# Chapter 7
# Travels with Epsilon in Sign and Space

**Louis H. Kauffman**

## Introduction

This paper is about the relationship of diagrams with mathematics.

Mathematics is replete with diagrams of all kinds such as the classical diagrams of Euclidean Geometry and the wilder diagrams of topology. Indeed, symbolisms in mathematics such as the Leibniz notations for integration and differentiation are themselves diagrams indicating the very processes that they represent.

$$\int_0^x f(t)\,dt = F(x)$$
$$dF(x)/dx = f(x)$$

What is less obvious is how certain forms can exhibit shape that links different areas of mathematics via a common structure that lives in the diagrams.

We study the linking of mathematical fields in this paper by examining first a magical diagrammatic for vector calculus, and then showing how it works and why it works by relating that formalism to the question of coloring maps and graphs in and out of the plane. In the course of this journey we shall have a trivalent vertex that we call the *epsilon*. Different ways of viewing the way the epsilon works and behaves shed light on the structure of dot products of vectors, cross products of vectors, multiple cross products, the structure of the quaternions and edge coloring problems for graphs that are equivalent to the Four Color Theorem. Once one has

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taken this journey, neither graphs and colorings nor vectors and their algebra are ever the same again. It all pivots on the epsilon.

The second journey in this paper is into diagrammatic knot theory. There we show how the diagrams of knot theory, decorated shadows of projections from three dimensions, are intimately related to non-associative algebras called quandles. The simplest quandle involves three colors and is, in its structure, very close to the coloring problems we have considered earlier in the paper. But now the associations are with topology and how the algebra helps uncover hidden topological properties.

The third journey examines the resolution of a diagrammatic singularity and finds a generalized epsilon and the Jacobi identity for Lie algebras hidden in the diagrams.

A longer tale can be told here, but we hope that this introduction to the ways of diagrams gives the reader a taste of this way to imagine the roots of mathematics.

In the first part of the paper. The author is in dialogue wth a fictional mathematician named RosePen. Professor RosePen is a figment of the author's imagination, influenced by the ideas, discoveries and inventions of Roger Penrose, John H Conway, George Spencer-Brown, Charles Sanders Peirce, Lewis Carroll and other great contributors to the diagrammatic interfaces in the making of sign and space.

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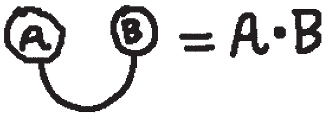
## A Magic Calculus of Vectors

I went to the CMF last year. That's the Convention on Mathematical Fictions. Sometimes we call it the CFM, the Convention on Fictional Mathematics. Well, call it what you will, we were still meeting in person then and sitting down to scraps of paper and scribbling funny geometries and strange equations. You remember how it was. And I met this guy RosePen and he sits me down and says. Look. You have to learn my graphical rules. They will change your life. I says - yeah, really? And he says Really! So I sat down at the bar with him in the Atlanta Ritz Carlton and he takes out a sheet of paper, a bit crumpled.

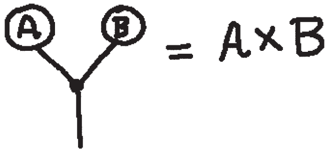
He makes the drawing you see in the figure below, and he says this is a vector.


I says, it looks like a blob with a line hanging on it. Yep! He says. That's a vector. And here is the dot product of two vectors. He draws two of his vectors and joins their arcs together like this .




I says. Hmm... I guess you are going to tell me that if a blob has no hanging strings, then it is a scalar? Right! He says. How did you know? I says, look you told me that thing there is a scalar product (dot product) and so I figured you joined those arcs to get rid of them. Well. He says. You are absolutely right! Can you figure out what would be the vector cross product?

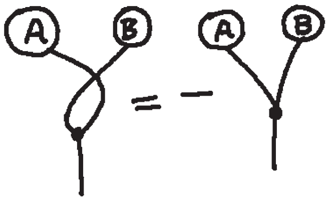
Aw, I says. Well, you have to combine two vectors to get a vector. I gather your vectors just have one arc attached to the blob. So I wager you need a trivalent node like this



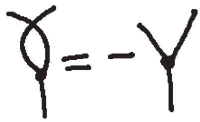

and you can run the arcs from your vector into two of the three lines on the tri-node and you will have a new blob with one arc! That's my guess for the cross product.

I couldn't help myself. I continued. I says: Look. You are gonna have to have that.

A x A is zero and that A x B is perpendicular to A and to B. And you are gonna need that A x B = – B x A. So there is a lot of work to be done here. I think we better start with A x B = – B x A. This is what you need!




