Dominator and Total Dominator Colorings in Graphs

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1 Introduction

A *dominating set* of a graph G is a set $S \subseteq V(G)$ such that every vertex in $V(G) \smallsetminus S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of cardinality $\gamma(G)$ is called a γ-*set of G*.

A *total dominating set*, abbreviated a TD-set, of a graph G with no isolated vertex is a set $S \subseteq V(G)$ such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A TD-set of G of cardinality $\gamma_t(G)$ is called a γ_t-set of G. Total domination is now well studied in graph theory. The literature on the subject of total domination in graphs has been surveyed and detailed in the book [19].

A *proper vertex coloring* of a graph G is an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, so that adjacent vertices are assigned distinct colors. A proper vertex coloring whose colors are taken from a set of k colors, usually the set $[k] = \{1, 2, \ldots, k\}$, is called a *proper k-coloring*. In a given coloring of G, a *color class* of the coloring is a set consisting of all those vertices assigned the same color. The *vertex chromatic number* of G, denoted $\chi(G)$, is the smallest positive integer k for which G has a proper k-coloring. A χ-*coloring* of G is a proper k-coloring of G that uses $\chi(G)$ colors. In what follows, we simply

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call a proper vertex coloring a *coloring*, and we refer to the vertex chromatic number as the *chromatic number*.

In this chapter, we combine the concept of domination (total domination) in graphs with the concept of colorings in graphs and study dominator colorings (respectively, total dominator colorings) of a graph. In Section 3, we formally define dominator colorings in graphs, and in Section 4, we formally define the analogous concept of total dominator colorings in graphs. In these sections, we present selected results on the so-called dominator chromatic number and total dominator chromatic number of a graph.

2   Graph Theory Notation

For completeness, we include some graph theory terminology that we will use in this chapter. A vertex of degree 1 is called a *leaf*, and its unique neighbor is called a *support vertex*. Two vertices v and w are *neighbors* in a graph G if they are adjacent, that is, if $vw \in E(G)$. The *open neighborhood* of a vertex v in G is the set of neighbors of v, denoted $N_G(v)$, whereas the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V(G)$ is the set of all neighbors of vertices in S, denoted $N_G(S)$, whereas the *closed neighborhood* of S is $N_G[S] = N_G(S) \cup S$. The *S-private neighborhood* of a vertex $v \in S$ is defined by $\mathrm{pn}_G(v, S) = \{w \in V(G) : N_G[w] \cap S = \{v\}\}$. Thus, $\mathrm{pn}_G(v, S) = N_G[v] \smallsetminus N_G[S \smallsetminus \{v\}]$. We note that if $v \in \mathrm{pn}_G(v, S)$, then the vertex v is isolated in the subgraph $G[S]$. A vertex outside the set S that belongs to the set $\mathrm{pn}_G(v, S)$ is called an *S-external private neighbor* of v. If the graph G is clear from the context, we omit the subscript G in the above definitions. For example, we write $N[v]$ and $N[S]$ rather than $N_G[v]$ and $N_G[S]$, respectively.

We denote a *complete graph* on n vertices by K_n, and we denote a *path* and *cycle* on n vertices by P_n and C_n, respectively. We denote a *complete bipartite graph* with partite sets of cardinality m and n by $K_{m,n}$. A *star* is a graph $K_{1,n}$ for some $n \geq 1$. A *double star* is a tree with exactly two (adjacent) non-leaf vertices. If one of these vertices is adjacent to ℓ_1 leaves and the other to ℓ_2 leaves, then we denote the double star by $S(\ell_1, \ell_2)$. By a *nontrivial graph*, we mean a graph of order at least two. The *corona* $\mathrm{cor}(G)$ of a graph G, also denoted $G \circ K_1$ in the literature, is the graph obtained from G by attaching a leaf v' to every vertex v of G. The 2-*corona* $G \circ P_2$ of G is the graph of order $3|V(G)|$ obtained from G by attaching a path of length 2 to each vertex of G so that the resulting paths are vertex-disjoint.

Given a graph F, a graph G is *F-free* if it does not contain any induced subgraph isomorphic to F. If G is $K_{1,3}$-free, then G is said to be *claw-free*. A graph is *chordal* if it contains no induced cycle of length 4 or more. A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set. A *universal vertex* in a graph is a vertex that is adjacent to every other vertex in the graph. A *clique* in G is a complete subgraph of G. The *clique number* of G, denoted $\omega(G)$, is the maximum cardinality of a clique in G.

A set of vertices in a graph G is a *packing* if the vertices in S are pairwise at distance at least 3 apart, that is, if u and v are arbitrary distinct vertices in S, then $d(u, v) \geq 3$. Equivalently, S is a packing if the closed neighborhoods of vertices in S are pairwise disjoint. A subset S of vertices in a graph G is an *open packing* if the open neighborhoods of vertices in S are pairwise disjoint. Further the set S is a *perfect packing* (respectively, a *perfect open packing*) if every vertex belongs to exactly one of the closed (respectively, open) neighborhoods of vertices in S. The *packing number* $\rho(G)$ (respectively, the *open packing number* $\rho^o(G)$) is the maximum cardinality of a packing (respectively, open packing) in G.

A vertex and an edge are said to *cover* each other in a graph G if they are incident in G. A *vertex cover* in G is a set of vertices that covers all the edges of G. The *vertex covering number* $\tau(G)$ (also denoted by $\beta(G)$ or $\mathrm{vc}(G)$ in the literature) is the minimum cardinality of a vertex cover in G. The independence number $\alpha(G)$ of a graph G is the maximum cardinality of an independent set in G.

3 Dominator Colorings

A vertex in a graph G *dominates* itself and all vertices adjacent to it. Further, a vertex is a *dominator* of a set S if it dominates every vertex in S. A *dominator coloring* of a graph G is a proper coloring of G with the additional property that every vertex in $V(G)$ dominates all vertices in at least one color class, that is, each vertex of the graph belongs to a singleton color class or is adjacent to every vertex of some (other) color class. The *dominator chromatic number* $\chi_d(G)$ of G is the minimum number of color classes in a dominator coloring of G. A χ_d-*coloring* of G is a dominator coloring of G that uses $\chi_d(G)$ colors.

The concept of a dominator coloring in a graph was birthed in the late 1970s when Cockayne, Hedetniemi, and Hedetniemi [9] defined the domatic number of a graph involving partitions into dominating sets. In 2006, Hedetniemi, Hedetniemi, and McRae [14] further studied the concept of dominator colorings in graphs. (We remark that these two papers are cited as [4] and [13], respectively, in the 2006 paper by Gera, Horton, and Rasmussen [13].) On March 15, 2004, Hedetniemi, Hedetniemi, Laskar, McRae, and Wallis [15] submitted a paper on dominator partitions in graphs, but due to the backlog in the journal at the time, the paper only appeared 5 years later! In 2006, Gera et al. [13] published a paper on dominator colorings in graphs, and in 2007, Gera [11, 12] continued the study of dominator colorings.

Since every vertex is a dominator of itself, the coloring of G that assigns a unique color to each vertex is a trivial dominator coloring of G. Thus, every graph G has a dominator coloring, and therefore the dominator chromatic number $\chi_d(G)$ is well-defined. Since every dominator coloring of G is a coloring of G, we have the following observation.

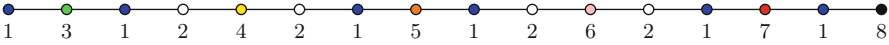**Observation 1** *For every graph G, we have $\chi(G) \leq \chi_d(G)$.*

Fig. 1 A χ_d-coloring of a path P_{17}

The simplest example to show that strict inequality may occur in Observation 1 is to take G to be a path P_4 given by $v_1v_2v_3v_4$. We note that $\chi(G) = 2$ and the unique 2-coloring of G has color classes $\{v_1, v_3\}$ and $\{v_2, v_4\}$. However, neither the vertex v_1 nor v_4 dominates any color class, implying that $\chi_d(G) \geq 3$. However, the 3-coloring of G with color classes $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2\}$, and $V_3 = \{v_4\}$ is a dominator coloring of G, noting that the vertices v_1 and v_3 dominate the color class V_2, the vertex v_2 dominates both color classes V_1 and V_2, and the vertex v_4 dominates its own color class V_3. Thus, $\chi_d(G) \leq 3$. Consequently, $\chi_d(G) = 3$. As shown by Theorems 2 and 3, the difference $\chi_d(G) - \chi(G)$ can be made arbitrarily large by taking, for example, G to be a path P_n or cycle C_n of sufficiently large order n.

We note that if G is a star $K_{1,k}$ where $k \geq 1$, then a proper 2-coloring of G is also a dominator coloring of G, and so $\chi(G) = \chi_d(G) = 2$. If $G = K_n$, then $\chi(G) = \chi_d(G) = n$. Hence, equality in Observation 1 is possible.

Gera et al. [13] determined the dominator chromatic number of a path P_n on n vertices. We note that $\chi_d(P_2) = \chi_d(P_3) = 2$. As observed earlier, $\chi_d(P_4) = 3$. It is a simple exercise to verify that $\chi_d(P_5) = 3$. If G is the path P_n: $v_1v_2 \ldots v_n$ where $n \geq 6$, let $f : V(G) \rightarrow \{1, 2, \ldots, 2 + \lceil \frac{n}{3} \rceil\}$ be the dominator coloring defined by

$$f(v_i) = \begin{cases} 1 & \text{when } n \ (\text{mod } 6) \in \{1, 3\} \\ 2 & \text{when } n \ (\text{mod } 6) \in \{0, 4\} \\ \lceil \frac{i}{3} \rceil + 2 & \text{when } n \ (\text{mod } 6) \in \{2, 5\}. \end{cases}$$

However if $n \equiv 1 \ (\text{mod } 3)$, then we redefine $f(v_n)$ to be the value $\lceil \frac{n}{3} \rceil + 2$. When $n = 16$, for example, the resulting dominator coloring is illustrated in Figure 1 where here $f(v_{16}) = \lceil \frac{16}{3} \rceil + 2 = 8$ (and where color 1 is blue, color 2 is white, color 3 is green, etc.).

Gera et al. [13] proved that the dominator coloring f defined above is a χ_d-coloring of the path P_n.

Theorem 2 ([13]) *For $n \geq 2$, we have*

$$\chi_d(P_n) = \begin{cases} 1 + \lceil \frac{n}{3} \rceil & \text{if } n \in \{2, 3, 4, 5, 7\} \\ 2 + \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$
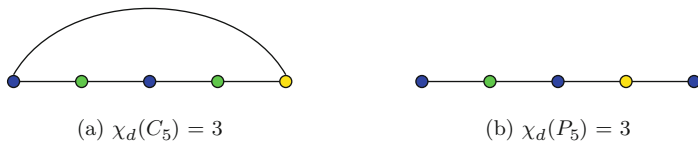
(a) $\chi_d(C_5) = 3$ (b) $\chi_d(P_5) = 3$

Fig. 2 χ_d-coloring of C_5 and P_5

In 2007, Gera [12] determined the dominator chromatic number of a cycle C_n on n vertices.

