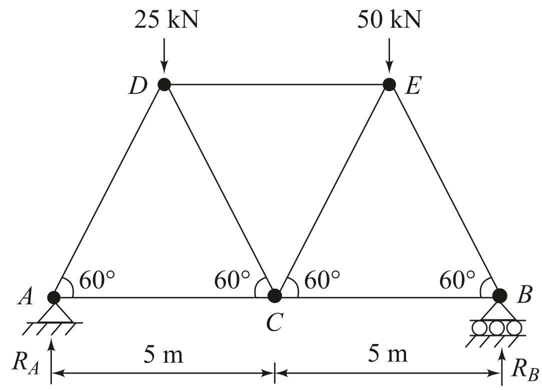


Fig. 13.4

Solution: *D* and *E* are the upper joints and *A*, *C* and *B* the lower joints of the truss. Two vertical loads of 25 kN and 50 kN are acting at joints *D* and *E* respectively.

Support reactions at *A* and *B*

Using $\Sigma M_A = 0$, we have

$$R_B \times (5+5) = 25 \times \frac{5}{2} + 50 \times \left(5 + \frac{5}{2}\right)$$

$$= \frac{875}{2}$$

or $$R_B = 43.75 \text{ kN } (\uparrow)$$

and $$R_A + R_B = 25 + 50 = 75 \text{ kN}$$

or $$R_A = 75 - R_B$$

$$= 31.25 \text{ kN } (\uparrow)$$

Forces in members *AD* and *AC*

Consider joint *A*. Assume a tension force in the desired member. The tension force is directed away from the joint under consideration (Fig. 13.5).

Using $\Sigma F_y = 0$, we have

$$31.25 + F_{AD} \sin 60^\circ = 0$$

or $$F_{AD} = -\frac{31.25}{\sin 60^\circ} = -36.08 \text{ kN} = 36.08 \text{ kN (C)}$$

Because of compressive force in the member *AD*, joint *A* is pushed by it and hence for equilibrium it must also push the joint *D*.

Using $\Sigma F_x = 0$, we have

$$F_{AC} + F_{AD} \cos 60° = 0$$

or $$F_{AC} = -F_{AD} \cos 60° = -(-36.08) \cos 60° = 18.04 \text{ kN (T)}$$

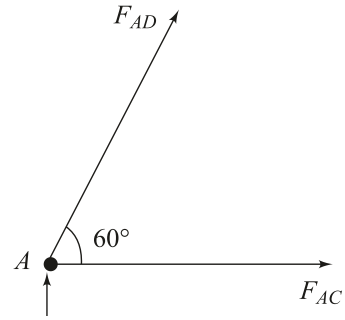


Fig. 13.5

Forces in members *DE* and *DC*

Consider joint *D*. Show all the forces acting on the joint (Fig. 13.6).

Using $\Sigma F_y = 0$, we have

$$25 + F_{DC} \cos 30° - F_{AD} \cos 30° = 0$$

$$25 + 0.866\, F_{DC} - 31.24 = 0$$

or $$F_{DC} = 7.21 \text{ kN (T)}$$

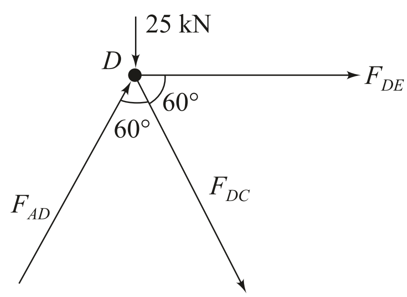


Fig. 13.6

Using $\Sigma F_x = 0$, we have

$$F_{AD} \sin 30° + F_{DE} + F_{DC} \sin 30° = 0$$

or $$18.04 + F_{DE} + 3.605 = 0$$

$$F_{DE} = -21.641 \text{ kN} = 21.641 \text{ kN (C)}$$

Forces in members *EC* and *EB*

Consider joint *E*. Forces acting on the joint are shown in Fig. 13.7. The force F_{DE} pushes joint *E* because it is compressive in nature.

Using $\Sigma F_y = 0$, we have

$$F_{EC} \sin 60° + F_{EB} \sin 60° + 50 = 0$$

or $$0.866F_{EC} + 0.866F_{EB} + 50 = 0 \quad ...(1)$$

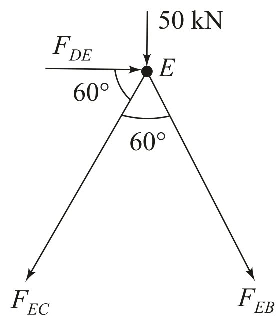


Fig. 13.7

Using $\Sigma F_x = 0$, we have

$$F_{DE} + F_{EB} \cos 60° - F_{EC} \cos 60° = 0$$

or $$21.645 + 0.5\, F_{EB} - 0.5\, F_{EC} = 0 \quad ...(2)$$

Solving equations (1) and (2), we get

$$F_{EB} = -\,50.51 \text{ kN} = 50.51 \text{ kN (C)}$$

and $$F_{EC} = -\,7.22 \text{ kN} = 7.22 \text{ kN (C)}$$

Force in member *CB*

Consider joint *C* (Fig. 13.8).

Using $\Sigma F_x = 0$, we have

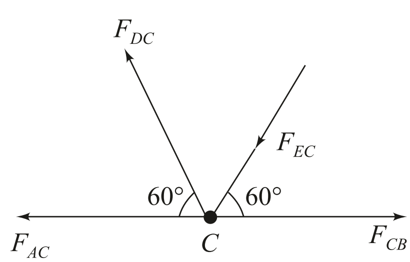


Fig. 13.8

$$F_{CB} - F_{AC} - F_{DC} \cos 60° - F_{EC} \cos 60° = 0$$

$$F_{CB} - 18.04 - 7.21 \times \cos 60° - 7.22 \cos 60° = 0$$

or $F_{CB} = 25.25$ kN (T)

The complete result is shown in Fig.13.9.

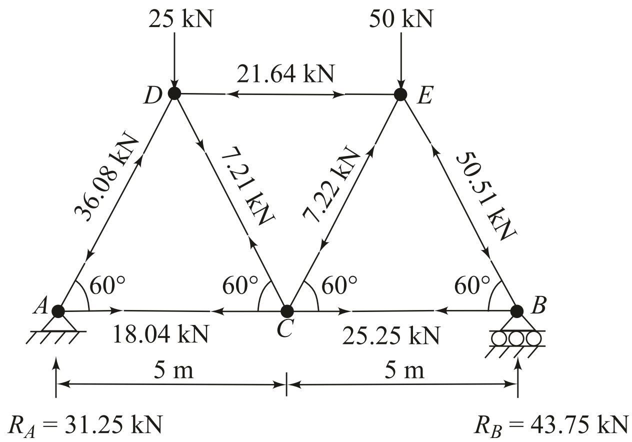


Fig. 13.9

Example 13.2

Find the forces in all the members of the loaded truss shown in Fig. 13.10. *D* and *F* are the middle points of *AE* and *BE* respectively.

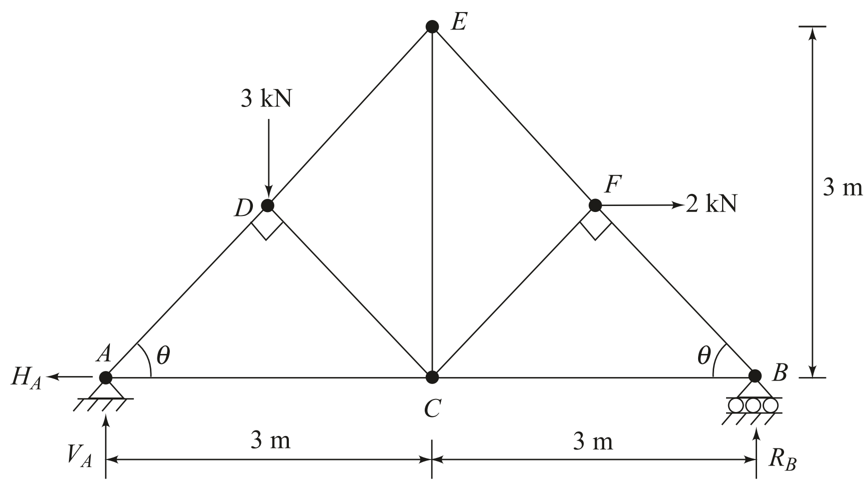


Fig. 13.10

Solution: The $\angle CAE$ can be calculated by considering $\Delta\, ACE$.

$$\tan\theta = \frac{3}{3} = 1$$

or

$$\theta = 45°$$

Support reactions at *A* and *B*

Using $\Sigma M_A = 0$, we have

$$R_B \times (3 + 3) - 2 \times 1.5 - 3 \times 1.5 = 0$$

or $$R_B = 1.25 \text{ kN } (\uparrow)$$

The vertical component of reaction at *A* is

$$V_A = 3 - R_B$$
$$= 3 - 1.25$$
$$= 1.75 \text{ kN } (\uparrow)$$

The horizontal component of reaction at *A* is

$$H_A = 2 \text{ kN } (\leftarrow)$$

Hence, the reaction at *A* is

$$R_A = \sqrt{V_A^2 + H_A^2}$$
$$= \sqrt{(1.75)^2 + (2)^2}$$
$$= 2.65 \text{ kN}$$

Direction of R_A

Refer Fig. 13.11.

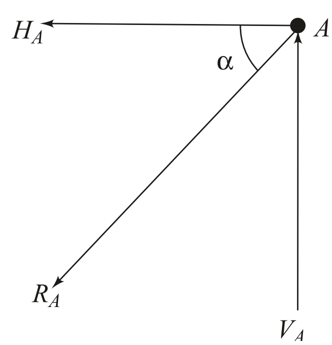


Fig. 13.11

$$\tan \alpha = \frac{V_A}{H_A} = \frac{1.75}{2.00}$$

or $$\alpha = 41.18°$$

Forces in members *AD* and *AC*

Consider joint *A*. Forces acting on the joint are shown in Fig. 13.12.

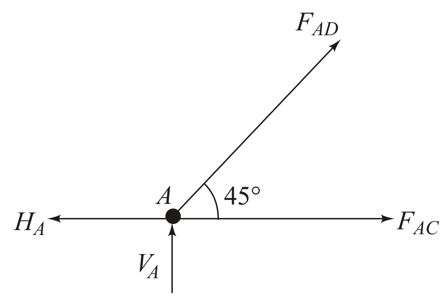


Fig. 13.12

Using $\Sigma F_y = 0$, we have

$$F_{AD} \sin 45° + V_A = 0$$
$$0.707\, F_{AD} + 1.75 = 0$$

or $$F_{AD} = -2.47 \text{ kN} = 2.47 \text{ kN (C)}$$

Using $\Sigma F_x = 0$, we have

$$F_{AC} + F_{AD} \cos 45° - H_A = 0$$
$$F_{AC} - 2.47 \times 0.707 - 2 = 0$$

or $$F_{AC} = 3.74 \text{ kN (T)}$$

Forces in members *DE* and *DC*

Consider joint D. Refer Fig. 13.13.

Using $\Sigma F_y = 0$, we have

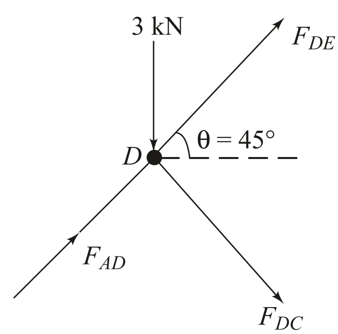


Fig. 13.13

$$3 - F_{DE} \sin 45° + F_{DC} \sin 45° - F_{AD} \sin 45° = 0$$
$$3 - 0.707 F_{DE} + 0.707 F_{DC} - 2.47 \times 0.707 = 0$$
$$1.25 - 0.707 F_{DE} + 0.707 F_{DC} = 0 \quad ...(1)$$

Using $\Sigma F_x = 0$, we have

$$F_{AD} \cos 45° + F_{DC} \cos 45° + F_{DE} \cos 45° = 0$$
$$2.47 \times 0.707 + 0.707 F_{DC} + 0.707\, F_{DE} = 0$$
$$1.75 + 0.707 F_{DC} + 0.707 F_{DE} = 0 \quad ...(2)$$

Solving equations (1) and (2), we get

$$F_{DC} = -2.12 \text{ kN} = 2.12 \text{ kN (C)}$$

and $$F_{DE} = -0.35 \text{ kN} = 0.35 \text{ kN (C)}$$

Forces in members *BF* and *BC*

Consider joint *B*. Refer Fig. 13.14.

Using $\Sigma F_x = 0$, we have

$$F_{BC} + F_{BF} \cos 45° = 0$$

$$F_{BC} + 0.707\, F_{BF} = 0 \qquad ...(3)$$

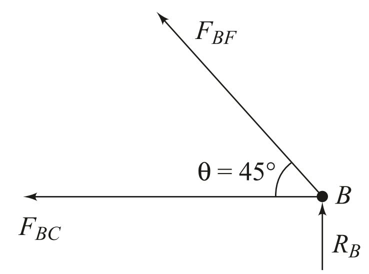


Fig. 13.14

Using $\Sigma F_y = 0$, we have

$$R_B + F_{BF} \sin 45° = 0$$

$$1.25 + 0.707\, F_{BF} = 0$$

$$F_{BF} = -1.76 \text{ kN} = 1.76 \text{ kN (C)}$$

Substituting F_{BF} in equation (3), we have

$$F_{BC} = -0.707\, F_{BF}$$

$$= -0.707 \times (-1.76) = 1.24 \text{ kN (T)}$$

Forces in members *FE* and *FC*

Consider joint F. Refer Fig. 13.15.

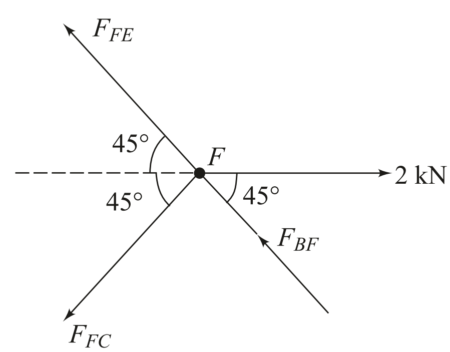


Fig. 13.15

Using $\Sigma F_x = 0$, we have

$$2 - F_{FC} \cos 45° - F_{FE} \cos 45° - F_{BF} \cos 45° = 0$$

$$\frac{2}{\cos 45°} - F_{FC} - F_{FE} - F_{BF} = 0$$

$$2.82 - F_{FC} - F_{FE} - 1.76 = 0$$

$$1.06 - F_{FC} - F_{FE} = 0 \qquad \text{...(4)}$$

Using $\Sigma F_y = 0$, we have

$$F_{FE} \sin 45° + F_{BF} \sin 45° - F_{FC} \sin 45° = 0$$

$$F_{FE} + F_{BF} - F_{FC} = 0$$

$$F_{FE} - F_{FC} + 1.76 = 0 \qquad \text{...(5)}$$

Solving equations (4) and (5), we get

$F_{FE} = -0.35$ kN = 0.35 kN (C)

and $F_{FC} = 1.41$ kN (T)

Force in member *EC*

Consider joint *E*. Refer Fig. 13.16.

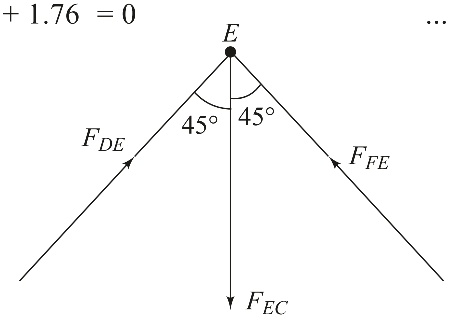


Fig. 13.16

Using $\Sigma F_y = 0$, we have

$$-F_{EC} + F_{DE} \sin 45° + F_{FE} \sin 45° = 0$$

$$-F_{EC} + 0.35 \times 0.707 + 0.35 \times 0.707 = 0$$

or $$F_{EC} = 2 \times 0.35 \times 0.707 = 0.49 \text{ kN (T)}$$

The complete result is shown in Fig. 13.17.

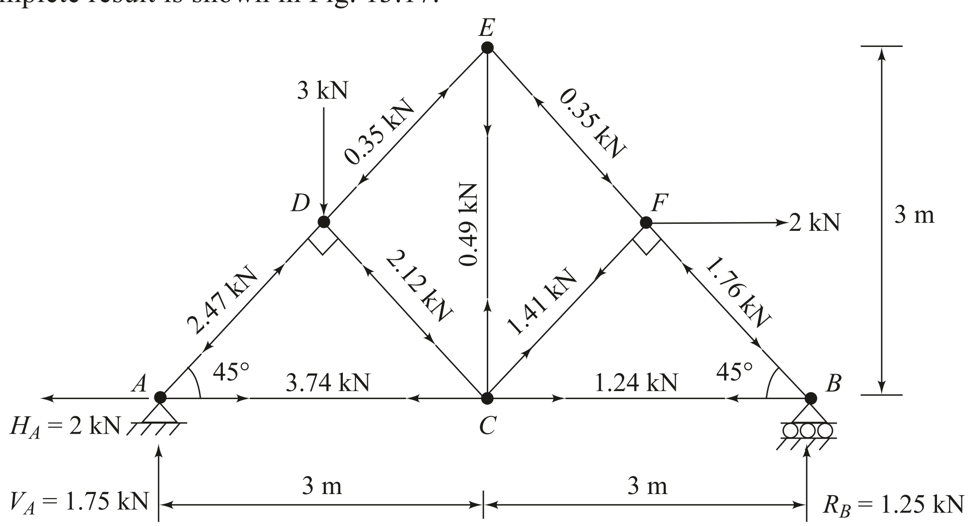


Fig. 13.17

Example 13.3

Find the forces in each member of the loaded truss shown in Fig. 13.18. All the forces are acting normal to *BD*.

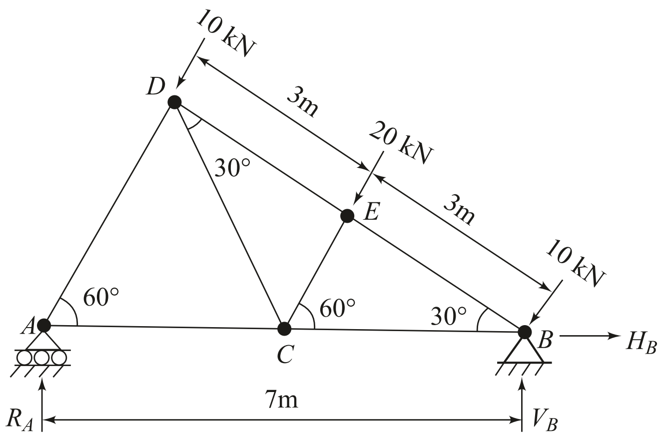


Fig. 13.18

Solution: **Support reactions at *A* and *B***

Using $\Sigma M_B = 0$, we have

$$R_A \times 7 = 20 \times 3 + 10 \times (3 + 3) = 120$$

or $$R_A = 17.14 \text{ kN } (\uparrow)$$

Resolving forces in the horizontal and vertical direction, we have

$$H_B = \text{Horizontal component of the reaction at } B$$
$$= (10 + 20 + 10) \cos 60° = 20 \text{ kN } (\rightarrow)$$
$$V_B = \text{Vertical component of the reaction at } B$$
$$= (10 + 20 + 10) \sin 60° - R_A = 17.5 \text{ kN } (\uparrow)$$

Hence, the reaction at *B* is

$$R_B = \sqrt{(20)^2 + (17.5)^2} = 26.57 \text{ kN}$$

Direction of R_B

Let R_B makes an angle θ with the horizontal as shown in Fig. 13.19.

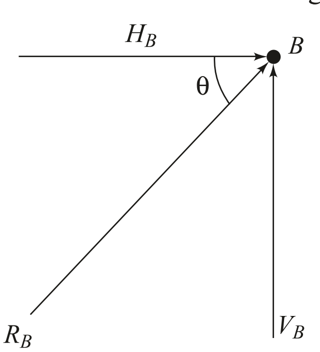


Fig. 13.19

$$\tan\theta = \frac{V_B}{H_B} = 0.875$$

Hence, $\theta = 41.18°$

Forces in members *AD* and *AC*

Consider joint A. Forces acting on the joint are shown in Fig. 13.20.

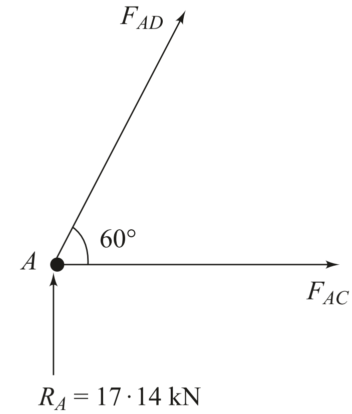


Fig. 13.20

Using $\Sigma F_y = 0$, we have

$$R_A + F_{AD} \sin 60° = 0$$

$$17.14 + 0.866\, F_{AD} = 0$$

or $F_{AD} = -\,19.79 \text{ kN} = 19.79 \text{ kN (C)}$

Using $\Sigma F_x = 0$, we have

$$F_{AC} + F_{AD} \cos 60° = 0$$

$$F_{AC} - 19.79 \cos 60° = 0$$

or $F_{AC} = 9.89 \text{ kN (T)}$

Forces in members *DC* and *DE*

Consider joint D. Refer Fig. 13.21.

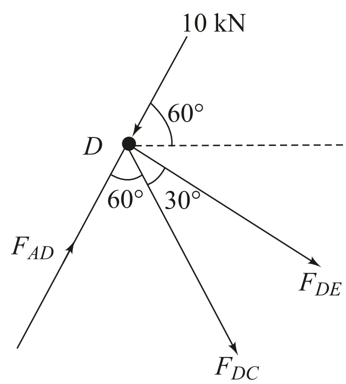


Fig. 13.21

Using $\Sigma F_x = 0$, we have

$$F_{AD} \cos 60° + F_{DC} \cos 60° + F_{DE} \cos 30° - 10 \cos 60° = 0$$

$$19.79 \times 0.5 + 0.5\, F_{DC} + 0.866\, F_{DE} - 5 = 0$$

$$0.5\, F_{DC} + 0.866\, F_{DE} + 4.895 = 0 \quad ...(1)$$

Using $\Sigma F_y = 0$, we have

$$F_{AD} \sin 60° - F_{DC} \sin 60° - F_{DE} \sin 30° - 10 \sin 60° = 0$$

$$19.79 \times 0.866 - 0.866\, F_{DC} - 0.5\, F_{DE} - 8.66 = 0$$

$$0.866\, F_{DC} + 0.5\, F_{DE} - 8.478 = 0 \quad ...(2)$$

Solving equations (1) and (2), we get

$$F_{DE} = -16.95 \text{ kN} = 16.95 \text{ kN } (C)$$

and

$$F_{DC} = 19.56 \text{ kN } (T)$$

Forces in members *BE* and *BC*

Consider joint *B*. Refer Fig. 13.22.

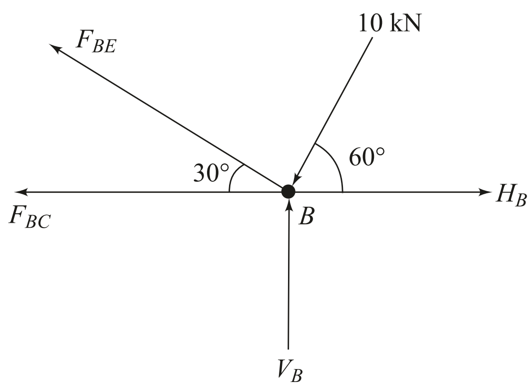


Fig. 13.22

Using $\Sigma F_x = 0$, we have

$$F_{BC} + F_{BE} \cos 30° + 10 \cos 60° = 0$$

$$F_{BC} + 0.866\, F_{BE} + 5 - H_B = 0$$

$$F_{BC} + 0.866\, F_{BE} + 5 - 20 = 0$$

$$F_{BC} + 0.866\, F_{BE} - 15 = 0 \qquad ...(1)$$

Using $\Sigma F_y = 0$, we have

$$F_{BE} \sin 30° - 10 \sin 60° + V_B = 0$$

$$0.5\, F_{BE} - 8.66 + 17.5 = 0$$

$$0.5\, F_{BE} + 8.84 = 0$$

or

$$F_{BE} = -17.68 \text{ kN} = 17.68 \text{ kN (C)}$$

From equation (1), we have

$$F_{BC} = 15 - 0.866\, F_{BE}$$

$$= 15 - 0.866 \times (-17.68) = 30.31 \text{ kN (T)}$$

Force in member *EC*

Consider joint *E*. Refer Fig. 13.23.

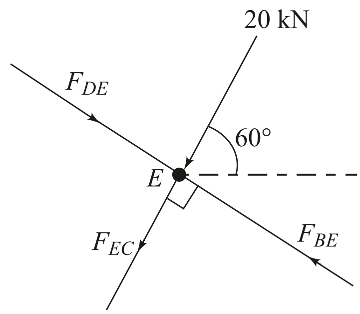


Fig. 13.23

$$F_{EC} + 20 = 0 \quad \text{(along the direction of 20 kN)}$$

or $$F_{EC} = -20 \text{ kN} = 20 \text{ kN (C)}$$

The complete result is shown in Fig. 13.24.

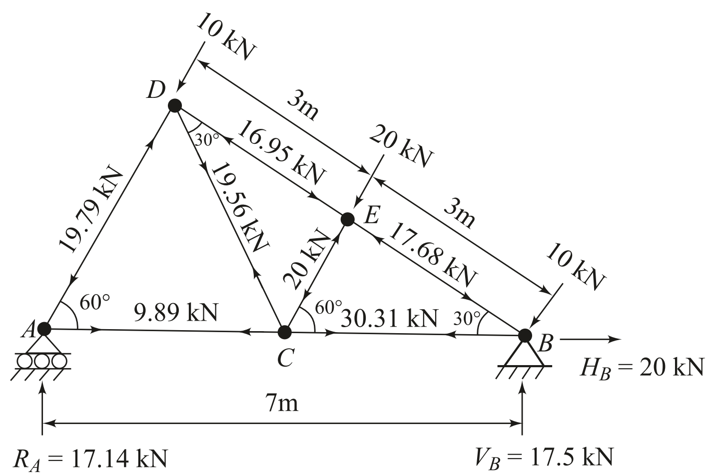


Fig. 13.24

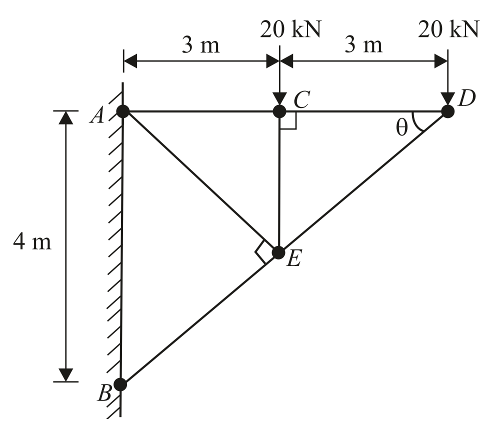


Fig. 13.25

Example 13.4

Find the forces in all the members of the loaded truss shown in Fig. 13.25.

Solution: Let $$\angle ADB = \theta$$

$$\tan\theta = \frac{4 \text{ m}}{6 \text{ m}} = 0.667$$

Hence, $$\theta = 33.7^{\circ}$$

Forces in members *CD* and *DE*

Consider joint D. Refer Fig. 13.26.

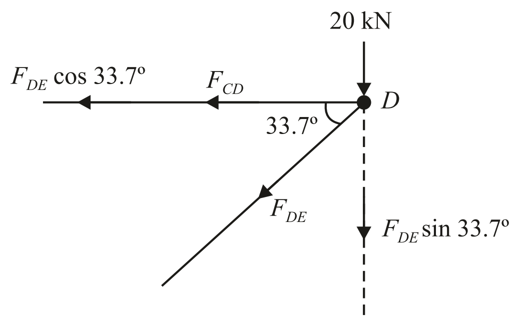


Fig. 13.26

Using $\Sigma F_x = 0$, we have

$$F_{CD} + F_{DE} \cos 33.7° = 0 \qquad \ldots(1)$$

Using $\Sigma F_y = 0$, we have

$$F_{DE} \sin 33.7° + 20 = 0$$

$$F_{DE} = -\frac{20}{\sin 33.7°} = -36.04 \text{ kN}$$

$$= 36.04 \text{ kN (C)}$$

Substituting F_{DE} in equation (1), we have

$$F_{CD} = -F_{DE} \cos 33.7°$$

$$= -(-36.04) \cos 33.7°$$

$$= 29.98 \text{ kN (T)}$$

Forces in members *AC* and *CE*

Consider joint C. Refer Fig. 13.27.

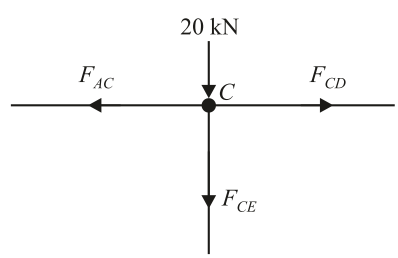


Fig. 13.27

Using $\Sigma F_x = 0$, we have

$$F_{AC} = F_{CD}$$

$$= 29.98 \text{ kN (T)}$$

Using $\Sigma F_y = 0$, we have

$$F_{CE} + 20 = 0$$

or
$$F_{CE} = -20 \text{ kN}$$
$$= 20 \text{ kN (C)}$$

Forces in members *AE* and *BE*

Consider joint E. Refer Fig. 13.28.

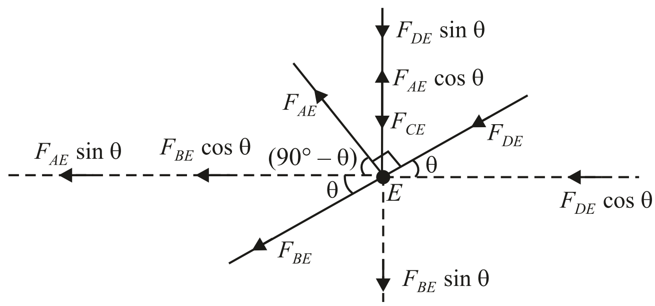


Fig. 13.28

Using $\Sigma F_x = 0$, we have

$$F_{AE} \sin\theta + F_{BE} \cos\theta + F_{DE} \cos\theta = 0$$

$$F_{AE} \sin 33.7^\circ + F_{BE} \cos 33.7^\circ = -F_{DE} \cos 33.7^\circ$$
$$= -(36.04) \times \cos 33.7^\circ$$
$$= -29.98$$

or
$$0.554\, F_{AE} + 0.832\, F_{BE} = -29.98 \quad \ldots(1)$$

Using $\Sigma F_y = 0$, we have

$$F_{DE} \sin\theta + F_{CE} + F_{BE} \sin\theta = F_{AE} \cos\theta$$

$$36.04 \sin 33.7^\circ + 20 + F_{BE} \sin 33.7^\circ = F_{AE} \cos 33.7^\circ$$

$$19.99 + 20 + 0.554\, F_{BE} = 0.832\, F_{AE}$$

$$39.99 + 0.554\, F_{BE} = 0.832\, F_{AE}$$

or
$$0.832\, F_{AE} - 0.554\, F_{BE} = 39.99 \quad \ldots(2)$$

Solving equations (1) and (2), we get

$$F_{BE} = -47.13 \text{ kN}$$
$$= 47.13 \text{ kN (C)}$$
$$F_{AE} = 16.66 \text{ kN (T)}$$

The complete result is shown in Fig. 13.29.

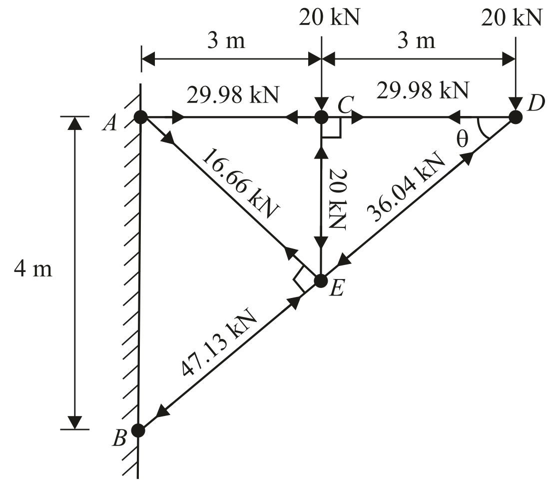


Fig. 13.29

Example 13.5

Find the forces in all the members of the loaded truss shown in Fig. 13.30.

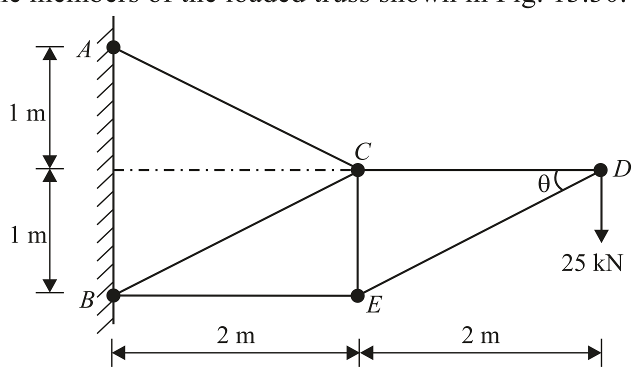


Fig. 13.30

Solution: In ΔCDE

$$\tan\theta = \frac{1\,\text{m}}{2\,\text{m}} = 0.5$$

Hence, $\theta = \tan^{-1}(0.5) = 26.56^\circ$

Forces in members *CD* and *DE*

Consider joint *D*. Refer Fig. 13.31.

Using $\sum F_x = 0$, we have

$$F_{CD} + F_{DE}\cos\theta = 0 \qquad \text{... (1)}$$

Using $\sum F_y = 0$, we have

$$F_{DE}\sin\theta + 25 = 0$$

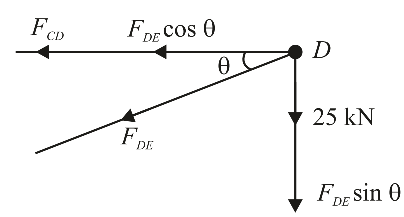


Fig. 13.31

or $$F_{DE} = -\frac{25}{\sin\theta} = -\frac{25}{\sin 26.56°}$$

$$= -55.91 \text{ kN} = 55.91 \text{ kN (C)}$$

From equation (1), we have

$$F_{CD} = -F_{DE}\cos\theta$$
$$= -(-55.91)\cos 26.56°$$
$$= 50 \text{ kN (T)}$$

Forces in members *CE* and *BE*

Consider joint *E*. Refer Fig. 13.32.

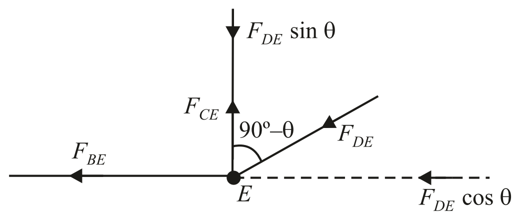


Fig. 13.32

Using $\sum F_x = 0$, we have

$$F_{BE} + F_{DE}\cos\theta = 0$$
$$F_{BE} = -F_{DE}\cos\theta$$
$$= -55.91\cos 26.56°$$
$$= -50 \text{ kN} = 50 \text{ kN (C)}$$

Using $\sum F_y = 0$, we have

$$F_{CE} = F_{DE}\sin\theta$$
$$= 55.91\sin 26.56°$$
$$= 24.99 \text{ kN (T)}$$

Forces in members *AC* and *BC*

Consider joint *C*. Refer Fig. 13.33.

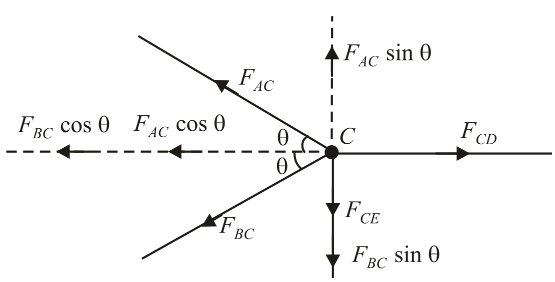


Fig. 13.33

Using $\sum F_x = 0$, we have

$$F_{BC} \cos \theta + F_{AC} \cos \theta = F_{CD}$$

$$F_{BC} \cos 26.56^\circ + F_{AC} \cos 26.56^\circ = 50$$

or
$$0.894\,(F_{BC} + F_{AC}) = 50$$

or
$$F_{BC} + F_{AC} = 55.92 \qquad \ldots(1)$$

Using $\sum F_y = 0$, we have

$$F_{BC} \sin \theta + F_{CE} = F_{AC} \sin \theta$$

$$(F_{BC} - F_{AC}) \sin \theta = -F_{CE}$$

$$(F_{BC} - F_{AC}) \sin 26.56^\circ = -24.99$$

$$F_{BC} - F_{AC} = -55.89$$

or
$$F_{AC} - F_{BC} = 55.89 \qquad \ldots(2)$$

Solving equations (1) and (2), we get

$$F_{AC} = 55.905 \text{ kN (T)}$$

and
$$F_{BC} = 0.015 \text{ kN} = 0 \text{ kN}$$

The complete result is shown in Fig. 13.34.

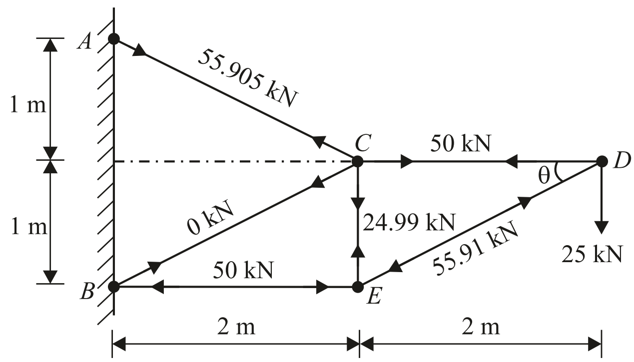


Fig. 13.34

Example 13.6

Find the forces in all the members of the loaded truss shown in Fig. 13.35.

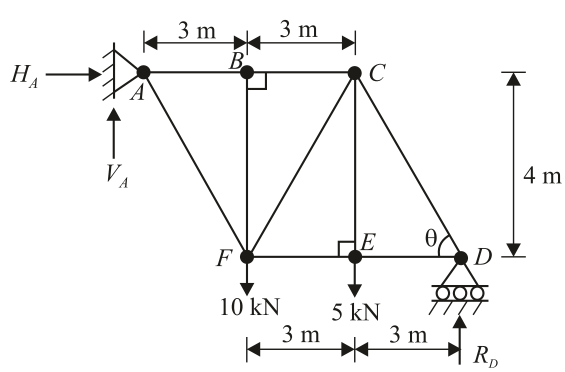


Fig. 13.35

Solution: Reactions at A are V_A and H_A, and reaction at D is R_D.

Using $\sum M_A = 0$, we have

$$R_D \times (3 + 3 + 3) = 10 \times 3 + 5 \times (3 + 3)$$

$$R_D \times 9 = 30 + 30$$

or $$R_D = 6.67 \text{ kN}$$

Now $$10 + 5 = R_D + V_A$$

or $$V_A = 15 - R_D$$

$$= 15 - 6.67 = 8.33 \text{ kN}$$

and $$H_A = 0$$

In Δ *CDE*

$$\tan\theta = \frac{4\text{m}}{3\text{m}} = 1.33$$

Hence, $$\theta = \tan^{-1}(1.33) = 53.06^\circ$$

Forces in members *CD* and *DE*

Consider joint D. Refer Fig. 13.36.

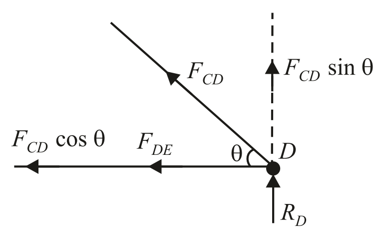


Fig. 13.36

Using $\sum F_x = 0$, we have

$$F_{DE} + F_{CD}\cos\theta = 0$$

$$F_{DE} + F_{CD}\cos 53.06^\circ = 0$$

or $$F_{DE} + 0.6\, F_{CD} = 0 \quad \ldots(1)$$

Using $\sum F_y = 0$, we have

$$F_{CD}\sin\theta + R_D = 0$$

$$F_{CD} = -\frac{R_D}{\sin\theta} = -\frac{6.67}{\sin 53.06^\circ}$$

$$= -8.345 \text{ kN}$$

$$= 8.345 \text{ kN (C)}$$

Using equation (1), we get

$$F_{DE} = -0.6\, F_{CD}$$

$$= -0.6 \times (-8.345)$$

$$= 5 \text{ kN (T)}$$

Forces in members *CE* and *FE*

Consider joint E. Refer Fig. 13.37.

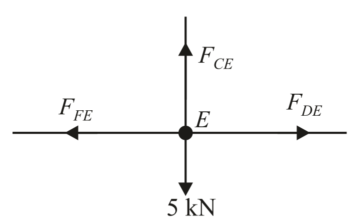


Fig. 13.37

Using $\sum F_x = 0$, we have

$$F_{FE} = F_{DE}$$

$$= 5 \text{ kN (T)}$$

Using $\sum F_y = 0$, we have

$$F_{CE} = 5 \text{ kN (T)}$$

Forces in members *BC* and *CF*

Consider joint C. Refer Fig. 13.38.

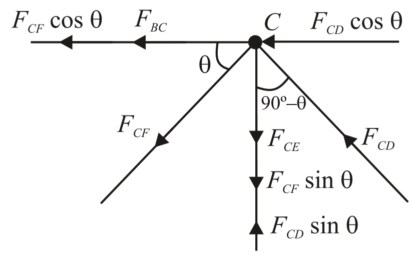


Fig. 13.38

Using $\sum F_x = 0$, we have

$$F_{CF}\cos\theta + F_{BC} + F_{CD}\cos\theta = 0$$

$$F_{CF}\cos 53.06^\circ + F_{BC} + F_{CD}\cos 53.06^\circ = 0$$

$$0.6\, F_{CF} + F_{BC} + 0.6\, F_{CD} = 0$$

$$0.6\, F_{CF} + F_{BC} + 0.6 \times 8.345 = 0$$

$$0.6\, F_{CF} + F_{BC} + 5 = 0$$

or $$0.6\, F_{CF} + F_{BC} = -5 \qquad \text{... (1)}$$

Using $\sum F_y = 0$, we have

$$F_{CE} + F_{CF}\sin\theta = F_{CD}\sin\theta$$

$$5 + F_{CF}\sin 53.06^\circ = F_{CD}\sin 53.06^\circ$$

$$5 + 0.799\, F_{CF} = 8.345 \times 0.799$$

$$5 + 0.799\, F_{CF} = 6.66$$

or $$F_{CF} = 2.07 \text{ kN (T)}$$

Using equation (1), we get

$$\begin{aligned} F_{BC} &= -5 - 0.6\, F_{CF} \\ &= -5 - 0.6 \times 2.07 \\ &= -6.24 \text{ kN} \\ &= 6.24 \text{ kN (C)} \end{aligned}$$

Forces in members *AB* and *BF*

Consider joint *B*. Refer Fig. 13.39.

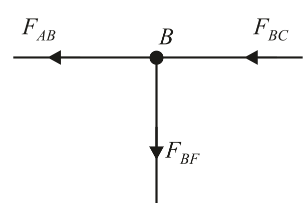


Fig. 13.39

Using $\sum F_x = 0$, we have

$$\begin{aligned} F_{AB} + F_{BC} &= 0 \\ F_{AB} &= -F_{BC} \\ &= -6.24 \text{ kN} \\ &= 6.24 \text{ kN (C)} \end{aligned}$$

Using $\sum F_y = 0$, we have

$$F_{BF} = 0 \text{ kN}$$

Force in member *AF*

Consider joint A, Refer Fig. 13.40.

Using $\sum F_x = 0$, we have

$$H_A + F_{AF} \cos\theta = F_{AB}$$

$$0 + F_{AF} \cos 53.06^\circ = 6.24$$

Hence, $$F_{AF} = 10.38 \text{ kN (T)}$$

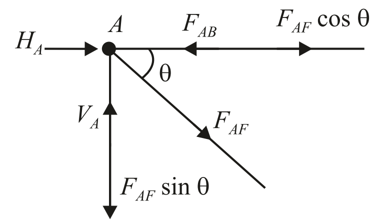


Fig. 13.40

The complete solution is shown in Fig. 13.41.

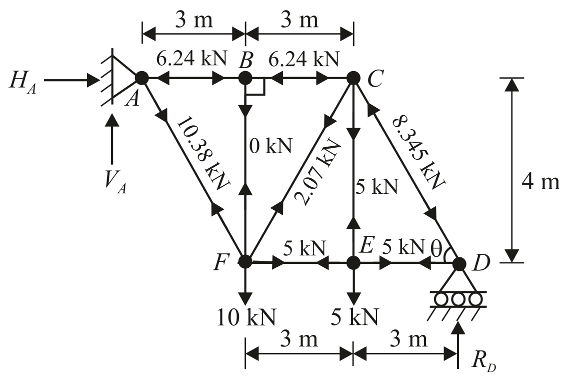


Fig. 13.41

Example 13.7

Find the forces in all the members of the loaded truss shown in Fig. 13.42.

***Solution*:**

In ΔABD $$\tan 30^\circ = \frac{AB}{AD}$$

or $$AD = \frac{2.5\text{ m}}{\tan 30^\circ} = 4.33 \text{ m}$$

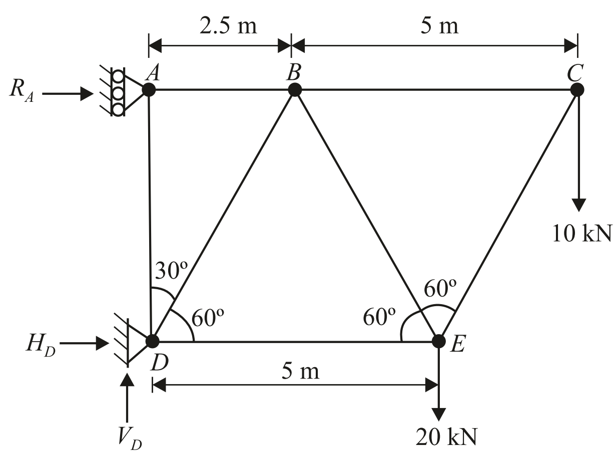


Fig. 13.42

Support reactions at *A* and *D*

$$R_A + H_D = 0$$

and $$V_D = 20 \text{ kN} + 10 \text{ kN} = 30 \text{ kN}$$

Taking moments of the forces about A, we have

$$-10 \times (2.5 + 5) - 20 \times 5 + H_D \times 4.33 = 0$$

or $$H_D = 40.41 \text{ kN}$$

and $$R_A = -40.41 \text{ kN}$$

The reaction R_A is acting opposite to the selected direction.

Forces in members *AB* and *AD*

Consider joint A. Refer Fig. 13.43.

$$F_{AB} = 40.41 \text{ kN (T)}$$

and $$F_{AD} = 0$$

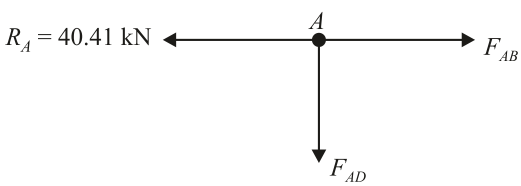


Fig. 13.43

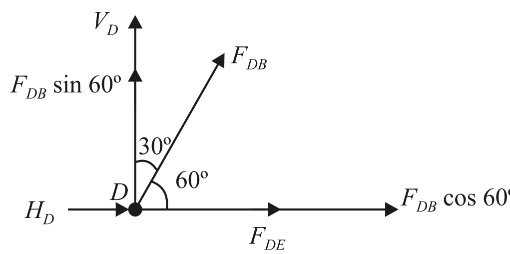


Fig. 13.44

Forces in members *DB* and *DE*

Consider joint D. Refer Fig. 13.44.

Using $\Sigma F_y = 0$, we have

$$V_D + F_{DB} \sin 60° = 0$$

Hence, $$F_{DB} = -\frac{V_D}{\sin 60°} = -\frac{30}{0.866}$$

$$= -34.64 \text{ kN}$$

$$= 34.64 \text{ kN (C)}$$

Using $\Sigma F_x = 0$, we have

$$H_D + F_{DE} + F_{DB} \cos 60° = 0$$

$$40.41 + F_{DE} + 0.5 \times (-34.64) = 0$$

or $$40.41 + F_{DE} - 17.32 = 0$$

Hence, $$F_{DE} = -23.09 \text{ kN}$$

$$= 23.09 \text{ kN (C)}$$

Forces in members *BC* and *BE*

Consider joint *B*. Refer Fig. 13.45.

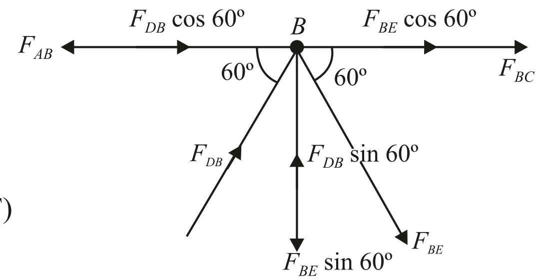


Fig. 13.45

Using $\Sigma F_y = 0$, we have

$$F_{BE} \sin 60° = F_{DB} \sin 60°$$

Hence, $F_{BE} = F_{DB} = 34.64$ kN (T)

Using $\Sigma F_x = 0$, we have

$$F_{BC} + F_{BE} \cos 60° + F_{DB} \cos 60° = F_{AB}$$

$$F_{BC} + 34.64 \times \cos 60° + 34.64 \times \cos 60° = 40.41 \text{ kN}$$

Hence, $F_{BC} = 5.77$ kN (T)

Force in member *EC*

Consider joint *E*. Refer Fig. 13.46.

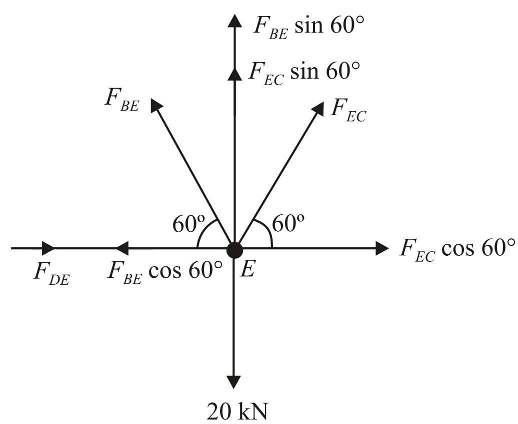


Fig. 13.46

Using $\Sigma F_x = 0$, we have

$$F_{EC} \cos 60° + F_{DE} - F_{BE} \cos 60° = 0$$

or $$0.5\, F_{EC} + 23.09 - 0.5 \times 34.64 = 0$$

Hence, $$F_{EC} = -\,11.55 \text{ kN}$$

$$= 11.55 \text{ kN (C)}$$

The complete result is shown in Fig. 13.47.

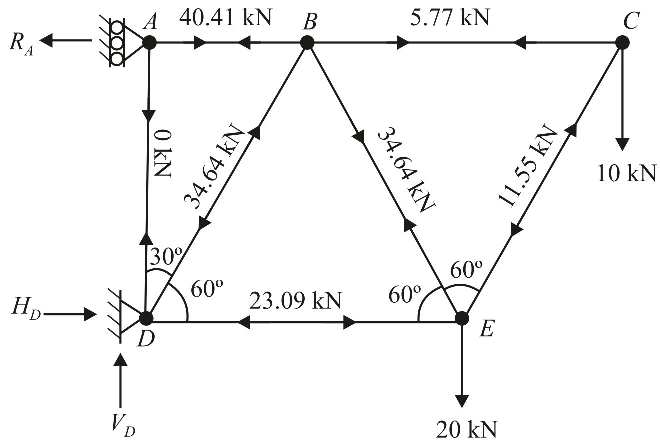


Fig. 13.47

Example 13.8

Find the forces in all the members of the loaded simply-supported truss shown in Fig. 13.48.

Solution:

Support reactions at *A* and *C*

Let H_A and V_A be the reactions at A, and R_C be the reaction at C.

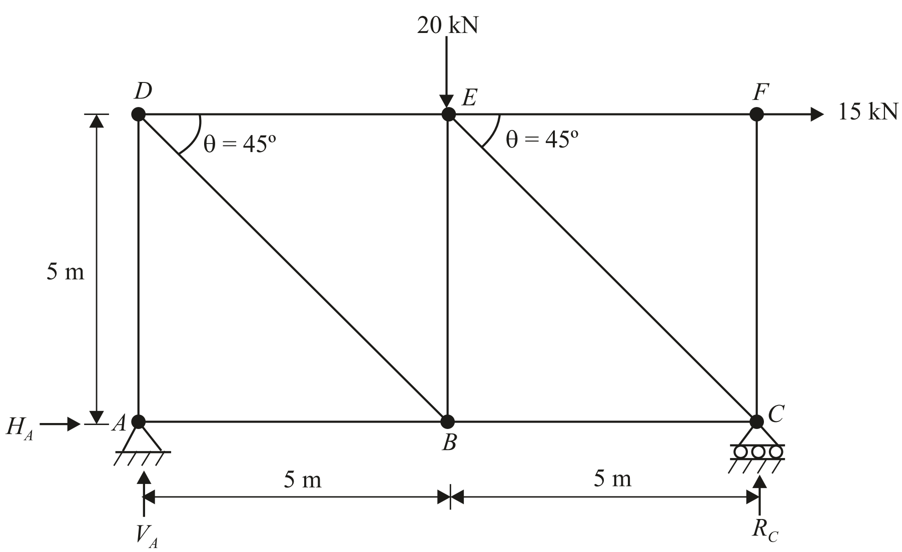


Fig. 13.48

$$V_A + R_C = 20 \qquad \text{... (1)}$$

and $$H_A + 15 = 0$$

Hence, $$H_A = -15 \text{ kN} = 15 \text{ kN } (\leftarrow)$$

The actual direction of H_A is opposite to the selected one.

Taking moments of forces about A, we have

$$R_c \times 10 = 20 \times 5 + 15 \times 5$$
$$= 100 + 75$$
$$= 175$$

Hence, $R_c = 17.5 \text{ kN } (\uparrow)$

From equation (1) $V_A = 20 - R_c = (20 - 17.5) \text{ kN} = 2.5 \text{ kN } (\uparrow)$

Forces in members *AD* and *AB*

Consider joint A. Refer Fig. 13.49.

$$F_{AB} = H_A = 15 \text{ kN (T)}$$

and $F_{AD} + V_A = 0$

or
$$F_{AD} = -V_A$$
$$= -2.5 \text{ kN}$$
$$= 2.5 \text{ kN (C)}$$

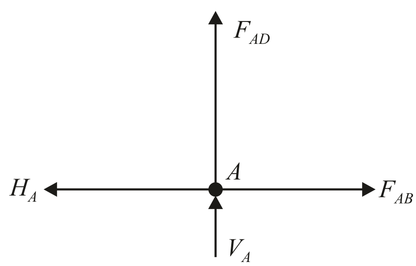


Fig. 13.49

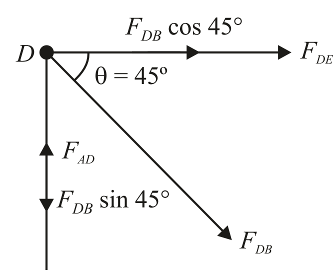


Fig. 13.50

Forces in members *DE* and *DB*

Consider joint D. Refer Fig. 13.50.

Using $\Sigma F_y = 0$, we have

$$F_{AD} = F_{DB} \sin 45°$$

Hence, $F_{DB} = \dfrac{F_{AD}}{\sin 45°} = 3.53 \text{ kN (T)}$

Using $\Sigma F_x = 0$, we have

$$F_{DE} + F_{DB} \cos 45° = 0$$

or
$$F_{DE} = -3.53 \times \cos 45°$$
$$= -2.5 \text{ kN}$$
$$= 2.5 \text{ kN (C)}$$

Forces in members *BE* and *BC*

Consider joint B. Refer Fig. 13.51.

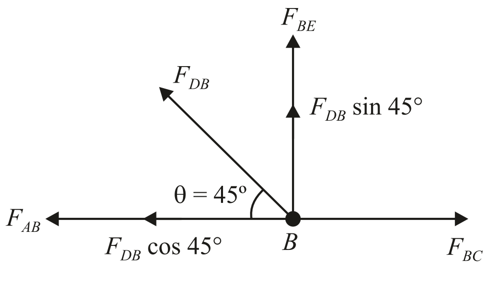


Fig. 13.51

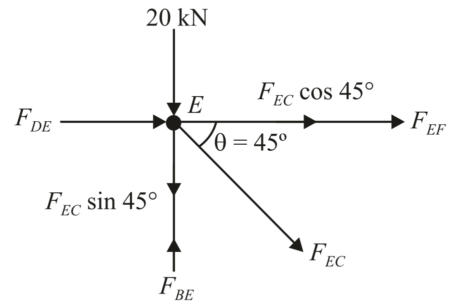


Fig. 13.52

Using $\Sigma F_x = 0$, we have

$$F_{BC} = F_{DB} \cos 45° + F_{AB}$$
$$= 3.53 \times \cos 45° + 15 = 17.5 \text{ kN (T)}$$

Using $\Sigma F_y = 0$, we have

$$F_{BE} + F_{DB} \sin 45° = 0$$

Hence, $F_{BE} = -3.53 \times \sin 45° = -2.5 \text{ kN} = 2.5 \text{ kN (C)}$

Forces in members *EF* and *EC*

Consider joint *E*. Refer Fig. 13.52.

Using $\Sigma F_y = 0$, we have

$$F_{EC} \sin 45° + 20 = F_{BE}$$

or $F_{EC} \sin 45° = F_{BE} - 20$

$$= 2.5 - 20 = -17.5$$

Hence, $F_{EC} = -24.75 \text{ kN} = 24.75 \text{ kN (C)}$

Using $\Sigma F_x = 0$, we have

$$F_{EF} + F_{DE} + F_{EC} \cos 45° = 0$$

or $F_{EF} = -F_{DE} - F_{EC} \cos 45°$

$$= -2.5 - (-24.75) \times \cos 45°$$
$$= 15 \text{ kN (T)}$$

Force in member *FC*

Consider joint *F*. Refer Fig. 13.53.

Using $\Sigma F_y = 0$, we have

$$F_{FC} = 0$$

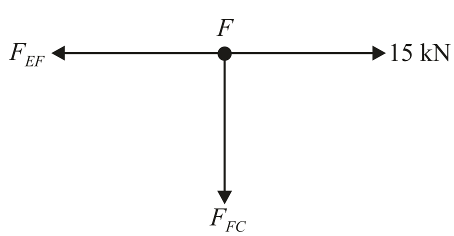


Fig. 13.53

The complete result is shown in Fig. 13.54.

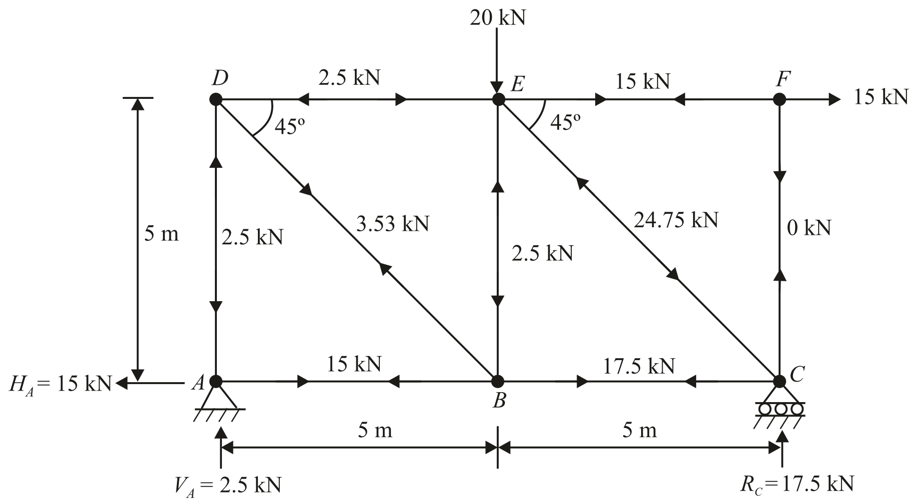


Fig. 13.54

Example 13.9

Determine the forces in each member of the truss shown in Fig. 13.55 considering each member weighs 50 N.

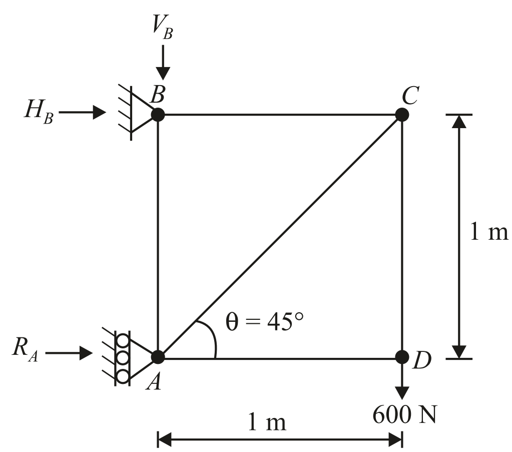


Fig. 13.55

Solution: The truss is supported at points A and B. There is only one reaction at A which is R_A, and two reactions at B include H_B (in the horizontal direction) and V_B (in the vertical direction). Weights of every member in the truss are equally divided on the connecting joints and act only in the vertical direction at the concerned joints. We will use the method of joints to determine the forces in the members.

Force consideration due to weight forces of members at various joints

Weights of all the members of the truss are equal to 50 N.

$$W_{AB} = W_{BC} = W_{CD} = W_{AD} = W_{AC} = 50\text{N}$$

Joint *A*

Three members *AB*, *AC* and *AD* are connected to the joint *A*. Four forces are acting at joint *A*. They are:

- the horizontal reaction at *A*, that is, R_A
- the one-half of weight of member *AB*, that is, $\dfrac{W_{AB}}{2} = \dfrac{50}{2} = 25$ N
- the one-half of weight of member *AC*, that is, $\dfrac{W_{AC}}{2} = \dfrac{50}{2} = 25$N
- the one-half of weight of member *AD*, that is, $\dfrac{W_{AD}}{2} = \dfrac{50}{2} = 25$N

The forces at joint *A* are shown in Fig. 13.56. The total force due to weights of the three members acting in the vertically downward direction is (25 + 25 + 25) N, that is, 75 N.

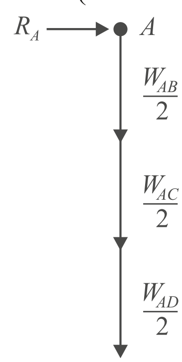


Fig. 13.56
Weight forces at *A*.

Joint *B*

Two members *AB* and *BC* are connected to the joint *B*. Four forces are acting at joint *B*. They are:

- the horizontal reaction at *B*, that is, H_B
- the vertical reaction at *B*, that is, V_B
- the one-half of weight of member *AB*, that is, $\dfrac{W_{AB}}{2} = \dfrac{50}{2} = 25$N
- the one-half of weight of member *BC*, that is, $\dfrac{W_{BC}}{2} = \dfrac{50}{2} = 25$N

The forces at joint *B* are shown in Fig. 13.57. The total force due to weights of the two members acting in the vertically downward direction is (25 + 25)N, that is, 50 N.

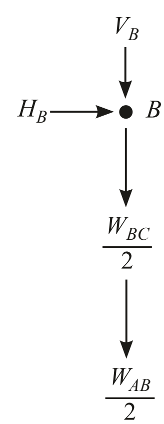


Fig. 13.57
Weight forces at *B*.

Joint *C*

Three members BC, AC and CD are connected to the joint C. The forces at the joint C are because of the weights of the members. They are:

- the one-half of weight of member BC, that is, $\dfrac{W_{BC}}{2} = \dfrac{50}{2} = 25\text{N}$
- the one-half of weight of member AC, that is, $\dfrac{W_{AC}}{2} = \dfrac{50}{2} = 25\text{N}$
- the one-half of weight of member CD, that is, $\dfrac{W_{CD}}{2} = \dfrac{50}{2} = 25\text{N}$

The forces at joint C are shown in Fig. 13.58. The total force due to weights of the three members acting in the vertically downward direction is (25 + 25 + 25) N, that is, 75 N.

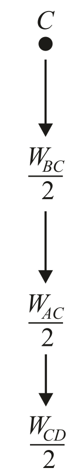


Fig. 13.58
Weight forces at C

Joint *D*

Two members AD and CD are connected to the joint D. Three forces are acting at joint D. They are:

- the externally applied load of 600 N
- the one-half of weight of member AD, that is, $\dfrac{W_{AD}}{2} = \dfrac{50}{2} = 25\text{N}$
- the one-half of weight of member CD, that is, $\dfrac{W_{CD}}{2} = \dfrac{50}{2} = 25\text{N}$

The forces at joint D are shown in Fig. 13.59. The total force due to weights of the two members acting in the vertically downward direction is (25 + 25) N, that is, 50 N.

There is no force acting in the horizontal direction at any joint because of the weight of any member.

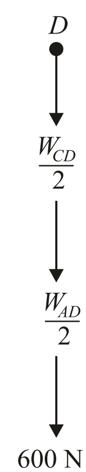


Fig. 13.59
Weight forces at D.

Support reactions at *A* and *B*

Using $\Sigma M_B = 0$, we have

$$R_A \times 1 = \text{Moment of weight forces at } C + \text{Moment of weight forces at } D + \text{Moment of the force 600N}$$

$$= \left(\frac{W_{BC}}{2} + \frac{W_{AC}}{2} + \frac{W_{CD}}{2}\right)\times 1 + \left(\frac{W_{CD}}{2} + \frac{W_{AD}}{2}\right)\times 1 + 600 \times 1$$

$$= (25 + 25 + 25) \times 1 + (25 + 25) \times 1 + 600 \times 1$$

$$= 75 + 50 + 600 = 725 \text{ N}$$

or $$R_A = 725 \text{ N } (\rightarrow)$$

Forces due to one-half weights of the members AB, AC and AD acting at joint A and forces due to one-half weights of the members AB and BC acting at joint B have the same line of action along AB in the downward direction, hence moments of all these forces are not considered.

Also $$R_A + H_B = 0$$

or $$H_B = -R_A = -725 \text{ N}$$
$$= 725 \text{ N } (\leftarrow)$$

Now

Weight forces at A + Weight forces at B + Weight forces at C + Weight forces at D + V_B + 600 = 0

$$\left(\frac{W_{AB}}{2}+\frac{W_{AC}}{2}+\frac{W_{AD}}{2}\right)+\left(\frac{W_{AB}}{2}+\frac{W_{BC}}{2}\right)+\left(\frac{W_{BC}}{2}+\frac{W_{AC}}{2}+\frac{W_{CD}}{2}\right)+\left(\frac{W_{CD}}{2}+\frac{W_{AD}}{2}\right)+V_B+600 = 0$$

$$(25 + 25 + 25) + (25 + 25) + (25 + 25 + 25) + (25 + 25) + V_B + 600 = 0$$

$$75 + 50 + 75 + 50 + V_B + 600 = 0$$

or $$850 + V_B = 0$$

Hence, $$V_B = -850 \text{ N} = 850 \text{ N } (\uparrow)$$

Forces in members *AD* and *CD*

Consider joint D. Refer Fig. 13.60. The one-half of weights of the members AD and CD, that is, $\frac{W_{AD}}{2}$ and $\frac{W_{CD}}{2}$ are assumed to act at D in the vertically downward direction.

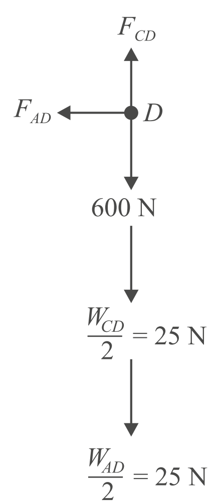


Fig. 13.60

Using $\Sigma F_x = 0$

$$F_{AD} = 0$$

Using $\Sigma F_y = 0$

$$F_{CD} = 600 + 25 + 25 = 650 \text{ N (T)}$$

Forces in members *BC* and *AC*

Consider joint *C*. Refer Fig. 13.61. The one-half of weights of the members *BC*, *AC* and *CD*, that is, $\frac{W_{BC}}{2}$, $\frac{W_{AC}}{2}$ and $\frac{W_{CD}}{2}$ are assumed to act at *C* in the downward direction.

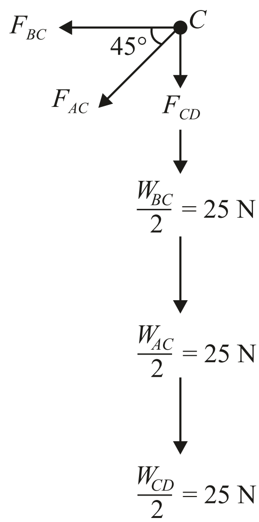


Fig. 13.61

Using $\Sigma F_x = 0$

$$F_{BC} + F_{AC} \cos 45^\circ = 0 \qquad \text{... (1)}$$

Using $\Sigma F_y = 0$

$$F_{CD} + F_{AC} \sin 45^\circ + 25 + 25 + 25 = 0$$

$$F_{CD} + F_{AC} \sin 45^\circ + 75 = 0 \qquad \text{... (2)}$$

From equation (2), we have

$$F_{AC} \sin 45^\circ = -(F_{CD} + 75)$$

$$= -(650 + 75) \text{ N} = -725 \text{ N}$$

Hence,

$$F_{AC} = -\frac{725}{\sin 45^\circ} \text{ N} = -1025.3 \text{ N}$$

$$= 1025.3 \text{ N (C)}$$

From equation (1), we have

$$F_{BC} = -F_{AC} \cos 45^\circ$$

$$= -(-1025.3) \cos 45^\circ = 725 \text{ N (T)}$$

Force in member *AB*

To find the force in the member *AB*, that is, F_{AB}, the joint *A* or the joint *B* can be considered. We are calculating F_{AB} using both joints.

When joint A is considered, the one-half of weights of three members AB, AC and AD, that is $\frac{W_{AB}}{2}$, $\frac{W_{AC}}{2}$ and $\frac{W_{AD}}{2}$ are assumed to act at A in the vertical direction as shown in Fig. 13.62.

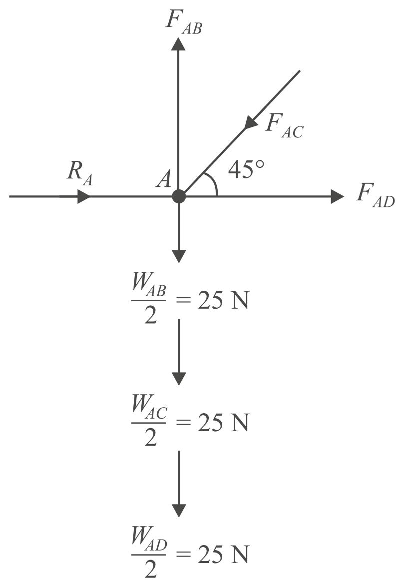


Fig. 13.62

Using $\Sigma F_y = 0$, we have

$$F_{AB} = F_{AC} \sin 45^{\circ} + 25 + 25 + 25 = (1025.3) \sin 45^{\circ} + 75 = 800 \text{ N (T)}$$

When joint B is considered, the one-half of weights of two members AB and BC, that is, $\frac{W_{AB}}{2}$ and $\frac{W_{BC}}{2}$ are assumed to act at B in the vertical direction as shown in Fig. 13.63.

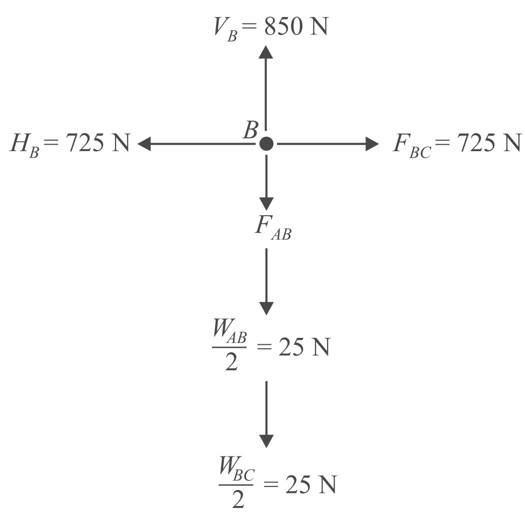


Fig. 13.63

Using $\Sigma F_y = 0$, we have

$$F_{AB} + 25 + 25 = V_B = 850$$

$$F_{AB} + 50 = 850$$

Hence, $$F_{AB} = (850 - 50)\text{ N} = 800\text{ N (T)} \qquad \text{(for check)}$$

The complete result is shown in Fig. 13.64.

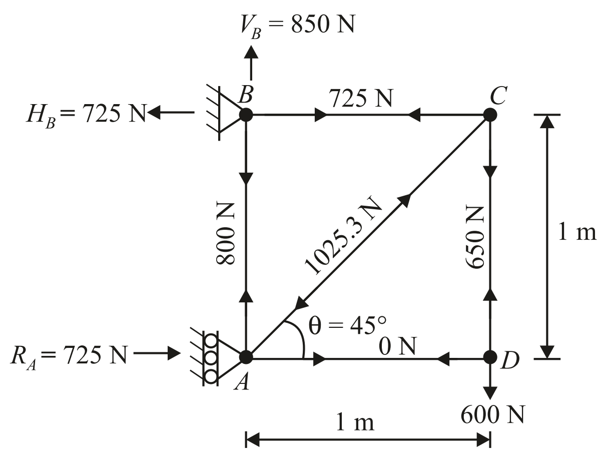


Fig. 13.64

13.4.2 Analysis of Trusses by Method of Sections

This method is based on dividing the entire truss into two parts by a section line. The section must pass through the members in which forces are required to be determined. The equilibrium of the part either to the left or to the right of the section is considered by treating it as a free body. A vertical or inclined section can be chosen. A horizontal section is usually avoided because it will cut many members of unknown forces. The section should be chosen in such a way that the members cut by the section are as less as possible and have maximum three unknowns. The method of joints is most effective when forces in all the members of a truss are required to be determined. Calculation of forces in various members can't be made independently. The method of sections can be used to find out forces in one or two desired members and is independent of calculation of forces in other members. The method is illustrated by the suitable examples.

Example 13.10

Find the forces in the members *BF* and *AF* of the loaded truss shown in Fig. 13.65.

Solution: Refer Fig. 13.65.

Support reactions at *A* and *B*

Using $\Sigma M_A = 0$, we have

$$R_B \times 5 = 3 \times 2.5 + 5 \times 3 = 7.5 + 15 = 22.5$$

or $$R_B = 4.5\text{ kN } (\uparrow)$$

Now $$V_A + 3 = R_B$$

or $$V_A = R_B - 3 = 4.5 - 3 = 1.5 \text{ kN } (\downarrow)$$

and $$H_A = 5 \text{ kN } (\leftarrow)$$

Forces in the desired members are initially assumed to be tension, that is, positive.

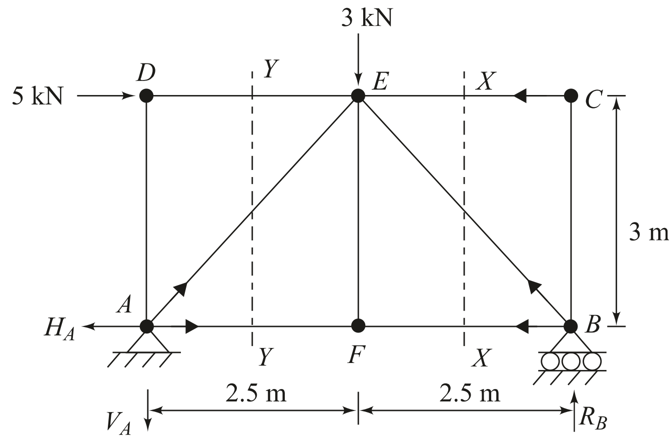


Fig. 13.65

Force in member *BF*

Take a section *XX* which cuts the desired member *BF* and consider the equilibrium of the part right to the section. Select *E* as centre of moment. Forces in the members *EC*, *EB* and *EF*, and 3 kN are passing through *E* and hence their moments about that point are zero. The only unknown force left is F_{BF}. F_{BF} is initially considered tension, that is, directed away from the joint *B*. Moment of the force in member *BC* about the point *E* is not considered. Why? Taking moments of the forces about *E*, we have

$$F_{BF} \times 3 = R_B \times 2.5$$

or $$F_{BF} = \frac{R_B \times 2.5}{3} = \frac{4.5 \times 2.5}{3}$$

$$= 3.75 \text{ kN (T)}$$ **Ans.**

Force in member *AF*

Take another section *YY* which cuts the member *AF* and consider the equilibrium of the part left to the section. Again *E* is considered as centre of moment. Moment of the force in member *AD* about the point *E* is not considered. Why? The moment equation is

$$F_{AF} \times 3 + V_A \times 2.5 = H_A \times 3$$

or $$F_{AF} = \frac{H_A \times 3 - V_A \times 2.5}{3}$$

$$= \frac{5 \times 3 - 1.5 \times 2.5}{3} = 3.75 \text{ kN (T)}$$ **Ans.**

(Forces 3 kN and 5 kN are passing through *E*, hence their moments are not considered.)

Example 13.11

Find the forces in the members *EF*, *DF* and *CD* of the loaded cantilever truss shown in Fig. 13.66.

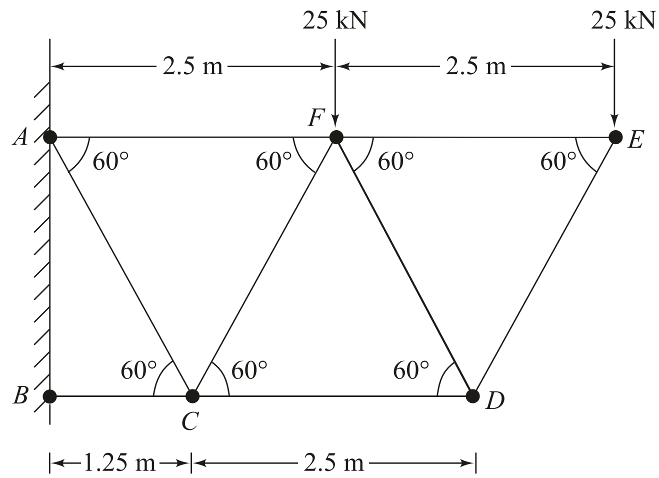


Fig. 13.66

Solution: Consider a section *XX* which cuts the members *CD*, *EF* and *DF* in which forces are desired (Fig. 13.67). Consider equilibrium of the part right to the section.

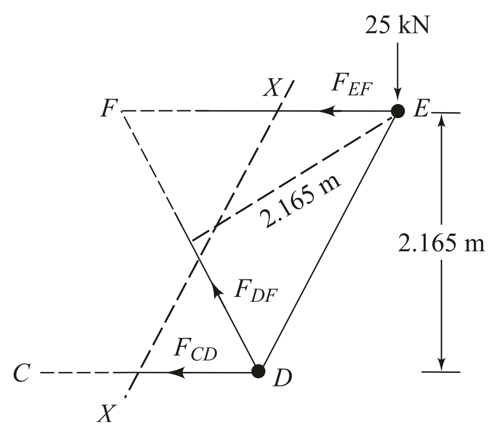


Fig. 13.67

Force in member *CD*

The perpendicular distance between *EF* and *CD* is 1.25 tan 60° = 2.165 m.

Taking moments of the forces about *F*, we have

$$F_{CD} \times 2.165 + 25 \times 2.5 = 0$$

or

$$F_{CD} = -28.86 \text{ kN}$$

$$= 28.86 \text{ kN (C)}$$ **Ans.**

Forces in members *EF* and *DF* are passing through *F* and hence their moments about that point are zero.

Force in member *EF*

Taking moments of the forces about *D*, we have

$$F_{EF} \times 2.165 - 25 \times 1.25 = 0$$

or $F_{EF} = 14.43$ kN (T) **Ans.**

Force in member *DF*

The perpendicular distance of point *E* from *FD* is

$$2.5 \sin 60° = 2.165 \text{ m}$$

Taking moments of the forces about *E*, we have

$$F_{DF} \times 2.165 + F_{CD} \times 2.165 = 0$$

$$F_{DF} \times 2.165 + (-28.86) \times 2.165 = 0$$

or $F_{DF} =$ 28.86 kN (T) **Ans.**

Forces 25 kN and F_{EF} are passing through *E*, hence their moments about that point are zero.

Example 13.12

Find the forces in the members *CD* and *DE* of the loaded truss shown in Fig. 13.68.

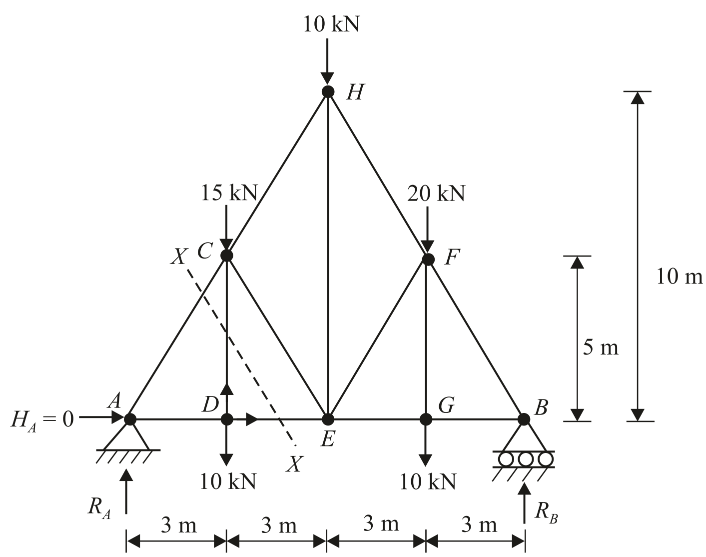


Fig. 13.68

Solution:

Support reactions at *A* and *B*

Using $\Sigma M_A = 0$, we have

$$R_B \times (3 + 3 + 3 + 3) = 10 \times 3 + 15 \times 3 + 10 \times (3 + 3) + 20 \times (3 + 3 + 3) + 10 \times (3 + 3 + 3)$$

$$= (30 + 45 + 60 + 180 + 90) \text{ kN} = 405 \text{ kN}$$

$$\text{or} \qquad R_B = 33.75 \text{ kN } (\uparrow)$$

$$\text{Now} \qquad R_A + R_B = (15 + 10 + 10 + 20 + 10) \text{ kN} = 65 \text{ kN}$$

$$\text{or} \qquad R_A = 65 - R_B = (65 - 33.75) \text{ kN} = 31.25 \text{ kN } (\uparrow)$$

Force in member *DE*

Consider a section *XX* , which cuts both members *CD* and *DE*. To determine force in member *DE,* choose *C* as the centre of moment. Moments of the forces in the members *AC* and *CD* about that point are zero and the only unknown force left is the force in the member *DE*. Considering equilibrium of the part left to the section, the equation of moment can be written as

$$R_A \times 3 = F_{DE} \times 5$$

$$\text{or} \qquad F_{DE} = \frac{R_A \times 3}{5} = \frac{31.25 \times 3}{5} \text{ kN}$$

$$= 18.75 \text{ kN (T)} \qquad \textbf{Ans.}$$

Force in member *CD*

To determine F_{CD}, choose *A* as the centre of moment. Forces in members *DE* and *AC* and reaction R_A are passing through the point *A,* hence their moments about that point are zero. Again considering the same section and the left part of the section, the moment equation is

$$F_{CD} \times 3 = 10 \times 3$$

$$\text{or} \qquad F_{CD} = 10 \text{ kN (T)} \qquad \textbf{Ans.}$$

Example 13.13

Find the forces in the members *IJ, JE* and *DE* of the loaded truss shown in Fig. 13.69.

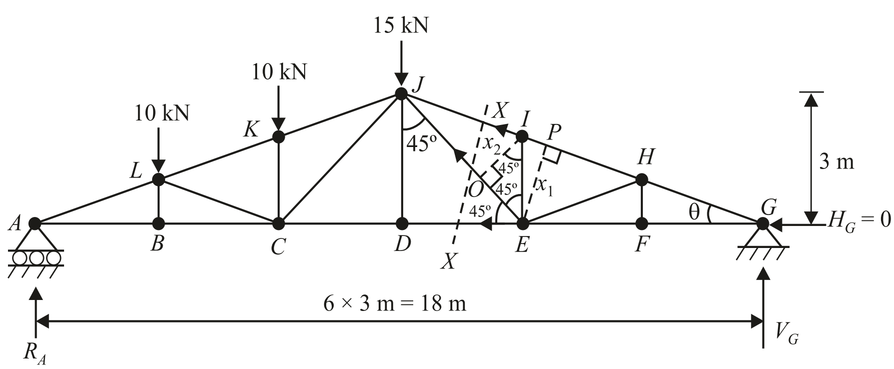


Fig. 13.69

Solution:

Support reactions at *A* and *G*

Using $\Sigma M_A = 0$, we have

$$V_G \times (3 + 3 + 3 + 3 + 3 + 3) = 10 \times 3 + 10 \times (3 + 3) + 15 \times (3 + 3 + 3)$$

$$= (30 + 60 + 135) \text{ kN} = 225 \text{ kN}$$

or $$V_G = \frac{225}{18} = 12.5 \text{ kN } (\uparrow)$$

Now $$R_A + V_G = (10 + 10 + 15) \text{ kN} = 35 \text{ kN}$$

or $$R_A = 35 - R_G = (35 - 12.5) \text{ kN} = 22.5 \text{ kN } (\uparrow)$$

Geometrical calculations

In Δ *DGJ* $$\tan\theta = \frac{3}{9}$$

Hence, $$\theta = 18.43°$$

In Δ *EGP* $$\sin\theta = \frac{x_1}{6}$$

or $$x_1 = 6 \sin 18.43° = 1.897 \text{ m}$$

In ΔEGI $$\tan\theta = \frac{IE}{6}$$

or $$IE = 6 \tan 18.43° = 2 \text{ m}$$

In ΔOEI $$\sin 45° = \frac{x_2}{IE}$$

or $$x_2 = 2 \times \sin 45° = 1.414 \text{ m}$$

Force in member *DE*

Consider a section *XX*, which cuts the members *IJ, JE* and *DE*. Choose *J* as the centre of moment, and consider the equilibrium of the part right to the section. Moments of the forces in the members *IJ* and *JE* about *J* are zero, since the forces are passing through that point. Hence, the moment equation is

$$F_{DE} \times 3 = R_G \times 9$$

or $$F_{DE} = \frac{12.5 \times 9}{3} = 37.5 \text{ kN (T)}$$ **Ans.**

Force in member *IJ*

This time choose *E* as the centre of moment, and consider the part right to the section. Forces in the member *DE* and *JE* have no contribution in the equation of moment. The moment equation is

$$F_{IJ} \times x_1 + R_G \times 6 = 0$$

or $$F_{IJ} = \frac{12.5 \times 6}{1.897} \text{ kN} = 39.53 \text{ kN (C)}$$ **Ans.**

Force in member *JE*

Choose *I* as the centre of moment, and consider the part right to the section. Force in member *IJ* has no contribution in the moment equation, given by

$$F_{JE} \times x_2 + F_{DE} \times IE = R_G \times 6$$

$$F_{JE} \times 1.414 + 37.5 \times 2 = 12.5 \times 6$$

or $$F_{JE} = 0$$ **Ans.**

Example 13.14

Find the forces in the members *CE, CD* and *BD* of the loaded truss shown in Fig. 13.70.