That twisted thing is B x A and you really need your trivalent vertex to satisfy the same identity!!



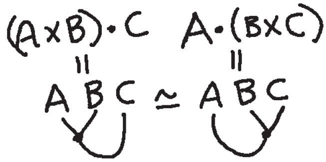

Twist two legs of that trivalent vertex of yours and thing changes sign. Now its ok because we will have A x A = – A x A and so A x A = 0. No sweat!

He looks at me with slitted eyes, a bit suspicious you know. And he says. You are exactly right. Nobody ever got this before. Are you from the CIA? Maybe I should just stop talking right here. Naw, I says. I never talk to the Cantorian Infinite Adepts. They are too theological for me.

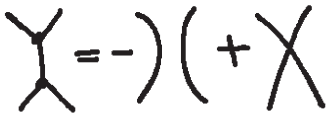
But look, I says, your system works too well! Look at (A x B).C where I use a period for the dot product. We get a clear proof that

$$(A \times B).C = A.(B \times C)$$

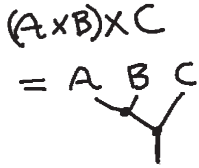
by just deforming your diagrams. Ha!




You catch on fast, he says. But now I will tell you the *secret.* We call the trivalent node our e*psilon*. And here is the e*psilon identity.*

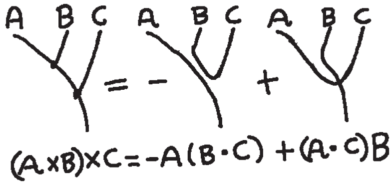


I shall initiate you into its vectorial secrets. You mean, I says, you can derive other identities from this secret identity. He smiled. A co-conspirator, I thought. Well, I decided to play along. So I says, Ok wise-guy lets try the notorious. Vector Triple Product: (A x B) x C. What will your smart diagrams do with this stumper? Here, don't tell me. I'll do it! There it is.




I look and look at this. And then I remember his epsilon identity and it re-forms in my mind, slightly deformed:

And I say, why not! You told me I could do this and I know you don't care if I deform it a little. Now I will put the blobs back on top. Aha! There it is. Yes!

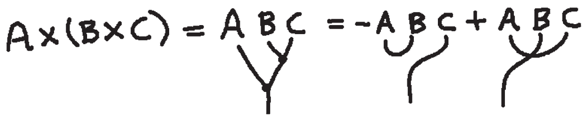



I put the blobs back and the epsilon identity became that familiar formula

$$(\mathrm{A\,x\,B})\,\mathrm{x\,C} = -\mathrm{A}(\mathrm{B.C}) + (\mathrm{A.C})\mathrm{B}$$

from our beloved vector calculus. And I says to RosePen. What the heck. How did you do that? That is a complicated geometrical formula and your diagrams make it fall out of nowhere. What is going on here. Are vectors really something other than what I thought they were? What planet are you from?

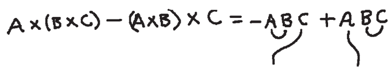
Then I decided to try something simpler. I says to RosePen what about the fact that A x B is perpendicular to A and to B? Can we see that? I know. I know. You are going to say that perpendicularity of V and W is defined by the eq. V.W = 0. Ok. Then I am supposed to prove that (A x B). A = 0. Oh wait. I don't even need the diagram. After all I just did show that A x A = 0 for any A. So (A x B). A = − (B x A). A = − B. (A x A) = 0 and we are done. Ok I am satisfied. Lets go back to your special epsilon identity. What about A x (B x C). We can do that and find that A x (B x C) = −A (B.C) + (A.C)B.




So I get this beautiful difference formula

$$(\mathrm{A\,x\,B})\,\mathrm{x\,C} - \mathrm{A\,x}(\mathrm{B\,xC}) = -\mathrm{A}(\mathrm{B.C}) + (\mathrm{A.B})\mathrm{C}.$$

And this shows very explicitly how the vector cross product operation is not associative.




RosePen intervened and said. Why don't you try for associativity? Can you make an associative product from these materials? I said. Wait. I remember the definition of *quaternion multiplication*. It is

$$UV = -A.B + A \text{ x } B.$$

You almost do cross product multiplication, but you add that scalar product.

Why don't you try it? He says.