Theorem 3 ([12]) *We have* $\chi_d(C_3) = 3$, $\chi_d(C_4) = 2$, *and* $\chi_d(C_5) = 3$, *while for* $n \geq 3$ *and* $n \notin \{4, 5\}$, *we have*

$$\chi_d(C_n) = \left\lceil \frac{n}{3} \right\rceil + 2.$$

We note that if H is a spanning proper subgraph of G, then a χ_d-coloring of G may not be a dominator coloring of H. As a simple example, the χ_d-coloring of the cycle C_5 shown in Figure 2(a) is not a dominator coloring of P_5, even though $\chi_d(C_5) = \chi_d(P_5) = 3$.

For disconnected graphs, we have the following upper and lower bounds on the dominator chromatic number.

Theorem 4 ([13]) *If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$ where $k \geq 2$, then*

$$k - 1 + \max \{\chi_d(G_i) \mid i \in [k]\} \leq \chi_d(G) \leq \sum_{i=1}^{k} \chi_d(G_i).$$

Proof Let $\mathcal{C}_i$ be a χ_d-coloring of G_i for all $i \in [k]$, where we can choose the colorings so that no two color classes uses the same color. Let $\mathcal{C}$ be the union of these k color classes, and so the restriction of $\mathcal{C}$ to the component G_i yields the χ_d-coloring $\mathcal{C}_i$ for all $i \in [k]$. The coloring $\mathcal{C}$ is a chromatic dominator coloring of G, and so

$$\chi_d(G) \leq |\mathcal{C}| = \sum_{i=1}^{k} |\mathcal{C}_i| = \sum_{i=1}^{k} \chi_d(G_i) = \sum_{i=1}^{k} \chi_d(G_i).$$

To prove the lower bound, consider a component of G with largest dominator chromatic number. Each of the remaining $k - 1$ components of G requires at least one additional color, since every vertex must be a dominator of some color class. Hence, $\chi_d(G) \geq k - 1 + \max \{\chi_d(G_i) \mid i \in [k]\}$. $\qquad\square$

That the lower bound of Theorem 4 is tight may be seen by taking G to be the vertex-disjoint union of $k \geq 2$ stars $K_{1,n}$, for some $n \geq 2$. Each component H of G has

$\chi_d(H) = 2$. Assigning to all leaves of G the same color and assigning to the central vertex of each of the k stars a unique color produce a dominator coloring of G using $k + 1 = (k - 1) + 2 = k - 1 + \max\{\chi_d(H) \mid H \text{ is a component of } G\}$ colors.

That the upper bound of Theorem 4 is tight may be seen by taking G to be the vertex-disjoint union of $k \geq 2$ copies of $K_{2,n}$ for some $n \geq 2$. Let $\mathcal{C}$ be a dominator coloring of G. Since every vertex must be a dominator of some color class, we note that each component of G has at least one color not used in any other color class. Suppose that some component H of G uses exactly one color, say color 1, not used in any other color class. If two or more vertices in H are colored 1, then no vertex in the color class associated with the color 1 is a dominator of any color class, a contradiction. Hence, exactly one vertex in H is colored 1. However, every vertex different from v and not adjacent to v in the component H is therefore not a dominator of any color class, a contradiction. Hence, each component of G has at least two colors unique to that component, implying that $\chi_d(G) \geq 2k = \sum_{i=1}^{k} \chi_d(G_i)$, where $G_1, G_2, \ldots, G_k$ denote the components of G.

3.1 Bounds on the Dominator Chromatic Number

By definition of a dominator coloring, we have the following observation.

Observation 5 *If v is an arbitrary vertex in a graph G, then in every dominator coloring of G, the closed neighborhood $N[v]$ of v contains a color class.*

Theorem 6 *If G is a graph, then $\chi_d(G) \geq \rho(G)$, with strict inequality if there is no perfect packing in G.*

Proof If S is a packing in G, then by Observation 5, the closed neighborhoods of vertices in S contain at least $|S|$ color classes, and so $\chi_d(G) \geq |S|$. Choosing S to be a maximum packing, we have that $\chi_d(G) \geq \rho(G)$. Further, if G does not have a perfect packing, then at least one additional color class is needed to contain the vertices that do not belong to the closed neighborhood of any vertex in S, and so $\chi_d(G) \geq \rho(G) + 1$. $\qquad\square$

The dominator chromatic number of a graph is related to its independence number as follows, where the independence number $\alpha(G)$ of a graph G is the maximum cardinality of an independent set in G.

Theorem 7 ([13, 15]) *If G is a connected graph of order n, then $\chi_d(G) \leq n + 1 - \alpha(G)$.*

Proof If $n = 1$, then the result is trivial since in this case $\chi_d(G) = n = \alpha(G) = 1$. Hence, we may assume that $n \geq 2$. Let I be a maximum independent set in G, and consider the coloring $\mathcal{C}$ that colors all vertices in I with the same color, and colors all remaining $n - \alpha(G)$ vertices each with a different color. Each vertex in $V(G) \smallsetminus I$ dominates the color class that contains it, noting that it is the unique vertex in that color class. By the connectivity of G and by the independence of the set I, every vertex in I has degree at least 1 and has all of its neighbor in $V(G) \smallsetminus I$. Therefore,
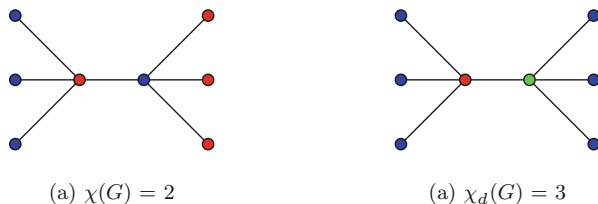
(a) $\chi(G) = 2$

(a) $\chi_d(G) = 3$

Fig. 3 A double star $G = S(3, 3)$

by our choice of the coloring $\mathcal{C}$, every vertex in I dominates every color class that contains one of its neighbors. Hence, $\mathcal{C}$ is a dominator coloring of G, implying that $\chi_d(G) \leq |\mathcal{C}| = n + 1 - |I| = n + 1 - \alpha(G)$. $\qquad\qquad\qquad\square$

That the bound of Theorem 7 is sharp may be seen by taking, for example, a double star $G = S(\ell_1, \ell_2)$. We note that the (unique) proper 2-coloring of the double star is a dominator coloring of G, since no leaf dominates a color class. Hence, $\chi_d(G) \geq 3$. However, coloring all the leaves with one color, and coloring the two central vertices (the non-leaf vertices) with distinct colors, produces a proper 3-coloring that is a dominator coloring. Hence, $\chi_d(G) = 3$. In this example, G has order $n = \ell_1 + \ell_2 + 2$ and $\alpha(G) = \ell_1 + \ell_2 = n - 2$, and so $\chi_d(G) = n + 1 - \alpha(G)$. In the special case when $G = S(3, 3)$, we illustrate the χ-coloring and χ_d-coloring of G in Figure 3(a) and 3(b), respectively.

As observed earlier, the coloring of a graph G of order n that assigns a unique color to each vertex is a trivial dominator coloring of G, and so $\chi_d(G) \leq n$. By Observation 1, if G is a connected graph on at least two vertices, then $\chi_d(G) \geq \chi(G) \geq 2$. We state these observations formally as follows.

Observation 8 *If G is a connected graph of order $n \geq 2$, then $2 \leq \chi_d(G) \leq n$.*

A characterization of graphs achieving equality in the lower and upper bounds of Observation 8 is given by the following result.

Theorem 9 ([11, 15]) *If G is a connected graph of order $n \geq 2$, then the following holds.*

(a) *$\chi_d(G) = 2$ if and only if G is a complete bipartite graph.*
(b) *$\chi_d(G) = n$ if and only if G is a complete graph.*

Proof Suppose that $\chi_d(G) = 2$. By Observation 1, $\chi(G) = 2$, implying that the 2-coloring of G is a dominator coloring of G. Let V_1 and V_2 be the two color classes of G. If $|V_i| = 1$ for some $i \in [2]$, then $G = K_{1,n-1}$, and the desired result follows. Hence, we may assume that $|V_i| \geq 2$ for $i \in [2]$. Thus, no vertex can be a dominator of its own color class, implying that every vertex in V_i is a dominator of the color class V_{3-i} for $i \in [2]$, that is, $G = K_{n_1,n_2}$ where $n_i = |V_i|$. Hence if $\chi_d(G) = 2$, then G is a complete bipartite graph. The converse is immediate.

Suppose next that $\chi_d(G) = n$. By Theorem 7, $\chi_d(G) \leq n + 1 - \alpha(G)$. If G is not a complete graph, then $\alpha(G) \geq 2$, implying that $\chi_d(G) \leq n - 1$, a contradiction. Hence, G must be a complete graph. The converse is immediate. $\square$

The dominator chromatic number of a graph is related to its domination number. For a given graph G, let $\mathcal{A}(G)$ denote the set of all γ-sets in G. We next present an upper bound on the dominator chromatic number of a graph.

Theorem 10 *If G is a connected graph, then*

$$\chi_d(G) \leq \gamma(G) + \min_{S \in \mathcal{A}(G)} \{\chi(G - S)\},$$

and this bound is tight.

Proof Let S be an arbitrary γ-set of G, and let $\mathcal{C}$ be a proper coloring of the graph $G - S$ using $\chi(G - S)$ colors. We extend the coloring $\mathcal{C}$ to a coloring of the vertices of G by assigning to each vertex in S a new and distinct color. Let $\mathcal{C}'$ denote the resulting coloring of G, and note that $\mathcal{C}'$ uses $\gamma(G) + \chi(G - S)$ colors. Since S is a dominating set of G, every vertex in $V(G) \smallsetminus S$ is adjacent to at least one vertex of S. Since the color class of $\mathcal{C}'$ containing a given vertex of S consists only of that vertex, each vertex in $V(G) \smallsetminus S$ is adjacent to every vertex of some color class in the coloring $\mathcal{C}'$. Further, each vertex of S is a dominator of its own (singleton) color class. Hence, $\mathcal{C}'$ is a dominator coloring of G using $\gamma(G) + \chi(G - S)$ colors. This is true for every γ-set of G. The desired upper bound now follows by choosing S to be a γ-set of G that minimizes $\chi(G - S)$. The bound is achieved, for example, by taking G to be a complete graph. $\square$

The proof of Theorem 10 yields the following more general result.

Theorem 11 *If G is a connected graph, and $D(G)$ denotes the set of all dominating sets of G, then*

$$\chi_d(G) \leq \min_{S \in D(G)} \{ |S| + \chi(G - S) \}.$$

Gera [11, 12] established the following upper and lower bounds on the dominator chromatic number of an arbitrary graph in terms of its domination number and chromatic number.

