Algorithms and Complexity of Power Domination in Graphs

Stephen T. Hedetniemi, Alice A. McRae, and Raghuveer Mohan

1 Power Domination in Graphs

Based on the early research of Baldwin, Mili, Boisen and Adapa in 1993 [6], Brueni in 1993 [13] and Boisen, Baldwin and Mili in 2000 [10], Haynes, Hedetniemi, Hedetniemi and Henning introduced the concept of power domination in graphs in 2002 [24], as follows. Electric power companies continuously monitor the state of their electrical power lines with the use of *phasor measurement units* or *PMUs*, which estimate the magnitude and phase angles of electrical phasor quantities such as voltage or current. These are placed at *substation buses* where transmission lines, loads, and generators are connected and transmission lines connecting two electrical nodes are represented by edges in a graph G. For more information on power domination, please see the excellent chapter by Dorbec in [17].

In the corresponding graph theoretical model, a set S of vertices is initially selected, at which PMUs are to be located.

A set S is said to be a *power dominating set* of a graph $G = (V, E)$ if every vertex and every edge in the system are monitored by S in accordance with the following rules:

R1. If a vertex v is in the set S, then both v and all edges incident to v are called *observed*.
R2. Every vertex incident with an observed edge is observed.
R3. If both vertices u and v of an edge uv are observed, so is the edge uv.

S. T. Hedetniemi
School of Computing, Clemson University, Clemson, SC, USA
e-mail: hedet@clemson.edu

A. A. McRae · R. Mohan (✉)
Computer Science Department, Appalachian State University, Boone, NC 28608, USA
e-mail: mcraeaa@appstate.edu; mohanr@appstate.edu

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R4. If all of the edges but one incident to an observed vertex v are observed, then the one final edge incident to v is observed.

A *power dominating set* of a graph $G = (V, E)$ is a set $S \subseteq V$ having the property that all vertices $v \in V$ and all edges $uv \in E$ are observed by the vertices in S.

The minimum cardinality of a power dominating set of a graph is called the *power domination number*, denoted $\gamma_P(G)$.

This definition immediately gives rise to the following decision problem.

POWER DOMINATING SET (PDS)
Instance: Graph $G = (V, E)$, positive integer $k > 1$
Question: Does G have a power dominating set of cardinality at most k?

In 2002 this was solved as follows.

Theorem 1 (Haynes et al.) *PDS is NP-complete when restricted to bipartite or chordal graphs.*

Proof Sketch. For the bipartite case, use a transformation from the well-known NP-complete problem 3-SAT.

3-SAT
Instance: A set $U = \{u_1, u_2, \ldots, u_n\}$ of Boolean variables and a set $C = \{C_1, C_2, \ldots, C_m\}$ of three-variable clauses, where each clause contains three distinct occurrences of either a variable u_i or its complement $\overline{u_i}$.
Question: Does C have a satisfying truth assignment, that is, an assignment of TRUE or FALSE to each Boolean variable, such that in each clause C_i at least one variable (or its complement) is assigned the value TRUE?

Given an instance (U, C) of 3-SAT, construct a bipartite graph $G(U, C)$ instance of PDS, as follows. For each variable u_i, construct a 4-cycle C_4, with two non-adjacent, variable vertices, labeled u_i and $\overline{u_i}$.

For each clause, for example, $C_i = \{u_i, \overline{u_j}, u_k\}$, create two non-adjacent vertices $C_{i,1}$ and $C_{i,2}$, and join both of these two vertices to the three vertices in the 4-cycles corresponding to u_i, $\overline{u_j}$, and u_k. Thus, each of the two vertices $C_{i,1}$ and $C_{i,2}$ will have degree 3 in the bipartite graph $G(U, C)$ so constructed (cf. Figure 1).

One can then show that (U, C) has a satisfying truth assignment if and only if the constructed bipartite graph $G(U, C)$ has a power dominating set S of cardinality at most $k = n$. Note that if S contains one vertex in a 4-cycle, either u_i or $\overline{u_i}$, then all four vertices and all four edges in the 4-cycle will be observed, using rules R1, R2, R4, and then R2. Thus, from a satisfying truth assignment, one can choose one variable vertex from each 4-cycle and observe all vertices and all edges in all 4-cycles. This will also suffice to observe all edges between a 4-cycle and a clause vertex, and all clause vertices, using rules R1, R2, and R3.