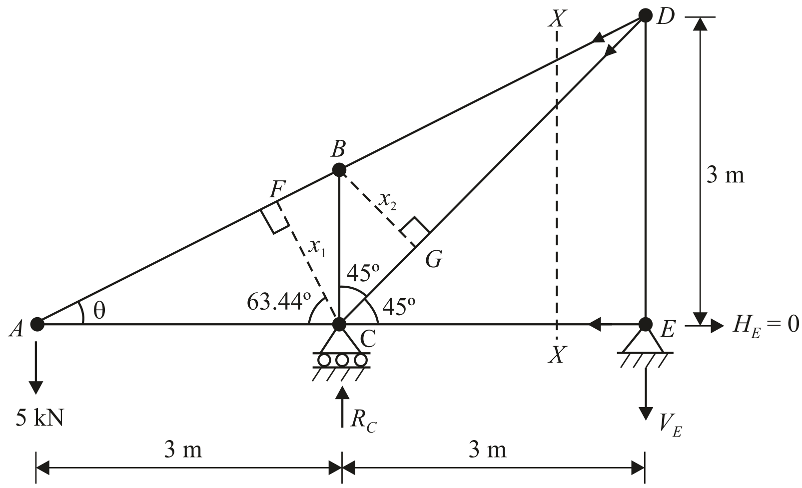


Fig. 13.70

Solution:

Support reactions at *C* and *E*

Using $\Sigma M_C = 0$, we have

$$V_E \times 3 = 5 \times 3$$

or $$V_E = 5 \text{ kN } (\downarrow)$$

Now $$R_C = 5 + V_E = (5 + 5) \text{ kN} = 10 \text{ kN } (\uparrow)$$

Geometrical calculations

In Δ *ADE* $$\tan\theta = \frac{3}{6}$$

Hence, $$\theta = 26.56°$$

On comparing Δ*s ABC* and *ADE*, we have

$$\frac{BC}{DE} = \frac{AC}{AE}$$

or $$BC = \frac{AC \times DE}{AE} = \frac{3 \times 3}{6} = 1.5 \text{ m}$$

In ΔCDE $$CD = \sqrt{3^2 + 3^2} = 4.24 \text{ m}$$

In ΔACF $$\angle ACF = 90° - \theta = (90 - 26.56)° = 63.44°$$

and $$\sin\theta = \frac{x_1}{AC}$$

or $$x_1 = AC \sin\theta = 3 \times \sin 26.56° = 1.34 \text{ m}$$

In ΔBCG $$\sin 45° = \frac{x_2}{BC}$$

or $$x_2 = BC \sin 45° = 1.5 \times \sin 45° = 1.06 \text{ m}$$

Force in member *CE*

Consider a section XX, which cuts all the desired members. Choose D as the centre of moment and consider right part of the section. The moment equation is

$$F_{CE} \times 3 = V_E \times 0 \qquad (V_E \text{ is passing through } D)$$

or $$F_{CE} = 0 \text{ kN}$$ **Ans.**

Force in member *CD*

Choose B as the centre of moment. The equation of moment is

$$F_{CE} \times BC + F_{CD} \times x_2 + R_E \times 3 = 0$$

$$0 + F_{CD} \times 1.06 + 5 \times 3 = 0 \qquad (\because F_{CE} = 0)$$

or $$F_{CD} = -\frac{15}{1.06} = -14.15 \text{ kN} = 14.15 \text{ kN (C)}$$ **Ans.**

Force in member *BD*

Choose C as the centre of moment. The equation of moment is

$$F_{BD} \times x_1 = V_E \times 3$$

or $$F_{BD} = \frac{3V_E}{x_1} = \frac{3 \times 5}{1.34} \text{kN} = 11.2 \text{ kN (T)}$$ **Ans.**

Example 13.15

Find the forces in all the members of the loaded truss shown in Fig. 13.71.

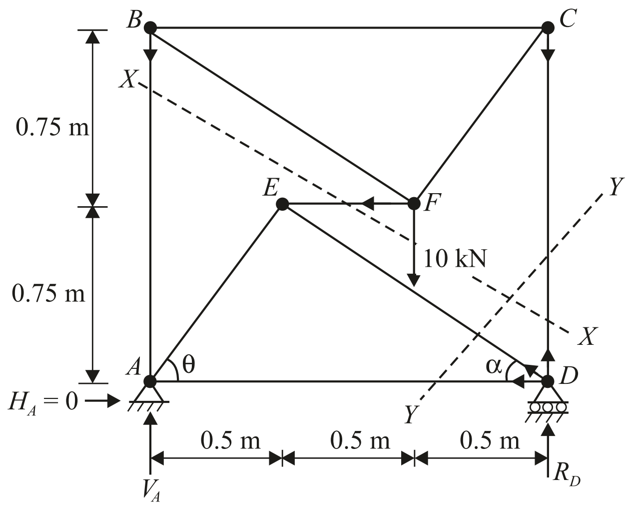


Fig. 13.71

Solution:

Support reactions at *A* and *D*

$$V_A + R_D = 10 \text{ kN} \quad \text{... (1)}$$

Using $\Sigma M_A = 0$, we have

$$R_D \times (0.5 + 0.5 + 0.5) = 10 \times (0.5 + 0.5)$$

Hence,
$$R_D = 6.66 \text{ kN } (\uparrow)$$

From equation (1), we have

$$V_A = (10 - 6.66) \text{ kN}$$
$$= 3.34 \text{ kN } (\uparrow)$$

Each joint consists of three unknown members, hence method of joints cannot be used at the start. Method of sections is used first to find forces in one of three members connected to every joint so that each joint now consists of two unknown members only, and the method of joints becomes applicable henceforth.

Take a section *XX* that cuts members *AB*, *EF* and *CD*, and consider the equilibrium of the right part of the truss. Direction of forces in unknown members are initially assumed to be tension, that is, directed away from the joints where members are connected. Take *A* as the centre of moment.

Using $\Sigma M_A = 0$, we have

$$F_{EF} \times 0.75 = 10 \times (0.5 + 0.5) + F_{CD} \times (0.5 + 0.5 + 0.5)$$

F_{AB} passes through A, hence does not appear in the equation of moment.

or $$0.75\,F_{EF} = 10 + 1.5\,F_{CD}$$

or $$F_{EF} - 2\,F_{CD} = 13.33 \qquad ...(1)$$

Now, take B as the centre of moment, and consider the equilibrium of the right part of the truss.

Using $\Sigma M_B = 0$, we have

$$F_{EF} \times 0.75 + 10 \times (0.5 + 0.5) + F_{CD} \times (0.5 + 0.5 + 0.5) = 0$$

All the moments are in the clockwise direction.

or $$0.75\,F_{EF} + 1.5\,F_{CD} = -\,10$$

or $$F_{EF} + 2\,F_{CD} = -\,13.33 \qquad ...(2)$$

Solving equations (1) and (2), we get

$$F_{EF} = 0$$

From equation (1), we get

$$F_{CD} = -\,6.66 \text{ kN}$$
$$= 6.66 \text{ kN (C)}$$

Now, take D as the centre of moment, and consider the equilibrium of the right part of the truss.

Using $\Sigma M_D = 0$, we have

$$F_{AB} \times (0.5 + 0.5 + 0.5) + 10 \times 0.5 + F_{EF} \times 0.75 = 0$$

All the moments are in the anti-clockwise direction.

or $$1.5\,F_{AB} + 5 + 0 \times 0.75 = 0 \qquad (\text{as } F_{EF} = 0)$$

or $$1.5\,F_{AB} + 5 = 0$$

Hence, $$F_{AB} = -\,3.33 \text{ kN}$$
$$= 3.33 \text{ kN (C)}$$

Now, take another section $Y\,Y$ that cuts members AD, DE and CD, and consider the equilibrium of the right part of the truss. Take E as the centre of moment.

Using $\Sigma M_E = 0$, we have

$$F_{AD} \times 0.75 = R_D \times (0.5 + 0.5) + F_{CD} \times (0.5 + 0.5)$$

F_{DE} passes through E, hence does not appear in the equation of moment.

or $$0.75\,F_{AD} = 6.66 \times 1 + (-\,6.66) \times 1$$
$$= 6.66 - 6.66$$
$$= 0$$

Hence, $$F_{AD} = 0$$

Force in member *AE*

Consider joint *A*. Refer Fig. 13.72.

$$\tan\theta = \frac{0.75\,\text{m}}{0.5\,\text{m}}$$

Hence, $\theta = 56.3°$

Using $\Sigma Fx = 0$, we have

$$F_{AE}\cos\theta + F_{AD} = 0$$

$$F_{AE}\cos 56.3° + 0 = 0 \quad (\text{as } F_{AD} = 0)$$

Hence, $F_{AE} = 0$

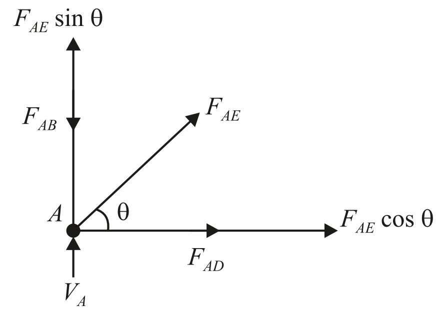


Fig. 13.72

Force in member *DE*

Consider joint *D*. Refer Fig. 13.73.

$$\tan\alpha = \frac{0.75\,\text{m}}{(0.5 + 0.5)\,\text{m}}$$

Hence, $\alpha = 36.8°$

Using $\Sigma Fx = 0$, we have

$$F_{DE}\cos\alpha + F_{AD} = 0$$

or $F_{DE}\cos 36.8° + 0 = 0 \quad (\text{as } F_{AD} = 0)$

Hence, $F_{DE} = 0$

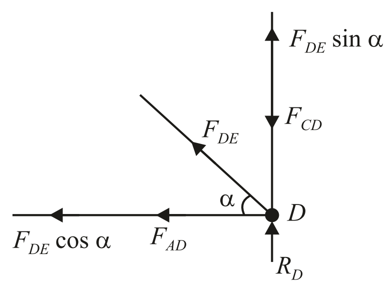


Fig. 13.73

Forces in members *BF* and *CF*

Consider joint *F*. Refer Fig. 13.74.

$$\tan\phi = \frac{0.75\,\text{m}}{0.5\,\text{m}}$$

Hence, $\phi = 56.3°$

and $$\tan\gamma = \frac{0.75\,\text{m}}{(0.5 + 0.5)\,\text{m}}$$

Hence, $\gamma = 36.8°$

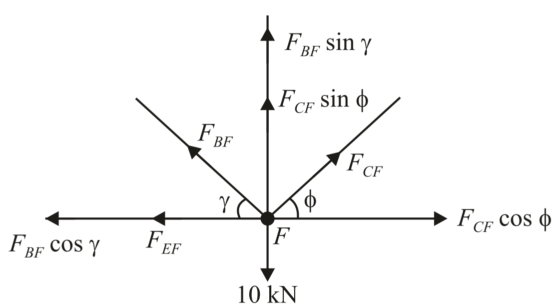


Fig. 13.74

Using $\Sigma F_x = 0$, we have

$$F_{BF} \cos \gamma + F_{EF} = F_{CF} \cos \phi$$

or $$F_{BF} \cos 36.8° + 0 = F_{CF} \cos 56.3° \quad \text{(as } F_{EF} = 0\text{)}$$

$$0.8\, F_{BF} = 0.554\, F_{CF}$$

or $$F_{BF} = 0.693\, F_{CF} \quad ...(3)$$

Using $\Sigma F_y = 0$, we have

$$F_{BF} \sin \gamma + F_{CF} \sin \phi = 10$$

or $$F_{BF} \sin 36.8° + F_{CF} \sin 56.3° = 10$$

$$0.6\, F_{BF} + 0.83\, F_{CF} = 10$$

$$0.6 \times 0.693\, F_{CF} + 0.83\, F_{CF} = 10 \quad \text{(using equation (3))}$$

$$0.415\, F_{CF} + 0.83\, F_{CF} = 10$$

or $$1.245\, F_{CF} = 10$$

Hence, $$F_{CF} = 8.03 \text{ kN (T)}$$

From equation (3), we get

$$F_{BF} = 0.693 \times 8.03$$

$$= 5.56 \text{ kN (T)}$$

Force in member *BC*

Consider joint *B*. Refer Fig. 13.75.

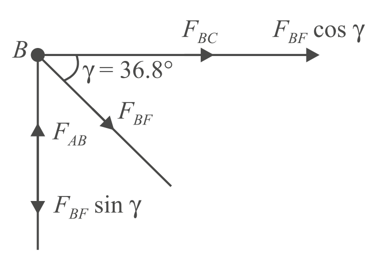


Fig. 13.75

Using $\Sigma F_x = 0$, we have

$$F_{BC} + F_{BF} \cos 36.8° = 0$$

or $$F_{BC} + 5.56 \times \cos 36.8° = 0$$

Hence, $$F_{BC} = -4.45 \text{ kN}$$

$$= 4.45 \text{ kN (C)}$$

The complete result is tabulated below.

S.No.	*Member*	*Force in Member* (kN)	*Nature of Force*
1.	*AB*	3.33	Compression
2.	*AD*	0	—
3.	*AE*	0	—
4.	*BC*	4.45	Compression
5.	*EF*	0	—
6.	*CF*	8.03	Tension
7.	*CD*	6.66	Compression
8.	*DE*	0	—
9.	*BF*	5.56	Tension

13.5 ZERO-FORCE MEMBERS

A member having no or zero force is called a zero force member. There are two important statements which tremendously help to find zero-force members in a truss without making any calculations. These statements are:

- For a non-loaded joint (joint having no external load and no support reaction) in a truss consisting of three members, if two members are collinear, the third member will be a zero-force member.
- For a non-loaded joint in a truss consisting of two members, if both members are non-collinear, then both members will be zero-force members.

An example is shown in Fig. 13.76, which illustrates the use of these statements to find out zero-force members.

In Fig. 13.76, *B*, *D*, *E*, *F*, *H* and *I* are non-loaded joints. Joint *I* consists of three members *AI*, *BI* and *HI*. Of these, *AI* and *HI* are collinear members, hence according to first statement, *BI* must be a zero-force member. Similarly, joint *H* consists of three members *HI*, *HG* and *HB* of which *HI* and *HG* are collinear members, hence *HB* must be a zero-force member. In the absence of zero-force members *BI* and *HB*, joint *B* consists of three members, namely *AB*, *BC* and *BG*. Out of these, *AB* and *BC* are collinear members, and hence *BG* must be a zero-force member. Joint *D* consists of two non-collinear members *CD* and *DE*. These are zero-force members according to second statement. Joint *E* consists of two non-collinear members *CE* and *EF*, assuming *DE* to be missing on account of its zero force nature. Therefore, *CE* and *EF* must be zero-force members. Again, joint *F* consists of two members *GF* and *CF* and both of them are non-collinear, assuming *EF* to be missing; therefore *GF* and *CF* must be zero-force members.

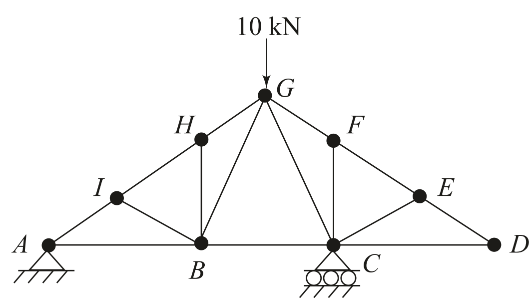


Fig. 13.76

SHORT ANSWER QUESTIONS

1. What is meant by a plane truss? How does a plane truss differ from a space truss?
2. Give a few examples of plane and space trusses.
3. How does a frame differ from a truss?
4. What is the difference between rigid and non-rigid truss?
5. Why is every member of a truss is called a two-force member?
6. Why is the self-weight of a member usually not considered in the analysis of a truss?
7. Which method is considered most suitable for finding forces in a few members of a truss?
8. What are the demerits of method of joints?
9. What are zero-force members? Why are they used?

MULTIPLE CHOICE QUESTIONS

1. Load is applied in a truss

(*a*) at the joint (*b*) on the member

(*c*) at joint and member both (*d*) at the joint as well as on the member.

2. Why is truss called a two-force member? Because

(*a*) two forces are applied on the members

(*b*) the member can either be subjected to tension or compression

(*c*) the member is under shear force and normal force

(*d*) none of these.

3. For a truss to be rigid, the condition is

(*a*) $2j = m - 3$ (*b*) $j = 2\,m + 3$ (*c*) $m = 2j - 3$ (*d*) $m = 2j + 3$.

where m = Number of members

j = Number of joints.

4. If a truss has more members than required to make it rigid, the truss is called

(*a*) under-rigid (*b*) over-rigid (*c*) rigid (*d*) non-rigid.

5. The method of joints is used to calculate the

(*a*) forces in all the members (*b*) forces in few members

(*c*) forces at the supports (*d*) bending moments in members.

6. Tension force in a member is assumed to be positive, if it is

(*a*) pushing the joint (*b*) pulling the joint

(*c*) bending the member (*d*) pushing the member.

7. The method of sections is useful, when

(*a*) forces in every member is required

(*b*) force in few members are required

(*c*) forces in few members as well as on the supports are required

(*d*) none of these.

ANSWERS

1. (*a*) **2.** (*b*) **3.** (*c*) **4.** (*b*) **5.** (*a*) **6.** (*b*) **7.** (*b*).

EXERCISES

1. Determine the forces in each member of the loaded truss shown in Fig. 13.77.

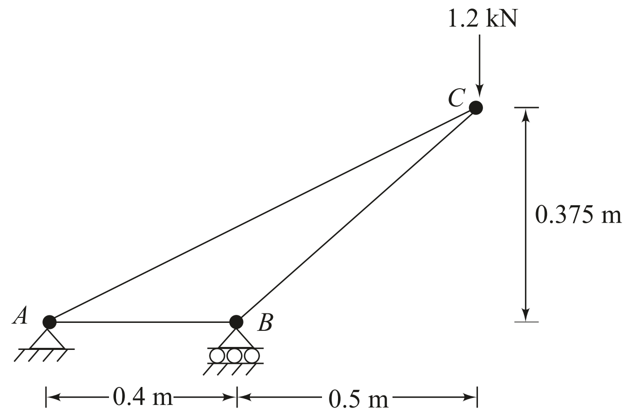


Fig. 13.77

(*Ans*. F_{AB} = 3.6 kN (C), F_{AC} = 3.9 kN (T), F_{BC} = 4.5 kN (C)).

2. Determine the forces in all the members of the loaded truss shown in Fig. 13.78.

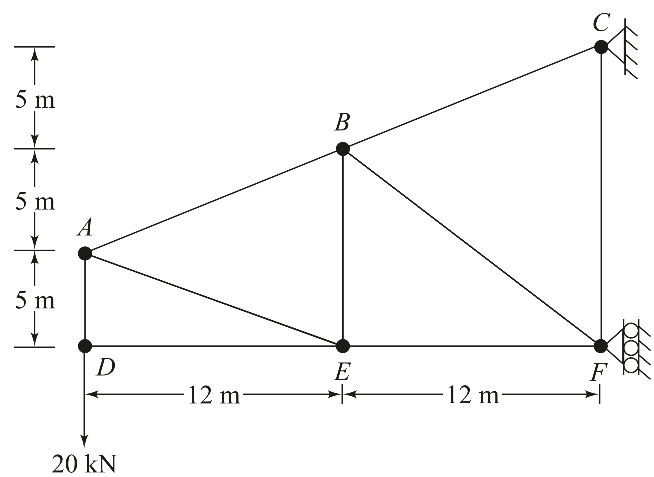


Fig. 13.78

(*Ans*. F_{AD} = 20 kN (T)
F_{AE} = 26 kN (C)
F_{DE} = 0
F_{AB} = 26 kN (T)
F_{BE} = 10 kN (T)
F_{BC} = 34.7 kN (T)
F_{BF} = 10.41 kN (C)
F_{EF} = 24 kN (C)
F_{CF} = 6.67 kN (T)).

3. Determine the zero-force members in the loaded truss shown in Fig. 13.79.

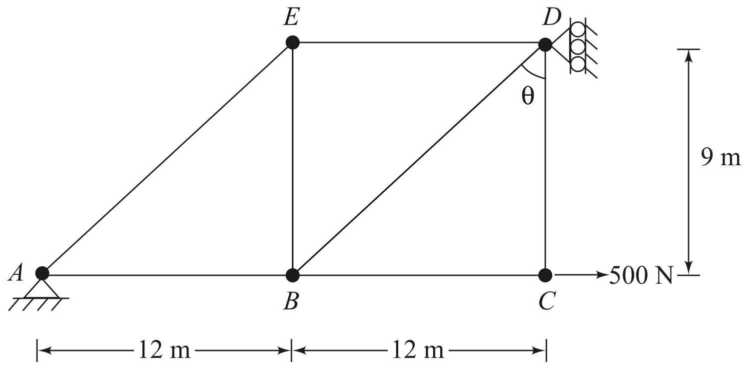


Fig. 13.79

(*Ans.* AE, DE, BD, CD).

4. Determine the forces in each member of the truss shown in Fig 13.80.

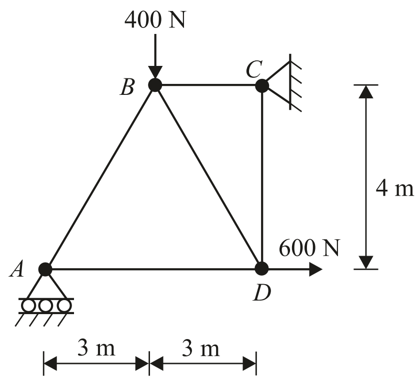


Fig. 13.80

(*Ans.* $V_A = 600$ N (↑), $H_C = 600$ N (←), $V_C = 200$ N (↓), $F_{AB} = 750$ N (C), $F_{AD} = 450$ N (T), $F_{BD} = 250$ N (T), $F_{CD} = 200$ N (C), $F_{BC} = 600$ N (C)).

5. Members AB and BC can support a maximum compressive force of 800 N, and members AD, DC and BD can support a maximum tensile force of 1500 N. If $y = 10$ m, determine the greatest load P the truss shown in Fig. 13.81 can support.

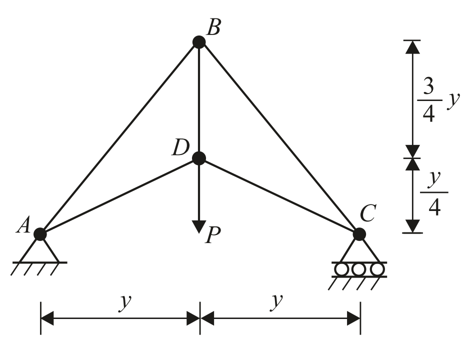


Fig. 13.81

(*Ans.* 849 N).

6. Determine the force in each member of the loaded truss consisting of three equilateral triangles as shown in Fig. 13.82.

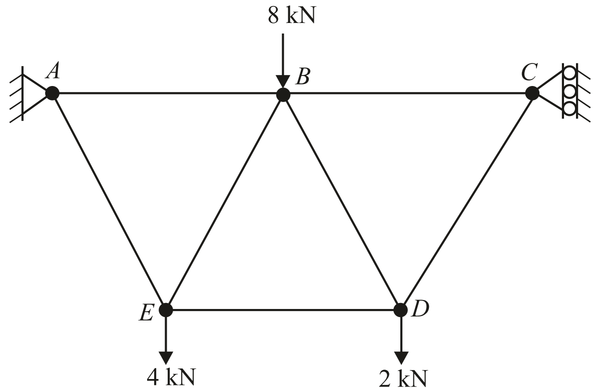


Fig. 13.82

(*Ans.* F_{AB} = 15.59 kN (C), F_{AE} = 8.66 kN(T),
F_{BC} = 15.01 kN (C), F_{BD} = 5.19 kN (C),
F_{BE} = 4.04 kN (C), F_{CD} = 7.50 kN(T),
F_{DE} = 6.35 kN (T)).

7. Determine the forces in the members *KC*, *KJ*, *CJ* and *JI* of the loaded truss shown in Fig.13.83.

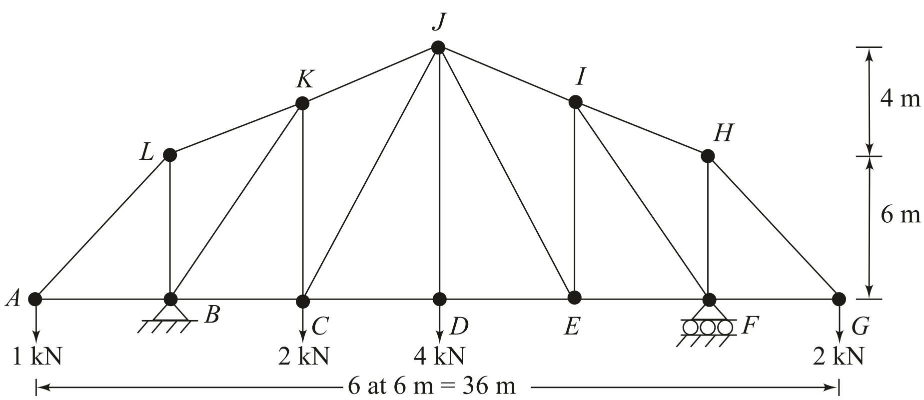


Fig. 13.83

(*Ans.* F_{KC} = 2.69 kN (T)
F_{KJ} = 1.78 kN (C)
F_{CJ} = 0.79 kN (C)
F_{JI} = 0.59 kN (C)).

8. Determine the forces in the members *HG*, *GC* and *CD* of the loaded truss shown in Fig. 13.84.

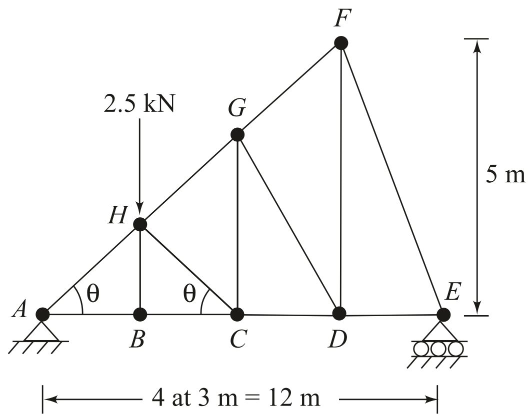


Fig. 13.84

(*Ans.* F_{HG} = 1.29 kN (C)
F_{GC} = 1.25 kN (T)
F_{CD} = 1.125 kN (T)).

9. Find the forces in the members *BC*, *BE* and *FE* of the loaded cantilever truss shown in Fig. 13.85.

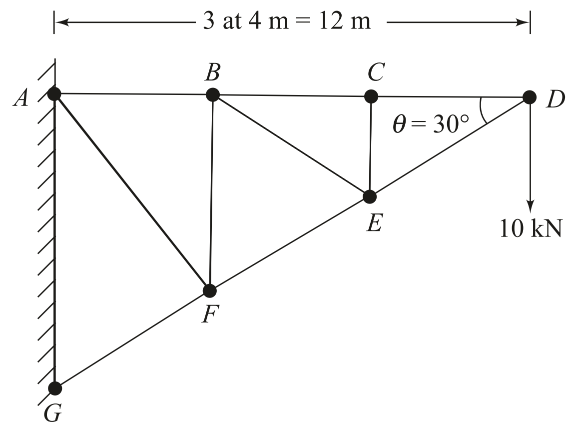


Fig. 13.85

(*Ans.* F_{BC} = 17.32 kN (T)
F_{BE} = 0
F_{FE} = 20 kN (C)).

10. Determine the force in member *BF* of the loaded truss shown in Fig. 13.86.

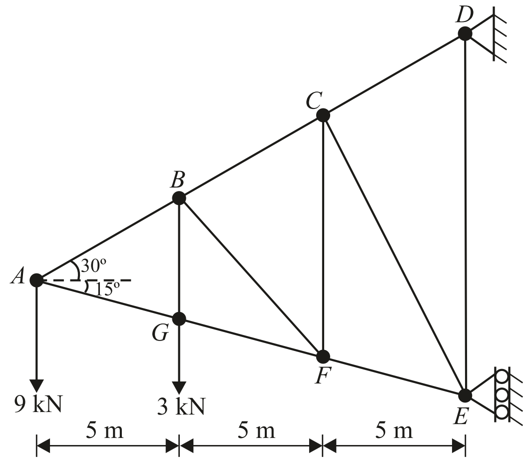


Fig. 13.86

(*Ans.* $F_{BF} = 2.66$ kN (C)).

11. Determine the forces in the members *AB* and *GH* of the loaded truss shown in Fig. 13.87.

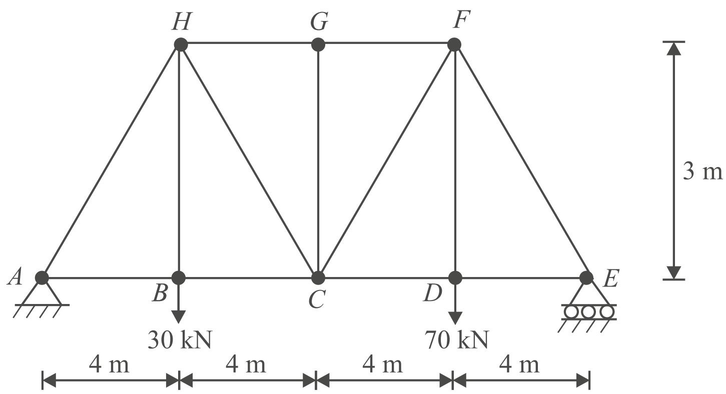


Fig. 13.87

(*Ans.* $F_{AB} = 53.33$ kN (T) , $F_{GH} = 66.67$ kN (C)).

12. Determine the forces in the members *BC* and *CD* of the simply supported truss shown in Fig. 13.88.

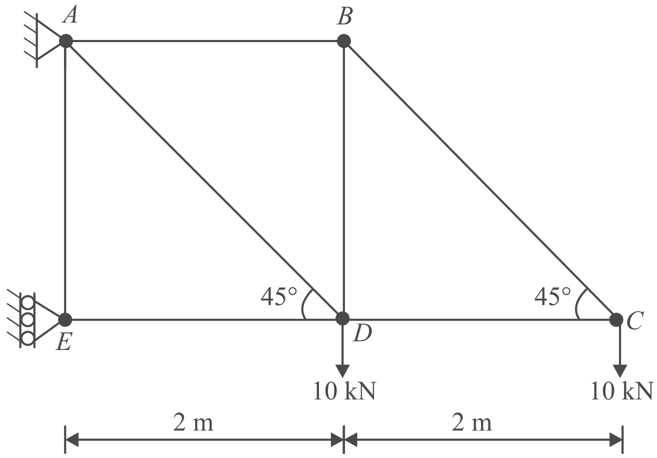


Fig. 13.88

(*Ans.* $F_{BC} = 14.14$ kN (T) , $F_{CD} = 10$ kN (C)).

13. Determine the forces in each member of the truss shown in Fig. 13.89 taking into consideration their weights of 50 N each.

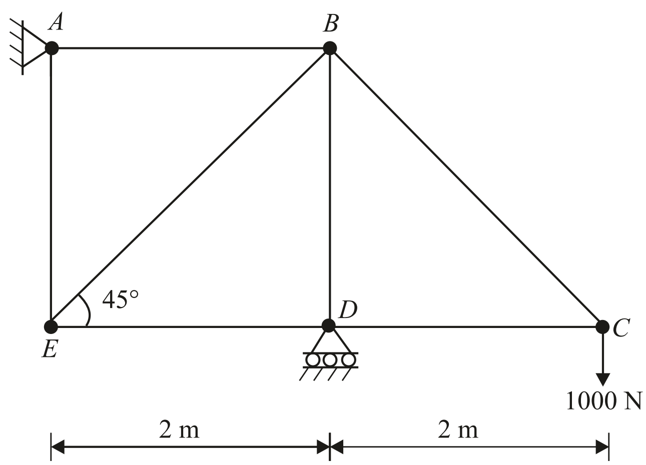


Fig. 13.89

(*Ans.* $V_A (\downarrow) = 925$ N, $H_A (\rightarrow) = 0$, $R_D (\uparrow) = 2275$ N,
$F_{AB} = 0$, $F_{BC} = 1485$ N (T),
$F_{CD} = 1050$ N (C), $F_{DE} = 1050$ N (C),
$F_{AE} = 975$ N (C), $F_{BD} = 2200$ N (C),
$F_{BE} = 1485$ N (T)).

❑❑❑

14 Combined Loadings

William John Macquorn Rankine (1820-1872)

William John Macquorn Rankine, born on 5 July 1820, was a Scottish civil engineer, physicist and mathematician. He is widely known for his contributions in both civil and mechanical engineering. Rankine was a founding contributor in thermodynamics, and developed the Rankine cycle, which gives the analysis of an ideal heat engine used in most of the power plants. He worked on the properties of steam, gases and vapours, and established relationship between saturated vapour pressure and temperature. He also established relationship between temperature, pressure and density of gas, and found expressions for the latent heat of evaporation of the liquid. In 1859, he proposed the Rankine scale of temperature, an absolute or thermodynamic scale whose degree is equal to a Fahrenheit degree. He served as the regius professor of civil engineering and mechanics at the University of Glasgow between 1855 and 1872. The Rankine method of earth pressure analysis in soil mechanics is named after him, which relates to the stabilization of retaining walls. He is also known for Rankine-Hugoniot equation, which governs the behavior of shock waves normal to the oncoming flow. He has been described as the 'father of engineering science' in recognition of his achievements.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- Why is combined loading analysis important?
- What is a stress element?
- What is meant by double eccentricity?
- Why is retaining wall so named?
- What is the safety condition to ensure no tension at the base of a retaining wall?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_14

14.1 INTRODUCTION

Structural members subjected to single type of loading are discussed in the previous chapters. For example, chapter 1 deals with an axially loaded bar under tension or compression, which produces normal stresses of tensile or compressive nature on the cross-section of the bar. The stress is constant at every section of a prismatic bar. Similarly, chapter 5 discusses the effect of pure bending in beams in which bending stresses of tensile and compressive nature are produced at the section of the beam. And chapter 7 discusses the torsional effect on the circular shaft in which the cross-section of the shaft is subjected to shear stress, which varies linearly from zero at the axis of the shaft to the maximum value at its surface. In all these cases, stresses produced in the members remain within the elastic limit of the material defined by the Hooke's law for which stress is proportional to strain.

However, in many engineering applications, members are subjected to more than one type of loading, thus forming the case of combined loadings. For example, a shaft in torsion may also be subjected to bending due to weights of the pulley, couplings, self-weight of the shaft and belt tension. Also a shaft experiences an axial load in addition to the twisting moment because of external axial loads, or due to the weight of the components attached to the shaft or due to thermal loading produced by the temperature change during service. Similarly, a beam may be subjected to the simultaneous action of bending moments and axial forces as shown in Fig. 14.1, where a cantilever beam *AB* is supported by a pin at *A* and the cable *CD*. The beam carries a vertical force *P* at its free end *B*,

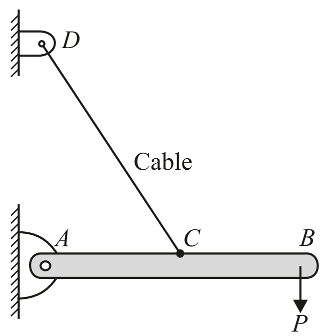


Fig. 14.1 A cantilever beam supported by a cable.

which causes bending and the cable produces axial force in the beam. The stress analysis of a member under combined loadings involves finding of stresses due to each load acting separately and then superimposing them, that is, adding them algebraically using the principle of superposition to find the resultant stresses caused by all the loadings acting simultaneously. In all these cases, the objective is to identify the stress elements in the members which are associated with high stress levels due to various loadings. Further principal stresses and maximum shear stresses are found by using the stress transformation relations, usually in the form of Mohr's circle. Again the Hooke's law is valid and the deformations produced are considered small.

14.2 COMBINED BENDING AND AXIAL LOADS

Many structural and machine members are subjected to combined loadings of axial forces and bending moments. Both loads produce normal stresses along the longitudinal direction of the members. The stresses are calculated separately due to each load and then added algebraically to find the combined stresses using the principle of superposition. Crane, timber beam and hacksaw are a few examples which involve combined stresses caused by axial force and bending moment.

Consider a simple beam loaded with a uniform load and an axial force P as shown in Fig. 14.2. The stress analysis is carried out by considering two points A and B on the cross-section of the beam. The point A lies on the bottom side and point B lies on the top side of the beam.

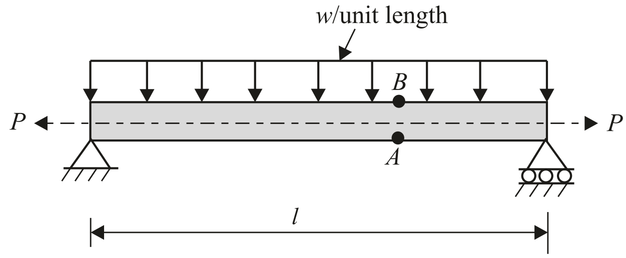


Fig. 14.2

Normal Stress due to Axial Force *P*

The axial force P produces direct tensile stress at the cross-section of the beam, given by

$$\sigma_d = \frac{P}{A} \quad \text{... (14.1)}$$

where A is the cross-sectional area of the beam. The direct stress is uniform across the cross-section of the beam.

Bending Stress due to Uniform Load *w*

The uniform load produces bending moment M, which in turn, produces normal stresses at points A and B, which are calculated by using flexure formula, given below.

$$\frac{\sigma_b}{y} = \frac{M}{I} = \frac{E}{R}$$

which gives

$$\sigma_b = \frac{M}{I} \cdot y = \frac{M}{S} \quad \text{... (14.2)}$$

where M = Bending moment due to uniform load

I = Moment of inertia of the beam's cross-section about the neutral axis

$= \frac{\pi}{64} d^4$ for a solid circular section

d = Diameter of the cross-section

y = Distance from the neutral axis

S = Section modulus of the cross-section

$= \frac{I}{y}$

The bending moment M is maximum at the midspan ($l/2$) of the beam and and its value is equal to $\frac{wl^2}{8}\cdot$ It produces tensile normal stress at A and compressive normal stress at B.

Resultant normal stresses

The resultant normal stresses due to combined loadings are calculated using the principle of superposition as follows:

Normal stress at point A is

σ_A = Normal tensile stress due to axial force P + Tensile bending stress due to uniform load w

$$= \frac{P}{A} + \frac{M}{S} \quad \text{... (14.3)}$$

Normal stress at point B is

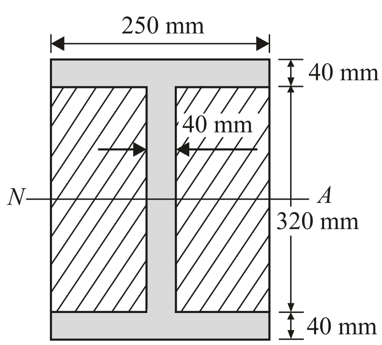
σ_B = Normal tensile stress due to axial force P + Compressive bending stress due to uniform load w

$$= \frac{P}{A} + \left(-\frac{M}{S}\right)$$

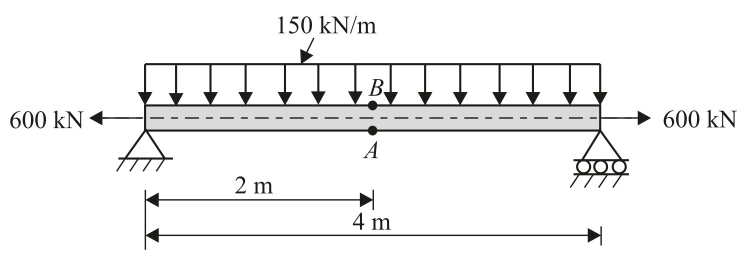
$$= \frac{P}{A} - \frac{M}{S} \quad \text{... (14.4)}$$

Example 14.1

A 4 m long simple steel beam having a symmetrical I-section is subjected to a uniform load of instensity 150 kN/m over its entire span and an axial tensile force of 600 kN as shown in Fig. 14.3. Determine the normal stresses at points A and B, and plot the normal stress variation between A and B.



(a) Beam's cross-section



(b) Loaded beam

Fig. 14.3

Solution: Given,

$$\text{Uniform load, } w = 150 \text{ kN/m} = 150 \times 10^3 \text{ N/m}$$

$$\text{Axial load, } P = 600 \text{ kN} = 600 \times 10^3 \text{ N}$$

$$\text{Length of the beam, } l = 4 \text{ m}$$

$$\text{Distance from the neutral axis, } y = \frac{400}{2} \text{ mm} = 200 \times 10^{-3} \text{ m}$$

The total area of cross-section of the beam is

$$\begin{aligned} A &= (250 \times 40) + (320 \times 40) + (250 \times 40) \\ &= 3.28 \times 10^4 \text{ mm}^2 \\ &= 3.28 \times 10^4 \times 10^{-6} \text{ m}^2 \\ &= 3.28 \times 10^{-2} \text{ m}^2 \end{aligned}$$

The moment of inertia of the beam's cross-section is calculated as

$$\begin{aligned} I &= I \text{ of rectangle } 250 \text{ mm} \times 400 \text{ mm} \\ &\qquad - 2\times (I \text{ of rectangle } 105 \text{ mm} \times 320 \text{ mm}) \\ &= \left[\frac{1}{12} \times 250 \times 400^3 - 2 \times \frac{1}{12} \times 105 \times 320^3\right] \times 10^{-12} \text{ m}^4 \\ &= 7.598 \times 10^{-4} \text{ m}^4 \end{aligned}$$

The beam is subjected to combined loadings of an axial tensile force and the bending moment caused due to uniform load. Axial force produces direct normal stress (tensile stress), which remains constant throughout the beam. The uniform load on the beam produces maximum bending moment at the centre of the beam.

The maximum bending moment due to uniform load is given as

$$\begin{aligned} M &= \frac{wl^2}{8} \\ &= \frac{150 \times 10^3 \times 4^2}{8} \\ &= 3 \times 10^5 \text{ N.m} \end{aligned}$$

It is the positive bending moment, which causes tension at point A and compression at point B.

The bending stress is found as

$$\begin{aligned} \sigma_b &= \frac{M}{I} \cdot y \\ &= \frac{3 \times 10^5 \times (200 \times 10^{-3})}{7.598 \times 10^{-4}} \times \frac{1}{10^6} \text{ MPa} \\ &= 78.97 \text{ MPa} \end{aligned}$$

Ans.

It is of equal magnitude at both points A and B, but is tensile at A and compressive at B.

The direct stress due to axial load is

$$\sigma_d = \frac{P}{A} = \frac{600\times10^3}{3.28\times10^{-2}}\times\frac{1}{10^6}\text{ MPa}$$

$$= 18.29\text{ MPa} \qquad \textbf{Ans.}$$

The direct stress is tensile and is always positive.

The normal stress due to combined loadings at A is

$$\sigma_A = \sigma_d + \sigma_b$$

$$= (18.29 + 78.97)\text{ MPa}$$

$$= 97.26\text{ MPa} \qquad \textbf{Ans.}$$

Hence, the tensile stress occurs at point A.

The normal stress due to combined loadings at point B is

$$\sigma_B = \sigma_d - \sigma_b$$

$$= (18.29 - 78.97)\text{ MPa}$$

$$= -60.68\text{ MPa} \qquad \textbf{Ans.}$$

Hence, the compressive stress occurs at point B.

The distributions of axial, bending and combined stresses across the cross-section of the beam are shown in Fig. 14.4. It is important to note that when the beam is subjected only to bending stresses, the neutral axis (NA) passes through the centroid of the cross-section. Further, when the beam is subjected to combined stresses due to both axial force and bending moment, the line of zero stress shifts upward.

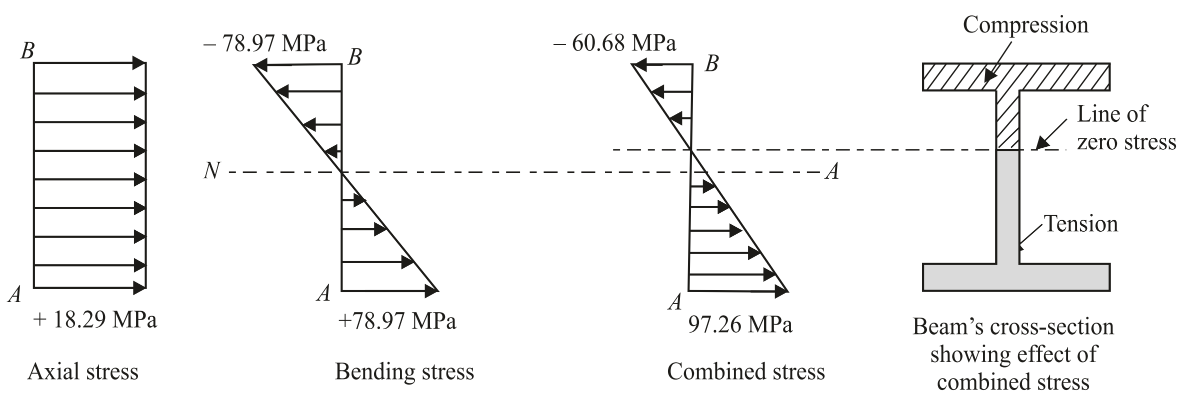


Fig. 14.4 Stress distribution across the cross-section.

14.3 COMBINED BENDING AND TORSION OF CIRCULAR SHAFTS

Let us consider a solid circular shaft of diameter d and length l be subjected to combined loadings of torsion and bending as shown in Fig. 14.5. The shaft is loaded by a twisting moment (torque) and a bending force P at its free end. These loads produce a bending moment M, a vertical shear force V and a twisting moment T at every cross-section of the shaft (Fig. 14.6), which in turn, individually produces stresses over the section.

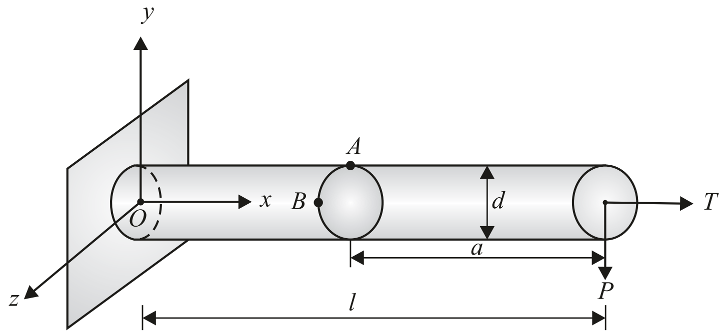


Fig. 14.5 A solid circular shaft under combined bending and torsion.

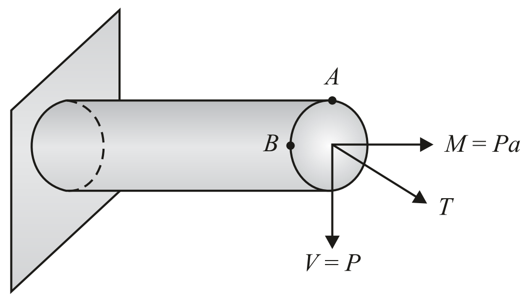


Fig. 14.6 Three loads (M, T and P) acting on the cross-section of the shaft at 'a'.

To start the stress analysis, we consider two points A and B on the cross-section of the shaft at a distance 'a' from its free end as shown in Fig. 14.5. Point A is on the top of the shaft and point B on the left side at the neutral axis.

Effect of twisting moment *T*

The torsional shear stress τ_1 produced due to twisting moment T is given as

$$\tau_1 = \frac{Tr}{J} = \frac{2T}{\pi r^3} = \frac{16T}{\pi d^3} \qquad \text{(using torsion formula) ... (14.5)}$$

where r = Radius of the shaft

J = Polar moment of inertia of the shaft cross-section

$$= \frac{\pi r^4}{2} = \frac{\pi}{32} d^4$$

d = Diameter of the shaft = $2r$

Since the torsional shear stress acts on the surface of the shaft, hence it acts at both points A and B. The shear stress acts in the horizontal direction (xz-plane) at point A and in the vertical direction at point B as B is located on the side of the shaft.

Effect of bending moment M

The bending moment M caused due to load P produces a bending tensile stress at point A, given by

$$\sigma_A = \frac{Mr}{I} = \frac{4M}{\pi r^3} = \frac{32M}{\pi d^3} \qquad \text{(using bending formula) ... (14.6)}$$

where I = Moment of inertia of the shaft cross-section about the neutral axis (the z-axis).

However, no bending stress is produced at point B by the bending moment M as point B is located on the neutral axis.

Effect of vertical shear force V

The vertical shear force arising from the bending force P produces no shear stress at point A as it is located at the top surface of the shaft, but produces the shear stress at point B, given by

$$\tau_2 = \frac{4V}{3A} = \frac{4V}{3\pi r^2} = \frac{16V}{3\pi d^2} \qquad \text{(using equation (5.32)) ....(14.7)}$$

where A = Cross-sectional area of the shaft

$$= \pi r^2$$

$$= \frac{\pi d^2}{4}$$

Representation of stresses

All the stresses produced by M, T and V are shown in Fig. 14.7.

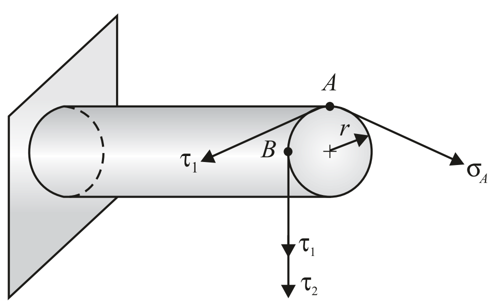
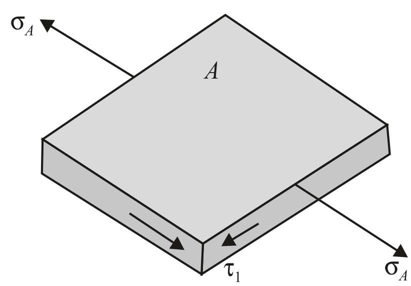


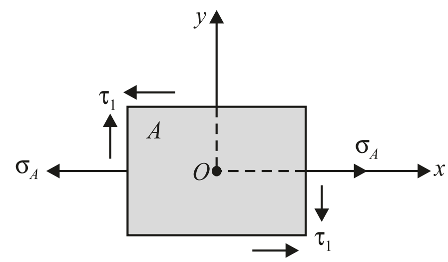
Fig. 14.7 Stresses acting at points A and B.

Stress elements at points *A* and *B*

The stresses σ_A and τ_1 acting at point *A* are shown in Fig. 14.8 (*a*) on a stress element *A*, which has been cut out from the top surface of the shaft at point *A*, and the corresponding plane stress condition is shown in Fig. 14.8 (*b*). For the purpose of calculating the principal stresses and the maximum shear stress, the *x* and *y* axes are drawn through the element, where *x*-axis is parallel to the longitudinal axis of the shaft and *y*-axis is horizontal. Hence, it is a case of plane stress condition in which $\sigma_x = \sigma_A$, $\sigma_y = 0$ and $\tau_{xy} = -\tau_1$.



(*a*) Stress element at point *A* (in horizontal plane)

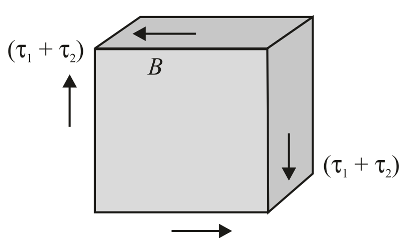


(*b*) Plane stress condition at point *A*

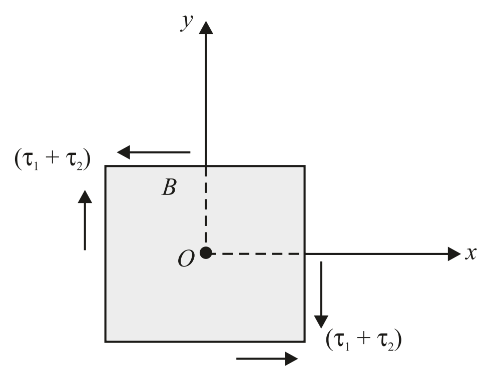
Fig. 14.8

Similarly, another element at point *B*, which has been cut out from the left side of the shaft at point *B* is shown in Fig. 14.9 (*a*). Only shear stress is acting on this element, which has the largest value of $(\tau_1 + \tau_2)$. Hence, the point *B* is in a state of pure shear. The plane stress system for the element *B* is shown in Fig. 14.9 (*b*).

Hence, $$\sigma_x = \sigma_y = 0 \text{ and } \tau_{xy} = -(\tau_1 + \tau_2)$$



(*a*) Stress element at point *B*
(in vertical plane)



(*b*) Plane stress condition at point *B*

Fig. 14.9

Principal stresses and maximum shear stress due to combined loadings

The maximum and minimum normal stresses at point A are the maximum and minimum principal stresses respectively, given as

$$\sigma_1 = \text{Maximum principal stress}$$

$$= \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{\sigma_A}{2} + \sqrt{\left(\frac{\sigma_A}{2}\right)^2 + \tau_1^2} \qquad (\sigma_x = \sigma_A \text{ and } \tau_{xy} = \tau_1)$$

On substituting the values of σ_A and τ_1 in the above equation, we have

$$\sigma_1 = \frac{1}{2} \times \frac{32M}{\pi d^3} + \sqrt{\left(\frac{1}{2} \times \frac{32M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16M}{\pi d^3} + \sqrt{\left(\frac{16M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right] \qquad \text{... (14.8)}$$

$$= \frac{32}{\pi d^3}\left[\frac{M + \sqrt{M^2 + T^2}}{2}\right]$$

$$= \frac{32M_e}{\pi d^3}$$

where $M_e = \frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right]$

= Equivalent bending moment, defined as the bending moment which acting alone, will produce the same maximum direct stress as produced by the combined bending moment and torque acting together.

and σ_2 = Minimum principal stress

$$= \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{\sigma_A}{2} - \sqrt{\left(\frac{\sigma_A}{2}\right)^2 + \tau_1^2}$$

$$= \frac{16}{\pi d^3}\left[M - \sqrt{M^2 + T^2}\right] \qquad \text{... (14.9)}$$

The maximum shear stress is calculated as

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{\sigma_A}{2}\right)^2 + \tau_1^2}$$

On substituting the values of σ_A and τ_1 in the above equation, we have

$$\tau_{max} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2} \quad \text{... (14.10)}$$

$$= \frac{16T_e}{\pi d^3}$$

where $T_e = \sqrt{M^2 + T^2}$

= Equivalent torque, defined as the torque which acting alone, will produce the same maximum shear stress as produced by the combined bending moment and torque acting together.

If θ be the angle of inclination of the principal plane with the transverse section of the shaft, measured in the anticlockwise direction, then

$$\tan 2\theta = \frac{\tau_1}{\left(\frac{\sigma_A}{2}\right)} = \frac{2\tau_1}{\sigma_A} = \frac{T}{M} \quad \text{(on substituting } \sigma_A \text{ and } \tau_1\text{)}$$

These maximum stresses, both principal and shear, can be compared with their design values to ensure the safety criteria of the shaft.

Critical points in the shaft

The normal stresses due to bending are maximum where maximum bending moment occurs. It means that the normal stresses are maximum, when element *A* is located at the support end of the shaft, where the bending moment *M* has its maximum value. Hence, points on the top and bottom of the shaft at its support end are the critical points. Another critical point is *B* itself, where the shear stress is maximum given by the sum of τ_1 and τ_2, but the bending stress σ_x is zero. The value of the shear stress does not change, if point *B* is moved anywhere along the shaft in the longitudinal direction.

Example 14.2

A hollow shaft of inside diameter equal to one-half of the outside diameter is subjected to a torque of 35 kN·m and a bending moment of 20 kN·m. If the maximum shear stress is limited to 70 MPa, find the diameter of shaft.

Solution: Given,

Torque, T = 35kN·m

Bending moment, M = 20 kN·m

Maximum shear stress, τ_{max} = 70 MPa = 70×10^6 Pa

Let

Outside diameter of the shaft $= d_o$

Inside diameter of the shaft $= d_i = \dfrac{d_o}{2}$ (Given)

Using equation (14.10), the maximum shear stress for a hollow shaft can be expressed as

$$\tau_{max} = \frac{16d_o}{\pi(d_o^4 - d_i^4)}\sqrt{M^2 + T^2}$$

$$70 \times 10^6 = \frac{16 \times 2d_i}{\pi[(2d_i)^4 - d_i^4)]}\sqrt{(20 \times 10^3)^2 + (35 \times 10^3)^2}$$

Solving for d_i, we get

$d_i = 73.12$ mm **Ans.**

and $d_o = 2d_i = 146.24$ mm **Ans.**

Example 14.3

An engine has an overhung crankshaft and its stroke is 300 mm. The centre-line of the crank-pin and the connecting rod is 200 mm distant from the centre of the supporting bearing. A thrust of 40 kN acts on the crank-pin at right angles to the crank. Determine the diameter of the shaft, if the stresses in tension and shear are not to exceed 70 MPa and 40 MPa respectively.

Solution: Given,

Maximum stress in tension, $\sigma = 70$ MPa $= 70 \times 10^6$ Pa

Maximum shear stress, $\tau_{max} = 40$ MPa $= 40 \times 10^6$ Pa

The bending moment at the bearing is given by

$$M = \frac{40 \times 10^3 \times 200}{10^3} \text{ N.m} = 8000 \text{ N.m}$$

The torque at the bearing is given as

$$T = \frac{40 \times 10^3 \times 150}{10^3} \text{ N.m} = 6000 \text{ N.m}$$

The equivalent bending moment is

$$M_e = \frac{1}{2}[M + \sqrt{M^2 + T^2}]$$

$$= \frac{1}{2}[8000 + \sqrt{(8000)^2 + (6000)^2}] = 9000 \text{ N.m}$$

The equivalent torque is $T_e = \sqrt{M^2 + T^2} = \sqrt{(8000)^2 + (6000)^2}$

$= 10000$ N.m

Let d be the diameter of the shaft.

Diameter on the basis of equivalent bending moment

Since $$\sigma = \frac{32M_e}{\pi d^3}$$

or $$70 \times 10^6 = \frac{32 \times 9000}{\pi d^3}$$

Solving for d, we get $d = 0.109$ m = 109 mm

Diameter on the basis of equivalent torque

Since $$\tau_{max} = \frac{16T_e}{\pi d^3}$$

or $$40 \times 10^6 = \frac{16 \times 10000}{\pi d^3}$$

Solving for d, we get $d = 0.108$ m = 108 mm

Selecting bigger of the two values, we have

$$d = 109 \text{ mm}$$ **Ans.**

Example 14.4

A shaft transmits 740 kW at 750 rpm. The maximum torque on the shaft exceeds its mean value by 80%. The shaft is supported in main bearings 3 metres apart and carries at its middle a flywheel weighing 50 kN. Due to fluid pressure on the piston, the shaft is also subjected at the point, where the flywheel is located, to a bending moment which may be taken as numerically equal to 80% of the mean twisting moment.

If the maximum allowable stresses for the material of the shaft in tension and shear are 65 MPa and 40 MPa respectively, calculate the diameters of the shaft based on these values.

Solution: Given,

Power transmitted by the shaft, $P = 740$ kW

Rotational speed of the shaft, $N = 750$ rpm

Maximum stress in tension, $\sigma = 65 \text{ MPa} = 65 \times 10^6$ Pa

Maximum shear stress, $\tau_{max} = 40 \text{ MPa} = 40 \times 10^6$ Pa

Length of the shaft, $l = 3$ m

Central load on the shaft, $W = 50 \text{ kN} = 50 \times 10^3$ N

The power transmitted by the shaft is given as

$$P = \frac{\pi N T_{av}}{30,000} \text{ kW}$$

$$740 = \frac{\pi \times 750 \times T_{av}}{30,000}$$

which gives $T_{av} = 9421.97$ N.m

The maximum torque is $T = 1.8 \times T_{av} = 1.8 \times 9421.97 = 16959.5$ N.m

The maximum bending moment due to central load W occurs at the centre of the shaft, given by

$$M_w = \frac{Wl}{4} = \frac{50 \times 10^3 \times 3}{4} = 37500 \text{ N.m}$$

The bending moment due to fluid pressure is

$$M_f = 0.8 \times T_{av} = 0.8 \times 9421.97$$
$$= 7537.57 \text{ N.m}$$

The total bending moment acting on the shaft is

$$M = M_w + M_f$$
$$= 37500 + 7537.57$$
$$= 45037.57 \text{ N.m}$$

The equivalent bending moment is given as

$$M_e = \frac{1}{2}[M + \sqrt{M^2 + T^2}]$$
$$= \frac{1}{2}[45037.57 + \sqrt{(45037.57)^2 + (16959.5)^2}]$$
$$= 46581.24 \text{ N.m}$$

The equivalent torque is given as

$$T_e = \sqrt{M^2 + T^2} = \sqrt{(45037.57)^2 + (16959.5)^2}$$
$$= 48124.91 \text{ N.m}$$

Diameter on the basis of M_e

Since $\sigma = \dfrac{32M_e}{\pi d^3}$

where d = Diameter of the shaft

or $65 \times 10^6 = \dfrac{32 \times 46581.24}{\pi d^3}$

Solving for d, we get $d = 0.194$ m = 194 mm **Ans.**

Diameter on the basis of T_e

Since $\tau_{max} = \dfrac{16T_e}{\pi d^3}$

or $40 \times 10^6 = \dfrac{16 \times 48124.91}{\pi d^3}$

Solving for d, we get $d = 0.183$ m = 183 mm **Ans.**

Example 14.5

An overhanging pulley of diameter 1 m and weighing 1 kN transmits 45 HP at 140 rpm, the sides of the belt being vertical. The ratio of tensions is 2 : 1, and the maximum tensile and shear stresses are limited to 120 MPa and 60 MPa respectively. Find the diameter of the shaft. The centre of the pulley is 0.35 m from the nearest bearing.

Solution: Given,

Power to be transmitted, $P = 45$ HP

Radius of the pulley, $r = 0.5$ m

Weight of the pulley, $W = 1$ kN $= 1000$ N

Revolutions per minute, $N = 140$

Ratio of the two tensions, $\frac{T_2}{T_1} = 2$

Maximum tensile stress, $\sigma = 120$ MPa $= 120 \times 10^6$ Pa

Maximum shear stress, $\tau_{max} = 60$ MPa $= 60 \times 10^6$ Pa

The power to be transmitted is

$$P = \frac{2\pi NT}{4500}$$

$$45 = \frac{2\pi \times 140 \times T}{4500}$$

or

$$T = 230.2 \text{ kgf.m}$$

$$= 2258.32 \text{ N.m} \qquad (1 \text{ kgf} = 9.81 \text{ N})$$

Now

$$T = (T_2 - T_1) \times r$$

$$= T_2\left(1 - \frac{T_1}{T_2}\right) \times r$$

or

$$2258.32 = T_2\left(1 - \frac{1}{2}\right) \times 0.5$$

Hence,

$$T_2 = 9033.28 \text{ N}$$

and

$$T_1 = \frac{T_2}{2} = \frac{9033.28}{2} = 4516.64 \text{ N}$$

The resultant force on the pulley is

$$R = T_1 + T_2 + W$$

$$= 4516.64 + 9033.28 + 1000$$

$$= 14549.92 \text{ N}$$

The maximum bending moment is

$$M = R \times 0.35$$

$$= 14549.92 \times 0.35 \text{ N.m}$$

$$= 5092.47 \text{ N.m}$$

Diameter on the basis of equivalent bending moment

The equivalent bending moment is

$$M_e = \frac{M + \sqrt{M^2 + T^2}}{2}$$

$$= \frac{5092.47 + \sqrt{(5092.47)^2 + (2258.32)^2}}{2}$$

$$= 5331.61 \text{ N.m}$$

Let d be the diameter of the shaft.

Now $$\sigma = \frac{32 M_e}{\pi d^3}$$

or $$120 \times 10^6 = \frac{32 \times 5331.61}{\pi d^3}$$

Solving for d, we get $$d = 0.0768 \text{ m} = 76.8 \text{ mm}$$

Diameter on the basis of equivalent torque

The equivalent torque is $$T_e = \sqrt{M^2 + T^2}$$

$$= \sqrt{(5092.47)^2 + (2258.32)^2}$$

$$= 5570.75 \text{ N.m}$$

Now $$\tau_{max} = \frac{16 T_e}{\pi d^3}$$

or $$60 \times 10^6 = \frac{16 \times 5570.75}{\pi d^3}$$

Solving for d, we get $$d = 0.0779 \text{ m} = 77.9 \text{ mm}$$

Of the two diameters, we choose the bigger diameter.

Hence, $d = 77.9$ mm **Ans.**

Example 14.6

A flywheel weighing 6 kN is mounted on a shaft 80 mm in diameter and midway between bearings 600 mm apart, in which the shaft may be assumed to be directionally free. If the shaft is transmitting 29.6 kW at 360 rpm, calculate the principal stresses and the maximum shearing stresses in the shaft at the ends of a vertical and a horizontal diameter in a plane close to that of the flywheel.

Solution: Given,

Weight of the flywheel, $W = 6 \text{ kN} = 6 \times 10^3$ N

Diameter of the shaft, $d = 80$ mm

Distance between the bearings, $l = 600$ mm

Power to be transmitted, $P = 29.6$ kW

Rotational speed of the shaft, $N = 360$ rpm

Since the flywheel is centrally mounted on the shaft, the situation is similar to the case, when a concentrated load is placed centrally on a simply supported beam. The maximum bending moment in this case is given as

$$M = \frac{Wl}{4} = \frac{6\times 10^3 \times 600}{4\times 10^3} \text{ N.m} = 900 \text{ N.m}$$

The maximum shear force is given as

$$V = \frac{W}{2} = \frac{6\times 10^3}{2} = 3000\ N$$

The torque can be calculated using power equation.

$$P = \frac{\pi NT}{30,000} \text{ kW}$$

or

$$29.6 = \frac{\pi\times 360\times T}{30,000}$$

Solving for T, we get $T = 785.16$ N.m

Calculation of stresses in the shaft at the ends of the vertical diameter

The stress due to bending is

$$\sigma = \frac{32M}{\pi d^3} = \frac{32\times 900}{\pi\times(80\times 10^{-3})^3}\times\frac{1}{10^6} \text{ MPa} = 17.9 \text{ MPa}$$

The stress due to torque is

$$\tau = \frac{16T}{\pi d^3} = \frac{16\times 785.16}{\pi\times(80\times 10^{-3})^3}\times\frac{1}{10^6} \text{ MPa} = 7.81 \text{ MPa}$$

The shear stress due to shear force V is zero.

Hence, the principal stresses are given by

$$\sigma_{1,2} = \frac{1}{2}[\sigma\pm\sqrt{\sigma^2+4\tau^2}] = \frac{1}{2}[17.9\pm\sqrt{(17.9)^2+4\times(7.81)^2}]$$

$$= 20.83 \text{ MPa and } -2.93 \text{ MPa}$$

The major principal stress is $\sigma_1 = 20.83$ MPa (Tensile) **Ans.**

The minor principal stress is $\sigma_2 = 2.93$ MPa (Compressive) **Ans.**

The maximum shear stress is

$$\tau_{max} = \frac{\sigma_1-\sigma_2}{2} = \frac{20.83-(-2.93)}{2}$$

$$= 11.88 \text{ MPa}$$ **Ans.**

Calculation of stresses in the shaft at the ends of the horizontal diameter

The shear stress due to torque remain the same.

$$\tau = 7.81 \text{ MPa}$$

The shear stress due to shear force V is given using equation (5.32) as

$$\tau' = \frac{4}{3} \times \frac{V}{A} = \frac{4}{3} \times \frac{3000}{\frac{\pi}{4} \times (80 \times 10^{-3})^2} \times \frac{1}{10^6} \text{ MPa}$$

$$= 0.795 \text{ MPa}$$

The stress due to bending is zero, because the distance of outermost layer from the neutral axis is zero.

Hence, the maximum shear stress is given as

$$\tau_{max} = \tau + \tau' = 7.81 + 0.795 = 8.605 \text{ MPa} \qquad \textbf{Ans.}$$

And the principal stresses are given as

$$\sigma_{1,2} = \pm\, \tau_{max} = \pm\, 8.605 \text{ MPa}$$

$$\sigma_1 = 8.605 \text{ MPa (Tensile)}$$

and $\sigma_2 = 8.605$ MPa (Compressive) **Ans.**

14.4 COMBINED TORSION AND AXIAL LOADS

Let us consider a solid circular shaft be subjected to the combined effects of torsion and axial load as shown in Fig. 14.10. The shaft of diameter d and length l is loaded by a twisting moment T and an axial load P. The state of stress in the shaft is a combination of the shear stress τ produced by the twisting moment T and the axial normal stress σ produced due to axial load P. As the surface of the shaft is the location of the most critically stressed part, hence we consider stress element on the surface at point A.

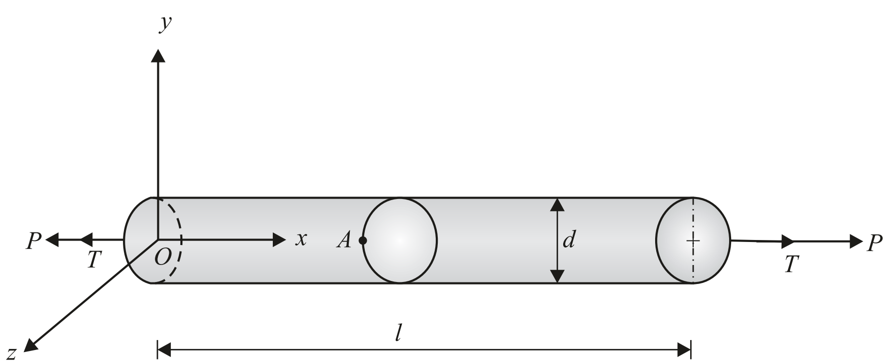


Fig. 14.10

Effect of twisting moment *T*

The twisting moment T produces torsional shear stress τ on the cross-section of the shaft, which varies linearly with the distance from the centre of the shaft and is maximum at the surface. The shear stress is given as

$$\tau = \frac{16}{\pi d^3} \qquad \text{(using torsion formula) ... (14.11)}$$

The shear stress acts in the vertical direction as point A is on the side surface of the shaft.

Effect of axial load *P*

The axial load P produces direct tensile stress σ_d at A, which is constant across the cross-section of the shaft and is given as

$$\sigma_d = \frac{P}{A} = \frac{4P}{\pi d^2} \qquad \text{...(14.12)}$$

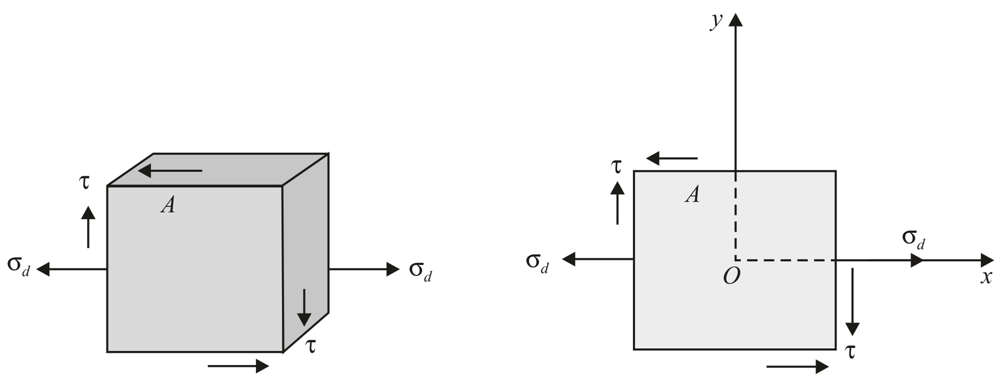
where A = Cross-sectional area of the shaft

$= \frac{\pi}{4} d^2$

d = Diameter of the shaft

Stress element at point *A*

The stresses σ_A and τ acting at point A are shown in Fig. 14.11 (*a*) on a stress element, which has been cut out from the side of the shaft at point A. The x and y axes are drawn through the element, where x-axis is parallel to the longitudinal axis of the shaft and y-axis is perpendicular to the longitudinal axis. Hence, it is a case of plane stress condition in which $\sigma_x = \sigma_A$, $\sigma_y = 0$ and $\tau_{xy} = -\tau$ as shown in Fig. 14.11(*b*).



(*a*) Stress element at point *A* (in vertical plane)

(*b*) Plane stress condition at point *A*

Fig. 14.11

Combined stress and Mohr's circle

The maximum shear stress due to combined loadings in the shaft is obtained as

$$\tau_{max} = R = \sqrt{\left(\frac{\sigma_d}{2}\right)^2 + \tau^2} \qquad \text{...(14.13)}$$

The Mohr's circle representation of the combined loadings is shown in Fig. 14.12, where R is the radius of the circle, which represents the maximum shear stress.

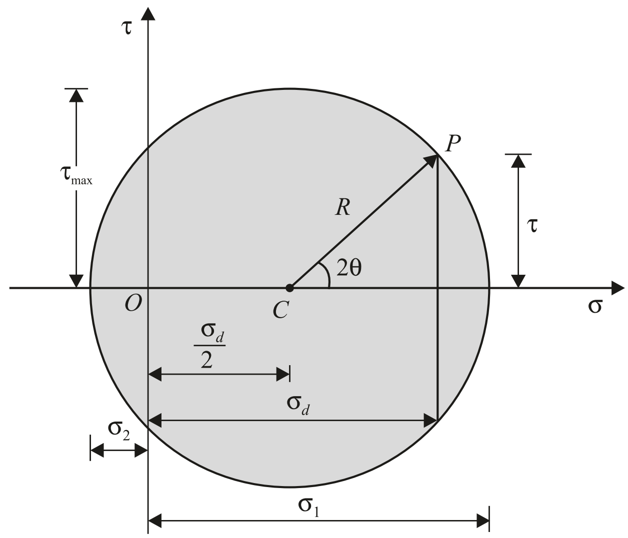


Fig. 14.12 Mohr's circle for combined torsion and axial loads.

Example 14.7

A 50 mm diameter shaft is subjected to an axial compressive load of 280 N and a twisting moment of 400 N.m. Find the principal stresses and the maximum shear stress due to the combined effect.

Solution: Given,

$$\text{Diameter of the shaft, } d = 50 \text{ mm} = 50 \times 10^{-3} \text{ m}$$

$$\text{Axial load, } P = 280 \text{ N}$$

$$\text{Twisting moment, } T = 400 \text{ N.m}$$

The axial load P produces direct compressive stress σ_d, which is distributed uniformly over the cross-section of the shaft, and is given as

$$\sigma_d = -\frac{P}{A} = -\frac{P}{\left(\frac{\pi}{4}d^2\right)} \qquad (A \text{ is the cross-sectional area})$$

$$= -\frac{280}{\frac{\pi}{4}\times(50\times10^{-3})^2} = -0.142 \text{ MPa}$$

$$= 0.142 \text{ MPa (Compressive)}$$

The twisting moment T produces shear stress across the section of the shaft, given as

$$\tau = \frac{16T}{\pi d^3} \qquad \text{(using torsion formula)}$$

$$= \frac{16\times400}{\pi\times(50\times10^{-3})^3} = 16.3 \text{ MPa}$$

Calculation of principal stresses

The maximum principal stress σ_1 is given as

$$\sigma_1 = \frac{\sigma_d}{2} + \frac{\sqrt{\sigma_d^2 + 4\tau^2}}{2}$$

$$= \frac{-0.142}{2} + \frac{\sqrt{(-0.142)^2 + 4 \times (16.3)^2}}{2}$$

$$= 16.23 \text{ MPa (Tensile)} \qquad \textbf{Ans.}$$

The minimum principal stress σ_2 is given as

$$\sigma_2 = \frac{\sigma_d}{2} - \frac{\sqrt{\sigma_d^2 + 4\pi^2}}{2}$$

$$= \frac{-0.142}{2} - \frac{\sqrt{(-0.142)^2 + 4 \times (16.3)^2}}{2} = -16.37 \text{ MPa}$$

$$= 16.37 \text{ MPa (Compressive)} \qquad \textbf{Ans.}$$

Calculation of maximum shear stress

The maximum shear stress τ_{max} is calculated as

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{16.23 - (-16.37)}{2} = 16.3 \text{ MPa} \qquad \textbf{Ans.}$$

14.5 COMBINED BENDING, TORSION AND DIRECT THRUST

Marine propeller shafts are usually subjected to direct thrust in addition to torsion and bending moment. The direct thrust is produced because of the compressive reaction of the water on the propeller as the craft is pushed forward. The bending moment is caused due to the self-weight of the shaft between bearings and the torque comes from the rotation of the shaft which is needed to produce the required power. The shaft at its every cross-section experiences these loads, which produce stresses separately.

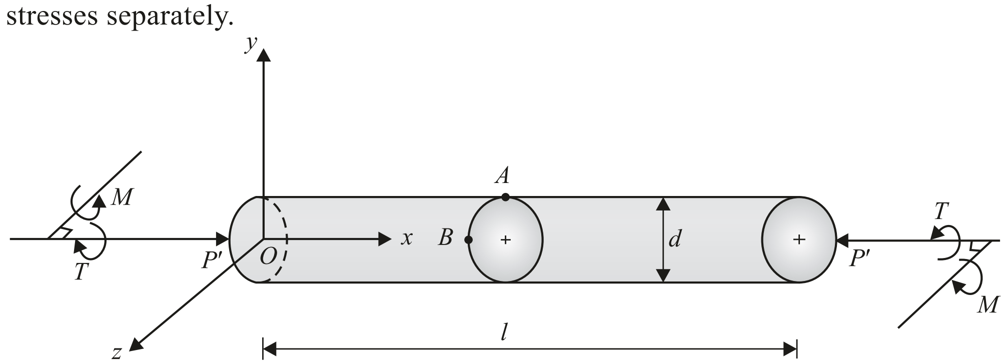


Fig. 14.13 A solid circular shaft under combined bending, torsion and direct thrust.

Let us consider a solid circular shaft of diameter d and length l being subjected to a twisting moment (torque) T, a bending moment M and an axial thrust P exerted at the end as shown in Fig. 14.13.

For stress analysis consideration, we select two points A and B on the cross-section of the shaft as shown in the figure. Point A is on the top surface of the shaft and point B on the left side at the neutral axis.

Effect of direct thrust

The axial thrust P' acts along the longitudinal axis (x-axis) of the shaft and produces a direct compressive stress σ_d, which is uniform across the cross-section of the shaft, and is given by equation (14.12) as

$$\sigma_d = \frac{P'}{A} = \frac{4P'}{\pi d^2}$$

where A is the cross-sectional area of the shaft. The direct stress has the same value at both points A and B.

Effect of twisting moment *T*

The torsional shear stress τ_1 produced due to twisting moment T is given by equation (14.5) as

$$\tau_1 = \frac{16T}{\pi d^3}$$

The shear stresses are produced at both points A and B as they are positioned on the surface of the shaft. The shear stress at A acts in the horizontal plane (xz-plane) and at B in the vertical direction.

Effect of bending moment *M*

The bending moment M produces tensile or compressive bending stress, given by

$$\sigma_b = \frac{Mr}{I} = \frac{4M}{\pi r^3} = \frac{32M}{\pi d^3} \qquad \text{(using bending formula) ... (14.14)}$$

where the symbols have their usual meanings. The maximum bending stress is produced at point A as it is farthest from the neutral axis, and there is no bending stress produced at point B because of its position at the neutral axis.

Effect of vertical shear force *V*

The vertical shear force V arises from the self-weight of the shaft, which produces no shear stress at point A because of its location at the top surface of the shaft. The shear stress produced by V at point B is given by equation (14.7) as

$$\tau_2 = \frac{4V}{3A} = \frac{4V}{3\pi r^2} = \frac{16V}{3\pi d^2} \qquad (A = \text{Cross-sectional area of the shaft})$$

$$= \pi r^2 = \frac{\pi}{4} d^2$$

Superimposition of normal stresses

The direct stress due to thrust is superimposed over the direct stress due to bending to obtain their combined effects taking due accounts of their positive or negative signs.

Representation of stresses

All the stresses acting at points *A* and *B* are shown in Fig. 14.14. The bending stress σ_b may be tensile or compressive. Here it is assumed to be compressive.

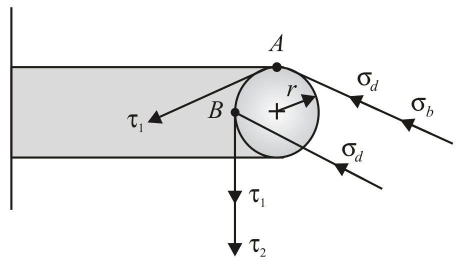
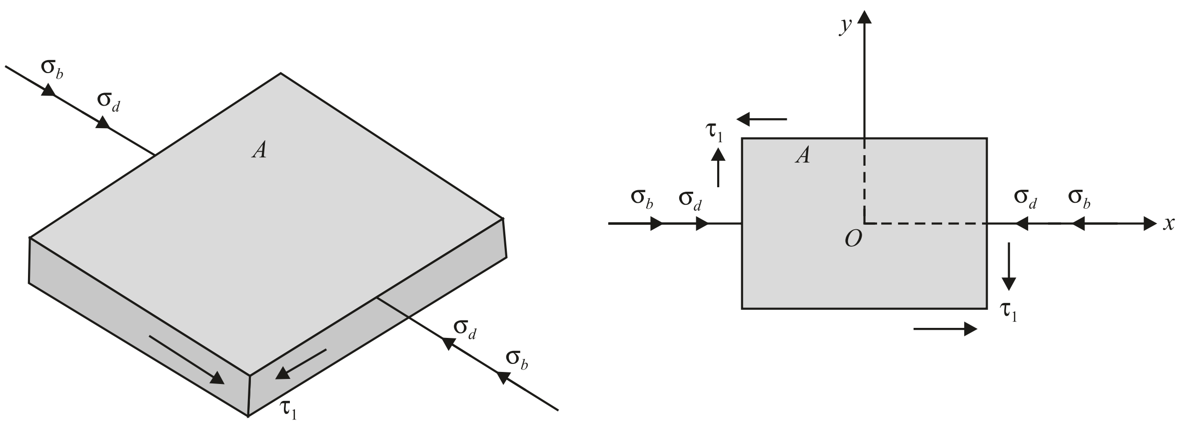


Fig. 14.14

Stress elements at points *A* and *B*

The stress element at *A* lies in the horizontal plane and is shown in Fig. 14.15 (*a*). The corresponding plane stress condition is shown in Fig. 14.15 (*b*). Here $\sigma_x = -(\sigma_d + \sigma_b)$, $\sigma_y = 0$ and $\tau_{xy} = -\tau_1$.

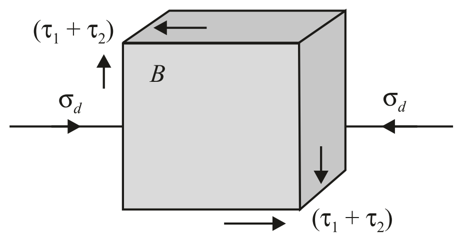


(*a*) Stress element at point *A* (in horizontal plane)

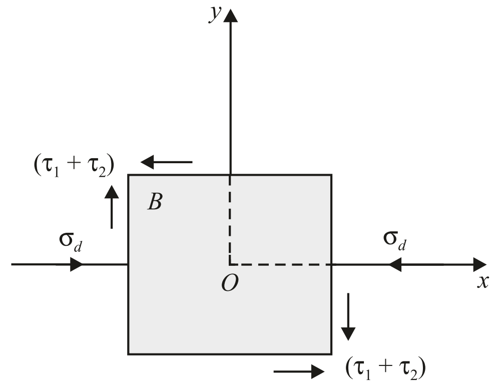
(*b*) Plane stress condition at point *A*

Fig. 14.15

The stress element at *B* lies in the vertical plane and is shown in Fig. 14.16 (*a*). The corresponding plane stress condition is shown in Fig. 14.16 (*b*). Here $\sigma_x = -\sigma_d$, $\sigma_y = 0$ and $\tau_{xy} = -(\tau_1 + \tau_2)$.



(*a*) Stress element at point *B* (in vertical plane)



(*b*) Plane stress condition at point *B*

Fig. 14.16

Critical points in the shaft

Point *A* which experiences higher stresses due to the combined effect of bending and thrust is the most critical point in the shaft. It means that point *A* or any other point just below point *A* on the bottom surface of the shaft can be critical points. Point *B* is not as critical as point *A* because it is subjected to only direct stress due to thrust, and has a smaller value compared to other stresses.

Principal stresses and maximum shear stress due to combined effect

At point *A*

The maximum and minimum normal stresses at point *A* are the maximum and minimum principal stresses respectively, given as

$$\sigma_1 = \text{Maximum principal stress}$$

$$= \frac{\sigma_x}{2} + \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

$$= \frac{-(\sigma_b + \sigma_d)}{2} + \frac{1}{2}\sqrt{\{-(\sigma_b + \sigma_d)\}^2 + 4\tau_1^2}$$

On substituting the values of σ_b, σ_d and τ_1, we have

$$\sigma_1 = \frac{-\left(\dfrac{32M}{\pi d^3} + \dfrac{4P'd}{\pi d^3}\right)}{2} + \frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3} + \frac{4P'd}{\pi d^3}\right)^2 + 4\left(\frac{16T}{\pi d^3}\right)^2}$$

where $$\sigma_d = \frac{4P'}{\pi d^2} = \frac{4P'd}{\pi d^3}$$

or $$\sigma_1 = -\frac{1}{2\pi d^3}(32M + 4P'd) + \frac{1}{2\pi d^3}\sqrt{(32M + 4P'd)^2 + 4(16T)^2}$$

$$= -\left[\frac{1}{\pi d^3}(16M + 2P'd)\right] + \frac{1}{\pi d^3}\sqrt{(8M + P'd)^2 + (16T)^2} \quad \text{... (14.15)}$$

and $\sigma_2 =$ Minimum principal stress

$$= -\left[\frac{1}{\pi d^3}(16M + 2P'd)\right] - \frac{1}{\pi d^3}\sqrt{(8M + P'd)^2 + (16T)^2} \quad \text{... (14.16)}$$

Now the maximum shear stress at point A is calculated as

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{1}{\pi d^3}\sqrt{(8M + P'd)^2 + (16T)^2} \quad \text{... (14.17)}$$

At point *B*

The maximum and minimum principal stresses at point B are given as

$$\sigma_1 = \text{ Maximum principal stress}$$

$$= \frac{\sigma_x}{2} + \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

$$= -\frac{\sigma_d}{2} + \frac{1}{2}\sqrt{(-\sigma_d)^2 + 4\{-(\tau_1 + \tau_2)\}^2}$$

On substituting the values of σ_d, τ_1 and τ_2, we have

$$\sigma_1 = -\frac{1}{2}\times\left(\frac{4P'd}{\pi d^3}\right) + \frac{1}{2}\sqrt{\left(\frac{4P'd}{\pi d^3}\right)^2 + 4\left(\frac{16T}{\pi d^3} + \frac{16Vd}{3\pi d^3}\right)^2}$$

$$\left(\text{as } \sigma_d = \frac{4P'}{\pi d^2} = \frac{4P'd}{\pi d^3} \text{ and } \tau_2 = \frac{16V}{3\pi d^2} = \frac{16Vd}{3\pi d^3}\right)$$

$$= -\frac{2P'd}{\pi d^3} + \frac{2}{\pi d^3}\sqrt{(P'd)^2 + 4\left(T + \frac{Vd}{3}\right)^2} \quad \text{...(14.18)}$$

and σ_2 = Minimum principal stress

$$= -\frac{2P'd}{\pi d^3} - \frac{2}{\pi d^3}\sqrt{(P'd)^2 + 4\left(T + \frac{Vd}{3}\right)^2} \quad \text{... (14.19)}$$

Now the maximum shear stress at point B is calculated as

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$= \frac{2}{\pi d^3}\sqrt{(P'd)^2 + 4\left(T + \frac{Vd}{3}\right)^2} \quad \text{... (14.20)}$$

Example 14.8

A propeller shaft of diameter 150 mm is subjected to a bending moment of 15 kN·m and an end thrust of 110 kN. It transmits 1500 kW of power at 200 rpm. Find the following parameters:

(*a*) the principal stresses and

(*b*) the maximum shear stress.

Solution: Given,

Diameter of the shaft d = 150 mm

Bending moment, M = 15 kN·m

End thrust, P' = 110 kN

Power being transmitted, P = 1500 kW

Revolutions per minute, N = 200

The power transmitted is $P = \frac{2\pi NT}{60 \times 1000}$

which gives $T = \frac{P \times 60 \times 1000}{2\pi N} = \frac{1500 \times 60 \times 1000}{2\pi \times 200} = 71619.72$ N·m

The shear stress due to torsion is given as

$$\tau = \frac{16T}{\pi d^3} = \frac{16 \times 71619.72}{\pi \times (150 \times 10^{-3})^3} \text{ N/m}^2 = 108 \text{ MPa}$$

The net direct stress due to the combined effect of bending moment and thrust is

$$\sigma_b - \sigma_d = \pm \frac{32M}{\pi d^3} - \frac{4P'}{\pi d^2}$$

$$= \pm \frac{32 \times 15 \times 10^3}{\pi \times (150 \times 10^{-3})^3} - \frac{4 \times 110 \times 10^3}{\pi \times (150 \times 10^{-3})^2} = \pm 45.27 \text{ MPa} - 6.22 \text{ MPa}$$

For tension side of the shaft

$$\sigma_{d'} = (+\,45.27 - 6.22) \text{ MPa} = 39.05 \text{ MPa}$$

The principal stresses are given as

$$\sigma_{1,2} = \frac{\sigma_{d'}}{2} \pm \frac{1}{2}\sqrt{\sigma_{d'}^2 + 4\tau^2} \qquad \text{... (1)}$$

$$= \frac{39.05}{2} \pm \frac{1}{2}\sqrt{(39.05)^2 + 4\times(108)^2}$$

$$= 19.52 \pm 109.75 = 129.27 \text{ MPa}, -90.23 \text{ MPa} \qquad \textbf{Ans.}$$

Hence, $\sigma_1 = 129.27$ MPa, $\sigma_2 = -90.23$ MPa

The maximum shear stress is

$$\tau_{\max_t} = \frac{\sigma_1 - \sigma_2}{2} = \frac{129.27 - (-90.23)}{2} = 109.75 \text{ MPa} \qquad \textbf{Ans.}$$

For compression side of the shaft

$$\sigma_{d'} = (-45.27 - 6.22) \text{ MPa} = -51.49 \text{ MPa}$$

The principal stresses are given as

$$\sigma_{1,2} = -\frac{51.49}{2} \pm \frac{1}{2}\sqrt{(-51.49)^2 + 4\times(108)^2} \qquad \text{(using equation (1))}$$

$$= -25.74 \pm 111.02 = 85.28 \text{ MPa}, -136.76 \text{ MPa} \qquad \textbf{Ans.}$$

Hence, $\sigma_1 = 85.28$ MPa, $\sigma_2 = -136.76$ MPa

The maximum shear stress is

$$\tau_{\max_c} = \frac{\sigma_1 - \sigma_2}{2} = \frac{85.28 - (-136.76)}{2} = 111.02 \text{ MPa} \qquad \textbf{Ans.}$$

Example 14.9

A solid shaft of diameter 180 mm transmits 2000 kW of power at 300 rpm and is subjected to a bending moment of 20 kN·m. Calculate the maximum permissible end thrust on the shaft, if maximum shear stress is limited to 75 MPa.

