A Primal-Dual Randomized Algorithm for the Online Weighted Set Multi-cover Problem

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Abstract. Given a ground set $\mathcal{U}$ of n elements and a family of m subsets $\mathcal{S} = \{S_i : S_i \subseteq \mathcal{U}\}$. Each subset $S \in \mathcal{S}$ has a positive cost $c(S)$ and every element $e \in \mathcal{U}$ is associated with an integer coverage requirement $r_e > 0$, which means that e has to be covered at least r_e times. The *weighted set multi-cover problem* asks for the minimum cost subcollection which covers all of the elements such that each element e is covered at least r_e times.

In this paper, we study the online version of the weighted set multi-cover problem. We give a randomized algorithm with competitive ratio $8(1+\ln m)\ln n$ for this problem based on the primal-dual method, which improve previous competitive ratio $12\log m \log n$ for the online set multi-cover problem that is the special version where each cost $c(S)$ is 1 for every subset S.

1 Introduction

The *weighted set multi-cover* problem is the generalization of the set cover problem, which is defined as follows. Given a ground set $\mathcal{U} = \{1, \ldots, n\}$ of n elements and a family of m subsets $\mathcal{S} = \{S_i : 1 \le i \le m\}$, where $S_i \subseteq \mathcal{U}$ for all i. Each subset $S \in \mathcal{S}$ has a positive cost $c(S)$ and every element $e \in \mathcal{U}$ is associated with an integer coverage requirement $r_e > 0$, which means that e has to be covered at least r_e times. The goal is to find a minimum cost subcollection that covers all of the elements such that each element e is covered at least specified times r_e. When all $r_e = 1$, the set multi-cover problem becomes the set cover problem. Let $R = \max_{e \in \mathcal{U}} r_e$. We assume that $R = O(n)$.

Similarly, the *online weighted set multi-cover* problem is the generalization of the online set cover problem, which is described as follows. An adversary gives elements and their coverage requirement to the algorithm from $\mathcal{U}$ one-by-one. When a new element e and its coverage requirement r_e are given, the algorithm has to cover it at least r_e times by choosing some sets of $\mathcal{S}$ containing it. We assume that the elements of $\mathcal{U}$ and the coverage requirement of elements and

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the members of $\mathcal{S}$ are known in advance to the algorithm, however, the set of elements given by the adversary is not known in advance to the algorithm. The objective is to minimize the total cost of the sets chosen by the algorithm.

The performance of an online algorithm is measured by the competitive ratio, which is defined as follows. Given an instance I of a minimization optimization problem M. Let $OPT(I)$ denote the optimum cost of off-line algorithms for instance I. If for each instance I of M, an online algorithm OA outputs a solution with cost at most $c \cdot OPT(I) + \alpha$, where α is a constant independent of the input sequence, then the competitive ratio of OA is c. If for each instance I of M, a randomized online algorithm ROA outputs a solution with expected cost at most $c \cdot OPT(I) + \alpha$, where α is independent of the input sequence, then the competitive ratio of ROA is c.

The set cover problem has wide application and is a well-known problem in algorithms and complexity. In [11], Karp shows that the set cover problem is NP-compete. Johnson [10] and Lovasz [13] give the greedy approximation algorithm for the unweighted set cover problem. Chvatal [7] proposes the greedy approximation algorithm for the weighted set cover problem. These greedy algorithms are of approximation ratio H_n, where $H_n = 1 + 1/2 + \ldots + 1/n$. Lund and Yannakakis show that the approximation ratio $O(\log n)$ for the set cover problem is essentially tight [14]. Later, Feige proves that it is impossible to have an approximation algorithm for the set cover problem with approximation ratio better than $O(\log n)$ [8]. Rajagopalan and Vazirani propose primal-dual RNC approximation algorithms for the set mullti-cover and covering integer programs problems [15]. Noga Alon et al. study the online set cover problem. Based on the techniques from computational learning theory, Noga Alon et al. propose a deterministic algorithm for this problem with competitive ratio $O(\log m \log n)$ [1]. The set cover problem is related to the budgeted maximum coverage problem, which is a flexible model for many applications [16–20].