Conversely, any power dominating set S for the constructed graph G must contain at least one vertex from each 4-cycle. One vertex per 4-cycle accounts for all k vertices in S. Since every clause vertex is duplicated, rule R4 cannot be used to observe clause vertices. Every clause vertex is observed through R1 by being
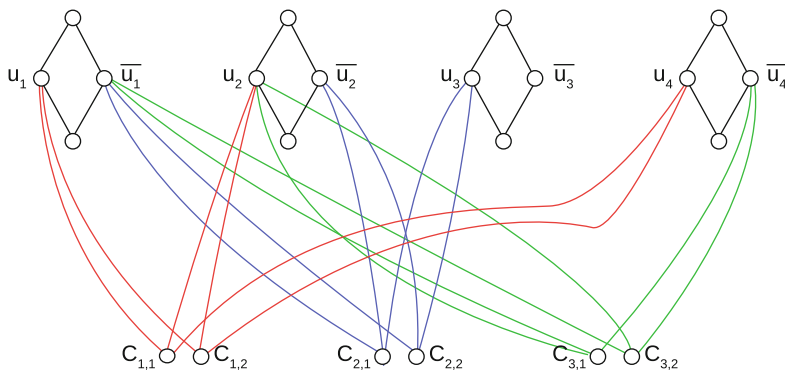
Fig. 1 Example of a constructed bipartite graph for proof of Theorem 1. The clauses are $C_1 = (u_1 \cup u_2 \cup u_4)$, $C_2 = (\bar{u_1} \cup \bar{u_2} \cup u_3)$, and $C_3 = (\bar{u_1} \cup u_2 \cup \bar{u_4})$

adjacent to vertices in S. A truth assignment for U exists by assigning a true value to the literals corresponding to the U vertices in S.

The fact that PDS remains NP-complete for chordal graphs can be shown by creating a clique of cardinality $2n$ among the $2n$ variable vertices, which creates a chordal graph. $\qquad \square$

Although Haynes et al. did not point this out, this same clique construction creates a split graph, that is, a graph whose vertices can be partitioned into two sets, $V = \{V_1, V_2\}$, such that V_1 induces a clique and V_2 is an independent set. Thus, we have as a corollary the following.

Corollary 1 *PDS is NP-complete when restricted to split graphs.*

When restricted to the family of trees, Haynes et al. develop a noteworthy result, as follows. Let T be a tree obtained from a single vertex x by attaching to x any number of paths, of any finite length. Such a tree is called a *spider*. The *spider number* $sp(T)$ of a tree T equals the minimum order k of a vertex partition $V = \{V_1, V_2, \ldots, V_k\}$ such that each subset V_i induces a spider.

For spiders the authors present the following two results.

Proposition 1 (Haynes et al.) *For any tree T, $\gamma_P(T) = 1$ if and only if T is a spider.*

Proof Sketch. Let $S = \{x\}$. Then by rules R1, R2, and R4, all vertices and edges of T will be observed. $\qquad \square$

Theorem 2 (Haynes et al.) *For any tree T, $\gamma_P(T) = sp(T)$.*

On the basis of these two results, Haynes et al. present a linear algorithm to compute the value $\gamma_P(T)$ for any tree T.

In 2005, [14] Brueni and Heath, following on the earlier master's thesis of Brueni [13], offered the following simpler vertex definition of power domination in graphs, which they prove is equivalent to the vertex-edge definition.

Rule B1. If a vertex $v \in S$, then v and all vertices in $N(v)$ are observed.

Rule B2. (Kirchhoff's Rule) If a vertex v is observed and there is a vertex $u \in N(v)$ that is the only unobserved vertex in $N(v)$, then vertex u is observed.

These two rules appear in several other papers in the following form.

For a connected graph G and a vertex set $S \subseteq V$, the set $M(S)$ of vertices *monitored* by S is defined recursively as follows:

(1) $M(S) \leftarrow S \cup N(S)$ (for every $v \in S$, v and all of its neighbors in $N(v)$ are monitored),

(2) While there is a vertex $v \in M(S)$ having exactly one unmonitored neighbor w, that is, $N(v) \cap (V(G) - M(S)) = \{w\}$, set $M(S) \leftarrow M(S) \cup \{w\}$.

