Optimal Matroid Bases with Intersection Constraints: Valuated Matroids, M-convex Functions, and Their Applications

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Abstract. For two matroids M_1 and M_2 with the same ground set V and two cost functions w_1 and w_2 on 2^V, we consider the problem of finding bases X_1 of M_1 and X_2 of M_2 minimizing $w_1(X_1) + w_2(X_2)$ subject to a certain cardinality constraint on their intersection $X_1 \cap X_2$. Lendl, Peis, and Timmermans (2019) discussed modular cost functions: They reduced the problem to weighted matroid intersection for the case where the cardinality constraint is $|X_1 \cap X_2| \leq k$ or $|X_1 \cap X_2| \geq k$; and designed a new primal-dual algorithm for the case where $|X_1 \cap X_2| = k$. The aim of this paper is to generalize the problems to have nonlinear convex cost functions, and to comprehend them from the viewpoint of discrete convex analysis. We prove that each generalized problem can be solved via valuated independent assignment, valuated matroid intersection, or M-convex submodular flow, to offer a comprehensive understanding of weighted matroid intersection with intersection constraints. We also show the NP-hardness of some variants of these problems, which clarifies the coverage of discrete convex analysis for those problems. Finally, we present applications of our generalized problems in matroid congestion games and combinatorial optimization problems with interaction costs.

Keywords: Valuated independent assignment · Valuated matroid intersection · M-convex submodular flow · Matroid congestion game · Combinatorial optimization problem with interaction costs

1 Introduction

Weighted matroid intersection is one of the most fundamental combinatorial optimization problems solvable in polynomial time. This problem generalizes

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a number of tractable problems including maximum-weight bipartite matching and minimum-weight arborescence. The comprehension of mathematical structures of weighted matroid intersection, e.g., Edmonds' intersection theorem [4] and Frank's weight splitting theorem [5], contributes to the development of polyhedral combinatorial optimization as well as matroid theory.

Recently, Lendl, Peis, and Timmermans [9] have introduced the following variants of weighted matroid intersection, in which a cardinality constraint is imposed on the intersection. Let V be a finite set, n a positive integer, and $[n] := \{1, 2, \ldots, n\}$. For each $i \in [n]$, let $M_i = (V, \mathcal{B}_i)$ be a matroid with ground set V and base family of $\mathcal{B}_i$, and w_i a modular function on 2^V. Let k be a nonnegative integer. The problems are formulated as follows.

$$\begin{array}{ll} \text{Minimize} & w_1(X_1) + w_2(X_2) \\ \text{subject to} & X_i \in \mathcal{B}_i \quad (i = 1, 2), \\ & |X_1 \cap X_2| = k. \end{array} \tag{1.1}$$

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^{n} w_i(X_i) \\ \text{subject to} & X_i \in \mathcal{B}_i \quad (i \in [n]), \\ & \left|\displaystyle\bigcap_{i=1}^{n} X_i\right| \leq k. \end{array} \tag{1.2}$$

They further discussed the following problem for polymatroids. Let $B_1, B_2 \subseteq \mathbf{Z}^V$ be the base polytopes of some polymatroids on the ground set V. Here, w_1 and w_2 are linear functions on $\mathbf{Z}^V$.

$$\begin{array}{ll} \text{Minimize} & w_1(x_1) + w_2(x_2) \\ \text{subject to} & x_i \in B_i \quad (i = 1, 2), \\ & \displaystyle\sum_{v \in V} \min\{x_v, y_v\} \geq k. \end{array} \tag{1.3}$$

Lendl et al. [9] showed that the problems (1.1)–(1.3) are polynomial-time solvable. They developed a new primal-dual algorithm for the problem (1.1), and reduced the problems (1.2) and (1.3) to existing tractable problems of weighted matroid intersection and *polymatroidal flow*, respectively. By this result, they affirmatively settled an open question on the polynomial-time solvability of the *recoverable robust matroid basis problem* [7].

The aim of this paper is to provide a comprehensive understanding of the result of Lendl et al. [9] in view of *discrete convex analysis* (*DCA*) [14,17], particularly focusing on *M-convexity* [11]. DCA offers a theory of convex functions on the integer lattice $\mathbf{Z}^V$, and M-convexity, a quantitative generalization of matroids, plays the central roles in DCA. M-convex functions naturally appear in combinatorial optimization, economics, and game theory [18,19].