Theorem 12 ([11, 12]) *Every graph G satisfies*

$$\max\{\gamma(G), \chi(G)\} \leq \chi_d(G) \leq \gamma(G) + \chi(G).$$

Proof By Observation 1, recall that $\chi(G) \leq \chi_d(G)$. To show that $\gamma(G) \leq \chi_d(G)$, consider a χ_d-coloring of G with color classes $V_1, \ldots, V_k$, where $k = \chi_d(G)$. Let v_i be an arbitrary vertex in the color class V_i for $i \in [k]$, and consider the set $D = \{v_1, \ldots, v_k\}$. Let v be an arbitrary vertex of G. By definition of a dominator coloring, the vertex v is a dominator of the color class V_i for at least one $i \in [k]$. In particular,

the vertex $v = v_k$ or the vertex v is adjacent to the vertex v_k. This is true for every vertex v of G, implying that D is a dominating set of G. Hence, $\gamma(G) \leq |D| = \chi_d(G)$. This establishes the desired lower bound. The upper bound follows from Theorem 10, noting that $\chi(G - S) \leq \chi(G)$ for every proper subset $S \subset V(G)$. □

Gera [12] established an intermediate value-type result for the dominator chromatic number and showed that for every triple (a, b, c) of integers where $1 \leq a \leq c$ and $2 \leq b \leq c$ is a *dominator realizable triple*, there exists a connected graph G such that $\gamma(G) = a$, $\chi(G) = b$, and $\chi_d(G) = c$.

That the lower bound of Theorem 12 is sharp may be seen by taking, for example, a complete bipartite graph G with both partite sets of cardinality at least 2. In this case, $\gamma(G) = \chi(G) = \chi_d(G) = 2$. To see that the upper bound is sharp, let G, for example, be a path P_n or a cycle C_n for some $n \geq 8$ even. In this case, $\gamma(G) = \lceil \frac{n}{3} \rceil$ and $\chi(G) = 2$, and so by Theorems 2 and 3, we have $\chi_d(G) = 2 + \lceil \frac{n}{3} \rceil = \chi(G) + \gamma(G)$.

3.2 Special Classes of Graphs

In this section, we consider the dominator chromatic number of certain classes of graphs.

3.2.1 Bipartite Graphs

As a special case of Theorem 12 when G is a bipartite graph, we have the following result.

Theorem 13 ([11, 12, 15]) *If G is a bipartite graph, then $\gamma(G) \leq \chi_d(G) \leq \gamma(G) + 2$.*

In order to characterize the graphs achieving equality in the lower bound of Theorem 13, we define a special subclass of bipartite graphs as follows.

Definition 1 A bipartite graph G is a *partially complete bipartite graph* if G can be obtained from the disjoint union of $k \geq 1$ complete bipartite graphs K_{x_i, y_i} with partite sets X_i and Y_i where $x_i = |X_i| \geq 2$ and $y_i = |Y_i| \geq 2$ for all $i \in [k]$ by adding edges between copies of these graphs so that the resulting graph is connected and the following conditions hold, where $X = \cup_{i=1}^{k} X_i$ and $Y = \cup_{i=1}^{k} Y_i$.

(a) For each set X_i where $i \in [k]$, there is no set $A \subseteq Y \setminus Y_i$ such that $|A \cap Y_j| = 1$ for all $j \in [k] \setminus \{i\}$ and the set A dominates the set X_i.
(b) For each set Y_i where $i \in [k]$, there is no set $A \subseteq X \setminus X_i$ such that $|A \cap X_j| = 1$ for all $j \in [k] \setminus \{i\}$ and the set A dominates the set Y_i.
(c) For each set X_i where $i \in [k]$, if $A \subseteq X_i$ dominates ℓ of the partite sets in Y, then $\ell \geq |A|$.

(d) For each set Y_i where $i \in [k]$, if $A \subseteq Y_i$ dominates ℓ of the partite sets in X, then $\ell \geq |A|$.

We note, for example, that every complete bipartite graph with both partite sets of cardinality at least 2 is a partially complete bipartite graph. In particular, $K_{2,n}$ is a partially complete bipartite graph for all $n \geq 2$.

Theorem 14 ([11]) *If G is a connected bipartite graph of order at least* 2, *then* $\gamma(G) = \chi_d(G)$ *if and only if G is a partially complete bipartite graph.*

Let Q_n be the n-dimensional hypercube, and so Q_n can be represented as the nth power of K_2 with respect to the Cartesian product operation $\square$, that is, $Q_1 = K_2$ and $Q_n = Q_{n-1} \square K_2$ for $n \geq 2$. Gera [11] established the following upper bound on the dominator chromatic number of an n-dimensional hypercube. The proof given in [11] is algorithmic in nature.

Theorem 15 ([11]) *For* $n \geq 2$, $\chi_d(Q_{n+1}) \leq \chi_d(Q_n) + \gamma(Q_n)$.

The following result established an upper bound on the dominator chromatic number of a connected bipartite graph in terms of its order.

Theorem 16 ([11]) *If G is a connected bipartite graph of order* $n \geq 2$, *then* $\chi_d(G) \leq \frac{1}{2}(n + 2)$, *and this bound is sharp.*

Proof Let X and Y be the partite sets of G, where $|X| \leq |Y|$. Coloring all vertices in Y with the same color and assigning a new color to each vertex of X produce a dominator coloring of G using $|X| + 1 \leq \frac{1}{2}n + 1$ colors. This establishes the desired upper bound.

That this bound is sharp may be seen by taking G to be the corona of an arbitrary connected bipartite graph F, and so $G = \mathrm{cor}(F)$. The graph G has order $n = 2|V(F)|$ and satisfies $\gamma(G) = |V(F)|$. Coloring all added vertices of degree 1 with the same color and assigning a new color to every vertex of F produce a dominator coloring of T using $|V(F)| + 1$ colors. Thus, $\chi_d(G) \leq |V(F)| + 1$. We note that each added vertex v of degree 1 either dominates its own class, in which case the vertex v is the only vertex of that color, or dominates the class of its unique neighbor, in which case its neighbor in F is the only vertex of that color. This implies that at least $|V(F)|$ vertices must receive a unique color. Since at least one additional color is needed for the remaining vertices of G, every dominator coloring of G uses at least $|V(F)| + 1$ colors. Thus, $\chi_d(G) \geq |V(F)| + 1$. As observed earlier, $\chi_d(G) \leq |V(F)| + 1$. Consequently, $\chi_d(G) = |V(F)| + 1 = \frac{1}{2}n + 1$. $\square$

3.2.2 Trees

Since no tree is a partially complete bipartite graph, we have the following consequence of Theorems 13 and 14.

Theorem 17 ([11]) *If T is a tree of order* $n \geq 2$, *then* $\chi_d(T) = \gamma(T) + 1$ *or* $\chi_d(T) = \gamma(T) + 2$.

By Theorem 2, if T is a path P_n where $n \geq 8$, then $\chi_d(T) = \lceil \frac{n}{3} \rceil + 2 = \gamma(T) + 2$. We note that if T is obtained from $k \geq 1$ vertex-disjoint copies of $K_{1,r}$ where $r \geq 2$ by adding a new vertex and joining it to the central vertex of each star, then $\chi_d(T) = k + 1 = \gamma(T) + 1$. More generally, if a tree T contains a γ-set D such that $V(T) \smallsetminus D$ is an independent set, then $\chi_d(T) = \gamma(T) + 1$, noting that we can color all vertices outside D with the same color and assign a new color to every vertex of D to produce a minimum dominator coloring of T using $\gamma(T) + 1$ colors. In particular, we note that both values for the dominator chromatic number in Theorem 17 are achievable for infinitely many trees. We say that a tree belongs to *dominator class i* if $\chi_d(T) = \gamma(T) + i$ for $i \in [2]$. It remains an open problem to characterize the dominator class 1 and dominator class 2 trees.

A sufficient condition for a tree to belong to dominator class 1 is the following.

Proposition 18 ([5, 15]) *If T is a nontrivial tree such that $\gamma(T) = \tau(T)$, then T belongs to dominator class* 1.

Proof Let D be a minimum vertex cover in T, and so $|D| = \tau(T) = \gamma(T)$. Since D is a vertex cover, the set $V \smallsetminus D$ is an independent set. Coloring all vertices in $V \smallsetminus D$ with the same color and assigning a new color to each vertex of D produce a dominator coloring of T using $|D| + 1 = \gamma(T) + 1$ colors. Thus, $\chi_d(T) \leq \gamma(T) + 1$. By Theorem 17, $\chi_d(T) \geq \gamma(T) + 1$. Consequently, $\chi_d(T) = \gamma(T) + 1$. $\square$

As observed in [5, 15], the converse of Proposition 18 is not true in general. For example, the tree T shown in Figure 4 belongs to dominator class 1, noting that $\chi_d(T) = 5 = \gamma(T) + 1$. However, $\gamma(T) = 4 < 5 = \tau(T)$. The 5-coloring shown in Figure 4 is a χ_d-coloring of the tree T.

In 2012, Boumediene Merouane and Chellali [4] provide a characterization of trees that belongs to dominator class 1.

Theorem 19 ([4]) *If T is a nontrivial tree, then $\chi_d(T) = \gamma(T) + 1$ if and only if there exists a γ-set, D, of T such that the set $V(T) \smallsetminus (D \cup N(A))$ is an independent set where $A = \{v \in D : pn(v, D) = \{v\}\}$, that is, A is the set of vertices in D, if any, that are isolated in $T[D]$ and have no D-external private neighbor.*

In practice, a tree may admit many minimum dominating sets, and it may not be easy to identify such a set satisfying the statement of Theorem 19. Therefore, in 2015, Boumediene Merouane and Chellali [5] provide a different characterization, which is more pleasing in the sense that it resulted in a polynomial time algorithm

Fig. 4 A χ_d-coloring of a tree T

(a) The tree T

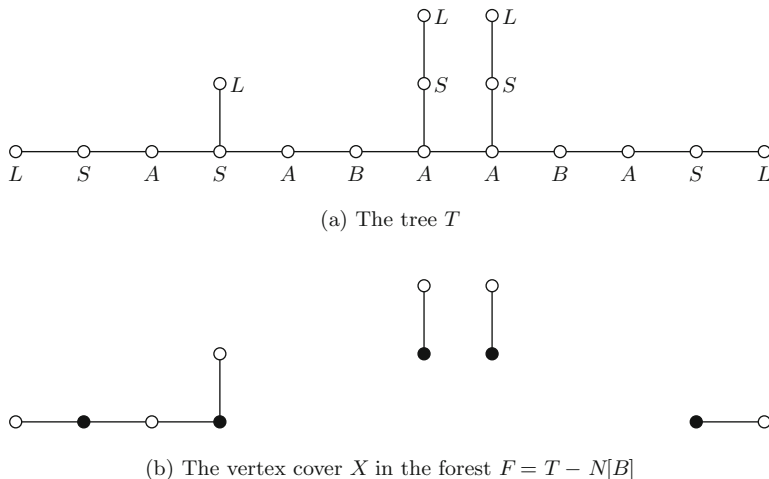(b) The vertex cover X in the forest $F = T - N[B]$
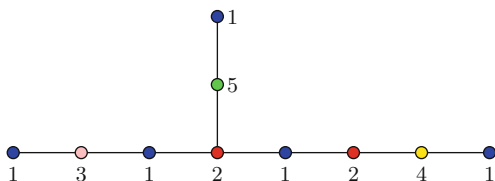
Fig. 5 A tree T and its associated forest F

for computing the dominator chromatic number for every nontrivial tree. In order to state this result, we need some additional terminology.