A set S is called a *power dominating set* of G if $M(S) = V(G)$. The *power domination number* $\gamma_p(G)$ is the minimum cardinality of a power dominating set of G. This contrasts with dominating sets S, which can only observe, or dominate, vertices at distance 1 from vertices in S. Power dominating sets are not "local" in that they can monitor or observe vertices arbitrarily far from vertices in S.

Brueni and Heath provide an $O(|V| + |E|)$ algorithm for computing the set of vertices observed by any set $S \subseteq V$ in any graph G.

In [14] Brueni and Heath prove the following results.

Theorem 3 (Brueni-Heath) *For any connected graph G of order $n = |V|$, $\gamma_P(G) \leq n/3$, and this bound is tight.*

Theorem 4 (Brueni and Heath) *PDS is NP-complete for planar bipartite graphs.*

Proof Sketch. Use a transformation from what is called PLANAR 3-SAT. To each Boolean variable $U = \{u_1, u_2, \ldots, u_n\}$, construct a 4-cycle, two non-adjacent vertices of which are labeled u_i and $\overline{u_i}$. From each of these two variable vertices, attach a leaf. For each clause C_j in $C = \{C_1, C_2, \ldots, C_m\}$, construct a K_2, one vertex of which is labeled c_j. Then add an edge between c_j and the three variable vertices the clause contains. It is assumed in PLANAR 3-SAT that this graph is always planar and it is by construction bipartite. One can then show that U, C has a satisfying truth assignment if and only if the planar bipartite graph $G(U, C)$ has a power dominating set of size at most n. $\square$

In 2005 [23] and again in 2008 [24], Guo, Niedermeier, and Raible show that the power dominating set (PDS) problem can be solved by a dynamic programming algorithm for graphs of bounded treewidth. Moreover, they simplify and extend several NP-completeness results, by showing that PDS remains NP-complete for planar graphs, circle graphs (intersection graphs of chords of a circle, where two vertices are adjacent if and only if the corresponding chords cross each other), and split graphs.

Guo et al. show that PDS, when parameterized by $|S| = k$, the size of a power dominating set, is W[2]-hard for general graphs and, like DOMINATING SET, is only $\Theta(\log n)$-approximable, meaning that it cannot be approximated any better than DOMINATING SET.
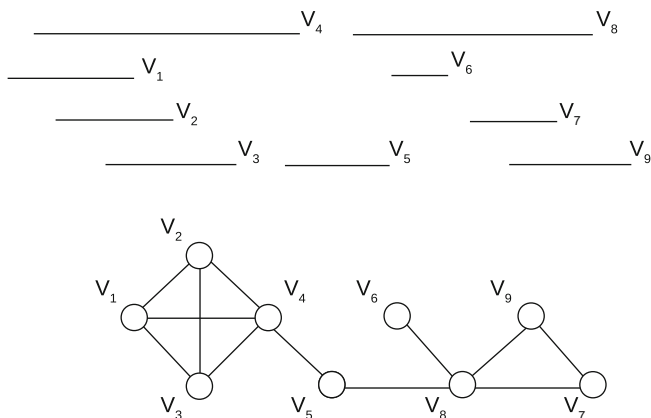
Fig. 2 Interval graphs

DOMINATING SET (DOMSET)
Instance: Graph $G = (V, E)$, positive integer k.
Question: Does G have a dominating set of size at most k?

Guo et al. were perhaps the first to discuss the non-locality of power domination. Whereas a dominating set can be verified by examining the neighborhoods of all vertices in a graph, the same check is not sufficient for verifying that a set is a connected dominating set. In this case one must verify that the subgraph induced by a dominating set is connected. However, verification that a set is a power dominating set is neither 1-local, as in the case of domination, nor a question of the subgraph induced by a power dominating set; rather, it is a question of the effect that a power dominating set has at arbitrary distances from vertices in the power dominating set.

In [23] Guo et al. present a linear algorithm for computing $\gamma_P(T)$ for any tree T that is simpler than the one given by Haynes et al. [25].

In 2008 [24] Guo, Niedermeier, and Raible construct a dynamic programming algorithm for computing $\gamma_P(G)$ for graphs of treewidth k. The running time of their algorithm is $O(c^{k^2}n)$, where c is a constant.