Ok. I will says I. I will define.

$$UV = -U \cdot V + U \times V$$

And quaternions are four dimensional vectors, so if U and V are three dimensional, then a quaternion is of the form a + V where a is a scalar. So we have

$$(a + U)(b + V) = ab + aV + bU + UV = ab + aV + bU - U.V + U \text{ x } V,$$

quite a mixture of scalar and vector products. It is no wonder that after the "quaternion wars" in the nineteenth century most applied mathematicians wanted to work separately with scalars, vectors, scalar products and vector products. But the quaternions get around, and they are really fundamental for understanding three and four dimensional space. Note that, from our formula above, we have that UU = -U.U, and so if U has length 1 we have UU = −1. We have a whole sphere's worth of square roots of minus one!

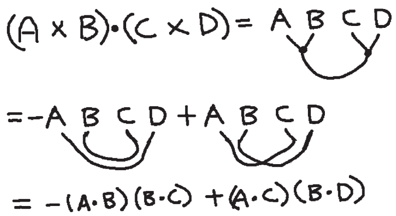
Well. In this case I won't bore you with the calculation showing that quaternion multiplication is associative. You'll see that it works out. If I, J and K are three perpendicular vectors of unit length so that II = J.J = K.K = 1, so we have

$$II = JJ = KK = IJK = -1,$$

the famous formula for the quaternions discovered by Sir William Rowan Hamilton in 1843. You know what he said about it:

> …an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashhed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; namely, ii = jj = kk = ijk = - 1 which contains the Solution of the Problem… (Altmann 1986)

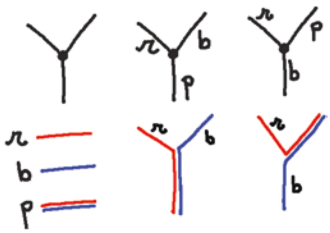
And I stopped for moment and then I said. Wow! Look at this one!!



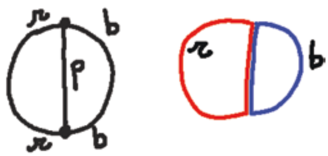

I turned to RosePen and I said. You had better explain what is going on here.

## RosePen's Explanation

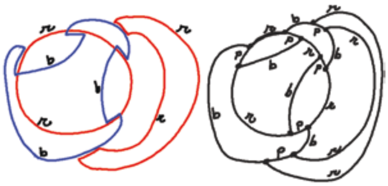
In order to explain this to you, RosePen said, I have to tell you about a problem that does not seem to have anything to do with three dimensional space or vectors or dot products. The problem I am concerned about is a problem of coloring the edges of a network with trivalent nodes, using three colors: red (r), blue (b) and purple (p). It is very convenient for me to think of purple as a superposition of red and blue and so I will write p = rb and make drawings like this.



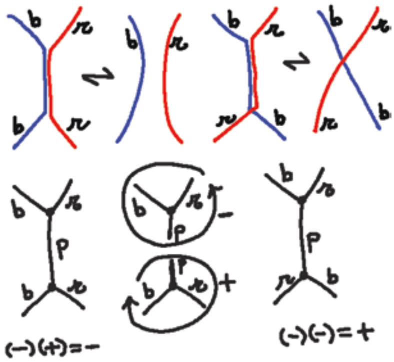

In this drawing you see that I color a red line red and a blue line blue, but I color a purple line by a combination of red and blue. The RULE for my coloring problem is that there must be three distinct colors at each node in the network. Thus at a trivalent node drawn in the plane, you will see the cyclic order of rbp or rpb, and I can make my drawings as illustrated using only red, blue and the superposition of red and blue that I call purple. Then the solution to a coloring problem looks like this.



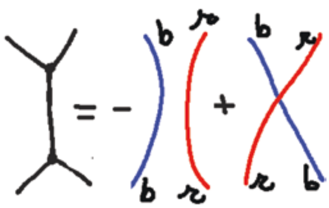

I can indicate the solution to a coloring problem by putting letters on the edges of the graph (the network),or I can color the edges. When I color them I have a collection of blue loops and red loops. The blue loops do not touch each other and the red loops do not touch each other. Red and blue loops can share segments of arc that correspond to purple edges in the network. Here is a more complex example.



It is from this coloring problem that I conceived of the epsilon identity, for you see that the parallel and crossed arcs arise naturally when one looks at the color interactions of two nodes.



