Chapter 11
Why Do Mathematicians Need Diagrams? Peirce's Existential Graphs and the Idea of Immanent Visuality

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Introduction

The topic of this chapter is the relationship between mathematical reasoning, diagrams, and everyday visual experience. My goal here is not to discuss either the external causes of this relationship or the variety of ways in which mathematicians actually use pictures and diagrams in their work. There is ample literature on both topics, including accounts of how the links between numerical and spatial representations are rooted in the same patterns of brain activity (Gracia-Bafalluy and Noël 2008; Hubbard et al. 2005), discussions of particular ways in which conceptual material and images are combined in mathematical reasoning (Loeb 2012; Lowrie and Kay 2001; Martinec and Salway 2005; Pinto and Tall 2002) and studies of the cases in which the application of diagrammatic representation proves to be especially conducive to teaching math (Bakker and Hoffmann 2005, Boaler 2016, Danesi 2016, pp. 92–108, Hegarty and Kozhevnikov 1999, Kucian et al. 2011, Legg 2017, Prusak 2012). The question I would like to ask here is more general: Why at all do mathematicians need to use diagrams, images and other visualizations in their work?

One way to approach this question is to say that pictures and diagrams play in mathematical proofs the role of auxiliary tools (Hanna 2007; Mumma 2010; Brown 1999). According to this view, pictures and diagrams are used by mathematicians in order to facilitate their reasoning and then translate those pictures and diagrams into a formal calculus. Although the diagrams are constructed as elaborate staged observations that make certain steps of a mathematical proof visually available, from this perspective, they do not constitute an independent mathematical language and are but partial and imprecise models designed for the purposes of informal demonstration only (Barker-Plummer 1997; Kulpa 2009). Accordingly, on this view,

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S. A. Costa et al. (eds.), *Mathematics (Education) in the Information Age*, Mathematics in Mind, https://doi.org/10.1007/978-3-030-59177-9_11

mathematicians do not actually build proofs directly on visual imagery, but rather use the latter to enhance the symbolic formalization of the former. This view does have a significant practical merit, as it proves suggestive of a variety of particular modes of use associated with diagrams in mathematics. Yet in spite of its practical merit, in terms of the question posed above, this view does not help much. From the more general perspective the question above represents, explaining the advantages of using something by saying that it is good for the purpose, is like explaining the effects of opium, as the doctor from Moliere's *Imaginary Invalid* famously puts it, by *virtus dormitiva*, or its *capacity* to do so. The immediate further questions, in this case, are "What is it exactly that makes diagrams, understood as tools, useful?" and "Why exactly are formal proofs not enough?"

Another answer to the same general question—the answer I am going to defend here—is to say that there is a tight relationship between the deductive character of mathematical reasoning and the very way mathematicians construct their diagrams. According to this view, all deductions, including mathematical ones, in order to be accomplished, require some sort of observation—and therefore, ipso facto involve visual experience. Charles S. Peirce, the principal proponent of this view, claimed that, although not all diagrammatic reasoning is mathematical in nature, there is no mathematical reasoning proper that is not diagrammatic (CP1: 54, CP2: 216, CP5: 148, where CP followed by volume number and then paragraph number refers to *Collected Papers of Charles S. Peirce* [Peirce 1931–1958]). Peirce also believed that, this being the case, mathematical diagrams could be construed not simply as partial supplementary aids to formal mathematical proofs, but as immediate visualizations of the deductive process as such. Peirce's view has two important consequences. The first consequence is that the very necessity of mathematical deductions should be considered internal to the diagrams mathematicians construct. The second consequence is that there has to be something about the very nature of ordinary visual experience that directly links the basic spatial relations supporting our visual integration, on the one hand, and our mathematical intuitions, on the one hand. Furthermore, Peirce was convinced that, if these two claims are a matter of fact, then it should be possible to construct a deductive mathematical language that would amount to a complete system of diagrammatic expression independent of formal symbolic proofs.