Solution: Given,

Diameter of the the shaft, $d = 180$ mm

Power being transmitted, $P = 2000$ kW

Revolutions per minute, $N = 300$

Bending moment, $M = 20$ kN·m

Maximum shear stress, $\tau_{\max} = 75 \text{ MPa} = 75 \times 10^6$ Pa

The power to be transmitted is given by

$$P = \frac{2\pi N\text{T}}{60 \times 1000}$$

or
$$T = \frac{P \times 60 \times 1000}{2\pi N}$$

$$= \frac{2000 \times 60 \times 1000}{2\pi \times 300} \text{ N·m} = 63662 \text{ N·m}$$

The shear stress is given as

$$\tau = \frac{16T}{\pi d^3} = \frac{16 \times 63662}{\pi \times (180 \times 10^{-3})^3} \text{ N/m}^2 = 55.6 \text{ MPa}$$

The maximum shear stress is given as

$$\tau_{max} = \frac{1}{2}\sqrt{\sigma_d^2 + 4\tau^2}$$

$$75 \times 10^6 = \frac{1}{2}\sqrt{\sigma_d^2 + 4 \times (55.6 \times 10^6)^2}$$

Solving for σ_d, we get

$$\sigma_d = 10^8 \text{ Pa} = 100 \text{ MPa}$$

The bending stress is

$$\sigma_b = \frac{32M}{\pi d^3} = \frac{32 \times 20 \times 10^3}{\pi \times (180 \times 10^{-3})^3} \text{ N/m}^2 = 35 \text{ MPa}$$

The stress due to end thrust is

$$\sigma_b - \sigma_d = (35 - 100) \text{ MPa}$$

$$= -65 \text{ MPa} = 65 \text{ MPa (Compressive)}$$

Hence, the end thrust is

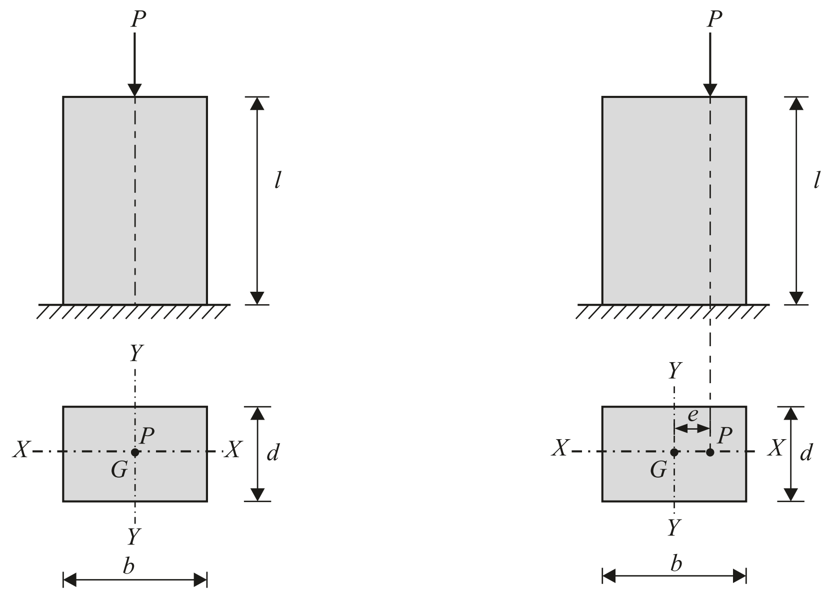
$$P' = 65 \times 10^6 \times \frac{\pi}{4} \times (180 \times 10^{-3})^2 = 1.65 \times 10^6 \text{ N}$$ **Ans**

14.6 OTHER CASES OF COMBINED LOADINGS

Some more general cases of combined loadings are considered here, in which members may be subjected to various other types of stresses, other than discussed before. The methods for finding the combined effect of all these stresses are similar. Examples under this category include eccentric loading, masonry dams, chimney and retaining walls.

14.6.1 Eccentric Loading on One Axis (Single Eccentricity)

Eccentric loading is a special case of combined bending and axial loads. For example, the eccentric load on a short column produces bending moment as well as direct compressive load. In Fig. 14.17 (*a*), a short column of cross-sectional area *A* and length *l* is subjected to an axial load *P*, which passes through the centroid *G* of the cross-section. The compressive normal stress produced in this case is equal to (*P*/*A*), which is distributed uniformly over the cross-section of the column.



(*a*) Axial load without eccentricity ($e = 0$) (*b*) Axial load with eccentricity '*e*'

Fig. 14.17

In many examples of engineering applications, the applied load does not pass through the centroid of the section. This case is shown in Fig. 14.17 (*b*), where the axial load P is applied at any point that lies on one of the centroidal axes of the cross-section, producing an eccentricity e from another centroidal axis. The direct stresses produced in this case are not distributed uniformly over the cross-section. At the same time, considerable bending effect is produced. In order to evaluate its effect, the eccentric load is replaced by an axial direct load P and a bending moment M about the axis, which equals to $(P \times e)$ as shown in Fig. 14.18. Hence, when a member is subjected

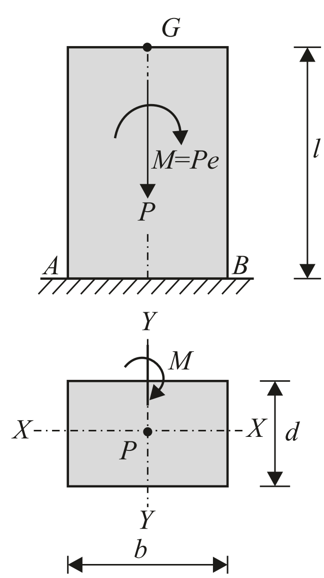


Fig. 14.18 Equivalence of eccentric loading.

to an eccentric load, it is equivalent to a member which is subjected to the combined loadings of axial load and bending. The axial load produces direct compressive stress σ_d which remains uniform throughout the section and equals to (P/A), where A is the cross-sectional area of the column. The bending moment M can be determined from the flexure formula as given below.

$$\frac{\sigma_b}{x} = \frac{M}{I_y}$$

which gives
$$\sigma_b = \frac{M}{\left(\frac{I_y}{x}\right)} = \frac{M}{S}$$

where
σ_b = Bending stress

S = Section modulus of the cross-section

I_y = Moment of inertia of the cross-section about y-axis

x = Distance from the neutral axis

Consider two points A and B at the base of the column as shown in Fig. 14.18. The bending moment M produces bending stresses, namely the maximum tensile stress at point A and the maximum compressive stress at point B. The normal stresses at points A and B can be determined by superimposing the direct and the bending stresses using the principle of superposition as given below.

The maximum normal stress at A is given as

$$\sigma_A = -\sigma_d + \sigma_b = -\frac{P}{A} + \frac{M}{S} \qquad \text{... (14.21)}$$

or
$$\sigma_A = -\frac{P}{A} + \frac{Pe}{S} \qquad \text{(substituting } M = Pe\text{)} \quad \text{... (14.22)}$$

The maximum normal stress at B is given as

$$\sigma_B = -\frac{P}{A} - \frac{M}{S} = -\frac{P}{A} - \frac{Pe}{S}$$

or
$$\sigma_B = -\left(\frac{P}{A} + \frac{Pe}{S}\right) \qquad \text{... (14.23)}$$

It is important to note that the normal stress at B is always compressive, while the normal stress at A may be compressive, tensile or zero depending upon the eccentricity of the load. Equation (14.23) gives the maximum compressive stress at B. If a material is required to resist only compression as in case of concrete, which is very weak in resisting tension, a condition of maximum eccentricity is obtained. This condition ensures that no tensile stress is developed anywhere in the members, and can be obtained by equating equation (14.22) to zero and solving for e.

Now let us consider a rectangular cross-section of the member having width b and depth d as shown in Fig. 14.19.

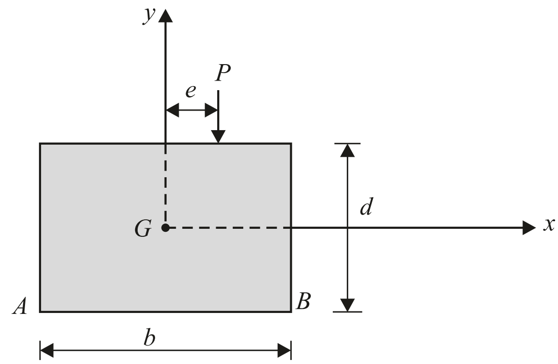


Fig. 14.19

The bending takes place about y-axis. The cross-sectional area A and the section modulus S are obtained as

$$A = bd$$

$$S = \frac{I_y}{x} = \frac{\frac{1}{12}db^3}{\left(\frac{b}{2}\right)} = \frac{db^2}{6}$$

Substituting the values of A and S in equations (14.22) and (14.23), we have

$$\sigma_A = -\frac{P}{bd} + \frac{Pe}{\left(\frac{db^2}{6}\right)} = \frac{P}{bd}\left(-1 + \frac{6e}{b}\right) \quad \text{... (14.24)}$$

and

$$\sigma_B = -\left[\frac{P}{bd} + \frac{Pe}{\left(\frac{db^2}{6}\right)}\right]$$

$$= -\frac{P}{bd}\left(1 + \frac{6e}{b}\right) \quad \text{... (14.25)}$$

For maximum eccentricity condition, put $\sigma_A = 0$ in equation (14.24). It gives

$$e = \frac{b}{6} \quad \text{... (14.26)}$$

This is the maximum eccentricity which ensures that the member is subjected to no tensile stress.

Example 14.10

A short piller of cross-section 500 mm × 400 mm shown in Fig. 14.20 is subjected to a compressive force of 75 kN with an eccentricity of 60 mm. Determine the maximum and minimum normal stresses induced in the cross-section.

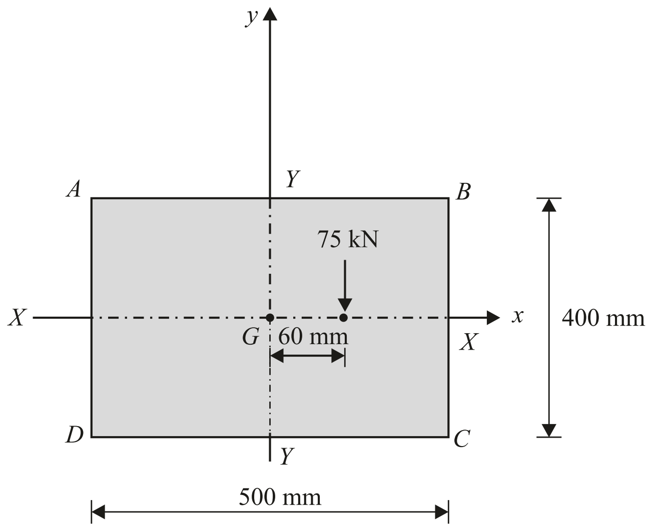


Fig. 14.20

Solution: Given,

Compressive load on the pillar, $P = 75$ kN

Eccentricity of the load, $e = 60$ mm

Width of the cross-section, $b = 500$ mm

Depth of the cross-section, $d = 400$ mm

The area A of the cross-section is found as

$$A = b \times d$$

$$= 500 \text{ mm} \times 400 \text{ mm} = 2 \times 10^5 \text{ mm}^2$$

The bending takes place about the y-axis, hence the moment of inertia I of the cross-section about the y-axis is given as

$$I_y = \frac{1}{12} \times 400 \times (500)^3 \text{ mm}^4 = 4.16 \times 10^9 \text{ mm}^4$$

Section modulus,
$$S = \frac{I_y}{\left(\frac{b}{2}\right)}$$

$$= \frac{4.16 \times 10^9}{\left(\frac{500}{2}\right)} = 1.67 \times 10^7 \text{ mm}^3$$

The bending moment M is given as

$$M = P \times e$$
$$= 75 \times 10^3 \times 60 \text{ N.mm}$$
$$= 4.5 \times 10^6 \text{ N.mm}$$

The direct stress due to compressive load P is

$$\sigma_d = -\frac{P}{A}$$
$$= -\frac{75\times10^3}{2\times10^5} \text{ N/mm}^2$$
$$= -0.375 \text{ N/mm}^2$$
$$= 0.375 \text{ N/mm}^2 \text{ (Compressive)}$$

The bending stress is

$$\sigma_b = \frac{M}{S} = \frac{4.5\times10^6}{1.67\times10^7} \text{ N/mm}^2$$
$$= 0.270 \text{ N/mm}^2$$

The bending stress is tensile on the face AD and compressive on the face BC.

The maximum stress on the face AD is given as

$$\sigma_{AD} = -\sigma_d + \sigma_b$$
$$= (-0.375 + 0.270) \text{ N/mm}^2$$
$$= -0.105 \text{ N/mm}^2$$ **Ans.**

The maximum stress on the face BC is given as

$$\sigma_{BC} = -\sigma_d - \sigma_b$$
$$= (-0.375 - 0.270) \text{ N/mm}^2$$
$$= -0.645 \text{ N/mm}^2$$ **Ans.**

Hence, both faces AD and BC are subjected to compressive stress.

14.6.2 Eccentric Loading on Two Axes (Double Eccentricity)

Single eccentricity is produced when the load is applied at any point on one of the centroidal axes of the section, but away from other centroidal axis. On the other hand, when the load is applied in such a manner that it does not lie on either of the centroidal axes, and is away from both axes, then it produces double eccentricity, that is, eccentricity is produced on both axes as shown in Fig. 14.21.

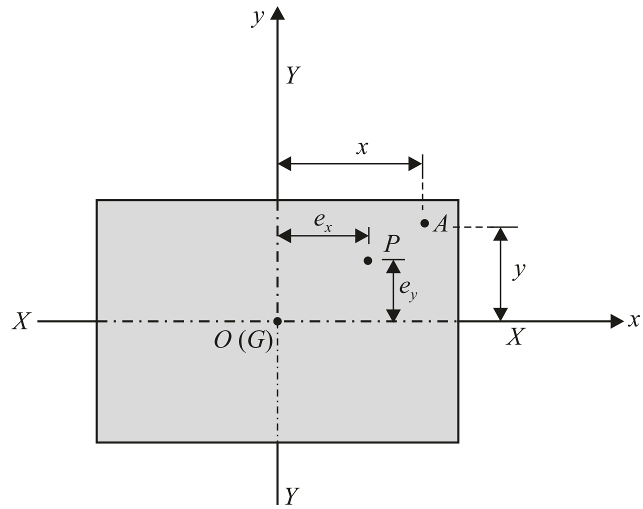


Fig. 14.21 Double eccentricity on a section.

In order to evaluate the effect of double eccentricity, the eccentric load P is replaced by an equivalent centroidal force and moments of the eccentric load about both centroidal axes. The equivalent centroidal force is P and its bending moments about the axes are M_x and M_y. Using the principle of superposition, the normal stresses due to axial load and bending moments are added algebraically to find the combined normal stress. For any point A (x, y) on the section, the equation of the combined normal stress σ is given as

$$\sigma = -\sigma_d \pm \sigma_b$$

$$= -\frac{P}{A} \pm \frac{M_x}{S_x} \pm \frac{M_y}{S_y} \qquad \text{... (14.27 (}a\text{))}$$

$$= -\frac{P}{A} \pm \frac{Pe_y}{\left(\frac{I_x}{y}\right)} \pm \frac{Pe_x}{\left(\frac{I_y}{x}\right)}$$

Hence,

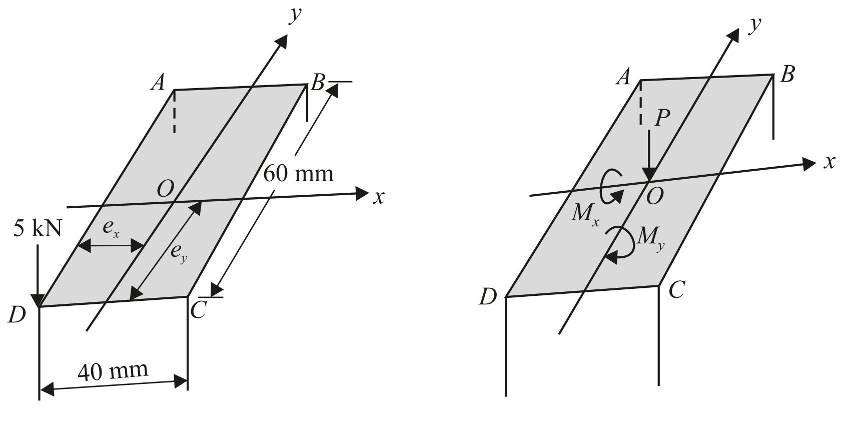
$$\sigma = -\frac{P}{A} \pm \frac{P\,y\,e_y}{I_x} \pm \frac{P\,x\,e_x}{I_y} \qquad \text{... (14.27 (}b\text{))}$$

where
σ_d = Direct stress due to axial load which is always compressive
σ_b = Bending stress along the two centroidal axes
A = Cross-sectional area of the section
M_x = Bending moment about the x-axis
= Pe_y
M_y = Bending moment about the y-axis
= Pe_x

S_x = Section modulus along the x-axis
S_y = Section modulus along the y-axis
I_x = Moment of interta of the section about the x-axis
I_y = Moment of interta of the section about the y-axis
e_x = Eccentricity on the x-axis
e_y = Eccentricity on the y-axis

Example 14.11

A load of 5 kN is acting at one of the corner points D of a rectangular section $ABCD$ as shown in Fig. 14.22 (a). Determine the normal stresses at all the corner points.



(a) Load with double eccentricity (b) Equivalent load P and moments M_x and M_y

Fig. 14.22

Solution: Given,

Eccentric load, $P = 5$ kN

Width of the cross-section, $b = 40$ mm $= 40 \times 10^{-3}$ m

Depth of the cross-section, $d = 60$ mm $= 60 \times 10^{-3}$ m

Eccentricity on x-axis, $e_x = \dfrac{40}{2} = 20$ mm $= 20 \times 10^{-3}$ m

Eccentricity on y-axis, $e_y = \dfrac{60}{2} = 30$ mm $= 30 \times 10^{-3}$ m

The eccentric load is converted into an axial compressive force P and bending moments M_x and M_y as shown in Fig. 14.22 (b).

The bending moments M_x and M_y are given as

$$M_x = P \times e_y$$
$$= 5 \times 30 \times 10^{-3} = 0.15 \text{ kN.m}$$
$$M_y = P \times e_x$$
$$= 5 \times 20 \times 10^{-3} = 0.1 \text{ kN.m}$$

The cross-sectional area A is given as

$$A = b \times d$$
$$= (40 \times 10^{-3}) \times (60 \times 10^{-3}) = 2.4 \times 10^{-3} \text{ m}^2$$

The moments of inertia of the cross-section about x and y axes are given as

$$I_x = \frac{1}{12} \times (40 \times 10^{-3}) \times \left(60 \times 10^{-3}\right)^3 = 7.2 \times 10^{-7} \text{ m}^4$$

$$I_y = \frac{1}{12} \times (60 \times 10^{-3}) \times (40 \times 10^{-3})^3 = 3.2 \times 10^{-7} \text{ m}^4$$

The section moduli of the cross-section about x and y axes are given as

$$S_x = \frac{I_x}{y}$$

$$= \frac{7.2 \times 10^{-7}}{30 \times 10^{-3}} = 2.4 \times 10^{-5} \text{ m}^3$$

$$S_y = \frac{I_y}{x}$$

$$= \frac{3.2 \times 10^{-7}}{20 \times 10^{-3}} = 1.6 \times 10^{-5} \text{ m}^3$$

Now the normal stresses at the corner points A, B, C and D of the cross-section are obtained using equation (14.27 (a)) as given below:

$$\sigma_A = -\frac{P}{A} + \frac{M_x}{S_x} - \frac{M_y}{S_y}$$

$$= -\frac{5}{2.4 \times 10^{-3}} + \frac{0.15}{2.4 \times 10^{-5}} - \frac{0.1}{1.6 \times 10^{-5}}$$

$$= (-2083.34 + 6250 - 6250) \text{ kN/m}^2$$
$$= -2083.34 \text{ kN/m}^2$$
$$= 2083.34 \text{ kN/m}^2 \text{ (Compressive)} \qquad \textbf{Ans.}$$

The direct compressive stress (P/A) is uniform across the cross-section.

$$\sigma_B = -\frac{P}{A} + \frac{M_x}{S_x} + \frac{M_y}{S_y}$$
$$= (-2083.34 + 6250 + 6250) \text{ kN/m}^2$$
$$= 10416.66 \text{ kN/m}^2 \text{ (Tensile)} \qquad \textbf{Ans.}$$

$$\sigma_C = -\frac{P}{A} - \frac{M_x}{S_x} + \frac{M_y}{S_y}$$
$$= (-2083.34 - 6250 + 6250) \text{ kN/m}^2 = -2083.34 \text{ kN/m}^2$$
$$= 2083.34 \text{ kN/m}^2 \text{ (Compressive)} \qquad \textbf{Ans.}$$

$$\sigma_D = -\frac{P}{A} - \frac{M_x}{S_x} - \frac{M_y}{I_y}$$
$$= (-2083.34 - 6250 - 6250) \text{ kN/m}^2 = -14583.34 \text{ kN/m}^2$$
$$= 14583.34 \text{ kN/m}^2 \text{ (Compressive)} \qquad \textbf{Ans.}$$

14.6.3 Biaxial Bending

The biaxial bending involves bending of a member along two axes. It occurs when the applied load is inclined at an angle with the vertical plane of symmetry (xy-plane) as shown in Fig. 14.23.

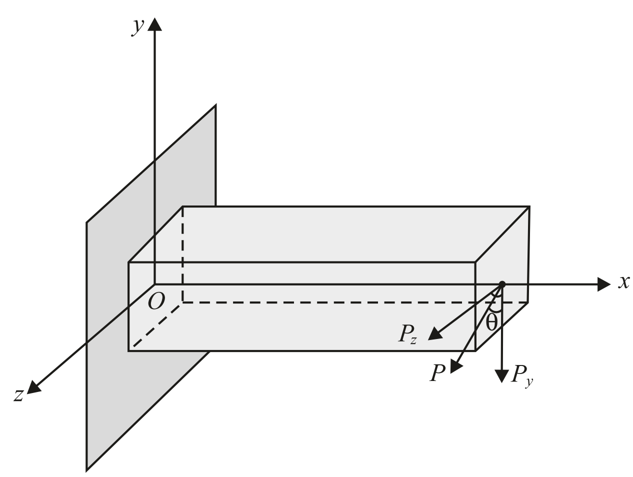


Fig. 14.23

A point load P is applied at the free end of the member at an angle θ with the y-axis of symmetry. To find the effect of the load, it is resolved into two components P_y and P_z along y and z axes respectively, which also happen to be the axes of symmetry of the section of the member. Both P_y and P_z contribute in the bending of the member by producing bending moments about the two axes. The load P_y bends the member about the horizontal axis (x-axis) and the load P_z bends the member about the vertical axis (y-axis). The bending stresses are produced by the bending moment components along the longitudinal direction. Using the principle of superposition, these stresses are added algebraically to find the combined bending stress. It is illustrated in the following example.

Example 14.12

A 4 m long simple beam having cross-section 200 mm × 250 mm carries a central point load of 15 kN and is supported at the ends in the tilted position as shown in Fig. 14.24. Determine the maximum bending stresses produced in the beam.

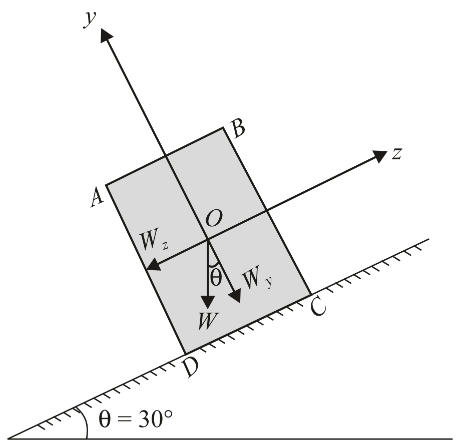


Fig. 14.24

Solution: Given,

Point load, $W = 15$ kN

Angle of tilt of the beam, $\theta = 30°$

Width of the beam, $b = 200$ mm

Depth of the beam, $d = 250$ mm

Calculation of Moments of Inertia

The moment of inertia of the cross-section of the beam about the z-axis is found as

$$I_z = \frac{1}{12} \times bd^3$$

$$= \frac{1}{12} \times 200 \times (250)^3$$

$$= 2.6 \times 10^8 \text{ mm}^4$$

$$= 2.6 \times 10^{8} \times 10^{-12} \text{ m}^4$$
$$= 2.6 \times 10^{-4} \text{ m}^4$$

The moment of inertia of the cross-section about the y-axis is given as

$$I_y = \frac{1}{12} \times db^3$$
$$= \frac{1}{12} \times 250 \times (200)^3$$
$$= 1.67 \times 10^{8} \text{ mm}^4$$
$$= 1.67 \times 10^{8} \times 10^{-12} \text{ m}^4$$
$$= 1.67 \times 10^{-4} \text{ m}^4$$

Calculation of section modulus

The section modulus of the cross-section along z and y axes are given as

$$S_z = \frac{I_z}{y}$$
$$= \frac{2.6 \times 10^{-4}}{\left(\frac{250 \times 10^{-3}}{2}\right)} = 2.08 \times 10^{-3} \text{ m}^3$$

$$S_y = \frac{I_y}{z}$$
$$= \frac{1.67 \times 10^{-4}}{\left(\frac{200 \times 10^{-3}}{2}\right)} = 1.67 \times 10^{-3} \text{ m}^3$$

Load and bending moment components

The point load W is resolved into components W_y and W_z along y and z axes respectively as

$$W_y = \text{W} \cos 30°$$
$$= 15 \times \cos 30°$$
$$= 13 \text{ kN}$$
$$W_z = W \sin 30°$$
$$= 15 \times \sin 30°$$
$$= 7.5 \text{ kN}$$

Since the maximum bending moment due to point load W is $\frac{Wl}{4}$, which occurs at the midspan of the beam, hence the bending moment components M_y and M_z about the axes are given as

$$M_y = \frac{W_z l}{4}$$

$$= \frac{7.5 \times 4}{4}$$

$$= 7.5 \text{ kN.m}$$

M_y produces tension on the face *AD* and compression on the face *BC* of the beam.

$$M_z = \frac{W_y l}{4} = \frac{13 \times 4}{4}$$

$$= 13 \text{ kN.m}$$

M_z produces tension in the layer *CD* and compression in the layer *AB* of the beam.

Calculation of bending stresses

The maximum bending stress due to bending about the *z*-axis is given as

$$\sigma_1 = \frac{M_z}{S_z}$$

$$= \frac{13 \times 10^3}{2.08 \times 10^{-3}} \times \frac{1}{10^6} \text{ MPa}$$

$$= 6.25 \text{ MPa}$$

The maximum bending stress due to bending about the *y*-axis is given as

$$\sigma_2 = \frac{M_y}{I_y}$$

$$= \frac{7.5 \times 10^3}{1.67 \times 10^{-3}} \times \frac{1}{10^6} \text{ MPa}$$

$$= 4.5 \text{ MPa}$$

Now the bending stresses at the corner points *A*, *B*, *C* and *D* of the cross-section of the beam at the midspan are found using the principle of superposition as

$$\sigma_A = -\sigma_1 + \sigma_2$$

$$= (-6.25 + 4.5) \text{ MPa}$$

$$= -1.75 \text{ MPa}$$

$$= 1.75 \text{ MPa (Compressive)}$$

$$\sigma_B = -\sigma_1 - \sigma_2$$

$$= (-6.25 - 4.5) \text{ MPa}$$

$$= -10.75 \text{ MPa}$$

$$= 10.75 \text{ MPa (Compressive)}$$

$$\sigma_C = +\sigma_1 - \sigma_2$$
$$= (+6.25 - 4.5) \text{ MPa}$$
$$= +1.75 \text{ MPa (Tensile)}$$
$$\sigma_D = +\sigma_1 + \sigma_2$$
$$= (+6.25 + 4.5) \text{ MPa}$$
$$= +10.75 \text{ MPa (Tensile)}$$

Hence,

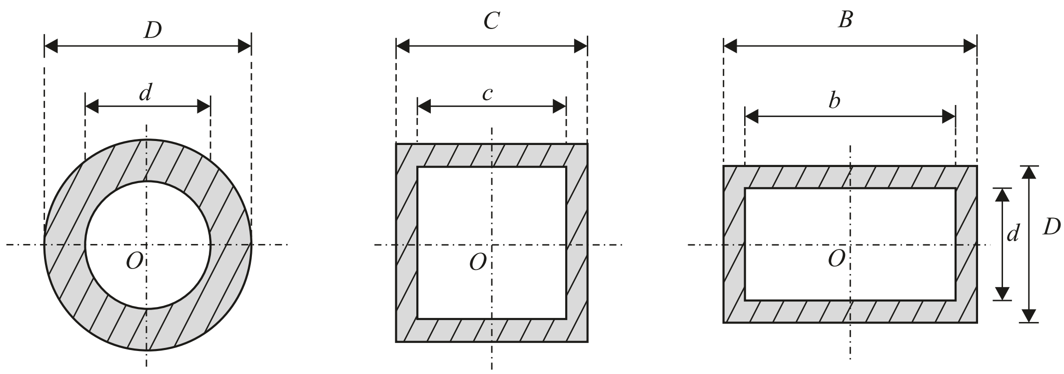
Maximum tensile bending stress = σ_D = 10.75 MPa **Ans.**

Maximum compressive bending stress = σ_B = 10.75 MPa **Ans.**

14.6.4 Loading on a Chimney

A masonry chimney is usually a vertical cylindrical structure, which is used to eject flue gases or smoke to the atmosphere. It has a hollow circular, square or rectangular cross-section. The chimney forms a case of combined bending and axial loads. It is subjected to horizontal wind pressure which causes bending and its self-weight produces direct stress. The normal stresses due to bending and direct loads are superimposed by using the principle of superposition taking due accounts of positive or negative signs of the stresses to find their combined effect.

The common cross-sections of a chimney are shown in Fig. 14.25.



(*a*) A circular section (*b*) A square section (*c*) A rectangular section

Fig. 14.25

Let us consider a cylindrical chimney of height H, outer diameter D and inner diameter d, being subjected to a wind pressure p as shown in Fig. 14.26. The wind pressure acts horizontally on the chimney and its intensity depends on the shape of the exposed projected area and height of the chimney. It tends to bend the chimney, and the maximum bending moment is produced at the base of the chimney, which forms the most critical point.

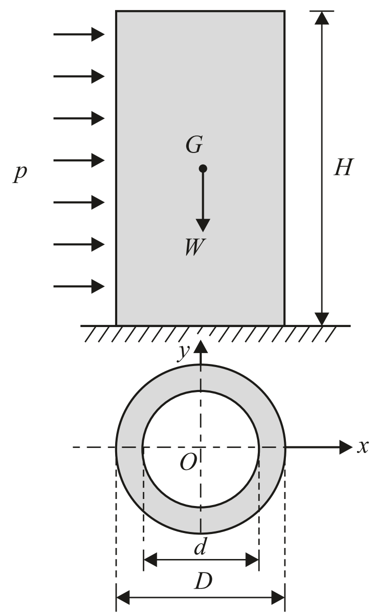


Fig. 14.26 A cylindrical chimney.

The force P produced by the wind pressure is found as

$$P = p \times A_p \times C \quad ...(14.28)$$

where p = Wind pressure = $K.y^n$

K = A constant

y = Height component

n = A positive exponent

A_p = Projected area

C = Coefficient of wind pressure or shape factor that varies between 0.5 and 1.44.

= 0.66 for circular section

= 1.0 for square or rectangular section

The wind force is assumed to act at one-half of the height of the chimney.

The maximum bending moment due to the wind force P occurs at the base of the chimney (similar to a cantilever beam loaded with a point load at the free end), given as

$$M = P \times \frac{H}{2} \quad ...(14.29)$$

The maximum bending stress, using flexure formula, can be expressed as

$$\sigma_b = \frac{M \times \left(\frac{D}{2}\right)}{I_y}$$

where I_y = Moment of inertia of the cross-section of the chimney about the centroidal axis (the y-axis)

$$= \frac{\pi}{64}(D^4 - d^4)$$

On substituting M and I_y in the equation of bending stress, we get

$$\sigma_b = \frac{16PHD}{\pi(D^4 - d^4)} \quad \text{...(14.30)}$$

Effect of self-weight

The self-weight W of the chimney acts through its centre of gravity (G). It produces direct stress, which is uniformly distributed across the cross-section of the chimney, and is always compressive, given as

$$\sigma_d = \frac{W}{A} = \frac{4W}{\pi(D^2 - d^2)} \quad \text{...(14.31)}$$

where σ_d = Direct compressive stress

A = Cross-sectional area of the chimney

$$= \frac{\pi}{4}(D^2 - d^2)$$

The self-weight of the chimney can be found as

$$W = \rho \times \frac{\pi}{4}(D^2 - d^2) \times H \times g \quad \text{...(14.32)}$$

where ρ = Density of the masonry material

g = Acceleration due to gravity = 9.8 m/s^2

W is usually expressed as unit weight (specific weight, γ) in kN/m^3 or N/m^3.

Example 14.13

A 25 m high masonry cylindrical chimney having outside diameter 2.5 m and inside diameter 1.25 m is subjected to a horizontal wind pressure that varies as $y^{2/3}$, where y is the height above the ground. If the unit weight of masonry is 22.4 kN/m^3, the coefficient of wind pressure is 0.6 and the pressure at a height of 30 m is 1.5 kN/m^2, determine the maximum and minimum stresses induced at the base of the chimney.

Solution: Given,

Height of the chimney, H = 25 m

Outside diameter of the chimney, D = 2.5 m

Inside diameter of the chimney, d = 1.25 m

Shape factor, C = 0.6

Unit weight of masonay γ_m = 22.4 kN/m^3

The cross-sectional area A of the chimney is found as

$$A = \frac{\pi}{4}(D^2 - d^2)$$

$$= \frac{\pi}{4}\left[(2.5)^2 - (1.25)^2\right] = 3.68 \text{ m}^2$$

The moment of inertia of the cross-section of the chimney about the centroidal axis is determined as

$$I_x = I_y = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}\left[(2.5)^4 - (1.25)^4\right] = 1.8 \text{ m}^4$$

Calculation of direct stress

The weight of the chimney W is given as

$$W = \text{Unit weight of masonry} \times \text{Volume of the chimney}$$

$$= \gamma_m \times A \times H$$

$$= 22.4 \times 3.68 \times 25 \text{ kN} = 2060.8 \text{ kN}$$

The direct stress σ_d due to the weight of the chimney is compressive, given as

$$\sigma_d = -\frac{W}{A} = -\frac{2060.8}{3.68}$$

$$= -560 \text{ kPa}$$

$$= 560 \text{ kPa (Compressive)}$$

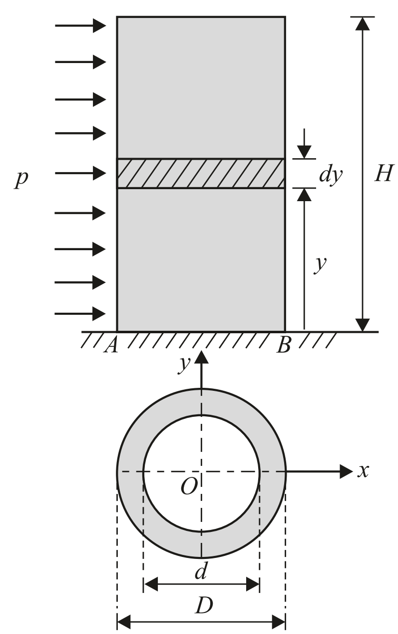


Fig. 14.27

Calculation of bending stress

The wind pressure can be expressed as

$$p = K.y^{2/3}$$

$$1.5 = K.(30)^{2/3}$$

which gives

$$K = 0.155$$

Hence, $$p = 0.155\, y^{2/3}$$

Now consider a chimney element of thickness dy at a height y from the base of the chimney as shown in Fig. 14.27. Two points A and B are considered at the base of the chimney.

The force exerted on the element by the wind pressure p can be expressed as

$$dP = p \times A_p \times C$$

$$= 0.155\, y^{2/3} \times 2.5 \times dy \times 0.6 \qquad (\text{as } A_p = D \times dy)$$

$$= 0.2325\, y^{2/3}\, dy$$

The moment of the force on the element about the base of the chimney is

$$dM = 0.2325\, y^{2/3}\, dy \times y$$
$$= 0.2325\, y^{5/3}\, dy$$

Hence, the total moment of the wind pressure about the base is given as

$$M = \int_0^{25} dM = \int_0^{25} 0.2325 y^{5/3} dy = 0.2325 \int_0^{25} y^{5/3} dy$$

$$= 0.2325 \left[\frac{y^{5/3+1}}{\left(\frac{5}{3}+1\right)} \right]_0^{25} = 0.2325 \left(\frac{y^{8/3}}{\frac{8}{3}} \right)_0^{25} = \frac{0.2325 \times 3}{8} \times 25^{8/3}$$

$$= 465.9 \text{ kN.m}$$

Now the bending stress σ_b is determined as

$$\sigma_b = \frac{M}{I_y} \times \left(\frac{D}{2} \right) = \frac{465.9}{1.8} \times \left(\frac{2.5}{2} \right) = 323.54 \text{ kPa}$$

The tensile bending stress is produced at A and compressive bending stress at B.

Hence,

The maximum normal stress at B is

$$\sigma_{\max} = \sigma_B = -\sigma_d - \sigma_b$$
$$= (-560 - 323.\ 54) \text{ kPa} = -883.54 \text{ kPa}$$
$$= 883.54 \text{ kPa (Compressive)}$$ **Ans.**

The minimum normal stress at A is

$$\sigma_{\min} = \sigma_A = -\sigma_d + \sigma_b = (-560 + 323.54) \text{ kPa}$$
$$= -236.46 \text{ kPa} = 236.46 \text{ kPa (Compressive)}$$ **Ans.**

14.6.5 Loading on a Dam

A dam is a vertical engineering structure, which is constructed across a river to retain water for the purpose of irrigation or electricity generation. It is usually of rectangular or trapezoidal section, and is made of earth, masonry or cement concrete.

Loading on a dam is another case of combined bending and axial loads. It is subjected to water pressure and its self-weight.

Let us consider a trapezoidal dam of unit length having cross-section *ABCD* as shown in Fig. 14.28 with the following details.

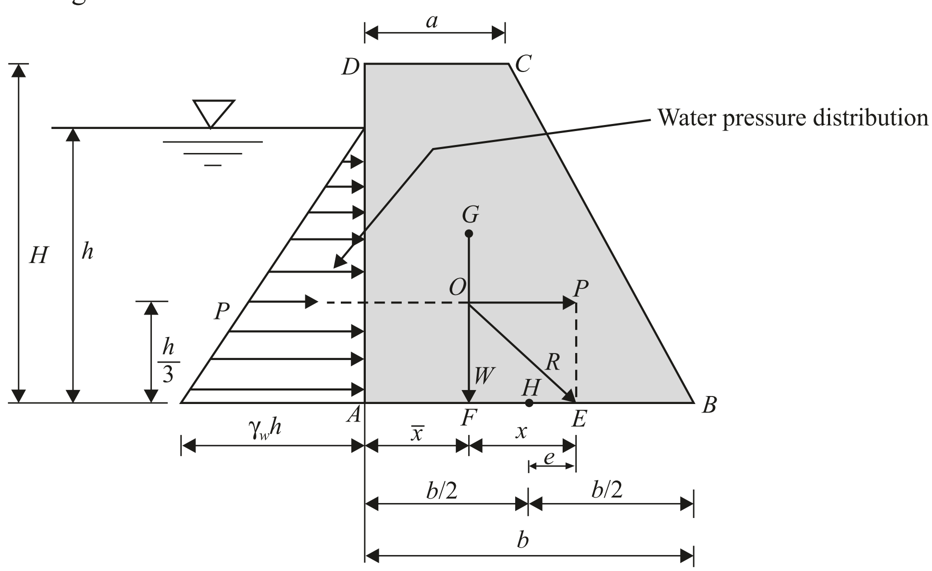


Fig. 14.28 A trapezoidal dam subjected to water pressure of height *h*.

Let

H = Height of the dam

l = Length of the dam = 1 m (assumed)

a = Top width of the dam

b = Base width of the dam

h = Height or depth of water retained by the dam

γ_w = Specific weight or unit weight of water

= $\rho_w g$ = 9800 N/m^3

ρ_w = Density of water = 1000 kg/m^3

g = Acceleration due to gravity = 9.8 m/s^2

γ_m = Specific weight or unit weight of the dam material

= $\rho_m g$

ρ_m = Density of the dam material

P = Water pressure

Weight consideration

The weight of the dam acts in the downward direction through its centre of gravity *G*. The distance of the C.G. from the vertical face *AD* of the dam is $\bar{x}$, given as

$$\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)} \qquad \text{(see Table 3.1)...(14.33)}$$

The weight per metre length of the dam is given as

$$W = \text{Specific weight of the dam material} \times \text{Volume of the dam}$$

$$= \gamma_m \times \frac{1}{2} \times (a+b) \times H \times 1 \qquad (l = 1 \text{ m})$$

$$= \frac{\gamma_m (a+b) H}{2} \qquad \text{... (14.34)}$$

The weight of the dam produces direct stress σ_d at the base, and is given as

$$\sigma_d = -\frac{W}{b \times l} = -\frac{W}{b \times 1} = -\frac{W}{b} \qquad \text{... (14.35)}$$

The stress is uniformly distributed across the base, and is always compressive.

Pressure consideration

Water exerts horizontal pressure on the dam. The pressure is zero at the free surface of water and increases linearly with increase in depth to become maximum at the base of the dam. The total water pressure P acts in the horizontal direction at one-third height of water (h/3) from the base of the dam, because of the triangular pressure distribution, and is given as

$$P = \text{Area of the triangular pressure distribution}$$

$$= \frac{1}{2} \times \gamma_w h \times h = \frac{\gamma_w h^2}{2} \qquad \text{...(14.36)}$$

In the above equation, the base of the pressure triangle is the water pressure at the base, which is $\rho_w g h = \gamma_w h$. P causes sliding of the dam. To balance it, the maximum frictional resistance μW is setup at the base of the dam, where μ is the coefficient of friction between the dam material and earth. Similarly, P also produces an overturning moment, which can overturn the dam about the point B, called toe. The weight of the dam W provides a restoring moment about B.

Combined effect

The weight W, which constitutes the stabilizing moment, can be transferred to act at points O, using the principle of transmissibility so that both forces W and P are acting at O, and their resultant force R cuts the base of the dam at point E.

Let $EF = x$ = Distance between the line of action of W and E

Using $\Sigma M_E = 0$, we have

$$W \times x = P \times \frac{h}{3}$$

which gives

$$x = \frac{Ph}{3W} \qquad \text{... (14.37)}$$

The vertical component of R, that is, the weight W, while acting at E forms a case of eccentric loading on the base along with producing a direct compressive stress σ_d at the base of the dam as given by equation (14.35). The eccentric loading produces bending moment, which in turn, produces bending stress at the base.

The bending moment M about the centre of the base, that is, point H is given as

$$M = We \qquad ...(14.38)$$

The bending stress σ_b produced on the base on 1 m length is given as

$$\sigma_b = \frac{M}{I}\cdot y \qquad \text{(using bending formula)}$$

$$= \frac{We \times \frac{b}{2}}{\frac{1}{12}\times 1 \times b^3}$$

$$= \frac{6We}{b^2} \qquad ...(14.39)$$

The bending stress is tensile at A and compressive at B.

Hence,

The maximum normal stress produced at B is

$$\sigma_{max} = -\sigma_d - \sigma_b \qquad ...(14.40)$$

$$= -\frac{W}{b} - \frac{6We}{b^2}$$

$$= -\frac{W}{b}\left(1 + \frac{6e}{b}\right) \qquad ...(14.41)$$

The minimum normal stress produced at A is

$$\sigma_{min} = -\sigma_d + \sigma_b \qquad ...(14.42)$$

$$= -\frac{W}{b} + \frac{6We}{b^2} = -\frac{W}{b}\left(1 - \frac{6e}{b}\right) \qquad ...(14.43)$$

The normal stress at B is always compressive, whereas the normal stress at A may be tensile or compressive both. The development of tensile normal stress at A is not desirable as the dam material cannot resist tension. At the same time, the maximum normal stress at B must be less than the permissible value of the compressive stress of the dam material, otherwise the dam will be crushed.

Eccentricity

The eccentricity e is expressed as

$$e = (\bar{x} + x) - \left(\frac{b}{2}\right) \qquad ... (14.44)$$

In case $(\bar{x} + x)$ is greater than $\left(\frac{b}{2}\right)$, the eccentricity is given by equation (14.44), and when $(\bar{x} + x)$ is less than $\left(\frac{b}{2}\right)$, then the eccentricity can be given as

$$e = \left(\frac{b}{2}\right) - (\bar{x} + x) \qquad ...(14.45)$$

For no tension to develop at point A on the base of the dam, equation (14.43) is equated to zero, which gives

$$e = \frac{b}{6} \qquad \text{...(14.46)}$$

Hence, for the dam to be safer in tension, $e \le \frac{b}{6}$.

Safety check for the dam

A dam is tested for four parameters, namely the maximum compressive stress at base, no tension condition at base, sliding and overturning. The following conditions should be fulfilled:

- The maximum compressive stress σ_{max} developed at the base of the dam, as given by equation (14.40), must be less than the permissible compressive stress of the dam material to prevent crushing of the dam.
- Eccentricity $e \le \frac{b}{6}$ to ensure that no tension is developed at the base of the dam.
- Factor of safety (n) against sliding = $\frac{\mu W}{P} \ge 1.5$, where μ is the coefficient of friction between dam material and earth.
- Factor of safety (n) against overturning = $\frac{\text{Resisting moment}}{\text{Overturning moment}} = \frac{W(b-\bar{x})}{P \times \left(\frac{h}{3}\right)} \ge 2$

Example 14.14

A masonry dam of trapezoidal section has a top width of 2.5 m, base width of 5 m and a height of 12 m. It retains water to a depth of 10 m. Determine the maximum and the minimum stress induced at the base. The specific weight of masonry is 20 kN/m^3 and the specific weight of water is 10 kN/m^3.

Solution: Refer Fig. 14.29.

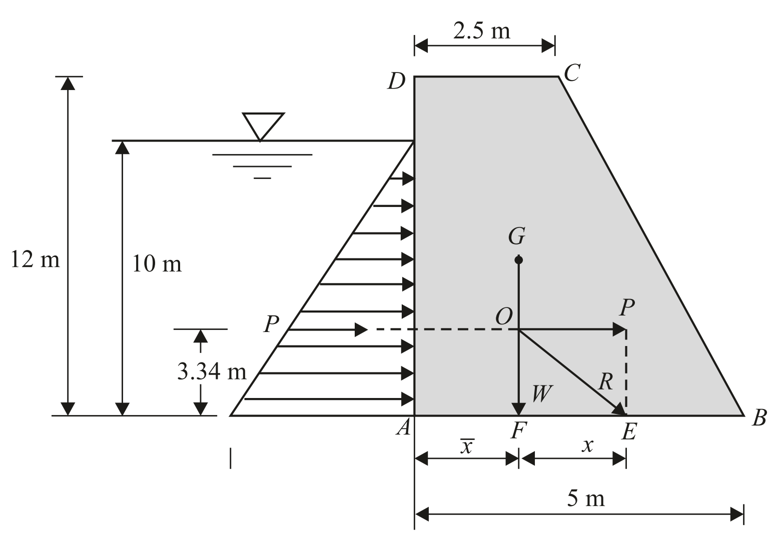


Fig. 14.29

Given,

Height of the dam, $H = 12$ m

Top width of the dam, $a = 2.5$ m

Base width of the dam, $b = 5$ m

Height of water retained, $h = 10$ m

Specific weight of water, $\gamma_w = 10 \text{ kN/m}^3 = 10 \times 10^3 \text{ N/m}^3$

Specific weight of masonry, $\gamma_m = 20 \text{ kN/m}^3 = 20 \times 10^3 \text{ N/m}^3$

Let us consider 1 m length of the dam.

The cross-sectional area of the dam is

$$A = \frac{1}{2} \times (a+b) \times H$$

$$= \frac{1}{2} \times (2.5+5) \times 12 = 45 \text{ m}^2$$

The moment of inertia of the base section per metre length of the dam is given as

$$I = \frac{1}{12} \times 1 \times b^3 = \frac{1}{12} \times 1 \times 5^3 = 10.42 \text{ m}^4$$

The weight per metre length of the dam is

$$W = \frac{\gamma_m (a+b) H}{2} = \frac{20 \times (2.5+5) \times 12}{2} = 900 \text{ kN}$$

The distance $\bar{x}$ at which the weight W acts, is given as

$$\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)} \qquad \text{(using equation (14.33))}$$

$$= \frac{(2.5)^2 + (2.5 \times 5) + (5)^2}{3(2.5+5)} = 1.944 \text{ m}$$

The horizontal water pressure P per metre length is obtained using equation (14.36) as

$$P = \frac{\gamma_w h^2}{2} = \frac{10 \times (10)^2}{2} = 500 \text{ kN}$$

P acts at $\frac{h}{3} = \frac{10}{3} = 3.34$ m from the base.

Using $\Sigma M_E = 0$, we have

$$W \times x = P \times 3.34$$

$$x = \frac{P \times 3.34}{W} = \frac{500 \times 3.34}{900} = 1.855 \text{ m}$$

Now $(\bar{x} + x) = 1.944 + 1.855 = 3.8$ m

and $\dfrac{b}{2} = \dfrac{5}{2} = 2.5$ m

The eccentricity e is given as

$$e = (\bar{x} + x) - \frac{b}{2} = 3.8 - 2.5 \text{ m} = 1.3 \text{ m}$$

The bending moment M is given as

$$M = We \qquad \text{(using equation (14.38))}$$

$$= 900 \times 1.3 = 1170 \text{ kN.m}$$

The bending stress is

$$\sigma_b = \frac{6We}{b^2} \qquad \text{(using equation (14.39))}$$

$$= \frac{6 \times 900 \times 1.3}{5^2} = 280.8 \text{ kN/m}^2 = 280.8 \text{ kPa}$$

The direct stress due to weight W is

$$\sigma_d = -\frac{W}{b} \qquad \text{(using equation (14.35))}$$

$$= -\frac{900}{5} = -180 \text{ kN/m}^2 = 180 \text{ kPa (Compressive)}$$

Hence,

The maximum normal stress at B is

$$\sigma_{max} = -(\sigma_d + \sigma_b) \qquad \text{(using equation (14.40))}$$

$$= -(180 + 280.8) \text{ kPa} = -460.8 \text{ kPa}$$

$$= 460.8 \text{ kPa (Compressive)}$$ **Ans.**

The minimum normal stress at A is

$$\sigma_{min} = -\sigma_d + \sigma_b \qquad \text{(using equation (14.42))}$$

$$= (-180 + 280.8) \text{ kPa}$$

$$= 100.8 \text{ kPa (Tensile)}$$ **Ans.**

14.6.6 Loading on Retaining Walls

Retaining walls are the engineering structures which are used to retain earth, soil, sand etc. and sustain their lateral pressures. Basement walls and dams are the examples of retaining walls. Loading on retaining walls is a case of combined bending and axial loads. The walls are subjected to active soil pressure that acts horizontally. The design analysis of the retaining walls involves ensuring their stability against sliding and overturning (rotation), and finding the maximum soil pressure developed at their base. The trapezoidal section is the most commonly used cross-section for the retaining walls. A trapezoidal retaining wall is shown in Fig. 14.30 showing all the relevant parameters.

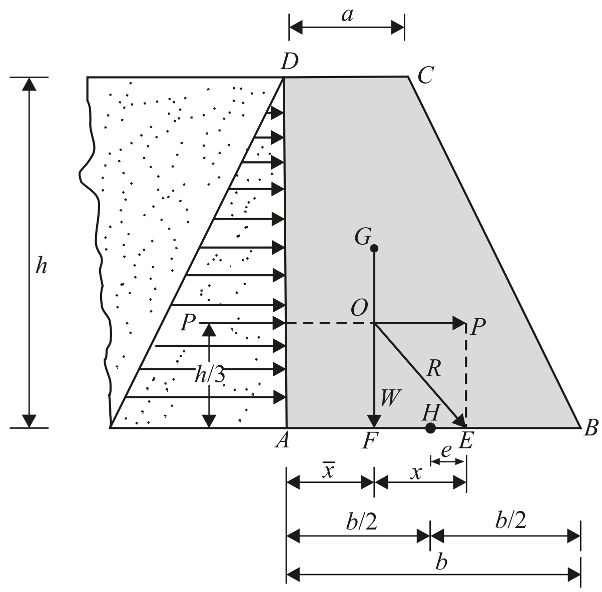


Fig. 14.30 A trapezoidal retaining wall.

The stress analysis of a retaining wall is similar to that of a dam. We have seen in case of a dam that water retained exerts horizontal pressure on the dam, but here it is soil which the wall retains, and exerts lateral active pressure on the vertical face of the retaining wall in the horizontal direction. It is equivalent to a single horizontal force that acts as a resultant, and is called lateral thrust.

Considering 1 m length of the retaining wall, the total lateral thrust P, according to Rankine's theory, is given as

$$P = \frac{\gamma_s h^2}{2} \cdot \frac{1-\sin\phi}{1+\sin\phi} \quad \text{...(14.47)}$$

where h = Height of the soil retained

= Height of the retaining wall (H)

γ_s = Specific weight or unit weight of the soil retained

ϕ = Angle of repose of soil

The soil pressure distribution diagram is triangular, hence the thrust P acts at one-third height of the pressure diagram, and has the tendency to rotate the wall about the point B, called toe. Resistance to this rotation is offered by the weight of the wall itself and by the weight of the soil above the base.

The distances $\bar{x}$ and x are found by using equations (14.33) and (14.37) respectively. Equations (14.35) and (14.39) are used to determine direct and bending stresses respectively. The maximum and minimum pressure intensities at the base of the wall are determined by using equations (14.40) or (14.41) and equations (14.42) or (14.43) respectively.

Example 14.15

A masonry retaining wall of trapezoidal section is 9 m high, 2 m wide at top and 4 m wide at the bottom. The vertical face of the wall retains soil of unit weight of 12 kN/m^3 to a full height. Determine the maximum and minimum pressure intensities at the base of the wall. The unit weight of masonry is 20 kN/m^3 and the angle of repose of soil is 30°.

Solution : Refer Fig. 14.31.

Given,

Height of the retaining wall,	H	$= 9$ m
Height of the soil retained,	h	$= H = 9$ m
Top width of the retaining wall,	a	$= 2$ m
Base width of the retaining wall,	b	$= 4$ m,
Unit weight of masnory,	γ_m	$= 20$ kN/m^3
Unit weight of soil,	γ_s	$= 12$ kN/m^3
Angle of repose of soil,	ϕ	$= 30°$

Let us consider 1 m length of the retaining wall. The weight of the retaining wall is given as

$$W = \gamma_m \times \frac{1}{2} \times (a+b) \times H \times 1$$

$$= 20 \times \frac{1}{2} \times (2+4) \times 9 = 540 \text{ kN}$$

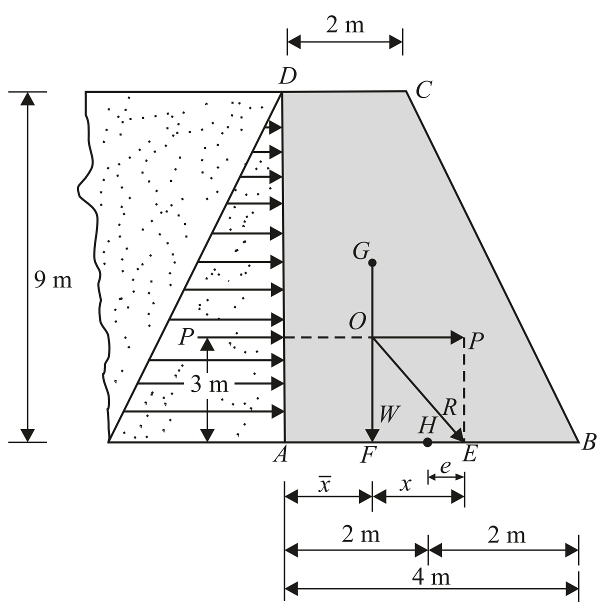


Fig. 14.31

The weight of the retaining wall acts at its centre of gravity G. The distance of the $C.G.$ from the vertical face AD of the wall is $\bar{x}$, given as

$$\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)} = \frac{2^2 + 2\times 4 + 4^2}{3(2+4)} = 1.56 \text{ m}$$

The lateral pressure of the soil exerted on the vertical face AD is

$$P = \frac{\gamma_s H^2}{2} \cdot \frac{1-\sin\phi}{1+\sin\phi}$$

$$= \frac{12\times 9^2}{2} \times \frac{1-\sin 30°}{1+\sin 30°} = 162 \text{ kN}$$

The lateral pressure P and weight W produce a resultant force R, which cuts the base at point E as shown in Fig. 14.31.

Using $\Sigma M_E = 0$, we have

$$W \times x = P \times \frac{H}{3}$$

or

$$x = \frac{P \times H}{3W}$$

$$= \frac{162\times 9}{3\times 540} = 0.9 \text{ m}$$

H is the midpoint of the base, that is, $AH = BH = \dfrac{b}{2} = \dfrac{4}{2} = 2$ m

Now $(\bar{x} + x) = (1.56 + 0.9)$ m $= 2.46$ m

The eccentricity e is given as

$$e = (\bar{x} + x) - \frac{b}{2}$$

$$= (2.46 - 2) \text{ m} = 0.46 \text{ m}$$

Point A on the base of the retaining wall experiences tension because of eccentric loading by the weight of the wall and point B experiences compression.

Hence,

The maximum pressure intensity produced at B is

$$\sigma_{max} = -\frac{W}{b}\left(1 + \frac{6e}{b}\right) \qquad \text{(using equation (14.41))}$$

$$= -\frac{540}{4}\left(1 + \frac{6\times 0.46}{4}\right) = -228.15 \text{ kN/m}^2$$

$$= 228.15 \text{ kN/m}^2 \text{ (Compressive)}$$ **Ans.**

The minimum pressure intensity produced at A is

$$\sigma_{min} = -\frac{W}{b}\left(1-\frac{6e}{b}\right) \qquad \text{(using equation (14.43))}$$

$$= -\frac{540}{4}\left(1-\frac{6\times 0.46}{4}\right) = -41.85 \text{ kN/m}^2$$

$$= 41.85 \text{ kN/m}^2 \text{ (Compressive)}$$ **Ans.**

Example 14.16

Determine the base width of a masonry trapezoidal retaining wall of 8 m height and 2 m top width to retain earth upto the top. Weight of soil and masonry are 18 kN/m^3 and 22 kN/m^3 respectively. Angle of repose of soil is 30°. Assume top of the earth is horizontal and backfill is vertical. Also, calculate the maximum base pressure at the known base width.

Solution: Refer Fig. 14.32.

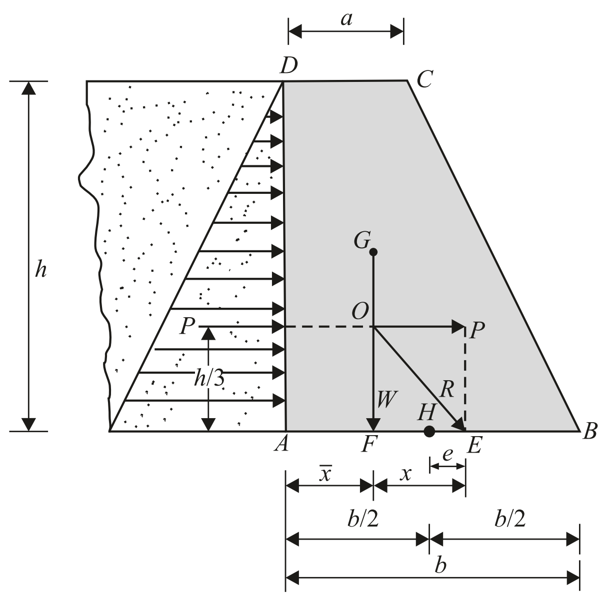


Fig. 14.32

Given,

Height of the retaining wall, $H = 8$ m

Height of the soil retained, $h = H = 8$ m

Top width of the retaining wall, $a = 2$ m

Unit weight of soil, $\gamma_s = 18$ kN/m^3

Unit weight of masonry, $\gamma_m = 22$ kN/m^3

Angle of repose of soil, $\phi = 30°$

Let b be the base width which ensures no tension at the base of the retaining wall. Consider 1 m length of the retaining wall.

The total thrust exerted by soil, which acts horizontally on the vertical face AD of the wall is

$$P = \frac{\gamma_s H^2}{2} \cdot \frac{1-\sin\phi}{1+\sin\phi}$$

$$= \frac{18\times 8^2}{2}\times\frac{1-\sin 30°}{1+\sin 30°} = 192 \text{ kN}$$

The weight of the retaining wall W is given as

$$W = \gamma_m \times \frac{1}{2}\times(a+b)\times H\times 1$$

$$= 22\times\frac{1}{2}\times(2+b)\times 8\times 1 = 88\ (2+b) \text{ kN}$$

The distance $\bar{x}$ is calculated as

$$\bar{x} = -\frac{a^2+ab+b^2}{3(a+b)}$$

$$= -\frac{2^2+2b+b^2}{3(2+b)} = \frac{4+2b+b^2}{3(2+b)}$$

Using $\Sigma M_E = 0$, we have

$$W\times x = P\times\frac{H}{3}$$

which gives

$$x = \frac{PH}{3W}$$

$$= \frac{192\times 8}{3\times 88\,(2+b)} = \frac{5.82}{(2+b)}$$

Now

$$(\bar{x}+x) = \frac{4+2b+b^2}{3(2+b)}+\frac{5.82}{(2+b)}$$

$$= \frac{4+2b+b^2+(3\times 5.82)}{3(2+b)} = \frac{21.46+2b+b^2}{3(2+b)}$$

The eccentricity e is given as

$$e = (\bar{x}+x)-\frac{b}{2} = \frac{21.46+2b+b^2}{3(2+b)}-\frac{b}{2}$$

For no tension to develop at the base of the retaining wall, we have

$$e = \frac{b}{6} \qquad \text{(using equation (14.46))}$$

$$\frac{21.46 + 2b + b^2}{3(2+b)} - \frac{b}{2} = \frac{b}{6}$$

$$\frac{21.46 + 2b + b^2}{3(2+b)} = \frac{b}{6} + \frac{b}{2} = \frac{2b}{3}$$

$$64.38 + 6b + 3b^2 = 12b + 6b^2$$

$$6b^2 - 3b^2 + 12b - 6b - 64.38 = 0$$

$$3b^2 + 6b - 64.38 = 0$$

which gives

$$b = \frac{-6 \pm \sqrt{(6)^2 - 4 \times 3 \times (-64.38)}}{2 \times 3}$$

$$= \frac{-6 \pm 28.43}{6}$$

$$= 3.74 \text{ m} \quad \text{or} \quad -5.74 \text{ m}$$

Negative value of b does not carry any meaning, therefore it is neglected.

Accept $b = 3.74$ m

Hence, the base width of the retaining wall is $b = 3.74$ m **Ans.**

Now the maximum compressive pressure at the base of the wall is given as

$$\sigma_{max} = \frac{W}{b}\left(1 + \frac{6e}{b}\right) \qquad \text{(using equation (14.41))}$$

$$= \frac{88(2+b)}{b}\left[1 + \frac{6e}{b}\right]$$

$$= \frac{88(2+3.74)}{3.74}\left[1 + \frac{6 \times 3.74}{3.74 \times 6}\right] \qquad \left(e = \frac{b}{6}\right)$$

$$= 270.11 \text{ kN/m}^2$$ **Ans.**

Example 14.17

A 5 m high masonry retaining wall has top width of 1 m and base width of 3 m. Upto what height a soil weighing 15 kN/m^3 can be retained by the wall so that the maximum pressure at the base is 1.2 times the minimum pressure at the base. The weight of masonry is 20 kN/m^3 and the angle of repose of soil is 30°.

Solution:

Refer Fig. 14.33.

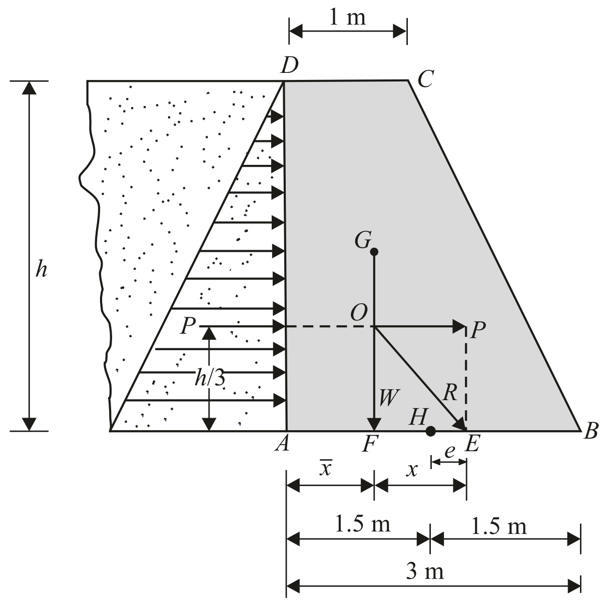


Fig. 14.33

Given,

Height of the retaining wall,	H	$= 5$ m
Top width of the retaining wall,	a	$= 1$ m
Base width of the retaining wall,	b	$= 3$ m
Unit weight of masonry,	γ_m	$= 20$ kN/m^3
Unit weight of soil,	γ_s	$= 15$ kN/m^3
Angle of repose of soil,	ϕ	$= 30°$

Let h be the height of soil retained against the vertical face AD of the retaining wall and consider 1 m length of the wall.

The distance $\bar{x}$ from the face AD, where the centre of gravity of the weight of the retaining wall acts, is given as

$$\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)} \qquad \text{(using equation (14.33))}$$

$$= \frac{1^2 + 1\times 3 + 3^2}{3(1+3)} = \frac{1+3+9}{3\times 4} = 1.083 \text{ m}$$

The total thrust due to pressure of soil of height h exerted on the face AD is given as

$$P = \frac{\gamma_s h^2}{2}\cdot\frac{1-\sin\phi}{1+\sin\phi}$$

$$= \frac{15h^2}{2} \times \frac{1 - \sin 30°}{1 + \sin 30°} = 2.5\ h^2$$

The weight of the retaining wall is

$$W = \gamma_m \times \frac{1}{2} \times (a + b) \times H \times 1$$

$$= 20 \times \frac{1}{2} \times (1 + 3) \times 5 \times 1 = 20 \times \frac{1}{2} \times 4 \times 5 = 200 \text{ kN}$$

Using $\sum M_E = 0$, we have

$$W \times x = P \times \frac{h}{3}$$

$$x = \frac{Ph}{3W} \qquad \text{...(1)}$$

On substituting P and W in equation (1), we get

$$x = \frac{2.5h^2 \times h}{3 \times 200} = \frac{2.5h^3}{600}$$

The eccentricity e is calculated as

$$e = (\bar{x} + x) - \frac{b}{2}$$

$$= 1.083 + \frac{2.5h^3}{600} - \frac{3}{2}$$

$$= \frac{2.5h^3}{600} - 0.417 \qquad \text{...(2)}$$

The maximum pressure (ignoring sign) exerted at the base of the retaining wall is

$$\sigma_{\max} = \frac{W}{b}\left(1 + \frac{6e}{b}\right) \qquad \text{(using equation (14.41))}$$

The minimum pressure (ignoring sign) exerted at the base of the retaining wall is

$$\sigma_{\min} = \frac{W}{b}\left(1 - \frac{6e}{b}\right) \qquad \text{(using equation (14.43))}$$

Given, $\sigma_{\max} = 1.2\ \sigma_{\min}$

or

$$\frac{W}{b}\left(1 + \frac{6e}{b}\right) = 1.2 \times \frac{W}{b}\left(1 - \frac{6e}{b}\right)$$

$$1+\frac{6e}{b} = 1.2-\frac{7.2e}{b}$$

$$\frac{6e}{b}+\frac{7.2e}{b} = 1.2 - 1$$

$$\frac{(6+7.2)e}{b} = 0.2$$

$$\frac{13.2e}{b} = 0.2$$

$$13.2\,e = 0.2 \times b$$

$$= 0.2 \times 3 = 0.6 \qquad \text{(on substituting } b\text{)}$$

which gives

$$e = 0.0454 \text{ m}$$

On substituting the value of e in equation (2), we have

$$0.0454 = \frac{2.5h^3}{600} - 0.417$$

or

$$\frac{2.5h^3}{600} = 0.0454 + 0.417 = 0.4624$$

$$h^3 = \frac{0.4624\times 600}{2.5} = 110.976$$

which gives

$$h = 4.8 \text{ m}$$

Hence, the height of soil filled up, $h = 4.8$ m **Ans.**

Example 14.18

A masonry retaining wall of trapezoidal section has a height of 4.5 m. Its top width is 0.6 m and base width is 2.4 m. The vertical face of the wall retains soil of unit weight of 12 kN/m^3 to a full height. Angle of repose of soil is 30°. If no tension in the wall is to be permitted, check the safety of the wall at the base. If unsafe, calculate the height upto which the wall at the top with a width of 0.6 m should be raised so that no tension develops at the base. Unit weight of masonry is 20 kN/m^3.

Solution: Refer Fig. 14.34.

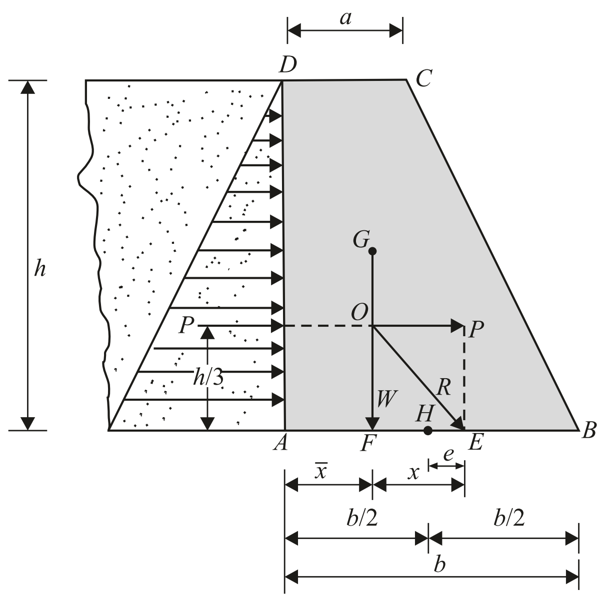


Fig. 14.34

Given,

Height of the retaining wall,	H	$= 4.5$ m
Height of the soil retained,	h	$= H = 4.5$ m
Top width of the retaining wall,	a	$= 0.6$ m
Base width of the retaining wall,	b	$= 2.4$ m
Unit weight of soil,	γ_s	$= 12$ kN/m^3
Unit weight of masonry,	γ_m	$= 20$ kN/m^3
Angle of repose of soil,	ϕ	$= 30°$.

Consider 1 m length of the retaining wall.

The total thrust P exerted by soil on the vertical face AD of the wall is given as

$$P = \frac{y_s H^2}{2} \cdot \frac{1-\sin\phi}{1+\sin\phi}$$

$$= \frac{12\times 4.5^2}{2}\times\frac{1-\sin 30°}{1+\sin 30°} = 40.5 \text{ kN}$$

The weight of the retaining wall W is given as

$$W = \gamma_m \times \frac{1}{2}\times(a+b)\times H\times 1$$

$$= 20\times\frac{1}{2}\times(0.6+2.4)\times 4.5\times 1 = 135 \text{ kN}$$

The distance $\bar{x}$ is calculated as

$$\bar{x} = \frac{a^2 + ab + b^2}{3(a+b)}$$

$$= \frac{(0.6)^2 + (0.6\times 2.4) + (2.4)^2}{3\times(0.6+2.4)} = 0.84 \text{ m}$$

Using $\Sigma M_E = 0$, we have

$$W \times x = P\times\frac{H}{3}$$

which gives $$x = \frac{PH}{3W} = \frac{40.5\times 4.5}{3\times 135} = 0.45 \text{ m}$$

The eccentricity e is given as

$$e = (\bar{x} + x) - \frac{b}{2}$$

$$= (0.84 + 0.45) - \frac{2.4}{2} = 0.09 \text{ m}$$

Safety check

For no tension to develop at the base of the retaining wall, we have

$$e = \frac{b}{6} = \frac{2.4}{6} = 0.4 \text{ m}$$

Since 0.09 m < 0.4 m, hence the wall is safe in tension at the base of the wall. **Ans.**

Example 14.19

A 6 m high masonry retaining wall of top width 1 m and base width 3 m retains earth level with the top on its vertical back. Safe bearing capacity of soil is 18 t/m^2. The coefficient of friction between wall and earth = 0.6, unit weight of soil is 1920 kgf/m^3 and unit weight of masonry is 2500 kgf/m^3. Angle of repose of soil is 30°. Examine the stability of the wall.

Solution:

Refer Fig. 14.35.

Given,

Height of the retaining wall, $H = 6$ m

Height of the soil retained, $h = H = 6$ m

Top width of the retaining wall, $a = 1$ m

Base width of the retaining wall, $b = 3$ m

Coefficient of friction between wall and earth, $\mu = 0.6$

Unit weight of soil, $\gamma_s = 1920 \text{ kgf/m}^3$

Unit weight of masonry, $\gamma_m = 2500 \text{ kgf/m}^3$

Angle of repose of soil, $\phi = 30°$

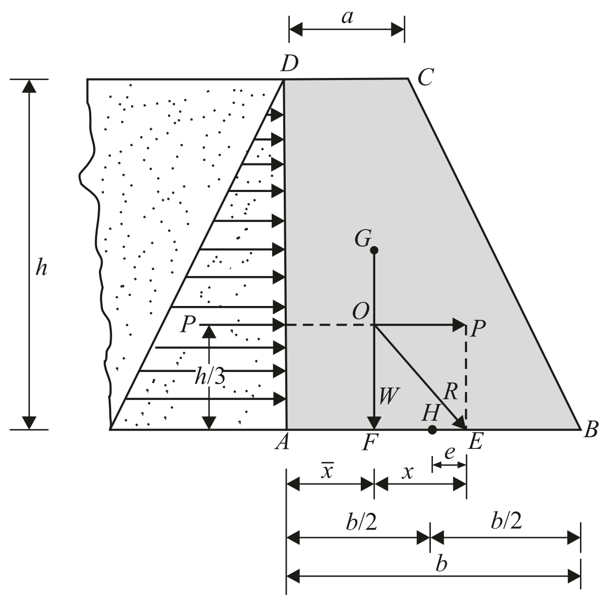


Fig. 14.35

Considering 1 m length of the retaining wall, the total lateral thrust P acting on the vertical face of the wall, according to Rankine's theory, is given as

$$P = \frac{y_s h^2}{2} \cdot \frac{1 - \sin\phi}{1 + \sin\phi}$$

$$= \frac{1920 \times 6^2}{2} \times \frac{1 - \sin 30°}{1 + \sin 30°} = 11520 \text{ kgf}$$

The weight of the retaining wall is given as

$$W = \gamma_m \times \frac{1}{2} \times (a + b) \times H \times 1$$

$$= 2500 \times \frac{1}{2} \times (1 + 3) \times 6 \times 1 = 30{,}000 \text{ kgf}$$

The weight of the retaining wall acts at its centre of gravity G, which is defined by the distance $\bar{x}$, given as

$$\bar{x} = \frac{a^2 + ab + b^2}{3(a + b)} = \frac{1^2 + 1 \times 3 + 3^2}{3(1 + 3)}$$

$$= \frac{1+3+9}{3\times 4} = 1.083 \text{ m}$$

Using $\Sigma M_E = 0$, we have

$$W \times x = P\times\frac{h}{3}$$

which gives

$$x = \frac{Ph}{3W}$$

$$= \frac{11520\times 6}{3\times 30{,}000} = 0.768 \text{ m}$$

The eccentricity e is calculated as

$$e = (\bar{x}+x)-\frac{b}{2}$$

$$= 1.083+0.768-\frac{3}{2} = 0.351 \text{ m}$$

Safety check against crushing

The maximum compressive stress developed at the base of the wall is given as

$$\sigma_{max} = \frac{W}{b}\left(1+\frac{6e}{b}\right) \qquad \text{(using equation (14.41))}$$

$$= \frac{30{,}000}{3}\left(1+\frac{6\times 0.351}{3}\right) = 17020 \text{ kgf/m}^2$$

The bearing capacity of soil is 18 t/m^2 = 18,000 kgf/m^2.

Since the maximum compressive stress developed at the base is less than the bearing capacity of soil, hence the retaining wall is safe against crushing.

Safety check against sliding

The maximum frictional resistance setup at the base of the retaining wall is

$$\mu W = 0.6 \times 30{,}000 = 18{,}000 \text{ kgf}$$

$$\text{The factor of safety } (n) = \frac{\text{Maximum frictional resistance } (\mu W)}{P}$$

$$= \frac{18{,}000}{11520} = 1.5625$$

Since $n > 1.5$, hence the retaining wall is safe against sliding.

Safety check against overturning

The resisting moment $= W\,(b-\bar{x})$

$= 30{,}000 \times (3 - 1.083) = 57510$ kgf.m

The overturning moment $= P \times \dfrac{h}{3}$

$= 11520 \times \dfrac{6}{3} = 23040$ kgf.m

The factor of safety (n) $= \dfrac{\text{Resisting moment}}{\text{Overturing moment}}$

$= \dfrac{57510}{23040} = 2.496$

Since $n > 2$, hence the retaining wall is safe against overturning.

Safety check against no tension at base

For no tension to develop at the base of the wall, we have

$$e = \frac{b}{6} = \frac{3}{6} = 0.5 \text{ m}$$

The calculated value of $e = 0.351$ m, which is less than 0.5 m. Hence, the wall is safe against no tension at the base.

Example 14.20

A masonry retaining wall of trapezoidal section and with a vertical face is 8 m high. Its top width is 1 m and bottom width is 4 m. The weight of masonry is 20 kN/m^3 and that of soil is 15 kN/m^3. The angle of repose of soil is 30°. If the wall retains earth to its full height against the vertical face, determine the stresses developed at the base. What is the additional height up to which soil can be retained so that no tension is developed at the base? Neglect soil on the top of the wall.

Solution:

Refer Fig. 14.36.

Given,

Height of the retaining wall, $H = 8$ m

Top width of the retaining wall, $a = 1$ m

Base width of the retaining wall, $b = 4$ m

Unit weight of masonry, $\gamma_m = 20$ kN/m^3

Unit weight of soil, $\gamma_s = 15$ kN/m^3

Angle of repose of soil, $\phi = 30°$

Initial height of soil retained $= h = H = 8$ m

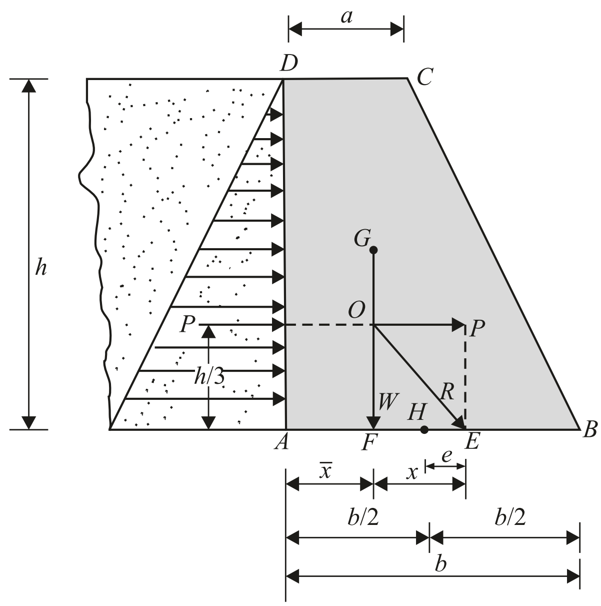


Fig. 14.36

Considering 1 m length of the retaining wall, the total thrust P acting on the vertical face of the retaining wall is given as

$$P = \frac{\gamma_s h^2}{2} \cdot \frac{1-\sin\phi}{1+\sin\phi} = \frac{15\times 8^2}{2} \times \frac{1-\sin 30°}{1+\sin 30°} = 160 \text{ kN}$$

The weight of the retaining wall is given as

$$W = \gamma_m \times \frac{1}{2} \times (a+b) \times H \times 1$$

$$= 20 \times \frac{1}{2} \times (1+4) \times 8 \times 1 = 400 \text{ kN}$$

The distance $$\bar{x} = \frac{a^2+ab+b^2}{3(a+b)} = \frac{1^2+1\times 4+4^2}{3(1+4)} = 1.4 \text{ m}$$

Using $\Sigma M_E = 0$, we have

$$W \times x = P \times \frac{h}{3}$$

which gives $$x = \frac{Ph}{3W} = \frac{160\times 8}{3\times 400} = 1.067 \text{ m}$$

The eccentricity e is found as

$$e = (\bar{x}+x) - \frac{b}{2} = 1.4 + 1.067 - \frac{4}{2} = 0.467 \text{ m}$$

Now $$\frac{b}{6} = \frac{4}{6} = 0.667 \text{ m}$$

Since $e = 0.467$ m < 0.667 m, hence the wall develops no tension at the base.

The maximum compressive stress developed at the base of the wall is given as

$$\sigma_{max} = \frac{W}{b}\left(1+\frac{6e}{b}\right) \quad \text{(using equation (14.41))}$$

$$= \frac{400}{4}\left(1+\frac{6\times 0.467}{4}\right)$$

$$= 170.05 \text{ kN/m}^2 \quad \textbf{Ans.}$$

The minimum stress developed at the base of the wall is given as

$$\sigma_{min} = -\frac{W}{b}\left(1-\frac{6e}{b}\right) \quad \text{(using equation (14.43))}$$

$$= -\frac{400}{4}\left(1-\frac{6\times 0.467}{4}\right) = -29.95 \text{ kN/m}^2$$

$$= 29.95 \text{ kN/m}^2 \text{ (Compressive)} \quad \textbf{Ans.}$$

Let h' be the additional height of the soil retained so as to develop no tension at the base of the wall.

Now the total height of the soil retained = $(h + h')$

The new total thrust P' is now given as

$$P' = \frac{\gamma_s (h+h')^2}{2}\cdot\frac{1-\sin\phi}{1+\sin\phi}$$

$$= \frac{15\times(8+h')^2}{2}\times\frac{1-\sin 30°}{1+\sin 30°} = 2.5\,(8+h')^2$$

P' acts at $\left(\frac{h+h'}{3}\right) = \left(\frac{8+h'}{3}\right)$ from the base of the wall.

The weight of the wall corresponding to its new height $(H + h')$ is given as

$$W' = y_m \times\frac{1}{2}\times(a+b)\times(H+h')\times 1$$

$$= 20\times\frac{1}{2}\times(1+4)\times(8+h')\times 1 = 50\,(8+h')$$

Using $\Sigma M_E = 0$, we have

$$W' \times x_1 = P'\times\frac{(h+h')}{3} \quad \text{(for additional height of soil)}$$

which gives
$$x_1 = \frac{P'(8+h')}{3W'}$$
$$= \frac{2.5(8+h')^2 \times (8+h')}{3 \times 50(8+h')}$$
$$= 0.0167\,(8 + h')^2 \qquad \text{(on substituting } P' \text{ and } W')$$

Now the eccentricity e is given as
$$e = (\bar{x} + x_1) - \frac{b}{2}$$
$$= 1.4 + 0.0167\,(8 + h')^2 - \frac{4}{2} \qquad \text{(as } \bar{x} = 1.4 \text{ m will remain same)}$$
$$= 0.0167\,(8 + h')^2 - 0.6$$

For no tension at the base of the wall, we have
$$e = \frac{b}{6}$$
$$0.0167\,(8 + h')^2 - 0.6 = 0.667$$
$$(8 + h')^2 = \frac{0.667 + 0.6}{0.0167} = 75.87$$

Solving, we get
$$h' = 0.71 \text{ m}$$

Hence, the additional height of the soil retained in order to have no tension at the base of the wall is 0.71 m. **Ans.**

SHORT ANSWER QUESTIONS

1. What is meant by combined loadings?
2. What is the difference between twisting moment and bending moment?
3. Give a few examples where a member is subjected to combined bending and axial load.
4. Give a few examples where a member is subjected to combined bending and torsion.
5. What are single and double eccentricity?
6. What are the usual cross-sections of a dam? What is the purpose of constructing a dam?
7. What are retaining walls? Give a few examples of retaining walls.

MULTIPLE CHOICE QUESTIONS

1. The eccentric load on a column can produce

(*a*) direct tensile stress

(*b*) direct compressive stress

(*c*) direct compressive stress and bending moment both

(*d*) direct compressive stress and shear stress both.

2. Consider the following statements about eccentric loading:

1. It is a special case of combined bending and axial loads.
2. Eccentricity occurs when the load line deviates from the axis of the member.
3. For no tension to develop at the base of a column, the value of the maximum eccentricity equals to one-sixth of the width of the section.
4. An eccentric load can be suitably replaced by a centric load and a moment.

Of these statements:

(*a*) 1 and 2 are true (*b*) 2 and 4 are true

(*c*) 2, 3 and 4 are true (*d*) 1, 2, 3 and 4 are true.

3. The maximum value of the eccentricity which ensures that no tension is developed at the base of a column, when subjected to an eccentric load, is equal to

(*a*) one-third of the width (*b*) one-half of the width

(*c*) one-sixth the width (*d*) one-fourth of the width.

4. A chimney is subjected to combined

(*a*) bending and torsion loads (*b*) bending and axial loads

(*c*) torsion and axial loads (*d*) bending, torsion and direct thrust.

5. A marine propeller shaft is subjected to combined

(*a*) bending and torsion loads (*b*) bending and axial loads

(*c*) torsion and axial loads (*d*) bending, torsion and direct thrust.

6. The horizontal water pressure at the base of a dam of height H is (γ_w = specific weight of water, h = height of water column)

(*a*) $\dfrac{\gamma_w h}{2}$ (*b*) $\gamma_w h$ (*c*) $\dfrac{\gamma_w h}{3}$ (*d*) $\gamma_w H$.

7. A dam is designed for

(*a*) maximum crushing stress (*b*) maximum bending stress

(*c*) maximum shear stress (*d*) maximum eccentricity.

8. Consider the following statements:

1. A retaining wall is used to sustain lateral pressures of soil or earth.
2. A retaining wall is subjected to both active and passive pressures.
3. Active and passive pressures act in opposite directions.
4. Active pressure tends to slide away the retaining wall from the retained earth.

Of these statements:

(*a*) 1 and 2 are true
(*b*) 1, 3 and 4 are true
(*c*) 1, 2 and 3 are true
(*d*) 1, 2, 3 and 4 are true.

9. The Rankine's earth pressure acting on a retaining wall is (γ = Unit weight of the earth, H = Height of the retaining wall, h = Height of soil filled up, ϕ = Angle of repose of soil)

(*a*) $\dfrac{\gamma h}{2} \cdot \dfrac{1-\sin\phi}{1+\sin\phi}$ (*b*) $\dfrac{\gamma H^2}{2} \cdot \dfrac{1-\sin\phi}{1+\sin\phi}$

(*c*) $\dfrac{\gamma h^2}{2} \cdot \dfrac{1-\sin\phi}{1+\sin\phi}$ (*d*) $\dfrac{\gamma h^2}{3} \cdot \dfrac{1-\sin\phi}{1+\sin\phi}$.

10. The horizontal thrust on the vertical face of the retaining wall acts at a height of (h = height of the retaining wall)

(*a*) $\dfrac{h}{2}$ above the base (*b*) $\dfrac{h}{3}$ above the base

(*c*) $\dfrac{h}{4}$ above the base (*d*) $\dfrac{2}{3}h$ above the base.