Recall that a leaf of a tree is a vertex of degree 1 and a vertex with a leaf neighbor is a *support vertex*. Given a tree T, let L and S be the set of leaves and support vertices of T, respectively. Further, let A be the set of vertices of T that are neither leaves nor support vertices, but have a support vertex as a neighbor, that is, if $v \in A$, then $v \notin L \cup S$, but the vertex v is adjacent to a vertex in S. Further, let B be the set of vertices that have all their neighbors in A but do not belong to A, that is, $B = \{v \in V \smallsetminus A : N(v) \subseteq A\}$. Let F be the forest obtained from T by deleting all vertices in $N[B]$, that is, $F = T - N[B]$, or, equivalently, F is the subgraph of T induced by the set $V(T) \smallsetminus N[B]$. Let X be a minimum vertex cover of F containing all support vertices of T (if X contains a leaf of T, we simply replace this leaf by its neighbor in T). To illustrate these definitions, consider the tree T shown in Figure 5(a). The label of each vertex represents one of the sets, namely, L, S, A, or B, that it belongs to, as shown in Figure 5(a). The associated forest $F = T - N[B]$ is illustrated in Figure 5(b), where the vertices in the vertex cover X are given by the darkened vertices.

We are now in a position to state the characterization of trees that belong to dominator class 1 as given in [5].

Theorem 20 ([5]) *If T is a nontrivial tree, then $\chi_d(T) = \gamma(T) + 1$ if and only if the following three conditions hold.*

(a) *B is a packing.*
(b) *$B \cup X$ is a γ-set of T.*
(c) *$\gamma(F) = \tau(F)$.*

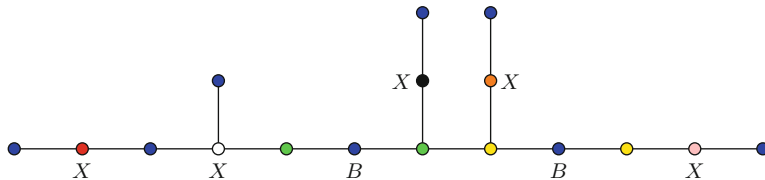**Fig. 6** A χ_d-coloring of the tree T

Suppose that T is a nontrivial tree satisfying the three conditions in the statement of Theorem 20. We color the vertices of T as follows.

- For each vertex v in B, we give a unique color to all vertices in $N(v)$.
- To each vertex in X, we give a unique (new) color.
- To all remaining vertices (including all vertices in $L \cup B$), we give the same, but new, color.

By condition (b), the set $X \cup B$ is a γ-set of T, and so $\gamma(T) = |B| + |X|$. Thus, the resulting coloring is a dominator coloring of T using $|B| + |X| + 1 = \gamma(T) + 1$ colors. Hence, $\gamma(T) + 1 \leq \chi_d(T) \leq |B| + |X| + 1 = \gamma(T) + 1$. Thus we must have equality throughout this inequality chain, implying that $\chi_d(T) = \gamma(T) + 1$.

To illustrate this coloring, consider the tree T shown earlier in Figure 5(a), where a vertex is labelled B or X if it belongs to the set B or X, respectively. To the one vertex in B, we color its two neighbors green, and to the other vertex in B, we color its two neighbors yellow. We color the five vertices in X with five new colors, namely, red, white, black, orange, and pink. Thereafter, we color all remaining vertices of T with a new color, namely, blue. The resulting coloring, illustrated in Figure 6, is a dominator coloring of T using $|B| + |X| + 1 = \gamma(T) + 1 = 8$ colors and is therefore a χ_d-coloring of T.

Based on Theorem 20, the authors in [5] give a quadratic time algorithm computing the dominator chromatic number of any nontrivial tree.

3.2.3 Chordal Graphs and Split Graphs

By Theorem 12, every graph G satisfies $\chi_d(G) \geq \gamma(G)$. In 2012, Chellali and Maffray [6] improved this bound by imposing certain structural restrictions on the graph.

Theorem 21 ([6]) *If G is a connected graph of order $n \geq 2$ that is C_4-free or is claw-free and different from C_4, then $\chi_d(G) \geq \gamma(G) + 1$.*

Since every chordal graph is C_4-free, as is every split graph, as an immediate consequence of Theorem 21, we have the following result.

Corollary 22 ([6]) *If G is a connected graph of order $n \geq 2$, then the following holds.*

(a) *If G is a chordal graph, then $\chi_d(G) \geq \gamma(G) + 1$.*
(b) *If G is a split graph, then $\chi_d(G) \geq \gamma(G) + 1$.*

We also remark that Theorem 17 follows immediately from Theorem 21, noting that every tree is, of course, C_4-free. Chellali and Maffray [6] characterized the split graphs that achieve equality in the bound of Corollary 22(b).

Theorem 23 ([6]) *If G is a connected split graph whose vertex set is partitioned into a clique Q and an independent set I such that Q is minimal, then $\chi_d(G) = \gamma(G) + 1$ if and only if every vertex of Q is a support vertex.*

3.2.4 Proper Interval Graphs and Block Graphs

In 2015, Panda and Pandey [32] study bounds on the dominator chromatic number for two important subclasses of chordal graphs, namely, proper interval graphs and block graphs.

A graph G is an *interval graph* if there exists a one-to-one correspondence between its vertex set and a family $\mathcal{F}$ of closed intervals in the real line, such that two vertices are adjacent if and only if their corresponding intervals intersect. Further, if no interval in $\mathcal{F}$ contains another interval in $\mathcal{F}$, then the graph G is called a *proper interval graph*. Panda and Pandey [32] establish the following lower and upper bounds for the dominator chromatic number of a proper interval graph in terms of its domination number and chromatic number. We note that the upper bound is a restatement of the result in Theorem 12.

Theorem 24 ([32]) *Every proper interval graph G satisfies*

$$\chi(G) + \gamma(G) - 2 \leq \chi_d(G) \leq \gamma(G) + \chi(G).$$

Moreover, all three values can be achieved by $\chi_d(G)$.

For a vertex v of G, the graph $G - v$ is the graph obtained from G by deleting v and deleting all edges of G incident with v. A vertex v is a *cut-vertex* of G if the number of components increases in $G - v$. A *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex of its own. Thus, a block is a maximal 2-connected subgraph of G. The number of vertices in a block is called the *order of the block*. Any two blocks of a graph have at most one vertex in common, namely, a cut-vertex. If a connected graph contains a single block, we call the graph itself a *block*. A *block graph* is a connected graph in which every block is a clique. A block containing exactly one cut-vertex is called an *end block*. A *non-complete block graph* has at least two end blocks. Panda and Pandey [32] generalized the result of Theorem 17 to the class of block graphs. (We note that every tree is a block graph, in which every block is a copy of K_2.)

Theorem 25 ([32]) *If G is a block graph of order at least 2 with k blocks where each block has the same order, then*

$$\chi_d(G) = \gamma(G) + \chi(G) - 1 \quad \text{or} \quad \chi_d(G) = \gamma(G) + \chi(G).$$

Further, both values can be achieved by $\chi_d(G)$.

To illustrate the tightness of the bounds, for $k \geq 1$, let $G_{k,1}$ be a block graph with $2k$ blocks $B_1, B_2, \ldots, B_{2k}$, each having order $s \geq 3$, and $2k - 1$ cut-vertices $v_1, v_2, \ldots, v_{2k-1}$ such that $V(B_i) \cap V(B_{i+1}) = \{v_i\}$ for all $i \in [2k-1]$. The resulting graph $G = G_{k,1}$ satisfies $\gamma(G) = k$, $\chi(G) = s$ and $\chi_d(G) = k + s - 1$. When $k = 3$ and $s = 3$, the block graph $G_{k,1}$ is illustrated in Figure 7. We color each vertex of the γ-set, $\{v_1, v_3, v_5\}$, of $G_{3,1}$ with a unique color (namely, green, yellow, and pink), and we 2-color the remaining vertices with two new colors (namely, blue and red). The resulting 5-coloring is a χ_d-coloring of G.

For $k \geq 3$, let $G = G_{k,2}$ be a block graph with $2k + 1$ blocks, $B_1, B_2, \ldots, B_{2k+1}$, each having order k, and $2k$ cut-vertices $v_1, v_2, \ldots, v_{2k}$. All vertices in block B_{2k+1} are cut-vertices, say $v_{k+1}, v_{k+2}, \ldots, v_{2k}$. For each $i \in [k]$, B_i is an end block, having exactly one cut-vertex v_i. For each j where $k + 1 \leq j \leq 2k$, block B_j has exactly two cut-vertices v_j and v_{j-k}. The resulting graph $G = G_{k,2}$ satisfies $\gamma(G) = k$, $\chi(G) = k$ and $\chi_d(G) = 2k$. When $k = 3$, the block graph $G_{k,2}$ is illustrated in Figure 8. We color each vertex of the γ-set, $\{v_1, v_2, v_3\}$, of $G_{3,1}$ with a unique color (namely, green, yellow, and pink), and we 3-color the remaining vertices with three new colors (namely, blue, red, and black). The resulting 6-coloring is a χ_d-coloring of G.

As a consequence of Theorem 25, we have the following result. We note that if G is a block graph, then the clique number $\omega(G)$ of G is the maximum order among all blocks in G.

Corollary 26 ([32]) *If G is a non-complete block graph that contains an end block of order $\omega(G)$, then $\chi_d(G) = \gamma(G) + \chi(G) - 1$ or $\chi_d(G) = \gamma(G) + \chi(G)$.*

Panda and Pandey [32] characterize the block graphs G for which one of the end blocks is of maximum size (namely, $\omega(G)$) and $\chi_d(G) = \gamma(G) + \chi(G) - 1$.

3.2.5 P_4-Free Graphs

Chellali and Maffray [6] determined the dominator chromatic number of the class of graphs that are P_4-free by exploiting the structure of these graphs, namely, that if
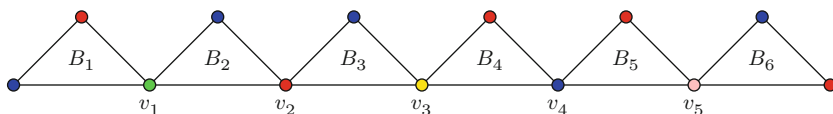

Fig. 7 A block graph $G_{3,1}$

Fig. 8 A block graph $G_{3,2}$

G is a P_4-free graph of order at least 2, then G or its complement $\overline{G}$ is disconnected (see Seinsche [34]). We mention that P_4-free graphs are also known as *cographs*.