In 2005 [33] Liao and Lee show that the PDS decision problem is NP-complete for split graphs, which are a subclass of chordal graphs. In [33] the authors present a linear algorithm for computing $\gamma_P(G)$ for interval graphs G (cf. Figure 2), provided the interval ordering of the graph is provided, and they show that if the interval ordering is not given, the algorithm with $O(n \log n)$ time complexity is asymptotically optimal, where n is the number of intervals. They also show that the same results hold for the class of proper circular arc graphs, where *circular arc graphs* are the intersection graphs of sets of arcs of a circle, and a circular arc graph is *proper* if no arc properly contains another arc (cf. Figure 3).
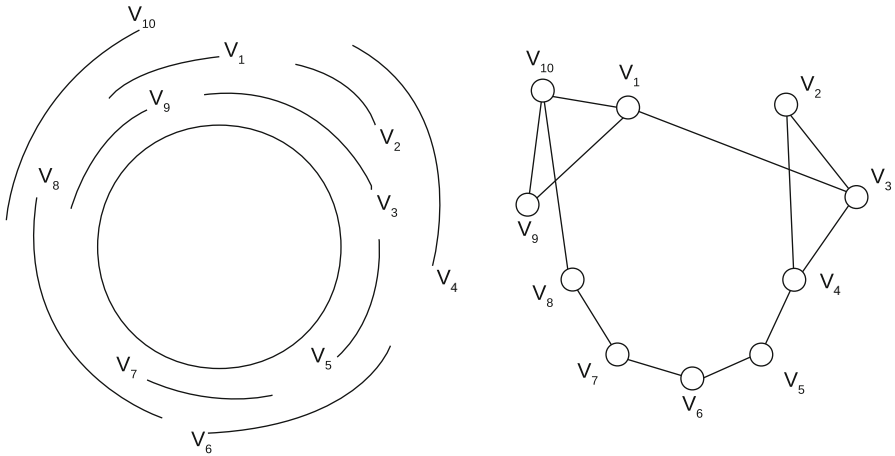
Fig. 3 Circular arc graphs

In 2006 [30] Kneis, Molle, Richter and Rossmanith present a linear-time algorithm for computing $\gamma_P(G)$ in graphs having bounded treewidth. They also prove the following.

Theorem 5 (Kneis et al.) *POWER DOMINATING SET is W[2]-hard for the parameter k, the number of PMUs, or the size of the power dominating set S.*

Proof. Use a reduction from DOMINATING SET, where the parameter is the size k of the dominating set. DOMINATING SET has been shown to be W[2]-complete by Downey and Fellows [20]. Given an instance of DOMINATING SET, a graph $G = (V, E)$ and a positive integer k, construct the corona $G' = G \circ K_1$ of G, by attaching a leaf v' to every vertex $v \in V$. Show that $\gamma(G) \leq k$ if and only if $\gamma_P(G') \leq k$.

We note at this point that the same corona reduction was independently used by Guo, Niedermeier, and Raible in [23] and in [24].

Let $S \subset V$ be a dominating set of G with $|S| \leq k$. Clearly, S dominates all vertices in G' except possibly some leaves v' attached to vertices v. Since all neighbors of any such vertex v except v' are power dominated by S, it follows that vertex v' is power dominated by S using Kirchhoff's Rule. Thus, the same set S is a power dominating set of G' with $|S| \leq k$.

Let $S \subseteq V'$ be a power dominating set of G' with $|S| \leq k$. If a leaf $v' \in S$, then $S' = S - \{v'\} \cup \{v\}$ is also a power dominating set of G' with $|S'| = |S| \leq k$. Thus, we may assume, without loss of generality, that $S \subseteq V$ and contains no leaves of G'. It remains to show that Kirchhoff's Rule never applies to any vertex $v \in V$, which means that S is also a dominating set of G.

Let $M(S) = N[S]$ be the set of vertices initially observed by the power dominating set S; thus, all vertices initially in $M(S)$ are either in S or are dominated by a vertex

in S. Let $U = V(G') - M(S)$ be the set of vertices that are unobserved, and assume that $U \neq \varnothing$.