11. The coefficient of active earth pressure used in the Rankine's theory of retaining wall is (ϕ = Angle of repose of soil)

(*a*) $\dfrac{1-\sin\phi}{1+\cos\phi}$ (*b*) $\dfrac{1+\sin\phi}{1-\cos\phi}$ (*c*) $\dfrac{1-\cos\phi}{1+\cos\phi}$ (*d*) $\dfrac{1-\sin\phi}{1+\sin\phi}$.

ANSWERS

1. (*c*) **2.** (*d*) **3.** (*c*) **4.** (*b*) **5.** (*d*) **6.** (*b*) **7.** (*a*) **8.** (*b*)
9. (*c*) **10.** (*b*) **11.** (*d*)

EXERCISES

1. A solid steel shaft of 100 mm diameter is subjected to a torque of 19.62 kN.m and a bending moment of 0.981 kN.m. Determine the maximum and minimum principal stresses and the maximum shear stress induced in the shaft. (*Ans*. 161.9 MPa (T), 61.7 MPa (C), 111.8 MPa).

2. A solid shaft of 50 mm diameter transmits 50 kW at 1000 rpm. The shaft is subjected to an end thrust of 50 kN and a bending moment *M*. If the maximum compressive stress in the shaft is not to exceed 100 MPa, find the maximum value of *M*. (*Ans.* 0.98 kN.m).

3. A solid shaft of diameter '*d*' is subjected to an axial thrust *P* and an axial torque *T*. Show that the value of the principal stresses at any point on the surface of the shaft can be expressed as:

$$\sigma_{1,\,2} = \frac{2P}{\pi d^2}\left[1 \pm \sqrt{1 + \frac{64T^2}{P^2 d^2}}\right]$$

4. A solid circular shaft of diameter 50 mm is subjected to a twisting moment of 50 kN.m and a bending moment of 100 kN.m. Determine the principal stresses and the maximum shear stress.

   (*Ans*. 8.63 kN/mm^2 (T), 0.481 kN/mm^2 (C), 4.56 kN/mm^2).

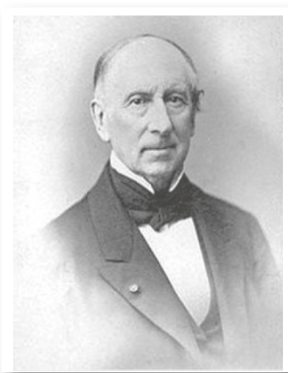
5. A solid rectangular column 200 mm wide and 150 mm thick carries a vertical load of 10 kN at an eccentricity of 50 mm in a plane bisecting the thickness. Determine the maximum and minimum stresses developed in the section. (*Ans*. 0.833 MPa (C), 0.161 MPa (T)).

6. A 20 m high cylindrical chimney having outside diameter 4 m and inside diameter 2.5m is subjected to a wind pressure of 1.8 kPa. If the unit weight of masonry is 22 kN/m^3 and the coefficient of wind pressure is 0.7, determine the maximum and minimum stresses induced at the base of the chimney. (*Ans*. σ_{max} = 629.15 kPa (C), σ_{min} = 250.6 kPa (C).

7. A 6 m high masonry retaining wall, trapezoidal in section, is 1 m wide at top. Its earth retaining face is vertical and smooth. The retained earth having unit weight of 1500 kgf/m^3 and angle of shearing resistance of 30° is level with the top of the wall. Assuming unit weight of masonry as 2100 kgf/m^3, calculate the minimum bottom width of the wall so that no tension is induced at the base. Also, calculate the maximum base pressure at this width. (*Ans***.** 2.63 m, 17390.9 kgf/m^2).

❑❑❑

15 Unsymmetrical Bending and Shear Centre



Augustin-Louis Cauchy
(1789-1857)

Augustin-Louis Cauchy, born on 21 August 1789, was a great French mathematician who worked under Lagrange, Laplace, Fourier and Poisson. He became a professor at the Ecole at the age of 27 and a member of the Academy of Sciences. He was a prolific writer who wrote approximately eight hundred research articles and five textbooks. His major works in pure mathematics were in group theory, number theory, series, integration, differential equation and analytical functions. He introduced the concept of stress as we know it today, developed the equations of theory of elasticity and introduced the notion of principal stresses and principal strains.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- What is a principal plane?
- What are the principal stresses?
- How is the maximum shear stress related to the principal stresses?
- What is the significance of shear centre?
- What is the difference between shear centre and centroid?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_15

15.1 SYMMETRICAL BENDING AND SIMPLE BENDING THEORY

The theories of pure bending of beams are restricted to beams having a plane of symmetry through their longitudinal axis and loads applied act in that plane. Bending takes place about an axis (called neutral axis), which is perpendicular to the plane of symmetry. It is assumed that beams are made of linearly elastic materials, which implies that neutral axis of the beam's cross-section passes through its centroid. A beam of this kind showing the related terminology is shown in Fig. 15.1.

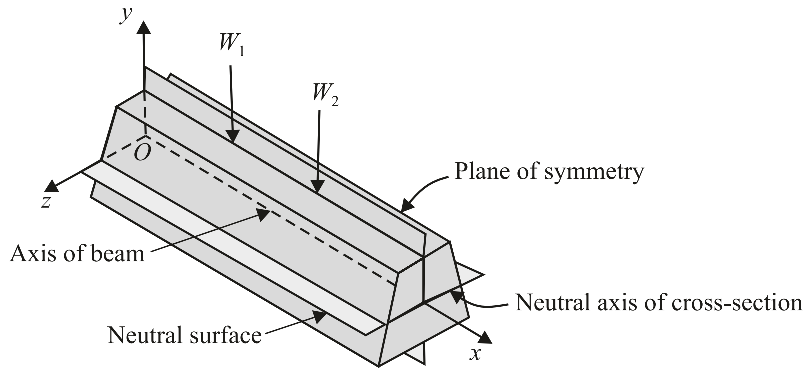


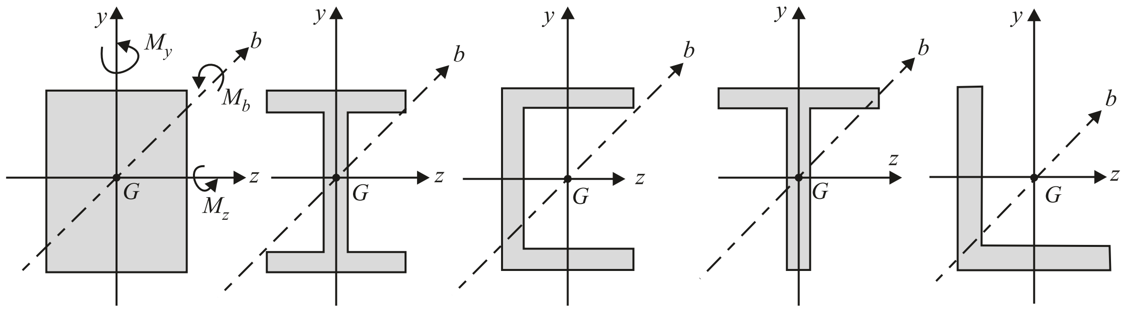
Fig. 15.1 A symmetrical beam with point loads acting in the plane of symmetry.

The plane of symmetry is xy-plane. The applied point loads W_1 and W_2 lie in the plane of symmetry (also called the plane of bending) and are pendicular to the axis of the beam (the x-axis), which lies somewhere in the plane of symmetry. The beam deflects only in the y direction. The y-axis (the axis of symmetry) and the z-axis (the neutral axis of beam's cross-section which is perpendicular to the y-axis) are the two principal axes of beam's cross-section, which are also the centrodial axes, as the centroid of the section passes through them. The second moments of area of a cross-section, about its principal axes are found to have maximum and minimum values, while the product second moment of area ($\int xydA$) is found to be zero. Hence, principal axes are the axes about which the product second moment of area is zero. Simple bending can then be taken as bending which takes place about a principal axis and moments are applied in a plane parallel to one such axis. Under these conditions, the normal bending stresses acting on the cross-sections vary linearly with the distance from the neutral axis, and are calculated from the flexure formula given by equation (5.5).

15.2 UNSYMMETRICAL BENDING

The unsymmetrical or asymmetrical bending of beams occurs when their cross-sections are not symmetric about any axis or when the beams are symmetric about one or two axes, but they are subjected to skew loading, that is, when the loads do not lie in a plane of symmetry.

Some cross-sections of beams having one and two axes of symmetry and subjected to skew loading are shown in Fig. 15.2, where y and z are the two principal axes, G is the centroid of the cross-section, and M_b is the applied moment about the b-axis produced due to skew loading. In general, moments are applied about a convenient axis in the cross-section and the plane containing the applied moment may not then be parallel to a principal axis. Beams in unsymmetric bending generally are subjected to bending moments acting about both principal axes of the cross-section. With



(*a*) Rectangular section (*b*) I-section (*c*) Channel section (*d*) T-section (*e*) Angle section

Fig. 15.2 Some singly and doubly symmetric sections with skew loading in which (*c*) and (*d*) are singly symmetric sections (*a*) and (*b*) are doubly symmetric sections and (*e*) is in a symmetric section.

unsymmetrical sections (for example angle sections, *Z*-sections etc.), the principal axes are not easily recognised. Because M_b is the moment about some axis *b* which is inclined to the principal axis *y* and *z*, the flexure formula is not directly applicable. Hence, M_b is resolved into its two components about the principal axes as M_y and M_z, where M_y acts in a horizontal plane and bends the member in that plane, whereas M_z acts in a vertical plane and bends the member in that plane. Bending is now assumed to take place simultaneously about the two principal axes, and the bending stresses are obtained from the flexure formula for each moment component acting separately. The total stresses can be obtained by superimposing the separate stresses, given by

$$\sigma_x = \text{Bending stress produced by } M_y + \text{Bending stress produced by } M_z$$

$$= \frac{M_y z}{I_y} + \frac{M_z y}{I_z} \quad \text{...(15.1)}$$

where

M_y and M_z = Bending moments about the principal axes *y* and *z* respectively

I_y and I_z = Second moments of area about respective principal axes *y* and *z*

y and *z* = Perpendicular distances from the two principal axes

It is important to note that of the two bending stresses in equation (15.1), one can be positive or negative depending upon its tensile or compressive nature.

15.3 DOUBLY SYMMETRIC BEAMS WITH SKEW OR INCLINED LOADS

The simplest case of unsymmetrical bending involves skew loading of a doubly symmetric beam. Fig. 15.3 shows a doubly symmetric (rectangular cross-sectioned) cantilever beam subjected to an inclined uniform load *w* per unit length over its full span.

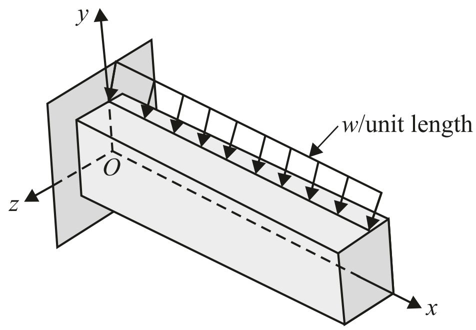


Fig. 15.3 A doubly symmetric cantilever beam with an inclined uniform load over its full span.

Another Fig. 15.4 shows a doubly symmetric cantilever beam subjected to an included point load W at its free end that acts at an angle θ to the positive y-axis.

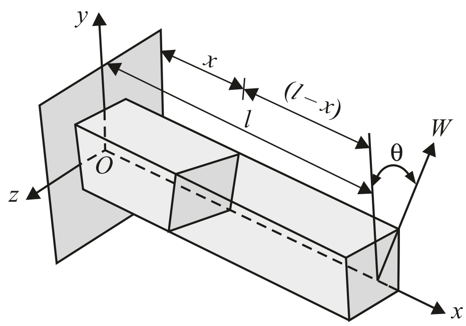


Fig. 15.4 A doubly symmetric cantilever beam with an inclined load at its free end.

Now consider the cross-section of the beam and resolve the inclined load W, and hence the bending moment M caused by the load W into its two components about each of the principal axes of the cross-section, one acting in each plane of symmetry (Fig. 15.5). The y-axis and z-axis are the principal axes of inertia and θ is the angle between M and the z-axis.

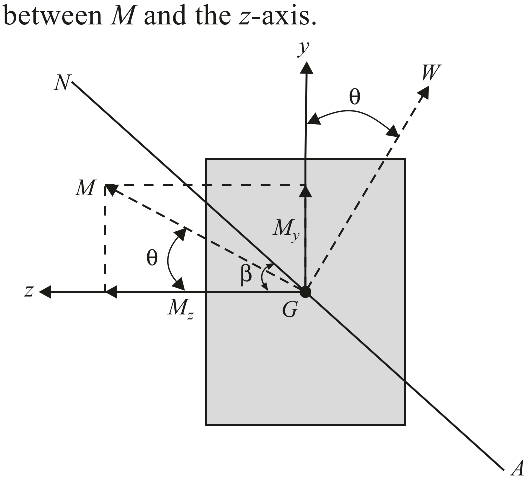


Fig. 15.5 Cross-section of the beam subjected to bending moments M_y and M_z.

On resolving the load W into its two components along y and z directions, we get

$$W_y = W\cos\theta \qquad \text{(along positive } y \text{ direction)...(15.2)}$$

and

$$W_z = W\sin\theta \qquad \text{(along negative } z \text{ direction)...(15.3)}$$

Similarly, on resolving the bending moment M into its two components M_y and M_z acting on a cross-section located at a distance x from the fixed support, we get

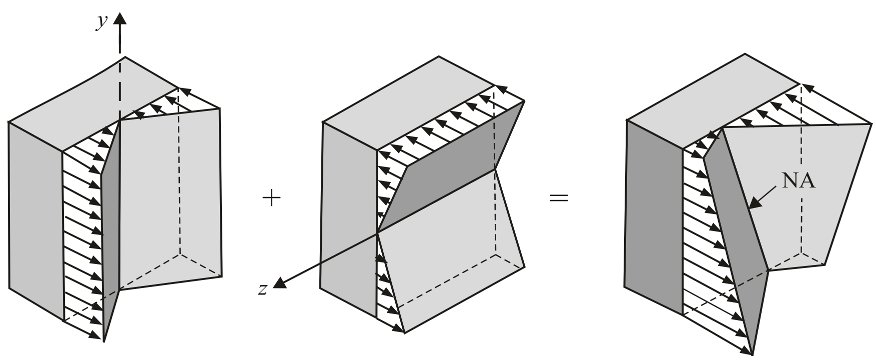
$$M_y = W_z(l-x) = (W\sin\theta)(l-x) = M\sin\theta \qquad \text{(along positive } y \text{ direction)...(15.4)}$$

and

$$M_z = W_y(l-x) = (W\cos\theta)(l-x) = M\cos\theta \qquad \text{(along positive } z \text{ direction)...(15.5)}$$

where l is the length of the beam.

Initially, since M is inclined to the principal axes, y and z, the flexure formula is not directly applicable. However, if M is resolved into its components M_y and M_z that act in the planes of symmetry of the beam, the flexure formula can be applied to each moment component separately to find the individual bending stress produced by each moment component and the final normal stresses are obtained by superimposing the stresses obtained from the flexural formula for each of the separate axes of the cross-section as shown in Fig. 15.6.



(a) Stress produced by M_y (a) Stress produced by M_z (c) Superimposition of stresses produced by M_y and M_z

Fig. 15.6 Normal stress distribution caused by unsymmetrical bending.

Position of the neutral axis

Let us consider a point P having coordinates y and z in the cross-section of the beam as shown in Fig. 15.7.

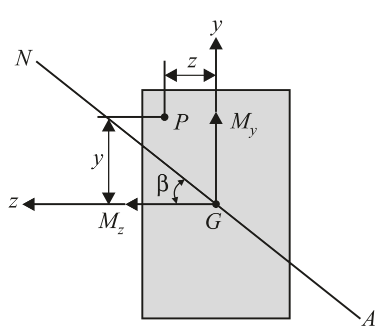


Fig. 15.7 Location of the neutral axis.

The positive bending moment M_y produces tension at P and the positive bending moment M_z produces compression. The total normal bending stress at P is given as

$$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z} \qquad \text{...(15.6)}$$

where I_y and I_z are the second moments of area of the beam's cross-sectional area with respect to y and z axes respectively.

In general, the neutral axis (NA) for unsymmetrical bending is not necessarily parallel to the bending moment M (Fig. 15.8). As the neutral axis of the cross-section is a line, where the bending stress is zero, its equation can be obtained by equating the bending stress σ_x to zero, given as

$$\frac{M_y z}{I_y} - \frac{M_z y}{I_z} = 0 \qquad \text{...(15.7)}$$

which gives

$$\frac{y}{z} = \frac{M_y I_z}{M_z I_y} \qquad \text{...(15.8)}$$

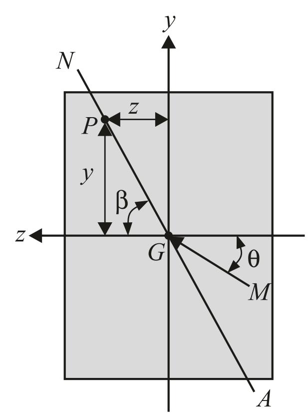


Fig. 15.8

Equation (15.7) represents a straight line. Hence, the neutral axis (NA) is an inclined line that passes through the centroid G of the cross-section. The angle β between the neutral axis and the z-axis is called slope angle of the neutral axis, defined as

$$\tan\beta = \frac{y}{z} = \frac{M_y I_z}{M_z I_y} \qquad \text{(using equation (15.8)...(15.9))}$$

where y/z is called slope of the neutral axis. Depending upon the magnitudes and directions of the bending moments, the angle β may vary from – 90° to + 90°.

Also $$\tan\beta = \frac{I_z}{I_y}\tan\theta \qquad \text{...(15.10)}$$

where $$\tan\theta = \frac{M_y}{M_z} \qquad \text{(using equations (15.4) and (15.5))}$$

It shows that the resultant bending moment M is at the angle θ from the z-axis. Consequently the resultant bending moment is perpendicular to the longitudinal plane containing the load W.

Equation (15.10) shows that unless we have symmetrical bending ($\theta = 0°$ or $90°$), the neutral axis will be parallel to the bending moment M, only if $I_y = I_z$. As the bending stress is proportional to the distance from the neutral axis, the maximum bending stress occurs at the point, which is farthest from the neutral axis. Hence, locating the neutral axis helps in finding the location of the maximum bending stress on a cross-section.

Relationship between neutral axis and inclination of applied load

From equation (15.10), it is clear that angle β is generally not equal to angle θ, which implies that the neutral axis (*NA*) is not perpendicular to the longitudinal plane carrying the applied load.

The angles β and θ may be equal in the following cases:

- When the load lies in the xy-plane ($\theta = 0°$ or $180°$) implying that the z-axis is the neutral axis.
- When the load lies in the xz-plane ($\theta = \pm 90°$) implying that the y-axis is the neutral axis.
- When the principal moments of inertia are equal implying that $I_y = I_z$.

In the last case, all the axes through the centroid are the principal axes, and have the same moment of inertia. The plane of loading irrespective of its direction, is always a principal plane, and the neutral axis is always perpendicular to it. The cross-sections for which this condition is valid include square and circular sections.

Example 15.1

A 1.2 m long wooden cantilever beam with rectangular cross-section 50 mm × 80 mm is subjected to a moment of 150 N.m in a plane forming an angle of 30° with the vertical as shown in Fig. 15.9. Determine (*a*) the maximum stress in the beam and (*b*) the angle that the neutral surface makes with the horizontal plane.

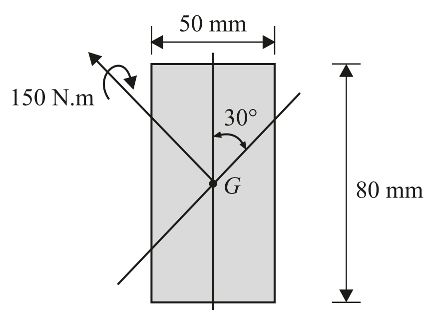


Fig. 15.9

Solution:

Given,

Length of the beam, $l = 1.2$ m

Width of the beam, $b = 50$ mm

Depth of the beam, $d = 80$ mm

Moment applied, $M = 150$ N.m

(*a*) **Maximum stress in the beam**

The moment M of 150 N.m is resolved into its components M_y and M_z along y and z directions respectively, which are the principal centroidal axes (Fig. 15.10). The moment component M_y acts in a horizontal plane and the moment component M_z in a vertical plane.

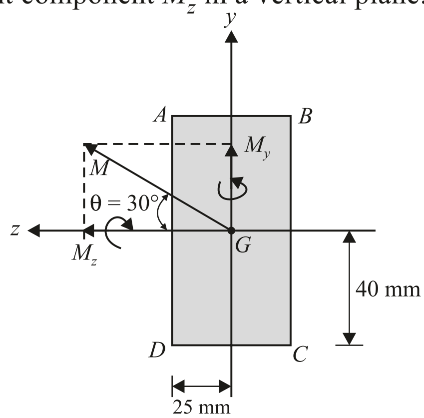


Fig. 15.10 Resolution of the applied moment into M_y and M_z.

The moment components M_y and M_z are calculated as

$$M_y = 150 \sin 30° = 75 \text{ N.m}$$

$$M_z = 150 \cos 30° = 129.9 \text{ N.m}$$

The moments of inertia of the beam's cross-sectional area about y and z axes are given as

$$I_y = \frac{1}{12} \times 80 \times 50^3 \times 10^{-12} \text{ m}^4 = 8.33 \times 10^{-7} \text{ m}^4$$

and

$$I_z = \frac{1}{12} \times 50 \times 80^3 \times 10^{-12} \text{ m}^4 = 2.13 \times 10^{-6} \text{ m}^4$$

The largest tensile stress due to M_z occurs along CD, given as

$$\sigma_1 = \frac{M_z}{I_z} \cdot y$$

$$= \frac{129.9 \times 40 \times 10^{-3}}{2.13 \times 10^{-6}} \times \frac{1}{10^6} \text{ MPa} = 2.44 \text{ MPa}$$

The largest tensile stress due to M_y occurs along AD, given as

$$\sigma_2 = \frac{M_y}{I_y} \cdot z$$

$$= \frac{75 \times 25 \times 10^{-3}}{8.33 \times 10^{-7}} \times \frac{1}{10^6} \text{ MPa}$$

$$= 2.25 \text{ MPa}$$

Now the largest tensile stress due to combined loading occurs at D, and its value is given as

$$\sigma_{max} = \sigma_1 + \sigma_2$$
$$= (2.44 + 2.25) \text{ MPa}$$
$$= 4.69 \text{ MPa}$$

The largest compressive stress has the same value equal to the largest tensile stress, and occurs at B.

Ans.

(*b*) **Angle made by the neutral surface with horizontal plane**

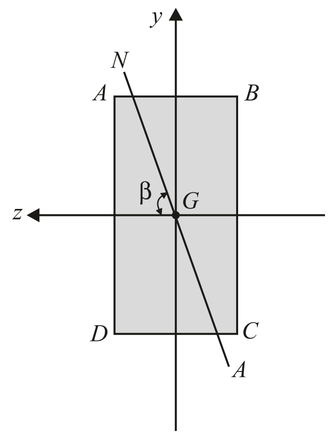


Fig. 15.11 Location of the neutral axis.

Let β be the angle made by the neutral axis with the horizontal plane, that is, the z-axis as shown in Fig. 15.11. It is given as

$$\tan\beta = \frac{I_z}{I_y}\tan\theta \qquad \text{(using equation (15.10))}$$

$$= \frac{2.13\times10^{-6}}{8.33\times10^{-7}}\times\tan 30° = 1.476$$

Hence, $\beta = 55.8°$ **Ans.**

Example 15.2

An I-section beam shown in Fig. 15.12 carries a bending moment of 35 kN.m inclined at 17° to the z-axis. Determine the following parameters using the data given below.

(*a*) the angle between the neutral axis and the z-axis and

(*b*) the largest bending stress acting on the section.

Take $I_z = 71.1 \times 10^6 \text{ mm}^4$, $I_y = 7.03 \times 10^6 \text{ mm}^4$

$S_z = 534 \times 10^3 \text{ mm}^3$, $S_y = 95.1 \times 10^3 \text{ mm}^3$.

Solution:

Given,

Bending moment, M = 35 kN.m

Angle between M and z-axis, $\theta = 17°$

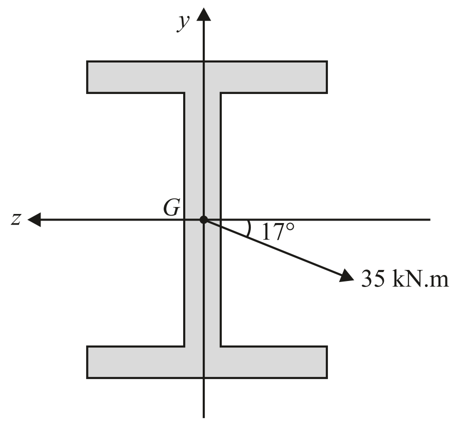


Fig. 15.12

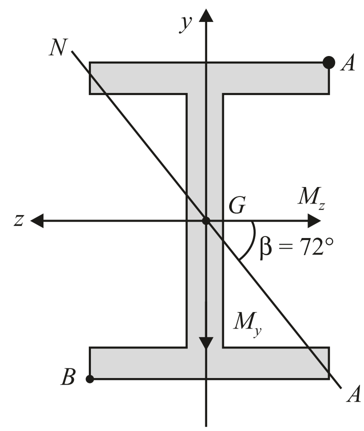


Fig. 15.13

(a) The angle β between the neutral axis (NA) and the z-axis is calculated as

$$\tan\beta = \frac{I_z}{I_y}\tan\theta \qquad \text{(using equation (15.10))}$$

$$= \frac{71.1\times10^6}{7.03\times10^6}\times\tan 17°$$

$$= 3.09$$

Hence, $\beta = 72°$ **Ans.**

(b) The bending moment of 35 kN.m is resolved into its components parallel to the principal axes y and z (Fig. 15.13), and are given as

$$M_z = -M\cos 17° \qquad \text{(along negative z-axis)}$$

$$= -35\times\cos 17°$$

$$= -33.47 \text{ kN.m}$$

$$M_y = -M\sin 17° \qquad \text{(along negative } y\text{-axis)}$$

$$= -35\times\sin 17°$$

$$= -10.23 \text{ kN.m}$$

The points A and B as shown in Fig. 15.13 are farthest from the neutral axis, where the largest bending stress occurs. It is to be noted that both components of the bending moment cause tension at A and compression at B. Hence, the largest bending stress produced on the section is given as

$$\sigma_{max} = \sigma_A = |\sigma_B| = \frac{|M_y|}{|I_y|}\cdot z + \frac{|M_z|}{|I_z|}\cdot y \qquad \text{(using equation (15.6))}$$

$$= \frac{|M_y|}{(I_y / z)} + \frac{|M_z|}{(I_z / y)}$$

$$= \frac{|M_y|}{S_y} + \frac{|M_z|}{S_z}$$

$$= \left[\frac{10.23\times10^3}{95.1\times10^3\times10^{-9}} + \frac{33.47\times10^3}{534\times10^3\times10^{-9}}\right] \text{Pa}$$

$$= 170.25 \times 10^6 \text{ Pa}$$

$$= 170.25 \text{ MPa}$$ **Ans.**

Example 15.3

A channel section shown in Fig. 15.14 is subjected to a bending moment of 3 kN.m at an angle 15° to the z-axis. Calculate the maximum tensile and compressive bending stresses at the section of the beam. Also, find the position of the neutral axis.

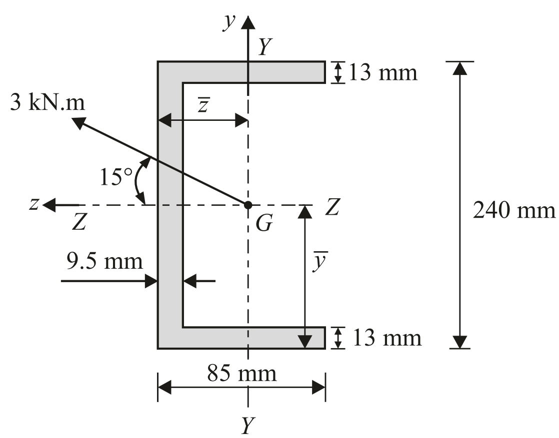


Fig. 15.14

The properties of the section are as follows:

$$I_{zz} = 3.6 \times 10^{-5} \text{ m}^4, I_{yy} = 2.48 \times 10^{-6} \text{ m}^4$$

$$\bar{z} = 22.3 \text{ mm}, \bar{y} = 120 \text{ mm}$$

Solution: Given,

Bending moment, M = 3 kN.m

Angle between M and the z-axis, θ = 15°

The z and y axes are the principal centroidal axes, as the centroid G of the section passes through them. The bending moment M is resolved into the components M_z and M_y along the z-axis and the y-axis respectively as shown in Fig. 15.15.

$$M_z = M\cos\theta = 3\times\cos 15° = 2.897 \text{ kN.m}$$

$$M_y = M\sin\theta = 3\times\sin 15° = 0.776 \text{ kN.m}$$

The stresses produced by the bending moment M are maximum at the points, which are located farthest from the neutral axis, for example, points A and B as shown in the figure.

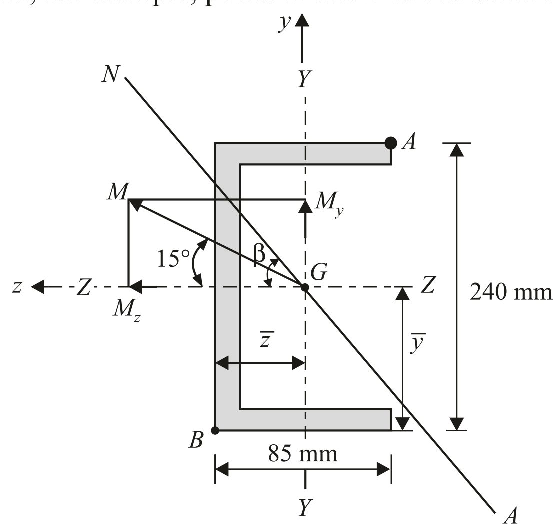


Fig. 15.15

The maximum bending stress at A is given as

$$\sigma_A = \frac{M_y z_1}{I_{yy}} - \frac{M_z . y_1}{I_{zz}} \quad \text{(using equation (15.6))}$$

$$= \frac{M_y[-(85-\bar{z})]}{I_{yy}} - \frac{M_z\bar{y}}{I_{zz}}$$

$$= \frac{0.776\times[-(85-22.3)]\times10^{-3}}{2.48\times10^{-6}} - \frac{2.897\times120\times10^{-3}}{3.6\times10^{-5}} \text{ kN/m}^2$$

$$= (-19619.03 - 9656.67) \text{ kN/m}^2 = -29275.7 \text{ kN/m}^2$$

$$= -29.27 \text{ MN/m}^2 = -29.27 \text{ MPa}$$

$$= 29.27 \text{ MPa (Compressive)}$$ **Ans.**

Hence, the maximum compressive bending stress is produced at point A.

The maximum bending stress at B is given as

$$\sigma_b = \frac{M_y z_2}{I_{yy}} - \frac{M_z y_2}{I_{zz}} \qquad \text{(using equation (15.6))}$$

$$= \frac{M_y \bar{z}}{I_{yy}} - \frac{M_z(-\bar{y})}{I_{zz}}$$

$$= \frac{0.776\times 22.3\times 10^{-3}}{2.48\times 10^{-6}} - \frac{2.897\times(-120\times 10^{-3})}{3.6\times 10^{-5}}\ \text{kN/m}^2$$

$$= (6977.74 + 9656.66)\ \text{kN/m}^2$$

$$= 16634.4\ \text{kN/m}^2 = 16.63\ \text{MN/m}^2$$

$$= 16.63\ \text{MPa (Tensile)}$$ **Ans.**

Hence, the maximum tensile bending stress is produced at point B.

Location of the neutral axis (NA)

The angle β made by the neutral axis (NA) with the z-axis is defined as

$$\tan\beta = \frac{I_{zz}}{I_{yy}}\tan\theta \qquad \text{(using equation (15.10))}$$

$$= \frac{3.6\times 10^{-5}\,\text{m}^4}{2.48\times 10^{-6}\,\text{m}^4}\times\tan 15^\circ = 3.889$$

Hence, $\beta = 75.58^\circ$ **Ans.**

Example 15.4

A T-shaped 5 m long cantilever beam shown in Fig. 15.16 is subjected to a transverse load W at its free end. The beam material yields according to the Tresca's yield criterion, when the maximum shear stress reaches 180 MPa. Determine the maximum load W carried by the beam.

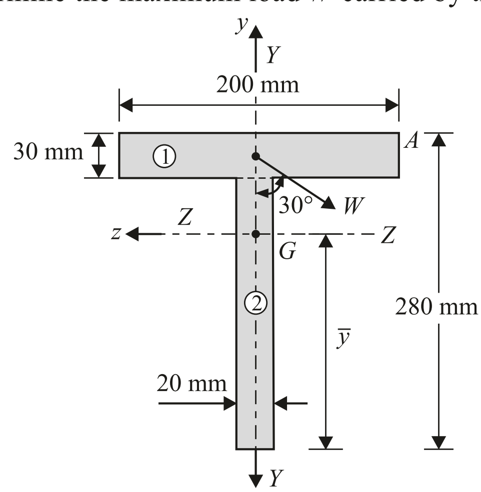


Fig. 15.16

Solution:

Given,

Length of the beam, $l = 5$ m

Maximum shear stress, $\tau_{max} = 180$ MPa

The y and z axes are the principal cerntroidal axes, as the centroid G of the section passes through them. The applied load W produces bending moment M, which is maximum (negative) at the fixed end of the beam. It is resolved into components M_y and M_z along the y and z axes respectively as shown in Fig. 15.17.

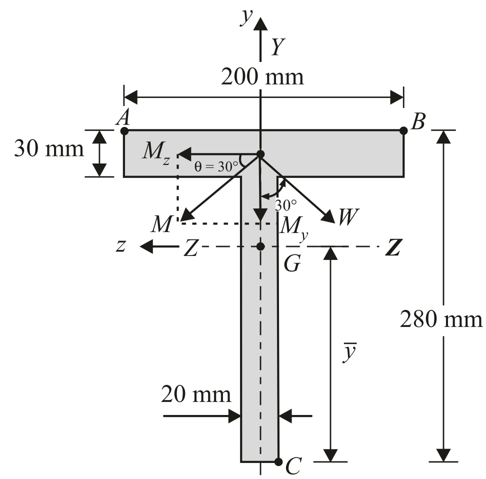


Fig. 15.17

When the load W is resolved, we get

$$W_y = W \cos \theta \qquad \text{(along negative } y \text{ direction)}$$

$$W_z = W \sin \theta \qquad \text{(along negative } z \text{ direction)}$$

When the bending moment M is resolved, we get

$$M_y = M \sin \theta \qquad \text{(along negative } y \text{ direction)}$$

$$= W_z \cdot l$$

$$= -W \sin \theta \times l$$

$$= -W \sin 30° \times l \qquad (\text{as } \theta = 30°)$$

$$= -W \times \frac{1}{2} \times 5 = -2.5\,W \qquad (\text{on substituting } l)$$

$$M_z = M \cos \theta \qquad \text{(along positive } z \text{ direction)}$$

$$= -W_y \cdot l$$

$$= -W \cos \theta \times l$$

$$= -W \cos 30° \times l$$

$$= -W \times \frac{\sqrt{3}}{2} \times l = -4.33\,W \qquad (\text{on substituting } l)$$

Both M_y and M_z are negative, as the bending moment is negative for a cantilever beam.

Position of the centroid *G*

The *T*-section is assumed to consist of two parts (1) and (2), and their areas of cross-section are given as

$$a_1 = 200 \times 30 = 6000 \text{ mm}^2$$

$$a_2 = (280 - 30) \times 20 = 5000 \text{ mm}^2$$

Total area of the *T*-section, $A = a_1 + a_2$

$$= (6000 + 5000) \text{ mm}^2 = 11000 \text{ mm}^2$$

Since the section is symmetrical about the y-axis, hence

$$\bar{z} = 0$$

and
$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{a_1 y_1 + a_2 y_2}{A}$$

$$= \frac{6000 \times 265 + 5000 \times 125}{11000} = 201.36 \text{ mm}$$

Hence, the centroid G of the section is located on the y-axis at a distance of 201.36 mm from the lower part of the web.

Calculation of moment of inertia

$$I_{zz} = I_{zz_1} + I_{zz_2}$$

$$= \left[\frac{1}{12} \times 200 \times 30^3 + 6000 \times (265 - 201.36)^2\right] + \left[\frac{1}{12} \times 20 \times 250^3 + 5000 \times (201.36 - 125)^2\right]$$

$$= 7.99 \times 10^7 \text{ mm}^4$$

$$= 7.99 \times 10^7 \times 10^{-12} \text{ m}^4 = 7.99 \times 10^{-5} \text{ m}^4$$

and
$$I_{yy} = I_{yy_1} + I_{yy_2}$$

$$= \frac{1}{12} \times 30 \times 200^3 + \frac{1}{12} \times 250 \times 20^3$$

$$= 2.016 \times 10^7 \text{ mm}^4 = 2.016 \times 10^7 \times 10^{-12} \text{ m}^4$$

$$= 2.016 \times 10^{-5} \text{ m}^4$$

$I_{xy} = 0$, as the beam is symmetrical about one of the two axes, here it is the y-axis.

The critical points in the cross-section include the positions located at points A, B and C as shown in Fig. 15.17. These are the locations where normal stresses due to bending moment are maximum because of their farthest positions from the neutral axis.

Calculation of bending stresses

The maximum bending stress at A is given as

$$\sigma_A = \frac{M_y \cdot z_1}{I_{yy}} - \frac{M_z \cdot y_1}{I_{zz}} \qquad ((z_1, y_1) \text{ is the coordinate of point } A)$$

$$= \frac{(-2.5W)\times 100\times 10^{-3}}{2.016\times 10^{-5}} - \frac{(-4.33W)\times(280-201.36)\times 10^{-3}}{7.99\times 10^{-5}}$$

$$(z_1 = 100 \text{ mm}, y = (280 - \bar{y}) \text{ mm})$$

$$= -12.4\times 10^3\, W + 4.26\times 10^3\, W = -8.14\times 10^3\, W$$

The maximum bending stress at B is given as

$$\sigma_B = \frac{M_y . z_2}{I_{yy}} - \frac{M_z . y_2}{I_{zz}} \qquad ((z_2, y_2) \text{ is the coordinate of point } B)$$

$$= \frac{(-2.5W)\times(-100\times 10^{-3})}{2.016\times 10^{-5}} - \frac{(-4.33W)\times(280-201.36)\times 10^{-3}}{7.99\times 10^{-5}}$$

$$(z_2 = -100 \text{ mm}, y_2 = 280 - \bar{y})$$

$$= 12.4\times 10^3\, W + 4\ .26\times 10^3\, W = 16.66\times 10^3\, W$$

The maximum bending stress at C is given as

$$\sigma_C = \frac{M_y . z_3}{I_{yy}} - \frac{M_z . y_3}{I_{zz}} \qquad ((z_3, y_3) \text{ is the coordinate of point } C)$$

$$= \frac{(-2.5W)\times(-10\times 10^{-3})}{2.016\times 10^{-5}} - \frac{(-4.33W)\times(-201.36)\times 10^{-3}}{7.99\times 10^{-5}}$$

$$(z_3 = -10 \text{ mm}, y_3 = (280 - \bar{y}) \text{mm})$$

$$= 1.241\times 10^3\, W - 10.91\times 10^3\, W$$

$$= -9.67\times 10^3\, W$$

These stresses are the values of the maximum principal stresses at the three points which are acting along the longitudinal axis of the beam, that is, the x-axis. The maximum bending stress occurs at B, and it is tensile in nature. The minimum principal stress at each point is zero.

According to Tresca's yield criterion, we have

$$\tau_{max} = \frac{(\sigma_1)_B - (\sigma_2)_B}{2}$$

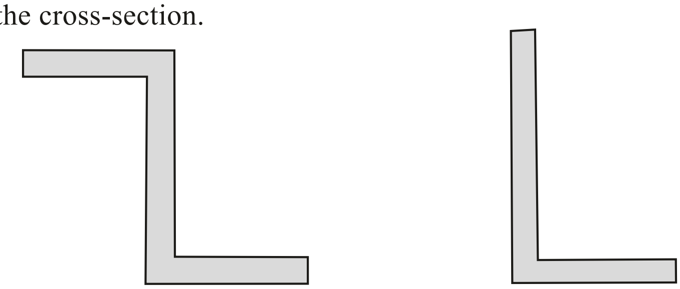
$$180 \times 10^6 = \frac{16.66 \times 10^3 W}{2} \qquad \text{(as } (\sigma_2)_B = 0\text{)}$$

Solving for W, we get

$$W = 21608.64 \text{ N} = 21.6 \text{ kN}$$ **Ans.**

15.4 PURE BENDING OF UNSYMMETRICAL BEAMS

Unsymmetrical beams are those beams, which have unsymmetrical sections or have no axis of symmetry. A few examples of unsymmetrical beams include z-section and angle-section beams as shown in Fig. 15.18. The bending of unsymmetrical sections removes the restriction of at least one axis of symmetry in the cross-section.



(*a*) Z-section (*b*) Angle-section

Fig. 15.18 Types of unsymmetrical beams.

Consider that an unsymmetrical beam is subjected to a load, which causes a bending moment M on the cross-section of the beam. While discussing the effect of this bending moment, we consider two perpendicular axes y and z, which happen to be the neutral axes of the cross-section through which the centroid G of the section passes, that is, both y and z axes are the principal centroidal axes of the cross-section (Fig. 15.19).

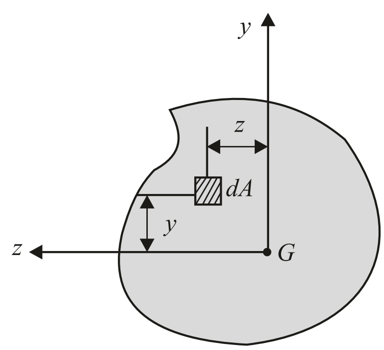


Fig. 15.19 Bending of unsymmetrical cross-section.

An element of area dA is now considered at a distance of y from the z-axis and at a distance of z from the y-axis as shown in the figure. The bending moment M is resolved into its components M_y and M_z along the axes y and z respectively as shown in Fig. 15.20.

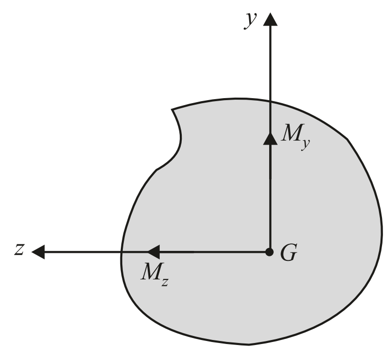


Fig. 15.20 Bending moment components M_y and M_z about y and z axes respectively.

While considering z-axis as the neutral axis, that is, one of the principal axes of inertia, the plane of bending is xy-plane and the beam deflects in that plane, and the bending moment M_z acts in that plane. On the other hand, when y-axis is the neutral axes, that is, the other principal axis of inertia, the plane of bending is xz-plane and the beam deflects in that plane, and the bending moment M_y acts in that plane. The bending of beam under these two conditions are considered separately. First consider that the bending takes place about the z-axis as the neutral axis.

Force acting on the element of area $dA = \sigma_x dA$

Total force acting on the entire cross-section $= \int \sigma_x \, dA$

where σ_x = Normal stress on the area dA

$= (-K_y)Ey$

$= -K_y Ey$

K_y = Curvature (negative) of the bent beam (equals to the reciprocal of the radius of curvature) in xy-plane

y = Distance from the z-axis

E = Young's modulus of the beam material

The bending moment of the total force about the z-axis is given as

$$M_z = -\int \sigma_x \, dAy$$

$$= K_y E \int y^2 dA \qquad \text{(on substituting } \sigma_x\text{)}$$

$$= K_y E I_z \qquad \text{...(15.11)}$$

where I_z = Second moment of area of the beam's cross-section about the z-axis

$$= \int y^2 dA$$

The bending moment of the total force about the y-axis is given as

$$M_y = \int \sigma_x \, dA \, z$$

$$= -K_y E \int y z \, dA \qquad \text{(on substituting } \sigma_x\text{)}$$

$$= -K_y E I_{yz} \qquad \text{...(15.12)}$$

where I_{yz} = Product of inertia of the beam's cross-section with respect to y and z axes.

$$= \int y z \, dA$$

z = Distance from the y-axis

In case of z-axis being the neutral axis, that is, the principal axis, the product of inertia I_{yz} is equal to zero, and the only bending moment left is M_z, which acts in the xy-plane and is represented by equation (15.11). Hence, bending of the unsymmetrical beam occurs in a manner analogous to that of a symmetrical beam.

Now consider that bending takes place about the y-axis as the neutral axis. The normal stress σ_x acting on the area dA is this time different. The new σ_x is given as

$$\sigma_x = (-K_z) E z$$

$$= -K_z E z$$

where K_z = Curvature (negative) of the bent beam in the xz-plane.

Force acting on the cross-section of the beam

$$= \int \sigma_x \, dA$$

$$= -\int K_z E z \, dA \qquad \text{(on substituting new } \sigma_x\text{)}$$

The bending moment components are now given as

$$M_y = \int \sigma_x \, dA \, z \qquad \text{(about the } y\text{-axis)}$$

$$= -K_z E \int z^2 dA \qquad \text{(on substituting new } \sigma_x\text{)}$$

$$= -K_z E I_y \qquad \text{... (15.13)}$$

where I_y = Second moment of area of the beam's cross-section about the y-axis.

$$= \int z^2 dA$$

Similarly,

$$M_z = -\int \sigma_x \, dA \, y \qquad \text{(about the } z\text{-axis)}$$

$$= K_z E \int y z \, dA \qquad \text{(on substituting new } \sigma_x\text{)}$$

$$= K_z E I_{yz} \qquad \text{... (15.14)}$$

In case of y-axis being the neutral axis, that is, the principal axis, the product of inertia I_{yz} is again zero, and the only bending moment left is M_y, which acts in the xz-plane and is represented by equation (15.13).

Hence, we conclude that for an unsymmetrical beam under pure bending, the plane in which the bending moment acts is perpendicular to the neutral surface only if y and z axes are the principal centroidal axes of the beam's cross-section, and the bending moment acts in one of the two principal planes (xy-plane or xz-plane). Therefore, if a bending moment acts in one of the principal planes, this plane will be the plane of bending and the usual bending theory and the flexure formula are applicable, which can be used to find the stresses due to the bending moments M_y and M_z acting separately, and then they are superimposed to find the stresses produced by the original bending moment M.

Sign conventions for beam curvature

The sign conventions for the curvature of the beam are shown in Fig. 15.21.

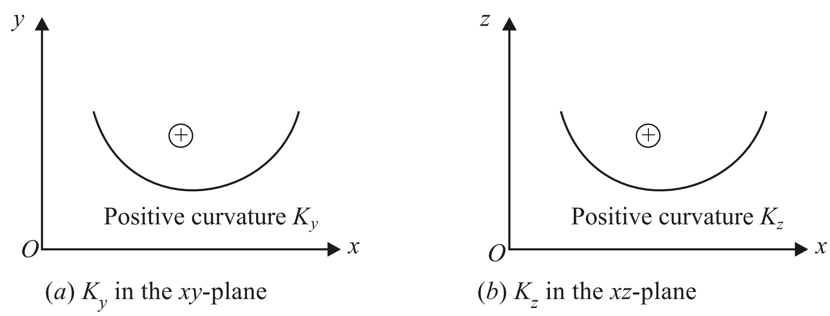


Fig. 15.21

Position of the neutral axis (NA)

The bending moment components M_y and M_z and the neutral axis (NA) are shown in Fig. 15.22. The bending moment components are expressed as

$$M_y = M \sin \theta \qquad ...(15.15)$$

and

$$M_z = M \cos \theta \qquad ...(15.16)$$

where θ is the angle between moment vector M and the z-axis, and β is the angle made by the neutral axis with the z-axis.

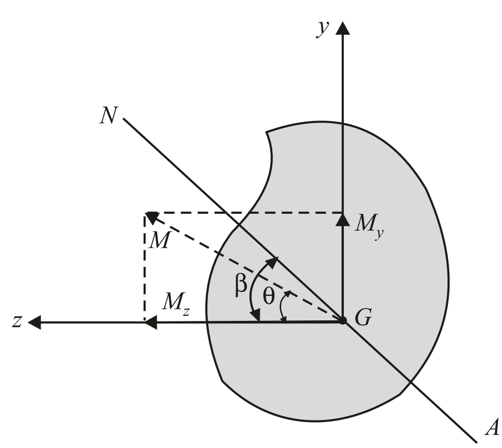


Fig. 15.22 Bending moment components M_y and M_z and the neutral axis (NA).

The superposition of the bending stresses produced by M_y and M_z acting separately gives the resultant stress at any point in the cross-section of the beam, given as

$$\sigma_x = \frac{M_y z}{I_y} - \frac{M_z y}{I_z} \quad \text{(using equation (15.6))}$$

$$= \frac{(M \sin\theta) z}{I_y} - \frac{(M \cos\theta) y}{I_z} \quad \text{...(15.17)}$$

where y and z are the coordinates of the point under consideration.

To know the position of the neutral axis, set $\sigma_x = 0$, and simplify.

$$\frac{\sin\theta}{I_y} z - \frac{\cos\theta}{I_z} y = 0$$

which gives

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{y}{z} \cdot \frac{I_y}{I_z}$$

or

$$\tan\theta = \tan\beta \frac{I_y}{I_z}$$

which gives

$$\tan\beta = \frac{I_z}{I_y} \tan\theta \quad \text{...(15.18)}$$

where

$$\tan\beta = \frac{y}{z}$$

Equation (15.18) clearly shows that two angles β and θ are not equal, which implies that the neutral axis is generally not perpendicular to the plane in which bending moment M acts.

15.5 DEFLECTION IN UNSYMMETRICAL BENDING

The deflections of unsymmetrical beams in the direction of the principal axes are determined using standard deflection formulae which are applicable for specified loading conditions. For a doubly symmetric beam with skew loads, the deflections are determined for each component of the load independently, and then the deflections are superimposed. For example, the deflection at the free end of a cantilever beam carrying a point load at its free end is given as

$$\delta = \frac{Wl^3}{3EI} \quad \text{...(15.19)}$$

where the symbols have their usual meanings. When using this condition for a skew-loaded cantilever beam as shown in Fig. 15.4, the deflections at the free end of the beam in the positive y direction and negative z direction are expressed as

$$\delta_y = \frac{(W \cos\theta) l^3}{3EI_z} \quad \text{...(15.20)}$$

and

$$\delta_z = \frac{(W \sin\theta) l^3}{3EI_y} \quad \text{...(15.21)}$$

where $W \cos \theta$ = Component of the load in positive y direction
$W \sin \theta$ = Component of the load in negative z direction
I_y = Moment of inertia of the beam's cross-sectional area about the principal y-axis
I_z = Moment of inertia of the beam's cross-sectional area about the principal z-axis
l = Length of the beam
E = Modulus of elasticity of the beam material

The total resultant deflection δ is then given by combining the two values δ_y and δ_z vectorially as

$$\delta = \sqrt{\delta_y^2 + \delta_z^2} \qquad ...(15.22)$$

The angle β between the resultant deflection and the y-axis is given as

$$\tan \beta = \frac{\delta_z}{\delta_y} = \frac{I_z}{I_y} \tan \theta \qquad ...(15.23)$$

Equation (15.23) is the same as equations (15.9) and (15.10). Hence, the resultant deflection lies in a plane that is perpendicular to the neural plane. This condition is shown in Fig. 15.23.

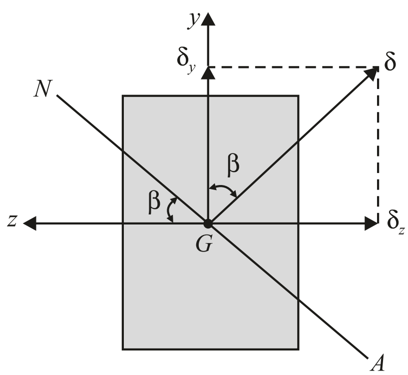


Fig. 15.23 Resultant deflection.

Example 15.5

A 1.5 m long cantilever beam having rectangular cross-section 60 mm × 90 mm is loaded at its free end with a point load of 7 kN at an angle 30° to the vertical as shown in Fig. 15.24. Determine (*a*) the position and magnitude of the greatest tensile stress in the cross-section and (*b*) the vertical deflection at the free end. Take E = 210 GPa.

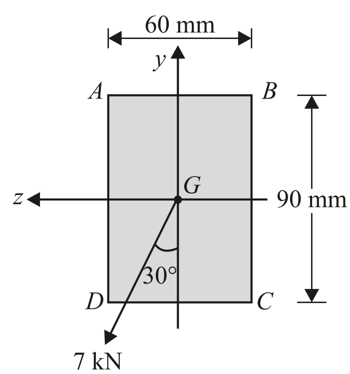


Fig. 15.24

Solution:

Given,

Load applied,	W	$= 7$ kN
Length of the beam,	l	$= 1.5$ m
Width of the beam,	b	$= 60$ mm
Depth of the beam,	d	$= 90$ mm
Modulus of elasticity,	E	$= 210$ GPa
		$= 210 \times 10^9$ Pa
Inclination of the load,	θ	$= 30°$

(*a*) **Maximum tensile stress**

It is the case of a doubly symmetric beam subjected to skew loading with its cross-section *ABCD* and centroid *G*, and the principal centroidal axes *y* and *z*. The load is resolved into its components parallel to the two major axis, that is, along the two principal centroidal axes, and then bending theory is applied simultaneously to both axes to find the bending stress on the cross-section.

The load of 7 kN, resulting moment M and its components M_y and M_z are shown in Fig. 15.25.

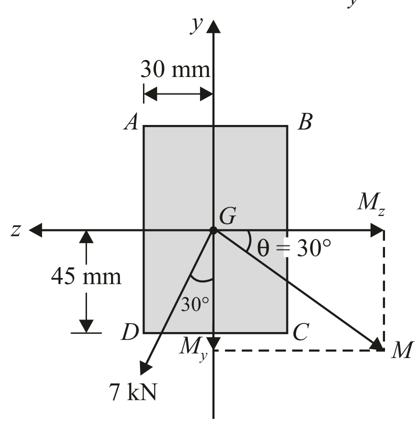


Fig. 15.25 Load and moment acting on the cross-section.

The load components are found as

$$W_y = 7 \cos 30° \qquad \text{(along negative } y \text{ direction)}$$
$$= 7 \times 0.866$$
$$= 6.062 \text{ kN}$$

and

$$W_z = 7 \sin 30° \qquad \text{(along positive } z \text{ direction)}$$
$$= 7 \times 0.5 = 3.5 \text{ kN}$$

The moment components are found as

$$M_y = M \sin \theta = W_z \times l \qquad \text{(along negative } y \text{ direction)}$$
$$= 3.5 \times 1.5$$
$$= 5.25 \text{ kN.m}$$

and $$M_z = M\cos\theta = W_y \times l \quad \text{(along negative } z \text{ direction)}$$

$$= 6.062 \times 1.5$$

$$= 9.093 \text{ kN.m}$$

The values of M_y and M_z are the maximum bending moments that occur at the fixed end of the beam.

The distances from the neutral axis are calculated as

$$z = 30 \text{ mm} = 30 \times 10^{-3} \text{ m}$$

$$y = 45 \text{ mm} = 45 \times 10^{-3} \text{ m}$$

The moments of inertia of the beam's cross-section about y and z axes are found as

$$I_y = \frac{1}{12} \times 90 \times 60^3 \times 10^{-12} \text{ m}^4$$

$$= 1.62 \times 10^{-6} \text{ m}^4$$

$$I_z = \frac{1}{12} \times 60 \times 90^3 \times 10^{-12} \text{ m}^4$$

$$= 3.645 \times 10^{-6} \text{ m}^4$$

The bending moment M_z produces tensile stress on AB and compressive stress on CD. Both stresses are equal, and their value is

$$\sigma_1 = \frac{M_z}{I_z} \cdot y = \frac{9.093 \times 45 \times 10^{-3}}{3.645 \times 10^{-6}}$$

$$= 112259.26 \text{ kN/m}^2 = 112.26 \text{ MPa}$$

The bending moment M_y produces tensile stress on BC and compressive stress on AD. Both stresses are equal, and their value is

$$\sigma_2 = \frac{M_y}{I_y} \cdot z = \frac{5.25 \times 30 \times 10^{-3}}{1.62 \times 10^{-6}}$$

$$= 97222.22 \text{ kN/m}^2 = 97.22 \text{ MPa}$$

Hence, the maximum tensile stress due to combined loading occurs at point B, where the two tensile stresses add, given by

Maximum tensile stress, $\sigma_{max} = \sigma_1 + \sigma_2$

$$= (112.26 + 97.22) \text{ MPa}$$

$$= 209.48 \text{ MPa}$$ **Ans.**

The compressive stress has the same value as tensile stress, and occurs at D.

(*b*) **Vertical deflection**

The negative deflection at the free end of a cantilever beam is given by

$$\delta = \frac{Wl^3}{3EI} \qquad \text{(using equation (15.19))}$$

In our case, the vertical deflection δ_v that occurs along y-axis, is given as

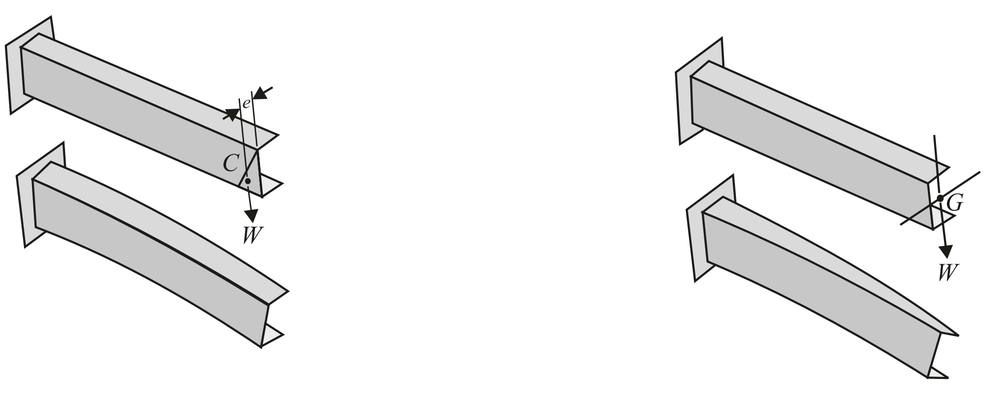
$$\delta_v = \frac{W_y l^3}{3EI_z} = \frac{6.062\times10^3\times(1.5)^3}{3\times210\times10^9\times3.645\times10^{-6}}\ \text{m}$$

$$= 8.9 \times 10^{-3}\ \text{m}$$

$$= 8.9\ \text{mm}$$ **Ans.**

15.6 SHEAR CENTRE

Lateral or transverse loads acting on a beam produce both shear forces and bending moments. The case of pure bending of beams is discussed in Chapter-5, where beams are subjected to only bending moments and the shear stress consideration is neglected. Also in the same chapter in section 5.7, case of shear stress in beams is discussed. The applied loads in the latter case act in a plane of symmetry. The concept of shear centre, also called flexural centre, is based on the effect of shear force on the cross-section of a beam without producing twisting, thus eliminating the restriction of pure bending. For pure bending theory to be applicable, the shear force must pass through the shear centre. It is possible only when the plane of external loading passes through the shear centre, and in that case no twisting takes place as shown in Fig. 15.26 (*a*). If the load does not act at the shear centre or acts at other points, then the bending will be accompanied by twisting, that is, the section of the beam will get twisted about the shear centre as shown in Fig. 15.26 (*b*). Because of this reason, the shear centre is sometimes also called centre of twist. Therefore, the shear centre is defined as a point in the cross-section of a beam, where if a lateral load is applied, it will bend the beam without twisting along the longitudinal axis.



(*a*) Bending of channel section without twisting, when the load W acts through the shear centre C.

(*b*) Bending of channel section with twisting, when the load W does not act at C but at G, the centroid of the section.

Fig. 15.26

The shear centre generally does not coincide with the centroid of the cross-section of a beam, but like the centroid, it lies on any axis of symmetry of the cross-section. It lies near to the centroid of the section and thus the section has high torsional rigidity, and so the effect of twisting can be safely neglected by applying the load at or near the centroid. Fig. 15.27 shows the cross-section of a singly symmetric beam, for example, an unequal flange *I*-section beam, which is symmetrical about *y*-axis. Both the centroid *G* and the shear centre *C* of the cross-section lie on *y*-axis, which is the axis of symmetry.

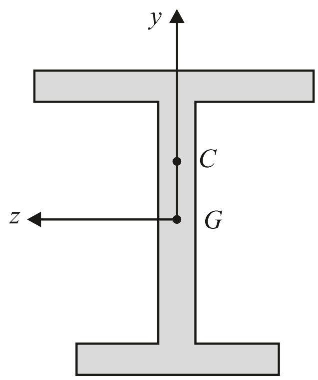


Fig. 15.27 Shear centre for an unequal I-section.

For a section which is symmetrical about two axes, the shear centre coincides with the centroid of the section Fig. 15.28 shows a doubly symmetric beam, for example, an equal flange I-section beam, which is symmetrical about both *y* and *z* axes. Both the shear centre *C* and the centroid *G* lie at the same point, which is the intersection point of the two axes of symmetry.

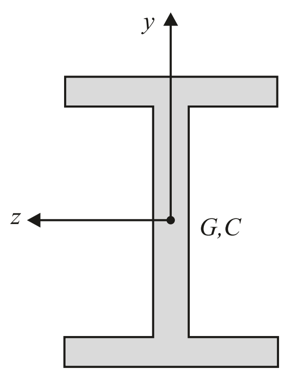


Fig. 15.28 Shear centre for an equal I-section.

For beams having unsymmetric cross-section or no axis of symmetry, such as the *Z*-section and the angle section, the location of the shear centre is more difficult.

The position of the shear centre is determined by using a term called eccentricity (e), which defines the distance by which the load line positions itself to act at the shear centre of the cross-section.

The shear stress τ at any point of the cross-section is obtained by using equation (5.19), given as

$$\tau = \frac{VQ}{It} = \frac{V}{It}A\overline{y} \qquad ...(15.24)$$

where $V =$ Vertical shear force or the load applied

$I =$ Moment of inertia of the section about the neutral axis

$t =$ Thickness of the section

$Q = A\bar{y} =$ First moment of area about the neutral axis

$A =$ Area of the section above the point, where the shear stress is τ

$\bar{y} =$ Distance of the centroid of the area from the neutral axis

The shear stresses are directed along the median line of the cross-section, parallel to the edges of the section, and are assumed to be of constant intensity across the thickness t of the section.

15.6.1 Shear Centre for a Channel Section

Consider a channel section having equal flanges with dimensions as shown in Fig. 15.29.

Let the section be subjected to a shear force V acting vertically downward and parallel to the web with an eccentricity e. C is the position of the shear centre. The neutral axis (NA) is the horizontal axis of symmetry of the section.

Area of the two equal flanges, $A_f = bt$

Area of the web, $A_w = \left(h - 2 \times \frac{t}{2}\right) w \approx hw$ (as t is very small)

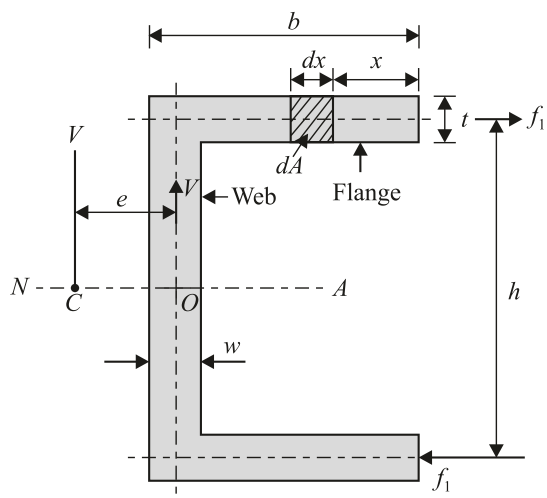


Fig. 15.29

The total shear force carried by the web must be equal to the applied vertical shear force V, thereby producing two equal and opposite shear forces in the flanges, say f_1. Considering their equilibrium and taking moments of the forces about O, we have

$$V \times e = f_1 \times \frac{h}{2} + f_1 \times \frac{h}{2} = f_1 \times h$$

which gives

$$e = \frac{f_1 \times h}{V} \qquad \text{...(15.25)}$$

The position of the shear centre is defined by e.

An elementary area dA is considered at a distance x from the right side in the top flange.

The shear stress in the top flange is given as

$$\tau = \frac{V}{It} A\bar{y} \qquad \text{(using equation (15.24))...(15.26)}$$

$$\left(A = t\,x \text{ and } \bar{y} = \frac{h}{2}\right)$$

$$= \frac{V}{It} \cdot t\,x \cdot \frac{h}{2}$$

$$= \frac{Vxh}{2I} \qquad \text{...(15.27)}$$

The shear force in the top flange is given as

$$f_1 = \tau.dA$$

$$= \int_0^b \tau \cdot (t\,dx) \qquad \text{...(15.28)}$$

where $dA = t\,dx =$ Elementary area at x

$b =$ Width of the flange

or $$f_1 = \int_0^b \frac{Vxh}{2I} tdx \qquad \text{(using equation (15.27)}$$

$$= \left(\frac{Vht}{2I}\right) \int_0^b xdx$$

$$= \frac{Vhb^2t}{4I} \qquad \text{...(15.29)}$$

An equal but opposite shear force is produced in the bottom flange.

Now $$I = \frac{wh^3}{12} + 2\left[\frac{1}{12}bt^3 + b \times t \times \left(\frac{h}{2}\right)^2\right]$$

$$= \frac{wh^3}{12} + \frac{1}{6}bt^3 + 2bt\left(\frac{h}{2}\right)^2$$

$$= \frac{wh^3}{12} + 2bt\left(\frac{h}{2}\right)^2$$

The middle term on account of its negligible value in comparison to other terms is neglected.

or $$I = \frac{1}{2}th^2b\left[1 + \frac{1}{6}\left(\frac{wh}{tb}\right)\right]$$

$$= \frac{1}{2}th^2b\left[1 + \frac{1}{6}\left(\frac{A_w}{A_f}\right)\right] \qquad \text{...(15.30)}$$

The shear force either in the top or bottom flange, after substituting equation (15.30) in equation (15.29), is expressed as

$$f_1 = \frac{Vhb^2t}{4 \times \frac{1}{2}th^2b\left[1+\frac{1}{6}\left(\frac{A_w}{A_f}\right)\right]}$$

$$= \frac{Vb}{2h}\left[\frac{1}{1+\frac{1}{6}\left(\frac{A_w}{A_f}\right)}\right] \qquad \text{...(15.31)}$$

From equation (15.25), we have

$$e = \frac{Vb}{2h}\left[\frac{1}{1+\frac{1}{6}\left(\frac{A_w}{A_f}\right)}\right]\times\frac{h}{V} \qquad \text{(on substituting } f_1 \text{ from equation (15.31))}$$

$$= \frac{b}{2}\left[\frac{1}{1+\frac{1}{6}\left(\frac{A_w}{A_f}\right)}\right] \qquad \text{...(15.32)}$$

This is the required expression for eccentricity.

Example 15.6

Find the shear centre for the section shown in Fig. 15.29, if b = 150 mm, t = 20 mm, w = 15 mm, h = 250 mm and $V = 2 \times 10^3$ N.

Solution:

Area of the web, $A_w = hw$

$= 250 \times 15$

$= 3750 \text{ mm}^2$

Area of the flange, $A_f = bt$

$= 150 \times 20 = 3000 \text{ mm}^2$

Using equation (15.32), we have

$$e = \frac{b}{2}\left[\frac{1}{1+\frac{1}{6}\left(\frac{A_w}{A_f}\right)}\right]$$

$$= \frac{150}{2}\left[\frac{1}{1+\frac{1}{6}\left(\frac{3750}{3000}\right)}\right] = 62 \text{ mm}$$ **Ans.**

Example 15.7

Find the position of the shear centre of the section shown in Figure 15.30.

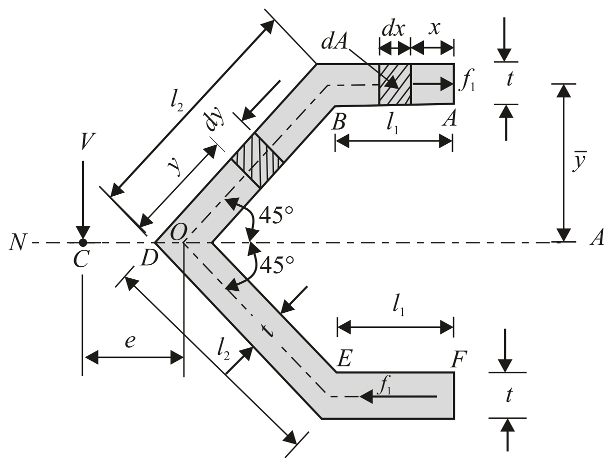


Fig. 15.30

Solution:

C is the position of the shear centre, where the shear force *V* acts vertically downward. *NA* is the neutral axis.

An elementary area *dA* is considered at a distance *x* from the right side in the top flange.

The shear forces in the two parts *AB* and *EF* are equal but opposite, say f_1. The shear forces in the parts *BD* and *DE* are not required to be calculated as they are passing through *O*, the point about which moments of forces are finally considered.

Now $$f_1 = \int \tau \, . \, dA$$

$$= \int_0^{l_1} \tau \, . \, t dx$$

$$= \int_0^{l_1}\left(\frac{V}{It}A\bar{y}\right)\cdot tdx \qquad \text{(using equation (15.24))...(15.33)}$$

Here Area, $A = x.\mathrm{t}$

$$\bar{y} = \frac{l_2}{\sqrt{2}}$$

$$I = 2 \times t \times l_1 \times (l_2 \sin 45°)^2 + 2\int_0^{l_2}(t\cdot dy)(y\sin 45°)^2$$

$$= tl_1 l_2^2 + t\int_0^{l_2} y^2 dy$$

$$= tl_1 l_2^2 + \frac{l_2^3}{3}t$$

$$= \frac{tl_2^2}{3}(3l_1 + l_2)$$

On substituting A, $\bar{y}$ and I in equation (15.33), we have

$$f_1 = \int_0^{l_1}\frac{V}{t}\cdot\frac{3}{tl_2^2(3l_1+l_2)}\cdot xt\cdot\frac{l_2}{\sqrt{2}}\cdot tdx$$

$$= \frac{3V}{\sqrt{2}l_2(3l_1+l_2)}\int_0^{l_1} xdx$$

$$= \frac{3V}{\sqrt{2}l_2(3l_1+l_2)}\left(\frac{x^2}{2}\right)_0^{l_1}$$

$$= \frac{3l_1^2V}{2\sqrt{2}l_2(3l_1+l_2)}$$

Taking moments of the shear forces about O, we have

$$V \times e = 2\times f_1\times\bar{y}$$

$$2\times f_1\times\frac{l_2}{\sqrt{2}}$$

$$= 2\times\frac{3l_1^2V}{2\sqrt{2}l_2(3l_1+l_2)}\times\frac{l_2}{\sqrt{2}}$$

$$= \frac{3l_1^2V}{2(3l_1+l_2)}$$

or $$e = \frac{3l_1^2}{2(3l_1 + l_2)} \quad ...(15.34)$$

This is the required expression for eccentricity.

Example 15.8

In Example 15.7, if l_1 = 150 mm, l_2 = 450 mm and V = 15 kN, find the value of e.

Solution:

Using equation (15.34), we have

$$e = \frac{3l_1^2}{2(3l_1 + l_2)}$$

$$= \frac{3 \times 150^2}{2(3 \times 150 + 450)}$$

$$= 37.5 \text{ mm}$$ **Ans.**

15.6.2 Shear Centre for an Equal-leg Angle Section

Consider an equal-leg angle section of uniform thickness as shown in Fig. 15.31. C is the point, where the vertical shear force V acts.

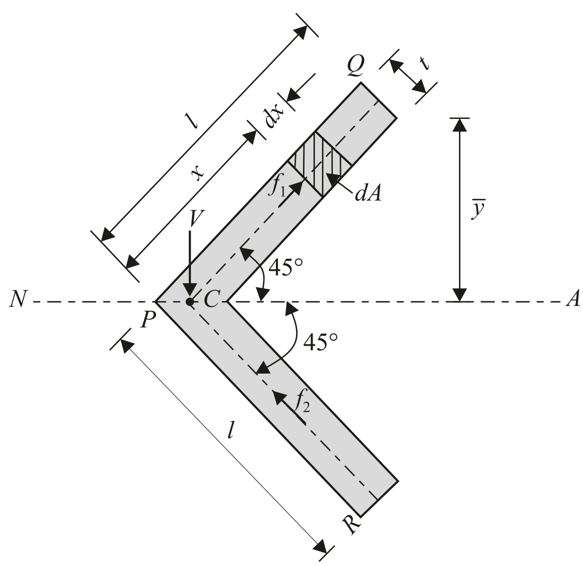


Fig. 15.31

An elementary area dA is considered at a distance x as shown in figure.

$$dA = t\,dx$$

Let V be the vertical shear force applied at the point C. f_1 and f_2 are the shear forces induced in the parts PQ and PR respectively. The shear stress developed in the section is τ.

Now

$$f_1 = \int \tau \, . \, dA$$

$$= \int \left(\frac{V}{It} A\bar{y} \right) . \, dA \qquad \text{(using equation (15.24))}$$

$$= \int_0^l \frac{V}{It} \cdot (l-x)t \cdot \left[\left\{ l - \frac{(l-x)}{2} \right\} \sin 45° \right] \cdot t dx$$

$$= \frac{Vt}{I} \int_0^l (l-x) \cdot \frac{(l+x)}{2} \cdot \frac{1}{\sqrt{2}} \cdot dx$$

$$= \frac{Vt}{2\sqrt{2}I} \int_0^l (l^2 - x^2) dx$$

$$= \frac{Vt}{2\sqrt{2}I} \left(l^2 x - \frac{x^3}{3} \right)_0^l$$

$$= \frac{Vt}{2\sqrt{2}I} \left(l^3 - \frac{l^3}{3} \right)$$

$$= \frac{Vt}{2\sqrt{2}I} \cdot \frac{2}{3} l^3$$

$$= \frac{Vtl^3}{3\sqrt{2}I} \qquad \text{... (15.35)}$$

The moment of inertia I is calculated as

$$I = 2 \int_0^l t \cdot dx \cdot (x \sin 45°)^2$$

$$= 2 \int_0^l t \cdot dx \cdot \frac{x^2}{2}$$

$$= t \int_0^l x^2 dx$$

$$= t \left(\frac{x^3}{3} \right)_0^l$$

$$= \frac{tl^3}{3}$$

Substituting I in equation (15.35), we have

$$f_1 = \frac{Vtl^3}{3\sqrt{2}} \times \frac{3}{tl^3} = \frac{V}{\sqrt{2}}$$

Similarly, $$f_2 = \frac{V}{\sqrt{2}}$$ (because of the symmetry of the section)

The resultant of f_1 and f_2
$$= \sqrt{f_1^2 + f_2^2}$$
$$= \sqrt{\left(\frac{V}{\sqrt{2}}\right)^2 + \left(\frac{V}{\sqrt{2}}\right)^2}$$
$$= V$$
= Applied vertical shear force at C

Hence, C is the position of the shear centre.

Example 15.9

Find the position of the shear centre of the section of a beam shown in Fig. 15.32.

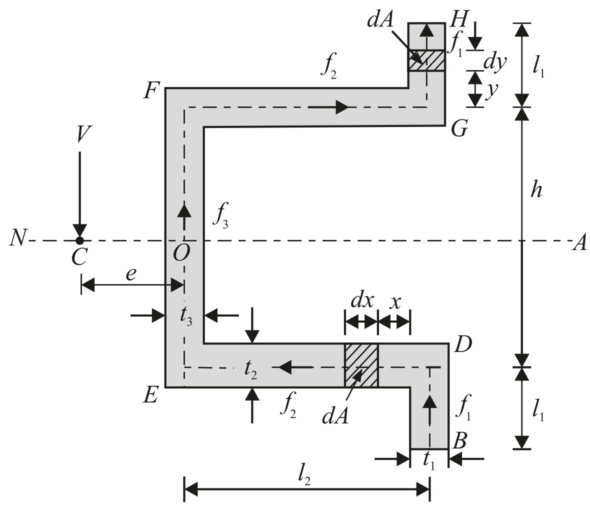


Fig. 15.32

Solution:

Let V be the applied vertical shear force acting at C. The shear stresses developed in each part of the section are shown in the figure. The shear stresses developed in the parts BD and GH are equal (say f_1) and in the parts DE and FG are equal (say f_2). f_3 is the shear force developed in the part EF. The respective thicknesses of the part BD (and GH), DE (and FG) and EF are t_1, t_2 and t_3 respectively. The shear force f_1 is expressed as

$$f_1 = \int \tau \, dA$$

$$= \int_0^{l_1} \frac{V}{It_1} A\bar{y} \cdot t_1 dy \qquad (dA = t_1.dy)$$

$$= \int_0^{l_1} \frac{V}{It_1} \cdot (l_1 - y)t_1 \cdot \left(\frac{h}{2} + y + \frac{l_1 - y}{2}\right) \cdot t_1 dy$$

$$= \frac{Vt_1}{I} \int_0^{l_1} (l_1 - y)\left(\frac{h}{2} + y + \frac{l_1 - y}{2}\right) dy$$

$$= \frac{Vt_1}{I} \int_0^{l_1} \left(\frac{l_1 h}{2} + l_1 y + \frac{l_1^2}{2} - \frac{l_1 y}{2} - \frac{hy}{2} - y^2 - \frac{l_1 y}{2} + \frac{y^2}{2}\right) dy$$

$$= \frac{Vt_1}{I} \left(\frac{l_1 hy}{2} + \frac{l_1 y^2}{2} + \frac{l_1^2 y}{2} - \frac{l_1 y^2}{4} - \frac{hy^2}{4} - \frac{y^3}{3} - \frac{l_1 y^2}{4} + \frac{y^3}{6}\right)_0^{l_1}$$

$$= \frac{Vt_1}{I} \left(\frac{l_1^2 h}{2} + \frac{l_1^3}{2} + \frac{l_1^3}{2} - \frac{l_1^3}{4} - \frac{hl_1^2}{3} - \frac{l_1^3}{3} - \frac{l_1^3}{4} + \frac{l_1^3}{6}\right)$$

$$= \frac{Vt_1}{I} \left(\frac{l_1^2 h}{4} + \frac{l_1^3}{3}\right) = \frac{Vt_1 l_1^2}{12I}(3h + 4l_1)$$

The shear force f_2 is found as

$$f_2 = \int \tau \, dA = \int_0^{l_2} \frac{V}{It_2} A\bar{y} \cdot t_2 dx$$

$$= \int_0^{l_2} \frac{V}{It_2} \times \left[l_1 . t_1 \left(\frac{h}{2} + \frac{l_1}{2}\right) + t_2 \cdot x \cdot \frac{h}{2}\right] t_2 dx$$

$$= \int_0^{l_2} \frac{V}{I} \left(\frac{l_1 t_1 h}{2} + \frac{l_1^2 t_1}{2} + \frac{t_2 x h}{2}\right) dx$$

$$= \frac{V}{I} \left(\frac{l_1 l_2 t_1 h}{2} + \frac{l_1^2 l_2 t_1}{2} + \frac{l_2^2 t_2 h}{4}\right) = \frac{V}{I} \left(\frac{l_1 l_2 t_1 h}{2} + \frac{l_1^2 l_2 t_1}{2} + \frac{l_2^2 t_2 h}{4}\right)$$

The shear force component f_3 is not required to be calculated as it is passing through the point O, the centre of moment.

Taking moments of the shear forces about O, we have

$$V \times e + 2 f_1 \times l_2 = f_2 \times \frac{h}{2} + f_2 \times \frac{h}{2} = f_2 \times h$$

or

$$V \times e = f_2 \times h - 2 f_1 \times l_2$$

$$= \frac{Vh}{I} \left(\frac{l_1 l_2 t_1 h}{2} + \frac{l_1^2 l_2 t_1}{2} + \frac{l_2^2 t_2 h}{4}\right) - \frac{Vt_1 l_1^2 l_2}{6I}(3h + 4l_1) \quad \text{(on substituting } f_1 \text{ and } f_2\text{)}$$

which gives
$$e = \frac{l_1 l_2 t_1 h^2}{2I} + \frac{l_2^2 t_1 h}{2I} + \frac{l_2^2 t_2 h^2}{4I} - \frac{l_1^2 l_2 t_1 h}{2I} - \frac{2 l_1^3 l_2 t_1}{3I}$$
$$= \frac{l_1 l_2 t_1}{I}\left(\frac{h^2}{2} - \frac{2 l_1^2}{3}\right) + \frac{t_2 l_2^2 h^2}{4I}$$

The moment of inertia of the section is found as
$$I = 2\times\frac{1}{12}\times t_1\times l_1^3 + 2\times l_1\times t_1\times\left(\frac{h}{2}+\frac{l_1}{2}\right)^2$$
$$+\ 2\times\frac{1}{12}\times l_2\times t_2^3 + 2\times l_2\times t_2\times\left(\frac{h}{2}\right)^2 + \frac{1}{12}\times t_3\times h^3$$

On substituting I in the equation of e, we find e in terms of l_1, l_2, t_1, t_2, t_3 and h.

Example 15.10

In Example 15.9, if $l_1 = 30$ mm, $l_2 = 90$ mm, $t_1 = 10$ mm, $t_2 = 8$ mm, $t_3 = 5$ mm and $h = 150$ mm, find the position of the shear centre.

Solution:

The moment of inertia of the section is found as
$$I = 2\times\frac{1}{12}\times 10\times 30^3 + 2\times 30\times 10\times\left(\frac{150}{2}+\frac{30}{2}\right)^2$$
$$+\ 2\times\frac{1}{12}\times 90\times 8^3 + 2\times 90\times 8\times\left(\frac{150}{2}\right)^2 + \frac{1}{12}\times 5\times 150^3$$
$$= (45000 + 4860000 + 7680 + 8100000 + 1406250)\ \text{mm}^4$$
$$= 14418930\ \text{mm}^4$$

The eccentricity e is given as
$$e = \frac{l_1 l_2 t_1}{I}\left(\frac{h^2}{2} - \frac{2 l_1^2}{3}\right) + \frac{t_2 l_2^2 h^2}{4I}$$
$$= \frac{30\times 90\times 10}{14418930}\left(\frac{150^2}{2} - \frac{2\times 30^2}{3}\right) + \frac{8\times 90^2\times 150^2}{4\times 14418930}$$
$$= 19.94 + 25.28$$
$$= 45.22\ \text{mm}$$

Hence, the shear centre is located on the neutral axis at a distance of 45.22 mm from the point O (towards left). **Ans.**

Example 15.11

Find the position of the shear centre for a section of uniform thickness as shown in Fig. 15.33.

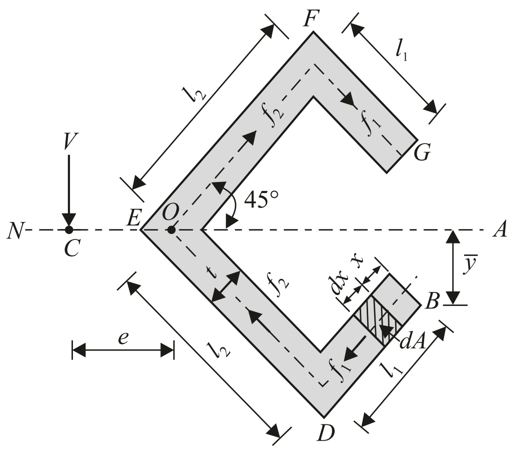


Fig. 15.33

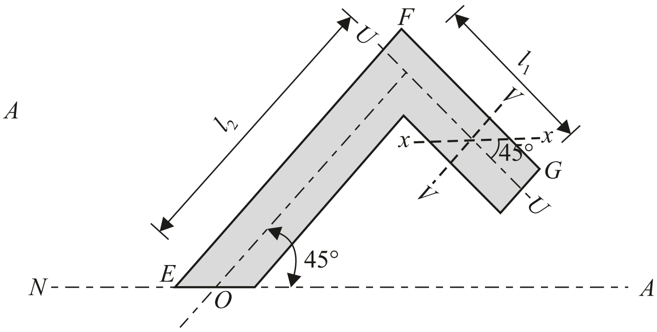


Fig. 15.34

Solution:

The applied vertical shear force is V acting at the shear centre C. The shear forces in the parts BD and FG are equal (say f_1) and in the parts DE and EF are equal (say f_2). The uniform thickness of the section is t.

The shear force f_1 is found as

$$f_1 = \int \tau \, dA$$

$$= \int_0^{l_1} \frac{V}{It} \cdot A\bar{y} \cdot t dx \qquad (dA = t dx)$$

$$= \int_0^{l_1} \frac{V}{It} (tx) \bar{y} \cdot t dx \qquad (A = tx) \ldots(1)$$

where

$$\bar{y} = l_2 \sin 45° - l_1 \sin 45° + \frac{x}{2} \sin 45°$$

$$= \frac{l_2}{\sqrt{2}} - \frac{l_1}{\sqrt{2}} + \frac{x}{2\sqrt{2}}$$

$$= \frac{2l_2 - 2l_1 + x}{2\sqrt{2}}$$

On substituting $\bar{y}$ in equation (1), we have

$$f_1 = \int_0^{l_1} \frac{V}{It} (tx) \cdot \left(\frac{2l_2 - 2l_1 + x}{2\sqrt{2}} \right) \cdot t dx$$

$$= \frac{Vt}{2\sqrt{2}I}\int_0^{l_1}(2l_2x - 2l_1x + x^2)dx$$

$$= \frac{Vt}{2\sqrt{2}I}\left(2l_2\cdot\frac{x^2}{2} - 2l_1\cdot\frac{x^2}{2} + \frac{x^3}{3}\right)_0^{l_1}$$

$$= \frac{Vt}{2\sqrt{2}I}\left(l_2x^2 - l_1x^2 + \frac{x^3}{3}\right)_0^{l_1}$$

$$= \frac{Vt}{2\sqrt{2}I}\left(l_2l_1^2 - l_1^3 + \frac{l_1^3}{3}\right)$$

$$= \frac{Vt}{2\sqrt{2}I}\left(\frac{3l_2l_1^2 - 3l_1^3 + l_1^3}{3}\right)$$

$$= \frac{Vt}{6\sqrt{2}I}(3l_2l_1^2 - 2l_1^3) \quad = \frac{Vtl_1^2}{6\sqrt{2}I}(3l_2 - 2l_1)$$

The shear force f_2 is not required to be determined as it passes through the point O, the centre of moment.

Calculation of moment of inertia

Consider the top half part of the section as shown in Fig. 15.34.

The moments of inertia of the parts *BD* and *FG* are first calculated about the axes *UU* (the horizontal centroial axis) and *VV* (the vertical centroial axis) as

$$I_{uu} = \frac{1}{12}\times l_1\times t^3$$

$$I_{vv} = \frac{1}{12}\times t\times l_1^3$$

Now

$$I_{xx} = I_{uu}\cos^2 45° + I_{vv}\sin^2 45°$$

$$= \frac{1}{12}\times l_1\times t^3\times\frac{1}{2} + \frac{1}{12}\times t\times l_1^3\times\frac{1}{2}$$

$$= \frac{l_1t^3}{24} + \frac{tl_1^3}{24}$$

$$= \frac{l_1t}{24}(t^2 + l_1^2)$$

Now the moment of inertia of the part *FG* about the neutral axis (*NA*) is found as

$$I_1 = I_{xx} + l_1\, t\left(l_2\sin 45° - \frac{l_1}{2}\sin 45°\right)^2 \qquad \text{(using parallel-axes theorem)}$$

$$= \frac{l_1 t}{24}(t^2 + l_1^2) + l_1 t\left(\frac{l_2}{\sqrt{2}} - \frac{l_1}{2\sqrt{2}}\right)^2$$

$$= \frac{l_1 t}{24}(t^2 + l_1^2) + \frac{l_1 t}{8}(2l_2 - l_1)^2$$

$$= \frac{l_1 t}{24}(t^2 + l_1^2) + \frac{l_1 t}{8}(4l_2^2 + l_1^2 - 4l_1 l_2)$$

$$= \frac{l_1 t^3 + l_1^3 t + t + 12 l_1 l_2^2 t + 3 l_1^3 t - 12 l_1^2 l_2 t}{24}$$

$$= \frac{l_1 t^3 + 4 l_1^3 t + 12 l_1 l_2^2 t - 12 l_1^2 l_2 t}{24}$$

$$= \frac{l_1 t}{24}(t^2 + 4l_1^2 + 12 l_2^2 - 12 l_1 l_2)$$

The moment of inertia of the part *BD* about the neutral axis (*NA*) is also equal to I_1.