Visual Representation

In order to fully appreciate the possibility of the independent visual language Peirce had in mind, and see how the idea of such language might help us answer, in the Peircean vein, the general question formulated above, we will need to understand why exactly Peirce the mathematician attached so much importance to diagrammatic expression. Although, as it is commonly recognized, mathematical reasoning was the heartbeat that pumped blood through the veins of Peirce's entire philosophical system, Peirce did not have a full-fledged philosophy of mathematics. For the lack of a systematic, self-explanatory account, then, we first need to look for the sources of Peirce's interest in diagrams that are external to his philosophy.

All of those sources are well known, yet have never been considered together. Meanwhile, such consideration, however brief, might prove very helpful. First, Peirce confessed that he had a strong *personal* habit of thinking by means of visual images, and that he was inclined to attribute this capacity to his mathematical mindset (MS 619: 8, 1909, where MS followed by manuscript number and then page number refers to the *Charles S. Peirce Papers*, Houghton Library, Harvard University). At the same time, visual experience, Peirce insisted, was at the core of ordinary linguistic competence. In one of the entries of his late diary, he makes a confession: "I do not think I ever reflect in words: I employ visual diagrams, firstly, because this way of thinking is my natural language of self-communion, and secondly, because I am convinced that it is the best system for the purpose" (MS 619: 8, 1909). As an educator, Peirce believed that it would be a good idea if some sort of diagrammatic logic were taught in schools prior to the grammar of any natural language (CP4: 619). In his correspondence over the years, Peirce confessed repeatedly that, to him personally, English was as foreign as any other tongue. Moreover, he linked his incapacity for linguistic expression to his left-handedness, which, as he explains one of his letters to a mathematician Cassius L. Keyser, in turn, framed his social interactions:

> But I am left-handed; and I often think that means that I do not use my brain in the way that the mass of men do, and that peculiarity betrays itself also in my ways of thinking. Hence, I have always labored under the misfortune of being thought "original." Upon a set subject, I am likely to write worse than any man of equal practice (quoted in Brent 1998: 43. As Brent (1998: 15) notes, Peirce in fact was able to use *both* of his hands in writing simultaneously. For example, he was able to shock his students by writing on the blackboard, ambidextrously and simultaneously, a logical or mathematical problem and its solution).

In an early draft of "A neglected argument for the reality of God" (1908), Peirce further clarifies the matter, stating that he was "accustomed to think of Reason and Authority as opposite ways of determining opinions, and to approve of the former alone" (MS 842, 180–181). According to one of Peirce's letters to his friend Victoria Welby, this attitude towards authority and social conventions in general might be partly explained by the fact that Peirce was "brought up with far too lose a rein," except that he "was forced to think hard and continuously" (Peirce 1958: 417). Another letter to Lady Welby contains a more extensive explanation that provides a useful overall link between Peirce's left-handedness, his troubles with written language, his disdain for conventionality and the meticulousness of his personal thinking habits:

> [A]s a boy I invented a language in which almost every letter of every word made a definite contribution to its signification. It involved a classification of all possible ideas; and I need not say that it was never completed…The grammar of my Language was, I need hardly say, modelled in a general way after the Latin Grammar as almost all ideas of grammar are to this day. It had, in particular, the Latin parts of speech; and it never dawned upon me that they could be other than they are in Latin. Since then I have bought Testaments in such languages as Zulu, Dakota, Hawaiian, Jagalu, Magyar…These studies have done much to broaden my ideas of language in general; but they have never made me a good writer,

> because my habits of thinking are so different from those of the generality of people. Besides I am left-handed (in the literal sense) which implies a cerebral development and connections of parts of the brain so different from those of right-handed people that the sinister is almost sure to be misunderstood and live a stranger to his kind, if not a misanthrope. This has, I doubt not, had a good deal to do with my devotion to the science of logic. Yet probably my intellectual left-handedness has been serviceable to my studies in that science. It has caused me to be thorough in penetrating the thoughts of my predecessors,—not merely their ideas as they understood them, but the potencies that were in them (Hardwick 1977: 95–96).

As all these letters and notes, taken together, suggest, in Peirce's case, the importance of visual experience extends beyond the bounds of purely theoretical concerns and has some implications in terms of his personal intellectual habits. Visual thinking, personal difficulties in dealing with written language, the nature of logic in general (and the model of a universal language in particular), left-handedness, and the tendency to disregard conventions happen to be intimately connected with each other.