In the areas of exact and approximation algorithms, the primal-dual method is one of powerful design methods. To our best of knowledge, the first time that the primal-dual method is used to the design of online algorithms is in Young' work about weighted paging [21], where he design an k-competitive online algorithm. In recent several years, Buchbinder and Naor have shown that the primal-dual method can be widely used to the design and analysis of online algorithms for many problems such as ski-rental, ad-auctions, routing and network optimization problems and so on [2–6].

In [12], Kuhnle et al. introduce the online set multi-cover problem and design randomized algorithms with $12 \log m \log n$-competitive ratio. In this paper, we study the online weighted set multi-cover problem. We present an $8(1+\ln m)\ln n$ competitive randomized algorithm for this problem based on the primal-dual method. Specially, when each cost $c(S)$ is 1 for every subset S, the online weighted set multi-cover problem become the online set multi-cover problem. Thus, our algorithm improve Kuhnle et al.'s competitive ratio for the online set multi-cover problem.

2 A Fractional Primal-Dual Algorithm For the Online Weighted Set Multi-cover Problem

In this section, we design a fractional algorithm for the online weighted set multi-cover problem via the primal-dual method. A fractional algorithm allows an element e is fractionally covered its f_S part by a set S such that $\sum_{e \in S} f_S = 1$.

First, the weighted set multi-cover problem can be formulated as a 0–1 integer program as follows.

$$\begin{array}{|ll|}\hline \text{Minimize} & \sum_{S \in \mathcal{S}} c(S) x_S \\ \text{Subject to} & \sum_{S : e \in S} x_S \geq r_e,\ e \in \mathcal{U} \\ & x_S \in \{0, 1\},\ S \in \mathcal{S} \\ \hline \end{array}$$

Its Linear Programs relaxation is as follows.

$$\begin{array}{|ll|}\hline \text{Minimize} & \sum_{S \in \mathcal{S}} c(S) x_S \\ \text{Subject to} & \sum_{S : e \in S} x_S \geq r_e,\ e \in \mathcal{U} \\ & -x_S \geq -1,\ S \in \mathcal{S} \\ & x_S \geq 0,\ S \in \mathcal{S} \\ \hline \end{array}$$

Its Dual Programs is as follows

$$\begin{array}{|ll|}\hline \text{Maximize} & \sum_{e \in U} r_e y_e - \sum_{S \in \mathcal{S}} z_S \\ \text{Subject to} & \sum_{e \in S} y_e - z_S \leq c(S),\ S \in \mathcal{S} \\ & y_e \geq 0,\ e \in \mathcal{U} \\ & z_S \geq 0,\ S \in \mathcal{S} \\ \hline \end{array}$$

In the following, we design the online fractional algorithm for the weighted set multi-cover problem via the primal-dual design method developed in recent years [2–6] (see Algorithm 2.1).

1: At time t, when an element e with coverage requirement r_e arrives:
2: If the primal constraints $\sum_{S : e \in S} x_S \geq r_e$ corresponding to e is satisfied, then do nothing.
3: Otherwise, do the following:
4: While $\sum_{S : e \in S} x_S < r_e$:
5: Continuously increase y_e .
6: If $x_S = 0$ and $(\sum_{e \in S} y_e) - z_S = c(S)$, then set $x_S \leftarrow \frac{1}{m}$.
7: If $\frac{1}{m} \leq x_S < 1$, then x_S increase by the following function: $x_S \leftarrow \frac{1}{m} \cdot \exp(\frac{1}{c(S)}[(\sum_{e \in S} y_e) - z_S - c(S)])$.
8: If $x_S = 1$, then z_S is increased at the same ratio as y_e.

Algorithm 2.1: The online fractional algorithm for the weighted set multi-cover problem.

Theorem 1. *The fractional online algorithm for the weighted set multi-cover problem is of competitive ratio* $2(1 + \ln m)$.

Proof. Let P denote the value of the objective function of the primal solution and D denote the value of the objective function of the dual solution. Initially, let $P = 0$ and $D = 0$. In the following, we prove three claims:

(1) The primal solution produced by the fractional algorithm is feasible.
(2) Every dual constraint in the dual program is violated by a factor of at most $(1 + \ln m)$.
(3) $P \leq 2D$.

By three claims and weak duality of linear programs, the theorem follows immediately.