Similarly, the moments of inertia of the parts *DE* and *EF* are first calculated about their centroidal axes *U′U′* and *V′V′* as

$$I_{u'u'} = \frac{1}{12} \times l_2 \times t^3$$

$$I_{v'v'} = \frac{1}{12} \times t \times l_2^3$$

Now $$I_{x'x'} = I_{u'u'} \cos^2 45° + I_{v'v'} \sin^2 45°$$

$$= \frac{1}{12} \times l_2 \times t^3 \times \frac{1}{2} + \frac{1}{12} \times t \times l_2^3 \times \frac{1}{2}$$

$$= \frac{l_2 t^3}{24} + \frac{t l_2^3}{24}$$

$$= \frac{l_2 t}{24}(t^2 + l_2^2)$$

Now the moment of inertia of the part *EF* about the neutral axis (*NA*) is found as

$$I_2 = I_{x'x'} + l_2 t\left(\frac{l_2 \sin 45°}{2}\right)^2 \qquad \text{(using parallel-axes theorem)}$$

$$= \frac{l_2 t}{24}(t^2 + l_2^2) + \frac{l_2 t \times l_2^2}{8}$$

$$= \frac{l_2 t^3 + t l_2^3 + 3 t l_2^3}{24}$$

$$= \frac{l_2 t^3 + 4tl_2^3}{24}$$

$$= \frac{l_2 t}{24}(t^2 + 4l_2^2)$$

The moment of inertia of the part *DE* about the neutral axis (*NA*) is also equal to I_2.
Now the moment of inertia of the whole section about the neutral axis (*NA*) is given as

$$I = 2I_1 + 2I_2$$

$$= \frac{l_1 t}{12}(t^2 + 4l_1^2 + 12l_2^2 - 12l_1 l_2) + \frac{l_2 t}{12}(t^2 + 4l_2^2) \quad \text{... (2)}$$

Taking moments of the shear forces about *O*, we have

$$V \times e = f_1 \times l_2 + f_1 \times l_2$$

$$= 2f_1 \times l_2 = 2 \times \frac{Vtl_1^2}{6\sqrt{2}I}(3l_2 - 2l_1) \times l_2$$

which gives $$e = \frac{tl_1^2 l_2}{3\sqrt{2}I}(3l_2 - 2l_1)$$

After substituting *I* from equation (2), we can find the value of *e* in terms of t_1, l_1 and l_2.

Example 15.12

In Example 15.11, if l_1 = 30 mm l_2 = 45 mm and t = 3 mm, find the position of the shear centre of the section.

Solution:

The moment of inertia of the section is found is

$$I = \frac{l_1 t}{12}(t^2 + 4l_1^2 + 12l_2^2 - 12l_1 l_2) + \frac{l_2 t}{12}(t^2 + 4l_2^2)$$

$$= \frac{30 \times 3}{12}(3^2 + 4 \times 30^2 + 12 \times 45^2 - 12 \times 30 \times 45) + \frac{45 \times 3}{12}(3^2 + 4 \times 45^2)$$

$$= 87817.5 + 91226.25 = 179043.75 \text{ mm}^4$$

The eccentricity *e* is given as

$$e = \frac{tl_1^2 l_2}{3\sqrt{2}I}(3l_2 - 2l_1) = \frac{3 \times 30^2 \times 45}{3\sqrt{2} \times 179043.75} \times (3 \times 45 - 2 \times 30) = 12 \text{ mm}$$

Hence, the shear centre is located on the neutral axis at a distance of 12 mm from the point *O* (towards left). **Ans.**

Example 15.13

Find the shear centre of the section of uniform thickness as shown in Fig. 15.35.

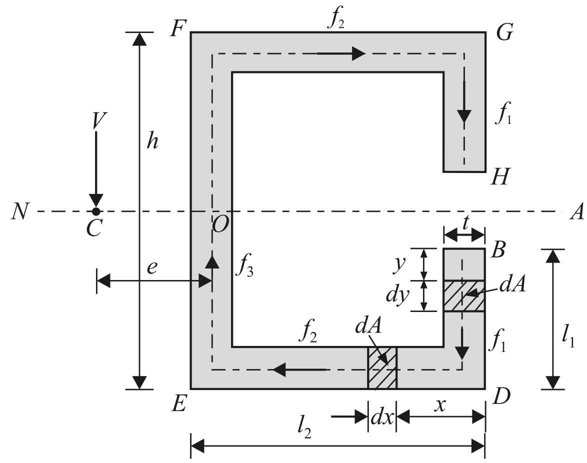


Fig. 15.35

Solution:

The vertical shear force V is applied at the point C, the shear centre, and t is the uniform thickness of the section. Equal shear forces are developed (say f_1) in the parts BD and GH. Similarly, the parts DE and FG have equal shear forces (say f_2). f_3 is the shear force developed in the part EF. e is the eccentricity and NA is the neutral axis.

The shear force f_1 is found as

$$f_1 = \int \tau \, dA$$

$$= \int_0^{l_1} \frac{V}{It} A\bar{y} \,.\, t dy \qquad (dA = t\,dy)$$

$$= \int_0^{l_1} \frac{V}{It} \cdot yt \cdot \left(\frac{h}{2} - l_1 + \frac{y}{2}\right) \cdot t dy \qquad (A = yt)$$

$$= \frac{Vt}{I}\left(\frac{h}{2}y - l_1 y + \frac{y^2}{2}\right) dy = \frac{Vt}{I}\left(\frac{hy^2}{4} - \frac{l_1 y^2}{2} + \frac{y^3}{6}\right)_0^{l_1}$$

$$= \frac{Vt}{I}\left(\frac{hl_1^2}{4} - \frac{l_1^3}{2} + \frac{l_1^3}{6}\right) = \frac{Vt}{I}\left(\frac{3hl_1^2 - 6l_1^3 + 2l_1^3}{12}\right)$$

$$= \frac{Vt}{I}\left(\frac{3hl_1^2 - 4l_1^3}{12}\right)$$

$$= \frac{Vtl_1^2}{I}\left(\frac{3h - 4l_1}{12}\right)$$

The shear force f_2 is found as

$$f_2 = \int \tau \, dA$$

$$= \int \frac{V}{It} A\bar{y} \cdot dA$$

$$= \int_0^{l_2} \frac{V}{It}\left[xt\left(\frac{h}{2}\right) + l_1 \cdot t\left(\frac{h}{2} - \frac{l_1}{2}\right)\right] \cdot t dx \qquad (A = xt \text{ and } dA = t\,dx)$$

$$= \frac{Vt}{I}\int_0^{l_2}\left(\frac{xh}{2} + \frac{l_1 h}{2} - \frac{l_1^2}{2}\right) dx = \frac{Vt}{I}\left(\frac{x^2 h}{4} + \frac{l_1 h x}{2} - \frac{l_1^2 x}{2}\right)_0^{l_2}$$

$$= \frac{Vt}{I}\left(\frac{l_2^2 h}{4} + \frac{l_1 l_2 h}{2} - \frac{l_1^2 l_2}{2}\right)$$

The shear force f_3 is not required to be determined as it passes through O, the point about which moment is calculated.

The moment of inertia of the section about the neutral axis (NA) is obtained as

$$I = 2\left[\frac{1}{12} \times t \times l_1^3 + l_1 \times t \times \left(\frac{h}{2} - \frac{l_1}{2}\right)^2\right]$$

$$+ 2 \times \left[\frac{1}{12} \times l_2 \times t^3 + l_2 \times t \times \left(\frac{h}{2}\right)^2\right] + \frac{1}{12} \times t \times h^3$$

$$= \frac{tl_1^3}{6} + \frac{l_1 th^2}{2} - l_1^2 th + \frac{l_1^3 t}{2} + \frac{l_2 t^3}{6} + \frac{l_2 th^2}{2} + \frac{th^3}{12}$$

$$= \frac{2}{3} tl_1^3 + \frac{l_1 th^2}{2} - l_1^2 th + \frac{l_2 t^3}{6} + \frac{l_2 th^2}{2} + \frac{th^3}{12}$$

Taking moments of the shear forces about the point O, we have

$$V \times e = 2 \times f_1 \times l_2 + 2 \times f_2 \times \frac{h}{2}$$

$$= 2 \times \frac{Vtl_1^2}{I}\left(\frac{3h - 4l_1}{12}\right) \times l_2 + 2 \times \frac{Vt}{I}\left(\frac{l_2^2 h}{4} + \frac{l_1 l_2 h}{2} - \frac{l_1^2 l_2}{2}\right) \times \frac{h}{2}$$

which gives $$e = \frac{tl_1^2 l_2}{6I}(3h - 4l_1) + \frac{th}{I}\left(\frac{l_2^2 h}{4} + \frac{l_1 l_2 h}{2} - \frac{l_1^2 l_2}{2}\right)$$

Substituting I in the above equation, we can find e in terms of given parameters.

Example 15.14

Find the position of the shear centre of an unequal *I*-section shown in Fig. 15.36.

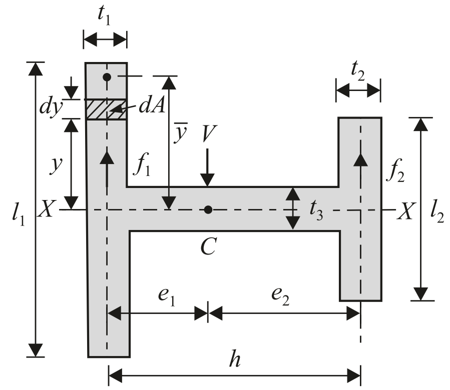


Fig. 15.36

Solution:

C is the position of the shear centre, where the vertical shear force V is applied. XX is the axis of symmetry of the section, and hence the shear centre lies on it. f_1 and f_2 are the shear forces developed in the two flanges of the section so that $V = f_1 + f_2$. The shear force developed in the web is not required to be determined as it passes through the axis of symmetry.

The shear force f_1 is found as

$$f_1 = \int \tau \, dA$$

$$= \int \frac{V}{It_1} A\bar{y} . dA$$

$$= \int_0^{l_1/2} \frac{V}{It_1}\left(\frac{l_1}{2} - y\right) t_1 \cdot \left\{\frac{1}{2}\left(\frac{l_1}{2} - y\right) + y\right\} \cdot t_1 dy$$

$$= \int_0^{l_1/2} \frac{Vt_1}{I}\left(\frac{l_1}{2} - y\right) \cdot \left(\frac{l_1}{4} + \frac{y}{2}\right) dy$$

$$= \frac{Vt_1}{I}\int_0^{l_1/2}\left(\frac{l_1^2}{8} - \frac{y^2}{2}\right) dy = \frac{Vt_1}{I}\left(\frac{l_1^2}{8} y - \frac{y^3}{6}\right)_0^{l_1/2}$$

$$= \frac{Vt_1}{I}\left(\frac{l_1^3}{16} - \frac{l_1^3}{48}\right) = \frac{Vt_1}{I}\left(\frac{3l_1^3 - l_1^3}{48}\right) = \frac{Vt_1 l_1^3}{24I}$$

Let I_1 = Moment of inertia of the left flange about the XX-axis

$$= \frac{1}{12} \times t_1 \times l_1^3$$

Now $$f_1 = \frac{VI_1}{2I} \qquad \text{(in terms of } I_1\text{)}$$

Similarly, f_2 is found as

$$f_2 = \frac{Vt_2 l_2^3}{24I}$$

Let I_2 = Moment of inertia of the right flange about the XX-axis

$$= \frac{1}{12} \times t_2 \times l_2^3$$

Now $$f_2 = \frac{VI_2}{2I} \qquad \text{(in terms of } I_2\text{)}$$

Taking moments of the shear forces about the point C, we have

$$f_1 \times e_1 = f_2 \times e_2$$

$$\frac{VI_1}{2I} \times e_1 = \frac{VI_2}{2I} \times e_2$$

$$I_1 e_1 = I_2 e_2$$

or $$\frac{e_1}{e_2} = \frac{I_2}{I_1} = \left(\frac{t_2}{t_1}\right) \times \left(\frac{l_2}{l_1}\right)^3$$ **Ans.**

SHORT ANSWER QUESTIONS

1. What is meant by unsymmetrical bending? How does it differ from symmetrical bending?
2. Name a few cross-sections, which are symmetrical about one axis only.
3. Name a few cross-sections, which are symmetrical about both axis only.
4. Name a few cross-sections, which are not symmetrical about any axis.
5. How is flexure formula applicable in case of unsymmetrical bending?
6. What is meant by skew loading?
7. How is the deflection of a beam found in unsymmetrical bending?
8. What is meant by a doubly symmetric beam? Give its a few examples.
9. What is shear centre? What does it signify?
10. Why is shear centre also called centre of twist?

MULTIPLE CHOICE QUESTIONS

1. Principal axes are those axes about which
(*a*) mass moment of inertia is zero
(*b*) second moment of area is zero
(*c*) product of inertia is zero
(*d*) polar moment of inertia is zero.

2. For method of superposition to be valid, the following assumption(s) should be valid:
1. The material must exhibit elastic behaviour.
2. The material must exhibit plastic behaviour.
3. The material must follow Hooke's law.
4. The deformation produced in the material should be very small.

Of these statements:
(*a*) 1 alone is true
(*b*) 1 and 3 are true
(*c*) 2 and 3 are true
(*d*) 1, 3 and 4 are true.

3. Which of the following statements is correct about the shear centre?
(*a*) For a section symmetrical about two axes, the shear centre lies at the centriod of the section.
(*b*) For a section symmetrical about one axis only, the shear centre lies along the axis of symmetry.
(*c*) Load is placed at the shear centre to avoid twisting of the section.
(*d*) All of these.

4. Which of the following sections has only one axis of symmetry?
(*a*) Channel section
(*b*) *T*-section
(*c*) Channel and *T*-section both
(*d*) *I*-section.

5. Symmetrical sections such as rectangular and *I*-sections have
(*a*) only one axis of symmetry
(*b*) two axes of symmetry
(*c*) three axes of symmetry
(*d*) no axis of symmetry.

6. The shear centre of a semi-circular ring of mean radius R is located at

(*a*) $\frac{2R}{\pi}$ from the centre of the ring

(*b*) $\frac{3R}{\pi}$ from the centre of the ring

(*c*) $\frac{4R}{\pi}$ from the centre of the ring

(*d*) $\frac{5R}{\pi}$ from the centre of the ring.

7. The position of the shear centre for a uniform thin-walled narrowly open circular section of radius R is defined by (e = Eccentricity)

(*a*) $e = R$ (*b*) $e = 2R$

(*c*) $e = 3R$ (*d*) $e = 4R$.

8. For an *I*-section symmetrical about both x and y axes, the shear centre lies at

(*a*) centroid of the top flange

(*b*) centroid of the bottom flange

(*c*) centroid of the web

(*d*) none of these.

9. For a channel section symmetrical about the x-axis, the shear centre lies at

(*a*) centroid of the top flange

(*b*) centroid of the bottom flange

(*c*) centroid of the web

(*d*) none of these.

10. The shear centre is also known as

(*a*) centroid (*b*) centre of twist

(*c*) centre of moment (*d*) none of these.

ANSWERS

1. (*c*) **2.** (*d*) **3.** (*d*) **4.** (*c*) **5.** (*b*) **6.** (*c*) **7.** (*b*) **8.** (*c*)
9. (*c*) **10.** (*b*).

EXERCISES

1. A 2 m long cantilever beam is constructed from 150 mm × 100 mm × 12 mm angle and arranged with its 150 mm leg vertical. If a vertical load of 5 kN is applied at the free end, passing through the shear centre of the section, determine the maximum tensile and compressive bending stresses set up across the section in the beam.

(*Ans*. 169 MPa (T), 204 MPa (C)).

2. A *T*-shaped 2 m long simple beam shown in Fig. 15.37 is subjected to a central point load of 4 kN inclined at 30° to the vertical and passing through the centroid of the section. Determine the following parameters:

(*a*) the maximum tensile and compressive bending stresses in the beam

(*b*) the position of the neutral axis and

(*c*) the maximum vertical deflection.

Take $E = 200$ GPa

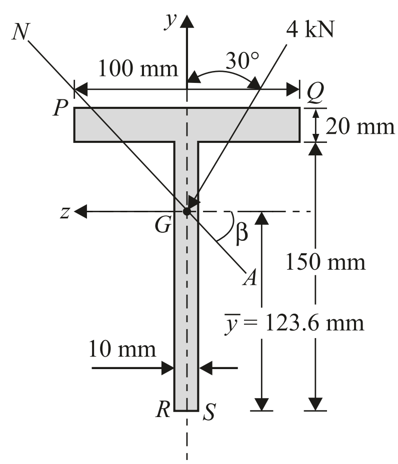


Fig. 15.37

(*Ans*. (*a*) σ_{max} (T) at R = 26.57 MPa, σ_{max} (*C*) at Q = 38.63 MPa.

(*b*) $\beta = 72°$ (*c*) 1.03 mm).

3. A *T*-shaped 2.5 m long simple beam shown in Fig. 15.38 is subjected to a central point load W inclined at 30° to the vertical and passing through the centroid of the section. If the maximum compressive and tensile bending stresses in the beam are not to exceed 75 MPa and 35 MPa respectively, find the maximum load W.

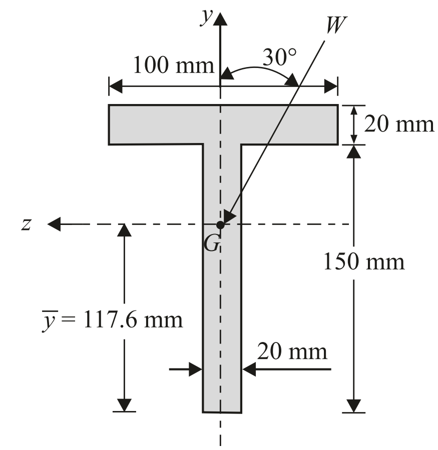


Fig. 15.38

(*Ans.* 5.82 kN).

4. Find the shear centre of the channel section shown in Fig. 15.39.

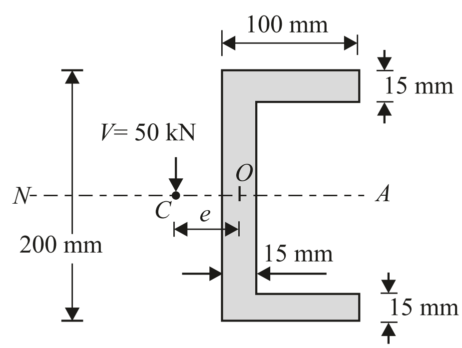


Fig. 15.39

(*Ans.* $e = 38.21$ mm).

5. Find the shear centre of the section shown in Fig. 15.40.

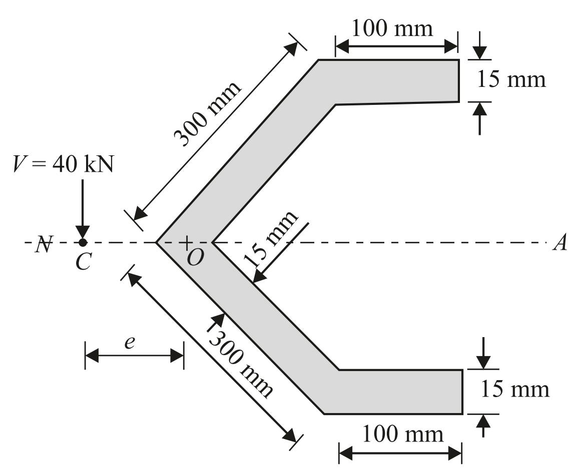


Fig. 15.40

(*Ans.* $e = 25$ mm).

6. Find the position of the shear centre of the section of a beam shown in Fig. 15.41.

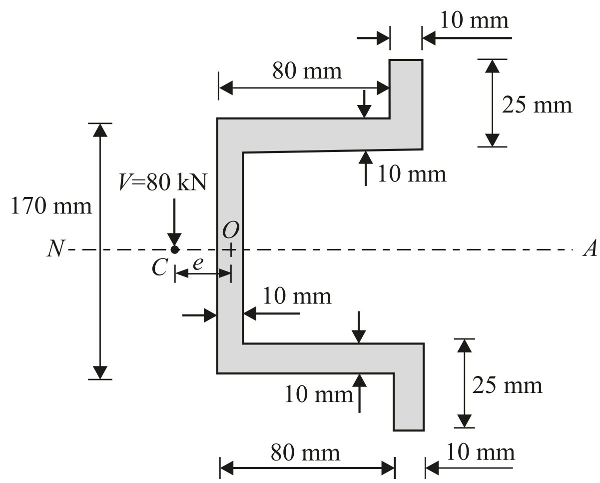
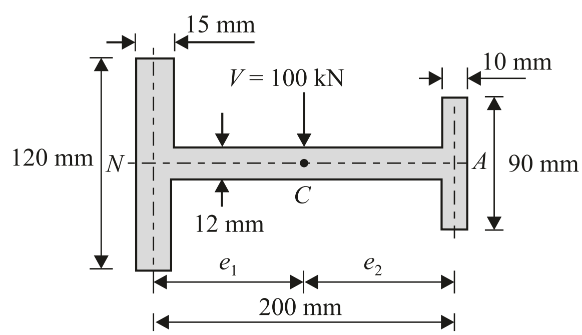


Fig. 15.41 (*Ans.* $e = 36$ mm).

7. Show that the shear centre of a semi-circular ring of mean radius R with uniform thickness is located at a distance of $\frac{4R}{\pi}$ from the centre of the ring.

8. Find the position of the shear centre of the unequal *I*-section shown in Fig. 15.42.



$\left(Ans.\ \frac{e_1}{e_2} = 0.281\right)$.

Fig. 15.42

9. Find the shear centre of the section shown in Fig. 15.43

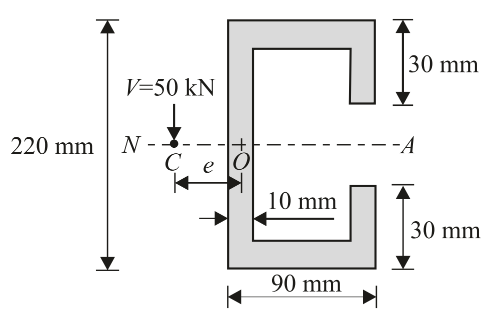
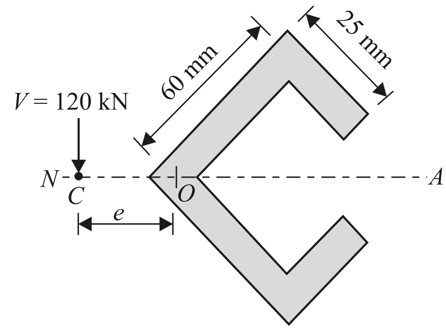


Fig. 15.43 (*Ans.* $e = 44.75$ mm).

10. Find the position of the shear centre at a section of uniform thickness (t = 5 mm) shown in Fig. 15.44.



(*Ans*. 8.846 mm).

Fig. 15.44

11. Show that the shear centre of a section in the form of a circular arc of mean radius R and of uniform thickness, subtending an angle 2α at the centre is given by

$$e = \frac{2R(\sin\alpha - \alpha\cos\alpha)}{(\alpha - \sin\alpha\cos\alpha)}$$

and hence find the shear centre of a narrowly open circular section. (*Ans*. e = 2R)

12. Find the shear centre of a section in the form of circular arc of mean radius 75 mm and of uniform thickness 5 mm subtending an angle of 60° at the centre. (*Ans*. e = 83.63 mm).

13. Find the shear centre of the section of a beam shown in Fig. 15.45.

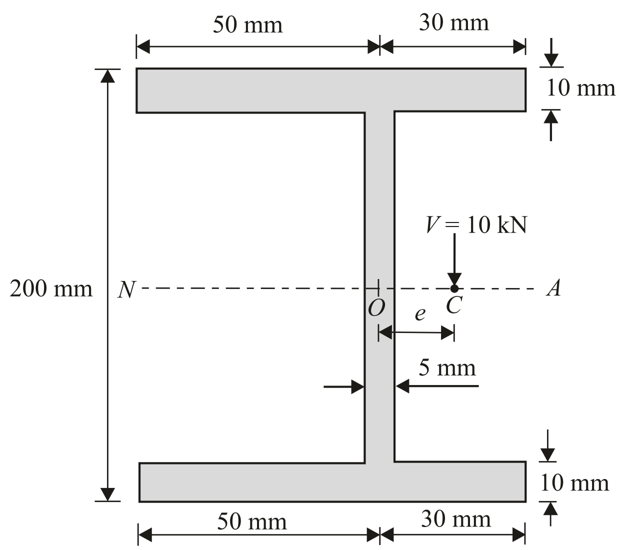


Fig. 15.45

(*Ans*. 8.55 mm).

❑❑❑

16 Fixed Beams

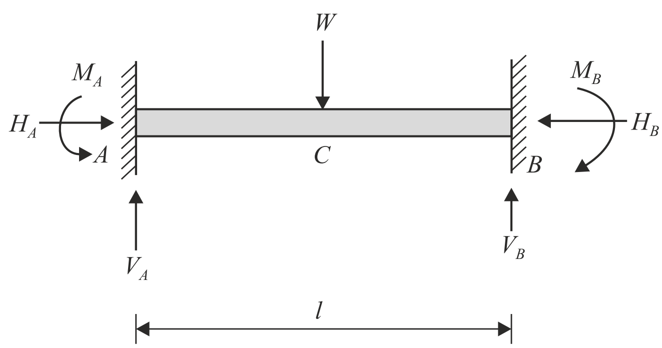


Figure shows a fixed beam subjected to a central point load W and fixing moments M_A and M_B.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- How is a fixed beam advantageous over a simple beam?
- How is the bending moment diagram of a fixed beam drawn?
- How does a fixed end support differ from a simple support?
- Differentiate between free moment diagram and fixing moment diagram?
- What is meant by sinking of a support?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_16

16.1 INTRODUCTION

A fixed beam is supported between two fixed ends. It is also called fixed-end beam or built-in beam or restrained beam. It is classified as a statically indeterminate beam, which involves more than three unknowns and the equilibrium equations of statics alone are not sufficient to determine the support reactions. It involves four unknowns and requires consideration of the deflection of the beam to find the additional unknown. A fixed beam with reactions at the support ends is shown in Fig. 16.1.

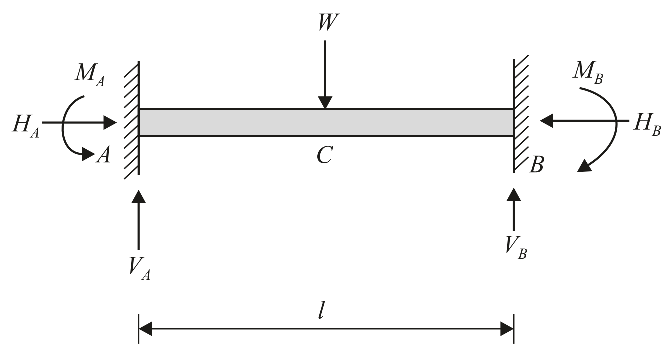
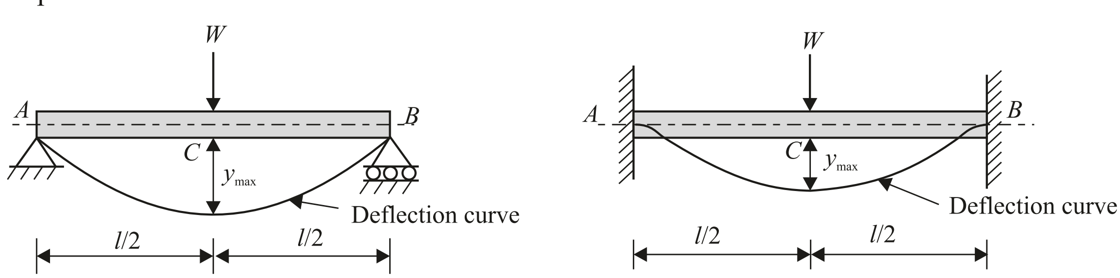


Fig. 16.1

Each support end consists of three reactions: two reactive forces and one reactive moment produced because of prevention of free rotation of the end. The reactions at A include forces H_A and V_A and moment M_A, and reactions at B include forces H_B and V_B and moment M_B. M_A and M_B are called fixing end moments. In case of only vertical load acting on the beam, both H_A and H_B are zero, and they only appear when inclined load acts on the beam.

The development of negative bending moment at the support ends of the beam is advantageous, as it helps to reduce the total bending moment. As a result, both bending stress and deflection are reduced as compared to a simple beam of same length carrying same load, implying that fixed beams are stronger and stiffer. The comparison of the deflection curves for two beams are shown in Fig. 16.2. But these beams need to be perfectly aligned at the supports, otherwise it can set up large stresses. At the same time, the end fixings are normally sensitive to vibrations and fluctuations in bending moments, when subjected to rolling loads as in case of bridges. The slope and deflection at both support points of the beam are zero.



(*a*) Simple beam

(*b*) Fixed beam

Fig. 16.2 Comparison of deflection curves for simple and fixed beams.

16.2 SHEAR FORCE AND BENDING MOMENT DIAGRAMS

The shear force (S.F) diagrams of a fixed beam are similar to that of a simple beam. The bending moment (B.M) diagram is constructed in two steps. In the first step, the fixed beam is treated as a simple beam of same length with the same load assuming to act on it and its bending moment diagram is drawn. The resulting diagram is called the free moment diagram, which consists of positive moments. In the second step, the effect of only end moments of the fixed beam is considered while drawing the bending moment diagram. This diagram is called the fixing moment diagram, which consists of negative moments. Now these two moment diagrams are superimposed to give the total bending moment diagram for the fixed beam. The bending moments are not zero at the ends of the beam as in case of a simple beam, but have always negative values. On the other hand, the middle portion of the beam has positive bending moments. The area of the free moment diagram is equal to the area of the fixing moment diagram, but of opposite sign. Also, the centroidal distance of both moment diagrams are equal from either support of the beam. A fixed beam has two points of contraflexure near each end.

16.3 FIXED BEAM CARRYING A CENTRAL POINT LOAD

Consider a fixed beam AB of length l carrying a point load W at the midpoint C as shown in Fig. 16.3(a).

Reactions at *A* and *B*

As only vertical load acts on the beam, hence the horizontal components of the reaction force at the support ends are zero.

$$H_A = H_B = 0$$

The vertical components of the reaction forces are now designated as

$$V_A = R_A \text{ and } V_B = R_B$$

Since the load is symmetrical about the centre of the beam, hence

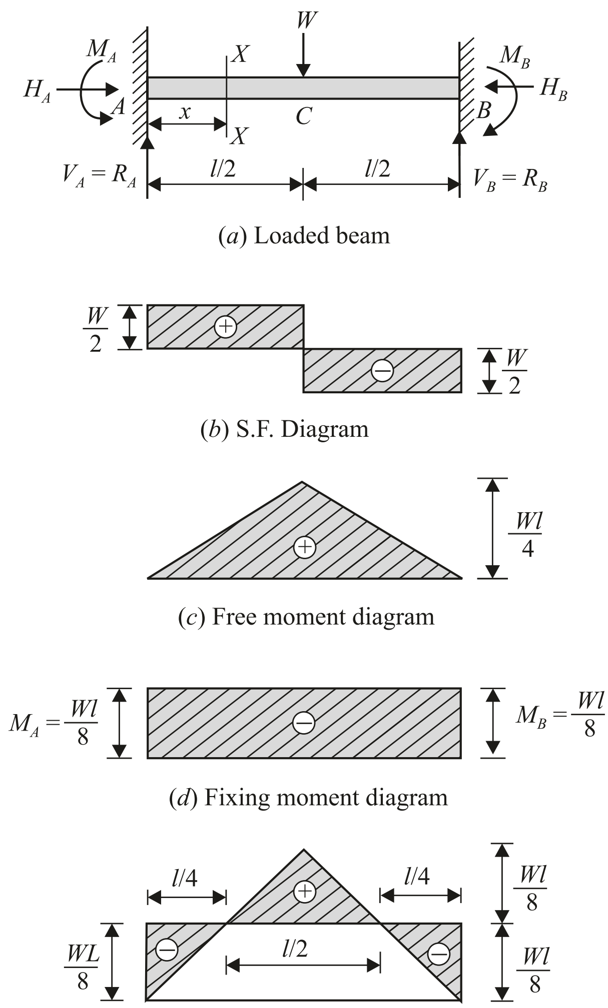
$$R_A = R_B = \frac{W}{2}$$

Calculations for shear forces

The shear force between A and C remains constant with the value $-\frac{W}{2}$ and just to the right of C changes to $-\frac{W}{2}$ and remains constant with this value between B and C. The shear force diagram is shown in Fig. 16.3 (b).

Fixing end moments at *A* and *B*

Let M_A and M_B be the fixing moments at A and B respectively. Since the load acts at the centre of the beam, hence $M_A = M_B$. Considering fixed beam to be a simple beam of length l, the free moment diagram is constructed, which is of triangular shape as shown in Fig. 16.3 (c). The maximum free moment is $+\frac{WL}{4}$ acting at the centre of the beam.



(*a*) Loaded beam

(*b*) S.F. Diagram

(*c*) Free moment diagram

(*d*) Fixing moment diagram

(*e*) Total B.M. Diagram

Fig. 16.3

Similarly, the fixing moment diagram is constructed using fixing moments at the support ends of the fixed beam, and it is of rectangular shapes as shown in Fig. 16.3 (*d*).

The area of the free moment diagram is obtained as

$$A_1 = \text{Area of the triangular moment distribution}$$

$$= \frac{1}{2} \times l \times \frac{Wl}{4}$$

$$= \frac{Wl^2}{8} \qquad \text{...(16.1)}$$

The area of the fixing moment diagram is obtained as

$$A_2 = -\text{ Area of the rectangular moment distribution}$$

$$= -M_A \times l \qquad \text{...(16.2)}$$

Now

$$A_1 = A_2$$

$$\frac{Wl^2}{8} = -M_A \times l$$

which gives

$$M_A = -\frac{Wl}{8} \qquad \text{...(16.3)}$$

Also

$$M_B = M_A = -\frac{Wl}{8} \qquad \text{...(16.4)}$$

Total B.M. Diagram

Using known values of M_A and M_B from equations (16.3) and (16.4) respectively, the free moment diagram and the fixing moment diagram are now superimposed to obtain the total bending moment diagram as shown in Fig. 16.3 (*e*).

Deflection of the beam

Consider a section *XX* in *AC* at a distance *x* from A as shown in Fig. 16.3(*a*). The equation of the moment at the section is

$$M_x = R_A x - M_A = \frac{W}{2}x - \frac{Wl}{8}$$

Using differential equation of flexure for slope and deflection, we have

$$EI\frac{d^2y}{dx^2} = M = M_x$$

or

$$EI\frac{d^2y}{dx^2} = \frac{W}{2}x - \frac{Wl}{8} \qquad \text{...(16.5)}$$

On integration, we get

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wlx}{8} + C_1 \qquad \text{...(16.6)}$$

where C_1 is the constant of integration.

The integration of equation (16.6) gives

$$EIy = \frac{Wx^3}{12} - \frac{Wlx^2}{16} + C_1x + C_2 \qquad \text{...(16.7)}$$

where C_2 is another constant of integration.

The boundary condition is

At A, where $x = 0,\ y = 0$

and $\dfrac{dy}{dx} = 0$

On substituting the boundary condition in equations (16.6) and (16.7), we get

$$C_1 = 0 \text{ and } C_2 = 0$$

Equations (16.6) and (16.7) after substituting C_1 and C_2 can now be expressed as

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{Wx^2}{4} - \frac{Wlx}{8}\right] \quad ...(16.8)$$

$$y = \frac{1}{EI}\left[\frac{Wx^2}{12} - \frac{Wlx^2}{16}\right] \quad ...(16.9)$$

Equation (16.8) is a slope equation and equation (16.9) is a deflection equation for the fixed beam. Because of symmetry of the loading, the maximum deflection occurs at the midspan C of the beam.

To find the deflection at C, put $x = \dfrac{l}{2}$ in equation (16.9).

$$y_C = y_{\max} = \frac{1}{EI}\left[\frac{W}{12}\times\left(\frac{l}{2}\right)^3 - \frac{Wl}{16}\times\left(\frac{l}{2}\right)^2\right]$$

$$= \frac{1}{EI}\left[\frac{Wl^3}{96} - \frac{Wl^3}{64}\right] = \frac{1}{EI}\left[\frac{2Wl^3 - 3Wl^3}{192}\right]$$

$$= -\frac{Wl^3}{192EI} \quad ...(16.10)$$

The negative sign shows the downward deflection.

16.4 FIXED BEAM CARRYING AN ECCENTRIC POINT LOAD

Let us consider a fixed beam AB of length l carrying a point load W acting at an eccentric point C as shown in Fig. 16.4 (*a*).

Fixing end moments at *A* and *B*

Let M_A and M_B be the fixing moments at A and B respectively. The free moment diagram is of triangular shape (Fig. 16.4 (*c*)) and the fixing moment diagram is of trapezoidal shape (Fig. 16.4 (*d*)).

The area of the free moment diagram is obtained as

$$A_1 = \text{Area of the triangular moment distribution}$$

$$= \frac{1}{2}\times l\times\frac{Wab}{l}$$

$$= \frac{Wab}{2} \quad ...(16.11)$$

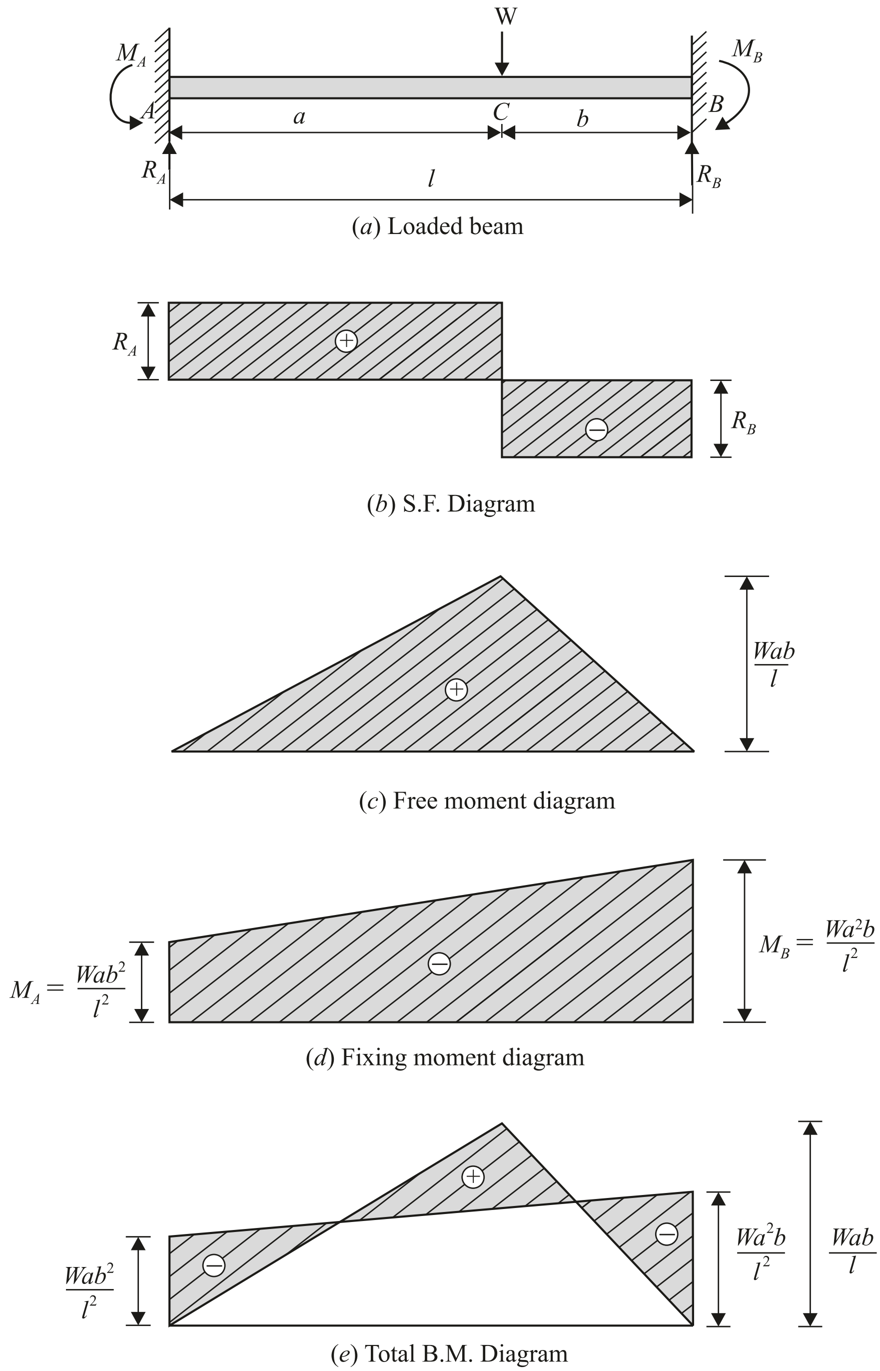

W
M_A
M_B
A
B
C
a
b
R_A
R_B
l
(a) Loaded beam
R_A
R_B
(b) S.F. Diagram
$\frac{Wab}{l}$
(c) Free moment diagram
$M_A = \frac{Wab^2}{l^2}$
$M_B = \frac{Wa^2b}{l^2}$
(d) Fixing moment diagram
$\frac{Wab^2}{l^2}$
$\frac{Wa^2b}{l^2}$
$\frac{Wab}{l}$
(e) Total B.M. Diagram

Fig. 16.4

The area of the fixing moment diagram is obtained as

$$A_2 = -\text{Area of the trapezoidal moment distribution}$$

$$= -\frac{1}{2}\times(M_A + M_B)\times l$$

$$= -\frac{(M_A + M_B)l}{2} \qquad ...(16.12)$$

Now $A_1 = A_2$

$$\frac{Wab}{2} = -\frac{(M_A + M_B)l}{2}$$

or

$$M_A + M_B = -\frac{Wab}{l} \qquad ...(16.13)$$

The centroidal distance of the free moment diagram from A is

$$x_1 = \frac{(a+l)}{3} \qquad \text{(see Table 3.1)...(16.14)}$$

The centroidal distance of the fixing moment diagram from A is

$$x_2 = \frac{l}{3}\times\frac{(M_A + 2M_B)}{(M_A + M_B)} \qquad \text{(see Table 3.1)... (16.15)}$$

Now $x_1 = x_2$

$$\frac{(a+l)}{3} = \frac{l}{3}\times\frac{(M_A + 2M_B)}{(M_A + M_B)}$$

$$(a + l)(M_A + M_B) = l(M_A + 2M_B)$$

$$aM_A + aM_B + lM_A + lM_B = lM_A + 2lM_B$$

$$aM_A + aM_B + lM_B - 2lM_B = 0$$

$$aM_A + aM_B - lM_B = 0$$

$$aM_A - (l - a)M_B = 0$$

or

$$aM_A - bM_B = 0 \qquad \text{(as } b = (l - a)\text{) ...(16.16)}$$

Solving equations (16.13) and (16.16), we get

$$M_A = -\frac{Wab^2}{l^2} \qquad ...(16.17)$$

$$M_B = -\frac{Wa^2b}{l^2} \qquad ...(16.18)$$

Total B.M. Diagram

Using known values of M_A and M_B, the free and fixing moment diagrams are superimposed to obtain the total bending moment diagram as shown in Fig. 16.4 (*e*).

Reactions at A and B

Let R_A and R_B be the vertical reactions at A and B respectively.

Taking moments about B, we have

$$R_A \times l - M_A - W \times b + M_B = 0$$

$$R_A = \frac{M_A - M_B + Wb}{l}$$

On substituting M_A and M_B from equations (16.17) and (16.18), we have

$$R_A = \frac{\frac{Wab^2}{l^2} - \frac{Wa^2b}{l^2} + Wb}{l}$$

$$= \frac{\frac{Wb\,(ab - a^2 + l^2)}{l^2}}{l}$$

$$= \frac{Wb}{l^3}(ab - a^2 + l^2)$$

$$= \frac{Wb}{l^3}\left[(ab - a^2 + (a+b)^2\right] \qquad \text{(as } l = (a+b))$$

$$= \frac{Wb}{l^3}(ab - a^2 + a^2 + b^2 + 2ab)$$

$$= \frac{Wb}{l^3}(3ab + b^2)$$

$$= \frac{Wb^2}{l^3}(3a + b)$$

$$= \frac{Wb^2}{l^3}(2a + l) \qquad \text{(putting } (a+b) = l) \quad ...(16.19)$$

Similarly

$$R_B = \frac{Wa^2}{l^3}(2b + l) \qquad ...(16.20)$$

Calculations for shear forces

Shear force at A is

$$V_A = +R_A$$

$$= +\frac{Wb^2}{l^3}(2a + l) \qquad ...(16.21)$$

The shear force between A and C remains constant with the value V_A.

The shear force just to the right of C is $-R_B$ and it remains constant with this value between B and C.

Shear force at B is

$$V_B = -R_B$$

$$= -\frac{Wa^2}{l^3}(2b+l) \qquad ...(16.22)$$

The shear force diagram is shown in Fig. 16.4 (b).

Deflection of the beam

The bending moment at any section at a distance x from A, using Macaulay's method, is given as

$$M_x = R_A x - M_A \Big| - W(x-a)$$

$$= \frac{Wb^2}{l^3}(2a+l)x - \frac{Wab^2}{l^2}\Bigg| - W(x-a)$$

Using differential equation of flexure for slope and deflection, we have

$$EI\frac{d^2y}{dx^2} = M = M_x$$

$$= \frac{Wb^2}{l^3}(2a+l)x - \frac{Wab^2}{l^2} - W(x-a) \qquad ...(16.23)$$

On integration, we get

$$EI\frac{dy}{dx} = \frac{Wb^2}{l^3}(2a+l)\times\frac{x^2}{2} - \frac{Wab^2}{l^2}x + C_1\Bigg| - \frac{W(x-a)^2}{2} \qquad ...(16.24)$$

where C_1 is the constant of integration.

The integration of equation (16.24) gives

$$EIy = \frac{Wb^2}{l^3}(2a+l)\times\frac{x^3}{6} - \frac{Wab^2}{l^2}\times\frac{x^2}{2} + C_1x + C_2\Bigg| - \frac{W(x-a)^3}{6} \qquad ...(16.25)$$

where C_2 is another constant of integration.

The boundary condition is

At A, where $x = 0$, $y = 0$

and $\frac{dy}{dx} = 0$

On substituting the boundary condition in equation (16.24), we have

$C_1 = 0$ (omitting the negative term within the bracket)

Equation (16.24) on substituting boundary condition gives

$$C_2 = 0 \qquad \text{(omitting the negative term within the bracket)}$$

On substituting C_1 in equation (16.24), we get

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{Wb^2x^2}{2l^3}(2a+l) - \frac{Wab^2x}{l^2}\right| - \left.\frac{W(x-a)^2}{2}\right] \qquad (16.26)$$

This is the required equation of slope for the fixed beam.

On substituting C_1 and C_2 in equation (16.25), we get

$$y = \frac{1}{EI}\left[\frac{Wb^2x^3}{6l^3}(2a+l) - \frac{Wab^2x^2}{2l^2}\right| - \left.\frac{W(x-a)^3}{6}\right] \qquad ...(16.27)$$

This is the required equation of deflection for the fixed beam.

Maximum deflection

To find the position of the maximum deflection of the fixed beam, equate the slope equation (16.26) to zero.

Considering the maximum deflection to occur in AC, we have

$$\frac{Wb^2x^2}{2l^3}(2a+l) = \frac{Wab^2x}{l^2} \qquad \text{(omitting the negative term within the bracket for } x < a\text{)}$$

which gives

$$x = \frac{2al}{2a+l} \qquad ...(16.28)$$

We observe that the value of x is less than a for all its values exceeding $l/2$, hence the maximum deflection occurs in AC.

On substituting the value of x in equation (16.27) and omitting the negative term within the bracket, we get the value of the maximum deflection as

$$y_{\max} = -\frac{2}{3EI} \times \frac{Wa^3b^2}{(2a+l)^2} \qquad ...(16.29)$$

$$= -\frac{2}{3EI} \times \frac{Wa^3b^2}{(3a+b)^2} \qquad ...(16.30)$$

The negative sign shows the downward deflection.

Deflection under the load W at C

The deflection under the load W can be obtained by putting $x = a$ in equation (16.27) as

$$y = -\frac{Wa^3b^3}{3EIl^3} \qquad ...(16.31)$$

16.5 FIXED BEAM CARRYING UNIFORMLY DISTRIBUTED LOAD (UDL) OVER THE ENTIRE SPAN

Consider a fixed beam *AB* of length *l* carrying a uniformly distributed load (*udl*) over its full span as shown in Fig. 16.5 (*a*).

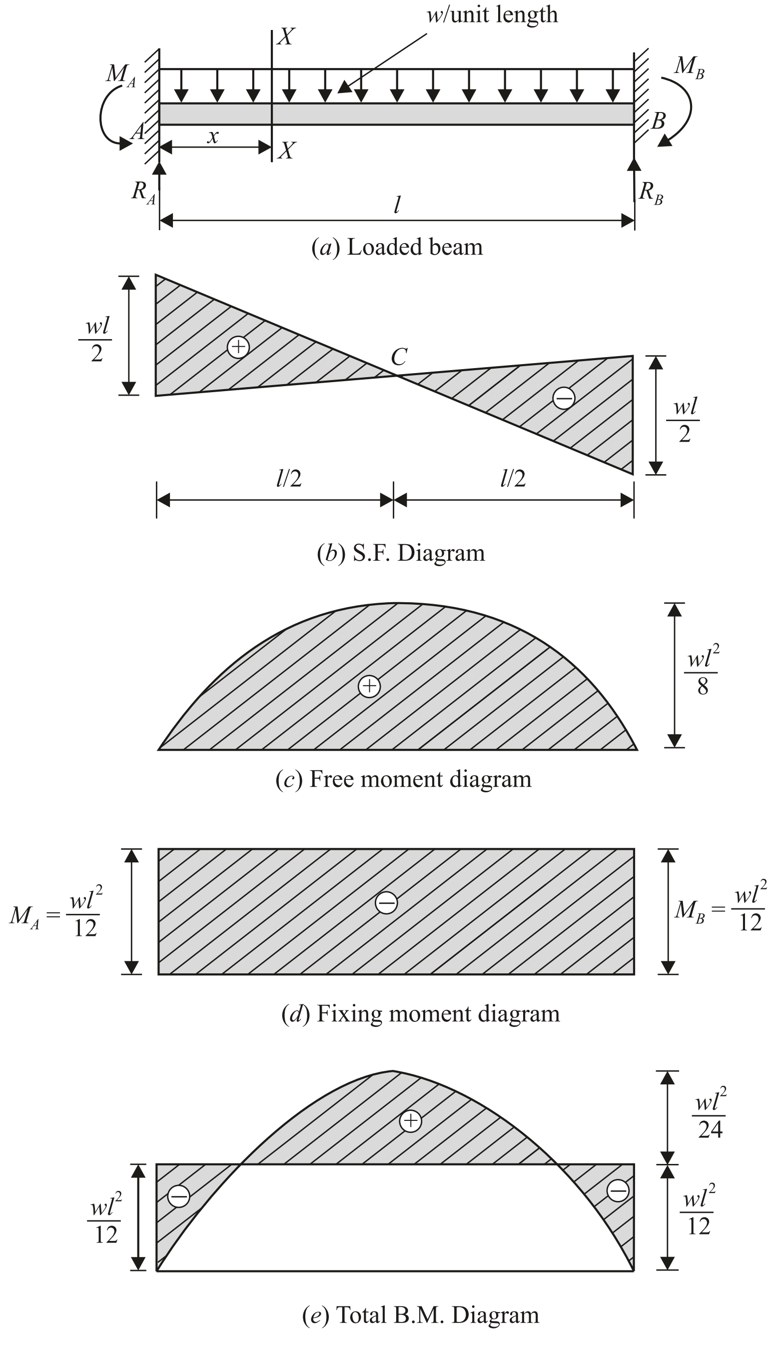


Fig 16.5

Reactions at *A* and *B*

Since the load *w*/unit length acts in the vertical direction, hence there are no horizontal reaction forces at both ends *A* and *B*. Let R_A and R_B are the vertical reaction forces at *A* and *B* respectively. Since the load is symmetrically placed on the beam, hence $R_A = R_B = \dfrac{wl}{2}$

Calculations for shear forces

The shear force at *A* is

$$V_A = +R_A$$

$$= +\frac{wl}{2}$$

The shear force at *C* is

$$V_C = +R_A - \frac{wl}{2}$$

$$= +\frac{wl}{2} - \frac{wl}{2} = 0$$

The shear force at *B* is

$$V_B = -R_B$$

$$= -\frac{wl}{2}$$

The shear force diagram is shown in Fig. 16.5 (*b*), and consists of two triangles of equal area but of opposite sign.

Fixing end moments at *A* and *B*

Let M_A and M_B be the fixing moments at *A* and *B* respectively.

The free moment diagram is of parabolic shape (Fig. 16.5 (*c*)) and the maximum bending moment is $\dfrac{wl^2}{8}$, which occurs at the midspan *C*. The fixing moment diagram is of rectangular shape (Fig. 16.5(*d*)).

The area of the free moment diagram is obtained as

$$A_1 = \text{Area of the parabolic moment distribution}$$

$$= \frac{4}{3} \times \frac{l}{2} \times \frac{wl^2}{8} \quad \text{(see Table 3.1)}$$

$$= \frac{wl^3}{12} \quad \text{...(16.32)}$$

The area of the fixing moment diagram is obtained as

$$A_2 = -\text{Area of the rectangular moment distribution}$$

$$= -M_A \times l$$

Now $$A_1 = A_2$$

$$\frac{wl^3}{12} = -M_A \times l$$

which gives $$M_A = -\frac{wl^2}{12} \qquad \text{...(16.33)}$$

Since the load is symmetrically placed on the beam, hence

$$M_B = M_A = -\frac{wl^2}{12} \qquad \text{...(16.34)}$$

Total B.M. Diagram

Using known values of M_A and M_B from equations (16.33) and (16.34) respectively, the free moment diagram and the fixing moment diagram are now superimposed to obtain the total bending moment diagram as shown in Fig. 16.5(*e*).

Point of contraflexure

Consider a section XX at a distance x from A as shown in Fig. 16.5 (*a*).

The moment at the section is

$$M_x = R_A x - M_A - \frac{wx^2}{2}$$

$$= \frac{wl}{2}x - \frac{wl^2}{12} - \frac{wx^2}{2} \qquad \text{...(16.35)}$$

At the point of contraflexure, the bending moment is zero.

$$M_x = 0$$

$$\frac{wlx}{2} - \frac{wl^2}{12} - \frac{wx^2}{2} = 0$$

$$x^2 - lx + \frac{l^2}{6} = 0$$

or $$6x^2 - 6lx + l^2 = 0$$

Solving for x, we get

$$x = \frac{-(-6l) \pm \sqrt{(-6l)^2 - 4 \times 6 \times l^2}}{2 \times 6}$$

$$= \frac{6l \pm 2\sqrt{3}l}{12} = 0.5\,l \pm 0.289\,l$$

or $$x = 0.789\,l \text{ or } 0.211\,l \qquad \text{...(16.36)}$$

Hence, there are two points of contraflexure of which one lies at a distance of 0.211 l from A and another at a distance of 0.789 l from A or at a distance of 0.211 l from B.

Deflection of the beam

Using equation (16.35) in the differential equation of flexure for slope and deflection, we have

$$EI\frac{d^2y}{dx^2} = M = M_x$$

$$= \frac{wlx}{2} - \frac{wl^2}{12} - \frac{wx^2}{2} \qquad \text{...(16.37)}$$

On integration, we get

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wl^2x}{12} - \frac{wx^3}{6} + C_1 \qquad \text{...(16.38)}$$

where C_1 is the constant of integration.

The boundary condition is

At A, where $x = 0$

$$\frac{dy}{dx} = 0$$

Using this boundary condition in equation (16.38), we get

$$C_1 = 0$$

On substituting C_1 in equation (16.38), we have

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{wlx^2}{4} - \frac{wl^2x}{12} - \frac{wx^3}{6}\right] \qquad \text{...(16.39)}$$

This is the equation of slope for the fixed beam.

The integration of equation (16.38) gives

$$EIy = \frac{wlx^3}{12} - \frac{wl^2x^2}{24} - \frac{wx^4}{24} + C_1x + C_2 \qquad \text{...(16.40)}$$

where C_2 is another constant of integration.

The other boundary condition is

At A, where $x = 0$

$y = 0$

Using this boundary condition in equation (16.40), we get

$$C_2 = 0$$

On substituting C_1 and C_2 in equation (16.40), we have

$$y = \frac{1}{EI}\left[\frac{wlx^3}{12} - \frac{wl^2x^2}{24} - \frac{wx^4}{24}\right] \qquad \text{...(16.41)}$$

This is the equation of deflection for the fixed beam.

Maximum deflection

Because of the symmetry of loading, the maximum deflection occurs at the midspan of the beam, and is calculated by putting $x = \frac{l}{2}$ in equation (16.41).

$$y_{\max} = \frac{1}{EI}\left[\frac{wl}{12}\times\left(\frac{l}{2}\right)^3 - \frac{wl^2}{24}\times\left(\frac{l}{2}\right)^2 - \frac{w}{24}\times\left(\frac{l}{2}\right)^4\right]$$

$$= \frac{1}{EI}\left[\frac{wl^4}{96} - \frac{wl^4}{96} - \frac{wl^4}{384}\right]$$

$$= -\frac{wl^4}{384EI} \qquad \text{...(16.42)}$$

The negative sign shows the downward deflection.

16.6 FIXED BEAM CARRYING UNIFORMLY VARYING LOAD

Case I When the load varies from w_1/unit length at the left end to w_2/unit length at the right end of the beam

Consider a fixed beam AB of length l, where the load varies uniformly from w_1/unit length at A to w_2/unit length at B as shown in Fig. 16.6.

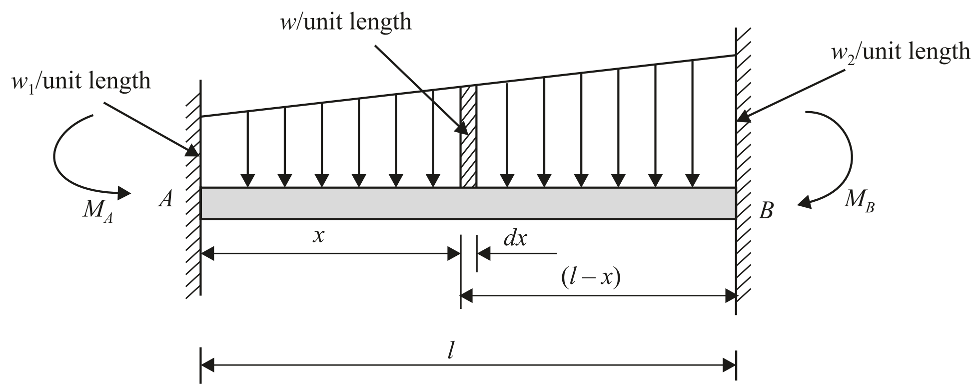


Fig. 16.6

A section of width dx is considered at a distance x from A, where the intensity of load is w/unit length.

Fixing end moments M_A and M_B

Load on the section is wdx.

This case is similar to the case in which a fixed beam is subjected to an eccentric load (wdx) acting at a distance x from A, for which the fixing end moments, M_A and M_B can be expressed as

$$M_A = -\frac{Wab^2}{l^2} \qquad \text{(using equation (16.17))}$$

$$= -\int_0^l \frac{(wdx) \times x \times (l-x)^2}{l^2}$$

$$= -\int_0^l \frac{wx(l-x)^2\,dx}{l^2} \quad \text{...(16.43)}$$

In the above expression, $a = x$, $b = (l - x)$ and $W = wdx$.

Similarly

$$M_B = -\frac{wa^2b}{l^2} \quad \text{(using equation (16.18))}$$

$$= -\int_0^l \frac{(wdx) \times x^2 \times (l-x)}{l^2}$$

$$= -\int_0^l \frac{w(l-x)x^2\,dx}{l^2} \quad \text{...(16.44)}$$

where $$w = w_1 + \frac{(w_2 - w_1)x}{l} \text{ or } w_2 - \frac{(w_2 - w_1)(l-x)}{l} \quad \text{...(16.45)}$$

Case II When the beam carries uniformly varying load over a certain part of the beam

Consider a fixed beam AB of length l, where the load varies uniformly from w_1/unit length at C to w_2/unit length at D, that is, the load varies uniformly over the part $CD = (b - a)$ as shown in Fig. 16.7.

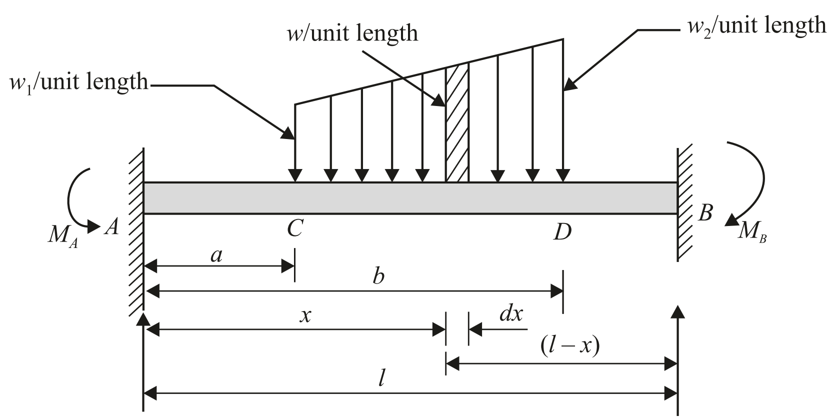


Fig. 16.7

Fixing end moments M_A and M_B

The fixing end moment M_A can be expressed as

$$M_A = -\int_{x=a}^{x=b} \frac{wx(l-x)^2\,dx}{l^2} \quad \text{...(16.46)}$$

The fixing end moment M_B can be expressed as

$$M_B = -\int_{x=a}^{x=b} \frac{w(l-x)x^2 dx}{l^2} \quad ...(16.47)$$

where $$w = w_1 + \frac{(w_2 - w_1)(x-a)}{(b-a)} \text{ or } w_2 - \frac{(w_2 - w_1)(b-x)}{(b-a)} \quad ...(16.48)$$

Case III When the load varies uniformly from zero at the left end to *w*/unit length at the right end of the beam

Consider a fixed beam *AB* of length *l* carrying a uniformly varying load, which varies from zero at *A* to *w*/unit length at *B* as shown in Fig. 16.8 (*a*).

Free moment diagram

To construct the free moment diagram, the fixed beam is treated to be a simple beam of same length *l* and carrying the same uniformly varying load, which varies from zero at *A* to *w*/unit length at *B*. Let $R_{A'}$ and $R_{B'}$ be the support reactions of the simple beam at *A* and *B* respectively.

Taking moments of the forces about *A*, we have

$$R_{B'} \times l = \frac{wl}{2} \times \frac{2l}{3}$$

which gives $$R_{B'} = \frac{wl}{3} (\uparrow)$$

Now $$R_{A'} + R_{B'} = \text{Total load on the beam}$$

$$= \frac{wl}{2}$$

Hence, $$R_{A'} = \frac{wl}{2} - R_{B'}$$

$$= \frac{wl}{2} - \frac{wl}{3}$$

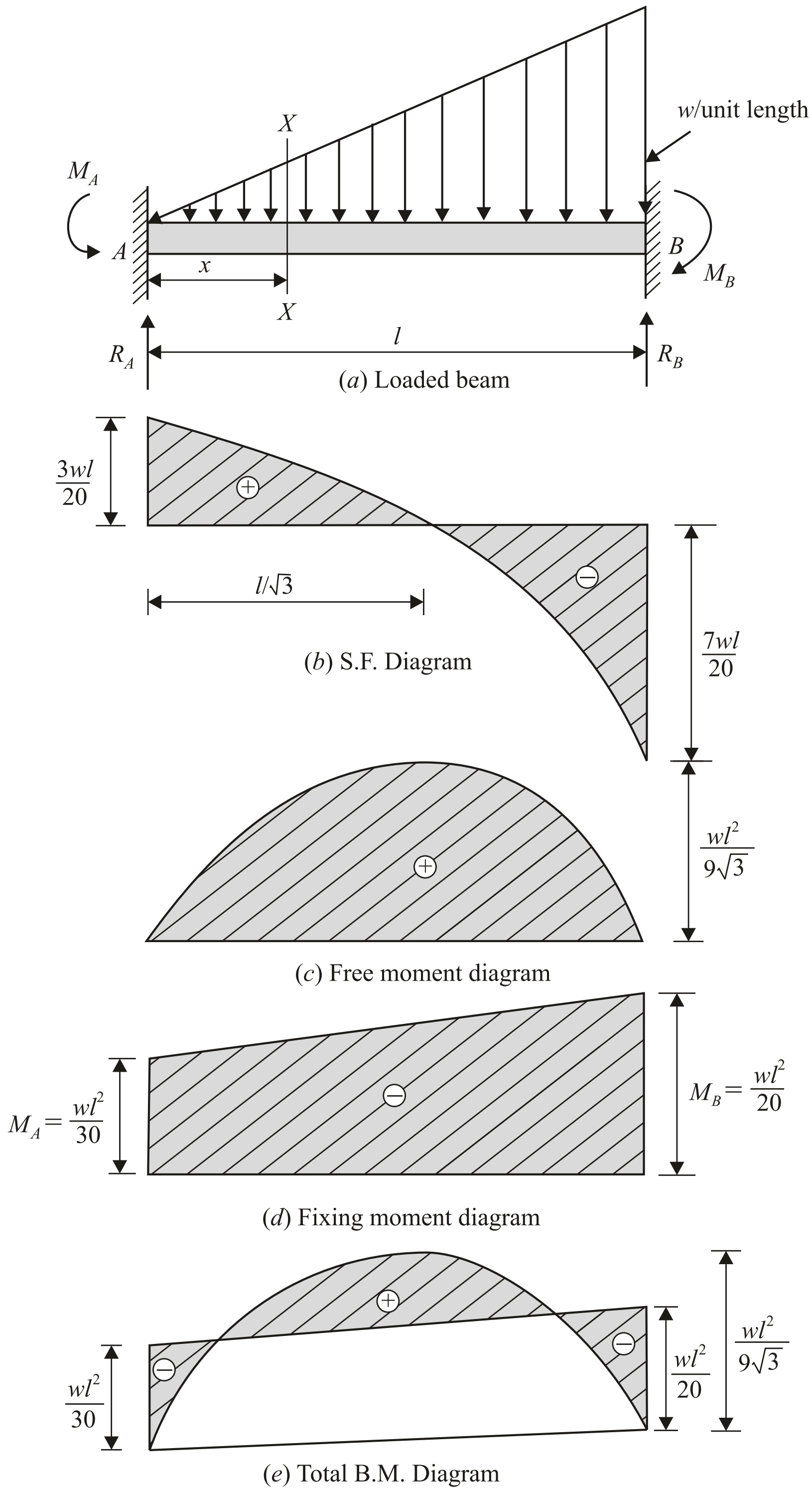
$$= \frac{wl}{6} (\uparrow)$$

The bending moment equation for the simple beam at the section *XX* at a distance *x* from *A* is given as

$$M_x = R_{A'} \times x - \text{Triangular load left to the section} \times \frac{x}{3}$$

$$= \frac{wl}{6} \times x - \frac{1}{2} \times x \times \frac{wx}{l} \times \frac{x}{3}$$

$$= \frac{wlx}{6} - \frac{wx^3}{6l} \quad ...(16.49)$$



w/unit length
X
M_A
A
x
X
B
M_B
R_A
l
R_B
(a) Loaded beam
$\frac{3wl}{20}$
$l/\sqrt{3}$
(b) S.F. Diagram
$\frac{7wl}{20}$
$\frac{wl^2}{9\sqrt{3}}$
(c) Free moment diagram
$M_A = \frac{wl^2}{30}$
$M_B = \frac{wl^2}{20}$
(d) Fixing moment diagram
$\frac{wl^2}{30}$
$\frac{wl^2}{20}$
$\frac{wl^2}{9\sqrt{3}}$
(e) Total B.M. Diagram

Fig. 16.8

For the maximum free bending moment, we have

$$\frac{dM_x}{dx} = 0$$

$$\frac{d}{dx}\left(\frac{wlx}{6} - \frac{wx^3}{6l}\right) = 0$$

or
$$\frac{wl}{6} - \frac{3wx^2}{6l} = 0$$

which gives

$$x = \frac{l}{\sqrt{3}}$$

Now the maximum free bending moment is obtained by putting the value of x in equation (16.49).

$$M_{\max} = \frac{wl}{6} \times \frac{l}{\sqrt{3}} - \frac{w}{6l} \times \left(\frac{l}{\sqrt{3}}\right)^3$$

$$= \frac{wl^2}{6\sqrt{3}} - \frac{wl^2}{18\sqrt{3}} = \frac{wl^2}{9\sqrt{3}}$$

The free moment diagram is shown in Fig. 16.8 (*c*).

The area of the free moment diagram can be obtained as

$$A_1 = \int_0^l M_x dx$$

$$= \int_0^l \left(\frac{wlx}{6} - \frac{wx^3}{6l}\right) dx \qquad \text{(using equation 16.49)}$$

$$= \left[\frac{wlx^2}{12} - \frac{wx^4}{24l}\right]_0^l = \frac{wl^3}{12} - \frac{wl^4}{24l}$$

$$= \frac{wl^3}{24} \qquad \text{...(16.50)}$$

The centroidal distance of the free moment diagram from A is

$$x_1 = \frac{1}{A_1}\int_0^l x \cdot M_x dx$$

$$= \frac{1}{\left(\frac{wl^3}{24}\right)}\int_0^l x\left(\frac{wlx}{6} - \frac{wx^3}{6l}\right) dx$$

$$= \frac{24}{wl^3}\int_0^l \left(\frac{wlx^2}{6} - \frac{wx^4}{6l}\right)dx$$

$$= \frac{24}{wl^3}\left[\frac{wlx^3}{18} - \frac{wx^5}{30l}\right]_0^l$$

$$= \frac{24}{wl^3}\left[\frac{wl^4}{18} - \frac{wl^4}{30}\right]$$

$$= \frac{24}{wl^3} \times \frac{2wl^4}{90} = \frac{8l}{15}$$

Fixing end moments at *A* and *B*

Let M_A and M_B be the fixing moments at A and B respectively. The area of the fixing moment diagram, which is trapezoidal in shape, can be found as

$$A_2 = -\text{Area of the trapezoidal moment distribution}$$

$$= -\frac{1}{2} \times (M_A + M_B) \times l$$

$$= -\frac{(M_A + M_B)l}{2} \qquad ...(16.51)$$

Now $$A_1 = A_2$$

$$\frac{wl^3}{24} = -\frac{(M_A + M_B)l}{2}$$

or $$M_A + M_B = -\frac{wl^2}{12} \qquad ...(16.52)$$

The centroidal distance of the fixing moment diagram from A is

$$x_2 = \frac{l}{3} \times \frac{(M_A + 2M_B)}{(M_A + M_B)} \qquad \text{(see Table 3.1)}$$

Now $$x_1 = x_2$$

$$\frac{8l}{15} = \frac{l}{3} \times \frac{(M_A + 2M_B)}{(M_A + M_B)}$$

$$5l\,(M_A + 2M_B) = 8l\,(M_A + M_B)$$

$$5M_A - 8M_A + 10\,M_B - 8M_B = 0$$

$$-3M_A + 2M_B = 0 \qquad ...(16.53)$$

Solving equations (16.52) and (16.53), we get

$$M_A = -\frac{wl^2}{30} \qquad ...(16.54)$$

$$M_B = -\frac{wl^2}{20} \qquad ...(16.55)$$

Using known values of M_A and M_B, the fixing moment diagram can be drawn as shown in Fig. 16.8 (*d*). Now the free moment diagram and the fixing moment diagram are superimposed to get the total bending moment diagram as shown in Fig. 16.8 (*e*).

Shear force diagram

Let R_A and R_B be the vertical reactions of the fixed beam at *A* and *B* respectively.

Taking moments about *B*, we have

$$R_A \times l - M_A - \frac{1}{2} \times l \times w \times \frac{l}{3} + M_B = 0$$

$$R_A \times l - \frac{wl^2}{30} - \frac{wl^2}{6} + \frac{wl^2}{20} = 0$$

or $$R_A \times l - \frac{9wl^2}{60} = 0$$

which gives

$$R_A = \frac{3wl}{20} (\uparrow) \qquad ...(16.56)$$

Now $$R_A + R_B = \text{Total load on the beam}$$

$$= \frac{wl}{2}$$

or $$R_B = \frac{wl}{2} - R_A$$

$$= \frac{wl}{2} - \frac{3wl}{20}$$

$$= \frac{7wl}{20} (\uparrow) \qquad ...(16.57)$$

Using known values of R_A and R_B, the shear force diagram can be drawn as shown in Fig. 16.8 (*b*). The variation of the shear force is parabolic.

Deflection of the beam

The bending moment equation for the fixed beam at the section XX at a distance x from A is given as

$$M_x = R_A x - M_A - \frac{1}{2} \times x \times \frac{wx}{l} \times \frac{x}{3}$$

$$= \frac{3wl}{20}x - \frac{wl^2}{30} - \frac{wx^3}{6l} \quad ...(16.58)$$

Using differential equation of flexure for slope and deflection, we have

$$EI\frac{d^2y}{dx^2} = M = M_x$$

$$= \frac{3wlx}{20} - \frac{wl^2}{30} - \frac{wx^3}{6l} \quad ...(16.59)$$

On integration, we have

$$EI\frac{dy}{dx} = \frac{3wlx^2}{40} - \frac{wl^2x}{30} - \frac{wx^4}{24l} + C_1 \quad ...(16.60)$$

where C_1 is the constant of integration.

The boundary condition is

At A, where $x = 0$

$$\frac{dx}{dy} = 0$$

Using this boundary condition in equation (16.60), we get

$$C_1 = 0$$

On substituting C_1 in equation (16.60), we have

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{3wlx^2}{40} - \frac{wl^2x}{30} - \frac{wx^4}{24l}\right] \quad ...(16.61)$$

This is the equation of slope for the fixed beam.

The integration of equation (16.60) gives

$$EIy = \frac{3wlx^3}{120} - \frac{wl^2x^2}{60} - \frac{wx^5}{120l} + C_1x + C_2 \quad ...(16.62)$$

where C_2 is another constant of integration.

The other boundary condition is

At A, where $x = 0$

$$y = 0$$

Using this boundary condition in equation (16.62), we get

$$C_2 = 0$$

On substituting C_1 and C_2 in equation (16.62), we have

$$y = \frac{1}{EI}\left[\frac{3wlx^3}{120} - \frac{wl^2x^2}{60} - \frac{wx^5}{120l}\right] \qquad ...(16.63)$$

This is the equation of deflection for the fixed beam.

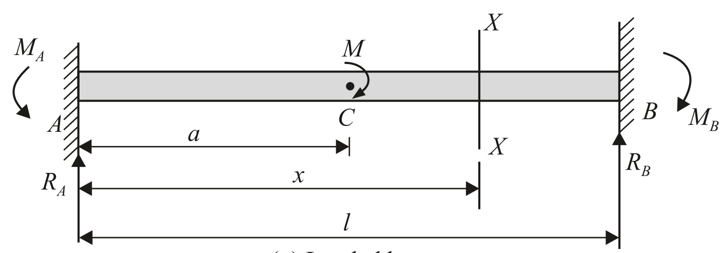
16.7 FIXED BEAM SUBJECTED TO A COUPLE

Consider a fixed beam AB of length l subjected to a couple M at a distance a from A as shown in Fig. 16.9.

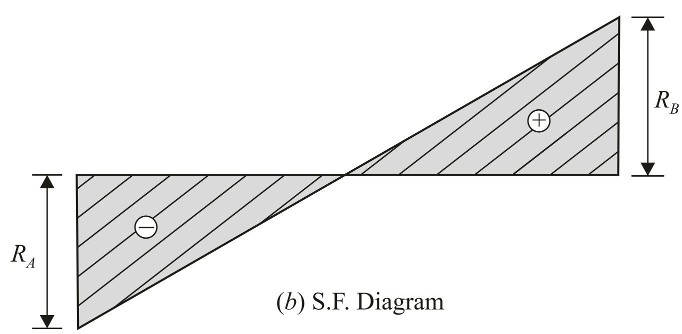
Reactions and fixing end moments at A and B

Let R_A and R_B be the vertical reactions, and M_A and M_B be the fixing moments at A and B respectively. Considering a section XX at a distance x from A, the bending moment at the section, using Macaulay's method, is given as

$$M_x = R_A\, x - M_A \mid + M\,(x - a)^\circ \qquad ...(16.64)$$



(*a*) Loaded beam



(*b*) S.F. Diagram

Fig. 16.9

Using differential equation of flexure for slope and deflection of the beam, we have

$$EI\,\frac{d^2y}{dx^2} = M = M_x$$

$$= R_A\,x - M_A \,|\, + M\,(x-a)^\circ \qquad ...(16.65)$$

On integration, we have

$$EI\,\frac{dy}{dx} = R_A\frac{x^2}{2} - M_A\,x + C_1 \Big| + M\,(x-a) \qquad ...(16.66)$$

where C_1 is the constant of integration.

The boundary condition is

At A, where $x = 0$

$$\frac{dy}{dx} = 0$$

Using this boundary condition in equation (16.66), we get

$$C_1 = 0 \qquad \text{(omitting the negative term within the bracket)}$$

On substituting C_1 in equation (16.66), we have

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{R_A x^2}{2} - M_A\,x \Big| + M\,(x-a)\right] \qquad ...(16.67)$$

The integration of equation (16.66), gives

$$EIy = \frac{R_A x^3}{6} - M_A\frac{x^2}{2} + C_1 x + C_2 \Big| + \frac{M(x-a)^2}{2} \qquad ...(16.68)$$

where C_2 is another constant of integration.

The boundary condition is

At A, where $x = 0$

$$y = 0$$

Using this boundary condition in equation (16.68), we have

$$C_2 = 0 \qquad \text{(omitting the negative term within the bracket)}$$

On substituting C_1 and C_2 in equation (16.68), we have

$$y = \frac{1}{EI}\left[\frac{R_A x^3}{6} - \frac{M_A x^2}{2} \Big| + \frac{M(x-a)^2}{2}\right] \qquad ...(16.69)$$

Now using another boundary condition, we have

At B, where $x = l$

$$\frac{dy}{dx} = 0$$

Equation (16.67) on substituting the above boundary condition gives

$$\frac{R_A l^2}{2} - M_A l + M(l-a) = 0$$

or $$M_A l = M(l-a) + \frac{R_A l^2}{2} \quad ...(1)$$

Again using boundary condition, we have

At B, where $x = l$

$y = 0$

Equation (16.69) on substituting the above boundary condition becomes

$$\frac{R_A l^3}{6} - \frac{M_A l^2}{2} + \frac{M(l-a)^2}{2} = 0$$

or $$\frac{M_A l^2}{2} = \frac{M(l-a)^2}{2} + \frac{R_A l^3}{6} \quad ...(2)$$

Solving equations (1) and (2), we get

$$R_A = -\frac{6Ma(l-a)}{l^3} \quad ...(16.70)$$

$$= \frac{6Ma(l-a)}{l^3}(\downarrow) \quad ...(16.71)$$

Substituting R_A in equation (1), we have

$$M_A = \frac{M(l-a)}{l} + \frac{R_A l}{2}$$

$$= \frac{M(l-a)}{l} + \frac{l}{2} \times \left(-\frac{6Ma(l-a)}{l^3}\right)$$

$$= \frac{M(l-a)}{l} - \frac{3Ma(l-a)}{l^2}$$

$$= \frac{Ml(l-a) - 3Ma(l-a)}{l^2}$$

$$= \frac{M(l-a)(l-3a)}{l^2} \quad ...(16.72)$$

Taking moments about B, we have

$$R_A \times l - M_A + M + M_B = 0$$

or $$M_B = -R_A l + M_A - M$$

On substituting the values of R_A and M_A in the above equation, we have

$$M_B = -\left[-\frac{6Ma(l-a)}{l^3}\right]\times l + \frac{M(l-a)(l-3a)}{l^2} - M$$

$$= \frac{6Ma(l-a)}{l^2} + \frac{M(l-a)(l-3a)}{l^2} - M$$

$$= \frac{M(l-a)}{l^2}[6a + l - 3a] - M$$

$$= \frac{M(l-a)}{l^2}(l+3a) - M$$

$$= \frac{Ml^2 + 3Mal - Mal - 3a^2M - Ml^2}{l^2}$$

$$= \frac{2Mal - 3a^2M}{l^2}$$

$$= \frac{Ma(2l-3a)}{l^2} \quad ...(16.73)$$

Using known values of M_A and M_B, the bending moment diagram can be drawn. The shape of the diagram will depend upon the value of M_A and M_B.

Now $\quad R_A + R_B = 0$

or $\quad R_B = -R_A$

$$= -\left[-\frac{6Ma(l-a)}{l^3}\right] \quad \text{(using equation (16.70))}$$

$$= \frac{6Ma(l-a)}{l^3}(\uparrow) \quad ...(16.74)$$

Using known values of R_A and R_B, the shear force diagram can be drawn as shown in Fig. 16.9 (*b*).

16.8 SINKING OF A SUPPORT

Consider a fixed beam *AB* of length *l* initially unloaded with its ends at the same level. Now it is considered that the end *B* sinks (moves in the downward direction) relative to the end *A* by an amount δ as shown in Fig. 16.10. The fixing moments M_A and M_B at the ends are equal in magnitude but opposite in sense. M_A is negative bending moment and M_B is positive bending moment. R_A and R_B are the reactions at *A* and *B* respectively. R_B acts downward.

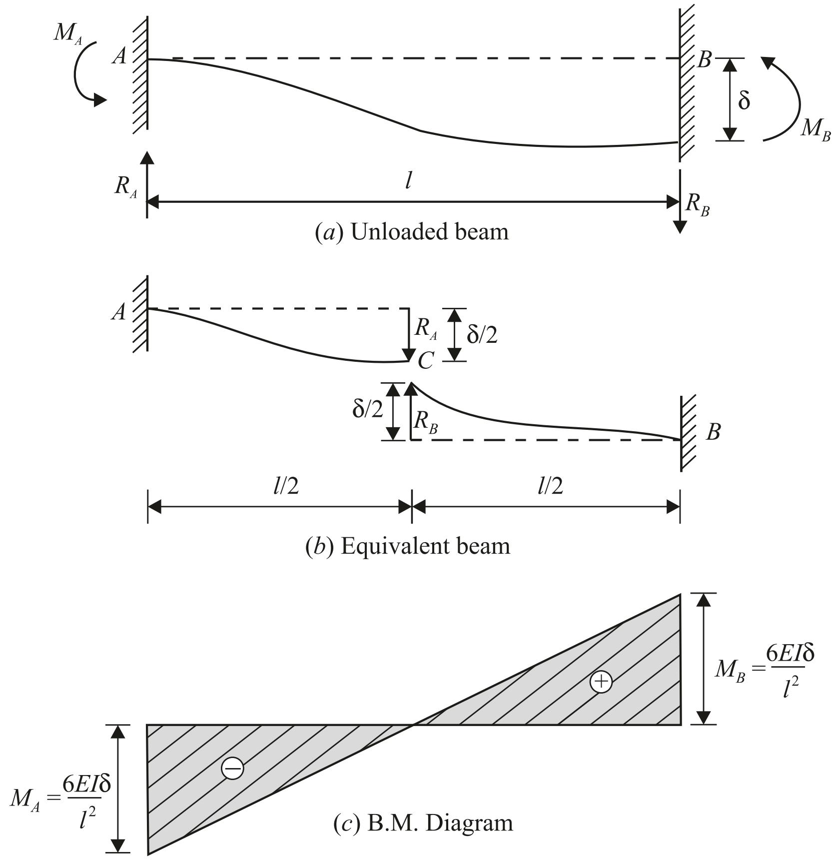


Fig. 16.10

Taking moments about B, we have

$$R_A \times l - M_A - M_B = 0$$

$$R_A \times l - 2M = 0 \qquad \text{(as } M_A = M_B = M\text{)}$$

or
$$R_A = \frac{2M}{l} (\uparrow) \qquad \text{...(16.75)}$$

Now $R_A + R_B$ = Total load on the beam = 0

or
$$R_B = -R_A$$

$$= -\frac{2M}{l}$$

$$= \frac{2M}{l} (\downarrow) \qquad \text{...(16.76)}$$

Now to find the end reactions in terms of known parameters, the fixed beam is considered to be split at the midspan so that it can be assumed to be made of two cantilevers of equal lengths $l/2$ producing equal deflections at their free ends under the reaction forces R_A and R_B as shown in Fig. 16.10 (*b*). The left cantilever *AC* is subjected to a downward point load R_A at its free and producing downward deflection $\delta/2$, whereas the right cantilever *BC* is subjected to an upward point load R_B at its free end producing upward deflection $\delta/2$.

Considering the left cantilever *AC*, its deflection can be expressed as

$$\frac{\delta}{2} = \frac{R_A\left(\frac{l}{2}\right)^3}{3EI} \qquad \left(\text{as } y = \frac{wl^3}{3EI}\right)$$

$$= \frac{R_A l^3}{24EI}$$

or

$$\delta = \frac{R_A l^3}{12EI} \qquad ...(16.77)$$

On substituting R_A, equation (16.77) transforms to

$$\delta = \frac{2M}{l} \times \frac{l^3}{12EI}$$

$$= \frac{Ml^2}{6EI} \qquad ...(16.78)$$

which gives

$$M = \frac{6EI\delta}{l^2} \qquad ...(16.79)$$

Now R_A and R_B can be expressed as

$$R_A = R_B = \frac{2M}{l} \qquad \text{(using equations (16.75) and (16.76))}$$

$$= \frac{12EI\delta}{l^3} \qquad \text{(using equations (16.79)} \quad ...(16.80)$$

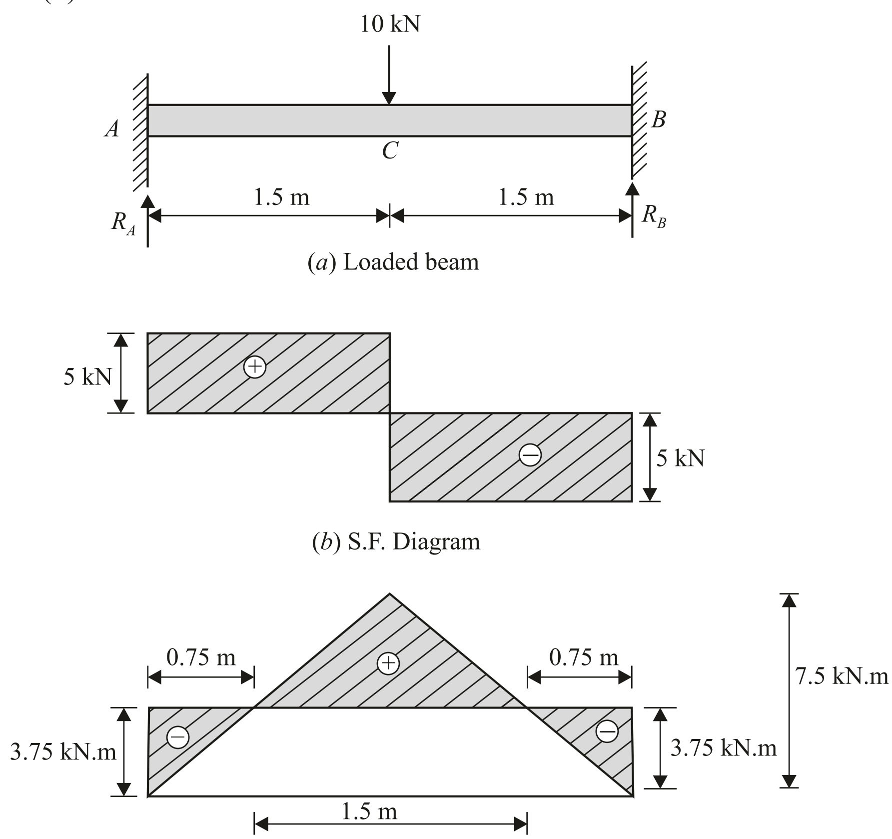
The bending moment diagram is shown in Fig. 16.10 (*c*). The centroid of the negative bending moment distribution lies at a distance of $(l/6)$ from *A* and that of positive bending moment distribution at a distance $\left(\frac{5l}{6}\right)$ from *A*.

Example 16.1

A fixed beam of length 3 m carries a point load of 10 kN at the mid span. Draw its shear force and bending moment diagrams, and also find the central deflection of the beam. Take $EI = 1500$ kN.m^2.

Solution:

Refer Fig 16.11 (*a*).