Peirce's preference for visual representations, of course, went beyond this knot of personal intellectual idiosyncrasies. From very early on in his career, both as a mathematician and as a philosopher, Peirce paid close attention to the role played in mathematical cognition by *maps*. As a mathematician, he was professionally involved in solving mathematical problems related to geological maps (CP7: 85), and proving the so-called "four color theorem" (CP2: 105 CP5: 490 NEM4: 216–222, where NEM refers to Peirce's *New Elements of Mathematics* [Peirce 1976]). He also developed a map projection known as the "quincuncial map," which represented a transformation of conformal stereographic projection and was one of the first maps created with an application of the theory of functions of a complex variable (Eisele 1963, Kiryushchenko 2012, Kiryushchenko 2015; W4: 68–71, where W followed by volume number and then page number refers to the *Chronological Edition* [Peirce 1982]). As a philosopher, Peirce considered diagrams in general as *maps of thought* (CP4: 530). He rejected the idea of likeness, or similarity as originating from the comparison of two simple, visually given qualities. Instead, he believed likeness to be the result of the application of a *mapping rule* describing a relation established between two sets, where a unique element of one set is *paired* with one single element of another set. Peirce's principal suggestion was that what underpins our perceptions of things as being *alike* is the isomorphism not of substances, but of *relations* (see also Stjernfelt 2007: 50–77, Paavola 2011). And he claimed that maps, together with geometric diagrams and algebraic equations, were the primary examples of such isomorphism (NEM4: xv, CP4: 530, Bradley 2004: 71–73). A mathematical function is routinely understood as a mapping relation between sets of numbers, which tells us how to go on with interpreting the dynamics of the function. According to Peirce, by analogy, a visual feature we perceive as common to, say, a portrait of a person and the person themselves, is a result of mapping one set of relations between facial features onto another, which reveals a character of the portrayed person based on the schematization of an anticipated facial change.

Diagrams

Another source of Peirce's interest in the role diagrams play in mathematical reasoning is the fact that Peirce, whose first degree was in chemistry from Lawrence Scientific School at Harvard, had a tendency to draw broad parallels between his graphical logic and the idea of chemical valence. In particular, he compared logical relations to chemical compounds. For instance, on his view, a *medad* (a relation whose arity is zero) is similar to a saturated chemical compound—such that may result, for instance, from joining two bonds of a bivalent radicle (CP3: 421), and a *dyad* is similar to one oxygen atom chemically bonded to two atoms of hydrogen, which constitutes a molecule of water, etc. The analogy Peirce drew between his logic of relations and chemical valences is well known and thoroughly studied (Parker 1998: 63–70, Roberts 1973: 17–25, Samway 1995). However, one historical aspect behind this analogy is rarely mentioned. Namely, it is that its source lies in the metamorphosis, which had taken place in chemistry in the mid-1840s, and which was triggered by the formulation of the chemical type theory.

The idea that the type theory and, later, the theory of valences brought about was that chemical compounds could be studied not as mixtures of actual substances, but as relational pictures, or diagrammatic schematizations of those substances. Chemists discovered that the relational structure of a molecule and transformations of chemical compounds could be *depicted* in a certain way, with the use of basic graphical conventions. Thus, it is the idea of chemical valences that actually gave birth to the first fully developed scientific language, which provided a diagrammatic projection of the (previously hidden) life of its natural object. This said, the reason why Peirce attached so much value to the analogy between his mathematical diagrammatic logic and the system of chemical valences is that he considered both logical and molecular graphs as messages capable of *saying what* the matter of fact is and, simultaneously, *showing how* it is to be interpreted. In both cases, seeing how the graphs develop into meaningful structures and understanding how this development works is one and the same process—or, better say, one and the same *act*.

Another source of inspiration for Peirce with respect to mathematical diagrams was his correspondence with Alfred Bray Kempe, a British mathematician best known for his proof of the four-colour theorem (later shown incorrect). In 1886, the Royal Society of London published Kempe's *Memoir on the theory of mathematical form*. In the opening paragraph of the *Memoir*, Kempe stated that his intention was to separate whatever is necessary for "exact or mathematical thought" from "the accidental clothing," as well as to offer an "exposition of fundamental principles" and "a description of some simple and uniform modes of putting the necessary matter in evidence" (Kempe 1886: 2). The fundamental principles of the mathematical thought, separated from the accidental geometrical, algebraic, and logical clothing, were presented by Kempe as a system of *diagrams* that consisted of spots connected by different types of lines. Kempe's diagrams were supposed to express the universal form of algebraic and geometrical representations that would reveal a deeper grammar of mathematical thinking common to both. Kempe sent Peirce a copy of the *Memoir*, and a few months later, in 1887, Peirce answered with some suggestions that caused Kempe to make revisions also published in the *Transactions of the Royal Society of London* later that year (W6: xlv).