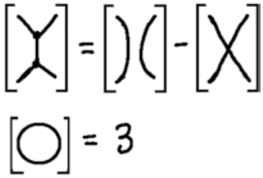

If the cyclic permuations of colors are opposite on the two nodes, then we can pull the purple superposition apart and get nearby uncrossed blue and red curves. If the cyclic permutations are the same, then we have a red arc crossing a blue arc. These are the only two structural possibilities for the color interaction of two nodes. Of course we have singled out purple for the sake of emphasis, but the same remarks would apply if the middle line were another color. So I label the case of parallel arcs with a minus sign to indicate that the two permutations are of opposite sign! And we get the epsilon identity as an expression of the coloring possiblitiies. I think of the identity in color like this:



And I discovered a most remarkable fact. *If I place a square root of negative unity at each node of the network and then expand the edges by the epsilon identity I can count the number of colorings of the net.*

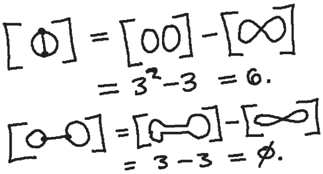
I says, to RosePen. If you put an i (with ii = −1) at every node, then you can just reverse the sign of the terms in the epsilon identity.

He says. Yes. That is what happens and I get a formula like this.




In applying this formula you erase an edge in two copies of the graph, and you replace the edge by two parallel arcs in one copy and by two crossed arcs in the other copy.

And here are two examples of color counts.




In the first case there are indeed six ways to color this graph. In the second case the graph is not colorable and the formula gives the correct answer zero. This graph is planar but it can be disconnected by removing an edge. There is a famous Theorem called the "Four Color Theorem" and it is equivalent to the statement:

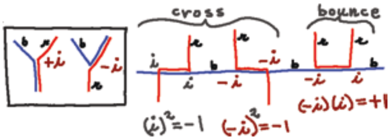
**Theorem** *A planar trivalent graph G that cannot be disconnected by removing an edge can be colored with three colors on its edges so that every node receives three distinct colors.*

This means that the formula [G] will always be non-zero for any such graph G.

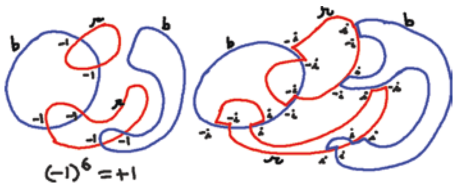
The Theorem does not have a simple proof. I am hoping that an analysis of this formula will yield a simple proof of the Theorem.

So I says. Ok. I see how you found the epsilon identity, but it is still a mystery to me what it has to do with three dimensional space. Is this some mysticism about your three colors?

RosePen replies. I had better say a few more words about coloring before we to back to vectors. Look at this diagram.



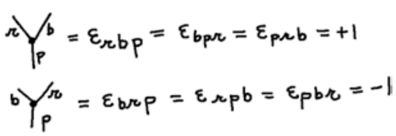

I have illustrated how each node of a coloring is assigned +i or –i according to the epsilon gives it +1 or − 1. Notice what happens when we have a red arc crossing a blue arc (or vice versa). Then one corner gets I and the the other gets –i. The product of (+i) and (−i) is (−1). So each time a red curve and a blue curve cross, my bookkeeping registers a negative one. If there is a bounce (no crossing) as shown in the figure, then we get a minus i and a plus i, so the product is one. Thus bounces contribute a + 1. Therefore if we have a coloring of a planar graph and we take the product of all my +i and –i contributions it will equal one – because curves in the plane intersect one another an even number of times (by the Jordan Curve Theorem). Here is an example for you to look at. This is why my sum [G] must always count one for each coloring of a plane trivalent graph.



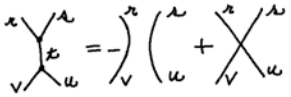

Now lets look more closely at the epsilon identity. I will make definitions. I let $\varepsilon[rst]$ be a a text symbol for the epsilon node with some specific assignment of values for r,s,t from among the colors r, b and p. Then I will define

$$\varepsilon\left[rbp\right]=\varepsilon\left[bpr\right]=\varepsilon\left[prb\right]=1$$
$$\varepsilon\left[rpb\right]=\varepsilon\left[pbr\right]=\varepsilon\left[brp\right]=-1$$

And $\varepsilon[rst]=0$ if any two of these labels are equal to one another. These are rules we have used in coloring.

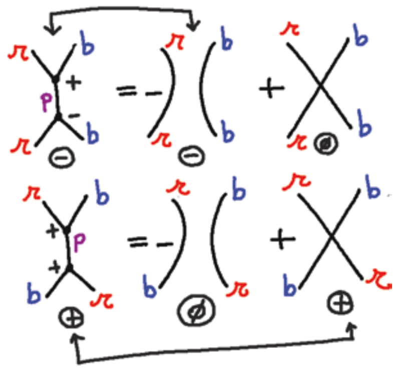



Then the epsilon identity becomes an algebraic statement about the values of the epsilon. It looks like this.