(*a*) Loaded beam

(*b*) S.F. Diagram

(*c*) B. M. Diagram

Fig. 16.11

Reactions at *A* and *B*

Let R_A and R_B be the vertical reactions at *A* and *B* respectively. Since the load is symmetrical on the beam, hence both vertical reactions are equal.

$$R_A = R_B = \frac{W}{2} = \frac{10}{2} = 5 \text{ kN } (\uparrow)$$

Calculations for shear forces

Shear force at *A* is

$$V_A = +\frac{W}{2} = +\frac{10}{2} = +5 \text{ kN}$$

The shear force between *A* and *C* remains constant at + 5 kN.

Shear force just to the right of C

$$= (+5 - 10) \text{ kN} = -5 \text{ kN}$$

The shear force between B and C remains constant at – 5 kN.

The shear force diagram is shown in Fig. 16.11 (*b*).

Fixing end moments at *A* and *B*

Let M_A and M_B be the fixing end moments at A and B respectively. The free moment diagram has triangular shape and the maximum value of the free moment is $\frac{Wl}{4} = \frac{10\times3}{4} = 7.5$ kN.m

Now
$$M_A = M_B = -\frac{Wl}{8} \quad \text{(using equation (16.4))}$$
$$= -\frac{10\times3}{8}$$
$$= -3.75 \text{ kN.m}$$

The bending moment diagram is shown in Fig. 16.11 (*c*).

The negative central maximum deflection is obtained by using equation (16.10) as

$$y_c = y_{\max} = \frac{Wl^3}{192EI}$$
$$= \frac{10\times(3)^3}{192\times1500}$$
$$= 9.375 \times 10^{-4} \text{ m}$$
$$= 0.9375 \text{ mm} \qquad \textbf{Ans.}$$

Example 16.2

A fixed beam of span 5 m carries two point loads each of 15 kN acting at a distance of 1 m from either end. Draw the shear force and bending moment diagrams, and also find the central deflection of the beam. Take $EI = 2000$ kN.m^2.

Solution:

Refer Fig. 16.12 (*a*).

Reactions at *A* and *B*

Let R_A and R_B be the support reactions at A and B respectively.

Since the load is symmetrically placed on the beam, hence

$$R_A = R_B = \frac{(15+15)}{2} = 15 \text{ kN } (\uparrow)$$

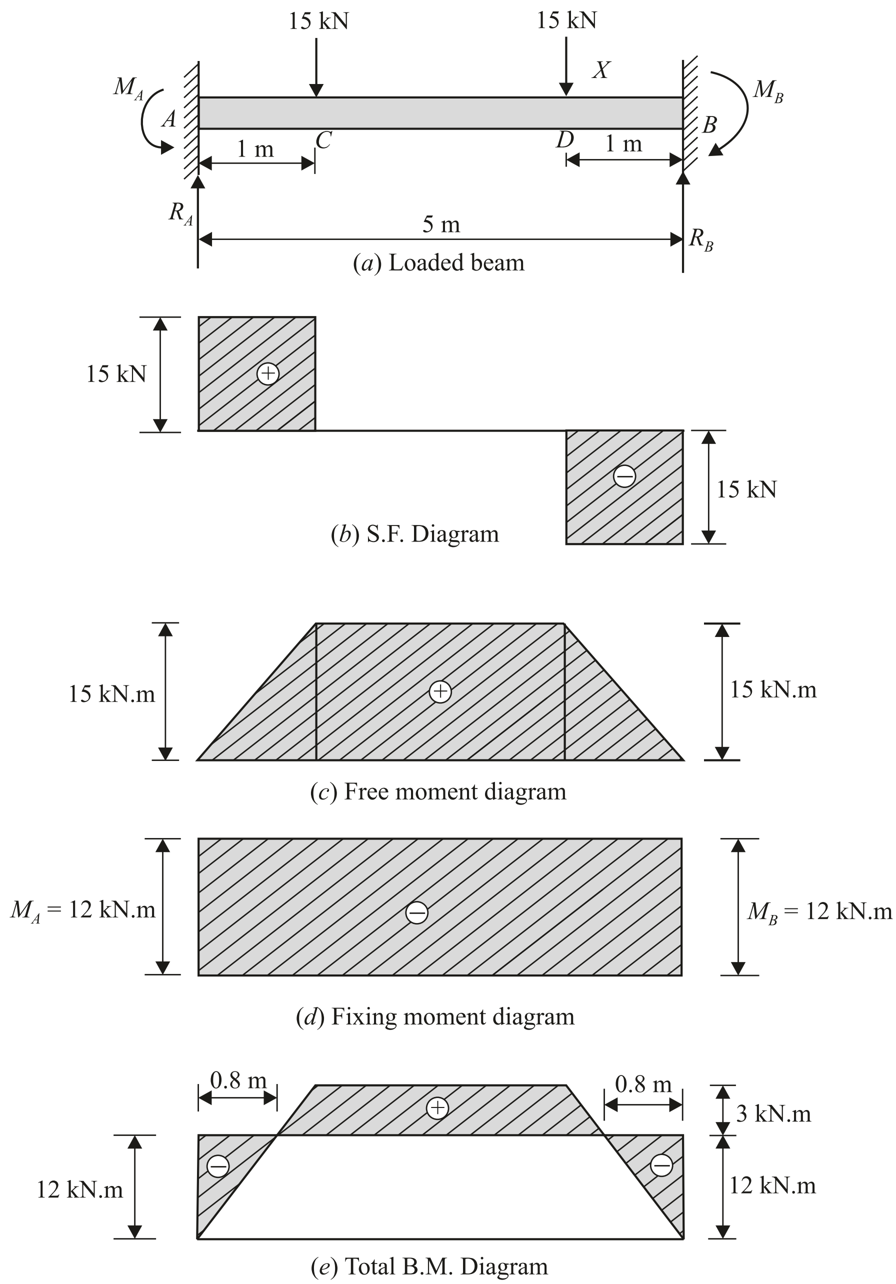


Fig. 16.12

Calculations for shear forces

Shear force at A is

$$V_A = +R_A = 15 \text{ kN}$$

The shear force between A and C remains constant at + 15 kN.

Shear force at B *is*

$$V_B = -R_B = -15 \text{ kN}$$

The shear force between B and D remains constant at – 15 kN.

There is no shear force between C and D. The *SFD* is shown in Fig. 16.11(*b*).

Fixing end moments at *A* and *B*

Let M_A and M_B be the fixing moments at the ends A and B of the beam respectively.

Because of symmetrical loading, the two fixing moments are same.

$$M_A = M_B$$

The fixing moment diagram is of rectangular spape (Fig. 16.11(*d*)) and the free moment diagram is of trapezoidal shape (Fig. 16.11(*c*)).

Free bending moment at $C = R_A \times 1$

$$= 15 \times 1 = 15 \text{ kN.m}$$

Free bending moment at $D = R_B \times 1$

$$= 15 \times 1 = 15 \text{ kN.m}$$

The bending moment between C and D remains constant at + 15 kN.m. The free moment diagram is shown in Fig. 16.12 (*c*).

The area of the free moment diagram is given as

$$A_1 = \text{Area of the trapezoidal moment distribution}$$

$$= \frac{1}{2}\times(5+3)\times 15$$

$$= 60 \text{ kN.m}^2$$

The area of the fixing moment diagram is given as

$$A_2 = -\text{Area of the rectangular moment distribution}$$

$$= -M_A \times 5$$

$$= -5M_A \text{ kN.m}^2$$

Now $$A_1 = A_2$$

or $$60 = -5M_A$$

$$M_A = -\frac{60}{5} = -12 \text{ kN.m}$$

Also $$M_B = M_A = -12 \text{ kN.m}$$

Using known values of M_A and M_B, the fixing moment diagram can be drawn as shown in Fig. 16.12 (*d*). The total bending moment diagram is shown in Fig. 16.12 (*e*).

Point of contraflexure

The total bending moment at a distance x in AC from A is given as

$$M_x = R_A x - M_A$$
$$= 15x - 12$$

For point of contraflexure, we have

$$M_x = 0$$
$$15x - 12 = 0$$

which gives $x = 0.8$ m

Since the load is symmetrically placed on the beam, hence there are two points of contraflexure, both occurring at 0.8 m from either end.

Deflection of the beam

The bending moment at any section XX at a distance x from A, using Macaulay's method, is given is

$$M_x = R_A x - M_A - 15(x-1) - 15(x-4)$$
$$= 15x - 12 - 15(x-1) - 15(x-4) \qquad ...(1)$$

Using differential equation of flexure for slope and deflection, we have

$$EI\frac{d^2y}{dx^2} = M = M_x$$
$$= 15x - 12 - 15(x-1) - 15(x-4) \qquad ...(2)$$

On integration, we have

$$EI\frac{dy}{dx} = \frac{15x^2}{2} - 12x + C_1 \Bigg| - \frac{15(x-1)^2}{2} \Bigg| - \frac{15(x-4)^2}{2} \qquad ...(3)$$

where C_1 is the constant of integration.

The boundary condition is

At A, where $x = 0$

$$\frac{dy}{dx} = 0$$

Using this boundary condition in equation (3), we get

$C_1 = 0$ (omitting the negative terms within the bracket)

On substituting C_1 in equation (3), we have

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{15x^2}{2} - 12x \Bigg| - \frac{15(x-1)^2}{2} \Bigg| - \frac{15(x-4)^2}{2}\right] \qquad ...(4)$$

The integration of equation (3) gives

$$EIy = \frac{15x^3}{6} - \frac{12x^2}{2} + C_1x + C_2 \left| -\frac{15(x-1)^3}{6} \right| -\frac{15(x-4)^3}{6} \qquad ...(5)$$

The boundary condition is

At A, where $x = 0$

$y = 0$

Using this boundary condition in equation (5), we get

$C_2 = 0$ (omitting the negative terms within the bracket)

On substituting C_1 and C_2 in equation (5), we have

$$y = \frac{1}{EI}\left[\frac{5x^3}{2} - 6x^2 \left| -\frac{5(x-1)^3}{2} \right| -\frac{5(x-4)^3}{2}\right] \qquad ...(6)$$

For maximum central deflection, put $x = 2.5$ m in equation (6) and omit the last term because of its negative value within the bracket.

$$y_{\max} = \frac{1}{2000}\left[\frac{5\times(2.5)^3}{2} - 6\times(2.5)^2 - \frac{5}{2}\times(2.5-1)^3\right]$$

$$= \frac{1}{2000}[39.06 - 37.5 - 8.43]$$

$$= \frac{1}{2000}\times 6.87 = -3.43\times10^{-3}\text{ m}$$

$= -3.43$ mm **Ans.**

The negative sign shows the downward deflection.

Example 16.3

A fixed beam of span 6 m carries a uniformly distributed load of intensity 15 kN/m over its entire span. Draw the shear force and bending moment diagrams, and also find the maximum deflection of the beam. Take $EI = 4000$ kN.m^2.

Solution: Given,

Length of the beam, $l = 6$ m

Uniformly distributed load, $w = 15$ kN/m

Flexural rigidity, $EI = 4000$ kN.m^2

The loaded fixed beam is shown in Fig. 16.13 (*a*).

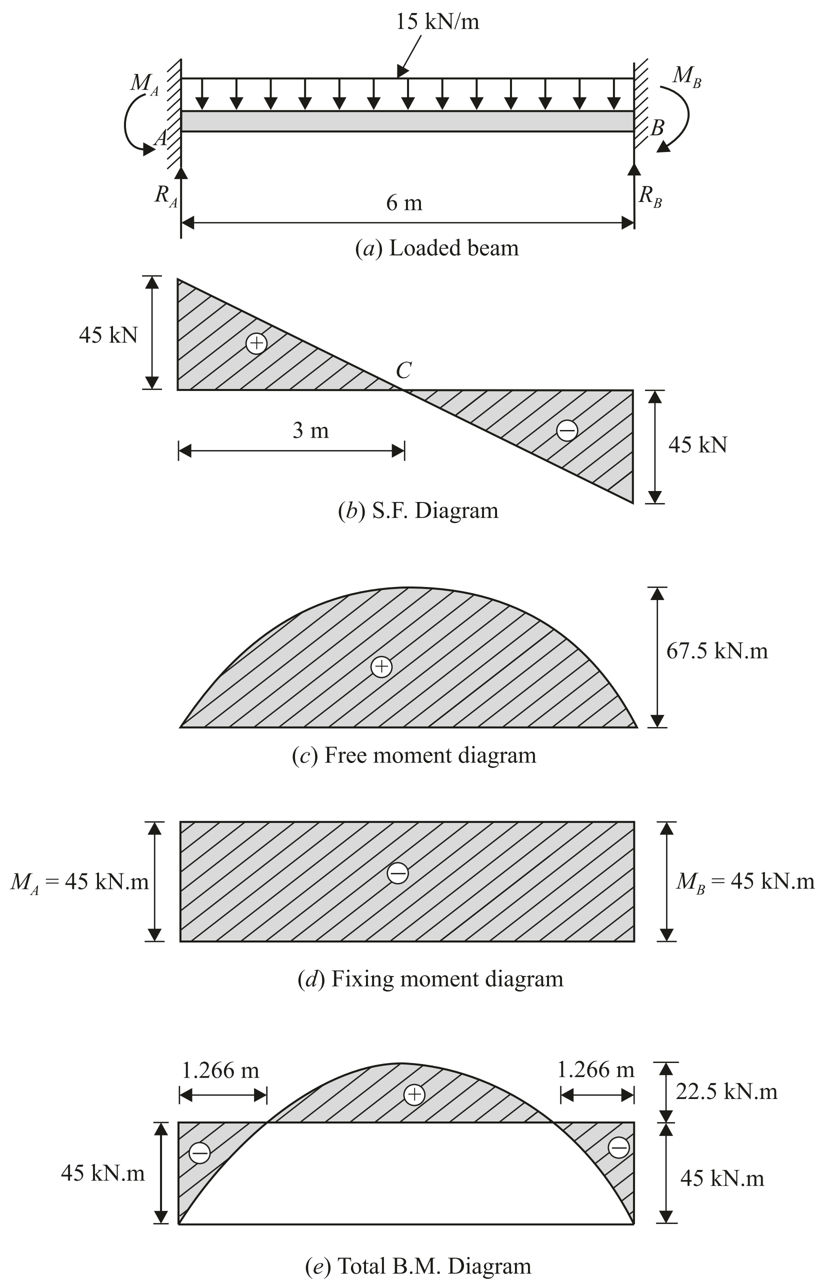


Fig. 16.13

Reactions at *A* and *B*

Since the load is symmetrically placed on the beam, hence it is equally distributed on the two support ends *A* and *B*.

$$R_A = R_B = \frac{wl}{2}$$

$$= \frac{15 \times 6}{2}$$

$$= 45 \text{ kN}$$

Both reactions are acting vertically upward.

Calculations for shear forces

Shear force at A is

$$V_A = +R_A$$
$$= +45 \text{ kN}$$

Shear force at C is

$$V_C = R_A - \frac{wl}{2}$$

$$= 45 - 45$$
$$= 0$$

Shear force at B is

$$V_B = -R_B$$
$$= -45 \text{ kN}$$

The shear force diagram is shown in Fig. 16.13 (*b*).

Fixing moments at *A* and *B*

Let M_A and M_B be the fixing moments at the ends A and B of the beam respectively.

Because of symmetry of load on the beam, we have

$$M_A = M_B = -\frac{wl^2}{12} \qquad \text{(using equation (16.34))}$$

$$= -\frac{15 \times 6^2}{12}$$

$$= -45 \text{ kN.m}$$

Using known values of M_A and M_B, the fixing moment diagram is constructed, and is shown in Fig. 16.13 (*d*).

The maximum value of the free bending moment at the centre of the simple beam is

$$\frac{wl^2}{8} = \frac{15\times 6^2}{8}$$
$$= 67.5 \text{ kN.m}$$

The free moment diagram is shown in Fig. 16.13 (*c*). The superposition of the fixing and free moment diagrams gives the total bending moment diagram as shown in Fig. 16.13 (*e*).

Point of contraflexure

There are two points of contraflexure of which one occurs at a distance of $0.211\, l = 0.211\times 6 = 1.266$ m from A and another occurs at a distance of 1.266 m from B or at a distance of $0.789\, l = 0.789 \times 6 = 4.734$ m from A.

Deflection of the beam

The negative central maximum deflection, using equation (16.42), is given as

$$y_{\max} = \frac{wl^4}{384EI}$$
$$= \frac{15\times 6^4}{384\times 4000} = 0.01265 \text{ m}$$
$$= 12.65 \text{ mm}$$ **Ans.**

Example 16.4

A fixed beam of length 4 m carries a point load of 20 kN at the midspan and a uniformly distributed load of 15 kN/m over the full span. Draw the shear force and bending moment diagrams and also find the maximum deflection of the beam. Take $EI = 3000$ kN.m^2.

Solution:

Given,

Length of the beam, $l = 4$ m

Point load, $W = 20$ kN

Uniform load, $w = 15$ kN/m

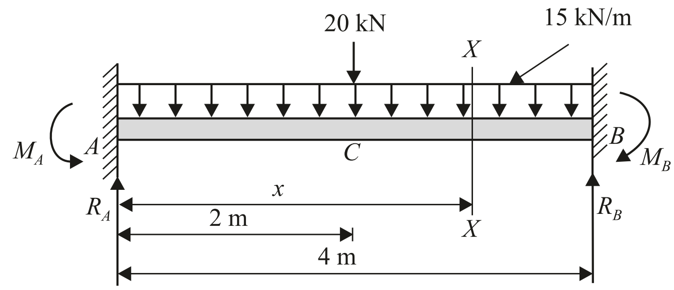
Flexural rigidity, $EI = 3000$ kN.m^2

The loaded beam is shown in Fig. 16.14 (*a*). The load 20 kN acts at the centre C of the beam.

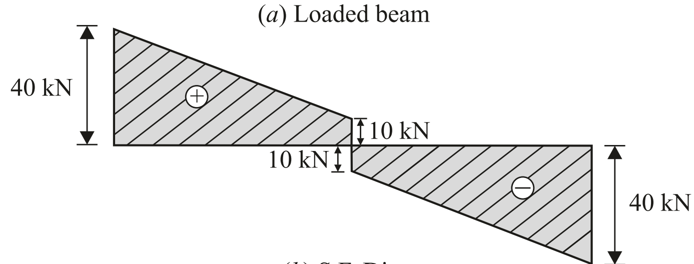
Reactions at *A* and *B*

Let R_A and R_B be the vertical reactions at A and B respectively. As the load is symmetrically placed on the beam, hence

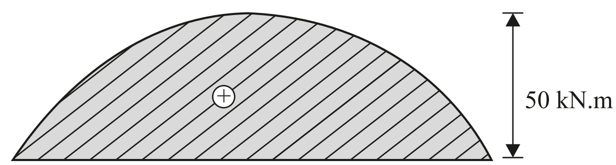
$$R_A = R_B = \frac{\text{Total load on the beam}}{2} = \frac{W + wl}{2}$$
$$= \frac{20 + (15\times 4)}{2} = 40 \text{ kN}$$



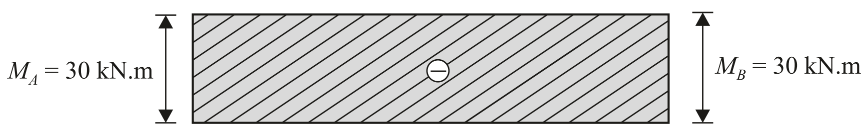
(a) Loaded beam



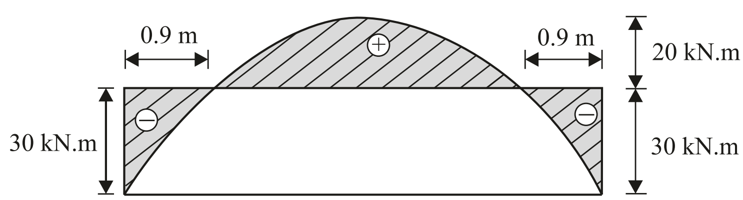
(b) S.F. Diagram



(c) Free moment diagram



(d) Fixing moment diagram



(e) Total B.M. Diagram

Fig. 16.14

Both reactions are acting vertically upward.

Calculations for shear forces

Shear force at A is

$$V_A = +R_A$$
$$= +40 \text{ kN}$$

Shear force just to the left of C

$$= +R_A - (15 \times 2)$$
$$= +40 - 30$$
$$= 10 \text{ kN}$$

Shear force just to the right of C

$$= +R_A - (15 \times 2) - 20$$
$$= 40 - 30 - 20$$
$$= -10 \text{ kN}$$

Shear force at B is

$$V_B = -R_B$$
$$= -40 \text{ kN}$$

The shear force diagram is shown in Fig. 16.14 (*b*).

Free bending moment

The maximum value of the free bending moment occurs at the midspan of the beam, and is equal to the sum of the maximum free moment due to point load and the maximum free moment due to uniform load, both acting separately.

$$(\text{B.M})_{\text{free}} = +\left(\frac{Wl}{4} + \frac{wl^2}{8}\right)$$
$$= +\left(\frac{20 \times 4}{4} + \frac{15 \times 4^2}{8}\right)$$
$$= +(20 + 30) = +50 \text{ kN.m}$$

The free moment curve is not continuous throughout, but is broken at the midspan, although it is symmetrical about the midspan. The free moment diagram is shown in Fig. 16.14 (*c*).

Fixing moments at *A* and *B*

Let M_A and M_B be the fixing moments at A and B respectively. As the load is symmetrically placed on the beam, the two fixing moments are equal. Each fixing end moment is equal to the sum of the fixing moment due to point load only and the fixing moment due to uniform load only.

$$M_A = M_B = -\left(\frac{Wl}{8} + \frac{wl^2}{12}\right)$$
$$= -\left(\frac{20 \times 4}{8} + \frac{15 \times 4^2}{12}\right)$$
$$= -(10 + 20)$$
$$= -30 \text{ kN.m}$$

Using known values of M_A and M_B, the fixing moment diagram can be constructed, and is shown in Fig. 16.14 (*d*).

Now the superposition of the free moment diagram and the fixing moment diagram gives the total bending moment diagram as shown in Fig. 16.14 (*e*).

Point of contraflexure

There are two points of contraflexure, both equidistant from either side. To find the point of contraflexure, we consider a section *XX* at a distance *x* from *A* as shown in Fig. 16.14 (*a*).

Using Macaulay's method, the bending moment at the section is given as

$$M_x = R_A x - M_A - w \times x \times \frac{x}{2} - W(x-2)$$

$$= 40\,x - 30 - \frac{15x^2}{2} - 20\,(x-2)$$

$$= 40\,x - 30 - 7.5x^2 - 20\,(x-2) \qquad \text{...(1)}$$

For point of contraflexure, $M_x = 0$

$$40\,x - 30 - 7.5x^2 \,|\!-20\,(x-2) = 0 \qquad \text{...(2)}$$

Since the left point of contraflexure lies in *AC* for which $x < 2$, hence

$$40\,x - 30 - 7.5\,x^2 = 0 \text{ (Omitting the negative term within the bracket of equation (2))}$$

or $\quad 7.5\,x^2 - 40\,x + 30 = 0$

which gives

$$x = \frac{40 \pm \sqrt{(-40)^2 - 4 \times 7.5 \times 30}}{2 \times 7.5}$$

$$= \frac{40 \pm 26.46}{15}$$

$$= 4.43 \text{ m or } 0.9 \text{ m}$$

The value of $x = 4.43$ m is not acceptable, hence the accepted value of $x = 0.9$ m from end *A*.

Hence, one point of contraflexure lies at a distance of 0.9 m from *A* and another point of contraflexure lies at a distance of 0.9 m from *B* (because of symmetry of load on the beam) or at a distance of (4 – 0.9) m = 3.1 m from *A*.

Deflection of the beam

Using differential equation of flexure for slope and deflection, we have

$$EI\frac{d^2y}{dx^2} = M = M_x$$

$$= 40x - 30 - 7.5x^2 \,|\, - 20(x-2) \qquad \text{(using equation (1))}$$

On integration, we get

$$EI\frac{dy}{dx} = \frac{40x^2}{2} - 30x + C_1 - \frac{7.5x^3}{3}\Bigg| - \frac{20(x-2)^2}{2}$$

$$= 20x^2 - 30x + C_1 - 2.5x^3 \,|- 10\,(x-2)^2 \qquad ...(3)$$

where C_1 is the constant of integration.

The integration of equation (3) gives

$$EIy = \frac{20x^3}{3} - \frac{30x^2}{2} + C_1x + C_2 - \frac{2.5x^4}{4} - \frac{10\,(x-2)^3}{3}$$

$$= 6.67\,x^3 - 15x^2 + C_1\,x + C_2 - 0.625x^4 \,|- 3.34\,(x-2)^3 \qquad ...(4)$$

The boundary conditions are

At A, where $x = 0$

$$\frac{dy}{dx} = 0 \text{ and } y = 0$$

On substituting the boundary conditions in equations (3) and (4), and omitting the negative term within the bracket, we get

$$C_1 = 0$$

and $C_2 = 0$

Equations (3) and (4) on substituting C_1 and C_2 are reduced to

$$\frac{dy}{dx} = \frac{1}{EI}\left[20x^2 - 30x - 2.5x^3\Big| - 10\,(x-2)^2\right] \qquad ...(5)$$

and

$$y = \frac{1}{EI}\left[6.67x^3 - 15x^2 - 0.625x^4\Big| - 3.34\,(x-2)^3\right] \qquad ...(6)$$

Equation (5) gives the slope and equation (6) gives the deflection for any value of x.

The maximum deflection occurs downward at the centre of the beam, for which $x = 2$ m.

On substituting $x = 2$ m in equation (6), we have

$$y_{\max} = \frac{1}{EI}\left[6.67 \times 2^3 - 15 \times 2^2 - 0.625 \times 2^4\right]$$

$$= \frac{1}{EI}[53.36 - 60 - 10]$$

$$= -\frac{1}{EI} \times 16.64 = -\frac{16.64}{3000}$$

$$= -5.55 \times 10^{-3} \text{ m} = -5.55 \text{ mm}$$ **Ans.**

The negative sign shows the downward deflection.

Alternative method to find the deflection

Alternatively, the deflection of the beam can be found by adding the maximum deflection due to point load and the maximum deflection due to uniform load, and is given as

$$y_{\max} = -\left[\frac{Wl^3}{192EI} + \frac{wl^4}{384EI}\right] = -\left[\frac{20\times 4^3}{192\times 3000} + \frac{15\times 4^4}{384\times 3000}\right]$$

$$= -(2.22 \times 10^{-3} + 3.33 \times 10^{-3})$$

$$= -5.55 \times 10^{-3} \text{ m} = -5.55 \text{ mm}$$

Example 16.5

A fixed beam of length 3 m carries a point load and a uniform load as shown in Fig. 16.15. Determine the end reactions, the fixing end moments and the deflection under the load of 25 kN. Take EI = 15000 kN.m^2.

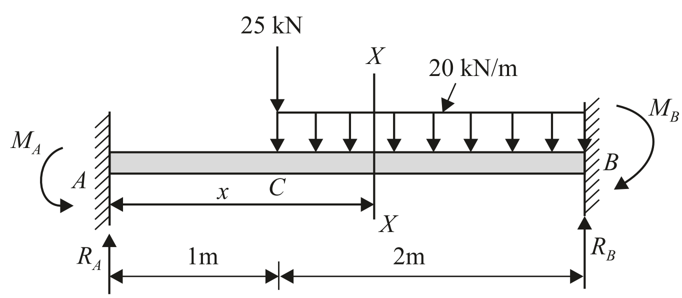


Fig. 16.15

Solution:

Given,

Length of the beam, l = 3 m

Point load, W = 25 kN

Uniform load, w = 20 kN/m

Let R_A and R_B be the end reactions, and M_A and M_B be the fixing end moments respectively.

Consider a section XX at a distance x from A as shown in Fig. 16.15.

Using Macaulay's method, the bending moment at the section can be expressed as

$$M_x = R_A x - M_A - 25(x-1) - 20(x-1) \times \frac{(x-1)}{2}$$

$$= R_A x - M_A - 25(x-1) - 10(x-1)^2 \qquad ...(1)$$

Using differential equation of flexure for slope and deflection, we have

$$EI\frac{d^2y}{dx^2} = M = M_x$$

$$= R_A x - M_A - 25(x-1) - 10(x-1)^2 \qquad ...(2)$$

On integration, we have

$$EI\frac{dy}{dx} = R_A \times \frac{x^2}{2} - M_A \times x + C_1 \Bigg| - \frac{25(x-1)^2}{2} \Bigg| - \frac{10(x-1)^3}{3} \qquad ...(3)$$

where C_1 is the constant of integration.

Integrating equation (3), we get

$$EIy = R_A \times \frac{x^3}{6} - M_A \times \frac{x^2}{2} + C_1 x + C_2 \Bigg| - \frac{25}{6} \times (x-1)^3 \Bigg| - \frac{10}{12} \times (x-1)^4 \qquad ...(4)$$

where C_2 is another constant of integration.

The boundary conditions are

At A, where $x = 0$

$$\frac{dy}{dx} = 0 \text{ and } y = 0$$

Equation (3) on using the boundary condition gives

$C_1 = 0$ (omitting the negative terms within the bracket)

Equation (4) on using the boundary condition gives

$C_2 = 0$ (omitting the negative terms with the bracket)

Equations (3) and (4) on substituting C_1 and C_2 are reduced to

$$\frac{dy}{dx} = \frac{1}{EI}\left[R_A \frac{x^2}{2} - M_A x \Bigg| - \frac{25(x-1)^2}{2} \Bigg| - \frac{10(x-1)^3}{3}\right] \qquad ...(5)$$

$$y = \frac{1}{EI}\left[R_A \frac{x^3}{6} - M_A \frac{x^2}{2} \Bigg| - \frac{25}{6}(x-1)^3 \Bigg| - \frac{5}{6}(x-1)^4\right] \qquad ...(6)$$

Again using the boundary condition, we have

At B, where $x = 3$ m

$y = 0$

From equation (6), we have

$$0 = \frac{1}{EI}\left[R_A \times \frac{3^3}{6} - M_A \times \frac{3^2}{2} - \frac{25}{6}(3-1)^3 - \frac{5}{6}(3-1)^4\right]$$

$$= \frac{1}{EI}\left[4.5R_A - 4.5M_A - 33.34 - 13.34\right]$$

or $4.5\ R_A - 4.5\ M_A = 46.68$ ...(7)

Again using the boundary condition, we have

At B, where $x = 3$ m

$$\frac{dy}{dx} = 0$$

From equation (5), we have

$$0 = \frac{1}{EI}\left[R_A \times \frac{3^2}{2} - M_A \times 3 - \frac{25}{2} \times (3-1)^2 + \frac{10}{3} \times (3-1)^3\right]$$

$$= \frac{1}{EI}[4.5R_A - 3M_A - 50 - 26.67]$$

or $$4.5\,R_A - 3M_A = 76.67 \qquad ...(8)$$

Solving equations (7) and (8), we get

$$M_A = 20 \text{ kN.m}$$ **Ans.**

$$R_A = 30.37 \text{ kN } (\uparrow)$$ **Ans.**

Now $$R_A + R_B = 25 + 20 \times 2$$

$$= 25 + 40 = 65 \text{ kN}$$

or $$R_B = 65 - R_A$$

$$= 65 - 30.37 \text{ kN}$$

$$= 34.63 \text{ kN } (\uparrow)$$ **Ans.**

Taking moments about A, we have

$$M_B - R_B \times 3 + 25 \times 1 + 20 \times 2 \times \left(1 + \frac{2}{2}\right) - M_A = 0$$

$$M_B - 34.63 \times 3 + 25 + 80 - 20 = 0$$

$$M_B - 103.89 + 25 + 80 - 20 = 0$$

$$M_B - 18.89 = 0$$

or $$M_B = 18.89 \text{ kN.m}$$ **Ans.**

M_A and M_B are actually negative moments.

Deflection of the beam

The deflection of the beam under the load of 25 kN acting at C can be obtained by putting $x = 1$ m in equation (6).

$$y = \frac{1}{EI}\left[30.37\times\frac{1^3}{6}-20\times\frac{1^2}{2}\right]$$

$$= \frac{1}{EI}(5.06-10)$$

$$= -\frac{1}{EI}\times 4.94$$

$$= -\frac{4.94}{15000} = -3.29\times 10^{-4}\text{ m}$$

$$= -0.329\text{ mm}$$ **Ans.**

The negative sign shows the downward deflection.

Example 16.6

A fixed beam AB of length 5 m is subjected to a uniformly varying load as shown in Fig. 16.16. Determine the fixing moments at A and B of the beam.

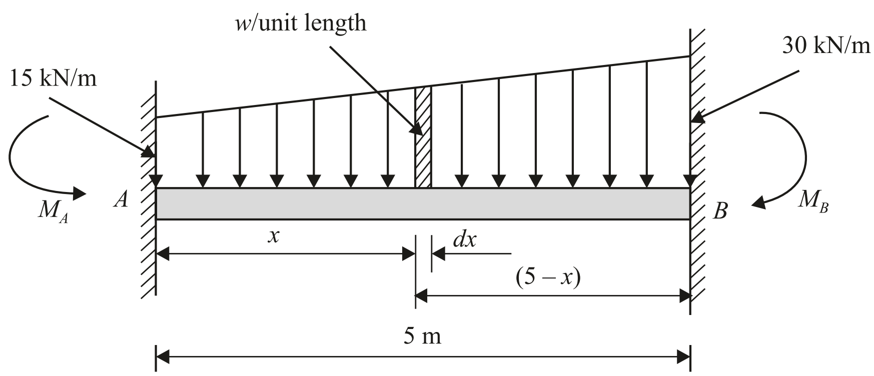


Fig. 16.16

Solution:

Given,

Length of the beam, $l = 5$ m

Uniform load at A, $w_1 = 15$ kN/m

Uniform load at B, $w_2 = 30$ kN/m

Consider a section of width dx of the beam at a distance x from A, where the intensity of load is w.

Load on the section $= wdx$

The fixing end moment at A, using equation (16.46), is given as

$$M_A = -\int_0^l \frac{wx(l-x)^2\,dx}{l^2} \quad \text{...(1)}$$

where $$w = w_1 + \frac{(w_2 - w_1)x}{l} \quad \text{(from equation (16.48))}$$

$$= 15 + \frac{(30-15)x}{5} = 15 + 3x \quad \text{...(2)}$$

On substituting w from equation (2) in equation (1), we have

$$M_A = -\int_0^5 \frac{(15+3x)\,x\,(5-x)^2\,dx}{5^2}$$

$$= -\frac{3}{25}\int_0^5 (5+x)\,x\,(5-x)^2\,dx$$

$$= -\frac{3}{25}\int_0^5 (5+x)\times x\times(25-10x+x^2)dx$$

$$= -\frac{3}{25}\int_0^5 (5x+x^2)\times(25-10x+x^2)dx$$

$$= -\frac{3}{25}\int_0^5 (125x-50x^2+5x^3+25x^2-10x^3+x^4)dx$$

$$= -\frac{3}{25}\int_0^5 (125x-25x^2-5x^3+x^4)dx$$

$$= -\frac{3}{25}\left[\frac{125x^2}{2}-\frac{25x^3}{3}-\frac{5x^4}{4}+\frac{x^5}{5}\right]_0^5$$

$$= -\frac{3}{25}\left[\frac{125\times 5^2}{2}-\frac{25\times 5^3}{3}-\frac{5\times 5^4}{4}+\frac{5^5}{5}\right]$$

$$= -\frac{3}{25}(1562.5-1041.67-781.25+625)$$

$$= -\frac{3}{25}(364.58)$$

$$= -43.75 \text{ kN.m}$$ **Ans.**

The fixing end moment at B, using equation (16.47), is given as

$$M_B = -\int_0^l \frac{w(l-x)x^2dx}{l^2} \qquad ...(3)$$

On substituting w from equation (2) in equation (3), we have

$$M_B = -\int_0^5 \frac{(15+3x)(5-x)x^2dx}{5^2}$$

$$= -\frac{3}{25}\int_0^5 (5+x)(5-x)x^2dx = -\frac{3}{25}\int_0^5 (25-x^2)x^2dx$$

$$= -\frac{3}{25}\int_0^5 (25x^2-x^4)dx = -\frac{3}{25}\left[\frac{25x^3}{3}-\frac{x^5}{5}\right]_0^5$$

$$= -\frac{3}{25}\left[\frac{25\times 5^3}{3}-\frac{5^5}{5}\right]$$

$$= -\frac{3}{25}(1041.67-625)$$

$$= -\frac{3}{25}\times 416.67$$

$$= -50 \text{ kN.m}$$ **Ans.**

Example 16.7

A fixed beam AB of length 4 m is subjected to a uniformly varying load as shown in Fig. 16.17. Determine the fixing moments at A and B.

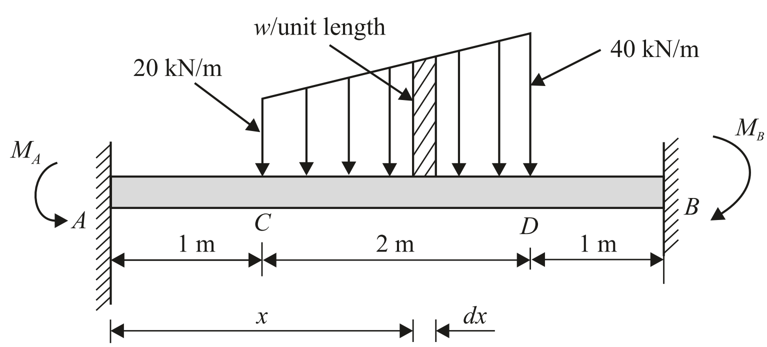


Fig. 16.17

Solution:

Given,

Length of the beam, $l = 4$ m

Uniform load at, C $w_1 = 20$ kN/m

Uniform load at D, $w_2 = 40$ kN/m

Distance, $a = 1$ m

Distance, $b = 3$ m

Consider a section of width dx of the beam at a distance x from A, where the intensity of load is w.

Load on the section is wdx.

The fixing moment at A, using equation (16.46), is given as

$$M_A = -\int_{x=a}^{x=b} \frac{wx(l-x)^2}{l^2}dx \qquad ...(1)$$

$$w = w_1 + \frac{(w_2 - w_1)(x-a)}{(b-a)} \qquad \text{(from equation (16.48))}$$

$$= 20 + \frac{(40-20)\times(x-1)}{(3-1)}$$

$$= 20 + 10\,(x-1)$$

$$= 10\,x + 10 \qquad ...(2)$$

On substituting w from equation (2) in equation (1), we have

$$M_A = -\int_1^3 \frac{(10x+10)\,x\,(4-x)^2\,dx}{4^2}$$

$$= -\frac{10}{16}\int_1^3 (x+1)\,x\,(4-x)^2\,dx$$

$$= -\frac{5}{8}\int_1^3 (x^2+x)(4-x)^2\,dx$$

$$= -\frac{5}{8}\int_1^3 (x^2+x)(16-8x+x^2)dx$$

$$= -\frac{5}{8}\int_1^3 (16x^2-8x^3+x^4+16x-8x^2+x^3)dx$$

$$= -\frac{5}{8}\int_1^3 (16x+8x^2-7x^3+x^4)dx$$

$$= -\frac{5}{8}\left[\frac{16x^2}{2}+\frac{8x^3}{3}-\frac{7x^4}{4}+\frac{x^5}{5}\right]_1^3$$

$$= -\frac{5}{8}\left[\frac{16\times 3^2}{2}+\frac{8\times 3^3}{3}-\frac{7\times 3^4}{4}+\frac{3^5}{5}-\frac{16\times 1^2}{2}-\frac{8\times 1^3}{3}-\frac{7\times 1^4}{4}-\frac{1^5}{5}\right]$$

$$= -\frac{5}{8}[72+72-141.75+48.6-8-2.67+1.75-0.2]$$

$$= -\frac{5}{8}\times 41.73$$

$$= -26.08 \text{ kN.m}$$ **Ans.**

The fixing end moment at B, using equation (16.47), is given as

$$M_B = -\int_{x=a}^{x=b} \frac{w(l-x)x^2 dx}{l^2} \quad \text{...(3)}$$

On substituting w from equation (2) in equation (3), we have

$$M_B = -\int_1^3 \frac{(10x+10)(4-x)x^2 dx}{4^2}$$

$$= -\frac{10}{16}\int_1^3 (x+1)(4-x)x^2 dx$$

$$= -\frac{10}{16}\int_1^3 (x^3+x^2)(4-x)dx$$

$$= -\frac{10}{16}\int_1^3 (4x^3-x^4+4x^2-x^3)\, dx$$

$$= -\frac{10}{16}\int_1^3 (4x^2+3x^3-x^4)dx$$

$$= -\frac{10}{16}\left[\frac{4x^3}{3}+\frac{3x^4}{4}-\frac{x^5}{5}\right]_1^3$$

$$= -\frac{10}{16}\left[\frac{4}{3}\times 3^3+\frac{3}{4}\times 3^4-\frac{1}{5}\times 3^5-\frac{4}{3}\times 1^3-\frac{3}{4}\times 1^4+\frac{1}{5}\times 1^5\right]$$

$$= -\frac{10}{16}[36+60.75-48.6-1.34-0.75+0.2]$$

$$= -\frac{10}{16}\times 46.26$$

$$= -28.91 \text{ kN.m}$$ **Ans.**

Example 16.8

A fixed beam of length 7 m has a rectangular cross-section of 80 mm × 170 mm. If its right end moves vertically through 12 mm, find the fixing moments and the reactions at the ends of the beam. Take $E = 210$ GPa.

Solution:

Given,

Length of the beam, $l = 7$ m

Width of the beam, $b = 80$ mm $= 80 \times 10^{-3}$ m

Depth of the beam, $d = 170$ mm $= 170 \times 10^{-3}$ m

Deflection of support, $\delta = 12$ mm $= 12 \times 10^{-3}$ m

Modulus of elasticity, $E = 210$ GPa $= 210 \times 10^{9}$ Pa

The moment of inertia of the cross-section of the beam is found as

$$I = \frac{1}{2}bd^3$$

$$= \frac{1}{12}\times(80\times10^{-3})\times(170\times10^{-3})^3$$

$$= 3.275 \times 10^{-5}\ \text{m}^4$$

The fixing moment M_A, using equation (16.79), is given as

$$M_A = \frac{6EI\delta}{l^2}$$

$$= \frac{6\times210\times10^9\times3.275\times10^{-5}\times12\times10^{-3}}{7^2}$$

$$= 10105.71\ \text{N.m}$$

$$= 10.10\ \text{kN.m}$$ **Ans.**

and $M_B = -M_A$

$= -10.10$ kN.m **Ans.**

The reactions at A and B are calculated as

$$R_A = \frac{12EI\delta}{l^3}$$ (using equation (16.80))

$$= \frac{12 \times 210 \times 10^9 \times 3.275 \times 10^{-5} \times 12 \times 10^{-3}}{7^3}$$

$$= 2887.34 \text{ N}$$

$$= 2.88 \text{ kN } (\uparrow) \qquad \textbf{Ans.}$$

and $$R_B = -R_A$$

$$= -2.88 \text{ kN}$$

$$= 2.88 \text{ kN } (\downarrow) \qquad \textbf{Ans.}$$

SHORT ANSWER QUESTIONS

1. How is a fixed beam defined?
2. How is a fixed beam different from a simple beam?
3. How does the fixing moment diagram differ from the free moment diagram?
4. What is total bending moment diagram?
5. What is meant by sinking of a support?

MULTIPLE CHOICE QUESTIONS

1. Consider the following statements about a fixed beam:

1. The fixing moments are negative moments.
2. The fixing moments are developed at the support ends.
3. The free moment diagram consists of positive moments.
4. The areas of the free moment diagram and the fixing moment diagram are equal in magnitude but opposite in sign.

Of these statements:

(*a*) 1 and 2 are true (*b*) 1, 2 and 3 are true

(*c*) 2 and 3 are true (*d*) 1, 2, 3 and 4 are true.

2. Which of the following statements about a fixed beam is wrong?

(*a*) Both slope and deflection at the support ends are zero.

(*b*) The centroidal distance of both the free and the fixing moment diagrams from either support are equal.

(*c*) The free moments are negative moments.

(*d*) The fixed beam is also called restrained beam.

3. Consider the following statements about a fixed beam:

1. It has fixed supports on both ends.
2. The slope and deflection at the support ends are zero.
3. It is a statically indeterminate beam.
4. It involves four unknown reaction components in the absence of axial forces.

Of these statements:

(*a*) 1 alone is true (*b*) 1, 2, 3 and 4 are true

(*c*) 1 and 3 are true (*d*) 1, 2 and 3 are true.

4. The free moment diagram of a fixed beam is constructed by treating it as a/an

(*a*) continuous beam (*b*) simple beam

(*c*) cantilever beam (*d*) overhanging beam.

5. For a fixed beam of length l carrying a central point load W, the fixing moments at both ends of the beam are given as

(*a*) $\frac{Wl}{8}, \frac{Wl}{4}$ (*b*) $\frac{Wl}{6}, \frac{Wl}{8}$

(*c*) $\frac{Wl}{8}, \frac{Wl}{8}$ (*d*) $\frac{Wl}{4}, \frac{Wl}{4}$.

6. For a fixed beam of length l carrying a uniformly distributed load (*udl*) of intensity w/meter over the full span, the fixing moments at both ends of the beam are given as

(*a*) $\frac{wl^2}{8}, \frac{wl^2}{8}$ (*b*) $\frac{wl^2}{12}, \frac{wl^2}{12}$

(*c*) $\frac{wl^2}{8}, \frac{wl^2}{12}$ (*d*) $\frac{wl^2}{4}, \frac{wl^2}{4}$.

7. For a fixed beam of length l carrying a central point load W, the maximum downward deflection is given as

(*a*) $\frac{Wl^2}{192EI}$ (*b*) $\frac{Wl^4}{384EI}$

(*c*) $\frac{Wl^3}{192EI}$ (*d*) $\frac{Wl^3}{384EI}$.

8. For a fixed beam of length l carrying a uniformly distributed load (*udl*) of intensity w/meter over the entire span, the maximum downward deflection is given as

(*a*) $\frac{wl^3}{192EI}$ (*b*) $\frac{wl^4}{384EI}$

(*c*) $\frac{wl^3}{384EI}$ (*d*) $\frac{wl^4}{192EI}$.

9. For a fixed beam of length l carrying a uniformly distributed load (*udl*) of intensity w/meter over the full span, the fixing moments at both ends of the beam are given as

(*a*) $\frac{wl^2}{8}, \frac{wl^2}{8}$ (*b*) $\frac{wl^3}{12}, \frac{wl^3}{12}$

(*c*) $\frac{wl^2}{12}, \frac{wl^2}{8}$ (*d*) $\frac{wl^2}{12}, \frac{wl^2}{12}$.

10. For a fixed beam of length l carrying a gradually varying load from zero at A to w per meter at B, the fixing moments are given as

(*a*) $M_A = \frac{wl^2}{15}, M_B = \frac{wl^2}{30}$ (*b*) $M_A = \frac{wl^2}{20}, M_B = \frac{wl^2}{30}$

(*c*) $M_A = \frac{wl^2}{30}, M_B = \frac{wl^2}{20}$ (*d*) $M_A = \frac{wl^2}{30}, M_B = \frac{wl^2}{15}$.

11. The reactions at the ends of a fixed beam, when its right end sinks by an amount δ, are given as

(*a*) $\dfrac{12EI\delta}{l^3}, \dfrac{6EI\delta}{l^3}$ (*b*) $\dfrac{6EI\delta}{l^3}, \dfrac{6EI\delta}{l^3}$

(*c*) $\dfrac{12EI\delta}{l^3}, \dfrac{12EI\delta}{l^3}$ (*d*) $\dfrac{12EI\delta}{l^2}, \dfrac{12EI\delta}{l^2}$.

12. The fixing end moments of a fixed beam, when its right end sinks by an amount δ, are given as

(*a*) $\dfrac{6EI\delta}{l^2}, \dfrac{12EI\delta}{l^2}$ (*b*) $\dfrac{6EI\delta}{l^2}, \dfrac{6EI\delta}{l^2}$

(*c*) $\dfrac{6EI\delta}{l^3}, \dfrac{6EI\delta}{l^3}$ (*d*) $\dfrac{12EI\delta}{l^2}, \dfrac{6EI\delta}{l^2}$.

ANSWERS

1. (*d*) **2.** (*c*) **3.** (*b*) **4.** (*b*) **5.** (*c*) **6.** (*b*) **7.** (*c*) **8.** (*b*)
9. (*d*) **10.** (*c*) **11.** (*c*) **12.** (*b*)

EXERCISES

1. A fixed beam of span 7 m carries a uniformly distributed load of 90 kN/m over the full span as well as a central point load of 30 kN. If the bending stress is limited to 90 MPa and the deflection is not to exceed 2.5 mm, find the depth of the cross-section of the beam. Take E = 210 GPa.

(*Ans*. 583 mm).

2. A fixed beam of span 7 m carries a uniformly distributed load of 20 kN/m over the one-half length from the left end along with a point load of 120 kN at 5 m from the left end. Find the reactions and the fixing moments at the ends and drawn the B.M. diagram of the beam.

(*Ans*. – 105.4kN, –148kN, 80.7kN.m, 109.3 kN.m).

3. A fixed beam of span 3 m carries two point loads of 10 kN each at a distance of 1 m from either of the support end. Draw the S.F. and B.M. diagrams of the beam. Is there any point of contraflexure? Also, find the central deflection. Take EI = 1200 kN.m^2.

(*Ans*. $R_A = R_B$ = 10 kN, $M_A = M_B$ = 6.67 kN.m, two points of contraflexure occur each at 0.67 m from either end, y_{max} = 1.7 mm).

4. A fixed beam of span 4 m carries a uniformly distributed load of 5 kN/m over the entire span as well as a central point load of 5 kN. Find the fixing end moments and the maximum deflection of the beam. Also, draw the S.F. and B.M. diagrams of the beam. Take EI = 2000 kN.m^2.

(*Ans*. $M_A = M_B$ = 9.17 kN.m, y_{max} = 2.5 mm).

5. A fixed beam of span 7 m carries a point load of 30 kN at a distance of 3m from the left end. Determine the fixing end moments, the position of contraflexure, and the deflection under the load. Take EI = 16000 kN.m^2.

(*Ans*. – 29.38 kN.m, – 22.04 kN.m, two points of coutraflexure: one at 1.615 m and another at 5.133 m both from left end, 3.15 mm).

6. A fixed beam *ABCD* is supported at points *A, D* and carries two points loads each of 5 kN acting at points *B* and *C*. The distances between various points are defined as: $AB = BC = CD$ = 1.8 m. Find the fixing end moments and the central deflection of the beam. Take EI = 1470 kN.m^2.

(*Ans*. $M_A = M_D$ = – 6 kN.m, y_{max} = 4.13 mm).

7. A couple of 4 kN.m is acting at the centre of a 3 m long fixed beam in the clockwise direction. Determine the reactions and the fixing moments at the ends of the beam.

(*Ans*. R_A = – 2kN, R_B = 2kN, M_A = –1kN.m, M_B = 1 kN.m).

8. A fixed beam of span 6 m is subjected to a uniformly varying load, which varies from zero at the left end to 2 kN/m at the right end of the beam. Determine the reactions and the fixing moments at the ends.

(*Ans*. R_A = 1.8 kN, R_B = 4.2 kN, M_A = – 2.4 kN.m, M_B = – 3.6 kN.m).

❑❑❑

17

Rotating Rings, Discs and Cylinders

Simeon Denis Poisson
(1781-1840)

Simeon Denis Poisson, born on 21 June 1781, was a great French mathematician, geometer and physicist. He worked under two famous mathematicians Pierre-Simon Laplace and Joseph-Louis Lagrange, and Sadi Carnot, who is called the father of thermodynamics, was his one of the famous students. Poisson is most known for applying mathematics to solve problems in electricity and magnetism, mechanics and other areas of physics. He is known for his work on definite integrals, electromagnetic theory and probability. In his Poisson equation, also known as potential theory equation, he corrected the Laplace's second order partial differential equation for potential. He is also known for Poisson's ratio, which is widely used in strength of materials. The Poisson distribution is extremely useful in the analysis of problems relating to radioactivity, traffic and random occurrence of events in time or space. In 1818, he was elected a fellow of the Royal Society and in 1823, a foreign member of the Royal Swedish Academy of Sciences. He is among the 72 people whose names are inscribed on the Eiffel Tower.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- Why is hoop stress also called circumferential stress?
- What is radial stress?
- Why is the area moment of inertia often called the second moment of area?
- What are the centroidal axes?
- Where is parallel-axes theorem used?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_17

17.1 INTRODUCTION

Components such as turbine shafts and discs while rotating at high speeds are subjected to large centrifugal forces, which in turn, produce large stresses that are distributed symmetrically about their axes of rotation. The stress analysis of these components is useful in their safe design so as to prevent their failure.

17.2 ROTATING RING

The force analysis of a thin rotating ring can also be applied to a thin rotating cylinder or rim-type flywheels.

Consider a thin ring or a thin cylinder rotating with a constant angular velocity ω rad/s about its axis as shown in Fig. 17.1.

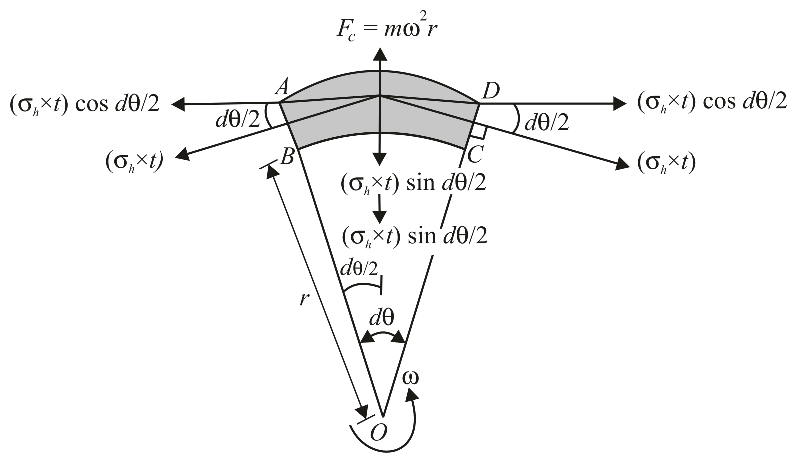


Fig. 17.1 A rotating ring.

Let r = Mean radius of the ring (or cylinder)

t = Thickness of the ring (or cylinder)

ρ = Density of the ring or (cylinder material)

Rotational motion produces centrifugal force on the circumference of the ring or on the walls of the cylinder, which in turn, produces hoop (or circumferential) stress σ_h. Since the thickness is very small, hence there is no variation of the hoop stress along the thickness, that is, the hoop stress may be assumed to be constant.

Now consider a small element *ABCD* of the ring or cylinder making an angle $d\theta$ at the centre as shown in the figure.

Forces on the element

The following three forces are acting on the element *ABCD* :

- The centrifugal force caused due to rotation acting radially outward
- The hoop tension force on the face *AB* caused due to hoop stress σ_h
- The hoop tension force on the face *CD* caused due to hoop stress σ_h

Centrifugal force

Considering unit length of the circumference of the element, the mass m of the element can be obtained as

$$\begin{aligned} m &= \text{Density} \times \text{Volume of the element} \\ &= \text{Density} \times \text{Area of the element} \times \text{Unit length} \\ &= \rho \times r d\theta \times t \times 1 \\ &= \rho r t \, d\theta \qquad \text{...(17.1)} \end{aligned}$$

The centrifugal force is given as

$$F_c = \frac{mV^2}{r} \qquad \text{...(17.2)}$$

$$= m\omega^2 r \qquad \text{...(17.3)}$$

$$= \rho r^2 \omega^2 t \, d\theta \qquad \text{(on substituting } m\text{)...(17.4)}$$

where V = Linear velocity

$= \omega r$

Hoop tension forces on faces *AB* and *CD*

The hoop tension forces act perpendicular to faces AB and CD are equal but opposite in direction; one is acting in the left direction and another in the right direction. Its magnitude is

$$\sigma_h \times t \times 1 = \sigma_h \times \text{t} \qquad \text{(assuming unit length)}$$

Now the hoop forces are resolved into horizontal and vertical components. The vertical components of the hoop forces acting on faces AB and CD are radially inward and both are equal to

$$\sigma_h \times t \times \sin\frac{d\theta}{2}$$

The horizontal component of the hoop force acting on face AB is directed leftward and the horizontal component on face CD is directed rightward and both are equal to

$$\sigma_h \times t \times \cos\frac{d\theta}{2}$$

Equilibrium of the element

The horizontal components of the hoop tension forces on the faces AB and CD are equal but opposite in direction, hence they cancel each other. Their vertical components are added as they are acting in the same direction.

Considering forces in the vertical direction, we find that the centrifugal force F_c is balanced by the sum of the vertical components of the hoop tension forces on the two faces.

$$F_c = \sigma_h \times t \times \sin\frac{d\theta}{2} + \sigma_h \times t \times \sin\frac{d\theta}{2}$$

$$\rho r^2 \omega^2 t \, d\theta = 2 \times \sigma_h \times t \times \sin\frac{d\theta}{2} \qquad \text{(using equation (17.4))}$$

$$= 2 \times \sigma_h \times t \times \frac{d\theta}{2} \quad \left(\text{as } d\theta \text{ is very small, hence } \sin\frac{d\theta}{2} \simeq \frac{d\theta}{2}\right)$$

which gives

$$\sigma_h = \rho\omega^2 r^2 \qquad \text{...(17.5)}$$

This is the required expression for the hoop stress in a thin rotating ring.

Example 17.1

The thin rim of a 900 mm diameter wheel is made of steel and weighs 7800 kg/m^3. Neglecting the effect of the spokes, how many revolutions per minute may it make, if the hoop stress is not to exceed 150 MPa. Also, find the increase in diameter of the wheel. Take E = 210 GPa.

Solution: Given,

Diameter of the wheel, $d = 900$ mm $= 900 \times 10^{-3}$ m

Density of the rim material, $\rho = 7800$ kg/m^3

Hoop stress, $\sigma_h = 150$ MPa $= 150 \times 10^6$ Pa

Modulus of elasticity, $E = 210$ GPa $= 210 \times 10^9$ Pa

The diameter of the rim is equal to the diameter of the wheel and its radius

$$r = \frac{d}{2} = \frac{900 \times 10^{-3}}{2} = 0.45 \text{ m}$$

The hoop stress is given by using equation (17.5) as

$$\sigma_h = \rho\omega^2 r^2$$

$$150 \times 10^6 = 7800 \times \omega^2 \times (0.45)^2$$

which gives

$$\omega = 308.167 \text{ rad/s}$$

Let N be the number of revolutions per minute, then

$$\omega = \frac{2\pi N}{60}$$

$$308.167 = \frac{2\pi N}{60}$$

which gives

$$N = \frac{308.167 \times 60}{2\pi}$$

$$= 2942.78$$

Ans.

Now the hoop strain is given as

$$\epsilon_h = \frac{\sigma_h}{E}$$

$$= \frac{150 \times 10^6}{210 \times 10^9} = 7.143 \times 10^{-4}$$

Hence, the increase in diameter of the wheel is

$$\text{Hoop strain} \times \text{Diameter} = 7.143 \times 10^{-4} \times 900$$

$$= 0.643 \text{ mm}$$ **Ans.**

Example 17.2

A rim-type flywheel of mean diameter 800 mm rotates at a speed of 2000 rpm. If the density of the material of the wheel is 8000 kg/m^3, find the hoop stress developed in the rim due to rotation.

Solution: Given,

Mean diameter of the wheel, d = 800 mm

Rotational speed, N = 2000 rpm

Density of the wheel material, ρ = 8000 kg/m^3

The mean radius of the wheel is given as

$$r = \frac{d}{2} = \frac{800}{2} = 400 \text{ mm}$$

$$= 400 \times 10^{-3} \text{ m}$$

The angular velocity of the wheel is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 2000}{60} = 209.44 \text{ rad/s}$$

Now the hoop stress is given by equation (17.5) as

$$\sigma_h = \rho\omega^2 r^2$$

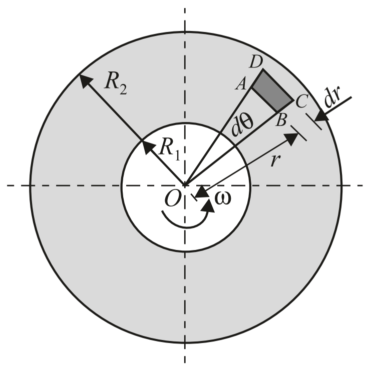
$$= 8000 \times (209.44)^2 \times (400 \times 10^{-3})^2$$

$$= 56.147 \times 10^6 \text{ Pa}$$

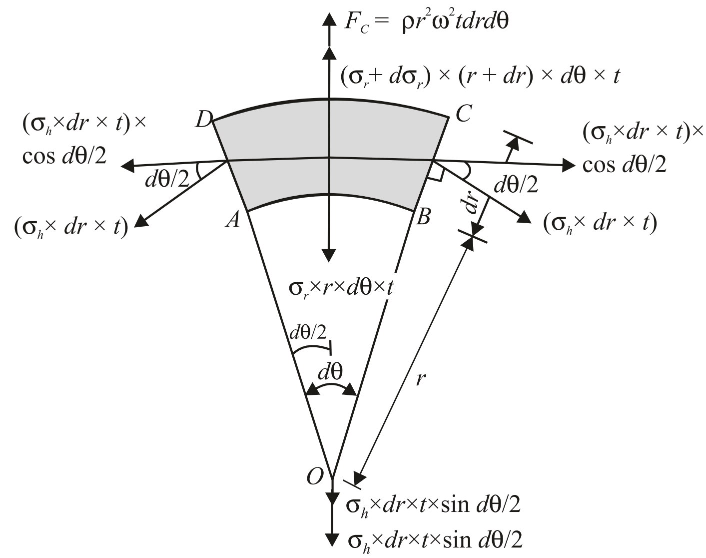
$$= 56.147 \text{ MPa}$$ **Ans.**

17.3 ROTATING THIN DISC

Consider a thin disc of inner radius r_1 and outer radius r_2 rotating at angular speed ω rad/s about its axis as shown in Fig. 17.2. The thickness of the disc is negligibly small, so there is no variation of stress across the thickness, and there is no axial stress (longitudinal stress) in the disc. The two stresses acting on the disc include hoop and radial.



(*a*) A rotating disc



(*b*) Forces acting on the element *ABCD*

Fig. 17.2

Let $t =$ Thickness of the disc

$\rho =$ Density of the disc material

Now consider an element *ABCD* of the disc of radial width dr at radius r subtending an angle $d\theta$ at the centre as shown in the figure.

Forces on the element

The following five forces are acting on the element *ABCD*:

- The centrifugal force caused due to rotation
- The radial force on face *AB* caused due to radial stress σ_r
- The radial force on face *CD* caused due to radial stress σ_r
- The hoop tension force on face *AD* caused due to hoop stress σ_h
- The hoop tension force on face *BC* caused due to hoop stress σ_h

Centrifugal force

The volume of the element is

$$(rd\theta) \times dr \times t$$

Now the mass of the element is

$$m = \text{Density} \times \text{Volume of the element}$$

$$= \rho \times (rd\theta) \times dr \times t$$

The centrifugal force acting on the element is

$$\begin{aligned} F_c &= m\omega^2 r \\ &= (\rho \times r d\theta \times dr \times t) \times \omega^2 \times r \\ &= \rho r^2 \omega^2 t \, d\theta dr \end{aligned} \qquad ...(17.6)$$

Hoop tension forces on faces *AD* and *BC*

The hoop tension forces act perpendicular to *AD* and *BC* and are equal to $\sigma_h \times dr \times t$.

Now the hoop forces are resolved into horizontal and vertical components. The vertical components of the hoop forces acting on faces *AD* and *BC* are radially inward and both are equal to

$$\sigma_h \times dr \times t \times \sin\frac{d\theta}{2} = \sigma_h \times dr \times t \times \frac{d\theta}{2} \qquad \left(\text{for small value of } d\theta,\ \sin\frac{d\theta}{2} \simeq \frac{d\theta}{2}\right)$$

The horizontal component of the hoop force acting on face *AD* is directed leftward and the horizontal component on face *BC* is directed rightward and both are equal to

$$\sigma_h \times dr \times t \times \cos\frac{d\theta}{2} = \sigma_h \times dr \times t \times \frac{d\theta}{2} \qquad \left(\text{for small value of } d\theta,\ \cos\frac{d\theta}{2} \simeq \frac{d\theta}{2}\right)$$

Radial forces on faces *AB* and *CD*

The radial force on AB is equal to $\sigma_r \times r d\theta \times t$. It acts radially inward.

The radial force on *CD* acting radially outward is

$$(\sigma_r + d\sigma_r) \times (r + dr) \times d\theta \times t$$

Equilibrium of the element

The horizontal components of the hoop tension forces on the faces *AD* and *BC* are equal but opposite in direction, hence they cancel each other. Their vertical components are added as they are acting in the same direction.

Balancing the forces in the radial direction, we have

$$\sigma_r \times r \times d\theta \times t + \sigma_h \times dr \times t \times \sin\frac{d\theta}{2} + \sigma_h \times dr \times t \times \sin\frac{d\theta}{2}$$
$$= \rho r^2 \omega^2 \, t dr d\theta + (\sigma_r + d\sigma_r) \times (r + dr) \times d\theta \times t$$

$$\sigma_r \times r \times d\theta \times t + 2\sigma_h \times dr \times t \times \frac{d\theta}{2} = \rho r^2 \omega^2 \, t dr d\theta + (\sigma_r + d\sigma_r) \times (r + dr) \times d\theta \times t$$

$$\left(\text{for small value of } d\theta,\ \sin\frac{d\theta}{2} = \frac{d\theta}{2}\right)$$

Eliminating $d\theta \times t$ from both sides of the equation, we have

$$\sigma_r \times r + \sigma_h \times dr = \rho r^2\omega^2\, dr + r\,\sigma_r + \sigma_r\, dr + rd\sigma_r + drd\sigma_r$$

Eliminating $r\sigma_r$ from both sides and neglecting $drd\sigma_r$ because of its small value, we have

$$\sigma_h\, dr = \rho r^2\omega^2\, dr + \sigma_r\, dr + rd\sigma_r$$

$$(\sigma_h - \sigma_r)\, dr = \rho r^2\omega^2\, dr + rd\sigma_r$$

Dividing by dr on both sides, we have

$$(\sigma_h - \sigma_r) = \rho r^2\omega^2 + r\frac{d\sigma_r}{dr} \qquad ...(17.7)$$

Strain in the element

Due to rotation, the radius of the disc increases. Let the radius r changes to $(r + u)$ and dr changes to $(dr + du)$.

Now the hoop strain is given as

$$\epsilon_h = \frac{\text{Final circumference} - \text{Initial circumference}}{\text{Initial circumference}}$$

$$= \frac{2\pi(r+u) - 2\pi r}{2\pi r}$$

$$= \frac{u}{r} \qquad ...(17.8)$$

And the radial strain is given as

$$\epsilon_r = \frac{\text{Final radial witdh} - \text{Initial radial width}}{\text{Initial radial width}}$$

$$= \frac{(dr + du) - dr}{dr}$$

$$= \frac{du}{dr} \qquad ...(17.9)$$

Also
$$\epsilon_h = \frac{\sigma_h}{E} - \nu\frac{\sigma_r}{E} \qquad ...(17.10)$$

$$\epsilon_r = \frac{\sigma_r}{E} - \nu\frac{\sigma_h}{E} \qquad ...(17.11)$$

On equating equations (17.8) and (17.10), we have

$$\frac{u}{r} = \frac{\sigma_h}{E} - \nu\frac{\sigma_r}{E}$$

$$= \frac{1}{E}\,(\sigma_h - \nu\sigma_r) \qquad ...(17.12)$$

On equating equations (17.9) and (17.11), we have

$$\frac{du}{dr} = \frac{\sigma_r}{E} - v\frac{\sigma_h}{E}$$

$$= \frac{1}{E}(\sigma_r - v\sigma_h) \qquad ...(17.13)$$

From equation (17.12), we get

$$E \times u = r \times (\sigma_h - v\sigma_r)$$

Differentiating w.r.t. r, we have

$$E\frac{du}{dr} = (\sigma_h - v\sigma_r) + \left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right) \times r$$

or

$$\frac{du}{dr} = \frac{1}{E}\left[(\sigma_h - v\sigma_r) + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right)\right] \qquad ...(17.14)$$

On equating equations (17.13) and (17.14), we have

$$\sigma_r - v\sigma_h = \sigma_h - v\sigma_r + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right)$$

$$\sigma_r + v\sigma_r = \sigma_h + v\sigma_h + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right)$$

$$\sigma_r(1 + v) = \sigma_h(1 + v) + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right)$$

or

$$(\sigma_h - \sigma_r)(1 + v) + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right) = 0 \qquad ...(17.15)$$

On substituting $(\sigma_h - \sigma_r)$ from equation (17.7) in equation (17.15), we have

$$\left(\rho r^2\omega^2 + r\frac{d\sigma_r}{dr}\right)(1 + v) + r\left(\frac{d\sigma_h}{dr} - v\frac{d\sigma_r}{dr}\right) = 0$$

$$\rho r^2\omega^2 + v\rho r^2\omega^2 + r\frac{d\sigma_r}{dr} + vr\frac{d\sigma_r}{dr} + r\frac{d\sigma_h}{dr} - vr\frac{d\sigma_r}{dr} = 0$$

Eliminating r from the above equation, we have

$$\rho r\omega^2 + v\rho r\omega^2 + \frac{d\sigma_r}{dr} + \frac{d\sigma_h}{dr} = 0$$

$$\rho r\omega^2(1 + v) + \frac{d\sigma_r}{dr} + \frac{d\sigma_h}{dr} = 0$$

or

$$\frac{d\sigma_r}{dr} + \frac{d\sigma_h}{dr} = -\rho r\omega^2(1 + v) \qquad ...(17.16)$$

On integration, we have

$$\sigma_r + \sigma_h = -\rho \times \frac{r^2}{2} \times \omega^2 (1 + v) + A$$

$$= \frac{-\rho r^2 \omega^2 (1+v)}{2} + A \quad \text{...(17.17)}$$

where A is a constant of integration.

Now subtracting equation (17.7) from equation (17.17), we get

$$2\sigma_r = \frac{-\rho r^2 \omega^2 (1+v)}{2} + A - \rho r^2 \omega^2 - r\frac{d\sigma_r}{dr}$$

$$2\sigma_r + r\frac{d\sigma_r}{dr} = -\rho r^2 \omega^2 \left(\frac{1+v}{2} + 1\right) + A$$

$$= -\rho r^2 \omega^2 \left(\frac{1+v+2}{2}\right) + A$$

$$= -\frac{\rho r^2 \omega^2 (3+v)}{2} + A \quad \text{...(17.18)}$$

Multiplying by r on both sides, we have

$$2 \times r \times \sigma_r + r^2 \frac{d\sigma_r}{dr} = -\frac{\rho r^3 \omega^2 (3+v)}{2} + A \times r$$

$$\frac{d}{dr}(r^2 \times \sigma_r) = -\frac{\rho r^3 \omega^2 (3+v)}{2} + A \times r$$

Integrating both sides, we get

$$r^2 \times \sigma_r = -\frac{\rho \omega^2 (3+v)}{2} \times \frac{r^4}{4} + A \times \frac{r^2}{2} + B$$

where B is another constant of integration.

Dividing by r^2 on both sides, we get

$$\sigma_r = -\frac{\rho \omega^2 r^2 (3+v)}{8} + \frac{A}{2} + \frac{B}{r^2}$$

$$= \frac{A}{2} + \frac{B}{r^2} - \frac{\rho \omega^2 r^2 (3+v)}{8} \quad \text{...(17.19)}$$

This is the required expression for the radial stress. The constants A and B can be determined by using suitable boundary conditions. Now substituting equation (17.19) in equation (17.17), we get

$$\frac{A}{2} + \frac{B}{r^2} - \frac{\rho \omega^2 r^2 (3+v)}{8} + \sigma_h = -\frac{\rho \omega^2 r^2 (1+v)}{2} + A$$

$$\sigma_h = \frac{\rho\omega^2 r^2 (3+v)}{8} - \frac{\rho\omega^2 r^2 (1+v)}{2} + A - \frac{A}{2} - \frac{B}{r^2}$$

$$= \rho\omega^2 r^2 \left(\frac{3+v}{8} - \frac{1+v}{2}\right) + \frac{A}{2} - \frac{B}{r^2}$$

$$= \rho\omega^2 r^2 \left(\frac{3+v-4-4v}{8}\right) + \frac{A}{2} - \frac{B}{r^2}$$

$$= \rho\omega^2 r^2 \times \frac{(-1-3v)}{8} + \frac{A}{2} - \frac{B}{r^2}$$

$$= -\frac{\rho\omega^2 r^2}{8}(1+3v) + \frac{A}{2} - \frac{B}{r^2}$$

$$= \frac{A}{2} - \frac{B}{r^2} - \frac{\rho\omega^2 r^2 (1+3v)}{8} \quad \text{...(17.20)}$$

This is the required expression for the hoop stress for a rotating thin disc. The constants A and B can be determined by using suitable boundary conditions.

17.3.1 Hoop and Radial Stresses in a Rotating Solid Disc

For a solid disc, there is no inner radius.

$$R_1 = 0 \text{ and } R_2 = R \text{ (say)}$$

When the value of $r = 0$ is substituted in equations (17.19) and (17.20), we find that the term $\frac{B}{r^2}$ becomes infinity, but the stresses cannot have infinite values. Hence to get finite value of stress, $B = 0$.

Equations (17.19) and (17.20) are now transformed to

$$\sigma_r = \frac{A}{2} - \frac{\rho\omega^2 r^2 (3+v)}{8} \quad \text{...(17.21)}$$

$$\sigma_h = \frac{A}{2} - \frac{\rho\omega^2 r^2 (1+3v)}{8} \quad \text{...(17.22)}$$

The boundary conditions is

At the outer radius, where $r = R$, the radial stress is

$$\sigma_r = 0$$

On substituting the boundary condition in equation (17.21), we get

$$\frac{A}{2} = \frac{\rho\omega^2 R^2 (3+v)}{8}$$

Now equation (17.21) on substituting the value of $\frac{A}{2}$ becomes

$$\sigma_r = \frac{\rho\omega^2 R^2 (3+v)}{8} - \frac{\rho\omega^2 r^2 (3+v)}{8}$$

$$= \frac{\rho\omega(3+v)}{8}(R^2 - r^2) \quad ...(17.23)$$

This is the required expression for the radial stress for a rotating thin solid disc. Equation (17.22) on substituting the value of $\frac{A}{2}$ becomes

$$\sigma_h = \frac{\rho\omega^2 R^2 (3+v)}{8} - \frac{\rho\omega^2 r^2 (1+3v)}{8}$$

$$= \frac{\rho\omega^2}{8}[(3+v)R^2 - (1+3v)r^2] \quad ...(17.24)$$

This is the required expression of the hoop stress for a rotating thin solid disc.

Hoop and radial stresses at the centre

At the centre of the solid disc, where $r = 0$, both hoop and radial stresses have equal maximum values, given by

$$\sigma_{r_{\max}} = \sigma_{h_{\max}} = \frac{\rho\omega^2 R^2 (3+v)}{8} \quad ...(17.25)$$

Hoop stress at outer radius

At the outer radius, the radial stress $\sigma_r = 0$, but the hoop stress is not zero.

The value of the hoop stress at the outer radius is obtained by putting $r = R$ in equation (17.24) as

$$\sigma_{h_{r=R}} = \frac{\rho\omega^2}{8}[R^2(3+v) - R^2(1+3v)]$$

$$= \frac{\rho\omega^2}{8}(3R^2 + vR^2 - R^2 - 3vR^2)$$

$$= \frac{\rho\omega^2}{8}(2R^2 - 2vR^2)$$

$$= \frac{\rho\omega^2}{8} \times 2R^2(1-v)$$

$$= \frac{\rho\omega^2 R^2}{4}(1-v) \quad ...(17.26)$$

The variation of the hoop and radial stresses in a rotating solid disc along the radius is shown in Fig. 17.3.

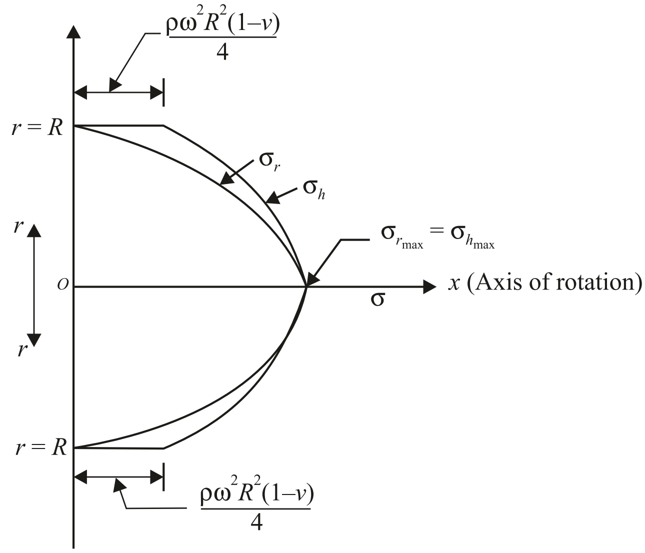


Fig. 17.3 Distribution of the hoop and radial stresses in a rotating solid disc.

17.3.2 Hoop and Radial Stresses in a Rotating Disc with a Central Hole

We have seen in case of a rotating solid disc, the constant of integration B is zero in order to have finite values of radial and hoop stresses. But in case of a disc with a central hole, B is not zero and its value is obtained using suitable conditions.

The radial stress σ_r is zero at both inner and outer radius of the disc.

i.e. at $r = R_1$, $\sigma_r = 0$

Also at $r = R_2$, $\sigma_r = 0$

From equation (17.19), we have

$$\sigma_r = \frac{A}{2} + \frac{B}{r^2} - \frac{\rho\omega^2 r^2(3+v)}{8}$$

$$0 = \frac{A}{2} + \frac{B}{R_1^2} - \frac{\rho\omega^2 R_1^2(3+v)}{8} \qquad ...(1)$$

and

$$0 = \frac{A}{2} + \frac{B}{R_2^2} - \frac{\rho\omega^2 R_2^2(3+v)}{8} \qquad ...(2)$$

Subtracting equation (2) from equation (1), we have

$$\frac{B}{R_1^2} - \frac{B}{R_2^2} + \frac{\rho\omega^2 R_2^2(3+v)}{8} - \frac{\rho\omega^2 R_1^2(3+v)}{8} = 0$$

or

$$\frac{B(R_2^2 - R_1^2)}{R_1^2 R_2^2} + \frac{\rho\omega^2(3+v)}{8}[R_2^2 - R_1^2] = 0$$

Eliminating $(R_2^2 - R_1^2)$, we have

$$\frac{B}{R_1^2 R_2^2} + \frac{\rho\omega^2(3+v)}{8} = 0$$

It gives

$$B = -\frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8} \qquad ...(17.27)$$

Now substituting the value of B in equation (1), we get

$$0 = \frac{A}{2} + \frac{1}{R_1^2} \times \left\{\frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8}\right\} - \frac{\rho\omega^2 R_1^2\,(3+v)}{8}$$

$$= \frac{A}{2} - \frac{(3+v)\rho\omega^2 R_2^2}{8} - \frac{(3+v)\rho\omega^2 R_1^2}{8}$$

$$= \frac{A}{2} - \frac{(3+v)\rho\omega^2}{8}[R_2^2 + R_1^2]$$

It gives

$$A = \frac{(3+v)\rho\omega^2(R_2^2 + R_1^2)}{4} \qquad ...(17.28)$$

Finally the values of A and B are substituted in equation (17.19) to get the expression for the radial stress σ_r as

$$\sigma_r = \frac{1}{2} \times \frac{(3+v)\rho\omega^2(R_2^2 + R_1^2)}{4} + \frac{1}{r^2} \times \left\{-\frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8}\right\} - \frac{\rho\omega^2 r^2(3+v)}{8}$$

$$= \frac{(3+v)\rho\omega^2(R_2^2 + R_1^2)}{8} - \frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8r^2} - \frac{\rho\omega^2 r^2(3+v)}{8}$$

$$= \frac{(3+v)\rho\omega^2}{8}\left[(R_2^2 + R_1^2) - \frac{R_1^2 R_2^2}{r^2} - r^2\right] \qquad ...(17.29)$$

This is the required expression for the radial stress for a rotating disc with a central hole.

To find the expression for the hoop stress, the values of A and B are substituted in equation (17.20).

$$\sigma_h = \frac{A}{2} - \frac{B}{r^2} - \frac{\rho\omega^2 r^2(1+3v)}{8}$$

$$= \frac{1}{2} \times \frac{(3+v)\rho\omega^2(R_2^2 + R_1^2)}{4} - \frac{1}{r^2} \times \left\{-\frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8}\right\} - \frac{\rho\omega^2 r^2(1+3v)}{8}$$

$$= \frac{(3+v)\rho\omega^2(R_2^2+R_1^2)}{8} + \frac{(3+v)\rho\omega^2 R_1^2 R_2^2}{8r^2} - \frac{\rho\omega^2 r^2(1+3v)}{8}$$

$$= \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1^2) + \frac{(3+v)R_1^2R_2^2}{r^2} - (1+3v)r^2\right] \qquad \text{...(17.30)}$$

This is the required expression for the hoop stress for a rotating disc with a central hole.

Maximum hoop stress (Hoop stress at inner radius)

Equation (17.30) suggests that with increase in the value of radius r, the hoop stress σ_h decreases and vice versa. Hence, σ_h is maximum when r is minimum, that is, when r approaches to R_1. Substituting $r = R_1$ in equation (17.30), we have

$$\sigma_{h_{\max}} = \sigma_{h_{r=R_1}} = \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1^2) + \frac{(3+v)R_1^2R_2^2}{R_1^2} - R_1^2(1+3v)\right]$$

$$= \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1^2) + (3+v)R_2^2 - R_1^2(1+3v)\right]$$

$$= \frac{\rho\omega^2}{8}\left[3R_2^2 + 3R_1^2 + vR_2^2 + vR_1^2 + 3R_2^2 + vR_2^2 - R_1^2 - 3vR_1^2\right]$$

$$= \frac{\rho\omega^2}{8}\left[6R_2^2 + 2vR_2^2 + 2R_1^2 - 2vR_1^2\right]$$

$$= \frac{\rho\omega^2}{8}\left[2R_2^2(3+v) + 2R_1^2(1-v)\right]$$

$$= \frac{\rho\omega^2}{4}\left[(3+v)R_2^2 + (1-v)R_1^2\right] \qquad \text{...(17.31)}$$

This is the required expression for the maximum hoop stress for a rotating disc with a central hole, which occurs at inner radius of the disc.

When R_1 approaches $R_2 = r$, we have from equation (17.31)

$$\sigma_h = \frac{\rho\omega^2}{4}[(3+v)r^2 + (1-v)r^2]$$

$$= \frac{\rho\omega^2}{4}[3r^2 + vr^2 + r^2 - vr^2] = \frac{\rho\omega^2}{4} \times 4r^2$$

$$= \rho\omega^2 r^2$$

The above expression is same as equation (17.5), which applies to a thin rotating ring or a thin rotating cylinder.

Hoop stress at outer radius

For hoop stress at outer radius, put $r = R_2$ in equation (17.30).

$$\sigma_{h_{r=R_2}} = \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1^2)+\frac{(3+v)R_1^2R_2^2}{R_2^2}-R_2^2(1+3v)\right]$$

$$= \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1^2)+(3+v)R_1^2-R_2^2(1+3v)\right]$$

$$= \frac{\rho\omega^2}{8}\left[3R_2^2+3R_1^2+vR_2^2+vR_1^2+3R_1^2+vR_1^2-R_2^2-3vR_2^2\right]$$

$$= \frac{\rho\omega^2}{8}\left[2R_2^2-2vR_2^2+6R_1^2+2vR_1^2\right]$$

$$= \frac{\rho\omega^2}{8}\left[2R_2^2(1-v)+2R_1^2(3+v)\right]$$

$$= \frac{\rho\omega^2}{4}\left[(1-v)R_2^2+(3+v)R_1^2\right] \qquad ...(17.32)$$

Maximum radial stress

To find the position of the maximum radial stress, we differentiate equation (17.29) with respect to r and equate it to zero.

$$\frac{d\sigma_r}{dr} = 0$$

$$\frac{d}{dr}\left[\frac{(3+v)\,\rho\omega^2}{8}\left\{(R_2^2+R_1^2)-\frac{R_1^2R_2^2}{r^2}-r^2\right\}\right] = 0$$

$$\frac{(3+v)\,\rho\omega^2}{8}\left[0-\frac{(-2)}{r^3}R_1^2R_2^2-2r\right] = 0$$

$$\frac{(3+v)\,\rho\omega^2}{8}\left[\frac{2R_1^2R_2^2}{r^3}-2r\right] = 0$$

which gives

$$r = \sqrt{R_1R_2} \qquad ...(17.33)$$

Substituting the value of r in equation (17.29), we get

$$\sigma_{r_{\max}} = \frac{(3+v)\rho\omega^2}{8}\left[(R_2^2+R_1^2)-\frac{R_1^2R_2^2}{R_1R_2}-R_1R_2\right]$$

$$= \frac{(3+v)\rho\omega^2}{8}\left[R_2^2+R_1^2-2R_1R_2\right]$$

$$= \frac{(3+v)\rho\omega^2}{8}(R_2-R_1)^2 \qquad ...(17.34)$$

This is the required expression for the maximum radial stress for a rotating disc with a central hole.

17.3.3 Hoop and Radial Stresses in a Rotating Disc with a Pin Hole at the Centre

In this case, R_1 tends to zero. Substituting $R_1 = 0$ and $R_2 = R$ in equations (17.31) and (17.34), we get the expressions for the maximum hoop and radial stresses for a rotating disc with a central pin hole.

$$\sigma_{h_{\max}} = \frac{(3+v)\rho\omega^2 R^2}{4} \qquad \text{...(17.35)}$$

and

$$\sigma_{r_{\max}} = \frac{(3+v)\rho\omega^2 R^2}{8} \qquad \text{...(17.36)}$$

Comparing equations (17.35) and (17.36), we get

$$\sigma_{h_{\max}} = 2 \times \sigma_{r_{\max}} \qquad \text{...(17.37)}$$

Also, when we compare equation (17.35) with equation (17.25), we observe that the maximum hoop stress for a rotating disc with a central pin hole is twice the maximum hoop stress for a rotating solid disc.

Example 17.3

A steel disc of diameter 800 mm rotates at 2500 rpm. Calculate the hoop and radial stresses developed at the centre and outer radius of the disc. The Poisson's ratio is 0.25 and the density of the disc material is 7800 kg/m^3.

Solution: Given,

Radius of the disc, $R = \frac{800}{2} = 400$ mm

$= 400 \times 10^{-3}$ m

Rotational speed, N $= 2500$

Poisson's ratio, $v = 0.25$

Density of the disc material, $\rho = 7800$ kg/m^3

The angular speed of the disc is given as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 2500}{60} = 261.8 \text{ rad/s}$$

Hoop stress at the centre

From equation (17.25), the hoop stress at the centre of the disc is maximum, and is given as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2 R^2(3+v)}{8}$$

$$= \frac{7800 \times (261.8)^2 \times (400 \times 10^{-3})^2 \times (3 + 0.25)}{8}$$

$$= 34.75 \times 10^6 \text{ N/m}^2$$

$$= 34.75 \text{ MPa}$$ **Ans.**

Hoop stress at outer radius

From equation (17.26), the hoop stress at the outer radius is given as

$$\sigma_{h_{r=R}} = \frac{\rho\omega^2 R^2 (1 - v)}{4}$$

$$= \frac{7800 \times (261.8)^2 \times (400 \times 10^{-3})^2 \times (1 - 0.25)}{4}$$

$$= 16.04 \times 10^6 \text{ N/m}^2$$

$$= 16.04 \text{ MPa}$$ **Ans.**

Radial stress at the centre

The radial stress at the centre is maximum, and is equal to the hoop stress at the centre.

$$\sigma_{r_{\max}} = \sigma_{h_{\max}} = 34.75 \text{ MPa}$$ **Ans.**

Radial stress at outer radius

The radial stress at the outer radius of the disc is zero.

$$\sigma_{r_{r=R}} = 0$$ **Ans.**

Example 17.4

A steel disc of diameter 250 mm has a central hole of diameter 50 mm, and rotates at 5000 rpm. Calculate the hoop stresses developed at the inner and outer radius of the disc. The Prisson's ratio is 0.25 and the density of the disc material is 7800 kg/m^3.