First, we prove the claim (1) as follows. Consider a primal constraint $\sum\limits_{S:e \in S} x_S \geq r_e$. In each *While* iteration (From line 5 to line 8 in the fractional algorithm), when this new primal constraint $\sum\limits_{S:e \in S} x_S \geq r_e$ becomes be satisfied, the variable x_S stop increasing its value and its value is not greater than 1. Upon x_S become 1, the fractional algorithm begin to increase z_S and y_e at the same ratio. After that, the increases of z_S and y_e cannot result in infeasibility.

Second, we prove the claim (2) as follows. Consider any dual constraint $\sum\limits_{e \in S} y_e - z_S \leq c(S)$. Since its corresponding variable x_S is not greater than 1, we get that:

$x_S = \frac{1}{m} \cdot \exp(\frac{1}{c(S)}[(\sum\limits_{e \in S} y_e) - z_S - c(S)]) \leq 1.$

So $\exp(\frac{1}{c(S)}[(\sum\limits_{e \in S} y_e) - z_S - c(S)]) \leq m.$

Then, $(\sum\limits_{e \in S} y_e) - z_S - c(S) \leq c(S) \ln m.$

Thus, we get that: $(\sum\limits_{e \in S} y_e) - z_S \leq c(S)(1 + \ln m).$

Third, we prove claim (3) as follows. The contribution to the primal cost consists of two parts. Let C_1 denote the contribution part which is from (6) of the fractional algorithm, where variables x_S are increased from $0 \rightarrow \frac{1}{m}$. Let C_2 denote the other contribution part which is from (7) of the fractional algorithm, where variables x_S are increased from $\frac{1}{m}$ up to at most 1 by the exponential function.

Bounding C_1: Let $\tilde{x}_S = \min(x_S, \frac{1}{m})$. We bound the term $\sum\limits_{S \in \mathcal{S}} c(S)\tilde{x}_S$. To do this, we need the following several facts.

First, from the fractional algorithm, we get that if $x_S > 0$, and therefore $\tilde{x}_S > 0$, then:

$$\sum_{e \in S} y_e - z_S \geq c(S). \tag{1}$$

We call (1) as the primal complementary slackness condition.

At the time t, let $B'(S) = \{S | x_S = 1, e \in S\}$. Then $|B'(S)| \leq r_e$ since otherwise the constraint at time t has been already satisfied and the fractional algorithm stops increasing the variable y_e. Thus, $(m-1)|B'(S)| \le (m-1)r_e$. So $\frac{m-|B'(S)|}{m} \le r_e - |B'(S)|$. Since $\tilde{x}_S \le \frac{1}{m}$, $\sum_{S \in \mathcal{S} \setminus B'(S)} \tilde{x}_S \le \frac{m-B'(S)}{m}$. Hence

$$\sum_{S \in \mathcal{S} \setminus B'(S)} \tilde{x}_S \le r_e - |B'(S)| \tag{2}$$

Also, it follows from the algorithm that if $z_S > 0$, then:

$$x_S \ge 1. \tag{3}$$

We call (2) as the dual complementary slackness and (3) as the second dual complementary slackness condition.

Using the primal and dual complementary slackness conditions, we show the following conclusions:

$$\sum_{S \in \mathcal{S}} c(S)\tilde{x}_S$$

$$\le \sum_{S \in \mathcal{S}} (\sum_{e \in S} y_e - z_S)\tilde{x}_S \tag{4}$$

$$= \sum_{S \in \mathcal{S}} (\sum_{e \in S} y_e \tilde{x}_S) - \sum_{S \in \mathcal{S}} z_S \tilde{x}_S \tag{5}$$

$$= \sum_{e} (\sum_{S: e \in S} \tilde{x}_S) y_e) - \sum_{S \in \mathcal{S}} z_S \tilde{x}_S \tag{6}$$

$$\le \sum_{e} r_e y_e - \sum_{S \in \mathcal{S}} z_S \tag{7}$$

Where inequality (4) follows from inequality (1) and equality (6) follows by changing the order of summation. As for the reason why inequality (7) holds, we consider some time t. At the time t when e with coverage requirement r_e arrive. From the fractional algorithm, we know that z_S is increased at the same ratio as y_e only when $x_S = 1$. Thus, $\frac{dy_e}{dt} = \frac{dz_S}{dt}$ only when $S \in B'(S)$. Hence, the increasing ratio of the left-hand side of (7) at the time t is $(\sum_{S \in \mathcal{S} \setminus B'(S)} \tilde{x}_S)\frac{dy_e}{dt}$. But, at the time t, the increasing ratio of the right-hand side of (7) is $(r_e - |B'(S)|)\frac{dy_e}{dt}$. By inequality (2), we get $(\sum_{S \in \mathcal{S} \setminus B'(S)} \tilde{x}_S)\frac{dy_e}{dt} \le (r_e - |B'(S)|)\frac{dy_e}{dt}$. So inequality (7) holds