Now that we have situated diagrammatic expression within Peirce's mathematical mindset and learned what areas of research beyond philosophy and pure mathematics conditioned Peirce's aptitude for visual thinking, we can see that all the interconnections between these areas and Peirce's personal intellectual idiosyncrasies are based on an amalgam of a few core ideas. These are the ideas of likeness as isomorphism, natural language, basic relational structure of things, and certain intellectual economy that prescribes us to pay attention to what is necessary while disregarding the accidental. As will transpire, what brings all these ideas together is the role that, according to Peirce, is played in mathematical reasoning by *observation*.

One of Peirce's entries for Mark Baldwin's *Dictionary of philosophy and psychology* (1901) reads as follows:

> In mathematical reasoning there is a sort of observation. For a geometrical diagram or array of algebraical symbols is constructed according to an abstractly stated precept, and between the parts of such diagram or array certain relations are observed to obtain, other than those which were expressed in the precept. These being abstractly stated, and being generalized, so as to apply to every diagram constructed according to the same precept, give the conclusion (CP2: 216).

Peirce further claims that, in any particular instance of mathematical reasoning (not only in the case of geometry, but also in the case of algebraic equations and syllogistic structures), "there must be something amounting to a diagram before the mind's eye," and that "the act of inference consists in observing a relation between parts of that diagram that had not entered into the design of its construction" (NEM4: 353, CP2: 279). *Inferring*, then, according to Peirce, is *observing* attentively what an experiment with a diagram brings about. To use one of Peirce's own examples, a particular case of Barbara syllogism, written down correctly, represents a simple diagram that clearly *shows* the relationship between the three terms involved, and, in doing so, actually *exhibits* the fact that the middle term of the syllogism occurs in both premises. Likewise, an algebraic equation is a rule that maps one relation between variables onto another in such a way that further manipulation could lead to the discovery of a series of new facts. Even a purely symbolic algebraic formalisation, then, is an *icon* that pictorially represents relations between the terms involved.

A simple geometrical example would be Pythagoras' theorem. The majority of the proofs of this theorem require that, in order to explain the relation among the three sides of a right triangle, a geometer should make a certain *rearrangement*. In the initial, Pythagoras's own version of the proof, it is the rearrangement of the four identical right triangles whose hypotenuses form a square. When describing the process of such rearrangement in some detail, Peirce adds that, in any other case similar to the two above, what we need is

> to set down, or to imagine, some individual and definite schema, or diagram—in geometry, a figure composed of lines with letters attached; in algebra an array of letters of which some are repeated. This schema is constructed so as to conform to a hypothesis set forth in general

> terms in the thesis of the theorem. Pains are taken so to construct it that there would be something closely similar in every possible state of things to which the hypothetical description in the thesis would be applicable, and furthermore to construct it so that …, although the reasoning is based upon the study of an *individual* schema, it is nevertheless *necessary*, that is, applicable to all possible cases (CP 4: 233; emphasis added).

In an unpublished work titled "Syllabus" (c. 1902), Peirce extrapolates this point about the link between manipulating images, deductive necessity and discovery of new truths to icons in general:

> For a great distinguishing property of the icon is that by the direct observation of it other truths concerning its object can be discovered than those which suffice to determine its construction. Thus, by means of two photographs a map can be drawn, etc. Given a conventional or other general sign of an object, to deduce any other truth than that which it explicitly signifies, it is necessary, in all cases, to replace that sign by an icon. This capacity of revealing unexpected truth is precisely that wherein the utility of algebraical formulae consists, so that the iconic character is the prevailing one (CP2: 279).