The formal definition of M-convex functions is given as follows. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *M-convex* if it satisfies the following generalization of the matroid exchange axiom: for all $x = (x_v)_{v \in V}$ and $y = (y_v)_{v \in V}$ with $x, y \in \mathrm{dom} f$, and all $v \in V$ with $x_v > y_v$, there exists $u \in V$ with $x_u < y_u$ such that $f(x) + f(y) \geq f(x - \chi_v + \chi_u) + f(y + \chi_v - \chi_u)$, where $\mathrm{dom} f$ denotes the effective domain $\{x \in \mathbf{Z}^V \mid f(x) < +\infty\}$ of f and χ_v the v-th unit vector for $v \in V$. In particular, if $\mathrm{dom} f$ is included in the hypercube $\{0,1\}^V$, then f is called a *valuated matroid*[1] [2,3].

We address M-convex (and hence nonlinear) generalizations of the problems (1.1)–(1.3). Let $\omega_1, \omega_2, \ldots, \omega_n$ be valuated matroids on 2^V, where we identify 2^V with $\{0,1\}^V$ by the natural correspondence between $X \subseteq V$ and $x \in \{0,1\}^V$; $x_v = 1$ if and only if $v \in X$.

– For the problem (1.1), by generalizing the modular cost functions w_1 and w_2 to valuated matroids, we obtain:

$$\begin{array}{ll} \text{Minimize} & \omega_1(X_1) + \omega_2(X_2) \\ \text{subject to} & |X_1 \cap X_2| = k. \end{array} \tag{1.4}$$

– For the problem (1.2), as well as generalizing $w_1, w_2, \ldots, w_n$ to valuated matroids, we generalize the cardinality constraint $|\bigcap_{i=1}^n X_i| \leq k$ to a matroid constraint. Namely, let $M = (V, \mathcal{I})$ be a new matroid, where $\mathcal{I}$ denotes its independent set family, and generalize (1.2) as follows.

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^n \omega_i(X_i) \\ \text{subject to} & \displaystyle\bigcap_{i=1}^n X_i \in \mathcal{I}. \end{array} \tag{1.5}$$

– It is also reasonable to take the cardinality constraint into the objective function. Let $w \colon V \to \mathbf{R}$ be a weight function. The next problem is a variant of the above problem.

$$\text{Minimize} \sum_{i=1}^n \omega_i(X_i) + w\left(\bigcap_{i=1}^n X_i\right). \tag{1.6}$$

– Let f_1 and f_2 be M-convex functions on $\mathbf{Z}^V$ such that $\mathrm{dom} f_1$ and $\mathrm{dom} f_2$ are included in $\mathbf{Z}_+^V$, where $\mathbf{Z}_+$ is the set of nonnegative integers. Also let $w \colon \mathbf{Z}^V \to \mathbf{R}$ be a linear function. The problem (1.3) is generalized as follows.

$$\begin{array}{ll} \text{Minimize} & f_1(x) + f_2(y) + w(\min\{x, y\}) \\ \text{subject to} & \displaystyle\sum_{v \in V} \min\{x_v, y_v\} \geq k, \end{array} \tag{1.7}$$

where $\min\{x, y\} := (\min\{x_v, y_v\})_{v \in V}$.

[1] The original definition of a valuated matroid is an *M-concave function*, i.e., the negative of an M-convex function, such that its effective domain is included in the hypercube.

Our main contribution is to show the tractability of the generalized problems (1.4)–(1.7):

Theorem 1. *There exist polynomial-time algorithms to solve the problems* (1.4)*,* (1.5)*,* (1.6) *for* $w \geq 0$*, and* (1.7) *for* $w \leq 0$*.*

The algorithm for the problem (1.4) is based on *valuated independent assignment* [12,13], that for (1.5) and (1.6) on *valuated matroid intersection* [12,13], and that for (1.7) on $M^\natural$*-convex submodular flow* [15]. It would be noteworthy that we essentially require the concept of valuated matroid intersection to solve the problem (1.6) even if ω_i is a modular function for each $i \in [n]$. That is, the problem (1.6) with modular functions ω_i ($i \in [n]$) is an interesting example which only requires matroids to define, but requires valuated matroids to solve. It might also be interesting that the problem (1.5) can be solved in polynomial time when $n \geq 3$, in spite of the fact that matroid intersection for more than two matroids is NP-hard.

We also demonstrate that the tractability of the problems (1.6) and (1.7) relies on the assumptions on w ($w \geq 0$ and $w \leq 0$, respectively), by showing the NP-hardness of the problems.