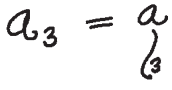


$$\sum_{t} \epsilon_{rst}\, \epsilon_{tuv} = -\delta^{r}_{v}\delta^{s}_{u} + \delta^{r}_{u}\delta^{s}_{v}$$

You have to stare at this formula for a while to see that it is actually very simple. The deltas are what we call Kronecker Deltas, delta[x,y] = 1 only if x = y and it is 0 otherwise. The sum on t in the formula above just amounts to taking the value different from both r and s or from t and u because our epsilon vanishes where there are equal indices and we only have three indices to work with. I will illustrate the actual cases for you below.

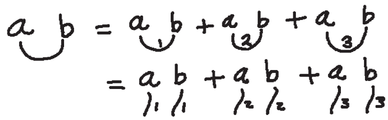


The upshot of this way of thinking of the epsilon identity in terms of indices is that we can interface it with vectors. What is a vector? I told you earlier to think of a vector as a blob with an arc hanging down, but the usual way to think of a vector is as a triple of numbers such as a = (a1, a2, a3). You can think of the line for the blob as a place to write the index so that for example:

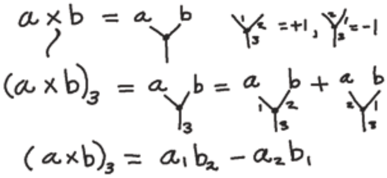


Then the dot product follows once we use the rule that *you must sum over all the possible index values for an arc that has no free ends.*

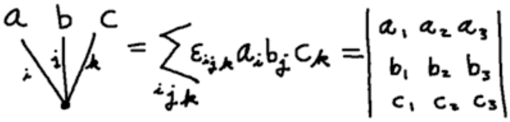
$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$



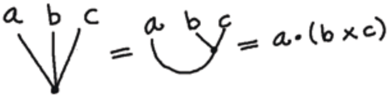
I hope you now see how our arc-connection diagram corresponds to the dot product. Just so, our diagram for the cross product is actually a definition for the cross product. I will calculate one component for you below and you will see that it is working!




In fact, he says, you see that the epsilon gives us the determinant just like this.




So we have that DET(a,b,c) = a.(b x c).




And there is a well known formula for the vector cross product that would formally put the perpendicular unit direction vectors I, J and K in the first row.

$$\begin{vmatrix} I & J & K \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = b \times c$$

Well, I thought about that, and I worked out the other two components of the vector cross product and it was all logically clear. So we really had proved all those identities and more by just drawing topological diagrams and using the diagrammatic epsilon identity. But it still seemed to me as mysterious as ever. Why should something like this work? I had never thought of vectors as topological before. Before this conversation with RosePen, I always thought of vectors as rigid arrows that could make angles with each other and that they were the underpinning of a corresponding geometry of lines and planes and sharp directions. So I asked him more questions.

I said. Well Professor RosePen, I still do not quite understand what is going on here.

Do you mean to tell me that properties of vectors are behind the questions about coloring graphs? Or do you mean that the properties of graph colorings are the subtle structure of vectors?

He smiled and said “Yes.”

## An Intermediate Epilogue

I had to go. And I am still puzzling about this connection to this day. Just yesterday I ran across a paper by Kauffman (Kauffman 1990) entitled “Map Coloring and the Vector Cross Product”. I could almost imagine what that might be about. I read it and continued to think about this colorful and disturbing way to look at vectors and vector calculus.

I have to tell you about this. Kauffman reformulated the coloring problem *entirely* in terms of the vector cross product! He turned it into some arcane property of perpendicularities. And I still don’t understand anything! You’ll see. Lets go back to I and J and K, ok? And we are looking at the cross product algebra so that I x I = J x J = K x K = 0, but I x J = K and J x I = − K and all that. It is just a way to talk about epsilon by now. But this is a weird algebra. It is not associative.

$$(\mathrm{I} \times \mathrm{J})\times \mathrm{J} = \mathrm{K} \times \mathrm{J} = -\mathrm{I},$$

$$\mathrm{I} \times (\mathrm{J} \times \mathrm{J}) = \mathrm{I} \times 0 = 0.$$

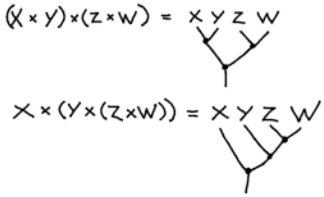
So Kauffman poses this problem. Suppose you take a product of some variables, any number of variables, like X, Y, Z, W and you associate it in two ways and write the equation stating that the result of the multiplication is the same for both associations. For example, you could write

$$(\mathrm{X} \times \mathrm{Y}) \times (\mathrm{Z} \times \mathrm{W}) = \mathrm{X} \times (\mathrm{Y} \times (\mathrm{Z} \times \mathrm{W})).$$

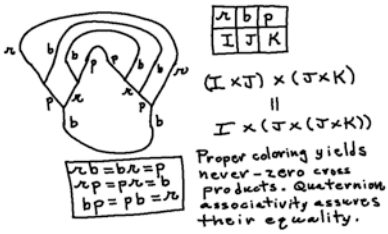
Kauffman then asks you to solve this equation, using only I or J or K for the values. You get to use a value for more than one variable, but ony get to use I or J or K.