Theorem 27 ([6]) *If G is a P_4-free graph, then the following holds.*

(a) *If G is connected, then $\chi_d(G) = \chi(G)$.*
(b) *If G is disconnected with $k \geq 2$ components and h of these components have a universal vertex, then either G has a component H with a universal vertex and satisfies $\chi(G) = \chi(H)$, in which case $\chi_d(G) = \chi(G) + 2k - h - 1$, or G has no such component, in which case $\chi_d(G) = \chi(G) + 2k - h - 2$.*

3.2.6 Other Classes

We mention that the dominator chromatic number of other classes of graphs has also been studied, including degree splitting graph of some graphs [22], dragon and lollipop graphs [30], wheel related graphs [35], the generalized Petersen graph [31], and Mycielskian graphs [1]. However, we do not define these classes of graphs here.

3.3 Graph Products

In this section, we present some results on the dominator chromatic number in Cartesian products of graphs. The Cartesian product $G \,\square\, H$ of graphs G and H is the graph with vertex set $V(G) \times V(H) = \{(g, h) : g \in V(G) \text{ and } h \in V(H)\}$, where two vertices (g_1, h_1) and (g_2, h_2) in the *Cartesian product $G \,\square\, H$ of graphs G and H are adjacent if either $g_1 = g_2$ and h_1h_2 is an edge in H or $h_1 = h_2$ and g_1g_2 is an edge in G.

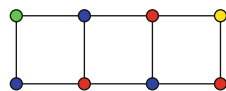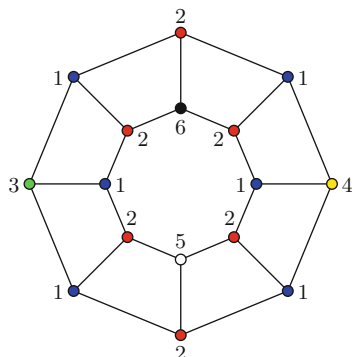**Fig. 9** A χ_d-coloring of $P_2 \,\square\, P_4$

Fig. 10 The circular ladder graph CL_8

In 2017, Chen, Zhao, and Zhao [8] determined the dominator chromatic number of Cartesian products of certain paths and cycles. The Cartesian product $P_m \,\square\, P_n$ of paths P_m and P_n is known as a $2 \times n$ grid graph.

Theorem 28 ([8]) $\chi_d(P_2 \,\square\, P_2) = 2$, $\chi_d(P_2 \,\square\, P_3) = \chi_d(P_2 \,\square\, P_3) = 4$, and for all $n \geq 5$,

$$\chi_d(P_2 \,\square\, P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 3.$$

A dominator coloring of the 2×4 grid graph, for example, using four colors is shown in Figure 9.

The Cartesian product of a cycle C_n on $n \geq 3$ vertices and a path P_2 on two vertices is called a *circular ladder graph* CL_n of order $2n$; that is, $CL_n = C_n \,\square\, K_2$ (cf. [23]). A circular ladder graph is also called a *cycle prism* in the literature. We note that CL_n is bipartite if and only if n is even. The circular ladder graph CL_n is also called the *n-prism* in the literature. For example, let $G = C_8 \,\square\, K_2$ be the circular ladder graph CL_8 shown in Figure 10. We note that G is a bipartite graph and $\gamma(G) = 4$, and so, by Theorem 13, $\chi_d(G) \leq 6$. As shown in the proof of Theorem 12, we can find a dominator coloring of G using six colors as follows. We first 2-color the vertices of G with the colors 1 and 2 (depicted as the colors blue and red in Figure 10), and thereafter we recolor the vertices of a γ-set of G with the colors 3, 4, 5, *and* 6 (depicted as the colors green, yellow, white, and black, respectively, in Figure 10). The resulting 6-coloring is a dominator coloring of G. Thus, $\chi_d(G) \leq 6$. Moreover, as shown in the proof of Theorem 29, $\chi_d(G) \geq \gamma(G) + 2 = 6$. Consequently, $\chi_d(CL_8) = 6$.

In 2015, Manjula and Rajeswari [29] claimed to have proven that $\chi_d(CL_n) = n + 1$ for all $n \geq 9$. This result is incorrect. The correct value for the dominator chromatic

Fig. 11 A χ_d-coloring of
CL$_3 = P_2 \,\square\, C_3$
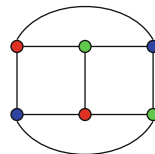
number of a circular ladder graph is given in Theorem 29 by Chen, Zhao, and Zhao [8].

Theorem 29 *The dominator chromatic number of the circular ladder graph* $\mathrm{CL}_n = C_n \,\square\, K_2$ *is given by* $\chi_d(\mathrm{CL}_3) = 3$ *and for all* $n \geq 4$ *as follows.*

$$
\chi_d(\mathrm{CL}_n) = \begin{cases} \frac{1}{2}(n+4) & \text{when } n \equiv 0 \,(\mathrm{mod}\ 4) \\ \frac{1}{2}(n+5) & \text{when } n \,(\mathrm{mod}\ 4) \in \{1, 3\} \\ \frac{1}{2}(n+6) & \text{when } n \equiv 2 \,(\mathrm{mod}\ 4). \end{cases}
$$

Proof Let $G = C_n \,\square\, K_2$ be the circular ladder graph CL_n where $n \geq 3$. A dominator coloring of CL_3 using three colors is shown in Figure 11, showing that $\chi_d(\mathrm{CL}_3) \leq 3$. By Observation 1, $\chi_d(\mathrm{CL}_3) \geq \chi(\mathrm{CL}_3) = 3$. Consequently, $\chi_d(\mathrm{CL}_3) = 3$. Hence in what follows, we let $n \geq 4$.

Let $x_1 x_2 \ldots x_n x_1$ and $y_1 x_2 \ldots y_n y_1$ be the two disjoint copies of the cycle C_n used to construct $\mathrm{CL}_n = C_n \,\square\, K_2$ and where $x_i y_i \in E(\mathrm{CL}_n)$ for $i \in [n]$. We note that $\gamma(G) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$, that is, $\gamma(G) = n/2$ if $n \equiv 0 \,(\mathrm{mod}\ 4)$, $\gamma(G) = n/2 + 1$ if $n \equiv 2 \,(\mathrm{mod}\ 4)$, and $\gamma(G) = (n+1)/2$ if $n \,(\mathrm{mod}\ 4) \in \{1, 3\}$.

We show firstly that $\chi_d(G) \leq \gamma(G) + 2$. If n is even, then we can apply Theorem 13 to yield $\chi_d(G) \leq \gamma(G) + 2$. Suppose that $n \equiv 3 \,(\mathrm{mod}\ 4)$. Thus, $n = 4k + 3$ for some $k \geq 0$. In this case, the set

$$
D = \bigcup_{i=0}^{k} \{x_{4i+1}, y_{4i+3}\}
$$

is a γ-set of D, noting that D is a dominating set of G and $|D| = 2(k+1) = (n+1)/2 = \gamma(G)$. We note that removing the set D from G produced a graph $G - D \cong P_{6k+4}$. We now 2-color the vertices of the path $G - D$, and thereafter we color each vertex of D with a unique color. The resulting coloring of G is a dominator coloring of G using $\gamma(G) + 2 = 2k + 4$ colors. Suppose next that $n \equiv 1 \,(\mathrm{mod}\ 4)$. Thus, $n = 4k + 1$ for some $k \geq 1$. In this case, we consider the set

$$
S = \bigcup_{i=0}^{k-1} \{x_{4i+2}, y_{4i+4}\}.
$$

We note that the set S is a packing in G. Further, the set S dominates all vertices of G, except for the two vertices y_1 and x_n (note that here $x_n = x_{4k+1}$). Further, we

note that $S \cup \{x_1\}$ is a γ-set of G, implying that $|S| = \gamma(G) - 1$. We note that with the set S as defined above, the graph $G - S$ can be obtained from a path P_{6k} on $6k$ vertices given by

$$P: \ y_1 y_2 y_3 x_3 x_4 x_5 \dots y_{4(k-1)+1} \ y_{4(k-1)+2} \ y_{4(k-1)+3} \ x_{4(k-1)+3} \ x_{4(k-1)+4} \ x_{4(k-1)+5}$$

that starts at the vertex y_1 and ends at the vertex x_{4k+1}, by adding the two vertices x_1 and y_{4k+1} and adding the four edges $x_1 y_1$, $x_1 x_{4k+1}$, $y_1 y_{4k+1}$, and $x_{4k+1} y_{4k+1}$. We now 2-color the vertices of the path P with the colors 1 and 2 and color x_1 and y_{4k+1} with the same new color 3 to produce a 3-coloring of $G - S$. Thereafter, we color the vertices of S with $|S| = \gamma(G) - 1$ new colors, one distinct color to each vertex. The resulting coloring is a dominator coloring of G using $\gamma(G) + 2$ colors. In particular, we note that y_1 and x_{4k+1} each are dominators of the color class $\{x_1, y_{4k+1}\}$ (whose vertices are colored 3), while every other vertex in $G - S$ is a dominator of the unique vertex in S that it is adjacent to. Further, each vertex v of S is a dominator of the color class that contains the vertex v (and is a singleton set consisting only of the vertex v).

To illustrate the above coloring, consider the case, for example, when $n = 9$. In this case, the set $S = \{x_2, y_4, x_6, y_8\}$ and is given by the set of darkened vertices in Figure 12(a). Further the 3-coloring of the graph $G - S$ is illustrated in Figure 12(b). We then extend this 3-coloring of $G - S$ to a 7-coloring of G by adding four new colors, one distinct color to each vertex in the set S. The resulting 7-coloring is a dominator coloring of G, implying that $\chi_d(\text{CL}_9) \leq 7$.