The claim is that the only vertices in U are leaves of G'. The only vertices that can be added to the set $M(S)$ of observed vertices are those which are observed using Kirchhoff's Rule. If there is a vertex $v \in V$ that is not in $M(S)$, then it has no neighbor in S. This means that vertex v has at least two unobserved neighbors, at least one unobserved neighbor in V and its leaf neighbor in G', since we are assuming that S contains no leaves of G'. Thus, v can never be observed, contradicting the assumption that S is a power dominating set of G'.   □

As observed by Guo et al., the corona operation preserves bipartiteness, planarity, and the property of being a circle graph (an intersection graph of chords of a circle). The corona operation also preserves the property of being a chordal bipartite graph. DOMINATING SET has been shown to be NP-complete for chordal bipartite graphs by H. Müller and A. Brandstädt in 1987 [36].

Corollary 2 *PDS is NP-complete for bipartite, planar, circle, and chordal bipartite graphs.*

Earlier we observed that a minor modification of the proof of Haynes et al. suffices to show that PDS is NP-complete for both chordal graphs and split graphs.

Guo, Niedermeier, and Raible have used a transformation from VERTEX COVER to show that PDS is NP-complete for split graphs [24].

In 2006, Xu, Kang, Shan, and Zhao [42] construct a linear algorithm for computing $\gamma_P(G)$ for any connected block graph G. A *block* of a graph G is a maximal 2-connected subgraph of G. A graph G is a *block graph* if and only if every block of G is a complete subgraph. Their algorithm is based in part on the following result.

Theorem 6 (Xu et al.) *If G is a block graph having at least one cut vertex, then G has a $\gamma_P(G)$-set in which every vertex is a cut vertex.*

The algorithm works by first constructing a decomposition tree of a block graph called the *refined cut tree*, as shown in Figure 4. This tree has one vertex for every block and one vertex for every cut vertex of G. An edge exists between a block vertex and a cut vertex if and only if the cut vertex is contained in that block. The *refined cut tree* is formed by re-labeling the blocks of the graph as *block vertices*. This tree can be constructed in linear time. The algorithm then roots the tree at a cut vertex and processes the cut vertices from the leaves up to the root. At each level, a power dominating set is constructed for that level by modifying the power dominating set constructed for the lower level.

Xu et al. also prove that the power domination number of a block graph of order n is no more than $n/3$, with equality if and only if the graph is the corona $G \circ K_2$ or $G \circ \overline{K_2}$ for some block graph G. Power domination in block graphs is also studied by Atkins, Haynes and Henning in 2006 [5].

In 2006 [43] Zhao, Kang, and Chang show that $\gamma_P(G) \leq n/3$ for any connected graph G of order $n \geq 3$ and $\gamma_P(G) \leq n/4$ for any connected claw-free cubic graph G
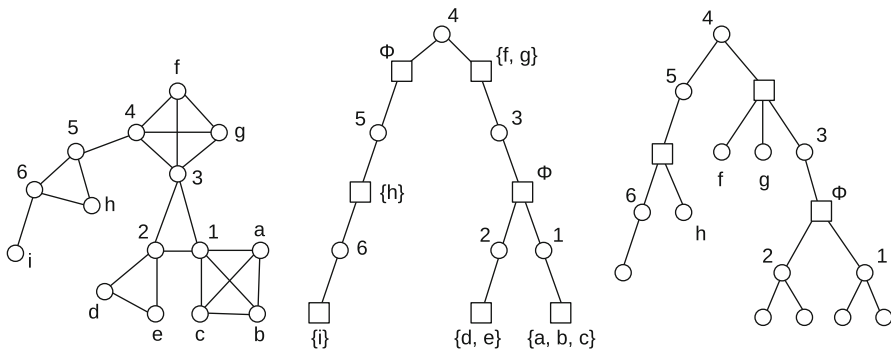
Fig. 4 Block graph cut trees (figure reproduced from [42])

of order *n*. It is interesting to note how many authors have noted the *n*/3 upper bound for various classes of graphs, for example, for trees [25] or block graphs [42].

In 2007 [26] Hon, Liu, Peng, and Tang present a linear algorithm for computing the power domination number of block-cactus graphs, which are defined as follows.

A *cut vertex* in a graph *G* is a vertex whose removal increases the number of connected components. A *cactus* is a graph in which every edge is a member of at most one cycle. Finally, a *block-cactus* graph is a graph in which every block is either a complete graph or a cycle. A linear algorithm for computing the power domination number of a block-cactus graph is based on the following observation.