Theorem 2. *The problems* (1.6) *and* (1.7) *are NP-hard in general even if* $w \leq 0$ *and* $m \geq 3$ *for* (1.6)*, and* $w \geq 0$ *and* $k = 0$ *for* (1.7)*.*

We then present applications of our generalized problems to *matroid congestion games* [1] and *combinatorial optimization problems with interaction costs* (*COPIC*) [8]. We show that computing the socially optimal state in a certain generalized model of matroid congestion games can be reduced to (a generalized version of) the problem (1.6), and thus can be done in polynomial time. We also reduce a certain generalized case of the COPIC with *diagonal costs* to (1.6) and (1.7), to provide a generalized class of COPIC which can be solved in polynomial time.

The proofs are omitted due to space constraint; see the full version for the proofs.

2 Algorithms

In this section, we provide polynomial-time algorithms for the problems (1.4)–(1.7) to prove Theorem 1. Theorem 2 is also shown in this section.

We first prepare several facts and terminologies on M-convex functions. For an M-convex function f, all members in $\mathrm{dom} f$ have the same "cardinality," that is, there exists some integer r such that $\sum_{v \in V} x_v = r$ for all $x \in \mathrm{dom} f$. We call r the *rank* of f. An $M^\natural$*-convex function* [20] is a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by the following weaker exchange axiom: for all $x = (x_v)_{v \in V}$ and $y = (y_v)_{v \in V}$ with $x, y \in \mathrm{dom}\, f$, and all $v \in V$ with $x_v > y_v$, it holds that $f(x) + f(y) \geq f(x - \chi_v) + f(y + \chi_v)$, or there exists $u \in V$ with $x_u < y_u$ such that $f(x) + f(y) \geq f(x - \chi_v + \chi_u) + f(y + \chi_v - \chi_u)$. From the definition, it is clear that $\mathrm{M}^\natural$-convex functions are a slight generalization of M-convex functions. Meanwhile, $\mathrm{M}^\natural$-convexity and M-convexity are essentially equivalent concepts (see e.g., [17]).

2.1 Reduction of (1.4) to Valuated Independent Assignment

This subsection provides a polynomial-time algorithm for solving the problem (1.4). In [9], the authors developed a new algorithm specific to (1.1). In this article, we give a reduction of the generalized problem (1.4) to a known tractable problem of *valuated independent assignment* [12,13], building upon the DCA perspective.

Let $G = (V, V'; E)$ be a bipartite graph, ω and ω' valuated matroids on 2^V and on $2^{V'}$, respectively, and w a weight function on E. Then the *valuated independent assignment problem* parameterized by an integer k, referred to as VIAP(k), is described as follows.

$$\text{VIAP}(k) \qquad \begin{array}{ll} \text{Minimize} & \omega(X) + \omega'(X') + w(F) \\ \text{subject to} & F \subseteq E \text{ is a matching of } G \text{ with } \partial F \subseteq X \cup X', \\ & |F| = k, \end{array}$$

where ∂F denote the set of endpoints of F.

We first consider the following variant of the problem (1.4):

$$\begin{array}{ll} \text{Minimize} & \omega_1(X_1) + \omega_2(X_2) \\ \text{subject to} & |X_1 \cap X_2| \geq k, \end{array} \tag{2.1}$$

in which the constraint $|X_1 \cap X_2| = k$ is replaced by $|X_1 \cap X_2| \geq k$. The problem (2.1) can be naturally reduced to VIAP(k) as follows. Set the input bipartite graph G of VIAP(k) by $(V_1, V_2; \{\{v_1, v_2\} \mid v \in V\})$, where V_i is a copy of V and $v_i \in V_i$ is a copy of $v \in V$ for $i = 1, 2$. We regard ω_i as a valuated matroid on 2^{V_i}. Set $w := 0$ for all edges. Then consider VIAP(k) for such G, ω_1, ω_2, and w. One can see that, if (X_1, X_2) is feasible for the problem (2.1), i.e., $|X_1 \cap X_2| \geq k$, then there is a matching F of G with $\partial F \subseteq X_1 \cup X_2$ and $|F| = k$, i.e., (X_1, X_2, F) is feasible for VIAP(k). On the other hand, if (X_1, X_2, F) is feasible for VIAP(k), then (X_1, X_2) is feasible for the problem (2.1). Moreover the objective value of a feasible solution (X_1, X_2) for the problem (2.1) is equal to that of any corresponding feasible solution (X_1, X_2, F) for VIAP(k) since w is identically zero.