In all the above cases, we have shown that $\chi_d(G) \leq \gamma(G) + 2$. Further one can readily show that $\chi_d(G) \geq \gamma(G) + 2$. We present a proof of the simplest case when $n \equiv 0 \pmod 4$ as an illustration. In this case, $n = 4k$ for some $k \geq 1$. Further, $\gamma(G) = 2k$, and every γ-set of G is a packing. Each vertex $v \in D$ either dominates its own class, in which case the vertex v is the only vertex of that color, or dominates a color class that is a subset of its neighborhood, $N(v)$. Since the sets $N[v] = N(v) \cup \{v\}$ are vertex-disjoint sets for all $v \in D$, this implies that at least $|D| = \gamma(G)$ vertices must receive a unique color. Since at least two additional colors are needed for the remaining vertices of G, every dominator coloring of G uses at least $\gamma(G) + 2$ colors. Hence, $\chi_d(G) \geq \gamma(G) + 2$ in this case when $n \equiv 0 \pmod 4$. Analogous arguments show that $\chi_d(G) \geq \gamma(G) + 2$ in the three other cases when $n \pmod 4 \in \{1, 2, 3\}$. We

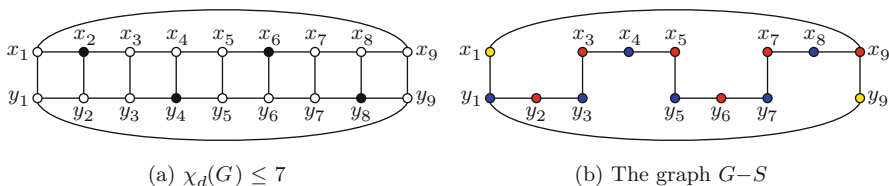(a) $\chi_d(G) \leq 7$ (b) The graph $G - S$

Fig. 12   A circular ladder graph $G = \text{CL}_9$

omit the details. Therefore, $\chi_d(G) = \gamma(G) + 2$. The desired result now follows from our earlier observation that $\gamma(G) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$. $\square$

We remark that Chen [7] continued the study of dominator chromatic number of Cartesian products of certain paths and cycles and considers the $3 \times n$ grid, $P_3 \square P_n$, the Cartesian product $P_3 \square C_n$. Two vertices (g_1, h_1) and (g_2, h_2) in the *direct product graph $G \times H$* of graphs G and H are adjacent if $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. They also consider the dominator chromatic number of the Cartesian product $K_m \square K_n$ for $m, n \geq 2$.

Paulraja and Handrasekar [33] determined the dominator chromatic number for some classes of graphs, such as the direct product $K_m \times K_n$ for $m, n \geq 2$, and the direct product $(K_m \circ K_1) \times K_r$ for $m, n \geq 2$. They also present results on the Cartesian product $K_n \square Q_r$ for $r \geq 3$, where Q_r is the *r*-dimensional hypercube. We omit these results here.

3.4   Dominator Partition Number

We discuss briefly in this section the dominator partition number of a graph. In their introductory paper, Hedetniemi, Hedetniemi, Laskar, McRae, and Wallis [15] define a *dominator partition* of a graph G as a coloring (not necessarily proper) with the property that every vertex in G is adjacent to all other vertices in some color class (including possibly its own). The *dominator partition number* of G, which they denote by $\pi_d(G)$, is the minimum number of color classes in a dominator partition of G. We note that every dominator coloring is a dominator partition, but not conversely. Thus, $\pi_d(G) \leq \chi_d(G)$ for all graphs G. Hedetniemi et al. [15] provide the following lower and upper bounds on the dominator partition number of a graph in terms of the minimum and maximum degrees.

Theorem 30   ([15]) *If G is a graph of order n, then*

$$\frac{n}{1 + \Delta(G)} \leq \pi_d(G) \leq n - \delta(G).$$

Hedetniemi et al. [15] showed that the dominator partition number of a graph is surprisingly one of two possible values.

Theorem 31 ([15]) *If G is a graph of order n, then $\pi_d(G) = \gamma(G)$ or $\pi_d(T) = \gamma(G) + 1$.*

Proof The proof of the lower bound $\pi_d(G) \geq \gamma(G)$ is identical to the proof we presented earlier of Theorem 12. To prove the upper bound $\pi_d(G) \leq \gamma(G) + 1$, let $D = \{v_1, \ldots, v_k\}$ be a γ-set of G. Since the partition $\pi = \{V_1, \ldots, V_{k+1}\}$ of V, where $V_i = \{v_i\}$ for $i \in [k]$ and where $V_{k+1} = V \setminus D$, is a dominator partition of G, we have that $\pi_d(T) \leq \gamma(G) + 1$. $\square$

3.5 Algorithmic and Complexity Results

In this section, we consider the algorithmic complexity of the problem of computing the dominator chromatic number of an arbitrary graph. Formally, we consider the following decision problem:

GRAPH DOMINATOR k-COLORABILITY

Input: A graph G, and an integer $k \geq 1$.
Question: Does G have a dominator k-coloring?

Hedetniemi et al. [17] showed that to determine if a graph G has a dominator 3-coloring can be computed in polynomial time.

Theorem 32 ([17]) GRAPH DOMINATOR 3-COLORABILITY *is solvable in polynomial time.*

To show that the GRAPH DOMINATOR k-COLORABILITY is NP-complete for $k \geq 4$, we give a transformation from GRAPH k-COLORABILITY:

GRAPH k-COLORABILITY

Input: A graph G, and an integer $k \geq 4$.
Question: Does G have a k-coloring?

Theorem 33 ([13, 15]) *Graph Dominator k-Colorability is NP-complete for general graphs, for $k \geq 4$.*

Proof Let k be an integer greater than 3. GRAPH DOMINATOR k-COLORABILITY is clearly in the class NP since we can efficiently verify that an assignment of colors to the vertices of G is both a proper coloring and that every vertex dominates some color class.

Next we transform an instance of GRAPH $(k-1)$-COLORABILITY to an instance of GRAPH DOMINATOR k-COLORABILITY. Given an instance of GRAPH $(k-1)$-COLORABILITY, a graph G, and a $k-1$ coloring of G, construct an instance of GRAPH DOMINATOR k-COLORABILITY as follows. Let G' be the graph obtained from G by adding a new vertex v to G and adding all edges joining v to every vertex of G. We now consider the instance given by the graph G' and a dominator k-coloring of G.

Let $\mathcal{C}$ be a $(k-1)$-coloring of G, and let $\mathcal{C}'$ be the k-coloring of G' obtained from the coloring $\mathcal{C}$ by assigning a new color to the vertex v. Thus, the color class containing v consists only of the vertex v. Since $\{v\} \subseteq N_{G'}[u]$ for every vertex in G',

every vertex in G' dominates some color class. Thus, C' is a dominator k-coloring of G'.

Conversely, suppose that G' has a dominator k-coloring C. Since v is adjacent to every other vertex in G', the vertex v is the only vertex in its color class. The removal of v produces a $(k-1)$-coloring of G.

It follows that G is $(k-1)$-colorable if and only if G' is dominator k-colorable.
$\square$

By Theorem 33, it is NP-complete to decide if a graph admits a dominator coloring with at most four colors. Chellali and Maffray [6] characterized the graphs G such that $\chi_d(G) \leq 3$ and showed that their characterization leads to a polynomial time recognition algorithm for such graphs. A rough estimate of the complexity of their algorithm is $O(n^8)$. We note that this result that the problem "$\chi_d(G) \leq 3$" can be solved in polynomial time is in contrast the problem "$\chi(G) \leq 3$," which in NP-complete.

In 2009, Hedetniemi et al. [15] and in 2011 Arumugam, Raja Chandrasekar, Misra, Philip, and Saurabh [2] studied algorithmic aspects of dominator colorings in graphs. They established the following complexity result.

Theorem 34 ([2, 15]) *For $k \geq 4$ an integer,* GRAPH DOMINATOR k-COLORABILITY, *is NP-complete for bipartite, chordal, planar, or split graphs.*

Arumugam et al. [2] complemented the above hardness results by showing that the GRAPH DOMINATOR COLORABILITY is fixed-parameter tractable in certain classes. Informally, a *parameterization* of a problem assigns an integer k to each input instance, and a parameterized problem is *fixed-parameter tractable*, abbreviated FPT, if there is an algorithm that solves the problem in time $f(k) \cdot |I|^{O(1)}$, where $|I|$ is the size of the input and f is an arbitrary computable function that depends only on the parameter k. (For a discussion on parameterized complexity, we refer the reader to the 2013 book by Downey and Fellows [10].)

A graph is an *apex graph* if there exists a vertex in G whose removal from G yields a planar graph. A family $\mathcal{F}$ of graphs is *apex minor-free* if there is a specific apex graph H such that no graph in $\mathcal{F}$ has H as a minor. As an example, planar graphs are apex minor-free since no planar graph has K_5 as a minor. Apex graphs play an important role in aspects of graph minor theory and are closed under the operation of taking minors, that is, contracting an edge or removing an edge or vertex leads to another apex graph.

As remarked in [2], for $k \geq 4$ an integer, GRAPH DOMINATOR k-COLORABILITY, is not fixed-parameter tractable in general graphs unless P$=$NP. However, the problem is fixed-parameter tractable in apex minor-free graphs (which include planar graphs) and chordal graphs.

Theorem 35 ([2]) *For $k \geq 4$ an integer, Graph Dominator k-Colorability, is fixed-parameter tractable on apex minor-free graphs and on chordal graphs.*

Arumugam et al. [2] show that for $k \geq 4$ an integer, GRAPH DOMINATOR k-COLORABILITY, can be solved in "fast" fixed-parameter tractable time in split graphs.

Theorem 36 ([2]) *For $k \geq 4$ an integer, Graph Dominator k-Colorability, can be solved in* $O(2^k \cdot n^2)$ *time on a split graph on n vertices.*

Arumugam et al. [2] pose the problem of whether for $k \geq 4$ an integer, GRAPH DOMINATOR k-COLORABILITY, can be solved in polynomial time on interval graphs.

4 Total Dominator Colorings

The total version of dominator coloring in a graph was studied by several authors. The concept of total dominator colorings in graphs was first defined in the manuscript by Hedetniemi, Hedetniemi, McRae, Rall, and Hedetniemi [16] dated July 9, 2009. Subsequently, Hedetniemi, Hedetniemi, Hedetniemi, McRae, and Rall [17] continued the study of total dominator colorings in graphs in their manuscript dated February 18, 2011. The first published papers on the topic appears to be the 2012 paper by Vijayalekshmi [36] and the 2015 paper by Kazemi [26].

Formally, a *total dominator coloring*, abbreviated TD-*coloring*, of a graph G with no isolated vertex is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of some other color class (different from its own color class). The *total dominator chromatic number* of G which we denote by χ_{td} (and denoted by $\chi_d^t(G)$ in [18, 26]) is the minimum integer k for which G has a TD-coloring with k colors. A χ_{td}-*coloring* of G is a coloring of G that uses $\chi_{td}(G)$ colors. Every total dominator coloring is a dominator coloring. Hence, we have the following observation.

Observation 37 *For every graph G without isolated vertices, we have* $\chi_d(G) \leq \chi_{td}(G)$.

Consider an arbitrary χ_{td}-coloring of G, and let S be a set consisting of one vertex from each of the resulting $\chi_{td}(G)$ color classes. Since every vertex in G is adjacent to every vertex of some color class (different from its own color class), the set S is a TD-set in G, implying that $\gamma_t(G) \leq |S| = \chi_{td}(G)$. Hence we have the following result, first observed by Vijayalekshmi [36] and Kazemi [26].

Observation 38 ([26, 36]) *For every graph G without isolated vertices,* $\gamma_t(G) \leq \chi_{td}(G)$.

Analogous results to Observation 8 and Theorem 9 hold for the total dominator chromatic number.