Thus, C_1 is at most D.

Bounding C_2 : At some time t, we show that the increase ΔC_2 is most ΔD in the same round.

$$\Delta C_2 = \sum_{S \in \mathcal{S}, \frac{1}{m} \le x_S < 1} c(S) \cdot \Delta x_S \tag{8}$$

From the line 7 of the fractional algorithm, we get that $\frac{dx_S}{dy_e} = \frac{1}{c(S)} \cdot x_S$. So, $\Delta x_S = \frac{1}{c(S)} \cdot x_S \cdot \Delta y_e$. Thus, we get that:

$$\Delta C_2 = (\sum_{S \in \mathcal{S}, \frac{1}{m} \leq x_S < 1} x_S) \cdot \Delta y_e \tag{9}$$

At the time t, the new primal constraints are not yet satisfied, so we get that: $\sum_{S \in \mathcal{S}, \frac{1}{m} \leq x_S < 1} x_S + \sum_{x_S = 1} 1 < r_e$. Thus, $\sum_{S \in \mathcal{S}, \frac{1}{m} \leq x_S < 1} x_S < r_e - \sum_{x_S = 1} 1$. Hence,

$$\Delta C_2 \leq (r_e - \sum_{x_S = 1} 1) \cdot \Delta y_e \tag{10}$$

From the line 8 of the fractional algorithm, $\Delta y_e = \Delta z_S$ when $x_S = 1$ in the same sound at the time t. So,

$$\Delta C_2 \leq r_e \cdot \Delta y_e - \sum_{x_S = 1} \Delta z_S = \Delta D \tag{11}$$

Thus, $C_2 \leq D$.

Hence, we get that $P = C_1 + C_2 \leq 2D$. So, claim (3) holds. Furthermore, the theorem holds. □

3 Randomized Algorithm for the Online Weighted Set Multi-cover Problem

In this section, we design a randomized algorithm for the online weighted set multi-cover problem with competitive ratio $8(1 + \ln m) \ln n$.

1: For each set $S \in \mathcal{S}$, $4 \ln n$ independently random variables $V(S, i)$ are uniformly chosen from $[0, 1]$ at random.
2: For every set $S \in \mathcal{S}$, let $\varepsilon(S) = \min_{i=1}^{4 \ln n} V(S, i)$.
3: At time t,a new element e and its cover requirement r_e arrives. Let c_e is the times that e has been covered at time t and let $u_e = r_e - c_e$. If $c_e \geq r_e$, then do nothing.
4: Otherwise, we use Algorithm 2.1 to compute the values of x_S in the unsatisfied primal constraint that corresponds to e, and let $\mathcal{C}$ denote the cover set, then do the following:
5: **for** $j = 1$ to u_e **do**
6: For all unchosen sets $S \in \mathcal{S} \backslash \mathcal{C}$ that appears in the unsatisfied primal constraint that corresponds to e, when $x_S \geq \varepsilon(S)$, take one of these sets to the cover $\mathcal{C}$.
7: $\mathcal{S} \leftarrow \mathcal{S} \backslash \{S\}$; $\mathcal{C} \leftarrow \mathcal{C} \cup S$.
8: **end for**

Algorithm 3.1: The randomized online algorithm for the weighted set multi-cover problem.

Theorem 2. *The randomized algorithm is of competitive ratio* $8(1 + \ln m) \ln n$.

Proof. First, we show that the randomized algorithm produces a feasible solution with high probability $1 - O(\frac{1}{n^2}) > \frac{1}{2}$.