Theorem 7 (Hon, Liu, Peng, Tang) *In any block-cactus graph, there exists a minimum power dominating set that contains only cut vertices.*

In 2006 [19] Dorfling and Henning determine the power domination numbers of $n \times m$ grid graphs.

Theorem 8 (Dorfling, Henning) *If $G = P_n \Box P_m$ is an $m \times n$ grid graph, where $1 \leq m \leq n$, then*

(i) $\gamma_P(G) = \lceil \frac{m+1}{4} \rceil$ *if $m \equiv 4 \mod 8$, and*
(ii) $\gamma_P(G) = \lceil \frac{m}{4} \rceil$ *otherwise.*

The authors show how to construct a minimum power dominating set *S* of a grid graph $G = P_m \Box P_n$, as follows: let $m = 8k + j$, where $0 \leq j \leq 7$ and vertices are represented by their row and column numbers, such as the vertex $v = (2, 3)$ in row 2, column 3.
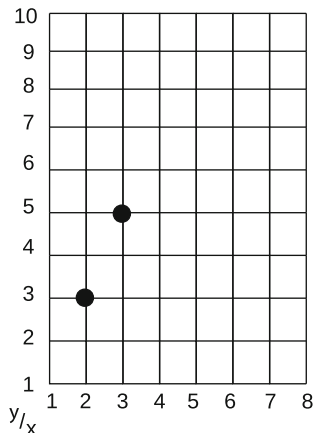
If $k = 0$, let $S' = \varnothing$; otherwise let $S' = \{(8i + 3, 2), (8i + 5, 3) : 0 \leq i \leq k - 1\}$. Then,

if $j = 0$, let $S = S'$.
if $j \in \{1, 2\}$, let $S = S' \cup \{(m, 1)\}$.
if $j = 3$ let $S = S' \cup \{(m - 1, 1)\}$.
if $j = 4$ let $S = S' \cup \{(m - 2, 1), (m - 1, 1)\}$.

Fig. 5 A minimum power dominating set of an 8×10 grid graph

if $j \in \{5, 6, 7\}$ let $S = S' \cup \{(m + 3 - j, 2), (m + 5 - j, 3)\}$.

Thus, for example, a minimum power dominating set of an 8×10 grid graph needs only two vertices, such as $v = (2, 3)$ and $w = (3, 5)$ (cf. Figure 5).

In 2007 [37] Pai, Chang and Wang provide a somewhat simpler algorithm for placing a minimum power dominating set in a grid graph than that given above by Dorfling and Henning.

In 2008 [39] and in 2012 [9] Raible and Fernau show that PDS is NP-hard on planar cubic graphs and design an $O^*(1.7548^n)$ algorithm for computing $\gamma_P(G)$ for arbitrary graphs. Their NP-completeness result can be briefly described as follows.

Theorem 9 (Binkele-Raible, Fernau) *PDS is NP-hard on planar cubic graphs.*

**Proof Sketch.** Use a reduction from PLANAR CUBIC VERTEX COVER, as follows:

VERTEX COVER
Instance: Graph $G = (V, E)$, positive integer k.
Question: Does G have a vertex cover of cardinality at most k, that is, a set $S \subseteq V$ such that for every edge $uv \in E$, either $u \in S$ or $v \in S$, that is, $|\{u, v\} \cap S| \geq 1$?

Given a planar cubic instance of VERTEX COVER, with $V = \{v_1, v_2, \ldots, v_n\}$, replace each vertex $v \in V$ with the graph in Figure 6. Notice that if G is planar and cubic, then the constructed graph G' is planar and cubic. One can then show that G has a vertex cover of cardinality at most k if and only if G' has a power dominating set of cardinality at most k. In the gadget in Figure 6, if a set S is a vertex cover, then all edges incident to a vertex v will have a vertex in S. This means that the vertices in S, when viewed as a power dominating set, will observe vertices q_1, c_{v1}, q_2, c_{v2}, q_3, c_{v3}. Once these have been observed, then all remaining vertices a_{v1}, z_v, a_{v2}, a_{v3} will be observed. Thus, $\gamma_P(G') \leq k$.