Neither side of your solution can be zero. You have to produce two equal non-zero products. Can you solve it for this example? Well in the example an answer is X = I, Y = J, Z = J, W = K. Try it! Kauffman claims to be able to solve all such equations in any number of variables.

It seems to be a tricky problem about combinations of perpendicularities. But that isn’t how Kauffman solves these problems. He uses the graphical calculus. Then we have:




He puts them together in one graph by taking the mirror image of one expression and tying it to the other.




Then you color the graph using r,b,p and read out a solution to the vector cross product equation by taking I for r, J for b and K for p. I kid you not. Since we have chosen a proper coloring of the graph of the two tied trees, all the partial products in the vector cross products will be non-zero. But this means that we can view these products as *quaternion products* (since in the quaternions the non-zero products of I and J and K are the same as the vector cross products). Thus the two associated products have to be equal *because the quaternions are associative*, and we are done! You can check that indeed

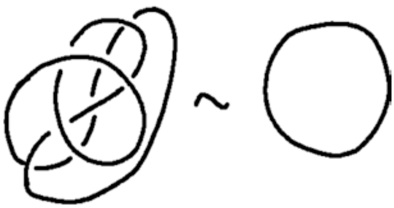
$$(\mathrm{I}\,\mathrm{x}\,\mathrm{J})\,\mathrm{x}\,(\mathrm{J}\,\mathrm{x}\,\mathrm{K}) = \mathrm{I}\,\mathrm{x}\,(\mathrm{J}\,\mathrm{x}\,(\mathrm{J}\,\mathrm{x}\,\mathrm{K})).$$

It turns out that the full coloring problem for arbitrary planar trivalent graphs is implied by the coloring of tied trees. This makes the Four Color Theorem (Appel and Haken 1977; Apprl et al. 1977; Heawood 1890) equivalent to this property of solutios to equations involving associated vector cross products. At this stage in mathematics we do not fully understand why maps can be colored (although there is a complex proof) and we do not fully understand the relationships among graphs, vector cross products and the quaternions. There is much to learn in this domain. Perhaps it will all become clear one day and we shall understand the whole story. For now, it is a fascinating ground for research. The relationship of particular mathematics with the geometry and topology of diagrams will become ever more important to the unity of mathematics and for the gesture that it makes to the unity of the world.

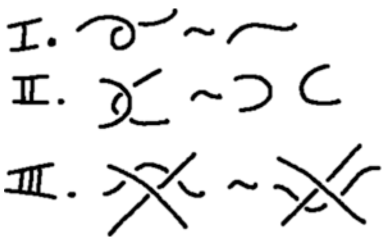
## Knots

Diagrams are often tied to specific contexts and so the best way to indicate the wider generality behind the examples of mathematical connection that we have drawn in this paper is to give another example of the phenomenon. In this case, I want to show, how by following the diagrams one can see a deep connection between knot theory and the mathematics certain algebras. I shall be a brief as possible, and start with the knot theory. In knot theory we use diagrams like this.

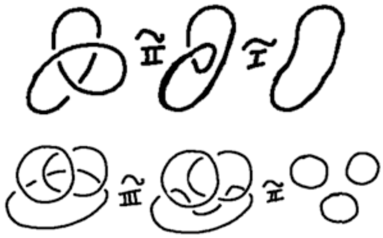
The diagram represents a curve in three dimensional space that goes under and over itself in the weaving pattern of the drawing. The diagram uses the well-known drawing convention that a broken line is a projection from space such that the unbroken arc that crosses the broken part is higher than the "broken arc" that proceeds underneath. We can represent topological movenments of knots (called istopies) by changes in the diagrams. For example, view the diagram below.



It should be clear to you that the complicated curve on the left can be undone and transformed to the unknotted loop on the right. In fact there is a system of moves on the diagrams that can accomplish this aim. The basic moves are shown below. These are called the *Reidemeister Moves* after Kurt Reidemeister, who wrote the first book on the theory of knots.