Solution: Given,

Outer radius of the disc, $R_2 = \frac{250}{2} = 125 \text{ mm} = 125 \times 10^{-3} \text{ m}$

Inner radius of the disc, $R_1 = \frac{50}{2} = 25 \text{ mm} = 25 \times 10^{-3} \text{ m}$

Rotation speed, $N = 5000$ rpm

Poisson's ratio, $v = 0.25$

Density of the disc material, $\rho = 7800$ kg/m^3

The angular speed of the disc is given as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 5000}{60} = 523.6 \text{ rad/s}$$

Hoop stress at inner radius

Substituting the values of ρ, ω, v, R_1 and R_2 in equation (17.31), we get the hoop stress at the inner radius of the disc as

$$\sigma_{h_{r=R_1}} = \frac{\rho\omega^2}{4}[(3+v)R_2^2 + (1-v)R_1^2]$$

$$= \frac{7800 \times (523.6)^2}{4} \times [(3+0.25) \times (125 \times 10^{-3})^2$$

$$+ (1-0.25) \times (25 \times 10^{-3})^2]$$

$$= 27.4 \times 10^6 \text{ N/m}^2$$

$$= 27.4 \text{ MPa}$$ **Ans.**

This is also the maximum value of the hoop stress, which occurs at the inner radius of the disc.

Hoop stress at outer radius

Substituting the values of ρ, ω, v, R_1 and R_2 in equation (17.32), we get the hoop stress at the outer radius of the disc as

$$\sigma_{h_{r=R_2}} = \frac{\rho\omega^2}{4}[(1-v)R_2^2 + (3+v)R_1^2]$$

$$= \frac{7800 \times (523.6)^2}{4} \times [(1 - 0.25) \times (125 \times 10^{-3})^2 + (3 + 0.25)$$

$$\times (25 \times 10^{-3})^2]$$

$$= 7.35 \times 10^6 \text{ N/m}^2$$

$$= 7.35 \text{ MPa}$$ **Ans.**

Example 17.5

A circular saw of thickness 5 mm and diameter 800 mm is secured upon a shaft of diameter 120 mm. The saw material has the density of 8100 kg/m^3 and the Prisson's ratio is 0.3. Calculate the permissible speed of the saw, if the allowable hoop stress is 250 MPa. Also, find the maximum value of the radial stress in the saw.

Solution: Given,

Inner radius of the saw, $R_1 = \frac{120}{2} = 60 \text{ mm} = 60 \times 10^{-3} \text{ m}$

$$\text{Outer radius of the saw, } R_2 = \frac{800}{2} = 400 \text{ mm} = 400 \times 10^{-3} \text{ m}$$

$$\text{Density of the saw material, } \rho = 8100 \text{ kg/m}^3$$

$$\text{Poisson's ratio, } \nu = 0.3$$

$$\text{Maximum hoop stress, } \sigma_{h_{max}} = 250 \text{ MPa} = 250 \times 10^6 \text{ Pa}$$

From equation (17.31), the maximum hoop stress is given as

$$\sigma_{h_{max}} = \frac{\rho\omega^2}{4}[(3+\nu)R_2^2 + (1-\nu)R_1^2]$$

Substituting values of the given parameters in the above equation, we have

$$250 \times 10^6 = \frac{8100 \times \omega^2}{4}[(3+0.3)\times(400\times10^{-3})^2 + (1-0.3)\times(60\times10^{-3})^2]$$

$$= 2025\ \omega^2\ [3.3 \times (400\times10^{-3})^2 + 0.7 \times (60 \times 10^{-3})^2]$$

$$= 1074.303\ \omega^2$$

or $$\omega^2 = \frac{250\times10^6}{1074.303}$$

which gives $$\omega = 482.4 \text{ rad/s}$$

Now $$\omega = \frac{2\pi N}{60}$$

or $$N = \frac{\omega\times60}{2\pi} = \frac{482.4\times60}{2\pi} = 4606.6 \text{ rpm}$$ **Ans.**

From equation (17.34), the maximum radial stress is given as

$$\sigma_{r_{max}} = \frac{(3+\nu)\rho\omega^2}{8}(R_2 - R_1)^2$$

$$= \frac{(3+0.3)\times8100\times(482.4)^2\times\{(400-60)\times10^{-3}\}^2}{8}$$

$$= 89.88 \times 10^6 \text{ N/m}^2 = 89.88 \text{ MPa}$$ **Ans.**

Example 17.6

A thin disc of diameter 900 mm has a central hole of diameter 100 mm. Calculate the maximum hoop stress developed in the disc, if the maximum radial stress is 25 MPa. The Poisson's ratio is 0.25.

Solution: Given,

$$\text{Outer radius of the disc, } R_2 = \frac{900}{2} = 450 \text{ mm}$$

$$= 450 \times 10^{-3} \text{ m}$$

$$\text{Inner radius of the disc, } R_1 = \frac{100}{2} = 50 \text{ mm}$$

$$= 50 \times 10^{-3} \text{ m}$$

$$\text{Poisson's ratio, } v = 0.25$$

$$\text{Maximum radial stress, } \sigma_{r_{\max}} = 25 \text{ MPa}$$

$$= 25 \times 10^6 \text{ N/m}^2$$

From equation (17.34), the maximum radial stress is given as

$$\sigma_{r_{\max}} = \frac{(3+v)\rho\omega^2}{8}(R_2 - R_1)^2$$

$$250 \times 10^6 = \frac{(3+0.25)\rho\omega^2}{8} \times \{(450 - 50) \times 10^{-3}\}^2$$

$$= 0.065\ \rho\omega^2$$

or

$$\rho\omega^2 = \frac{25\times10^6}{0.065}$$

$$= 3.84615 \times 10^8 \quad \text{...(1)}$$

Now the maximum hoop stress, using equation (17.31), is given as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{4}[(3+v)R_2^2 + (1-v)R_1^2]$$

$$= \frac{3.84615\times10^8}{4}\ [(3 + 0.25) \times (450 \times 10^{-3})^2 + (1 - 0.25) \times (50 \times 10^{-3})^2]$$

$$= 63.46 \times 10^6 \text{ N/m}^2$$

$$= 63.46 \text{ MPa}$$ **Ans.**

Example 17.7

A steel disc of diameter 300 mm has a central hole of diameter 100 mm and it rotates at 4000 rpm. Taking Poisson's ratio as 0.3 and the density of the disc material to be 7800 kg/m^3, find the following parameters :

(*a*) the hoop stress at the inner and outer radius of the disc

(*b*) the radius at which the radial stress is maximum and

(*c*) the maximum radial stress.

Solution: Given,

$$\text{Outer radius of the disc, } R_2 = \frac{300}{2} = 150 \text{ mm} = 150 \times 10^{-3} \text{ m}$$

$$\text{Inner radius of the disc, } R_1 = \frac{100}{2} = 50 \text{ mm} = 50 \times 10^{-3} \text{ m}$$

Poisson's ratio, v = 0.3

Rotational speed, N = 4000 rpm

Density of the disc material, ρ = 7800 kg/m^3

The angular speed of the disc is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 4000}{60} = 418.88 \text{ rad/s}$$

(*a*) **Hoop stress at inner radius**

The hoop stress at the inner radius is the maximum value of the hoop stress, and is given by equation (17.31) as

$$\sigma_{h_{r=R_1}} = \sigma_{h_{\max}} = \frac{\rho\omega^2}{4}[(3+v)R_2^2 + (1-v)R_1^2]$$

$$= \frac{7800 \times (418.88)^2}{4} \times [(3 + 0.3) \times (150 \times 10^{-3})^2 + (1 - 0.3) \times (50 \times 10^{-3})^2]$$

$$= 26 \times 10^6 \text{ N/m}^2$$

$$= 26 \text{ MPa}$$ **Ans.**

Hoop stress at outer radius

The hoop stress at the outer radius is given by equation (17.32) as

$$\sigma_{h_{r=R_2}} = \frac{\rho\omega^2}{4}\left[(1-v)R_2^2 + (3+v)R_1^2\right]$$

$$= \frac{7800 \times (418.88)^2}{4} \times [(1 - 0.3) \times (150 \times 10^{-3})^2 + (3 + 0.3) \times (50 \times 10^{-3})^2]$$

$$= 8.211 \times 10^6 \text{ N/m}^2$$

$$= 8.211 \text{ MPa}$$ **Ans.**

(*b*) The radius at which the radial stress is maximum, is given by equation (17.33) as

$$r = \sqrt{R_1 R_2}$$

$$= \sqrt{(50 - 10^{-3}) \times (150 \times 10^{-3})}$$

$$= 0.0866 \text{ m}$$

$$= 86.6 \text{ mm}$$ **Ans.**

(*c*) The maximum radial stress is given by equation (17.34) as

$$\sigma_{r_{max}} = \frac{(3+\nu)\rho\omega^2}{8}(R_2 - R_1)^2$$

$$= \frac{(3+0.3)\times 7800\times(418.88)^2}{8} \times \{(150 - 50) \times 10^{-3}\}^2$$

$$= 5.645 \times 10^6 \text{ N/m}^2$$

$$= 5.645 \text{ MPa}$$ **Ans.**

Example 17.8

A circular disc of outside diameter 500 mm has a central hole and rotates at a uniform speed about an axis through its centre. The diameter of the hole is such that the maximum stress due to rotation is 85% of that in a thin ring whose mean diameter is also 500 mm. If both disc and ring are made of the same material and rotate at the same speed, determine (*a*) the diameter of the central hole and (*b*) the speed of rotation, if the allowable stress in the disc is 90 MPa. Take the Poisson's ratio of 0.3 and the density of both disc and ring material as 7800 kg/m^3.

Solution: Given,

Mean radius of the thin ring, $r = \frac{500}{2}$ mm

$= 250 \times 10^{-3}$ m

Outside radius of the disc, $R_2 = \frac{500}{2}$ mm

$= 250 \times 10^{-3}$ m

Density of the disc and ring material,

$\rho = 7800$ kg/m^3

Poisson's ratio $\nu = 0.3$

Maximum hoop stress in the disc,

$\sigma_{h_{max}} = 90$ MPa

$= 90 \times 10^6$ Pa

Let σ_h be the maximum hoop stress in the thin ring.

Given $\sigma_{h_{max}} = 0.85 \times \sigma_h$

Hence $\sigma_h = \frac{\sigma_{h_{max}}}{0.85}$

For thin ring

The hoop stress in the thin ring is given as

$$\sigma_h = \rho\omega^2 r^2 \quad \text{(using equation (17.5))}$$

$$\frac{\sigma_{h_{max}}}{0.85} = \rho\omega^2 r^2 \qquad \text{(on substituting } \sigma_h\text{)}$$

$$\frac{90 \times 10^6}{0.85} = 7800 \times \omega^2 \times (250 \times 10^{-3})^2$$

$$\omega^2 = \frac{90 \times 10^6}{7800 \times 0.85 \times (250 \times 10^{-3})^2}$$

$$= 217194.6$$

which gives

$$\omega = 466 \text{ rad/s}$$

Now the rotational speed N is given as

$$N = \frac{60\omega}{2\pi}$$

$$= \frac{60 \times 466}{2\pi}$$

$$= 4450 \text{ rpm}$$ **Ans.**

For hollow disc

The maximum hoop stress is given as

$$\sigma_{h_{max}} = \frac{\rho\omega^2}{4}[(3+v)R_2^2 + (1-v)R_1^2] \qquad \text{(using equation (17.31))}$$

$$90 \times 10^6 = \frac{7800 \times 217194.6}{4} \times [(3 + 0.3) \times (250 \times 10^{-3})^2 + (1 - 0.3)\, R_1^2]$$

$$= 4.23 \times 10^8 \times [3.3 \times (250 \times 10^{-3})^2 + 0.7 \times R_1^2]$$

$$\frac{90 \times 10^6}{4.23 \times 10^8} = 0.20625 + 0.7\, R_1^2$$

$$0.2128 = 0.20625 + 0.7\, R_1^2$$

$$R_1^2 = \frac{0.2128 - 0.20625}{0.7} = 9.36 \times 10^{-3}$$

which gives

$$R_1 = 0.09674 \text{ m}$$

$$= 96.74 \text{ mm}$$

Hence, the diameter of the central hole $= 2 \times R_1$

$$= 2 \times 96.74$$

$$= 193.5 \text{ mm}$$ **Ans.**

Example 17.9

A steel disc of uniform thickness and of diameter 800 mm has a pin hole at the center. Calculate the maximum hoop stress developed in the disc, if it rotates at 3000 rpm. The Poisson's ratio is 0.25 and the density of the disc material is 7800 kg/m^3.

Solution: Given,

$$\text{Radius of the disc, } R = \frac{800}{2} = 400 \text{ mm}$$

$$= 400 \times 10^{-3} \text{ m}$$

$$\text{Rotational speed, } N = 3000 \text{ rpm}$$

$$\text{Poisson's ratio, } v = 0.25$$

$$\text{Density of the disc material, } \rho = 7800 \,\pi\text{kg/m}^3$$

The angular speed of the disc is found as

Now
$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 3000}{60} = 314.16 \text{ rad/s}$$

From equation (17.35), the maximum hoop stress is given as

$$\sigma_{h_{\max}} = \frac{(3+v)\rho\omega^2 R^2}{4}$$

$$= \frac{(3+0.25)\times 7800 \times (314.16)^2 \times (400\times 10^{-3})^2}{4}$$

$$= 10^8 \text{ Pa}$$

$$= 100 \text{ MPa}$$ **Ans.**

Example 17.10

A thin uniform steel disc of diameter 500 mm rotates at 2000 rpm. Calculate the maximum principal stress induced in the disc and also plot the distribution of the hoop stress and the radial stress along the radius of the disc. Take Poisson's ratio as 0.3 and the density of the disc material is equal to 7800 kg/m^3.

Solution: Given,

$$\text{Radius of the disc, } R = \frac{500}{2} = 250 \text{ mm}$$

$$= 250 \times 10^{-3} \text{ m}$$

Rotational speed, N = 2000 rpm

Poisson's ratio, v = 0.3.

Density of the disc material, ρ = 7800 kg/m^3.

The angular speed of the disc is obtained as

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 2000}{60} = 209.44 \text{ rad/s}$$

The maximum hoop stress is also the maximum principal stress, which can be obtained by using equation (17.25) as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2 R^2 (3+v)}{8} = \frac{7800 \times (209.44)^2 \times (250 \times 10^{-3})^2 \times (3+0.3)}{8} = 8.821 \times 10^6 \text{ N/m}^2 = 8.821 \text{ MPa}$$

Ans.

Distribution of the hoop stress

The hoop stress is given by equation (17.24) as

$$\sigma_h = \frac{\rho\omega^2}{8} [(3 + v) R^2 - (1 + 3v) r^2]$$

Now we select different values of the radius r and determine the corresponding hoop stresses using the above equation. The distribution of the stresses is shown in Table 17.1.

Table 17.1 Distribution of the hoop stress

r (mm)	0	50	100	150	200	250
σ_h (MPa)	8.821	8.618	8.00	7.00	5.570	3.742

Distribution of the radial stress

The radial stress is given by equation (17.23) as

$$\sigma_r = \frac{\rho\omega^2 (3+v)}{8} (R^2 - r^2)$$

The values of the radial stresses corresponding to the selected values of the radius are determined using the above equation, which are shown in Table 17.2.

Table 17.2 Distribution of the radial stress

r (mm)	0	50	100	150	200	250
σ_r (MPa)	8.821	8.468	7.409	5.645	3.175	0

Plotting of the hoop and radial stresses

The values of the radius are plotted on x-axis and the values of the hoop and radial stresses on y-axis. The resulting curves for the two stresses are shown in Fig. 17.4.

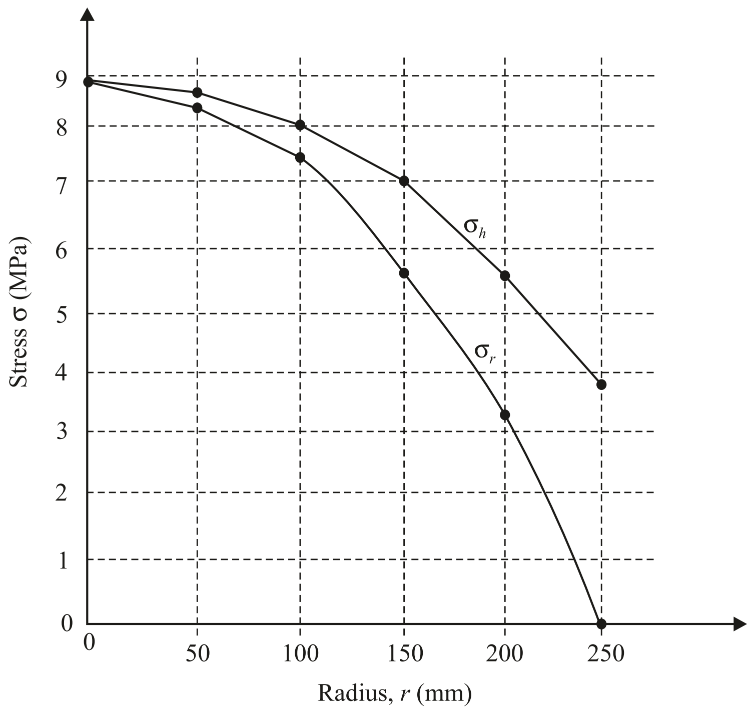


Fig. 17.4

Example 17.11

A steel disc of diameter 400 mm has a central hole of diameter 100 mm and rotates at 8000 rpm. Taking Poisson's ratio as 0.3 and the density of the disc material to be 7800 kg/m^3, plot the distribution of the hoop and radial stresses along the radius of the disc.

Solution: Given,

$$\text{Outer radius of the disc, } R_2 = \frac{400}{2} = 200 \text{ mm} = 200 \times 10^{-3} \text{ m}$$

$$\text{Inner radius of the disc, } R_1 = \frac{100}{2} = 50 \text{ mm}$$
$$= 50 \times 10^{-3} \text{ m}$$

Poisson's ratio, $v = 0.3$

Rotational speed, $N = 8000$ rpm

Density of the disc material, $\rho = 7800$ kg/m^3

The angular speed of the disc is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 8000}{60}$$

$$= 837.76 \text{ rad/s}$$

Distribution of the hoop stress

The hoop stress is given by equation (17.30) as

$$\sigma_h = \frac{\rho\omega^2}{8}\left[(3+v)(R_2^2+R_1)^2 + \frac{(3+v)R_1^2R_2^2}{r^2} - (1+3v)r^2\right]$$

Now we select different values of the radius r and determine the corresponding hoop stresses using the above equation. The distribution of the stresses is shown in Table 17.3.

Table 17.3 Distribution of the hoop stress

r (mm)	50	100	150	200
σ_h (MPa)	183.05	105.55	76.75	49.61

Distribution of the radial stress

The radial stress is given by equation (17.29) as

$$\sigma_r = \frac{(3+v)\rho\omega^2}{8}\left[(R_2^2+R_1^2) - \frac{R_1^2R_2^2}{r^2} - r^2\right]$$

The values of the radial stresses corresponding to the selected values of the radius are determined using the above equation, which are shown in Table 17.4. The radius at which radial stress is maximum is $\sqrt{R_1R_2} = \sqrt{(50\times10^{-3})\times(200\times10^{-3})} = 0.1$ m $= 100$ mm.

Table 17.4 Distribution of the radial stress

r (mm)	50	100	150	200
σ_r (MPa)	0	50.80	35.13	0

Plotting of the hoop and radial stresses

The values of the radius are plotted on x-axis and the values of the hoop and radial stresses on y-axis. The resulting curves for the two stresses are shown in Fig. 17.5.

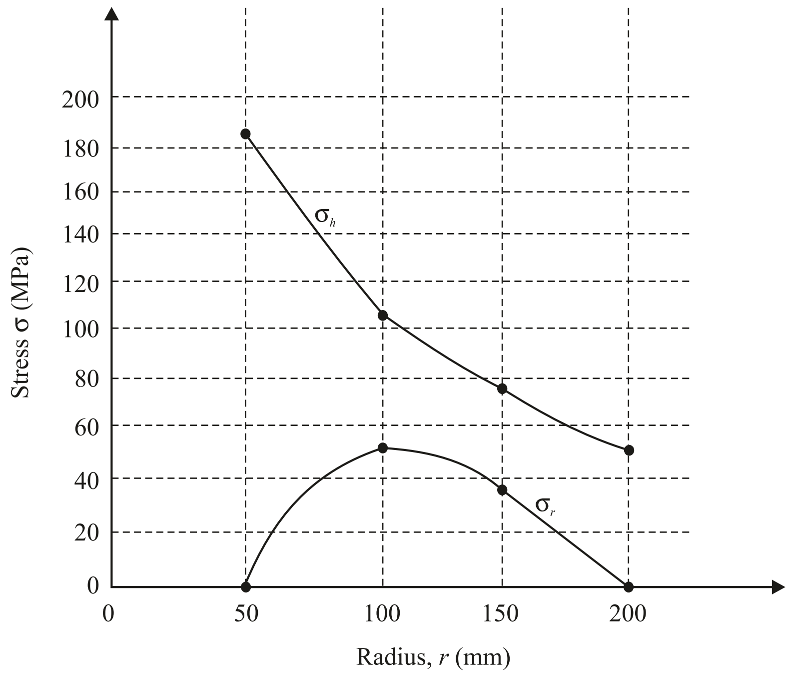


Fig. 17.5

17.4 ROTATING DISC OF UNIFORM STRENGTH

In case of a disc with uniform thickness, the hoop and radial stresses are not uniform and they vary along the radius of the disc. On the other hand, a disc of uniform strength has equal values of hoop and radial stresses at every radius. Thickness of such a disc is not uniform and varies along the axis. Analysis of such a disc is useful in the design of turbine blades rotating at high speeds and are required to be subjected to constant stress conditions to prevent their premature failure.

Consider a rotating disc of uniform strength which is subjected to equal hoop and radial stresses, that is, $\sigma_h = \sigma_r = \sigma$ and the stresses do not vary with radius. Now consider an element $ABCD$ of radial width dr of the disc at a radius r from the axis of rotation making an angle $d\theta$ at the center as shown in Fig. 17.6.

Let t = Thickness of the disc at radius r

$t + dt$ = Thickness of the disc at radius $(r + dr)$

t_o = Thickness of the disc at radius $r = 0$, that is, at the axis of rotation

ω = Angular speed of the disc

σ = Equal hoop and radial stresses

ρ = Density of the disc material

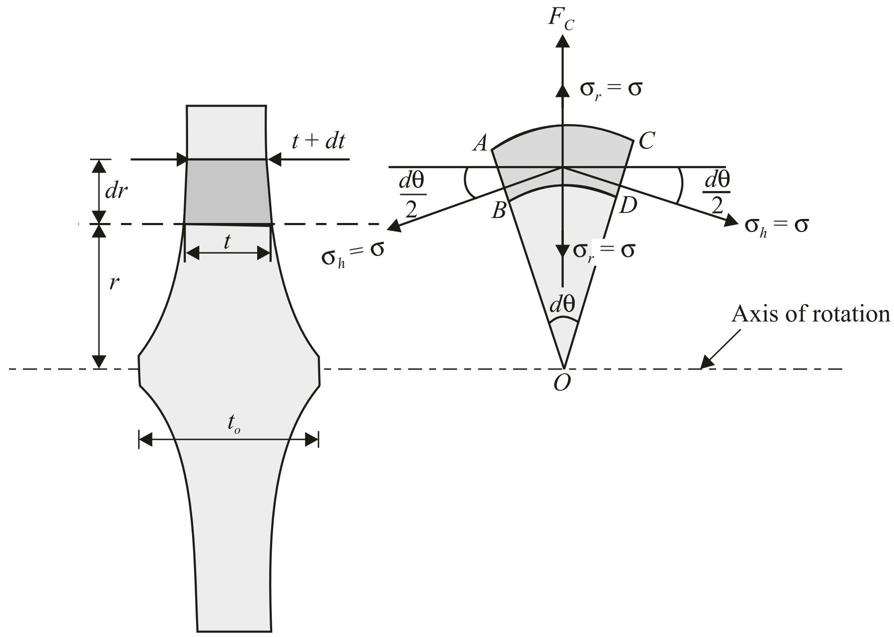


Fig. 17.6

Forces on the element

The following forces are acting on the element *ABCD* :

- The centrifugal force caused due to rotation acting radially outward
- The hoop tension force on the face *AB* caused due to hoop stress σ
- The hoop tension force on the face *CD* caused due to hoop stress σ
- The radial force on the face *BD* caused due to radial stress σ
- The radial force on the face *AC* caused due to radial stress σ

Centrifugal force

The mass *m* of the element can be obtained as

$$\begin{aligned} m &= \text{Density} \times \text{Volume of the element} \\ &= \rho \times (rd\theta \times dr \times t) \\ &= \rho rtd\theta dr \qquad \text{...(17.38)} \end{aligned}$$

The centrifugal force is given as

$$\begin{aligned} F_c &= \frac{mV^2}{r} \\ &= m\omega^2 r \qquad (\text{as } V = \omega r) \\ &= (\rho rtd\theta dr \times \omega^2 r) \qquad (\text{on substituting } m) \\ &= \rho t\omega^2 r^2 d\theta dr \qquad \text{...(17.39)} \end{aligned}$$

Hoop tension forces on faces *AB* and *CD*

The hoop tension forces act perpendicular to *AB* and *CD* and are equal to $\sigma_h \times dr \times t$.

Now the hoop forces are resolved into horizontal and vertical components. The vertical components of the hoop forces acting on faces *AB* and *CD* are radially inward and both are equal to $\sigma \times dr \times t \times \sin \dfrac{d\theta}{2}$ The horizontal component of the hoop forces acting on face *AB* is directed leftward and the horizontal component on face *CD* is directed rightward, and both are equal to $\sigma \times dr \times t \times \cos \dfrac{d\theta}{2}$

Radial force on face *BD*

Radial force on $BD = \sigma \times rd\theta \times t$, and its acts radially inward.

Radial force on face *AC*

Radial force on $AC = \sigma \times (r + dr)d\theta \times (t + dt)$, and it acts radially outward.

Equilibrium of the element

The horizontal components of the hoop tension forces on the faces *AB* and *CD* are equal but opposite in direction, hence they cancel each other. The vertical components are added as they are acting in the same direction (radially inward).

Considering forces in the radial direction, we have

$$\sigma \times rd\theta \times t + \sigma \times dr \times t \times \sin \frac{d\theta}{2} + \sigma \times dr \times t \times \sin \frac{d\theta}{2} = F_c + \sigma \times (r + dr)\, d\theta \times (t + dt)$$

or $$\sigma \times rd\theta \times t + 2\sigma \times dr \times t \times \sin \frac{d\theta}{2} = F_c + \sigma \times (r + dr)\, d\theta \times (t + dt)$$

Substituting the value of F_c from equation (17.39) in the above equation and equating $\sin \dfrac{d\theta}{2} = \dfrac{d\theta}{2}$ as $d\theta$ is very small, we get

$$\sigma \times rd\theta \times t + 2\sigma \times dr \times t \times \frac{d\theta}{2} = \rho t\omega^2 r^2 d\theta dr + \sigma rtd\theta + \sigma rd\theta dt + \sigma tdrd\theta + \sigma drd\theta dt$$

Eliminating $d\theta$ from both sides, we get

$$\sigma \times r \times t + \sigma tdr = \rho t\omega^2 r^2 dr + \sigma rt + \sigma rdt + \sigma tdr + \sigma drdt$$

Now eliminating σrt and σtdr from both sides and neglecting $(drdt)$ as being the product of two smaller quantities, we find

$$0 = \rho t\omega^2 r^2 dr + \sigma rdt$$

or $$-\sigma rdt = \rho t\omega^2 r^2 dr$$

$$\frac{dt}{t} = -\frac{\rho\omega^2 rdr}{\sigma}$$

On integration, we have

$$\log_e t = -\frac{\rho\omega^2 r^2}{2\sigma} + \log_e A \qquad \text{...(17.40)}$$

where $\log_e A$ is a constant of integration.

The boundary condition is

when $r = 0$

$t = t_o$

Substituting the boundary condition in equation (17.40), we get

$$\log_e t_o = \log_e A$$

which gives $A = t_o$

Equation (17.40) on substituting A becomes

$$\log_e t = -\frac{\rho\omega^2 r^2}{2\sigma} + \log_e t_o$$

$$\log_e t - \log_e t_o = -\frac{\rho\omega^2 r^2}{2\sigma}$$

$$\log_e \left(\frac{t}{t_o}\right) = -\frac{\rho\omega^2 r^2}{2\sigma}$$

$$\frac{t}{t_o} = e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

which gives

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)} \qquad \text{...(17.41)}$$

This is the required expression for the thickness of the disc, which varies according to the given value of the radius r.

Example 17.12

A turbine rotor is to be designed for uniform strength for a tensile stress of 150 MPa . The rotor runs at 6000 rpm and its thickness at the centre is 90 mm. If the density of the material of the rotor is 7800 kg/m^3, determine the thickness of the rotor at a radius of 400 mm.

Solution: Given,

Uniform stress, $\sigma = 150$ MPa

$= 150 \times 10^6$ Pa

Rotational speed, $N = 6000$ rpm

Thickness of the rotor at the center, $t_o = 90$ mm

$= 90 \times 10^{-3}$ m

Density of the rotor material, $\rho = 7800$ kg/m^3

Radius at the required thickness, $r = 400$ mm

$$= 400 \times 10^{-3} \text{ m}$$

The angular speed of the rotor is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 6000}{60}$$

$$= 628.32 \text{ rad/s}$$

From equation (17.41), the expression for the thickness is given as

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

$$= 90 \times 10^{-3} \times e^{\left(-\frac{7800 \times (628.32)^2 \times (400 \times 10^{-3})^2}{2 \times 150 \times 10^6}\right)}$$

$$= 90 \times 10^{-3} \times e^{-1.642}$$

$$= 90 \times 10^{-3} \times 0.193$$

$$= 0.01742 \text{ m}$$

$$= 17.42 \text{ mm}$$ **Ans.**

Example 17.13

The minimum thickness of a steam turbine rotor is 10 mm at a radius of 200 mm and is required to be designed for uniform strength under rotational conditions for a stress of 180 MPa. It runs at 10,000 rpm and its material weighs 7800 kg/m^3. Determine the thickness of the rotor at a radius of 40 mm.

Solution: Given,

Rotational speed of the rotor, N = 10,000 rpm

Radius at the desired thickness, r = 40 mm

$= 40 \times 10^{-3}$ m

Uniform stress, σ = 180 MPa

$= 180 \times 10^6$ Pa

Density of the rotor material, ρ = 7800 kg/m^3

The angular speed of the rotor is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 10,000}{60}$$

$$= 1047.2 \text{ rad/s}$$

The thickness expression is given as

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

At $r = 200$ mm, $t = 10$ mm. Substituting these values in the above equation, we have

$$10 \times 10^{-3} = t_o \times e^{\left(-\frac{7800 \times (1047.2)^2 \times (200 \times 10^{-3})^2}{2 \times 180 \times 10^6}\right)}$$

$$= t_o \times e^{-0.9504}$$

$$= t_o \times 0.3866$$

which gives

$$t_o = 0.02586 \text{ m}$$

$$= 25.86 \text{ mm}$$

Again using the thickness equation and substituting the value of t_o, we have

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

$$= 0.02586 \times e^{\left(-\frac{7800 \times (1047.2)^2 \times (40 \times 10^{-3})^2}{2 \times 180 \times 10^6}\right)}$$

$$= 0.02586 \times e^{-0.038}$$

$$= 0.02586 \times 0.9627$$

$$= 0.0249 \text{ m}$$

$$= 24.9 \text{ mm}$$ **Ans.**

Example 17.14

A steam turbine rotor is 160 mm diameter below the blade ring and 5 mm thick, and runs at 30,000 rpm. If the material of the rotor weighs 7800 kg/m^3 and the allowable stress is 160 MPa, what is the thickness of the rotor at a radius of 60 mm and at the centre? Assume uniform strength condition.

Solution: Given,

Rotational speed of the rotor,

$$N = 30{,}000 \text{ rpm}$$

Radius at the desired thickness, $r = 60$ mm

$$= 60 \times 10^{-3} \text{ m}$$

Uniform stress, $\sigma = 160$ MPa

$$= 160 \times 10^6 \text{ Pa}$$

Density of the rotor material, $\rho = 7800$ kg/m^3

The angular speed of the rotor is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 30,000}{60} = 3141.6 \text{ rad/s}$$

The thickness expressions is given as

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

where t_o is thickness of the rotor at the center, that is, at $r = 0$.

Now at $r = 80$ mm, $t = 5$ mm

Using these values in the above equation, we have

$$5 \times 10^{-3} = t_o \times e^{\left(-\frac{7800\times(3141.6)^2\times(80\times10^{-3})^2}{2\times160\times10^6}\right)}$$

$$= t_o \times e^{-1.53966}$$

$$= t_o \times 0.2144$$

which gives

$$t_o = 0.0233 \text{ m}$$

$$= 23.3 \text{ mm}$$ **Ans.**

Again using the thickness equation for $r = 60$ mm, we have

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

$$= 23.3 \times 10^{-3} \times e^{\left(-\frac{7800\times(3141.6)^2\times(60\times10^{-3})^2}{2\times160\times10^6}\right)}$$

$$= 23.3 \times 10^{-3} \times e^{-0.866}$$

$$= 23.3 \times 10^{-3} \times 0.4206 = 9.8 \times 10^{-3} \text{ m}$$

$$= 9.8 \text{ mm}$$ **Ans.**

Example 17.15

A steel turbine disc is to be designed so that between radii of 250 mm and 400 mm, the radial and hoop stresses are required to be constant at 60 MPa, when running at 3000 rpm. If the axial thickness is 12 mm at the outer edge of this zone, what should it be at the inner edge? The density of the disc material is 7800 kg/m^3.

Solution: Given,

Uniform stress, $\sigma = 60$ MPa

$= 60 \times 10^6$ Pa

Rotational speed, $N = 3000$ rpm

Density of the disc material, $\rho = 7800$ kg/m^3.

The angular speed of the disc is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 3000}{60}$$

$$= 314.16 \text{ rad/s}$$

The thickness expression is given as

$$t = t_o \times e^{\left(-\frac{\rho\omega^2 r^2}{2\sigma}\right)}$$

Now at $r = 400$ mm, $t = 12$ mm. Substituting these values in the above equation, we have

$$12 \times 10^{-3} = t_o \times e^{\left(-\frac{7800 \times (314.16)^2 \times (400 \times 10^{-3})^2}{2 \times 60 \times 10^6}\right)}$$

$$= t_o \times e^{-1.0264}$$

$$= t_o \times 0.3583$$

which gives

$$t_o = 0.0335 \text{ m}$$

$$= 33.5 \text{ mm}$$

Again using thickness equation for $r = 250$ mm, we have

$$t = 0.0335 \times e^{\left(-\frac{7800 \times (314.16)^2 \times (250 \times 10^{-3})^2}{2 \times 60 \times 10^6}\right)}$$

$$= 0.0335 \times e^{-0.40095}$$

$$= 0.0335 \times 0.6697$$

$$= 0.0224 \text{ m}$$

$$= 22.4 \text{ mm}$$ **Ans.**

17.5 ROTATING LONG CYLINDER

The force analysis of a rotating thick cylinder is similar to that of a rotating thin disc except the introduction of axial stress. Hence three stresses acting on a rotating thick cylinder include hoop stress (σ_h), radial stress (σ_r) and axial stress, also called longitudinal stress (σ_l). It is assumed that the transverse sections of the cylinder remain plane even at high speeds of rotation, which implies that longitudinal strain is constant. At the same time, all the three stresses acting on the cylinder are principal stresses.

Consider a small element *ABCD* of the cylinder at a distance *r* and of radial thickness *dr* subtending an angle $d\theta$ at the center as shown in Fig. 17.7.

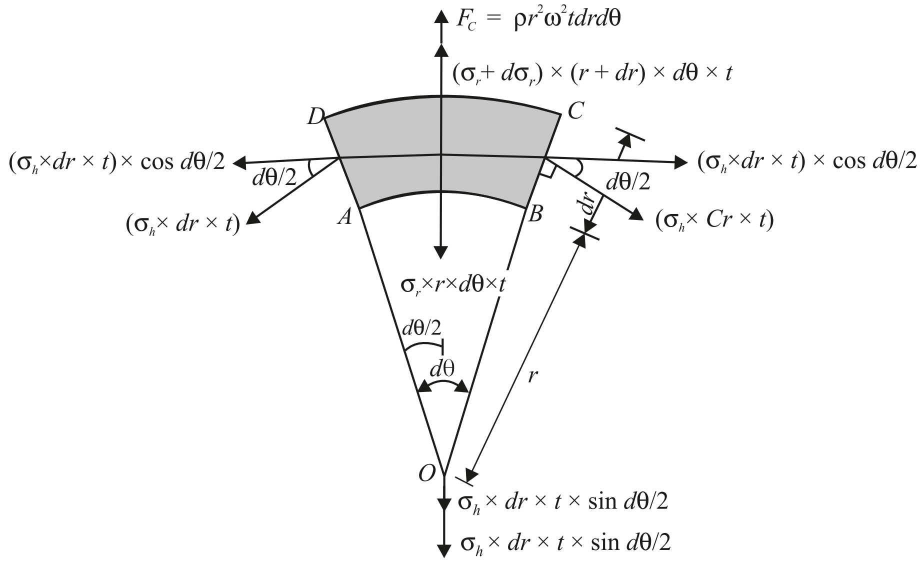


Fig. 17.7

Let $\in_h$ = Hoop strain, also called circumferential strain

$\in_r$ = Radial strain

$\in_l$ = Longitudinal strain

ω = Angular speed of rotation of the cylinder

v = Poisson's ratio

E = Modulus of elasticity of the cylinder

The strains produced by various stresses are obtained as

$$\in_h = \frac{\sigma_h}{E} - v\frac{\sigma_r}{E} - v\frac{\sigma_l}{E}$$

$$= \frac{1}{E}\ [\sigma_h - v\,(\sigma_r + \sigma_l)] \qquad ...(17.42)$$

$$\in_r = \frac{\sigma_r}{E} - v\frac{\sigma_h}{E} - v\frac{\sigma_l}{E}$$

$$= \frac{1}{E}\ [\sigma_r - v\,(\sigma_h + \sigma_l)] \qquad ...(17.43)$$

and

$$\in_l = \frac{\sigma_l}{E} - v\frac{\sigma_h}{E} - v\frac{\sigma_r}{l}$$

$$= \frac{1}{E}\ [\sigma_l - v\,(\sigma_h + \sigma_r)] \qquad ...(17.44)$$

Due to rotation, the radius of the cylinder increases. Let the radius r changes to $(r + u)$ and dr changes to $(dr + du)$.

Now the hoop strain is also expressed as

$$\in_h = \frac{2\pi(r+u) - 2\pi r}{2\pi r}$$

$$= \frac{u}{r} \quad \text{...(17.45)}$$

The radial strain is also expressed as

$$\in_r = \frac{(dr+du) - dr}{dr}$$

$$= \frac{du}{dr} \quad \text{...(17.46)}$$

Comparing equations (17.42) and (17.45), we have

$$\in_h = \frac{u}{r} = \frac{1}{E}\,[\sigma_h - v\,(\sigma_r + \sigma_l)]$$

which gives

$$Eu = r\,[\sigma_h - v\,(\sigma_r + \sigma_l)] \quad \text{...(17.47)}$$

Comparing equations (17.43) and (17.46), we have

$$\in_r = \frac{du}{dr} = \frac{1}{E}\,[\sigma_r - v\,(\sigma_h + \sigma_l)] \quad \text{...(17.48)}$$

Differentiating equation (17.47) with respect to r, we get

$$\frac{du}{dr} = \frac{1}{E}\,[\sigma_h - v\,(\sigma_r + \sigma_l)] + \frac{1}{E}\left[r\frac{d\sigma_h}{dr} - vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right)\right] \quad \text{...(17.49)}$$

Comparing equations (17.48) and (17.49), we get

$$\sigma_r - v\,(\sigma_h + \sigma_l) = \sigma_h - v\,(\sigma_r + \sigma_l) + r\frac{d\sigma_h}{dr} - vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right)$$

$$\sigma_r - v\sigma_h - v\sigma_l = \sigma_h - v\sigma_r - v\sigma_l + r\frac{d\sigma_h}{dr} - vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right)$$

Cancelling $v\sigma_l$ from both sides of the equation and rearranging the terms, we get

$$\sigma_r + v\sigma_r = \sigma_h + v\sigma_h + r\frac{d\sigma_h}{dr} - vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right)$$

$$\sigma_r\,(1 + v) = \sigma_h\,(1 + v) + r\frac{d\sigma_h}{dr} - vr\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_l}{dr}\right)$$

$$(1+\nu)(\sigma_r-\sigma_h) = r\frac{d\sigma_h}{dr} - \nu r\left(\frac{d\sigma_r}{dr}+\frac{d\sigma_l}{dr}\right)$$

$$(1+\nu)(\sigma_r-\sigma_h) - r\frac{d\sigma_h}{dr} + \nu r\left(\frac{d\sigma_r}{dr}+\frac{d\sigma_l}{dr}\right) = 0 \quad \text{...(17.50)}$$

Since $\in_l$ is constant, hence from equation (17.44), we have

$$\in_l = \frac{1}{E}\,[\sigma_l - \nu(\sigma_h+\sigma_r)] = \text{Constant}$$

or $\sigma_l - \nu(\sigma_h+\sigma_r) = \text{Constant}$ (as E is a constant)

Differentiating with respect to r, we have

$$\frac{d\sigma_l}{dr} - \nu\left(\frac{d\sigma_h}{dr}+\frac{d\sigma_r}{dr}\right) = 0$$

Multiplying all the terms by r, we get

$$r\frac{d\sigma_l}{dr} - \nu r\left(\frac{d\sigma_h}{dr}+\frac{d\sigma_r}{dr}\right) = 0$$

or

$$r\frac{d\sigma_l}{dr} = \nu r\left(\frac{d\sigma_h}{dr}+\frac{d\sigma_r}{dr}\right) \quad \text{...(17.51)}$$

From equation (17.50), we have

$$(1+\nu)(\sigma_r-\sigma_h) - r\frac{d\sigma_h}{dr} + \nu r\frac{d\sigma_r}{dr} + \nu r\frac{d\sigma_l}{dr} = 0$$

Substituting equation (17.51) in the above equation, we get

$$(1+\nu)(\sigma_r-\sigma_h) - r\frac{d\sigma_h}{dr} + \nu r\frac{d\sigma_r}{dr} + \nu^2 r\left(\frac{d\sigma_h}{dr}+\frac{d\sigma_r}{dr}\right) = 0$$

$$(1+\nu)(\sigma_r-\sigma_h) - r\frac{d\sigma_h}{dr} + \nu r\frac{d\sigma_r}{dr} + \nu^2 r\frac{d\sigma_h}{dr} + \nu^2 r\frac{d\sigma_r}{dr} = 0$$

$$(1+\nu)(\sigma_r-\sigma_h) - r\frac{d\sigma_h}{dr}(1-\nu^2) + \nu r\frac{d\sigma_r}{dr}(1+\nu) = 0$$

Eliminating $(1+\nu)$ from all the terms, we get

$$(\sigma_r-\sigma_h) - r\frac{d\sigma_h}{dr}(1-\nu) + \nu r\frac{d\sigma_r}{dr} = 0$$

$$(\sigma_h-\sigma_r) + r(1-\nu)\frac{d\sigma_h}{dr} - \nu r\frac{d\sigma_r}{dr} = 0 \quad \text{...(17.52)}$$

The equilibrium equation of the element can be obtained in a similar manner as in case of a rotating thin disc, which is given by equation (17.7) as

$$(\sigma_h-\sigma_r) = \rho r^2\omega^2 + r\frac{d\sigma_r}{dr} \quad \text{(from equation (17.7))}$$

Substituting the above in equation (17.52), we get

$$\rho r^2\omega^2 + r\frac{d\sigma_r}{dr} + r(1-\nu)\frac{d\sigma_h}{dr} - \nu r\frac{d\sigma_r}{dr} = 0$$

Eliminating r from all the terms and simplifying, we have

$$\rho r\omega^2 + \frac{d\sigma_r}{dr}(1-\nu) + (1-\nu)\frac{d\sigma_h}{dr} = 0$$

or

$$\frac{d\sigma_h}{dr} + \frac{d\sigma_r}{dr} = -\frac{\rho r\omega^2}{(1-\nu)} \qquad ...(17.53)$$

Integration of equation (17.53) gives

$$\sigma_h + \sigma_r = -\frac{\rho r^2\omega^2}{2(1-\nu)} + A \qquad ...(17.54)$$

where A is a constant of integration.

Now subtracting equation (17.7) from equation (17.54), we get

$$\sigma_h + \sigma_r - \sigma_h + \sigma_r = -\frac{\rho r^2\omega^2}{2(1-\nu)} + A - \rho r^2\omega^2 - r\frac{d\sigma_r}{dr}$$

$$2\sigma_r + r\frac{d\sigma_r}{dr} = -\rho r^2\omega^2\left[\frac{1}{2(1-\nu)} + 1\right] + A$$

$$= -\rho r^2\omega^2\left[\frac{1+2-2\nu}{2(1-\nu)}\right] + A$$

$$= -\frac{\rho r^2\omega^2}{2}\left(\frac{3-2\nu}{1-\nu}\right) + A$$

Multiplying by r on both sides, we get

$$2r\sigma_r + r^2\frac{d\sigma_r}{dr} = -\frac{\rho r^3\omega^2}{2}\left(\frac{3-2\nu}{1-\nu}\right) + Ar$$

$$\frac{d}{dr}(r^2 \times \sigma_r) = -\frac{\rho r^3\omega^2}{2}\left(\frac{3-2\nu}{1-\nu}\right) + Ar \qquad ...(17.55)$$

On integration, we have

$$r^2 \times \sigma_r = -\frac{\rho r^4\omega^2}{8}\left(\frac{3-2\nu}{1-\nu}\right) + \frac{Ar^2}{2} + B$$

where B is another constant of integration.

Dividing throughout by r^2, we get

$$\sigma_r = \frac{A}{2} + \frac{B}{r^2} - \frac{\rho r^2\omega^2}{8}\left(\frac{3-2\nu}{1-\nu}\right) \qquad ...(17.56)$$

This is the required expression for the radial stress for a rotating thick cylinder. The constants A and B can be determined by using suitable boundary conditions.

Substituting equation (17.56) in equation (17.54), we can obtain the value of σ_h.

$$\sigma_h + \frac{A}{2} + \frac{B}{r^2} - \frac{\rho r^2 \omega^2}{8}\left(\frac{3-2v}{1-v}\right) = -\frac{\rho r^2 \omega^2}{2(1-v)} + A$$

or

$$\begin{aligned}\sigma_h &= A - \frac{A}{2} - \frac{B}{r^2} + \frac{\rho r^2 \omega^2}{8}\left(\frac{3-2v}{1-v}\right) - \frac{\rho r^2 \omega^2}{2(1-v)} \\ &= \frac{A}{2} - \frac{B}{r^2} + \frac{\rho r^2 \omega^2}{2(1-v)}\left(\frac{3-2v}{4} - 1\right) \\ &= \frac{A}{2} - \frac{B}{r^2} + \frac{\rho r^2 \omega^2}{2\,(1-v)}\left(\frac{3-2v-4}{4}\right) \\ &= \frac{A}{2} - \frac{B}{r^2} - \frac{\rho r^2 \omega^2}{8}\left(\frac{1+2v}{1-v}\right) \qquad \text{...(17.57)}\end{aligned}$$

This is the required expression for the hoop stress for a rotating thick cylinder. The constants A and B can be determined by using suitable boundary conditions.

17.5.1 Hoop and Radial Stresses in a Rotating Solid Cylinder or a Solid Shaft

When we put $r = 0$ in equations (17.56) and (17.57), we see that stresses become infinite. Hence for the meaningful values of the two stresses, the constant B has to be zero. Now the stresses are expressed as

$$\sigma_r = \frac{A}{2} - \frac{\rho r^2 \omega^2}{8}\left(\frac{3-2v}{1-v}\right) \qquad \text{...(17.58)}$$

and

$$\sigma_h = \frac{A}{2} - \frac{\rho r^2 \omega^2}{8}\left(\frac{1+2v}{1-v}\right) \qquad \text{...(17.59)}$$

These are the expressions for the radial and hoop stresses respectively at the centre of a rotating solid cylinder.

At the surface of the cylinder, where $r = R_2$ ($= R$ say),

$$\sigma_r = 0$$

Substituting σ_r in equation (17.58), we have

$$0 = \frac{A}{2} - \frac{\rho R^2 \omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

which gives

$$\frac{A}{2} = \frac{\rho R^2 \omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

On putting $\frac{A}{2}$ in equation (17.58), we have

$$\sigma_r = \frac{\rho R^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right) - \frac{\rho r^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left(R^2 - r^2\right) \qquad ...(17.60)$$

This is the required expression for the radial stress for any value of r.

The expression for the hoop stress on substituting the value of $\frac{A}{2}$ in equation (17.59) becomes

$$\sigma_h = \frac{\rho\omega^2 R^2}{8}\left(\frac{3-2v}{1-v}\right) - \frac{\rho\omega^2 r^2}{8}\left(\frac{1+2v}{1-v}\right) \qquad ...(17.61)$$

Maximum radial stress

The radial stress is maximum at the centre of the cylinder, that is, at $r = 0$. Putting $r = 0$ in equation (17.60), we get the expression for the maximum radial stress as

$$\sigma_{r_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right) \qquad ...(17.62)$$

Maximum hoop stress

The hoop stress is also maximum at the centre of cylinder, that is, at $r = 0$. Putting $r = 0$ in equation (17.61), we get the expression for the maximum hoop stress as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right) \qquad ...(17.63)$$

Hence,

$$\sigma_{r_{\max}} = \sigma_{h_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right) \qquad ...(17.64)$$

17.5.2 Hoop and Radial Stresses in a Rotating Hollow Cylinder

For a hollow cylinder

$$\sigma_r = 0 \quad \text{at} \quad r = R_1$$

Also

$$\sigma_r = 0 \quad \text{at} \quad r = \mathrm{R}_2$$

Substituting these boundary conditions in equation (17.56), we have

$$0 = \frac{A}{2} + \frac{B}{R_1^2} - \frac{\rho\omega^2 R_1^2}{8}\left(\frac{3-2v}{1-v}\right) \qquad \text{(where } r = R_1\text{)} \quad ...(1)$$

and

$$0 = \frac{A}{2} + \frac{B}{R_2^2} - \frac{\rho\omega^2 R_2^2}{8}\left(\frac{3-2v}{1-v}\right) \qquad \text{(where } r = R_2\text{)} \quad ...(2)$$

Subtracting equation (2) from equation (1), we have

$$B\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)+\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_2^2-R_1^2) = 0$$

$$B\left(\frac{R_2^2-R_1^2}{R_1^2R_2^2}\right)+\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_2^2-R_1^2) = 0$$

Eliminating $(R_2^2-R_1^2)$, we have

$$\frac{B}{R_1^2R_2^2} = -\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

which gives

$$B = -\frac{\rho\omega^2R_1^2R_2^2}{8}\left(\frac{3-2v}{1-v}\right) \quad \text{...(17.65)}$$

Substituting the value of B in equation (1), we have

$$0 = \frac{A}{2}+\frac{1}{R_1^2}\times\left[-\frac{\rho\omega^2R_1^2R_2^2}{8}\left(\frac{3-2v}{1-v}\right)\right]-\frac{\rho\omega^2R_1^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{A}{2}-\frac{\rho\omega^2R_2^2}{8}\left(\frac{3-2v}{1-v}\right)-\frac{\rho\omega^2R_1^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{A}{2}-\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2)$$

which gives

$$A = \frac{\rho\omega^2}{4}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2) \quad \text{...(17.66)}$$

Now Substituting the values of A and B in equation (17.56), we get the expression for the radial stress as

$$\sigma_r = \frac{1}{2}\times\frac{\rho\omega^2}{4}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2)+\frac{1}{r^2}\times\left[-\frac{\rho\omega^2R_1^2R_2^2}{8}\left(\frac{3-2v}{1-v}\right)\right]$$

$$-\frac{\rho r^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2)-\frac{\rho\omega^2R_1^2R_2^2}{8r^2}\left(\frac{3-2v}{1-v}\right)-\frac{\rho r^2\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[R_1^2+R_2^2-\frac{R_1^2R_2^2}{r^2}-r^2\right] \quad \text{...(17.67)}$$

This is the required expression for the radial stress for a long rotating hollow cylinder.

Substituting the values of A and B in equation (17.57), we get the expression for the hoop stress as

$$\sigma_h = \frac{1}{2}\times\frac{\rho\omega^2}{4}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2)-\frac{1}{r^2}\times\left[-\frac{\rho\omega^2 R_1^2R_2^2}{8}\left(\frac{3-2v}{1-v}\right)\right]$$

$$-\frac{\rho r^2\omega^2}{8}\left(\frac{1+2v}{1-v}\right)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2+R_2^2)+\frac{\rho\omega^2R_1^2R_2^2}{8r^2}\left(\frac{3-2v}{1-v}\right)-\frac{\rho r^2\omega^2}{8}\left(\frac{1+2v}{1-v}\right)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[R_1^2+R_2^2+\frac{R_1^2R_2^2}{r^2}\right]-\frac{\rho\omega^2r^2}{8}\left(\frac{1+2v}{1-v}\right) \quad ...(17.68)$$

This is the required expression for the hoop stress for a long rotating hollow cylinder.

Maximum hoop stress

Equation (17.68) suggests that the hoop stress is maximum where r is minimum, that is, at $r = R_1$. Putting $r = R_1$ in equation (17.68), we obtain the value of the maximum hoop stress for the hollow cylinder as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[R_1^2+R_2^2+\frac{R_1^2R_2^2}{R_1^2}\right]-\frac{\rho\omega^2R_1^2}{8}\left(\frac{1+2v}{1-2v}\right)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2+2R_2^2)-\frac{\rho\omega^2R_1^2}{8}\left(\frac{1+2v}{1-v}\right) \quad ...(17.69)$$

Maximum radial stress

To find the location of the maximum radial stress, we differentiate equation (17.67) with respect to r and equate it to zero.

$$\frac{d\sigma_r}{dr} = 0$$

$$\frac{d}{dr}\left[\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left(R_1^2+R_2^2-\frac{R_1^2R_2^2}{r^2}-r^2\right)\right] = 0$$

$$\frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[0+0-\frac{R_1^2R_2^2\times(-2)}{r^3}-2r\right] = 0$$

or $$\frac{2R_1^2R_2^2}{r^3}-2r = 0$$

which gives

$$r = \sqrt{R_1R_2} \quad ...(17.70)$$

Hence, the radial stress is maximum at $r = \sqrt{R_1 R_2}$. Substituting this value in equation (17.67), we obtain the maximum value of the radial stress for a hollow cylinder as

$$\sigma_{r_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left(R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{R_1 R_2} - R_1 R_2\right)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2 + R_2^2 - 2R_1 R_2)$$

$$= \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_2 - R_1)^2 \qquad \text{...(17.71)}$$

This is the required expression for the maximum radial stress for a hollow cylinder.

Example 17.16

Determine the maximum hoop stress in a long cast iron solid cylinder of diameter 400 mm, which rotates at 2000 rpm about its axis. It weighs 7200 kg/m^3 and the Poisson's ratio is 0.3.

Solution: Given,

Radius of the solid cylinder, $R = \frac{400}{2}$ mm

$= 200 \times 10^{-3}$ m

Rotational speed, N = 2000 rpm

Density of the cylinder material,

ρ = 7200 kg/m^3

Poisson's ratio, v = 0.3

The angular speed of the cylinder is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 2000}{60}$$

$= 209.44$ rad/s

The maximum hoop stress is obtained using equation (17.63) as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{7200 \times (209.44)^2 \times (200 \times 10^{-3})^2}{8} \times \left(\frac{3 - 2 \times 0.3}{1 - 0.3}\right)$$

$= 5.41 \times 10^6$ Pa

$= 5.41$ MPa **Ans.**

Example 17.17

Calculate the maximum hoop stress in a long hollow cylinder of inside diameter 40 mm and outside diameter 200 mm rotating at 3000 rpm. The density of the cylinder material is 7800 kg/m^3 and the Poisson's ratio is 0.25.

Solution: Given,

Inside radius of the hollow cylinder, $R_1 = \dfrac{40}{2}$ mm

$= 20 \times 10^{-3}$ m

Outside radius of the hollow cylinder, $R_2 = \dfrac{200}{2}$ mm

$= 100 \times 10^{-3}$ m

Rotational speed, $N = 3000$ rpm

Density of the cylinder material, $\rho = 7800$ kg/m^3

Poisson's ratio, $v = 0.25$

The angular speed of the cylinder is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 3000}{60}$$

$$= 314.16 \text{ rad/s}$$

The maximum hoop stress is obtained using equation (17.69) as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2 + 2R_2^2) - \frac{\rho\omega^2 R_1^2}{8}\left(\frac{1+2v}{1-v}\right)$$

$$= \frac{7800 \times (314.16)^2}{8} \times \left(\frac{3-2\times 0.25}{1-0.25}\right) \times \left[(20\times 10^{-3})^2 + 2\times(100\times 10^{-3})^{-2}\right]$$

$$- \frac{7800\times(314.16)^2 \times (20\times 10^{-3})^2}{8} \times \left(\frac{1+2\times 0.25}{1-0.25}\right)$$

$= 6543578.3 - 76983.3$

$= 6466595$ Pa

$= 6.46$ MPa **Ans.**

Example 17.18

Compare the peripheral velocities for the same maximum intensity of stress of (*a*) a solid cylinder (*b*) a solid thin disc and (*c*) a thin ring, if they are made of the same material. Take velocity of the ring as unity and the Poisson's ratio 0.3.

Solution:

For a solid cylinder

The maximum hoop stress is

$$\sigma_{h_{max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right) \qquad \text{(using equation (17.64))}$$

Since the peripheral velocity is

$$V = \omega R$$

Hence, the hoop stress equation for the solid cylinder is

$$\sigma_{h_{max}} = \frac{\rho V_1^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{\rho V_1^2}{8}\times\left(\frac{3-2\times 0.3}{1-0.3}\right)$$

$$= 0.428\ \rho\ V_1^2$$

or

$$V_1^2 = \frac{\sigma_{h_{max}}}{0.428\rho}$$

$$= 2.33\times\frac{\sigma_{h_{max}}}{\rho} \qquad \text{...(1)}$$

For a solid thin disc

The maximum hoop stress is

$$\sigma_{h_{max}} = \frac{\rho\omega^2 R^2(3+v)}{8} \qquad \text{(using equation (17.25))}$$

$$= \frac{\rho V_2^2(3+v)}{8} \qquad \text{(as } V_2 = \omega R\text{)}$$

$$= \frac{\rho V_2^2\times(3+0.3)}{8}$$

$$= 0.4125\ \rho\ V_2^2$$

or

$$V_2^2 = \frac{\sigma_{h_{max}}}{0.4125\rho}$$

$$= 2.42\times\frac{\sigma_{h_{max}}}{\rho} \qquad \text{...(2)}$$

For a thin ring

The maximum hoop stress is

$$\sigma_{h_{max}} = \rho\omega^2 R^2 \qquad \text{(using equation (17.5))}$$

$$= \rho V_3^2 \qquad (\text{as } V_3 = \omega R)$$

or

$$V_3^2 = \frac{\sigma_{h_{max}}}{\rho} \qquad ...(3)$$

Hence,

$$V_1^2 : V_2^2 : V_3^2 = 2.33 : 2.42 : 1$$

which gives

$$V_1 : V_2 : V_3 = 1.526 : 1.556 : 1$$ **Ans.**

Example 17.19

A long solid cylinder of diameter 500 mm is rotating at 3500 rpm. Taking Poisson's ratio as 0.3 and the density of the cylinder material to be 7500 kg/m^3, find (*a*) the maximum stress developed in the cylinder and (*b*) plot the distribution of the hoop and radial stresses along the radius of the cylinder.

Solution: Given,

Radius of the cylinder, $R = \frac{500}{2}$ mm $= 250 \times 10^{-3}$ m

Rotational speed, $N = 3500$ rpm

Poisson's ratio, $v = 0.3$

Density of the cylinder material, $\rho = 7500$ kg/m^3

The angular speed of the cylinder is obtained as

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 3500}{60} = 366.52 \text{ rad/s}$$

The maximum hoop and radial stresses are equal and are given by equation (17.64) as

$$\sigma_{h_{max}} = \sigma_{r_{max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)$$

$$= \frac{7500 \times (366.52)^2 \times (250 \times 10^{-3})^2}{8} \times \left(\frac{3 - 2 \times 0.3}{1 - 0.3}\right)$$

$$= 26.98 \times 10^6 \text{ Pa} = 26.98 \text{ MPa}$$ **Ans.**

Distribution of the hoop stress

The hoop stress is given by equation (17.61) as

$$\sigma_h = \frac{\rho\omega^2 R^2}{8}\left(\frac{3-2v}{1-v}\right) - \frac{\rho\omega^2 r^2}{8}\left(\frac{1+2v}{1-v}\right)$$

Now we select different values of the radius r and determine the corresponding hoop stresses using the above equation. The distribution of the hoop stresses are shown in Table 17.5.

Table 17.5 Distribution of the hoop stress

r (mm)	0	50	100	150	200	250
σ_h (MPa)	26.98	26.26	24.10	20.50	15.47	8.99

Distribution of the radial stress

The radial stress is given by equation (17.60) as

$$\sigma_r = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R^2 - r^2)$$

The values of the radial stresses corresponding to the selected values of the radius r are determined using the above equation, which are shown in Table 17.6.

Table 17.6 Distribution of the radial stress

r (mm)	0	50	100	150	200	250
σ_r (MPa)	26.98	25.90	22.67	17.27	9.71	0

Plotting of the hoop and radial stresses

The values of the radius are plotted on x-axis and the values of the hoop and radial stresses on y-axis. The resulting curves for the two stresses are shown in Fig. 17.8.

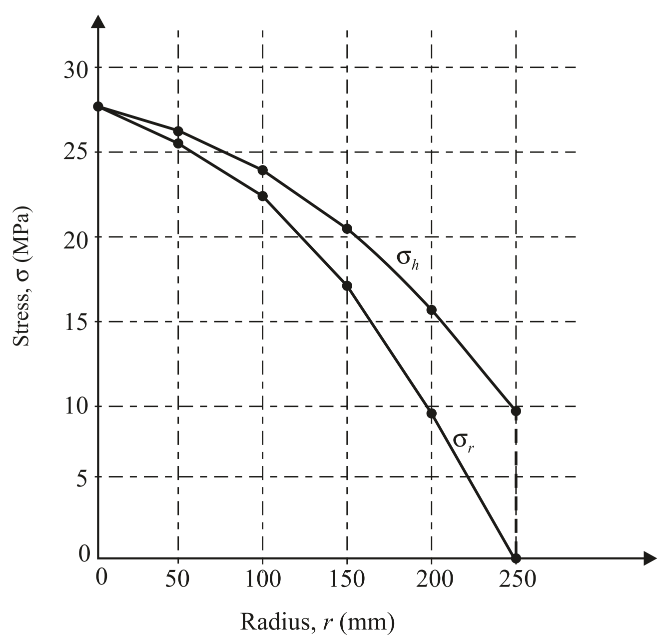


Fig. 17.8

Example 17.20

A long hollow cast iron cylinder of inside diameter 50 mm and outside diameter 300 mm is rotating at 6000 rpm. Taking Poisson's ratio as 0.3 and density of the cylinder material to be 7200 kg/m^3, find (*a*) the maximum hoop stress (*b*) the radius at which the radial stress is maximum (*c*) the maximum radial stress and (*d*) plot the distribution of the hoop and radial stresses along the radius of the cylinder.

Solution: Given,

Inside radius of the hollow cylinder, $R_1 = \dfrac{50}{2}$ mm

$= 25 \times 10^{-3}$ m

Outside radius of the hollow cylinder, $R_2 = \dfrac{300}{2}$ mm

$= 150 \times 10^{-3}$ m

Poisson's ratio, $v = 0.3$

Density of the cylinder material, $\rho = 7200$ kg/m^3

Rotational speed, $N = 3000$

The angular speed of the cylinder is obtained as

$$\omega = \frac{2\pi N}{60}$$

$$= \frac{2\pi \times 6000}{60}$$

$$= 628.32 \text{ rad/s}$$

(*a*) The maximum hoop stress is obtained using equation (17.69) as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_1^2 + 2R_2^2) - \frac{\rho\omega^2 R_1^2}{8}\left(\frac{1+2v}{1-v}\right)$$

$$= \frac{7200\times(628.32)^2}{8}\times\left(\frac{3-2\times0.3}{1-0.3}\right)\times[(25\times10^{-3})^2 + 2\times(150\times10^{-3})^2]$$

$$-\frac{7200\times(628.32)^2}{8}\times\left(25\times10^{-3}\right)^2\times\left(\frac{1+2\times0.3}{1-0.3}\right)$$

$= 55580232 - 507582.03$

$= 55072650$ Pa

$= 55.07$ MPa **Ans.**

(*b*) The radius at which the radial stress is maximum, is given by equation (17.70) as

$$r = \sqrt{R_1 \times R_2}$$

$$= \sqrt{25\times10^{-3}\times150\times10^{-3}}$$

$$= 0.0612 \text{ m}$$

$$= 61.2 \text{ mm}$$ **Ans.**

(*c*) The maximum radial stress is given by equation (17.71) as

$$\sigma_{r_{max}} = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)(R_2 - R_1)^2$$

$$= \frac{7200\times(628.32)^2}{8}\times\left(\frac{3-2\times0.3}{1-0.3}\right)\times(150\times10^{-3}-25\times10^{-3})^2$$

$$= 19034326 \text{ Pa}$$

$$= 19.03 \text{ MPa}$$ **Ans.**

(*d*) **Distribution of the hoop stress**

The hoop stress is given by equation (17.68) as

$$\sigma_h = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[R_1^2 + R_2^2 + \frac{R_1^2 R_2^2}{r^2}\right] - \frac{\rho\omega^2 r^2}{8}\left(\frac{1+2v}{1-v}\right)$$

Now we select different values of the radius r and determine the corresponding hoop stresses using the above equation. The distribution of the hoop stresses is shown in Table 17.7.

Table 17.7 Distribution of the hoop stress

r (mm)	25	50	75	100	125	150
σ_h (MPa)	55.07	33.00	26.65	21.76	16.58	10.66

Distribution of the radial stress

The radial stress is given by equation (17.67) as

$$\sigma_r = \frac{\rho\omega^2}{8}\left(\frac{3-2v}{1-v}\right)\left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2\right]$$

The values of the radial stresses corresponding to the selected values of the radius r are determined using the above equation, which are shown in Table 17.8.

Table 17.8 Distribution of the radial stress

r (mm)	25	50	75	100	125	150
σ_r (MPa)	0	18.27	18.27	14.28	8.04	0

Plotting of the hoop and radial stresses

The values of the radius are plotted on x-axis and the values of the hoop and radial stresses on y-axis. The resulting curves for the two stresses are shown in Fig. 17.9.

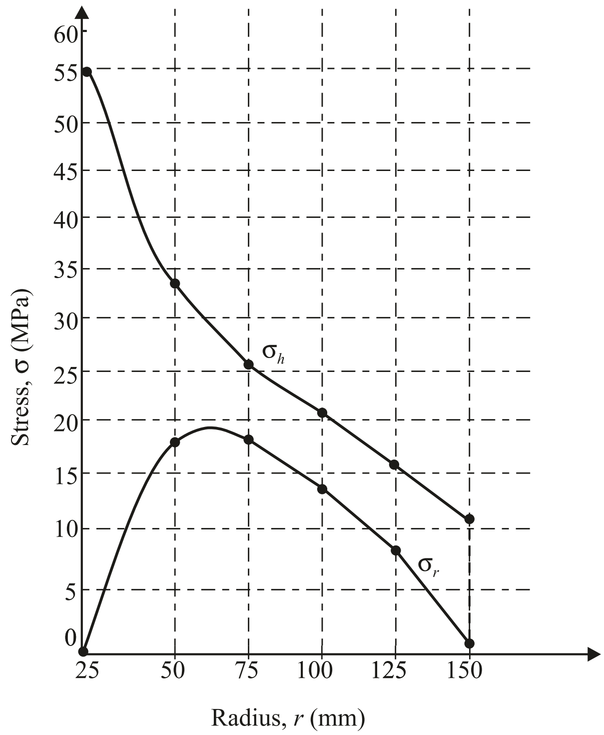


Fig. 17.9

SHORT ANSWER QUESTIONS

1. What are the forces that act on a rotating ring?
2. How do hoop stress produce?
3. Name the stresses that act on a rotating thin disc.
4. What is meant by a disc of uniform strength?
5. How does a rotating disc of uniform thickness differ from a rotating disc of uniform strength?
6. What are the stresses that act on a rotating long cylinder?

MULTIPLE CHOICE QUESTIONS

1. The rotational speed N (rpm) and the angular velocity ω (rad/s) are related as

(a) $N = \frac{2\pi\omega}{60}$ (b) $\omega = \frac{2\pi N}{60}$ (c) $\omega = \frac{\pi N}{60}$ (d) $\omega = \frac{3\pi N}{60}$.

2. The expression for the hoop stress for a thin rotating ring is given as (ρ = Density, ω = Angular speed and r = Radius)

(a) $\rho\omega r^2$ (b) $\rho\omega^2 r$ (c) $\rho\omega^2 r^2$ (d) $\rho^2\omega^2 r^2$.

3. The expression for the hoop stress for a solid rotating disc at any radial distance r is given as (ρ = Density, ω = Angular speed, v = Poisson's ratio, R = Radius).

(a) $\frac{\rho\omega(3+v)}{8}(R^2 - r^2)$ (b) $\frac{\rho\omega^2(1+v)}{8}(R^2 - r^2)$

(c) $\frac{\rho\omega^2}{8}[(3+v)R^2 - (1+3v)r^2]$ (d) $\frac{\rho\omega^2}{8}[(3+v)R^2 - (1+3v)r^2]$.

4. The expression for the radial stress for a solid rotating disc at any radial distance r is given as (ρ = Density, ω = Angular speed, v = Poisson's Ratio, R = Radius).

(a) $\frac{\rho\omega(3+v)}{8}(R^2 - r^2)$ (b) $\frac{\rho\omega^2(1+v)}{8}(R^2 - r^2)$

(c) $\frac{\rho\omega^2}{8}[(3+v)R^2 - (1+3v)r^2]$ (d) $\frac{\rho\omega^2}{8}[(3+v)R^2 - (1+3v)r^2]$.

5. The hoop and radial stresses at the centre of a solid rotating disc are expressed as (ρ = Density, ω = Angular speed, R = Radius and v = Poisson's ratio)

(a) $\frac{\rho\omega^2R^2(3+v)}{8}, \frac{\rho\omega^2R^2(1+v)}{8}$ (b) $\frac{\rho\omega^2R^2(3+v)}{8}, \frac{\rho\omega^2R^2(3+v)}{8}$

(c) $\frac{\rho\omega^2R^2(1-v)}{4}, \frac{\rho\omega^2R^2(3+v)}{8}$ (d) $\frac{\rho\omega^2R^2(1-v)}{4}, \frac{\rho\omega^2R^2(1+v)}{8}$.

6. The hoop stress at the outer radius of a solid rotating disc is (ρ = Density, ω = Angular speed, r = Radius and v = Poisson's ratio)

(a) $\frac{\rho\omega^2R^2(3+v)}{8}$ (b) $\frac{\rho\omega^2R^2(1+v)}{4}$ (c) $\frac{\rho\omega^2R^2(1-v)}{4}$ (d) $\frac{\rho\omega^2R^2(1+v)}{8}$.

7. The maximum radial stress in case of a hollow disc occurs at a radial distance equal to (R_1 = Inner radius and R_2 = Outer radius)

(a) $\sqrt{R_1}$ (b) $\sqrt{R_2}$ (c) $\sqrt{R_1R_2}$ (d) $\sqrt{2R_1R_2}$.

8. The maximum value of the radial stress for a hollow disc is (ρ = density, ω = Angular speed, v = Poisson's ratio, R_2 = Outer radius and R_1 = Inner radius)

(*a*) $\dfrac{(1+v)\rho\omega^2}{4}(R_2 - R_1)^2$ (*b*) $\dfrac{(3+v)\rho\omega^2}{8}(R_2^2 - R_1^2)^2$

(*c*) $\dfrac{(1+v)\rho\omega^2}{8}(R_2 - R_1)^2$ (*d*) $\dfrac{(3+v)\rho\omega^2}{8}(R_2^2 - R_1^2)^2$.

9. The expressions for the hoop and radial stresses in a rotating disc with a central pin hole are (ρ = Density, v = Poisson's ratio, ω = Angular speed and R = Radius)

(*a*) $\dfrac{(3+v)}{8}\rho\omega^2R^2, \dfrac{(3+v)\rho\omega^2R^2}{4}$ (*b*) $\dfrac{(1+v)\rho\omega^2R^2}{4}, \dfrac{(3+v)\rho\omega^2R^2}{8}$

(*c*) $\dfrac{(2+v)\rho\omega^2R^2}{4}, \dfrac{(3-v)\rho\omega^2R^2}{8}$ (*d*) $\dfrac{(3+v)\rho\omega^2R^2}{4}, \dfrac{(3+v)\rho\omega^2R^2}{8}$.

10. Consider the following statements :

1. The radial stress is zero at both inner and outer radius of a hollow rotating disc.
2. Both radial and hoop stresses at the centre of a solid rotating disc are maximum and equal.
3. The radial stress at the outer radius of a solid rotating disc is zero.
4. The maximum hoop stress for a rotating disc with a central pin hole is twice the maximum hoop stress for a rotating solid disc.

Of these statements:

(*a*) 1 and 2 are true (*b*) 1, 2 and 4 are true

(*c*) 2 and 3 are true (*d*) 1, 2, 3 and 4 are true.

11. Consider the following statements about a disc of uniform strength :

1. The hoop and radial stresses do not vary along the radius of the disc.
2. It has maximum thickness at the centre.
3. It has uniform thickness throughout.
4. Its thickness decreases gradually towards its outer edge.

Of these statements :

(*a*) 1 alone is true (*b*) 1, 2 and 4 are true

(*c*) 1 and 2 are true (*d*) 1 and 3 are true.

12. Consider the following statements about a rotating long cylinder :

1. It involves three stresses, namely hoop, radial and axial.
2. The longitudinal strain is constant.
3. All the stresses are principal stresses.
4. The radial stress is zero at the surface of the cylinder

Of these statements :

(*a*) 1 and 2 are true (*b*) 1, 3 and 4 are true

(*c*) 2 and 4 are true (*d*) 1, 2, 3 and 4 are true.

13. The maximum radial stress in case of a solid long rotating cylinder is (ρ = Density, ω = Angular speed, R = Radius, v = Poisson's ratio)

(*a*) $\dfrac{\rho\omega^2 R^2}{4}\left(\dfrac{3-v}{1-v}\right)$ (*b*) $\dfrac{\rho\omega^2 R^2}{4}\left(\dfrac{1-v}{3-2v}\right)$

(*c*) $\dfrac{\rho\omega^2 R^2}{8}\left(\dfrac{3-2v}{1-v}\right)$ (*d*) $\dfrac{\rho\omega^2 R^2}{8}\left(\dfrac{1-v}{3-2v}\right)$.

14. The maximum hoop stress in case of a solid long rotating cylinder is (ρ = Density, ω = Angular speed, R = Radius, v = Poisson's ratio)

(*a*) $\dfrac{\rho\omega^2 R^2}{8}\left(\dfrac{1-v}{3-2v}\right)$ (*b*) $\dfrac{\rho\omega^2 R^2}{8}\left(\dfrac{3-2v}{1-v}\right)$

(*c*) $\dfrac{\rho\omega^2 R^2}{4}\left(\dfrac{3-2v}{1-v}\right)$ (*d*) $\dfrac{\rho\omega^2 R^2}{4}\left(\dfrac{1-v}{3-2v}\right)$.

15. The maximum radial stress in case of a hollow long rotating cylinder is

(*a*) $\dfrac{\rho\omega^2}{4}\left(\dfrac{3-2v}{1-v}\right)(R_2-R_1)^2$ (*b*) $\dfrac{\rho\omega^2}{8}\left(\dfrac{1-v}{3-2v}\right)(R_2-R_1)^2$

(*c*) $\dfrac{\rho\omega^2}{8}\left(\dfrac{3-2v}{1-v}\right)(R_2-R_1)^2$ (*d*) $\dfrac{\rho\omega^2}{8}\left(\dfrac{3-2v}{1-v}\right)(R_2^2-R_1^2)$.

ANSWERS

1. (*b*) **2.** (*c*) **3.** (*c*) **4.** (*a*) **5.** (*b*) **6.** (*c*) **7.** (*c*) **8.** (*b*)

9. (*d*) **10.** (*d*) **11.** (*b*) **12.** (*d*) **13.** (*c*) **14.** (*b*) **15.** (*c*)

EXERCISES

1. A uniform thin disc of diameter 600 mm has a central hole of diameter 100 mm. Determine the maximum hoop stress induced in the disc, if the maximum radial stress is not to exceed 15 MPa. Take Poisson's ratio as 0.25. (*Ans.* 43.48 MPa).

2. A uniform thin disc of diameter 700 mm has a pin hole at the centre. Determine the maximum hoop stress induced in the disc, if it rotates at 3000 rpm. Take Poisson's ratio as 0.3 and the density of the disc material to be 7800 kg/m^3. (*Ans.* 77.8 MPa).

3. A hollow steel disc of uniform thickness has outer diameter 500 mm and inner diameter 200 mm and it rotates at 3000 rpm. Taking Poisson's ratio as 0.3 and the density of the disc material to be 7800 kg/m^3, find (*a*) the maximum hoop and radial stresses induced in the disc and (*b*) the radius at which the radial stress is maximum.

(*Ans.* (*a*) 41.04 MPa, 7.14 MPa (*b*) 158.11 mm).

4. A thin uniform steel disc of diameter 300 mm rotates at 4000 rpm. Calculate the maximum hoop stress induced in the disc and plot the distribution of the hoop and radial stresses along the radius of the disc. Take Poisson's ratio as 0.25 and the density of the disc material equals to 7800 kg/m^3. (*Ans.* 12.5 MPa).

5. Derive the expression for the hoop stress at the outer radius of a solid disc of radius R, which rotates at ω rad/s and is made of material having a density ρ and Poisson's ratio v. Hence, prove that the hoop stress reduces to zero, if the Poisson's ratio tends to unity.

6. A long thick cylinder of inner diameter 150 mm and outer diameter 450 mm rotates at 4000 rpm. Find the hoop stresses at its inner and outer surfaces. Take the Poisson's ratio of 0.3 and the density of the cylinder material as 7470 kg/m^3.

(*Ans.* 57.9 MPa, 11.9 MPa).

7. A steam turbine rotor is 150 mm diameter below the blade ring and 5 mm thick, and runs at 35,000 rpm. What is the thickness of the rotor at a radius of 50 mm and at the centre ? Take the allowable stress of 150 MPa and the density of the rotor material to be 7800 kg/m^3. Assume uniform strength condition.

(*Ans.* 14.9 mm, 35.7 mm).

8. A long solid steel cylinder of diameter 400 mm is rotating at 5000 rpm. Taking Poisson's ratio as 0.3 and the density of the material of the cylinder to be 7800 kg/m^3, find (*a*) the maximum stress developed in the cylinder and (*b*) plot the distribution of the hoop and radial stresses along the radius of the cylinder.

(*Ans.* 36.66 MPa).

9. A long hollow cast iron cylinder of inside diameter 60 mm and outside diameter 300 mm is rotating at 3600 rpm. Taking Poisson's ratio as 0.3 and the density of the material of the cylinder to be 7200 kg/m^3, find the following parameters :

(*a*) the maximum hoop stress

(*b*) the radius at which the radial stress is maximum, and

(*c*) the maximum radial stress.

(*Ans.* (*a*) 19.87 MPa (*b*) 67.08 mm (*c*) 6.31 MPa).

10. Prove that the maximum hoop stress at the centre of a long rotating solid cylinder is given as

$$\sigma_{h_{\max}} = \frac{\rho\omega^2 R^2}{8}\left(\frac{3-2v}{1-v}\right)$$

where ρ = Density of the cylinder material

ω = Angular speed of the cylinder

R = Mean radius of the cylinder

v = Poisson's ratio.

❑❑❑

18 Mechanical Testing of Materials

Johan August Brinell
(1849-1925)

Johan August Brinell, born on 21 November 1849, was a Swedish mechanical engineer. He invented a method, called Brinell hardness test in 1900, which is the oldest method of hardness testing commonly used today. The hardness of the material is indicated by a number called Brinell hardness number (BHN). He is also known for describing the failure mechanism of material surfaces, popularly called Brinelling. He became a member of the Royal Swedish Academy of Sciences in 1902 and a member of the Royal Swedish Academy of Engineering Sciences in 1919.

LEARNING OBJECTIVES

After reading this chapter, you will be able to answer some of the following questions:

- How does Brinell hardness differ from Rockwell hardness?
- Why is a fatigue test conducted?
- What are the different stages of creep?
- How is a compression test different from a tension test?
- Why is a torsion test conducted?

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5_18

18.1 INTRODUCTION

The mechanical properties of materials have intensive applications in manufacturing processes and to the service life of components. They help us in estimating forces required in forming processes and in predicting the behaviour of materials in shaping processes. Mechanical properties depend on several factors, such as temperature, rate of deformation, surface condition, environment and type of material. Important mechanical properties include tension, compression, hardness, torsion, bending, fatigue, creep, impact etc. Numerous tests have been developed, by which these properties can be measured. These tests are discussed below under different headings.

18.2 HARDNESS TEST

Three types of hardness tests are important, namely Brinell test, Rockwell test and Vickers test. These tests are based on finding a number, more commonly known as a hardness number. Higher number is usually indicative of increased hardness of the material.

18.2.1 Brinell Test

The Brinell test was introduced by J.A. Brinell in 1900. The test specimen, usually in the form of a circular disc, is pressed with a load of 500 kg, 1500 kg and 3000 kg against a hardened steel or tungsten carbide ball of 10 mm diameter, producing an impression on the specimen. The usual time of pressing the specimen is 15 seconds. Subsequently the load is released and the diameter of impression, usually a hemispherical cavity, is measured with the help of a travelling microscope. A number, called Brinell Hardness Number (BHN), is used to measure the hardness of the material, defined by

$$\text{BHN} = \frac{\text{Load applied in kgf}}{\text{Area of the spherical indentation in mm}^2}$$

$$= \frac{P}{\frac{\pi D}{2}\left[D - \sqrt{D^2 - d^2}\right]} \qquad \text{...(18.1)}$$

where D = Diameter of the steel ball

d = Diameter of indentation

The standard unit of BHN is kgf/mm^2. It should be ensured that the specified load is applied on the test specimen slowly and gradually, and not beyond the defined limit. When the load is removed from the specimen, the deformation *i.e.*, indentation has elastic recovery tendency to a little extent. When the plastic deformation has already occurred in the specimen, then its diameter is measured for the calculation of BHN. The steel ball gives satisfactory measurement, when BHN is less than 500. For higher than this value, steel ball suffers distortion. For thin sheets, the results are reliable if its thickness is at least ten times the depth of impression.

18.2.2 Rockwell Test

The Rockwell test was developed by S.P. Rockwell in1922. It is based on measuring depth of penetration instead of diameter of indentation as in case of Brinell test. The indentor is pressed against the surface of test specimen, first with a minor load and then with a major load. The difference in the depths of penetration is a measure of hardness of the material. The test uses three scales for measuring the hardness of the materials. Rockwell *A* is used for case hardened materials; Rockwell *B* for soft materials, such as mild steel, brass and aluminum; and Rockwell *C* for hard materials, such as high carbon steel, high speed steel and tool steels. *A* diamond indentor, having 120º included angle, is used in case of Rockwell *A* and *C*; and a hardened steel ball indentor of 0.0625 inch diameter for Rockwell *B*. The load to be applied during testing of materials also varies accordingly to the grade of the Rockwell. For example, Rockwell *A* uses a 60 kg load, Rockwell *B*, a 100 kg load and Rockwell *C*, a 150 kg load.

18.2.3 Vickers Test

The Vickers test was developed in 1922, and is also known as diamond pyramid hardness test. It uses a 136º pyramid-shaped diamond indentor on a square base. On pressing the indentor against the surface of test specimen, it produces a square-shaped indentation. The test uses a load varying between 1 kg and 120 kg. The hardness, usually expressed in terms of a number, called Vickers Pyramid Number (VPN) is expressed as

$$\text{VPN} = \frac{\text{Load appied in kgf}}{\text{Surface area of the pyramidal indentation in mm}^2}$$

$$= \frac{1.854\,P}{D^2} \qquad \text{...(18.2)}$$

where P = Load on the test specimen

D = Diagonal of the indent

18.3 FATIGUE

The behaviour of a manufactured part during its expected service life is an important consideration. The wings of an aircraft, the crankshaft of an automobile engine, and gear teeth in machinery are all subjected to static, as well as fluctuating (cyclic or periodic) loads. Their excessive values can lead to the formation of cracks, which may result in failure of components and the property responsible for it is called fatigue. Fatigue failure has brittle nature and probably account for nearly 90% of all mechanical fractures. A fatigue test is conducted by subjecting the test specimen under various states of stress, usually in alternate tension and compression mode or torsion. During testing, stress amplitudes (*S*), and the number of cycles (*N*) required to produce the failure is recorded. The stress amplitude represents maximum stress (tensile and compressive both) to which the specimen is subjected. A graph is plotted between *S* and *N* and the resulting figure is called *S-N* curve. These curves are based on complete reversal of the stress *i.e.*, maximum tension, maximum compression, maximum tension and so on as in case of bending a wire alternatively in two directions, just opposite to each other. The test is also performed on a rotating shaft, with a constant downward load. The endurance limit, also known by other names, such as endurance strength or fatigue limit, is the maximum stress before fatigue failure, regardless of the number of stress application cycles. Endurance limit depends on the ultimate tensile strength of the metals. For example, the endurance limit of steel is about one-half of its tensile strength.

18.4 CREEP

A turbine disk and the blades in the jet engine of an aircraft and high-temperature pressure vessels are subjected to high stresses and temperature. These components undergo creep during their use over a period of time. In creep, components elongate permanently under applied stresses, which ultimately lead to components' failure. It is a very slow process but of great importance from design point of view. Nuclear fuel components and tools and dies, when subjected to hot-working operations, such as forging and extrusion also undergo creep.

It has been observed in old houses that a window glass has more thickness at its bottom than at its top part. It is because of the reason that the glass has undergone creep by its own weight over many years.

Creep is the characteristics of metals, and certain nonmetallic materials, such as thermoplastics and rubber also show this property. It can occur at room temperature or at any temperature. For example, lead shows this property at room temperature, when subjected to a constant tension load.

A creep test is conducted on a test specimen, usually a lead wire of length 1m and diameter 2-3 mm, by subjecting it to a constant tension load at room temperature, and measuring the change in length, over a regular interval of time. Initially, the extension in the wire is measured at regular intervals of 15 seconds and after sometime, say 10 or 15 minutes, at interval of 1 minute. Initially, the strain rate is high and it gradually decreases to a constant value. The specimen finally breaks by necking with a conical shaped fracture.

A creep curve (Fig. 18.1), which is usually a plot between strain and time, has three distinct stages, namely primary, secondary and tertiary. The creep rate increases with temperature and the applied load.

The primary and tertiary stages are shot-lived, with higher creep rates; and secondary stage is rather longer, where creep rate is somewhat linear.

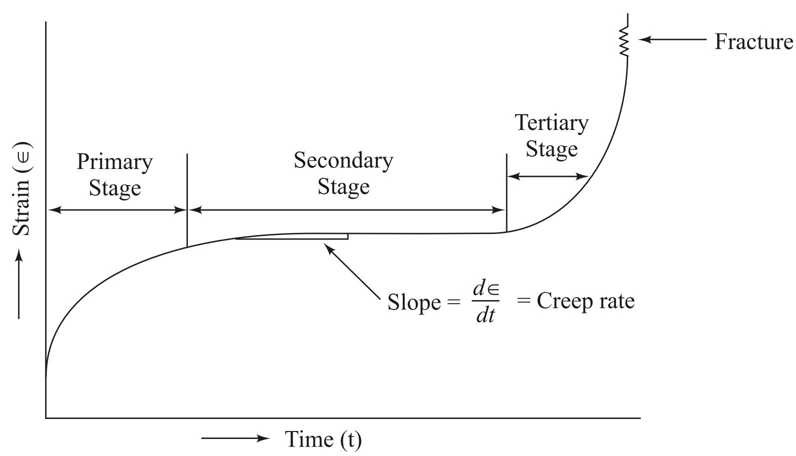


Fig. 18.1 A Creep curve.

18.5 TENSION TEST

The tension test is the most commonly used test to know various mechanical properties such as strength, ductility and toughness of the material. It is a static test, conducted on a 10 or 12 mm test specimen with gage length 50 or 60 mm, where the test specimen is subjected to uniaxial tension. The changes are reported over the specified gage length.