Theorem 39 ([26, 36]) *If G is a connected graph of order $n \geq 2$, then* $2 \leq \chi_{td}(G) \leq n$. *Moreover, the following holds.*

Fig. 13 A χ_{td}-coloring of a path P_{14}

(a) $\chi_{\mathrm{td}}(G) = 2$ if and only if G is a complete bipartite graph.
(b) $\chi_{\mathrm{td}}(G) = n$ if and only if G is a complete graph.

For disconnected graphs, we have the following upper and lower bounds on the total dominator chromatic number.

Theorem 40 ([36]) *If G is a disconnected graph with nontrivial components G_1, G_2, ..., G_k where $k \geq 2$, then*

$$2k - 2 + \max\{\chi_{\mathrm{td}}(G_i) \mid i \in [k]\} \leq \chi_{\mathrm{td}}(G) \leq \sum_{i=1}^{k} \chi_{\mathrm{td}}(G_i).$$

We remark that the total dominator chromatic number of a path and cycle is incorrectly determined in [26]. To state the total dominator chromatic number of a path P_n and a cycle C_n on n vertices, we shall need the following well-known result (see [19]).

Observation 41 *For $n \geq 3$, if $G \in \{P_n, C_n\}$, then we have*

$$\gamma_t(G) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor,$$

that is, $\gamma_t(G) = \frac{n}{2}$ if $n \equiv 0$ (mod 4), $\gamma_t(G) = \frac{n}{2} + 1$ if $n \equiv 2$ (mod 4), and $\gamma_t(G) = \frac{n+1}{2}$ for n odd.

Theorem 42 ([18]) *For $n \geq 2$, we have*

$$\chi_{\mathrm{td}}(P_n) = \begin{cases} \gamma_t(P_n) & \text{for } n \in \{2, 3, 6\} \\ \gamma_t(P_n) + 1 & \text{for } n \in \{4, 5, 7, 9, 10, 11, 14\} \\ \gamma_t(P_n) + 2 & \text{otherwise.} \end{cases}$$

For example, a χ_d-coloring of the path P_{14} (using $\gamma_t(P_{14}) + 1 = 8 + 1 = 9$ colors) is illustrated in Figure 13.

Thus, by Observation 41 and Theorem 42, we have the following closed formula for the total dominator chromatic number of a path of large order.

Theorem 43 ([18]) *For $n \geq 15$, $\chi_{\mathrm{td}}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor + 2$.*

For $n \geq 16$, we define next a $\chi_{\mathrm{td}}(P_n)$-coloring, $\mathcal{C}_n^*$, of a path P_n as follows. Let G be the path $v_1 v_2 \ldots v_n$, where $n \geq 16$. For each vertex v_i where $i \equiv 2, 3$ (mod 4), assign a unique color. For each vertex v_i where $i \equiv 1$ (mod 4), assign a new additional color, say 1. For each vertex v_i where $i \equiv 0$ (mod 4), assign a further additional
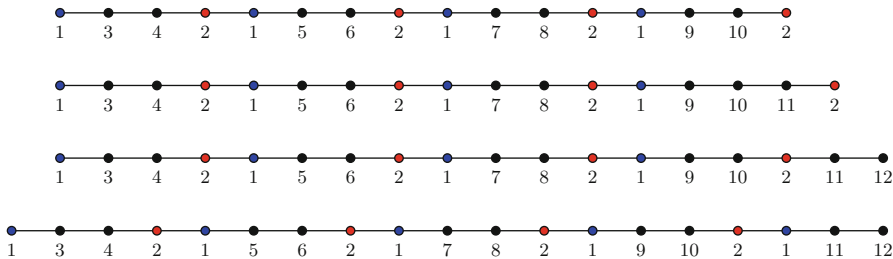
Fig. 14 A χ_{td}-coloring of a paths P_{16}, P_{17}, P_{18}, and P_{19}

color, say 2. Let $\mathcal{C}_n$ denote the resulting coloring. We now define a coloring $\mathcal{C}_n^*$ as follows. If $n \equiv 0$, 3 (mod 4), let $\mathcal{C}_n^* = \mathcal{C}_n$. If $n \equiv 1$ (mod 4), then recolor the vertex v_{n-1} (currently colored with color 2) with a new distinct color, and let $\mathcal{C}_n^*$ denote the resulting modified coloring. If $n \equiv 2$ (mod 4), then recolor the vertex v_{n-1} (currently colored with color 1) with a new distinct color, and let $\mathcal{C}_n^*$ denote the resulting modified coloring. The coloring $\mathcal{C}_n^*$ when $n \in \{16, 17, 18, 19\}$, for example, is illustrated in Figure 14. The darkened vertices in this coloring of $\mathcal{C}_n^*$ in Figure 14 form a γ_t-set of the path. A new color is assigned to each darkened vertex in the path.

Theorem 44 ([18]) $\chi_{td}(C_3) = 3$, $\chi_{td}(C_4) = 2$, and $\chi_{td}(C_{11}) = 8$. For all other values of $n \geq 5$, we have $\chi_{td}(C_n) = \chi_{td}(P_n)$.

4.1 Bounds on the Total Dominator Chromatic Number

By definition of a total dominator coloring, we have the following observation.

Observation 45 If v is an arbitrary vertex in a graph G without isolated vertices, then in every dominator coloring of G, the open neighborhood $N(v)$ of v contains a color class.

Theorem 46 If G is a graph without isolated vertices, then $\chi_{td}(G) \geq \rho^o(G)$, with strict inequality if there is no perfect packing in G.

Proof If S is an open packing in G, then by Observation 45, the open neighborhoods of vertices in S contain at least $|S|$ color classes, and so $\chi_{td}(G) \geq |S|$. Choosing S to be a maximum open packing, we have that $\chi_{td}(G) \geq \rho^o(G)$. Further, if G does not have a perfect open packing, then at least one additional color class is needed to contain the vertices that do not belong to the open neighborhood of any vertex in S, and so $\chi_{td}(G) \geq \rho^o(G) + 1$. $\qquad\square$

If H is any connected graph of order $k \geq 1$, then the 2-corona $G = H \circ P_2$ satisfies $\rho^o(G) = 2k = \chi_{td}(G)$, illustrating the existence of graphs G that contain a perfect
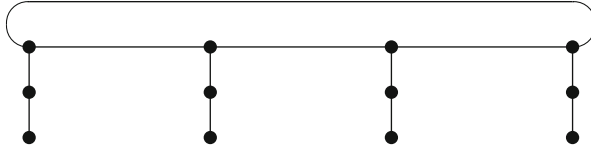
Fig. 15 The graph $C_4 \circ P_2$

open packing and satisfy $\rho^o(G) = \chi_{\mathrm{td}}(G)$. The graph $C_4 \circ P_2$, for example, is shown in Figure 15 (here, $H = C_4$).

If a graph G contains a perfect open packing, then it is not necessarily true that $\rho^o(G) = \chi_{\mathrm{td}}(G)$. The simplest example illustrating this is a path $G = P_4$, with $\rho^o(G) = 2$ and $\chi_{\mathrm{td}}(G) = 3$. More generally, if $G = P_n$ where $n \equiv 0 \pmod 4$ and $n \geq 8$, then G has a perfect open packing and $\rho^o(G) = \gamma_t(G)$. However in this case, by Theorem 42, we have $\chi_{\mathrm{td}}(G) = \gamma_t(G) + 2 = \rho^o(G) + 2$.

For a given graph G, let $\mathcal{A}_t(G)$ denote the set of all γ_t-sets in G. We next present an upper bound on the total dominator chromatic number of a graph.

Theorem 47 ([18, 26]) *If G is a connected graph without isolated vertices, then*

$$\chi_{\mathrm{td}}(G) \leq \gamma_t(G) + \min_{S \in \mathcal{A}_t(G)} \{\chi(G - S)\},$$

and this bound is tight.

Proof Let S be an arbitrary γ_t-set of G, and let $\mathcal{C}$ be a proper coloring of the graph $G - S$ using $\chi(G - S)$ colors. We now extend the coloring $\mathcal{C}$ to a coloring of the vertices of G by assigning to each vertex in S a new and distinct color. Let $\mathcal{C}'$ denote the resulting coloring of G, and note that $\mathcal{C}'$ uses $\gamma_t(G) + \chi(G - S)$ colors. Since S is a TD-set of G, every vertex in G is adjacent to at least one vertex of S. Since the color class of $\mathcal{C}'$ containing a given vertex of S consists only of that vertex, each vertex in G is adjacent to every vertex of some (other) color class in the coloring $\mathcal{C}'$. Hence, $\mathcal{C}'$ is a TD-coloring of G using $\gamma_t(G) + \chi(G - S)$ colors. This is true for every γ_t-set of G. The desired upper bound now follows by choosing S to be a γ_t-set of G that minimizes $\chi(G - S)$. The bound is achieved, for example, by taking G to be a complete graph. As shown in [18], the bound is also tight for infinitely many trees. $\square$

The proof of Theorem 47 yields the following more general result.

Theorem 48 *If G is a connected graph without isolated vertices, and* $\mathrm{TD}(G)$ *denotes the set of all total dominating sets of G, then*

$$\chi_{\mathrm{td}}(G) \leq \min_{S \in \mathrm{TD}(G)} \{|S| + \chi(G - S)\}.$$

We observe that $\chi(G - S) \leq \chi(G)$ for every proper subset $S \subset V(G)$. This observation, together with the results of Observations 37 and 38, gives us the

following analogous result to Theorem 12, thereby establishing upper and lower bounds on the total dominator chromatic number of an arbitrary graph in terms of its total domination number and chromatic number.

Theorem 49 ([26, 36]) *Every graph G without isolated vertices satisfies*

$$\max\{\gamma_t(G), \chi(G)\} \leq \chi_{\mathrm{td}}(G) \leq \gamma_t(G) + \chi(G).$$

4.2 Special Classes of Graphs

In this section, we consider the total dominator chromatic number of certain classes of graphs.

4.2.1 Bipartite Graphs

As a special case of Theorem 49 when G is a bipartite graph, we have the following result.

Theorem 50 ([26, 36]) *If G is a bipartite graph, then $\gamma_t(G) \leq \chi_{\mathrm{td}}(G) \leq \gamma_t(G) + 2$.*

For each $t \in \{0, 1, 2\}$, an infinite family $\mathcal{G}_t$ of bipartite graphs such that each graph $G \in \mathcal{G}_t$ satisfies $\chi_{\mathrm{td}}(G) = \gamma_t(G) + t$ is constructed in [18] as follows.

Let $\mathcal{G}_0$ be the family of graphs G without isolated vertices that contain a TD-set S that is a perfect open packing in G and such that the neighborhood of each edge e in $G[S]$ induces a complete bipartite graph in G, that is, if $e = uv$ is an edge in $G[S]$, then the subgraph of G induced by the neighborhood, $N[e]$, of e is a complete bipartite graph K_{n_1, n_2} where $d(u) = n_1$ and $d(v) = n_2$. Let $G \in \mathcal{G}_0$. As an example, if H is an arbitrary graph, then the graph $G = H \circ P_2$ belongs to the family $\mathcal{G}_0$ since the set $S = V(G) \smallsetminus V(H)$ is a TD-set that is a perfect open packing in G and the neighborhood of each edge e in $G[S]$ induces a complete bipartite graph $K_{1,2}$ in G.