Thus the problem (2.1) can be solved in polynomial time in the following way based on the augmenting path algorithm for VIAP(k) [12,13]; see also [16, Theorem 5.2.62]. Let X_1 and X_2 be the minimizers of ω_1 and ω_2, respectively, which can be found in a greedy manner.

Step 1: If $|X_1 \cap X_2| \geq k$, then output them and stop. Otherwise, let $X_1^j := X_1$ and $X_2^j := X_2$, where $j := |X_1 \cap X_2| < k$.

Step 2: Execute the augmenting path algorithm for VIAP(k). Then we obtain a sequence $\left((X_1^j, X_2^j), (X_1^{j+1}, X_2^{j+1}), \ldots, (X_1^\ell, X_2^\ell)\right)$ of solutions, where $\left|X_1^{j'} \cap X_2^{j'}\right| = j'$ for $j' = j, j+1, \ldots, \ell$. If $\ell < k$, then output "the problem (2.1) is infeasible." If $\ell \geq k$, then output (X_1^k, X_2^k).

The above approach directly leads to the following algorithm for the problem (1.4). Again let X_1 and X_2 be the minimizers of ω_1 and ω_2, respectively.

Case 1: If $|X_1 \cap X_2| \leq k$, then execute the augmenting path algorithm for VIAP(k), and let $\left((X_1^j, X_2^j), (X_1^{j+1}, X_2^{j+1}), \ldots, (X_1^\ell, X_2^\ell)\right)$ be the sequence of solutions obtained in the algorithm. If $\ell < k$, then output "the problem (1.4) is infeasible." If $\ell \geq k$, then output (X_1^k, X_2^k).

Case 2: If $|X_1 \cap X_2| > k$, then let r be the rank of ω_2 and $\overline{\omega_2}(X) := \omega_2(V \setminus X)$ for $X \subseteq V$, which is the *dual* valuated matroid of ω_2. Then apply Case 1, where VIAP(k) is replaced by VIAP($r - k$) for $(G, w, \omega_1, \overline{\omega_2})$.

Remark 1. If we are given at least three valuated matroids, then the problems (2.1) (and hence (1.4)) will be NP-hard, since it can formulate the matroid intersection problem for three matroids. ■

2.2 Reduction of (1.5) and (1.6) to Valuated Matroid Intersection

In this subsection, we give polynomial-time algorithms for solving the problems (1.5) and (1.6) by reducing them to *valuated matroid intersection*. This is the following generalization of weighted matroid intersection problem: Given two valuated matroids ω and ω' on 2^V, minimize the sum $\omega(X) + \omega'(X)$ for $X \subseteq V$. It is known [12,13] that valuated matroid intersection is polynomially solvable.

In order to reduce our problems to valuated matroid intersection, we need to prepare two valuated matroids for each problem. One valuated matroid is common in the reductions of the problems (1.5) and (1.6), which is defined as follows. Let $\coprod_{i \in [n]} V$ be the discriminated union of n copies of V. We denote by $(X_1, X_2, \ldots, X_n)$ a subset $\coprod_{i \in [n]} X_i$ of $\coprod_{i \in [n]} V$, where $X_i \subseteq V$ for each $i \in [n]$. Let us define $\tilde{\omega}$ by the disjoint sum of $\omega_1, \omega_2, \ldots, \omega_n$. That is, $\tilde{\omega}$ is a function on $2^{\coprod_{i \in [n]} V}$ defined by $\omega(X_1, X_2, \ldots, X_n) := \omega_1(X_1) + \omega_2(X_2) + \cdots + \omega_n(X_n)$ for $(X_1, X_2, \ldots, X_n) \subseteq \coprod_{i \in [n]} V$. It is a valuated matroid with rank $r := \sum_{i=1}^n r_i$, where r_i is the rank of ω_i.

We then provide the other valuated matroid used in the reduction of the problem (1.5). Define a set system $\tilde{M} = (\coprod_{i \in [n]} V, \tilde{\mathcal{B}})$ by

$$\tilde{\mathcal{B}} = \left\{ (X_1, X_2, \ldots, X_n) \;\middle|\; X_i \subseteq V (i \in [n]), \bigcap_{i=1}^n X_i \in \mathcal{I}, \sum_{i=1}^n |X_i| = r \right\}.$$

It is clear that (1.5) is equivalent to the problem of minimizing the sum of $\tilde{\omega}$ and $\delta_{\tilde{\mathcal{B}}}$, where $\delta_{\tilde{\mathcal{B}}}$ denotes the indicator function of $\tilde{\mathcal{B}}$, namely, $\delta_{\tilde{\mathcal{B}}}(X_1, X_2, \ldots, X_n) := 0$ if $(X_1, X_2, \ldots, X_n) \in \tilde{\mathcal{B}}$ and $\delta_{\tilde{\mathcal{B}}}(X_1, X_2, \ldots, X_n) := +\infty$ otherwise. We now have the following lemma.