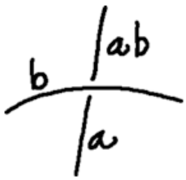



Here are two examples of unknotting and unlinking using the moves.

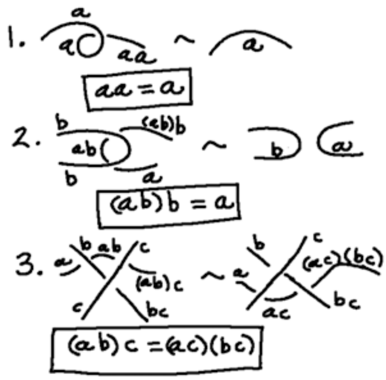



In the first case, we use the II move and then the I move to unknot this single curve. In the second case we use a III move to simplify the rings, and then three II moves to undo them completely. The second example is interesting because it actually needs the III move to be undone.

Now I will show you a way to related algebra to these diagrams. We will have a way to "multiply" elements a and b, denoted ab. And we shall label arcs in the knot and link diagrams by these elements. When an arc a under-crosses another arc b, then the exiting arc will be labeled by the product ab as shown below.




See the diagrams below.




We want the labeling to respect the Reidemeister moves and this leads to algebra rules:

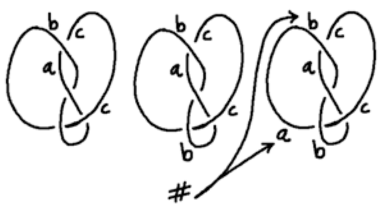
1. aa = a.
2. (ab)b = a
3. (ab)c = (ac)(bc).

An algebra that satisfies these rules is called a *quandle*. Here is a very simple example of a quandle. We shall have three algebra elements a,b and c. And we shall have the rules aa = a, bb = b, cc = c and ab = ba = c, ac = ca = b, bc = cb = a. In other words, any single element combines with itself to produce itself. And two distinct element combine to produce the third element. Indeed this algebra is similar to our colring rules for r,b and p but there the colors combine with themselves by different rules.

Note that (ab)c = cc = c while (ac)(bc) = ba = c. So we have (ab)c = (ac)(bc) as desired for the third Reidemeister move. You can check the other cases easily. For example, (aa)b = ab = c and (ab)(ab) = cc = c. We will use this three color algebra {a,b,c} to color knots and links! Here is a coloring of the trefoil knot.


The trefoil is correctly colored by our rules and this means that any diagram obtained from the trefoil by Reidemeister moves will inherit a coloring from this coloring that still has all three colors. (Think about this and you will see that it is so!). But this means that the trefoil can not be unknotted. For if it could then we would have transformed it to the unknot, and the unknot can only be colored with one color. So we have proved that the trefoil knot is knotted by using coloring.

Not every knot can be three-colored. For example, the figure eight knot cannot be so colored as the diagram below demonstrates. We start the coloring with two distinct colors a and b, propagate a c. Then the c interacts with a b and produces an a on a line already labeled with b. This contradiction shows that the figure eight knot cannot be three colored. This means that we have just proved that the figure eight knot is not isotopic to the trefoil knot, but we shall have to work harder to prove that the figure eight knot is actually knotted!




This can be done by using five colors and a more complex quandle but that is a story for another time.

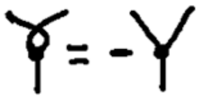
I will end with one more kind of conclusion that we can draw from uncolorability. Consider the famous Borommean Rings as shown below. They are a link of three components. If you remove one of the rings, the other two come apart. We want to prove that the three rings cannot come apart. To do this, I give you an exercise. Prove that the Borommean Rings cannot be colored with three colors! You can verify this in a fashion similar to what we did with the figure eight knot. Now I will assume that you did this exercise and you are convinced that there is no way to color the rings.

But if the rings could come apart, then there would be a sequence of Reidemeister moves from the Borommean Rings to three unlinked rings. You can color each one of three unlinked rings with one of three different colors. The moves that got you from the Borormmean rings to the unlinked rings could be reversed and you would have a sequence of Reidemeister moves from the three unlinked rings to the Borommean rings. Each move would result in a three colored link,starting from the three unlinked colored rings. So in the end you would have to find a three coloring of the Borommean rings. That is a contradiction. Therefore the Borommean rings are linked. Is this not an amazing argument? (Nanyes 1993; Adams 1994).