Let $\mathcal{G}_1$ be the family of graphs that can be obtained from a graph H without isolated vertices by attaching any number of pendant edges, but at least one, to each vertex of H. For example, if H is an arbitrary isolate-free graph, then the corona $G = H \circ P_1$ of H belongs to the family $\mathcal{G}_1$.

Let $\mathcal{G}_2$ be the family of all paths P_n and cycles C_n, where $n \equiv 0 \pmod 4$ and $n \geq 8$.

Theorem 51 ([18]) *The following holds.*

(a) *If $G \in \mathcal{G}_0$, then $\chi_{\mathrm{td}}(G) = \gamma_t(G)$.*
(b) *If $G \in \mathcal{G}_1$, then $\chi_{\mathrm{td}}(G) = \gamma_t(G) + 1$.*
(c) *If $G \in \mathcal{G}_2$, then $\chi_{\mathrm{td}}(G) = \gamma_t(G) + 2$.*

4.2.2 Trees

Recall that the dominator chromatic number of a tree is one of two values (see
Theorem 17). However, the total dominator chromatic number of a tree is one of
three values. By Theorem 50, if T is a tree, then $\gamma_t(T) \leq \chi_{td}(T) \leq \gamma_t(T) + 2$. Further,
there are infinitely many trees T for which $\chi_{td}(T) = \gamma_t(T) + i$ for each $i \in [2]_0 = \{0,$
$1, 2\}$.

Theorem 52 ([26, 36]) *If G is a tree, then $\gamma_t(T) \leq \chi_{td}(T) \leq \gamma_t(T) + 2$.*

The following properties of χ_{td}-colorings in a tree T are established in [18]. We
say that a color class C in a given TD-coloring $\mathcal{C}$ of G is *free* if each vertex of G is
adjacent to every vertex of some color class different from C.

Theorem 53 ([18]) *If T is a nontrivial tree, then the following holds.*

(a) *If $\gamma_t(T) = \chi_{td}(T)$, then no $\chi_{td}(T)$-coloring contains a free color class.*
(b) *If $\chi_{td}(T) = \gamma_t(T) + 1$, then there exists a $\chi_{td}(T)$-coloring that contains a free color class.*
(c) *If $\chi_{td}(T) = \gamma_t(T) + 2$, then there exists a $\chi_{td}(T)$-coloring that contains two free color classes.*

The trees T satisfying $\gamma_t(T) = \chi_{td}(T)$ are characterized in [18]. Let $\mathcal{T}$ be the
family of trees constructed as follows. Let $\mathcal{T}$ consist of the tree P_2 and all trees that
can be obtained from a disjoint union of $k \geq 1$ stars each of order at least 3 by adding
$k - 1$ edges joining leaf vertices in such a way that the resulting graph is connected
and the center of each of the original k stars remains a support vertex.

Theorem 54 ([18]) *If T is a nontrivial tree, then $\gamma_t(T) = \chi_{td}(T)$ if and only if $T \in$
$\mathcal{T}$.*

In [18] a tight upper bound on the total dominator chromatic number of a tree
in terms of its order is established, and the trees with maximum possible total
dominator chromatic number are characterized. For this purpose, let $\mathcal{F}$ be the family
of all trees T that can be obtained from a tree H of order at least 2 by selecting
an arbitrary edge $e = uv$ in H and attaching a path of length 2 to each vertex of
$V(H) \setminus \{u, v\}$ so that the resulting paths are vertex-disjoint. We call H the *underlying
tree* of T. A tree in the family $\mathcal{F}$ with underlying tree $H = P_5$, for example, is
illustrated in Figure 16 (here the vertices of H are depicted by the darkened vertices).

Theorem 55 ([18]) *If T is a tree or order $n \geq 2$, then $\chi_{td}(T) \leq \frac{2}{3}(n + 1)$, with
equality if and only if $T \in \mathcal{F}$.*

Fig. 16 A tree in the family
$\mathcal{F}$
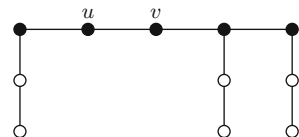
4.2.3 Mycielskian of a Graph

Let G be a graph without isolated vertices and with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The Mycielskian $M(G)$ is the graph obtained from G by adding n new vertices $u_1, u_2, \ldots, u_n$ and an additional vertex v and then adding the edges vu_i for all $i \in [n]$. Further, for each edge v_iv_j of G, we add the edges u_iv_j and v_iu_j to complete the construction of $M(G)$. For example, if $G = K_2$, then $M(G) = C_5$. If $G = C_5$, then $M(G)$ is the Grötzsch graph. Kazemi [24] proved that the dominator chromatic number of the Mycielskian of a graph is one of two values.

Theorem 56 ([24]) *If G is a graph without isolated vertices, then*

$$\chi_{\text{td}}(M(G)) = \chi_{\text{td}}(G) + 1 \quad \text{or} \quad \chi_{\text{td}}(M(G)) = \chi_{\text{td}}(G) + 2.$$

4.2.4 Circulants

Jalilolghadr, Kazemi, and Khodkar [20] studied total dominator colorings of circulant graphs $C_n(a, b)$ with two "jump sequences." For $n \geq 3$, let $1 \leq a_1 < \cdots a_k \leq \lfloor n/2 \rfloor$, and let $S = \{a_1, \ldots, a_k\}$. The graph G with vertex set $V(G) = [n]$ and edge set

$$E(G) = \{\{i, j\} : |i - j| \equiv a_i \pmod{n} \text{ for some } i \in [k]\}$$

is called a *circulant graph* with jump sequence S and denoted $C_n(S)$ or $C_n(a_1, \ldots, a_k)$. We note that $C_n(S)$ is a k-regular graph. Jalilolghadr et al. [20] prove the following result.

Theorem 57 ([20]) *If G is a circulant graph $C_n(a, b)$ where $n \geq 6$, $\gcd(a, n) = 1$ and $a^{-1}b \equiv 3 \pmod{n}$, then*

$$\chi_{\text{td}}(G) = \begin{cases} 2\lceil \frac{n}{8} \rceil & \text{for } n \in \{8, 9, 10\} \\ 2\lceil \frac{n}{8} \rceil + 1 & \text{for } n \equiv 1 \pmod 8 \text{ or } n = 11 \\ 2\lceil \frac{n}{8} \rceil + 2 & \text{otherwise.} \end{cases}$$

4.2.5 Central Graphs

Kazemi and Kazemnejad [28] studied the total dominator chromatic number of central graphs, where they define the *central graph $C(G)$* of a graph G as the graph obtained from G by subdividing every edge of G exactly once and adding all edges joining two vertices that were not adjacent in G. Among other results, they proved the following.

Theorem 58 ([28]) *If G is a connected graph of order $n \geq 4$, then the following holds.*

(a) $\chi_{td}(C(G)) \geq \frac{2}{3}n + 1$.
(b) $\chi_{td}(C(G)) \leq n + \lceil \frac{k}{2} \rceil$ where k is the order of a longest path in G.
(c) $\chi_{td}(C(G)) \leq n + 1$ if $\Delta(G) \leq n - 2$.
(d) $\chi_{td}(C(G)) \leq n + \lceil \frac{n}{2} \rceil$, with equality if and only if $G \cong K_n$.

4.3   Graph Products

In this section, we present some results due to Kazemi [25] on the total dominator chromatic number in Cartesian products ($\square$) and direct products ($\times$) of two graphs.

Theorem 59 ([25]) *If G and H are two graphs without isolated vertices, then*

$$\chi_{td}(G \times H) \leq \chi_{td}(G) \cdot \chi_{td}(H).$$

Theorem 60 ([25]) *For $q \geq p \geq 2$, if G is a complete p-partite graph and H is a complete q-partite graph, then $\chi_{td}(G \times H) = p + 2$. In particular, $\chi_{td}(K_p \times K_q) = p + 2$.*

Theorem 61 ([25]) *If G and H are two graphs without isolated vertices, then*

$$\max\{\chi_{td}(G), \chi_{td}(H)\} \leq \chi_{td}(G \square H) \leq \min\{\chi_{td}(G) \cdot n(H), \chi_{td}(H) \cdot n(G)\}.$$

Theorem 62 ([25]) *If G is a graph without isolated vertices, then $\chi_{td}(G) \leq \chi_{td}(G \square K_2) \leq 2\chi_{td}(G)$.*

4.4   Algorithmic and Complexity Results

We consider in this section the problem of finding the total dominator coloring number of an arbitrary graph. Formally, we consider the following decision problem:

GRAPH TOTAL DOMINATOR k-COLORABILITY

Input: A graph G, and an integer $k \geq 4$.
Question: Does G have a total dominator k-coloring?

An identical proof to that of Theorem 33 can be used to show that the GRAPH TOTAL k-DOMINATOR COLORABILITY is NP-complete for general graphs by transforming it from an instance of GRAPH DOMINATOR k-COLORABILITY.

Theorem 63 ([17, 26]) GRAPH TOTAL DOMINATOR k-COLORABILITY *is NP-complete for general graphs, for $k \geq 4$.*

5 Concluding Comments

In this chapter, we have surveyed selected results on the dominator chromatic number and total dominator chromatic number of a graph. Other results can be found, for example, in [3, 21, 27]. We close with a small list of open problems.

Problem 1 Find graphs, or classes of graphs, G satisfying the following.

(a) $\chi_d(G) = \gamma(G)$.
(b) $\chi_d(G) = \chi(G)$.
(c) $\chi_d(G) = \gamma(G) + \chi(G)$.
(d) $\chi_{td}(G) = \gamma_t(G)$.
(e) $\chi_{td}(G) = \chi(G)$.
(f) $\chi_{td}(G) = \gamma_t(G) + \chi(G)$.

Problem 2 Characterize the nontrivial trees T satisfying the following.

(a) $\gamma_t(T) = \chi_{td}(T) + 1$.
(b) $\gamma_t(T) = \chi_{td}(T) + 2$.

Problem 3 Characterize the graphs G satisfying $\chi_d(G) = \chi_{td}(G)$.

Problem 4 Determined the dominator chromatic number and the total dominator chromatic number of the $m \times n$ grid graph, $P_m \,\square\, P_n$, for all $m, n \geq 2$.

Problem 5 For any dominator (or total dominator) coloring, one can construct a so-called *dominator digraph* (*total dominator digraph*, respectively) which is an orientation of some of the edges of G such that for every vertex u, you orient the edge uv from u to v if u dominates the color class of vertex v. We note that for dominator colorings, this digraph will contain loops, if a vertex forms a singleton color class. However, the total dominator digraph will have no loops. We also note that these digraphs will have some unoriented edges which can be deleted. Study the resulting dominator digraphs and total dominator digraphs.

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