A year later, in lecture VI of his Harvard *Lectures on pragmatism* (1903), Peirce goes as far as to claim:

> *All* necessary reasoning without exception is diagrammatic. That is, we construct an icon of our hypothetical state of things and proceed to observe it. This observation leads us to suspect that something is true, which we may or may not be able to formulate with precision, and we proceed to inquire whether it is true or not. For this purpose, it is necessary to form a plan of investigation and this is the most difficult part of the whole operation. We not only have to select the features of the diagram which it will be pertinent to pay attention to, but it is also of great importance to *return again and again to certain features*. Otherwise, although our conclusions may be correct, they will not be the particular conclusions at which we are aiming (CP 5.162; emphasis added).

Based on these, as well as other, more complicated examples, Peirce further shows that it is never the case that, in solving a problem, simply thinking in general terms is enough. "It is necessary," he says, "that something should be *done*. In geometry, subsidiary lines are drawn. In algebra, permissible transformations are made. Thereupon, the faculty of observation is called into play. Some relation between the parts of the schema is remarked" (CP 4:23, Hull 2017: 149; Joswick 1988: 113).

As Peirce notes, any one of Euclid's theorems is first formulated in abstract terms. However, in the *Elements*, such abstract statement, from which only some trivial truths may be deduced, is followed by the construction of a geometrical figure, and then, upon observation, the initial statement is reformulated in new terms; this time—with reference to the figure constructed. This, in turn, is followed by modifying the figure (by moving certain parts of it, or adding new lines, or both), and ascertaining whether the modifications hold good relative to the second formulation. Once this is done, Peirce says, the words, "which had to be demonstrated," follow without any further restatement of the result in abstract terms. As he further notes,

> [i]n like manner when we have finished a process of thinking, and come to the logical criticism of it, the first question we ask ourselves is "What did I conclude?" To that we answer

> with some *form of words,* probably. Yet we had probably not been thinking in any such form—certainly not, if our thought amounted to anything…What the process of thinking may have been *has nothing to do with this question* (CP2: 55, emphasis added).

There is, in the end of all this construction and rearrangement, a moment at which the result is *shown* by the speediest way possible, and after which thought can only idle in creating trivial corollaries. Again, according to Peirce, geometry represents only one of many possible cases in which this ultimate point is revealed. In fact, in any logical process whatsoever, Peirce says,

> [w]hen we contemplate the premiss, we mentally *perceive* that that being true the conclusion is true. … Since the conclusion becomes certain, there is some state at which it becomes *directly* certain. Now this no symbol can show; for a symbol is an indirect sign depending on the association of ideas. Hence, *a sign directly exhibiting the mode of relation is required* (CP4: 75, emphasis added).

According to Peirce, mathematics can discover new regularities due to the following two features that diagrams exhibit. First, because there is always an array of possible transformations, which are implied by the very way a given diagram is constructed, and all of which will never be enacted. Second, because, due to the essential indeterminacy of perception, we cannot predict in advance what particular transformations out of the array will in fact be enacted, and what the ultimate result of those transformations will be (Stjernfelt 2007: 81–83). What these two features imply is that mathematics essentially is an observation-based *activity*, a habit-driven, and yet creative *practice* rather than a static deductive grammar that supplies rules for the contemplation of abstract mathematical forms (Campos 2009; Hull 2017). Within mathematical reasoning as a practice, visual imagination has a three-fold role to play. First, a mathematician forms a *skeletonized* iconic representation, a diagram, whether geometrical or algebraic, of the facts he is interested in considering. The principal purpose of the initial skeletonization, Peirce says, "is to strip the significant relations of all disguise," so that "only one kind of concrete clothing is permitted—namely, such as, whether from habit or from the constitution of the mind, has become so familiar that it decidedly aids in tracing the consequences of the hypothesis (CP3: 559). Second, a mathematician observes this diagrammatic picture until, at some point, "a hypothesis suggests itself that there is a certain relation between some of its parts." Third, he experiments upon the diagram in order to test his hypothesis, so that "it is *seen* that the conclusion is compelled to be true by the conditions of the construction of the diagram" (CP2: 278, CP3: 560, Joswick 1988: 108–109). Mathematicians, thus, use some basic features of spatial representation to construct skeletonized images, or diagrams, such that certain changes in the relations between parts of those diagrams and a further analysis thereof reveal the necessary deductive force of the argument the diagrams represent. Considered in this vein, the diagrams are not just illustrations of the reasoning process; they are the process itself, visualized. And the only *authority* that we have in this case is not symbolic conventions, but the reasoning process itself, immediately visually present.