Lemma 1. *$\tilde{M}$ is a matroid with the base family $\tilde{\mathcal{B}}$.*

It follows from Lemma 1 that the function $\delta_{\tilde{\mathcal{B}}}$ is a valuated matroid, and we thus conclude that the problem (1.5) can be reduced to valuated matroid intersection.

Remark 2. If we replace the constraint $\bigcap_{i=1}^{n} X_i \in \mathcal{I}$ in (1.5) by $\bigcap_{i=1}^{n} X_i \in \mathcal{B}$, where $\mathcal{B}$ is the base family of some matroid, then the problem will be NP-hard even if $m = 2$, since it can formulate the matroid intersection problem for three matroids. ■

We next provide another valuated matroid that is used in the reduction of the problem (1.6). A *laminar convex function* [17, Section 6.3], which is a typical example of an M$^\natural$-convex function, plays a key role here. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *laminar convex* if f is representable as

$$f(x) = \sum_{X \in \mathcal{L}} g_X \left(\sum_{v \in X} x_v \right) \quad \left(x \in \mathbf{Z}^V\right),$$

where $\mathcal{L} \subseteq 2^V$ is a laminar family on V, and for each $X \in \mathcal{L}$, $g_X : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$ is a univariate discrete convex function, i.e., $g_X(k+1) + g_X(k-1) \geq 2g_X(k)$ for every $k \in \mathbf{Z}$. As mentioned above, a laminar convex function is M$^\natural$-convex.

Now define a function $\tilde{w}$ on $2^{\coprod_{i \in [n]} V}$ by $\tilde{w}(X_1, X_2, \ldots, X_n) := w(\bigcap_{i=1}^{n} X_i)$ for $(X_1, X_2, \ldots, X_n) \subseteq \coprod_{i \in [n]} V$. It is clear that (1.6) is equivalent to the problem of minimizing the sum of $\tilde{\omega}$ and the restriction of $\tilde{w}$ to $\{(X_1, X_2, \ldots, X_n) \subseteq \coprod_{i \in [n]} V \mid \sum_{i=1}^{n} |X_i| = r\}$. For $\tilde{w}$, the following holds.

Lemma 2. *The function $\tilde{w}$ is a laminar convex function on $2^{\coprod_{i \in [n]} V}$ if $w \geq 0$.*

By Lemma 2, the restriction of $\tilde{w}$ to $\{(X_1, X_2, \ldots, X_n) \subseteq \coprod_{i \in [n]} V \mid \sum_{i=1}^{n} |X_i| = r\}$ is a valuated matroid on $2^{\coprod_{i \in [n]} V}$ if $w \geq 0$. Indeed, it is known [20] that, for an M$^\natural$-convex function f and an integer r, the restriction of f to a hyperplane $\{x \in \mathbf{Z}^V \mid \sum_{v \in V} x_v = r\}$ is an M-convex function with rank r, if its effective domain is nonempty. Thus the problem (1.6) can be formulated as the valuated matroid intersection problem for $\tilde{\omega}$ and $\tilde{w}$, establishing the tractability of the problem (1.6) in case of $w \geq 0$.

On the other hand, if $w \leq 0$ and $m \geq 3$, then the problem (1.6) is NP-hard, since it can formulate the matroid intersection problem for three matroids.

Remark 3. As mentioned in Sect. 1, the problem (1.6) with $w \geq 0$ does not fall into the weighted matroid intersection framework even if all functions are modular, while it can be reduced to valuated matroid intersection. That is, the concept of M-convexity is crucial for capturing the tractability of (1.6) even when all functions are modular. ■