The analysis of the test is carried out by knowing two parameters: one called strain, defined as the ratio of elongation and original gage length; and another, called stress, defined as the ratio of load and original cross-sectional area of the test specimen.

A universal testing machine (UTM) is usually used to conduct the tension test.

18.6 COMPRESSION TEST

There are many manufacturing operations such as forging, rolling and extrusion in which material is subjected to compression.

The compression test is conducted by applying a compressive load on a solid cylindrical specimen by keeping it between two flat dies. As the load is increased, there is decrease in the height of specimen and its starts bulging out with maximum cross-sectional area in the middle. At the maximum load, the specimen fails, as it loses its resistance. Again, stress-strain curve is plotted, as is done in case of tension test, but compression test is more difficult to conduct. This is because of the fact that, some energy is lost in overcoming friction between the specimen and the dies at its two ends, and it necessitates the use of increased load during the test.

The true stress-true strain curves are similar for tension and compression tests conducted on a ductile material, but the results vary for brittle materials. Brittle materials are more stronger in compression than in tension as opposite to ductile materials, which are more stronger in tension than in compression.

18.7 STIFFNESS TEST

Stiffness of a material is indicated by its modulus of elasticity, E. The higher value of modulus of elasticity is indicative of higher stiffness of the material. The modulus of elasticity is the ratio of stress to strain in the elastic region, given as

$$E = \frac{\text{Stress}}{\text{Strain}} = \frac{\sigma}{\in}$$

As strain has no unit, units of E and σ are the same. Diamond has higher stiffness compared to carbides, tungsten, titanium or steel. Generally, a harder material has higher stiffness.

18.8 TORSION TEST

Torsion is associated with shear strength, which in turn, is connected to the modulus of rigidity (shear modulus, G) of the material. Higher modulus of rigidity means increased shear strength. The torsion test is used to measure the shear modulus of material and also shear strain encountered with various applications, such as during punching of holes in sheet metals and metal cutting.

The test may be conducted on a thin tabular specimen or a solid shaft of certain length by subjecting them to a twisting moment *i.e.* torque.

The test is based on finding the ratio of torque (T) and angular twist (θ) in radian, from the curve, drawn between the two, and using polar moment of inertia (J) and length (l) of the test specimen.

From torsion formula, we have

$$\frac{T}{J} = \frac{G\theta}{l}$$

or
$$G = \left(\frac{T}{\theta}\right)\cdot\left(\frac{l}{J}\right) \quad \text{...(18.3)}$$

where
$$J = \frac{\pi}{32}d^4 \quad (d \text{ is diameter of the solid shaft})$$

The shear stress τ is expressed as

$$\tau = \frac{\text{Shear Force }(F_s)}{\text{Shear Area}(A_s)} = \frac{T}{2\pi r^2 t} \quad \text{...(18.4)}$$

where r = Average radius of the tube

t = Thickness of the tube

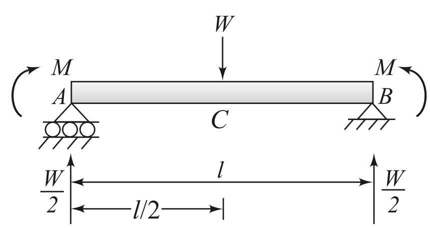
Again from torsion formula, we have

Shear strain,
$$\phi = \frac{\theta r}{l} \quad \text{...(18.5)}$$

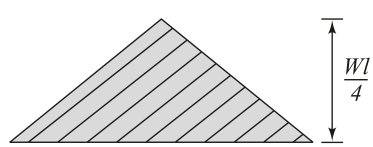
Dividing equation (18.4) by equation (18.5), we get the value of shear modulus of the shaft material.

18.9 BEND TEST

Bending finds extensive use in shaping materials to make them useful for many applications. The bend test is conducted on a brittle test specimen of rectangular cross-section, supported at its ends and vertically loaded with one or two point loads.



(*a*) Loaded beam in three-point bending

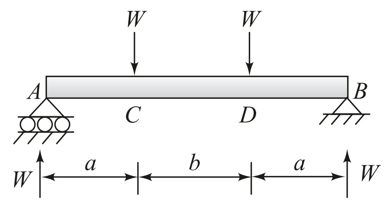


(*b*) Bending moment diagram

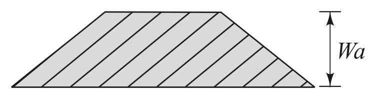
Fig.18.2 A three-point bend test.

(Maximum bending moment, $\frac{Wl}{4}$ occurs at centre of the beam)

In case, it uses one point load, the test is called three-point bend test (Fig. 18.2) and on using two point loads, the test is called four-point bend test (Fig. 18.3). During bending, the upper surface of the specimen (a beam) is subjected to compression and its lower surface to tension.



(*a*) Loaded beam in four-point bending



(*b*) Bending moment diagram

Fig.18.3 A four-point bend test.

(Maximum bending moment, *Wa* is constant for '*b*' part of the beam)

18.10 IMPACT TEST

An impact load is the example of dynamic loading, where a load is suddenly applied, as in case of drop forging. Its effect is much greater than a steady load of same magnitude. The impact test is used to measure the toughness of material, a property, which indicates its capacity to store strain energy before it finally breaks; and is characterized by high strength and high ductility. A ductile material with voids and discontinuity in its structure, behaves as a brittle material, and this is not indicated during its tension test. A brittle material requires less energy to break and hence lacks toughness than pure ductile material. The impact test is particularly useful in finding the ductile-brittle transition characteristics of materials. There are two commonly used impact tests, namely charpy and izod.

In *Charpy test*, a notched test specimen of size 55 mm × 10 mm × 10 mm is supported at its ends and a swinging pendulum is used to provide the impact load (Fig. 18.4). Here, the potential energy of the pendulum is converted into kinetic energy during the release of the load. The load thus applied at the notch portion breaks the specimen, and the energy required is absorbed by it.

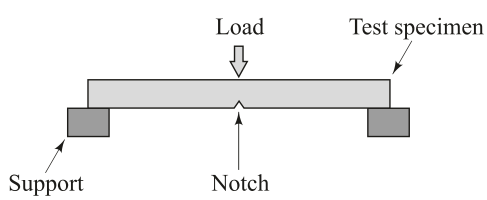


Fig.18.4 A Charpy test setup.

In *Izod test*, a test specimen of size 75 mm × 10 mm × 10 mm is supported as a cantilever beam (Fig. 18.5). The swinging pendulum is allowed to strike at the notched portion of the specimen. The energy needed to break the specimen is obtained, which measures its impact toughness.

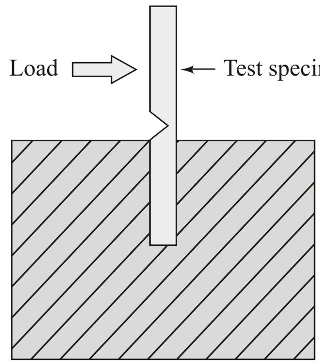


Fig. 18.5 An Izod test setup.

SHORT ANSWER QUESTIONS

1. What is meant by mechanical property of a material?
2. Name a few important mechanical properties of a material.
3. How is the hardness of a material measured in Brinell test?
4. How does Brinell test differ from Rockwell test?
5. How is the hardness of a material measured in Rockwell test?
6. How does creep differ from fatigue?
7. Why is tension test conducted?
8. How does Charpy test differ from Izod test?

MULTIPLE CHOICE QUESTIONS

1. Consider the following parameters:

1. Hardness 2. Toughness 3. Density 4. Specific gravity

Which of the above parameters can be classified as mechanical property?

(*a*) 1 and 2 (*b*) 1 and 3 (*c*) 2 and 3 (*d*) 3 and 4.

2. Consider the following statements:

1. Brinell test uses a 10 mm diameter steel ball indentor.
2. Brinell test measures depth of indentation on the test specimen.
3. Rockwell *A* uses a test load of 100 kg.
4. The standard unit of BHN is N/m^2.

Of these statements:

(*a*) 1 alone is true (*b*) 1 and 2 are true

(*c*) 1, 2 and 3 are true (*d*) 1, 2 and 4 are true.

3. Consider the following statements:

1. Brinell test uses spherical impression on the test specimen.
2. Vickers test uses a 136° pyramid-shaped diamond indentor.
3. Vickers test produces a square-shaped indentation.
4. Rockwell *C* uses a test load of 100 kg.

Of these statements:

(*a*) 1 and 2 are true (*b*) 2 and 3 are true

(*c*) 1, 2 and 3 are true (*d*) 1, 2 and 4 are true.

4. Which of the following mechanical properties is connected to time?

(*a*) Hardness (*b*) Fatigue (*c*) Creep (*d*) Tension.

5. Consider the following statements:

1. Creep is a time-dependent phenomenon.
2. Endurance limit is dependent on tensile strength of the material.
3. Impact test measures toughness.
4. Resistance against scratching is called hardness.

Of these statements:

(*a*) 1 and 3 are true (*b*) 1, 3 and 4 are true

(*c*) 2, 3 and 4 are true (*d*) 1, 2, 3 and 4 are true.

6. Endurance limit of steel is

(*a*) equal to its tensile strength (*b*) equal to one-half of its tensile strength

(*c*) equal to two-third of its tensile strength (*d*) independent of its tensile strength.

7. Consider the following statements about creep:

1. It increases with increase in temperature.
2. It decreases with increase in temperature.
3. It is independent of temperature.
4. It increases with applied load.

Of these statements:

(*a*) 1 alone is true
(*b*) 1 and 4 are true
(*c*) 3 and 4 are true
(*d*) 2 and 4 are true.

8. Consider the following statements:

1. In three-point bend test, two point loads are used.
2. In four-point bend test, the bending moment diagram is a trapezium.
3. Izod test uses a vertically placed test specimen.
4. Impact test measures hardness of a material.

Of these statements:

(*a*) 1 and 2 are true
(*b*) 2 and 3 are true
(*c*) 3 and 4 are true
(*d*) 2, 3 and 4 are true.

9. Consider the following statements:

1. For a test specimen loaded in the three-point bend test, the bending moment diagram is a triangle.
2. In bend test, the upper surface of the specimen is subjected to compression and the lower surface to tension.
3. Charpy test uses a horizontally placed test specimen.
4. A compression test is more difficult to conduct than a tension test.

Of these statements:

(*a*) 1 and 2 are true
(*b*) 1 and 3 are true
(*c*) 1, 2 and 4 are true
(*d*) 1, 2, 3 and 4 are true.

ANSWERS

1. (*a*) **2.** (*a*) **3.** (*c*) **4.** (*c*) **5.** (*d*) **6.** (*b*) **7.** (*b*) **8.** (*b*) **9.** (*d*).

❑❑❑

Model Multiple Choice Questions for Competitive Examinations

1. Consider the following statements:
   1. *Bending moment* producing downward concavity is termed as negative bending moment.
   2. Positive bending moment is also called *sagging*.
   3. Simply supported, overhanging and cantilever beams are categorized as statically *determinate* beams.
   4. A *simple beam* has two hinged supports.

   Of these statements:

   (*a*) 1 and 2 are true

   (*b*) 1, 2 and 3 are true

   (*c*) 2, 3 and 4 are true

   (*d*) 1, 3 and 4 are true.
2. Consider the following statements:
   1. *Roller* and *pinned* supports are termed as simple supports.
   2. Cantilever, continuous and fixed beams are called statically *indeterminate* beams.
   3. A *cantilever* beam has two fixed supports.
   4. A *continuous* beam has more than two supports.

   Of these statements:

   (*a*) 1 and 2 are true (*b*) 1 and 3 are true

   (*c*) 1 and 4 are true (*d*) 2 and 3 are true.
3. Consider the following statements:
   1. Statically *determinate* beams involve only three unknowns.
   2. A *fixed* support involves two reaction force components and a reaction moment.
   3. As compared to a cantilever beam, propped cantilever beam has shorter length.
   4. *Triangular loading* is an example of uniformly varying load.

   Of these statements:

   (*a*) 1 and 3 are true

   (*b*) 2 and 4 are true

   (*c*) 1, 2 and 3 are true

   (*d*) 1, 2 and 4 are true.
4. Consider the following statements:
   1. The shear force diagram (SFD) consists of horizontal straight lines in case of point loads and inclined straight lines in case of uniformly distributed load (*udl*).
   2. At point of inflexion, shear force is zero.
   3. The weight of a simple beam is considered to act at one support only.
   4. A beam is subjected to axial forces in case of inclined loads acting on it.

   Of these statements:

   (*a*) 1 and 2 are true (*b*) 2 and 3 are true

   (*c*) 1 and 4 are true (*d*) 1, 2 and 4 are true.
5. Match **List I** with **List II** and select the correct answer using the codes given below the lists:

List I (Beams)	**List II (Conditions)**
A. Simple beam	1. More than two supports
B. Fixed beam	2. One roller and one pinned support
C. Continuous beam	3. Confined between two rigid supports
D. Cantilever beam	4. Statically determinate beam

D. K. Singh, *Strength of Materials*,
https://doi.org/10.1007/978-3-030-59667-5

Codes:	**A**	**B**	**C**	**D**
(*a*)	2	3	4	1
(*b*)	2	3	1	4
(*c*)	4	3	1	2
(*d*)	2	4	1	3.

6. *Uniformly distributed load* **(***udl* **)** is expressed as
   (*a*) N (*b*) N/m
   (*c*) kg/m (*d*) N.m.
7. A *roller support* allows
   (*a*) Free movement in vertical plane
   (*b*) Free movement in horizontal plane
   (*c*) Free movement in vertical and horizontal planes both
   (*d*) No free movement in any direction.
8. Consider the following statements about *simple bending theory:*
   1. The beam material is isotropic and homogeneous.
   2. The elastic modulus of beam material is more in tension than in compression.
   3. The radius of curvature is small.
   4. The loads are applied in a plane perpendicular to the plane of bending.

   Of these statements:
   (*a*) 1 and 2 are true (*b*) 1 alone is true
   (*c*) 2 and 3 are true (*d*) 2, 3 and 4 are true.
9. Consider the following statements:
   1. *Bending stress* in a beam is maximum at its extreme faces and minimum (zero) at the neutral axis.
   2. Bending stresses can be tensile or compressive.
   3. *Shear stress* in a beam is maximum at neutral axis and zero at its extreme faces.
   4. Shear stress variation for a circular section of a beam is linear in nature.

   Of these statements:
   (*a*) 1 and 4 are true
   (*b*) 2 and 4 are true
   (*c*) 1, 2 and 3 are true
   (*d*) 3 and 4 are true.
10. For a beam being subjected to transverse loads, the shear force at a certain section is zero, then *bending moment* at that section will be
    (*a*) zero (*b*) maximum
    (*c*) minimum (*d*) maximum or minimum.
11. Consider the following statements:
    1. *Section modulus* and *elastic modulus* are the same.
    2. The SI unit of section modulus is N/m.
    3. *Bending stress* is also called *flexural strength.*
    4. The SI unit of bending stress is *pascal.*

    Of these statements:
    (*a*) 1 and 2 are true
    (*b*) 2 and 3 are true
    (*c*) 1, 3 and 4 are true
    (*d*) 3 and 4 are true.
12. Match **List I** with **List II** and select the correct answer using the codes give below the lists:

List I	**List II**
A. Moment of inertia	1. Tensile stress
B. Elongation	2. Modulus of rupture
C. Neutral axis	3. Zero shear stress
D. Top fibre	4. Zero longitudinal stress

Codes:	A	B	C	D
(*a*)	2	1	3	4
(*b*)	1	2	3	4
(*c*)	3	4	1	2
(*d*)	2	1	4	3.

13. A beam is designed to support
    (*a*) horizontal loads
    (*b*) vertical loads
    (*c*) vertical loads and moments
    (*d*) horizontal loads and moments.
14. The validity of torsion formula for *pure torsion* indicates that
    (*a*) The shaft is subjected to bending stress only
    (*b*) The shaft is subjected to shear stress only
    (*c*) The shaft is subjected to combined bending and shear stresses
    (*d*) The shaft is not subjected to any stress.
15. If a circular shaft is subjected to a torque T and a bending moment M, the ratio of the maximum shear stress to the maximum bending moment is given by
    (*a*) $2M/T$ (*b*) $T/2M$
    (*c*) $2T/M$ (*d*) $M/2T$.
16. Consider the following statements:
    1. *Stress* and *pressure* have the same unit.
    2. Stress is related to internal force, whereas pressure to external force.
    3. *Normal* and *shear* stresses are the same.
    4. *Shear strain* is produced by direct force.

    Of these statements:
    (*a*) 1 alone is true (*b*) 1 and 2 are true
    (*c*) 2 and 3 are true (*d*) 3 and 4 are true.
17. Consider the following statements:
    1. *Tensile* and *compressive* stresses are normal stresses.
    2. Tension force is related to increase in length.
    3. *True stress* is based on original cross-sectional area of the test specimen.
    4. *Normal strain* is the ratio of change in length to instantaneous length.

    Of these statements:
    (*a*) 1 and 2 are true
    (*b*) 1, 2 and 3 are true
    (*c*) 1, 2 and 4 are true
    (*d*) 2, 3 and 4 are true.
18. Consider the following statements:
    1. *Percentage elongation* and *percentage reduction of area* are the measures of ductility.
    2. *Poisson's ratio* is the ratio of axial strain to transverse strain.
    3. A *prismatic* bar is a bar of varying cross-section.
    4. *Elastic modulus* and *shear modulus* are related as $E = 2G(1 + \nu)$, where ν = Poisson's ratio.

    Of these statements:
    (*a*) 1 and 2 are true (*b*) 1 and 4 are true
    (*c*) 2 and 3 are true (*d*) 2 and 4 are true.
19. Consider the following statements:
    1. *Natural strain* and *normal strain* are the same.
    2. Strain has no unit.
    3. The value of *offset strain* is usually 0.2%.
    4. Normal strain is produced by a *shear force.*

    Of these statements:
    (*a*) 1 and 2 are true
    (*b*) 1, 2 and 4 are true
    (*c*) 2 and 3 are true
    (*d*) 3 and 4 are true.
20. *Stress-strain analysis* is conducted to know which of the following properties of material?
    (*a*) physical properties
    (*b*) optical properties
    (*c*) magnetic properties
    (*d*) mechanical properties.

21. If the diameter of a *shaft* subjected to torque alone is doubled, then the horse power can be increased to
    (*a*) 2P (*b*) 4P
    (*c*) 8P (*d*) 16P.
22. A *prismatic* bar has
    (*a*) maximum ultimate strength
    (*b*) maximum yield strength
    (*c*) uniform cross-section
    (*d*) varying cross-section.
23. A material showing *similar* elastic properties in all the directions is called
    (*a*) elastic material
    (*b*) isotropic material
    (*c*) plastic material
    (*d*) viscous material.
24. A *load* applied at the centre of a carriage spring to straighten all its leaves is called
    (*a*) yield load
    (*b*) proof load
    (*c*) safe load
    (*d*) ultimate load.
25. Two *close coiled springs* of stiffness *K* and 2*K* are arranged in series in one case and in parallel in another case. The ratio of stiffness of spring connected in series to parallel is
    (*a*) 2 / 3 (*b*) 1 / 9
    (*c*) 2 / 9 (*d*) 1 / 3.
26. A higher value of *flexural rigidity* is indicative of
    (*a*) higher stiffness and lower deflection
    (*b*) lower stiffness and lower deflection
    (*c*) lower hardness and higher deflection
    (*d*) lower deflection only.
27. The specimen rod under the *tension test* has the following parameters:
    (*a*) gauge length = 50 mm, diameter = 15 mm
    (*b*) gauge length = 60 mm, diameter = 12 mm
    (*c*) gauge length = 25 mm, diameter = 25 mm
    (*d*) gauge length = 75 mm, diameter = 20 mm.
28. The *working stress* is obtained by
    (*a*) multiplying ultimate stress with the factor of safety
    (*b*) dividing ultimate stress by the factor of safety
    (*c*) multiplying yield stress with the factor of safety
    (*d*) adding ultimate stress to the factor of safety.
29. Consider the following statements regarding a beam of uniform cross-section simply supported at its ends and carrying a concentrated load at one of its third point:
    1. Its deflection under the load will be maximum.
    2. The bending moment under the load will be maximum.
    3. The deflection at the mid-point of the span will be maximum.
    4. The slope at the nearer support will be maximum.

    Of these statements:
    (*a*) 1 and 3 are true (*b*) 2 and 4 are true
    (*c*) 1 and 2 are true (*d*) 3 and 4 are true.
30. Which of the following statements is true for *linear* strain?
    1. It is a ratio of two lengths.
    2. It is a dimensionless quantity.
    3. It measures deformation produced in the material.

    Of these statements:
    (*a*) 1 and 2 (*b*) 1 and 3
    (*c*) 2 and 3 (*d*) 1, 2 and 3.

31. *Thermal strain* varies
   (*a*) inversely proportional to change in temperature
   (*b*) directly proportional to change in temperature
   (*c*) inversely proportional to the square of change in temperature
   (*d*) directly proportional to normal strain.
32. The failure criteria for *ductile* materials is based on the following factor:
   (*a*) ultimate strength
   (*b*) shear strength
   (*c*) yield strength
   (*d*) limit of proportionality.
33. The failure criteria for *brittle* materials is based on the following factor:
   (*a*) Ultimate strength
   (*b*) Shear strength
   (*c*) Yield strength
   (*d*) Limit of proportionality.
34. *Stress-strain curves* are obtained by conducting the following tests on the materials:
   1. impact test 2. torsion test
   3. tension test 4. shear test
   Of these:
   (*a*) 1 alone is true (*b*) 2 and 3 are true
   (*c*) 1 and 4 are true (*d*) 3 alone is true.
35. The *limit of proportionality* of a material is the
   (*a*) minimum value of stress for which the stress is still proportional to the strain
   (*b*) maximum value of stress for which the stress is still proportional to the strain
   (*c*) average value of stress for which the stress is still proportional to the strain
   (*d*) average value of strain.
36. The materials become harder due to *strain hardening.* Stain hardening in case of structural steel occurs
   (*a*) between yield strength and ultimate strength
   (*b*) between limit of proportionality and yield strength
   (*c*) between ultimate strength and fracture point
   (*d*) at yield point.
37. Structural steel forms neck before it breaks. *Neck formation* starts
   (*a*) before limit of proportionality
   (*b*) after yield strength
   (*c*) before ultimate strength
   (*d*) at ultimate strength.
38. *Fatigue* failure occurs at a stress
   (*a*) higher than the static breaking strength
   (*b*) equal to the static breaking strength
   (*c*) much lower than the static breaking strength
   (*d*) equal to yield strength.
39. Which of the following materials has *zero* ductility ?
   (*a*) cast iron (*b*) brass
   (*c*) chalk (*d*) steel.
40. Which of the following statements is true **?**
   (*a*) When transverse strain increases, axial strain also increases.
   (*b*) When transverse strain decreases, axial strain also decreases
   (*c*) With increase in transverse strain, axial strain decreases.
   (*d*) There is no relationship between transverse strain and axial strain.
41. The length, co-efficient of thermal expansion and Young's modulus of bar *A* are twice that of bar *B,* If the temperature of both bars is increased by the same amount while preventing an expansion, then the ratio of stress developed in bar *A* to that in bar *B* will be
   (*a*) 2 (*b*) 4
   (*c*) 8 (*d*) 16.

42. If all the dimensions of a prismatic bar of certain cross-section suspended freely from the ceiling of a roof are doubled, then the total elongation produced by its own weight will increase
(*a*) eight times (*b*) four times
(*c*) three times (*d*) two times.

43. Two bars, one of material A and the other of material B of same lengths are tightly secured between two unyielding walls. Co-efficient of thermal expansion of bar A is more than that of bar B. When temperature rises, the stresses induced are
(*a*) tension in both materials
(*b*) tension in A and compression in B
(*c*) compression in A and tension in B
(*d*) compression in both A and B.

44. A bar of diameter 30 mm is subjected to a tensile load such that the measured extension on a gauge length of 200 mm is 0.09 mm and the change in diameter is 0.0045 mm, then the Poisson's ratio will be
(*a*) 1/4 (*b*) 1/3
(*c*) 1/5 (*d*) 1/2.

45. In terms of bulk modulus (K) and modulus of rigidity (G), the *Poisson's ratio* can be expressed as
(*a*) $(3K - 2G) / (6K + 4G)$
(*b*) $(3K + 4G) / (6K - 4G)$
(*c*) $(3K - 2G) / (6K + 2G)$
(*d*) $(3K + 2G) / (6K - 2G)$.

46. Match **List I** with **List II** and select the correct answer using the codes given below the lists:

List I (Bar)	**List II** (Elongation)
A. Uniform bar	1. $4PL/pDdE$
B. Square tapered bar	2. PL/AE
C. Circular tapered bar	3. PL/DdE

[where A = Cross-sectional area
D, d = Diameters or sides at two points
E = Elastic modulus
L = Length]

Codes:	A	B	C
(*a*)	2	3	1
(*b*)	3	2	1
(*c*)	1	2	3
(*d*)	2	1	3.

47. The *property* of a material to undergo large uniform elongation before fracture (in tension), is called
(*a*) superelasticity (*b*) superplasticity
(*c*) viscoelasticity (*d*) viscoplasticity.

48. A given material has Young's modulus E, modulus of rigidity G and Poisson's ratio 0.25. The ratio of *Young's modulus* to *modulus of rigidity* of the material is
(*a*) 3.75 (*b*) 3.0
(*c*) 2.5 (*d*) 1.5.

49. For *mild steel,* the ratio of Young's modulus of elasticity in tension and compression is equal to
(*a*) 0.5 (*b*) 1.0
(*c*) 1.2 (*d*) 1.3.

50. *Ductile* fracture generally takes place along planes on which the shear stress is
(*a*) maximum (*b*) minimum
(*c*) positive (*d*) negative.

51. *Fatigue* failure is basically of
(*a*) brittle nature
(*b*) ductile nature
(*c*) combination of brittle and ductile nature
(*d*) none of these.

52. Consider the following statements about *co-efficient of thermal expansion:*
1. It is independent of the temperature change.
2. It is a material constant.
3. It has the unit of per degree Celsius.

Of these statements:
(*a*) 1 alone is true (*b*) 2 and 3 are true
(*c*) 1 and 3 are true (*d*) 1, 2 and 3 are true.

53. In *tension test,* fracture takes place along a crystallographic plane, on which the normal tensile stress is maximum. Such plane is called

(*a*) shear plane (*b*) neutral plane
(*c*) cleavage plane (*d*) normal plane.

54. The percentage reduction in area during the *tension test* on a cast iron test specimen is

(*a*) 5 to 10 % (*b*) 10 to 15 %
(*c*) 0 to 3 % (*d*) 0 to 5 %.

55. The phenomenon under which the strain in a material varies under constant stress is called

(*a*) strain hardening
(*b*) Bauschinger's effect
(*c*) creep
(*d*) fatigue.

56. A material loaded in tension beyond yield point is unloaded and then loaded in compression. Its yield strength in compression is found to be reduced. This effect is known as

(*a*) inelasticity (*b*) Bauschinger's effect
(*c*) hysteresis effect (*d*) fatigue.

57. The usual value of *gauge length* is

(*a*) 100 mm (*b*) 75 mm
(*c*) 50 mm (*d*) 25 mm.

58. In an experiment, it is found that the bulk modulus of a material is equal to its shear modulus. The *Poisson's ratio* is

(*a*) 0.125 (*b*) 0.250
(*c*) 0.375 (*d*) 0.500.

59. For a material, Poisson's ratio is 0.25. The ratio of *elastic modulus* to *shear modulus* is

(*a*) 2.55 (*b*) 2.5
(*c*) 3.0 (*d*) 1.5.

60. The ratio of *Elastic modulus* to *Bulk modulus* for the Poisson's ratio of 0.25 is

(*a*) 2.55 (*b*) 2.5
(*c*) 3.0 (*d*) 1.5.

61. The limiting values of *Poisson's ratio* are

(*a*) 0 to (+ 0.5) (*b*) 0 to (– 0.5)
(*c*) 1 to (+ 0.5) (*d*) –1 to (+ 0.5).

62. Consider the following statements:

1. The proportional sign *thermal stress* is expressed as $E.\alpha.\,\Delta T$, where α represents thermal co-efficient and ΔT, the difference in temperatures and E, the modulus of elasticity.
2. The *volumetric strain* is the algebraic sum of normal strains.
3. The value of *gauge length* is usually 50 mm.
4. *Neck formation* begins just after yield point.

Of these statements:

(*a*) 1 and 2 are true
(*b*) 1, 2 and 3 are true
(*c*) 3 and 4 are true
(*d*) 4 alone is true.

63. A *ratio* of moment carrying capacity of a circular beam of diameter D and square beam of side D is

(*a*) $\pi/4$ (*b*) $3\pi/8$
(*c*) $\pi/3$ (*d*) $3\pi/16$.

64. Match **List I** with **List II** and select the correct answer using the codes given below the lists:

List I	List II
A. Shear centre	1. Tension
B. Principal plane	2. Slope
C. Fixed end	3. Shear stress
D. Middle third rule	4. Twisting

Codes:	A	B	C	D
(*a*)	4	3	2	1
(*b*)	3	1	4	2
(*c*)	4	1	2	3
(*d*)	4	2	3	1.

65. The value of Poisson's ratio for *cork* is
    (*a*) 0.25 (*b*) 0.30
    (*c*) 0 (*d*) 0.50.

66. A beam of channel cross-section with vertical web loaded with a concentrated load at mid-span in a plane perpendicular to the plane of symmetry passing through the centroid subjected to
    1. bending moment
    2. twisting moment
    3. shear force
    4. axial force

    Of these:
    (*a*) 1 and 2 are true
    (*b*) 1, 2 and 3 are true
    (*c*) 1 and 3 are true
    (*d*) 4 alone is true.

67. The *bending stress* in terms of bending moment (M) and section modulus (S) is expressed as
    (*a*) $2M/S$ (*b*) M/S
    (*c*) S/M (*d*) $M/2S$.

68. A beam has a *triangular* cross-section having base b and altitude h. If a section of the beam is subjected to a shear force V, then the shear stress at the level of neutral axis in the cross-section is given by
    (*a*) $4V/3bh$ (*b*) $3V/4bh$
    (*c*) $8V/3bh$ (*d*) $3V/8bh$.

69. Consider the following statements about *close-coiled* helical spring:
    1. The plane of the coil is normal to the axis of the spring.
    2. The angle of helix is large.
    3. The plane of coil and the axis of spring are closely attached.

    Of these statements:
    (*a*) 1 alone is true (*b*) 1, 2 and 3 are true
    (*c*) 2 and 3 are true (*d*) 3 alone is true.

70. Consider the following statements about *spring constant*:
    1. It is the force required to produce unit torque.
    2. It is the force required to produce a deformation of one unit length in the spring.
    3. It is the force required to produce unit angular twist.

    Of these statements:
    (*a*) 1 alone is true (*b*) 2 alone is true
    (*c*) 2 and 3 are true (*d*) 3 alone is true.

71. Match **List I** with **List II** and select the correct answer using the codes given below the lists:

List I (Loaded Cantilever beam)	List II (Shape of B.M. Diagram)
A. Linearly varying load from zero at its free end to maximum at the fixed end	1. Parabola
B. Uniformly distributed load over entire span	2. Rectangle
C. Concentrated load at its free end	3. Cubic parabola
D. Free end is subjected to a couple	4. Triangle

Codes:	A	B	C	D
(*a*)	1	2	3	4
(*b*)	4	3	2	1
(*c*)	3	1	4	2
(*d*)	1	3	4	2.

72. The variation of *shear stress* with respect to radius in a *circular shaft* is shown by a
    (*a*) parabola (*b*) cubic curve
    (*c*) straight line (*d*) hyperbola.

73. The effect of *arching* a beam is to
 (*a*) make the bending moment uniform throughout.
 (*b*) reduce the bending moment throughout
 (*c*) increase the bending moment throughout
 (*d*) increase the shear force throughout.

74. If a solid circular shaft and a hollow circular shaft have the same *torsional strength*, then
 1. the weight of the hollow shaft will be less than that of the solid shaft.
 2. the external diameter of the hollow shaft will be greater than that of the solid shaft.
 3. the stiffness of the hollow shaft will be equal to that of the solid shaft.

 Of these statements:
 (*a*) 1, 2 and 3 are true
 (*b*) 2 and 3 are true
 (*c*) 1 and 2 are true
 (*d*) 1 alone is true.

75. A shaft runs at 150 rpm under a torque of 1500 N-m. The power transmitted is
 (*a*) 15π kW (*b*) 10π kW
 (*c*) 7.5π kW (*d*) 5π kW.

76. Consider the following statements:
 1. The hoop stress acts along the circumferential direction and is tensile in nature.
 2. The longitudinal stress acts along the longitudinal direction and is equal to one-half of the hoop stress.
 3. The inside fluid pressure in the pressure vessels is much smaller as compared to hoop and longitudinal stresses.

 Of these statements:
 (*a*) 1, 2 and 3 are true
 (*b*) 2 and 3 are true
 (*c*) 1 and 2 are true
 (*d*) 1 alone is true.

77. The design of the *cylindrical shell* is based on
 (*a*) bending stress (*b*) longitudinal stress
 (*c*) hoop stress (*d*) shear stress.

78. Which of the following statements is true about a *thin cylinder*?
 (*a*) Hoop stress is one-half of the longitudinal stress.
 (*b*) Longitudinal stress is constant across the thickness.
 (*c*) Hoop stress is constant across the thickness.
 (*d*) Hoop stress is equal to longitudinal stress.

79. Which of the following statements is true about the effective length of a *pinned ended* column?
 (*a*) It is equal to its actual length.
 (*b*) It is one-half of its actual length.
 (*c*) It is two times of its actual length.
 (*d*) It is $1/\sqrt{2}$ times of its actual length.

80. Which of the following statements is true about the effective length of a *fixed ended* column?
 (*a*) It is equal to its actual length.
 (*b*) It is one-half of its actual length.
 (*c*) It is two times of its actual length.
 (*d*) It is $1/\sqrt{2}$ times of its actual length.

81. The *critical load* is the load at which the column
 (*a*) breaks
 (*b*) loses its strength
 (*c*) buckles
 (*d*) can take minimum load.

82. Which of the following statements is true about the *buckling* of column ?
 (*a*) It usually occurs about the axis w.r.t which the moment of inertia is the maximum.
 (*b*) It usually occurs about the axis w.r.t which the moment of inertia is the least.
 (*c*) It usually occurs about the axis w.r.t which the moment of inertia is zero.
 (*d*) It is independent of moment of inertia.

83. Consider the following statements about the *Johnston formula* of column:
    1. It is a semi-empirical formula.
    2. It involves inelastic buckling.
    3. It is basically an equation of a parabola.

    Of these statements:
    (*a*) 1, 2 and 3 are true
    (*b*) 2 and 3 are true
    (*c*) 1 and 2 are true
    (*d*) 1 alone is true.
84. Consider the following statements:
    1. On planes having maximum and minimum principal stresses, there will be no tangential stress.
    2. The shear stresses on mutually perpendicular planes are numerically equal.
    3. The maximum shear stress is equal to one-half of the difference of the maximum and minimum principal stresses.

    Of these statements:
    (*a*) 1, 2 and 3 are true
    (*b*) 1 and 2 are true
    (*c*) 1 and 3 are true
    (*d*) 1 alone is true.
85. The *Johnston formula* of column is used for
    (*a*) short columns
    (*b*) long columns
    (*c*) short and intermediate columns both
    (*d*) intermediate columns.
86. A cast-iron pipe of 1 m diameter is required to withstand a 200 m head of water. If the limiting tensile stress of the pipe material is 20 MPa, then the thickness of the pipe will be
    (*a*) 25 mm (*b*) 50 mm
    (*c*) 75 mm (*d*) 100 mm.
87. In a rectangular element being subjected to two like principal tensile stresses in two mutually perpendicular directions x and y, the maximum shear stress would occur along the
    (*a*) plane normal to x-axis
    (*b*) plane normal to y-axis
    (*c*) plane at 45 degree to y-direction
    (*d*) plane at 45 degree and 135 degree to y-direction.
88. Two closed thin vessels, one cylindrical and other spherical with equal internal diameter and wall thickness are subjected to equal internal fluid pressure. The ratio of hoop stresses in the cylindrical to that of spherical vessel is
    (*a*) 4.0 (*b*) 2.0
    (*c*) 1.0 (*d*) 0.5.
89. Which of the following statements is true about *a continuous* beam?
    (*a*) It is supported on three or more roller supports.
    (*b*) It is supported on three or more hinge supports.
    (*c*) It is supported on one hinge support and two or more roller supports.
    (*d*) It requires no support.
90. Consider the following statements:
    1. The slope of the shear force diagram (SFD) at any section of the beam is equal to the load intensity at that section.
    2. The slope of the shear force diagram (SFD) at any section of the beam is equal to the bending moment at that section.
    3. The slope of the bending moment diagram (BMD) at any section of the beam is equal to the shear force at that section.
    4. The slope of the bending moment diagram (BMD) at any section of the beam is equal to the load intensity at that section.

    Of these statements:
    (*a*) 1 alone is true (*b*) 2 and 3 are true
    (*c*) 1 and 3 are true (*d*) 4 alone is true.

91. *Poisson's ratio* is defined as a ratio of
    (*a*) axial stress to shear stress
    (*b*) longitudinal strain to lateral strain
    (*c*) lateral strain to longitudinal strain
    (*d*) axial stress to bending stress.
92. For a 12 mm diameter steel rod test specimen, the suitable *gauge length* is
    (*a*) 24 mm (*b*) 36 mm
    (*c*) 72 mm (*d*) 60 mm.
93. The *stress* produced on a surface normal to the load applied is called
    (*a*) shear stress (*b*) bending stress
    (*c*) normal stress (*d*) axial stress.
94. The *deformation* of a uniform section bar subjected to an axial pull P is given by (where symbols have their usual meanings)
    (*a*) PL/AE (*b*) $2PL/AE$
    (*c*) $PL/2AE$ (*d*) $PL/3AE$.
95. *Tensile load* results in
    (*a*) contraction (*b*) elongation
    (*c*) bending (*d*) twisting.
96. *Factor of safety* is defined as a ratio of
    (*a*) shear stress to working stress
    (*b*) bending stress to shear stress
    (*c*) ultimate stress to working stress
    (*d*) working stress to ultimate stress.
97. The relationship between E and G is
    (*a*) $E = 2G(1 - \nu)$ (*b*) $E = 2G(1 + \nu)$
    (*c*) $E = 2G(1 - 2\nu)$ (*d*) $E = 2G(1 + 2\nu)$.
98. The relationship between E and K is
    (*a*) $E = 3K(1 - 2\nu)$ (*b*) $E = 3K(1 + 2\nu)$
    (*c*) $E = 2K(1 - 2\nu)$ (*d*) $E = 2K(1 + 2\nu)$.
99. The relationship between E, G and K is
    (*a*) $E = 3\,KG/(2K + G)$
    (*b*) $E = 9\,KG/(3K + G)$
    (*c*) $E = 5\,KG/(2K + G)$
    (*d*) $E = 9\,KG/(3E + K)$.
100. *Shear stress*
    (*a*) acts normal to the surface
    (*b*) acts tangential to the surface
    (*c*) is equal to the tensile stress
    (*d*) is equal to the compressive stress.
101. *Modulus of rigidity* is defined as a ratio of
    (*a*) shear strain to volumetric strain
    (*b*) shear stress to shear strain
    (*c*) normal stress to shear strain
    (*d*) normal stress to linear strain.
102. During the *tightening* of a nut on a bolt, the stress induced in the bolt is
    (*a*) compressive (*b*) shear
    (*c*) tensile (*d*) bending.
103. Stresses are said to be *compound*, when
    (*a*) normal and shear stresses are acting simultaneously
    (*b*) torsion and bending stresses are acting simultaneously
    (*c*) normal and bending stresses are acting simultaneously
    (*d*) shear and bending stresses are acting simultaneously.
104. *Principal planes* are planes of
    (*a*) maximum shear stress
    (*b*) minimum shear stress
    (*c*) maximum normal stress
    (*d*) zero shear stress.
105. The *principal stresses* are basically
    (*a*) shear stresses (*b*) bending stresses
    (*c*) normal stresses (*d*) hoop stresses.
106. The *planes* of maximum shear stress are located at the following angle to the principal planes
    (*a*) 90° (*b*) 45°
    (*c*) 60° (*d*) 30°.

107. The *principal planes* are separated by
   (*a*) 90° (*b*) 45°
   (*c*) 60° (*d*) 180°.
108. The *maximum shear stress* is equal to
   (*a*) one-half the algebraic difference of principal stresses
   (*b*) difference of principal stresses
   (*c*) sum of principal stresses
   (*d*) algebraic difference of principal stresses.
109. For *uniaxial* loading condition, the maximum shear stress is equal to
   (*a*) uniaxial stress
   (*b*) two times the uniaxial stress
   (*c*) three times the uniaxial stress
   (*d*) one-half of uniaxial stress.
110. The *radius* of Mohr's circle indicates
   (*a*) maximum principal stress
   (*b*) minimum principal stress
   (*c*) maximum shear stress
   (*d*) minimum shear stress.
111. In case one principal stress is zero, the other principal stress is equal to
   (*a*) maximum principal stress
   (*b*) two times the maximum shear stress
   (*c*) maximum shear stress
   (*d*) three times the maximum shear stress.
112. The *maximum* bending moment, when a point load W is acting at the free end of a cantilever beam of length L, is
   (*a*) $WL/2$ (*b*) $WL/4$
   (*c*) $WL/3$ (*d*) WL.
113. The variation of bending moment for a *cantilever* beam carrying a *udl* of intensity w/unit length over its entire span is shown by a/an
   (*a*) straight line
   (*b*) second degree parabola
   (*c*) third degree parabola
   (*d*) ellipse.
114. The *bending* stress is proportional to
   (*a*) moment of inertia
   (*b*) modulus of elasticity
   (*c*) its distance from the neutral axis
   (*d*) radius of curvature.
115. For a *complex* stress system, the total number of principal planes is
   (*a*) two (*b*) four
   (*c*) three (*d*) five.
116. The *bending* stress is maximum at the
   (*a*) neutral axis
   (*b*) top layer of beam
   (*c*) bottom layer of beam
   (*d*) none of these.
117. The ratio of *maximum shear stress* and *average shear stress* for a triangular section is
   (*a*) 0.66 (*b*) 1.33
   (*c*) 1.5 (*d*) 0.75.
118. Compared to the bending deformation, the *shear deformation* is
   (*a*) large (*b*) small
   (*c*) very large (*d*) zero.
119. The *bending* stress is zero at the
   (*a*) neutral axis
   (*b*) top layer of beam
   (*c*) bottom layer of beam
   (*d*) none of these.
120. The *shear stress* varies in direct proportion to
   (*a*) moment of inertia about the neutral axis
   (*b*) width of the beam
   (*c*) distance between neutral axis and centroid of the area above the neutral axis
   (*d*) normal stress.
121. The shear stress is maximum, where
   (*a*) bending stress is minimum
   (*b*) bending stress is maximum
   (*c*) bending stress is zero
   (*d*) bending moment is positive.

122. The shear stress at the neutral axis of a *rectangular section* is
 (*a*) average shear stress
 (*b*) maximum shear stress
 (*c*) minimum shear stress
 (*d*) none of these.
123. The *modulus of section* is a ratio of
 (*a*) moment of inertia and bending stress
 (*b*) moment of inertia and the distance from the neutral axis
 (*c*) moment of inertia and modulus of elasticity
 (*d*) moment of inertia and modulus of rigidity.
124. The *strength* of a beam depends upon
 (*a*) modulus of elasticity
 (*b*) bending moment
 (*c*) section modulus
 (*d*) radius of curvature.
125. The deflected neutral surface of a beam after bending is called
 (*a*) deflected surface (*b*) bent surface
 (*c*) elastic curve (*d*) plastic curve.
126. The *bending equation* is valid for a beam subjected to
 (*a*) bending moment and no shear force
 (*b*) combined bending and shear force
 (*c*) shear force only
 (*d*) shear stress only.
127. A *composite* beam is made of
 (*a*) more than one material
 (*b*) more than one cross-section
 (*c*) plastic material
 (*d*) composite material.
128. The deflection produced by bending is
 (*a*) equal to deflection produced by shear
 (*b*) less than deflection produced by shear
 (*c*) greater than deflection produced by shear
 (*d*) unpredictable.
129. The *flexural rigidity* is the product of
 (*a*) modulus of elasticity and mass moment of inertia
 (*b*) modulus of rigidity and area moment of inertia
 (*c*) modulus of rigidity and mass moment of inertia
 (*d*) modulus of elasticity and area moment of inertia.
130. Shear stress variation across a *rectangular* section is
 (*a*) hyperbolic (*b*) parabolic
 (*c*) circular (*d*) elliptical.
131. The *slope* and *deflection* at the fixed end of a cantilever beam are
 (*a*) zero, maximum
 (*b*) zero, zero
 (*c*) maximum, minimum
 (*d*) maximum, zero.
132. The *slope* and *deflection* at the centre of a simple beam carrying a central point load are
 (*a*) Zero, zero
 (*b*) Zero, maximum
 (*c*) Maximum, zero
 (*d*) Minimum, maximum.
133. The torsion equation is
 (*a*) $T/I = G\theta/L = \tau/r$
 (*b*) $T/J = G\theta/L = \tau/r$
 (*c*) $T/J = \mathrm{G}\theta/L = \tau/D$
 (*d*) $J/T = G\theta/L = \tau/r$.
134. Which of the following methods uses *Mohr's theorem* for finding slope and deflection of a beam?
 (*a*) Macaulay's method
 (*b*) Integration method
 (*c*) Moment area method
 (*d*) Conjugate beam method.

135. The *slope* at any section of a beam is equal to which parameter of the *conjugate* beam?
(*a*) bending moment
(*b*) slope
(*c*) deflection
(*d*) shear force.

136. The *deflection* at any section of a beam is equal to which parameter of the *conjugate* beam?
(*a*) bending moment
(*b*) slope
(*c*) deflection
(*d*) shear force.

137. The *conjugate* beam method is the most suitable method for finding
(*a*) slope and deflection of a uniform sectional beam
(*b*) slope and deflection of a non-uniform sectional beam
(*c*) slope of a uniform sectional beam
(*d*) deflection of a uniform sectional beam.

138. According to *moment area* method, change in slope between any two sections of a beam is equal to
(*a*) moment of area of (M/EI) diagram between two sections
(*b*) area of bending moment diagram between two sections
(*c*) area of (M/EI) diagram between two sections
(*d*) area of shear force diagram between two sections.

139. According to *moment area method*, deflection at any section of a beam w.r.t. a reference point is equal to
(*a*) moment of area of (M/EI) diagram between section and reference point
(*b*) area of bending moment diagram between section and reference point
(*c*) area of (M/EI) diagram between section and reference point
(*d*) area of shear force diagram between section and reference point.

140. If the diameter of a circular sectional beam is doubled, its *deflection* is reduced by
(*a*) 16 times (*b*) 4 times
(*c*) 8 times (*d*) 32 times.

141. For a *shaft* being subjected to a torque T, variation of shear stress w.r.t its radius is
(*a*) linear (*b*) parabolic
(*c*) hyperbolic (*d*) cubic curve.

142. The *shear stress* produced in a circular shaft due to pure torsion is
(*a*) directly proportional to the radius of the shaft
(*b*) inversely proportional to the diameter of the shaft
(*c*) inversely proportional to the radius of the shaft
(*d*) directly proportional to normal stress.

143. The *torsional rigidity* is defined as the
(*a*) ratio of torque and angle of twist
(*b*) product of polar moment of inertia and modulus of rigidity
(*c*) sum of modulus of rigidity and angle of twist
(*d*) ratio of torque and polar moment of inertia.

144. *Shear stress* for a shaft being subjected to a torque T is minimum at
(*a*) half of radius from the axis
(*b*) axis of the shaft
(*c*) equal radial distances from the axis
(*d*) its both ends.

145. *Shear strain* in a circular shaft varies
(*a*) linearly with the distance from the axis of the shaft
(*b*) linearly with the square of the distance from the axis of the shaft
(*c*) inversely proportional to the distance from the axis of the shaft
(*d*) inversely proportional to the square of the distance from the axis of the shaft.

146. The *shear strain* is maximum
    (*a*) at a distance equal to one-third from the axis of the shaft
    (*b*) at the centre of the shaft
    (*c*) on the surface of the shaft
    (*d*) at a distance equal to one-half from the axis of the shaft.
147. The *stiffness* of the spring is defined as aratio of
    (*a*) load and angle of twist
    (*b*) load and deflection
    (*c*) load and strain energy
    (*d*) load and strain.
148. The *deformation* produced in the spring is said to be
    (*a*) semi-elastic
    (*b*) plastic
    (*c*) elastic
    (*d*) visco-elastic.
149. The *maximum strain energy* stored in a body at the elastic limit is called
    (*a*) resilience
    (*b*) modulus of resilience
    (*c*) proof resilience
    (*d*) potential energy.
150. In a *close-coiled* helical spring,
    (*a*) plane of the coil and axis of the spring are closely attached
    (*b*) angle of helix is large
    (*c*) plane of the coil is normal to the axis of the spring
    (*d*) deflection is small.
151. A *conical helical spring* is used, where
    (*a*) space is a problem
    (*b*) more stiffness is required
    (*c*) more load is to be taken
    (*d*) less deflection is required.
152. The *load-deflection curve* of a spring is a straight line, only if
    (*a*) spring is stressed up to yield point
    (*b*) spring is stressed up to ultimate point
    (*c*) spring is stressed up to failure point
    (*d*) spring is stressed within the elastic limit.
153. The *Wahl's correction factor* is introduced to
    (*a*) increase the number of coils in the spring
    (*b*) take care of extra load on the spring
    (*c*) take care of the curvature of spring wire
    (*d*) take care of extra stiffness in the spring.
154. The *spring index* is defined as a ratio of
    (*a*) load to deflection
    (*b*) mean coil diameter to spring wire diameter
    (*c*) load to angle of twist
    (*d*) mean coil diameter to length of spring wire.
155. Energy stored in a material during its deformation is called
    (*a*) elastic energy
    (*b*) plastic energy
    (*c*) strain energy
    (*d*) potential energy.
156. For two springs being connected in series, the following statement is correct?
    (*a*) Deflection produced in the equivalent spring is the sum of the deflections produced in the individual spring.
    (*b*) Total weight is the sum of the weights acting separately on the two springs.
    (*c*) Equivalent stiffness is the sum of the individual stiffnesses.
    (*d*) Equivalent stiffness is the product of the individual stiffnesses.
157. For two springs being connected in parallel, the following statement is correct?
    (*a*) Equivalent load is the sum of the individual loads.

(*b*) Equivalent deflection is the sum of the individual deflections.

(*c*) Equivalent stiffness is the sum of the individual stiffnesses.

(*d*) Equivalent deflection is the product of the individual deflections.

158. The *stress* produced by a suddenly applied load is how many times the stress produced by the gradually applied load ?

(*a*) four times (*b*) three times

(*c*) two times (*d*) eight times.

159. The most important point in the consideration of a material's failure is

(*a*) ultimate point (*b*) yield point

(*c*) failure point (*d*) elastic limit.

160. The *maximum principal stress theory* is also known as

(*a*) Haigh's theory

(*b*) St. Venant's theory

(*c*) Rankine's theory

(*d*) Von Mises's theory.

161. The *Hooke's law* is valid for

(*a*) brittle materials

(*b*) ductile materials

(*c*) isotropic materials

(*d*) isotropic and homogeneous materials.

162. The *maximum shear stress theory* gives better results for

(*a*) brittle materials

(*b*) ductile materials

(*c*) brittle and ductile materials both

(*d*) amorphous materials.

163. The *maximum shear stress* at the elastic limit for a body being subjected to a stress σ is

(*a*) 2σ (*b*) $\sigma/3$

(*c*) $\sigma/2$ (*d*) $\sigma/5$.

164. According to the *Tresca's theory*, failure occurs when the

(*a*) major principal stress exceeds the elastic limit stress

(*b*) maximum principal strain exceeds the elastic limit strain

(*c*) maximum shear stress exceeds the maximum shear stress at the elastic limit

(*d*) distortion energy per unit volume exceeds the distortion energy per unit volume at the elastic limit.

165. Which one of the groups consisting of two theories is said to be modern, because of their closeness to experimental values?

(*a*) Rankine's and St. Venant's theories

(*b*) Tresca's and Rankine's theories

(*c*) Tresca's and von Mises theories

(*d*) Rankine's and von Mises theories.

166. The *Euler's formula* is valid for

(*a*) short columns

(*b*) medium columns

(*c*) long columns

(*d*) short and long columns both.

167. The *Rankine-Gordon formula* is valid for

(*a*) short columns

(*b*) medium columns

(*c*) long columns

(*d*) short and long columns both.

168. A structural member subjected to an axial compressive load is called a

(*a*) strut (*b*) beam

(*c*) shaft (*d*) lever.

169. The ratio of equivalent length of a column to its least radius of gyration is known as

(*a*) factor of safety

(*b*) Poisson's ratio

(*c*) slenderness ratio

(*d*) moment of inertia.

170. The effective length of a column with *both ends fixed* is equal to

(*a*) the actual length

(*b*) one-half of the actual length

(*c*) two times the actual length
(*d*) $1/\sqrt{2}$ times the actual length.

171. The slenderness ratio is less than 30 for
(*a*) short columns
(*b*) long columns
(*c*) medium columns
(*d*) short and medium columns both.

172. A *short* column fails mainly due to
(*a*) buckling
(*b*) compressive stress
(*c*) combined effect of buckling and compressive stress
(*d*) tensile stress.

173. Consider the following statements:
1. In *three-point* bend test, two point loads are used.
2. In *four-point* bend test, the bending moment diagram is a trapezium.
3. *Izod* test uses a vertically placed test specimen.
4. *Impact* test measures hardness of a material.

Of these statements:
(*a*) 1 and 2 are true (*b*) 2 and 3 are true
(*c*) 3 and 4 are true (*d*) 2, 3 and 4 are true.

174. The crippling stress varies
(*a*) directly proportional to slenderness ratio
(*b*) inversely proportional to slenderness ratio
(*c*) inversely proportional to the cubic power of slenderness ratio
(*d*) inversely proportional to the square of slenderness ratio.

175. The *radius of gyration* of a circular section of diameter 50 mm is
(*a*) 25 mm (*b*) 50 mm
(*c*) 12.5 mm (*d*) 20 mm.

176. For a long column, the *slenderness ratio* is greater than
(*a*) 30 (*b*) 90
(*c*) 120 (*d*) 200.

177. For *Euler's formula* to be valid, crippling stress of the column is
(*a*) more than its yield strength
(*b*) less than its yield strength
(*c*) equal to its yield strength
(*d*) equal to its ultimate strength.

178. Consider the following statements:
1. For a test specimen loaded in *three-point* bend test, the bending moment diagram is a triangle.
2. In the *bend test*, the upper surface of the specimen is subjected to compression and the lower surface to tension.
3. *Charpy test* uses a horizontally placed test specimen.
4. A *compression* test is more difficult to conduct than a tension test.

Of these statements:
(*a*) 1 alone is true
(*b*) 1 and 3 are true
(*c*) 1 and 2 are true
(*d*) 1, 2, 3 and 4 are true.

179. Consider the following statements:
1. *Creep* is a time-dependent phenomenon.
2. *Endurance limit* is dependent on tensile strength of the material.
3. *Impact test* measures toughness.
4. Resistance against scratching is called *hardness*.

Of these statements:
(*a*) 1 and 3 are true
(*b*) 1, 3 and 4 are true
(*c*) 3 and 4 are true
(*d*) 1, 2, 3 and 4 are true.

180. Consider the following statements:
1. *Brinell test* uses a spherical impression on the test specimen.
2. *Vickers test* uses a 136 degree pyramid-shaped diamond indentor.

3. *Vickers test* produces a square-shaped indentation.
4. *Rockwell C* uses a test load of 100 kg.

Of these statements:

(*a*) 1 and 2 are true (*b*) 1, 2 and 3 are true
(*c*) 2 and 3 are true (*d*) 1, 2 and 4 are true.

181. Which of the following mechanical properties is connected to *time* ?

(*a*) hardness (*b*) fatigue
(*c*) creep (*d*) tension.

182. Consider the following statements:

1. *Brinell test* uses a 10 mm diameter steel ball indenter.
2. *Brinell test* measures depth of indentation on the test specimen.
3. *Rockwell A* uses a test load of 100 kg.
4. The standard unit of *BHN* is N/m^2.

Of these statements:

(*a*) 1 and 2 are true (*b*) 1 alone is true
(*c*) 2 and 3 are true (*d*) 1, 2 and 4 are true.

183. For a thin cylindrical shell of diameter *d* and thickness *t*, being subjected to a fluid pressure *p***,** the *hoop stress* is given by

(*a*) $pd/3t$ (*b*) $pd/8t$
(*c*) $pd/2t$ (*d*) pd/t.

184. Consider the following parameters:

1. Hardness
2. Toughness
3. Density
4. Specific gravity

Which of the above parameters can be placed under *mechanical property* ?

(*a*) 1 and 2 (*b*) 1 and 3
(*c*) 2 and 3 (*d*) 3 and 4.

185. **Creep**

1. increases with increase in temperature
2. decreases with increase in temperature
3. is independent of temperature
4. increases with applied load

Of these:

(*a*) 1 alone is true (*b*) 1 and 4 are true
(*c*) 3 and 4 are true (*d*) 2 and 4 are true.

186. For a *thin-walled shell***,** the diameter-thickness ratio is

(*a*) less than 20 (*b*) greater than 20
(*c*) equal to 20 (*d*) equal to 10.

187. For a *thick-walled shell*, the diameter-thickness ratio is

(*a*) less than 20 (*b*) greater than 20
(*c*) equal to 20 (*d*) equal to 10.

188. For a *thin cylindrical shell* of diameter *d* and thickness **t**, being subjected to a fluid pressure *p*, the hoop stress is given by

(*a*) $pd/3t$ (*b*) $pd/8t$
(*c*) $pd/2t$ (*d*) $pd/4t$.

189. For a *thin cylindrical shell* of diameter *d* and thickness *t*, being subjected to a fluid pressure *p*, longitudinal stress is given by

(*a*) $pd/3t$ (*b*) $pd/8t$
(*c*) $pd/2t$ (*d*) $pd/4t$.

190. The hoop stress is also known as

(*a*) longitudinal stress
(*b*) circumferential stress
(*c*) bending stress
(*d*) compressive stress.

191. Thin and thick walled *pressure vessels* are subjected to two types of stresses, namely

(*a*) hoop stress and longitudinal stress
(*b*) tensile stress and compressive stress
(*c*) bending stress and shear stress
(*d*) hoop stress and bending stress.

192. For a thin cylindrical shell, *longitudinal stress* is equal to

(*a*) hoop stress
(*b*) two times the hoop stress
(*c*) three times the hoop stress
(*d*) one-half of hoop stress.

193. The *hoop stress* is considered as
   (*a*) minor principal stress
   (*b*) major principal stress
   (*c*) bending stress
   (*d*) compressive stress.

194. The difference of the hoop stress and the longitudinal stress, for a *thin cylindrical shell* of diameter d and thickness t, being subjected to a pressure p, is
   (*a*) $pd/4t$ (*b*) $pd/16t$
   (*c*) $pd/8t$ (*d*) $pd/3t$.

195. The maximum shear stress in case of a *thin cylindrical shell* of diameter d and thickness t, being subjected to a pressure p, is
   (*a*) $pd/4t$ (*b*) $pd/16t$
   (*c*) $pd/8t$ (*d*) $pd/3t$.

196. Which is the correct relationship ?
   (*a*) Gauge pressure + Atmospheric pressure = Absolute pressure
   (*b*) Gauge pressure – Vaccum pressure = Absolute pressure
   (*c*) Gauge pressure + Absolute pressure = Atmospheric pressure
   (*d*) Gauge pressure + Vacuum pressure = Absolute pressure.

197. For a *thin spherical shell*
   (*a*) hoop stress is two times the longitudinal stress
   (*b*) longitudinal stress is two times the hoop stress
   (*c*) hoop stress is equal to one-half of the longitudinal stress
   (*d*) hoop and longitudinal stresses are equal.

198. The *hoop stress* and the *longitudinal stress* act at the following angle to each other.
   (*a*) 45 degree
   (*b*) 60 degree
   (*c*) 90 degree
   (*d*) 180 degree.

199. The hoop strain for a *thin cylindrical shell* of diameter d, thickness t, Poisson's ratio $\boldsymbol{\nu}$ and being subjected to a pressure p, is
   (*a*) $pd\,(1 - \nu) / 4tE$ (*b*) $pd\,(1 - 2\nu) / 4tE$
   (*c*) $pd\,(2 - \nu) / 4tE$ (*d*) $pd\,(1 + \nu) / 4tE$.

200. The longitudinal strain for a *thin cylindrical shell* of diameter d, thickness t, Poisson's ratio $\boldsymbol{\nu}$ and being subjected to a pressure p, is
   (*a*) $pd\,(1 - \nu) / 4tE$ (*b*) $pd\,(1 - 2\nu) / 4tE$
   (*c*) $pd\,(2 - \nu) / 4tE$ (*d*) $pd\,(1 + \nu) / 4tE$.

201. The volumetric strain for a *thin cylindrical shell* of diameter d, thickness t, Poisson's ratio $\boldsymbol{\nu}$ and being subjected to a pressure p, is
   (*a*) $pd\,(5 - 3\nu) / 4tE$
   (*b*) $pd\,(5 - 4\nu) / 3tE$
   (*c*) $pd\,(5 - 4\nu) / 4tE$
   (*d*) $pd\,(5 + 4\nu) / 4tE$.

202. The expression for hoop stress for a *thin spherical shell* of diameter d, thickness t, Poisson's ratio $\boldsymbol{\nu}$ and being subjected to a pressure p, is
   (*a*) $pd/2t$ (*b*) $pd/3t$
   (*c*) $pd/8t$ (*d*) $pd/4t$.

203. For a *thin cylindrical shell* with hemispherical ends, the ratio of thicknesses of the cylindrical part and hemispherical ends, being subjected to equal hoop stress, is
   (*a*) 0.5 (*b*) 1.5
   (*c*) 2.0 (*d*) 2.5.

204. In case of a *thin cylindrical shell* with hemispherical ends, if no distortion occurs at the junction of the two parts, the ratio of thicknesses of the cylindrical part and hemispherical ends, for a Poisson's ratio n, is given as
   (*a*) $(1 - n) / (2 - n)$ (*b*) $(2n - 1) / (1 - n)$
   (*c*) $(2 - n) / (1 - n)$ (*d*) $(1 + n) / (2 - n)$.

205. The unit of the constant A in the *Lame's equation* is
   (*a*) newton (*b*) newton-meter
   (*c*) newton-sec (*d*) pascal.

206. *Lame's equations* are used to find
   (*a*) bending and hoop stresses
   (*b*) hoop and radial stresses
   (*c*) hoop and longitudinal stresses
   (*d*) axial and bending stresses.
207. *Lame's equations* are applicable in case of
   (*a*) thin-walled pressure vessels
   (*b*) thick-walled pressure vessels
   (*c*) both thin- and thick-walled pressure vessels
   (*d*) members of elliptical section.
208. In case of only internal pressure acting in the cylinder, maximum radial stress is equal to
   (*a*) one-half of internal pressure
   (*b*) two-third of internal pressure
   (*c*) internal pressure
   (*d*) one-third of internal pressure.
209. The unit of the constant *B* in the *Lame's equation* is
   (*a*) newton (*b*) newton-meter
   (*c*) newton-sec (*d*) pascal.
210. In case of only *internal* pressure acting in the cylinder, maximum radial stress occurs at
   (*a*) internal surface of the cylinder
   (*b*) external surface of the cylinder
   (*c*) centre of the cylinder
   (*d*) none of these.
211. The minimum radial stress occurs at the following location of a *thick cylinder*, when subjected to internal pressure only
   (*a*) internal surface of the cylinder
   (*b*) external surface of the cylinder
   (*c*) centre of the cylinder
   (*d*) none of these.
212. The relationship among *volumetric* strain, *hoop* strain and *longitudinal* strain is
   (*a*) Hoop strain = 2 × Volumetric strain + Longitudinal strain
   (*b*) Volumetric strain = 2 × Hoop strain + Longitudinal strain
   (*c*) Longitudinal strain = 2 × Hoop strain + Volumetric strain
   (*d*) Longitudinal strain = 3 × Hoop strain + Volumetric strain.
213. The hoop stress variation in a *thick cylinder* is shown by a
   (*a*) straight line (*b*) parabola
   (*c*) ellipse (*d*) hyperbola.
214. The cylinders are *compounded* to
   (*a*) increase the strength of the cylinder
   (*b*) increase the pressure bearing capacity of the cylinder
   (*c*) make the distribution of hoop stress uniform
   (*d*) make the distribution of longitudinal stress uniform.
215. In a thick cylinder, the radial stress at inner surface is
   (*a*) independent of fluid pressure
   (*b*) more than fluid pressure
   (*c*) less than fluid pressure
   (*d*) equal to fluid pressure.
216. In a thick cylinder, the radial stress at outer surface is usually
   (*a*) more than zero (*b*) less than zero
   (*c*) zero (*d*) not defined.
217. *Neck formation* starts after
   (*a*) yield point (*b*) ultimate point
   (*c*) elastic limit (*d*) failure point.
218. *Poisson ratio* is a ratio of
   (*a*) longitudinal strain to lateral strain
   (*b*) lateral strain to longitudinal strain
   (*c*) lateral strain to volumetric strain
   (*d*) lateral strain to shear strain.
219. Which of the following materials has zero value of *Poisson ratio*?
   (*a*) Cork (*b*) Cement concrete
   (*c*) Rubber (*d*) Wood.

220. *Tensile load* produces

(*a*) contraction (*b*) elongation

(*c*) bending (*d*) no effect.

221. The highest stress that a material can withstand for a specified length of time without excessive deformation is called

(*a*) fatigue strength

(*b*) endurance strength

(*c*) creep strength

(*d*) creep rupture strength.

222. Match **List I** with **List II** and select the correct answer using the codes given below the lists:

List I (Beam's Cross-section)	**List II** (Maximum Shear Stress)
A. Rectangular	1. $\frac{4}{3}\tau_{av}$
B. Thin circular tube	2. $\frac{3}{2}\tau_{av}$
C. Circular	3. $2\ \tau_{av}$

Codes:	A	B	C
(*a*)	1	2	3
(*b*)	2	1	3
(*c*)	2	3	1.

223. The equation representing shear stress distribution in a beam, is given by

(*a*) $\frac{F_s}{IA} b\bar{y}$ (*b*) $\frac{F_s}{Ab} I\bar{y}$

(*c*) $\frac{F_s}{Ib} A\bar{y}$ (*d*) $\frac{F_s}{I\bar{y}} Ab$.

(where

F_s = Shear stress

I = Moment of inertia of beam's cross-section

A = Cross-sectional area

b = Width of cross-section

$\bar{y}$ = Distance of centroid of area from the neutral axis)

224. The equivalent bending moment in case of a shaft being subjected to bending moment M and twisting moment T is

(*a*) $\sqrt{M^2 + T^2}$

(*b*) $\frac{1}{2}\sqrt{M^2 + T^2}$

(*c*) $\frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right]$

(*d*) $\frac{1}{2}\left[M - \sqrt{M^2 + T^2}\right]$.

225. The equivalent torque in case of a shaft being subjected to bending moment M and twisting moment T is

(*a*) $\frac{1}{2}\sqrt{M^2 + T^2}$

(*b*) $\sqrt{M^2 + T^2}$

(*c*) $\frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right]$

(*d*) $\frac{1}{2}\left[M - \sqrt{M^2 + T^2}\right]$.

226. A shaft of diameter *d* is subjected to bending moment *M* and twisting moment *T*. The developed principal stresses will be

(*a*) $\pm\frac{16}{\pi d^3}\sqrt{M^2 + T^2}$

(*b*) $\frac{16}{\pi d^3}[M \pm \sqrt{M^2 + T^2}]$

(*c*) $\pm\frac{32}{\pi d^3}\sqrt{M^2 + T^2}$

(*d*) $\frac{32}{\pi d^3}[T \pm \sqrt{M^2 + T^2}]$.

227. A section of solid circular shaft with diameter d is subjected to bending moment M and torque T. The maximum principal stress at the section is given as

(a) $\frac{16}{\pi d^3}\sqrt{M^2+T^2}$

(b) $\frac{32}{\pi d^3}\sqrt{M^2+T^2}$

(c) $\frac{16}{\pi d^3}[M+\sqrt{M^2+T^2}]$

(d) $\frac{32}{\pi d^3}[M+\sqrt{M^2+T^2}]$.

228. The strain energy stored in a solid shaft subjected to maximum shear stress τ_{max} is given by

(a) $\frac{\tau_{max}^2}{2G} \times V$ (b) $\frac{\tau_{max}^2}{3G} \times V$

(c) $\frac{\tau_{max}^2}{4G} \times V$ (d) $\frac{\tau_{max}}{4G} \times V$.

229. The strain energy stored in a hollow shaft subjected to maximum shear stress τ_{max} is given by

(a) $\frac{\tau_{max}^2}{2G} \times V$ (b) $\frac{\tau_{max}^2}{4G} \times V$

(c) $\frac{\tau_{max}}{2G} \times V$ (d) $\frac{\tau_{max}}{4G} \times V$.

230. The maximum tensile stress induced in a shaft of diameter d subjected to equivalent bending moment M_e and equivalent torque T_e is given as

(a) $\frac{16\,M_e}{\pi d^3}$ (b) $\frac{16\,T_e}{\pi d^3}$

(c) $\frac{32\,M_e}{\pi d^3}$ (d) $\frac{32\,T_e}{\pi d^3}$.

231. The maximum shear stress induced in a shaft of diameter d subjected to equivalent bending moment M_e and equivalent torque T_e is given as

(a) $\frac{16\,M_e}{\pi d^3}$ (b) $\frac{16\,T_e}{\pi d^3}$

(c) $\frac{32\,T_e}{\pi d^3}$ (d) $\frac{32\,M_e}{\pi d^3}$.

232. Luders linesare connected to
(a) brittle materials
(b) ductile materials
(c) both brittle and ductile materials
(d) plastic materials.

233. Poisson's ratiorelates
(a) axial strain and shear strain
(b) lateral strain and shear strain
(c) lateral strain and longitudinal strain
(d) lateral strain and volumetric strain.

234. A prismatic baris a bar of
(a) circular cross section
(b) rectangular cross section
(c) uniform cross section
(d) non-uniform cross section.

235. Hooke's lawis related to
(a) stress and strain
(b) work and energy
(c) coplanar forces
(d) collinear forces.

236. Consider the following statements about theoretical and actual stresses:

1. The theoretical stress is based on the actual cross sectional area.
2. The actual stress is based on the original cross sectional area.
3. The actual stress is lesser than the theoretical stress.
4. The theoretical stress is lesser than the actual stress.

Of these statements:
(a) 1 and 4 are true
(b) 2 and 3 are true
(c) 1, 2 and 4 are true
(d) 4 alone is true.

237. Consider the following statements:

1. Elastic limit and limit of proportionality are same.
2. Yield point indicates the elastic state of the material.
3. Actual stress is always higher than the theoretical stress.
4. Ultimate point indicates the highest load bearing capacity of the material.

Of these statements:

(*a*) 1 and 2 are true
(*b*) 1, 2 and 3 are true
(*c*) 1, 3 and 4 are true
(*d*) 3 and 4 are true.

238. Shear modulus is also called

(*a*) Bulk modulus of elasticity
(*b*) Modulus of rigidity
(*c*) Modulus of elasticity
(*d*) Torsional rigidity.

239. Combined stresses include

(*a*) axial and bending
(*b*) torsion and bending
(*c*) torsion and axial
(*d*) all of the above.

240. Which of the following methods is employed to find the combined stresses produced by two or more different types of loading?

(*a*) Moment area method
(*b*) Method of superposition
(*c*) Macaulay's method
(*d*) Flexure method.

241. A circular shaft is usually subjected to the following stresses:

(*a*) axial and torsion
(*b*) axial and bending
(*c*) torsion and bending
(*d*) none of the above.

242. Structural members are usually subjected to stresses produced by

(*a*) axial load only
(*b*) axial load and bending moment
(*c*) axial load and torsion
(*d*) bending moment and torsion.

243. A hack saw may be subjected to stresses produced by

(*a*) axial and torsion
(*b*) axial and bending
(*c*) torsion and bending
(*d*) bending only.

244. Consider the following statements:

1. Maximum stress must be within the elastic limit.
2. Deformations produced should be small.
3. Load should always act in the vertical direction only.

While using method of superposition, which of the above statements should be valid?

(*a*) 1 alone (*b*) 1 and 2
(*c*) 2 and 3 (*d*) 1, 2 and 3.

245. In case of thick-walled cylindrical pressure vessels, the wall thickness exceeds the inner radius by

(*a*) less than 5%
(*b*) more than 5%
(*c*) more than 10%
(*d*) less than 10%.

246. Pure torsion produces direct stresses on planes inclined to the shaft at the following angle:

(*a*) 30° (*b*) 45°
(*c*) 60° (*d*) 90°.

ANSWERS

1. (*b*)	2. (*c*)	3. (*d*)	4. (*c*)	5. (*b*)	6. (*b*)	7. (*b*)	8. (*b*)
9. (*c*)	10. (*d*)	11. (*d*)	12. (*d*)	13. (*b*)	14. (*b*)	15. (*b*)	16. (*b*)
17. (*c*)	18. (*b*)	19. (*c*)	20. (*d*)	21. (*c*)	22. (*c*)	23. (*b*)	24. (*b*)
25. (*c*)	26. (*a*)	27. (*b*)	28. (*b*)	29. (*b*)	30. (*d*)	31. (*b*)	32. (*c*)
33. (*a*)	34. (*d*)	35. (*b*)	36. (*a*)	37. (*d*)	38. (*c*)	39. (*c*)	40. (*c*)
41. (*b*)	42. (*b*)	43. (*c*)	44. (*b*)	45. (*c*)	46. (*a*)	47. (*b*)	48. (*c*)
49. (*b*)	50. (*a*)	51. (*a*)	52. (*d*)	53. (*c*)	54. (*c*)	55. (*c*)	56. (*b*)
57. (*c*)	58. (*a*)	59. (*b*)	60. (*d*)	61. (*d*)	62. (*b*)	63. (*d*)	64. (*a*)
65. (*c*)	66. (*c*)	67. (*b*)	68. (*c*)	69. (*a*)	70. (*b*)	71. (*c*)	72. (*c*)
73. (*b*)	74. (*c*)	75. (*c*)	76. (*a*)	77. (*c*)	78. (*c*)	79. (*a*)	80. (*b*)
81. (*c*)	82. (*b*)	83. (*a*)	84. (*c*)	85. (*c*)	86. (*b*)	87. (*d*)	88. (*b*)
89. (*c*)	90. (*c*)	91. (*c*)	92. (*d*)	93. (*c*)	94. (*a*)	95. (*b*)	96. (*c*)
97. (*b*)	98. (*a*)	99. (*b*)	100. (*b*)	101. (*b*)	102. (*c*)	103. (*a*)	104. (*d*)
105. (*c*)	106. (*b*)	107. (*a*)	108. (*a*)	109. (*d*)	110. (*c*)	111. (*b*)	112. (*d*)
113. (*b*)	114. (*c*)	115. (*c*)	116. (*d*)	117. (*c*)	118. (*b*)	119. (*a*)	120. (*c*)
121. (*c*)	122. (*b*)	123. (*b*)	124. (*c*)	125. (*c*)	126. (*a*)	127. (*a*)	128. (*c*)
129. (*d*)	130. (*b*)	131. (*b*)	132. (*b*)	133. (*b*)	134. (*c*)	135. (*d*)	136. (*a*)
137. (*b*)	138. (*c*)	139. (*a*)	140. (*a*)	141. (*a*)	142. (*a*)	143. (*b*)	144. (*b*)
145. (*a*)	146. (*c*)	147. (*b*)	148. (*c*)	149. (*c*)	150. (*c*)	151. (*a*)	152. (*d*)
153. (*c*)	154. (*b*)	155. (*c*)	156. (*a*)	157. (*a*)	158. (*c*)	159. (*b*)	160. (*c*)
161. (*d*)	162. (*b*)	163. (*c*)	164. (*c*)	165. (*c*)	166. (*c*)	167. (*b*)	168. (*a*)
169. (*c*)	170. (*b*)	171. (*a*)	172. (*b*)	173. (*b*)	174. (*d*)	175. (*c*)	176. (*c*)
177. (*b*)	178. (*d*)	179. (*d*)	180. (*b*)	181. (*c*)	182. (*b*)	183. (*c*)	184. (*a*)
185. (*b*)	186. (*b*)	187. (*a*)	188. (*c*)	189. (*d*)	190. (*b*)	191. (*a*)	192. (*d*)
193. (*b*)	194. (*a*)	195. (*c*)	196. (*a*)	197. (*d*)	198. (*c*)	199. (*c*)	200. (*b*)
201. (*c*)	202. (*d*)	203. (*c*)	204. (*c*)	205. (*d*)	206. (*b*)	207. (*b*)	208. (*c*)
209. (*a*)	210. (*a*)	211. (*b*)	212. (*b*)	213. (*b*)	214. (*b*)	215. (*d*)	216. (*c*)
217. (*b*)	218. (*b*)	219. (*a*)	220. (*b*)	221. (*c*)	222. (*c*)	223. (*c*)	224. (*c*)
225. (*b*)	226. (*b*)	227. (*c*)	228. (*c*)	229. (*a*)	230. (*c*)	231. (*b*)	232. (*b*)
233. (*c*)	234. (*c*)	235. (*a*)	236. (*d*)	237. (*d*)	238. (*b*)	239. (*d*)	240. (*b*)
241. (*c*)	242. (*b*)	243. (*b*)	244. (*b*)	245. (*c*)	246. (*b*)		

❑❑❑

Appendix A

IMPORTANT MATHEMATICAL RELATIONS

ALGEBRA

A quadratic equation having highest power of variable 2 is represented by

$$ax^2 + bx + c = 0$$

Its two roots are expressed as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The mathematical expression for binomial theorem is given as

(*a*) $(1 + x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \frac{n(n-1)\,(n-2)}{\lfloor 3} x^3 + \cdots$

where $|x| < 1$ and n can be any number; positive, negative or a fraction. In case, n is a positive integer, the expansion will have $(n + 1)$ terms and in other cases, the number of terms will be infinite.

(*b*) $(1 + x)^n = 1 + nx$, when $|x|$ is very small.

TRIGONOMETRY

Quadrant System (Fig. A.1)

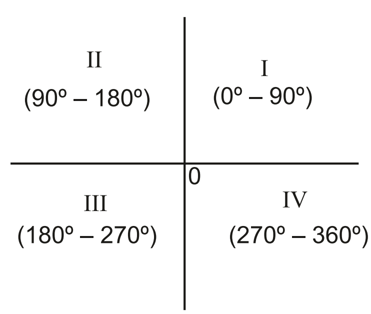


Fig. A.1 Four quadrants with their angle ranges.

D. K. Singh, *Strength of Materials*,
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Ist quadrant has the angle range of 0° – 90° and all the trigonometrical ratios namely sin θ, cos θ and tan θ, and their respective reciprocals cosec θ, sec θ and cot θ are positive in this quadrant.

IInd quadrant has the angle range of 90° – 180°. sin θ and its reciprocal cosec θ are positive and rest are negative in this quadrant.

IIIrd quadrant has the angle range of 180° – 270°. tan θ and its reciprocal cot θ are positive and rest are negative in this quadrant.

IVth quadrant has the angle range of 270° – 360°. cos θ and its reciprocal sec θ are positive and rest are negative in this quadrant.

$$\sin(90° + \theta) = \cos\theta$$

$$\cos(90° + \theta) = -\sin\theta$$

$$\tan(90° + \theta) = -\cot\theta$$

$$\sin(180° + \theta) = -\sin\theta$$

$$\cos(180° + \theta) = -\cos\theta$$

$$\tan(180° + \theta) = \tan\theta$$

$$\sin(270° + \theta) = -\cos\theta$$

$$\cos(270° + \theta) = \sin\theta$$

$$\tan(270° + \theta) = -\cot\theta$$

$$\sin(90° - \theta) = \cos\theta$$

$$\cos(90° - \theta) = \sin\theta$$

$$\tan(90° - \theta) = \cot\theta$$

$$\sin(180° - \theta) = \sin\theta$$

$$\cos(180° - \theta) = -\cos\theta$$

$$\tan(180° - \theta) = -\tan\theta$$

$$\sin(270° - \theta) = -\cos\theta$$

$$\cos(270° - \theta) = -\sin\theta$$

$$\tan(270° - \theta) = \cot\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

T-Ratios of Some Standard Angles

Angle (θ)	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$	1	$\frac{3}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
cos θ	1	$\frac{3}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	1
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	α	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

Important Trigonometrical Relations

$$\pi \text{ radian} = 180°$$

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta}$$

$$\cos\theta = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{1}{\cot\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

Note: The angle θ is taken to be positive, if it is measured anticlockwise and is negative if measured clockwise.

$$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$\operatorname{cosec}^{-1}\frac{1}{x} = \frac{\cot^{-1}\sqrt{1-x^2}}{x}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\cot(A \pm B) = \frac{\cot B . \cot A \mp 1}{\cot B \pm \cot A}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\tan 15° = 2-\sqrt{3}$$

$$\cot 15° = 2+\sqrt{3}$$

$$\sin 18° = \frac{\sqrt{5}-1}{4} = \cos 72°$$

$$\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°$$

$$\sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°$$

$$\cos 36° = \frac{\sqrt{5}+1}{4} = \sin 54°$$

Important Logarithmic Relations

$$\log (mn) = \log m + \log n$$

$$\log\left(\frac{m}{n}\right) = \log m - \log n$$

$$\log m^n = n \log m$$

$$\log_b m = \frac{\log_a m}{\log_a b}$$

CO-ORDINATE GEOMETRY

Conic Section

Conic section or conic is the locus of a point moving in a plane in such a way that the ratio of its distances from a fixed point and a fixed straight line is always constant.

The fixed point is called the *focus* and fixed straight line, the *directrix*. The ratio of the two distances is called *eccentricity* (e). It is always less than 1 for ellipse, equal to 1 for parabola and greater than 1 for hyperbola.

The line passing through the focus and perpendicular to the *directrix* is called the *axis*. The point at which the conic cuts its axis is called the *vertex*.

Standard Equation of the Ellipse (Fig. A.2)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a > b \text{ and } b^2 = a^2 (1 - e^2)$$

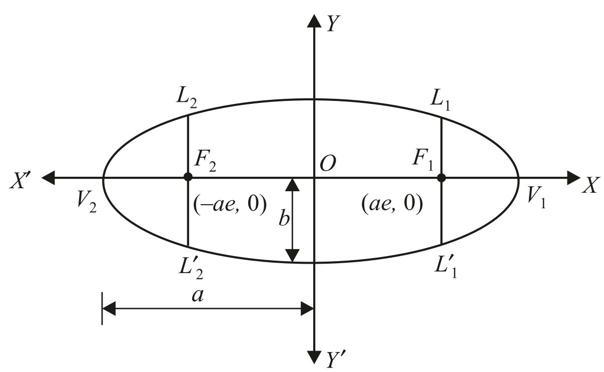

Fig. A.2 An ellipse.

V_1 and V_2 are vertices.

a is semi-major axis.

b is semi-minor axis.

F_1 and F_2 are focii.

$L_1 L_1'$ and $L_2 L_2'$ are latus recta.

Equation of a Circle

Standard Forms

1. $$(x - h)^2 + (y - k)^2 = r^2$$

where (h, k) is the centre and r is the radius.

2. $$x^2 + y^2 = r^2$$

where origin (0, 0) is the centre and r is the radius of the circle.

General Form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where centre is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

Diameter Form

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

where (x_1, y_1) and (x_2, y_2) are the co-ordinates of the ends of the diameter.

Parametric Form

$$x = h + r\cos\theta$$
$$y = k + r\sin\theta,\ 0 \le \theta \le 2\pi$$

where centre is (h, k) and r is the radius.

Standard Equation of the Parabola (Fig. A.3)

$$y^2 = 4ax$$

The parabola is symmetrical about the x-axis.

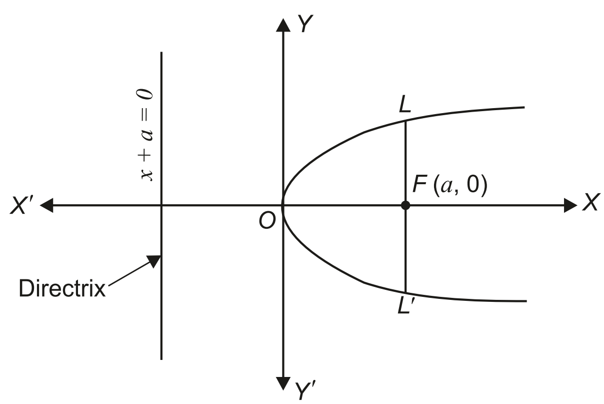

Fig. A.3 A parabola.

Focus is $F(a, 0)$.

Vertex is origin $O(0, 0)$.

XOX' is the axis of parabola.

LL' is the latus rectum $= 4a$.

Standard Equation of the Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

CALCULUS

Important Differential Relations

- If $y = f(x)$,

  then $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

- $\dfrac{d(c)}{dx} = 0$, where c = Constant

- If $y = uv$ and $u = f(x)$, $v = f(x)$,

  then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

- If $y = \dfrac{u}{v}$ and $u = f(x)$, $v = f(x)$,

  then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

- If $y = u \pm v \pm w$ and u, v and w are functions of x,

  then $\dfrac{dy}{dx} = \dfrac{du}{dx} \pm \dfrac{dv}{dx} \pm \dfrac{dw}{dx}$

- If $y = x^n$, where n is a real number

  then $\dfrac{dy}{dx} = nx^{n-1}$

- If $y = u^n$, and $u = f(x)$

  then $\dfrac{dy}{dx} = nu^{n-1}\dfrac{du}{dx}$

- $\dfrac{d(\sin x)}{dx} = \cos x$, $\dfrac{d(\cos x)}{dx} = -\sin x$

- $\dfrac{d(\tan x)}{dx} = \sec^2 x$, $\dfrac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

- $\dfrac{d(\sec x)}{dx} = \sec x \tan x$, $\dfrac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x$

- $\dfrac{d(e^x)}{dx} = e^x$

- $\dfrac{d(\log_e x)}{dx} = \dfrac{1}{x}$
- $\dfrac{d(a^x)}{dx} = a^x \log_e a$
- $\dfrac{d(\log_a x)}{dx} = \dfrac{1}{x} \log_a e$

Important Integral Relations

- $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c, n \neq -1$
- $\int e^x dx = e^x + c$
- $\int a^x dx = \dfrac{a^x}{\log_e a} + c$
- $\int \sin x \, dx = -\cos x + c$
- $\int \cos x \, dx = \sin x + c$
- $\int \sec^2 x \, dx = \tan x + c$
- $\int \text{cosec}^2 x \, dx = -\cot x + c$
- $\int \sec x \tan x \, dx = \sec x + c$
- $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x + c$
- $\int \dfrac{1}{1+\sin x} dx = \tan x - \sec x + c$
- $\int \dfrac{1}{1-\sin x} dx = \tan x + \sec x + c$
- $\int \dfrac{1}{1+\cos x} dx = -\cot x + \text{cosec}\, x + c$
- $\int \dfrac{1}{1-\cos x} dx = -(\cot x + \text{cosec}\, x) + c$

Note: In all the above integrals, c is the constant of integration.

Appendix B

IMPORTANT SI UNITS

Quantity	**Unit**
Acceleration	m/s^2
Angle	rad
Angular acceleration	rad/s^2
Angular velocity	rad/s
Area	m^2
Density	kg/m^3
Energy or Work	J (joule = N.m)
Force	N
Frequency	Hz
Impulse	N.s
Length	m
Mass	kg
Moment	N.m
Power	W (watt = J/s)
Pressure or Stress	Pa (pascal = N/m^2)
Time	s
Velocity	m/s
Volume	m^3

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IMPORTANT CONVERSIONS

1 kgf	=	9.81 N
1 lbf	=	4.448 N
1 cm	=	10 mm
1 m	=	100 cm = 1000 mm
1 kg	=	1000 g
1 lb	=	0.453 kg
1 hp	=	740 W
1 Psi (lb/inch^2)	=	6.895 kPa
1 GPa	=	10^9 Pa
1 MPa	=	10^6 Pa
1 kPa	=	10^3 Pa
1 cm^2	=	10^{-4} m^2
1 mm^2	=	10^{-6} m^2
1 km	=	1000 m
1 gal	=	3.785 litre

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Subject Index

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