Recall that mathematical diagrams show relations that are constitutive of their objects and that, at the same time, can be manipulated so that new truths about their objects are discovered. Although a diagram is constructed "according to an abstractly stated precept" (CP2: 216), not all possible relations between the parts of the diagram are initially predefined in the precept. In this respect, diagrammatic expressions, if sufficiently conventionalized, are like any other language in that the array of possible interpretations presupposed by their initial construction always exceeds the array of new interpretations available, given our current goals and our point of view. Some Peirce scholars (Ambrosio 2014: 257) extrapolate this link between iconicity and the generative aspect of language on *any* representation:

> The very process of constructing an icon matters for Peirce, as it reveals the very respects in which a particular sign stands for its object. What seems to emerge from Peirce's account is that the very relation of representation is itself the result of a process of discovery: 'constructing' an icon amounts to discovering, and selecting, relevant respects in which a representation captures salient features of the object it stands for.

Given this, it is not surprising that Peirce himself consistently links iconicity and language. In particular, he claims, for instance, that "in the syntax of every language there are *logical icons* of the kind that are aided by conventional rules" (CP2: 280; emphasis added). The suggestion here is that language is capable of conveying and storing information not only because it symbolically encodes this information and refers to appropriate external objects, but also due to the fact that its syntax iconically frames our perception. On this view, the very order of meaning to some extent depends on the visual schematisms set up by the general syntactic arrangement of a given language. On this view, in a sense, the way we put organize the symbols we use in writing reflects the way we *think*.

In using diagrams, what we have is, as it were, a system of keyholes, through which we see something only because we do not see all the rest. However, what is peculiar about the use of diagrams in *mathematics* is that, even though the possibilities are limitless, the mathematician is capable of anticipating changes between the parts of a given diagrams that are characterized by *necessity*. Peirce admits that the nature of this relationship between novelty and necessity presents an unresolved problem. He claims that, "how the mathematician can guess in advance what changes to make is a mystery" (NEM4: 215). However, one might speculate that this capacity has something to do with the interplay between two Peircean distinctions: the deductive force of mathematical reasoning vs. the compulsive force of perception, and the active power of the imagination vs. the passive receptivity of perception. Taken together, the distinctions constitute part of the reason observation, according to Peirce, is always involved in mathematical reasoning. And the link between the two, again, is provided by an analogy between *perceptual* and *mathematical* judgments:

> We speak of *hard facts*. We wish our knowledge to conform to hard facts. Now, the "hardness" of fact lies in the insistency of the percept, its entirely irrational insistency... But this factor is not confined to the percept. We can know nothing about the percept … except through the perceptual *judgment*, and this likewise compels acceptance without any assignable reason. This indefensible compulsiveness of the perceptual judgment is precisely what

> constitutes the cogency of mathematical demonstration. One may be surprised that I should pigeon-hole mathematical demonstration with things unreasonably compulsory. But it is the truth that the nodus of any mathematical proof consists precisely in a judgment in every respect similar to the perceptual judgment except only that instead of referring to a percept forced upon our perception, it refers to an imagination of our creation. There is no more why or wherefore about it than about the perceptual judgment, "This which is before my eyes looks yellow" (CP7: 659).

The analogy, as Peirce further describes it, is rather intricate and by no means self-evident. The receptivity of perception is passive, while the imagination is an active capacity. What Peirce is saying here is that perceptual content forced on the passive receptivity of perception and an imprint produced by my own active power of imagination share the same phenomenological quality. Just as a percept is forced upon our perceptive capacity, a mathematical truth is forced upon our imagination; there is no "why or wherefore" about either of the two. In the latter case, there is a parallelism between the internal imaginative experimentation with diagrams (the capacity to *predict* the dynamic pattern of future changes) and external visual perception based on the capacity to *adapt* to the ever-changing environment. This analogy between the diagrammatic mathematical visuality and our ordinary, everyday visual experience now finally needs to be clarified.