Consider any an element e, assume that it appears at time t. let A_i denote the event that e isn't covered in the i-th round from 5-th to 7-th line in the randomized algorithm. Let $\mathcal{S}_{t_i}$ denote the unchosen sets of $\mathcal{S}$ and $\mathcal{C}_{t_i}$ denote the chosen sets at the beginning of in the i-th round. $c(e, t_i)$ denote the number of e has been covered at the beginning of in the i-th round.

Then, we compute the probability that A_i occurs. Consider any $j(1 \leq j \leq 4 \ln n)$, let D_j denote the event that e is not covered due to j, which means that for all unchosen sets $S \in \mathcal{S}_{t_i}$ and $e \in S$, none of the value of $V(S, j)$ is less than x_S. Thus, $Pr(A_i = 1) = \bigcap\limits_{1 \leq j \leq 4 \ln n} Pr(D_j = 1)$

The probability $Pr(V(S, i) \leq x_S)$ is x_S. So $Pr(D_j = 1) = \prod\limits_{S \in \mathcal{S}_{t_i} | e \in S} (1 - x_S)$. Since $1 - x \leq exp(-x)$, we get that: $Pr(D_j = 1) \leq exp(-\sum\limits_{S \in \mathcal{S}_{t_i} | e \in S} x_S)$. Since all x_S consist of a fractional solution after the fractional algorithm, we get that $\sum\limits_{S \in \mathcal{S} : e \in S} x_S \geq r_e$. Thus, $\sum\limits_{S \in \mathcal{S}_{t_i} | e \in S} x_S + \sum\limits_{S \in \mathcal{C}_{t_i} | e \in S} x_S \geq r_e$. So $\sum\limits_{S \in \mathcal{S}_{t_i} | e \in S} x_S \geq r_e - \sum\limits_{S \in \mathcal{C}_{t_i} | e \in S} x_S = r_e - c(e, t_i)$. Hence, $Pr(D_j = 1) \leq exp(-\sum\limits_{S \in \mathcal{S}_{t_i} | e \in S} x_S) \leq exp(-n_i)$, where $n_i = r_e - c(e, t_i)$. So, $Pr(D_j = 1) \leq exp(-1)$. Hence, $Pr(A_i = 1) \leq (exp(-1))^{4 \ln n} = exp(-4 \ln n) = \frac{1}{n^4}$.

So, the probability that e is not covered r_e times is $Pr(A_1 = 1 \vee \ldots \vee A_{u_e} = 1) \leq \sum_{i=1}^{u_e} Pr(A_i = 1) \leq \sum_{i=1}^{u_e} \frac{1}{n^4} = \frac{n_e}{n^4} \leq \frac{r_e}{n^4} \leq \frac{R}{n^4} \leq \frac{O(n)}{n^4} = O(\frac{1}{n^3})$.

By the union bound that the probability of union events is at most the sum of the probability of each event, the probability that there is an element e which is not covered r_e times is at most $n \times O(\frac{1}{n^3}) = O(\frac{1}{n^2})$ since there are at most n elements.

Hence, the randomized algorithm produces a feasible solution with high probability $1 - O(\frac{1}{n^2}) > \frac{1}{2}$.

Second, we show that the expected cost of the solution of randomized algorithms is $O(\log n)$ times the fractional solution.

Let B_i denote the event that $V(S, i) \leq x_S$. Then, $Pr(B_i = 1) = x_S$. The probability that the set S is chosen to the solution is at most the probability that there exists an $i, 1 \leq i \leq 4 \ln n$, such that $V(S, i) \leq x_S$.

Thus, the probability that S is chosen to the solution is at most the probability of $\bigcup_{i=1}^{4 \ln n} B_i$. By the union bound this probability is at most the sum of the probabilities of the different events, which is $4 x_S \ln n$. Therefore, using the linearity of expectation, the expected cost of the solution is at most $4 \ln n$ times the cost of the fractional solution.

By Theorem 1, the cost of the fractional solution is $2(1 + \ln m)$ times the optimal solution. So the competitive ratio of the randomized algorithm is $8(1 + \ln m) \ln n$. □

4 Conclusion

In this paper, we have studied the online version of the weighted set multi-cover problem. We have proposed a $8(1+\ln m)\ln n$-competitive randomized algorithm for this problem based on the primal-dual method. An interesting open problem is to design deterministic algorithms for the online weighted set multi-cover problem.

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