2.3 Reduction of (1.7) to M$^\natural$-Convex Submodular Flow

In this subsection, we prove that the problem (1.7) with $w \leq 0$ can be solved in polynomial time by reducing it to *M$^\natural$-convex submodular flow*, which is defined as follows. Let f be an M$^\natural$-convex function on $\mathbf{Z}^V$ and $G = (V, A)$ a directed graph endowed with an upper capacity $\overline{c} : A \to \mathbf{R} \cup \{+\infty\}$, a lower capacity $\underline{c} : A \to \mathbf{R} \cup \{-\infty\}$, and a weight function $w : A \to \mathbf{R}$. For a flow $\xi \in \mathbf{R}^A$, define its boundary $\partial\xi \in \mathbf{R}^V$ by $\partial\xi(v) := \sum\{\xi(a) \mid a \in A,\ a\,\mathrm{enters}\,v\,in\,\mathrm{G}\} - \sum\{\xi(a) \mid a \in A,\ a\,\mathrm{leaves}\,v\,in\,G\}$ for $v \in V$. The *M*$^\natural$*-convex submodular flow problem* for (f, G) is the following problem with variable $\xi \in \mathbf{R}^A$:

$$\begin{array}{ll} \text{Minimize} & f(\partial\xi) + \sum_{a\in A} w(a)\xi(a) \\ \text{subject to} & \underline{c}(a) \le \xi(a) \le \overline{c}(a), \\ & \partial\xi \in \mathrm{dom} f. \end{array}$$

It is known [15] that the M$^\natural$-convex submodular flow problem can be solved in polynomial time.

The problem (1.7) with $w \le 0$ can be reduced to M$^\natural$-convex submodular flow in the following way. Let r_1 and r_2 be the rank of f_1 and f_2, respectively. We define univariate functions g_1 and g_2 on $\mathbf{Z}$ by

$$g_1(p) := \begin{cases} 0 & \mathrm{if}\, p \le r_2 - k, \\ +\infty & \text{otherwise}, \end{cases} \qquad g_2(q) := \begin{cases} 0 & \mathrm{if}\, q \le r_1 - k, \\ +\infty & \text{otherwise}. \end{cases}$$

Then define the function h on $\mathbf{Z}^{V\sqcup\{p\}\sqcup V\sqcup\{q\}}$ by the disjoint sum of f_1, g_1 with the simultaneous coordinate inversion and f_2, g_2, i.e.,

$$h(x, p, y, q) := (f_1(-x) + g_1(-p)) + (f_2(y) + g_2(q)) \quad x, y \in \mathbf{Z}^V\ (\mathrm{and}\, p, q \in \mathbf{Z}).$$

It is not difficult to see that h is M$^\natural$-convex. We then construct a directed bipartite graph $G = (\{x_v\}_{v\in V} \cup \{p\}, \{y_v\}_{v\in V} \cup \{q\}; A)$ endowed with a weight function $\hat{w}\colon A \to \mathbf{R}$ defined by

$$\begin{aligned} A &:= \{(x_v, y_v) \mid v \in V\} \cup \{(p, y_v) \mid v \in V\} \cup \{(x_v, q) \mid v \in V\}, \\ \hat{w}(a) &:= \begin{cases} w(v) & \mathrm{if}\, a = (x_v, y_v), \\ 0 & \text{otherwise}, \end{cases} \qquad (a \in A). \end{aligned}$$

Here we identify the vertices of G with the variables of f. Consider the following instance of the M$^\natural$-convex submodular flow problem:

$$\begin{array}{ll} \text{Minimize} & h(\partial\xi) + \sum_{a\in A} \hat{w}(a)\xi(a) \\ \text{subject to} & \xi(a) \ge 0 \quad (a \in A), \\ & \partial\xi \in \mathrm{dom} h. \end{array} \tag{2.2}$$

The following lemma shows that the problem (1.7) with $w \le 0$ is reduced to the problem (2.2), and thus establishes its tractability.

Lemma 3. *The problem* (1.7) *with* $w \le 0$ *is equivalent to the problem* (2.2)*.*

The NP-hardness of the problem (1.7) with $w \ge 0$ and $k = 0$ follows from the fact that it can formulate the problem of minimizing $f_1(x_1) + f_2(x_2)$ subject to $\sum_{v\in V} \min\{x_v, y_v\} = 0$, whose NP-hardness has been shown in [9].