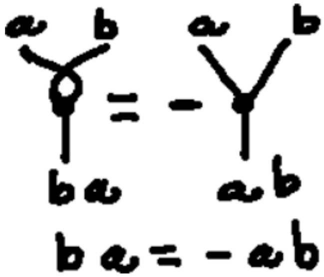
Algebra and diagrams and their mathematical interpretaions interact in a multitude of ways that give rise to new ways to think about geometry, topology, algebra, combinatorics and indeed the entire mathematical universe.

## The Roots of Lie Algebra

And now we return to the form of the epsilon. Let be given a trivalent vertex with sign change under permutation as we have had it from the beginning.




And following our penchant to look at algebra in relation to diagrams let there be an algebra **L** so that the product of elements of **L** is indicated by the vertex.




Now contemplate a singular vertex as show below. In this singular vertex two arcs meet at a singular point along the arrow base-line.

There are three natural resolutions of this singularity and we have put them into a diagrammatic equation below.

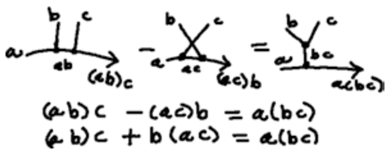
This way to put the resolutions of this singularity into an equational pattern tells a nice algebraic story. In the algebra story we see that the equation is

$$(\mathrm{ab})\mathrm{c} - (\mathrm{ac})\mathrm{b} = \mathrm{a}(\mathrm{bc})$$

and that this can be changed by using b(ac) = −(ac)b to

$$(\mathrm{ab})\mathrm{c} + \mathrm{b}(\mathrm{ac}) = \mathrm{a}(\mathrm{bc}).$$

This is called the *Jacobi Identity*.




An algebra that satisfies the Jacobi Identity and the anticommutativity of ab = −ba for all a and b in the algebra, is called a *Lie Algebra*. Lie algebras (Kauffman 2012; Bourbaki 1989) are ubiquitous in mathematics and indeed very closely related to the original epsilon of our paper and with the quandles in the knot theory, and more generally to knot theory in fundamental ways. It is quite surprising to meet the Jacobi identity as an expression of the resolution of a simple graphical singularity. Another story relates this combinatorics with the Reidemeister Moves (Kauffman 2012), but we will tell that tale another time.

But we cannot resist ending where we began and recount a little more of that conversation between RosePen and myself at the bar in the Ritz-Carlton. RosePen says to me: Are you familiar with Lie algebras? And I say, only a little. I know that the vector cross products form a Lie algebra and they satisfy the Jacobi Identity:

$$a \times (b \times c) = (a \times b) \times c + b \times (a \times c).$$

Well. He says. You can verify that Jacobi Identity by using the epsilon identity. I would not want to spoil the fun of it for you. Do it when you get back to your hotel room and before the rope tricks start this evening. I did, and I am sure the reader would like to do this as well. Once this exercise is completed the reader will see clearly that, enticing as it is, the epsilon is just the tip of the iceberg of a pattern to continues into Lie algebras, group theory, symmetry and beyond.

## Epilogue

Some references may be useful to the reader. Much of the material in this paper can be found in the author's book "Knots and Physics" (Kauffman 2012) and in his papers (Kauffman 1990, 1992, 2005, 2016). The origin of the diagrammatics of vectors can be found in the work of Roger Penrose (Penrose 1971) and certain key insights and their diagrams are in the work of G. Spencer-Brown (Spencer-Brown 1979, 1997). For the coloring problem the reader can consult (Appel and Haken 1977; Kauffman 1990, 2016; Penrose 1971; Spencer-Brown 1979; Heawood 1890). For Lie algebras, a good start is (Stillwell 2008).

In this paper Professor RosePen is a fictional character who takes on some of the ideas and mathematical attitudes of Roger Penrose, George Spencer-Brown, John Horton Conway and the Author.

I have included some of my favorite mathematical tricks in this paper. The intent however is to go beyond tricks and ask about the nature of the sort of relationships that we have seen here. There are many more relationships of this kind. My field, topology, is full of them, and I am sure that other mathematicians in other fields would have many examples of their own. All of these examples use a diagram or some geometry to pivot between one conceptual domain and another. These diagrams give us an excuse to shift from one point of view to another and to find that the two points of view are related by the structure of the diagram and the meanings that are associated with it.

One can think about this situation as an allegory for a search for relationship that is mediated by a special place where the meeting can be accomplished. That place, the place of the diagram, is a multiplicity that is a unity where the multiplicity resides in the many interpretations that the diagram can receive, and the unity resides in the act of making the diagram, a making that can be accomplished and reenacted by any one who wishes to come to the understanding that the diagram offers. It is in the making that the many becomes the one and the one becomes the many.

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