Existential Graphs

As has been argued above, according to Peirce, spatial imagination and abstract reasoning are involved in the process of manipulating diagrams not as two distinct mental faculties, but as two aspects of the same activity put to work together. The point is aptly summarized in Hull (2017: 147): "Peirce's conception of a diagram is fundamentally and inseparably both conceptual *and* spatial insofar as reasoning by diagrams engages the continuum of spatial extension in the reasoning process."

Mathematics, then, is a practice that makes use of a set of specific cognitive mechanisms in order to creatively schematize together the general and the particular, abstractions and images, thought and action. A mathematician is capable of conceptualizing and, *in doing so*, directly observing the world as an ordered variety of forms of relations, because in mathematical thought, visual integration and conceptual syntheses are as *mutually interdependent* as two sides of a sheet of paper. Naturally, this mutual interdependence of general concepts and individual images should have a medium. Therefore, there should exist the possibility for a diagrammatic language that, in using simple graphical conventions, would embody the unity of the visual and the conceptual, of the perceptual dynamics and logical inference. This, Peirce believed, should be a language capable of visually representing thinking as it happens—thinking *in actu* (CP4: 6).

The possibility of such language, introduced by Peirce in late 1890s as "Existential Graphs," is contingent on two facts. First, according to Peirce, any perception can only be of a change. Just as there is no feeling of one's skin when a feather is not drawn across it, there is no proper vision of an object without some perception of the corresponding stereotypical motion. A cheetah chasing its prey, a butterfly in flight, a pencil bouncing off the table—each of these moving images loads our perception with habitual expectations, without which the visual integration necessary for our grasping those objects as such, would not be complete. Accordingly, if motion of an object does not just tell us where an object is going, but helps us recognize the object as such, it shall do so whether this object is a an animal, a pencil, a mathematical function, or a thought itself. If mathematics, expressed in a system of diagrams, or graphs, is to borrow from the architecture of ordinary visual recognition, then, in order to capture thought in action, to grasp the *continuity* of thinking, what we need to work out is not just a set of graphical conventions, but also a corresponding set of moves. In short, we need a system of *moving* pictures in order to turn thought into a proper object of study.

The second fact is this. Peirce, I believe, would admit that visual perceptions are inferential. Simply seeing something as "red" requires the capacity to apply the concept "red." Besides, acquiring such a concept involves a long history of piecemeal adjustments and readjustments, gradually habitualized intakes and responses to various objects in various circumstances. And this requires mastering some inferential skills. Moreover, according to Peirce, there is no sharp line of demarcation between perceptual judgment and hypothetical (or abductive) inference. Both amount to an act of a fallible insight assembling different elements that were present in our minds before. In the case of abduction, "it is the idea of putting together what we had never before dreamed of putting together which flashes the new suggestion before our contemplation" (CP5: 181). Perceptual judgment, in turn, "is the result of a process…not sufficiently conscious to be controlled, or, to state it more truly, not controllable and therefore not fully conscious" (ibid.). Peirce's system of graphs represents a move in the opposite direction: with the help of simple graphic conventions, the graphs make inferences a matter of visual perception.

On the one hand, then, we have images supported by the inferential ties that hold together our linguistic competence. On the other hand, we have inferences encoded visually. There is thus an exchange between the external, inferentially informed imagery of ordinary perception, and the immanent, diagrammatic imagery of mathematical thought. In using a set of basic spatial intuitions, Peirce's graphs *show* how inferences work. Meanwhile, the cognitive mechanisms that allow us to make those intuitions into the moving objects that the graphs are, are the same as those that shape our ordinary perception. To put it slightly differently, manipulating the graphs, which leads to the discovery of new truths, is based on the same perceptual dynamics that characterizes ordinary vision. But the deductive force of a conclusion, which results from the manipulation, is revealed due to the visuality that is immanent to the mechanisms of inference.

Final Remark

To conclude, Peirce's graphs represent an intricate knot of relations between written language, ordinary visual experience, necessary mathematical reasoning, and imaginative experimentation. While an ordinary person is content with the passive exercise of external perception only, a mathematician makes a good use of the interplay between the external visuality of objects and the immanent visuality of inferences in order to combine the creativity of mathematical thinking and the robust deductive necessity of its results.

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