3 Applications

In this section, we present two applications of our generalized problems (1.6) and (1.7). One is for *matroid congestion games*, and the other for *combinatorial optimization problems with interaction costs.*

3.1 Socially Optimal States in Valuated Matroid Congestion Games

A *congestion game* [21] is represented by a tuple $(N, V, (\mathcal{B}_i)_{i \in N}, (c_v)_{v \in V})$, where $N = \{1, 2, \ldots, n\}$ is a set of players, V is a set of resources, $\mathcal{B}_i \subseteq 2^V$ is the set of strategies of a player $i \in N$, and $c_v \colon \mathbf{Z}_+ \to \mathbf{R}_+$ is a nondecreasing cost function associated with a resource $v \in V$. Here $\mathbf{R}_+$ is the set of nonnegative real numbers. A *state* $\mathcal{X} = (X_1, X_2, \ldots, X_n)$ is a collection of strategies of all players, i.e., $X_i \in \mathcal{B}_i$ for each $i \in N$. For a state $\mathcal{X} = (X_1, X_2, \ldots, X_n)$, let $x_v(\mathcal{X})$ denote the number of players using v, i.e., $x_v(\mathcal{X}) = |\{i \in N \mid v \in X_i\}|$. If $\mathcal{X}$ is clear from the context, $x_v(\mathcal{X})$ is abbreviated as x_v. In a state $\mathcal{X}$, every player using a resource $v \in V$ should pay $c_v(x_v)$ to use v, and thus the total cost paid by a player $i \in N$ is $\sum_{v \in X_i} c_v(x_v)$.

The importance of congestion games is appreciated through the fact that the class of congestion games coincides with that of *potential games*. Rosenthal [21] proved that every congestion game is a potential game, and conversely, Monderer and Shapley [10] proved that every potential game is represented by a congestion game with the same potential function.

We show that, in a certain generalized model of *matroid congestion games with player-specific costs*, computing *socially optimal states* reduces to (a generalized version of) the problem (1.6). A state $\mathcal{X}^* = (X_1^*, X_2^*, \ldots, X_n^*)$ is called *socially optimal* if the sum of the costs paid by all the players is minimum, i.e.,

$$\sum_{i \in N} \sum_{v \in X_i^*} c_v(x_v(\mathcal{X}^*)) \leq \sum_{i \in N} \sum_{v \in X_i} c_v(x_v(\mathcal{X}))$$

for any state $\mathcal{X} = (X_1, X_2, \ldots, X_n)$. In a *matroid congestion game*, the set $\mathcal{B}_i \subseteq 2^V$ of the strategies of each player $i \in N$ is the base family of a matroid on V. For matroid congestion games, a socially optimal state can be computed in polynomial time if the cost functions are *weakly convex* [1,23], while it is NP-hard for general nondecreasing cost functions [1]. A function $c \colon \mathbf{Z}_+ \to \mathbf{R}$ is called *weakly convex* if $(x+1) \cdot c(x+1) - x \cdot c(x)$ is nondecreasing for each $x \in \mathbf{Z}_+$. In a *player specific-cost* model, the cost paid by a player $i \in N$ for using $v \in V$ is represented by a function $c_{i,e} \colon \mathbf{Z}_+ \to \mathbf{R}_+$, which may vary with each player.

We consider the following generalized model of congestion games with player-specific costs. In a state $\mathcal{X} = (X_1, X_2, \ldots, X_n)$, the cost paid by a player $i \in N$ is

$$\omega_i(X_i) + \sum_{v \in X_i} d_v(x_v),$$

where $\omega_i\colon 2^V \to \mathbf{R}_+$ is a monotone set function and $d_v\colon \mathbf{Z}_+ \to \mathbf{R}_+$ is a non-decreasing function for each $v \in V$. This model represents a situation where a player $i \in N$ should pay $\omega_i(X_i)$ regardless of the strategies of the other players, as well as $d_v(x_v)$ for every resource $v \in X_i$, which is an additional cost resulting from the congestion on v. It is clear that the standard model of congestion games is a special case where $\omega_i(X_i) = \sum_{v\in X_i} c_v(1)$ for every $i \in N$ and every $X_i \in \mathcal{B}_i$, and

$$d_v(x) = \begin{cases} 0 & (x = 0), \\ c_v(x) - c_v(1) & (x \geq 1). \end{cases}$$

In this model, the sum of the costs paid by all the players is equal to

$$\sum_{i\in N} \omega_i(X_i) + \sum_{v\in V} x_v \cdot d_v(x_v). \tag{3.1}$$

The following lemma is also straightforward to see.

Lemma 4. *The following are equivalent.*

- c_v *is weakly convex.*
- d_v *is weakly convex.*
- $x \cdot d_v$ *is discrete convex.*

By Lamma 4, if c_v (or d_v) is weakly convex, then the function $\sum_{v\in V} x_v \cdot d_v(x_v)$ is laminar convex.

The solution for the problem (1.6), or the DCA perspective for (1.6), provides a new insight on this model of cost functions in matroid congestion games. In addition to the weak convexity of d_v ($v \in V$), this model allows us to introduce some convexity of the cost function ω_i. Namely, we assume that ω_i is a valuated matroid for every $i \in N$. Then, computing the optimal state, i.e., minimizing (3.1), is naturally viewed as the valuated matroid intersection problem for the valuated matroid $\sum_{i\in N} \omega_i(X_i)$ and the laminar convex function $\sum_{v\in V} x_v \cdot d_v(x_v)$ as in the problem (1.6). Thus it can be done in polynomial time.

3.2 Combinatorial Optimization Problem with Interaction Costs

Lendl, Ćustić, and Punnen [8] introduced a framework of *combinatorial optimization with interaction costs* (*COPIC*), described as follows. For two sets V_1 and V_2, we are given cost functions $w_1\colon V_1 \to \mathbf{R}$ and $w_2\colon V_2 \to \mathbf{R}$, as well as *interaction costs* $q\colon V_1 \times V_2 \to \mathbf{R}$. The objective is to find a pair of feasible sets $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$ minimizing

$$w_1(X_1) + w_2(X_2) + \sum_{u\in X_1} \sum_{v\in X_2} q_{uv}.$$

We focus on the *diagonal COPIC*, where V_1 and V_2 are identical and $q_{uv} = 0$ if $u \neq v$. We further assume that the feasible sets are the base families of matroids. That is, we are given two matroids $(V, \mathcal{B}_1)$ and $(V, \mathcal{B}_2)$, and a pair (X_1, X_2) of subsets of V is feasible if and only if $X_1 \in \mathcal{B}_1$ and $X_2 \in \mathcal{B}_2$. In summary, the problem is formulated as

$$\begin{array}{ll} \text{Minimize} & w_1(X_1) + w_2(X_2) + q(X_1 \cap X_2) \\ \text{subject to} & X_i \in \mathcal{B}_i \quad (i = 1, 2). \end{array} \tag{3.2}$$

If w_1 and w_2 are identically zero and $q \geq 0$, then the problem (3.2) amounts to finding a socially optimal state in a two-player matroid congestion game, and thus can be solved in polynomial time [1]. Lendl et al. [8] showed the solvability on the case where the interaction cost q may be arbitrary.

Now we can discuss another direction of generalization; the costs $w_1(X_1)$ and $w_2(X_2)$ are valuated matroids. This is a special case of the problem (1.6) or the problem (1.7), and thus can be solved in polynomial time when $q \geq 0$ or $q \leq 0$.

4 Discussions

In this paper, we have analyzed the complexity of several types of minimization of the sum of valuated matroids (or M-convex functions) under intersection constraints. For the following standard problem of this type, its complexity is still open even when the cardinality constraint $|X_1 \cap X_2| = k$ is removed and ω_1, ω_2 are modular functions on the base families of some matroids:

$$\begin{array}{ll} \text{Minimize} & \omega_1(X_1) + \omega_2(X_2) + w(X_1 \cap X_2) \\ \text{subject to} & |X_1 \cap X_2| = k, \end{array}$$

where ω_1 and ω_2 are valuated matroids on 2^V, w is a modular function on 2^V, and k is a nonnegative integer.

The above problem seems similar to VIAP(k), but is essentially different; the problem of this type formulated by VIAP(k) is

$$\begin{array}{ll} \text{Minimize} & \omega_1(X_1) + \omega_2(X_2) + w(F) \\ \text{subject to} & F \subseteq X_1 \cap X_2, \\ & |F| = k. \end{array}$$

Only the following cases are known to be tractable:

- w is identically zero. This case is equivalent to the problem (1.4).
- $w \geq 0$ and the cardinality constraint $|X_1 \cap X_2| = k$ is removed. This case is a subclass of the problem (1.6).
- $w \leq 0$ and $|X_1 \cap X_2| = k$ is replaced by $|X_1 \cap X_2| \geq k$. This case is a subclass of the problem (1.7).
- $|X_1 \cap X_2| = k$ is removed and ω_1, ω_2 are the indicator functions of the base families of some matroids. This case has been dealt with Lendl et al. [8]; see Sect. 3.2.

Another possible direction of research would be to generalize our framework so that it includes computing the socially optimal state of *polymatroid congestion games* [6,22], as we have done for matroid congestion games in Sect. 3.1.

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