It only remains to show that starting with a power dominating set S' of G' of cardinality at most k, there will be a vertex cover of G of cardinality at most k. This

Fig. 6 Gadget for cubic
planar graphs (figure
reproduced from Binkele [9])
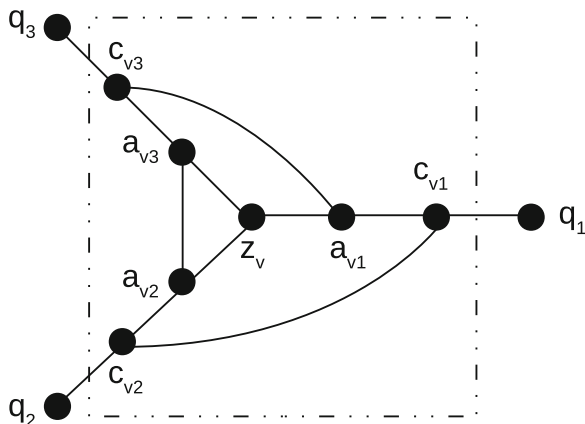
is based on the observations that (i) any minimum power dominating set of G' will contain at most one vertex from any gadget; (ii) if a gadget contains a vertex, then it must be the z_v vertex; and (iii) if a gadget T_v contains no vertices in S', then the a_{vi} vertices are indirectly observed by all three of the c_{vi} vertices and the z_v vertex is indirectly observed by the a_{v1} vertex. $\qquad\square$

Binkele-Raible and Fernau conclude their paper by presenting an 11-page, highly technical, exact algorithm, for computing the value of $\gamma_P(G)$ for any graph G, whose running time is $O^*(1.7548^n)$ and which when applied to cubic graphs has a running time of $O^*(1.6212^n)$. We refer the interested reader to [9] for the details.

In 2009 [3] Aazami and Stilp present several approximation results for PDS, in particular demonstrating for the first time a "gap" in the approximation guarantees between DOMINATING SET and PDS, especially since it has been shown that DOMINATING SET has an $O(\log n)$ approximation guarantee, and it has been shown that no polynomial algorithm can give a better approximation guarantee. Indeed, Feige [21] has shown that DOMINATING SET is even hard to approximate within a ratio of $(1 - \varepsilon)\, n$. Notable as well is their introduction of power domination applied to directed graphs.

Aazami and Stilp present a transformation from the MIN-REP decision problem to PDS which shows that PDS cannot be approximated within a factor of $2^{log^{1-\epsilon} n}$, unless $NP \subseteq DTIME(n^{polylog(n)})$, since the same result holds for the MIN-REP problem, which can be stated as follows.

MIN-REP

Instance: (i) Bipartite graph $G = (A, B, E)$; (ii) a partition of the two partite sets $A = \{A_1, A_2, \ldots, A_j\}$ and $B = \{B_1, B_2, \ldots, B_j\}$ into sets of equal size, that is, the sets A_i and B_j all have the same size, say N; and (iii) a positive integer k.

Question: Does there exist a set $C \subset A \cup B$, with $|C| \leq k$, such that whenever there exist vertices $u \in A_i$ and $v \in B_j$ where $uv \in E$, there exist a vertex $a \in C \cap A_i$ and a vertex $b \in C \cap B_j$ with $ab \in E$?

For undirected graphs, the authors introduce the notion of *strong regions*, meaning sets of vertices which contain vertices which must appear in a $\gamma_P(G)$-set, as a means of obtaining lower bounds on the size of an optimum solution for PDS. Using this idea, they develop an algorithm for finding power dominating sets that have an approximation guarantee of $O(k)$ for graphs of treewidth k. The algorithm requires as input the partial-k tree decomposition and runs in time $O(n^3)$, independent of k.

Since it is known that planar graphs have treewidth $O(\sqrt{n})$, the Aazami-Stilp algorithm provides an $O(\sqrt{n})$ approximation for PDS. The authors then show that their methods cannot improve on this $O(\sqrt{n})$ approximation guarantee.

Aazami and Stilp also describe a simple algorithm with an approximation guarantee of $O(\frac{n}{\log n})$ for the PDS problem. Their algorithm can be described as follows. Partition the vertices of a graph G into $k = \log n$ equal-sized sets $\{V_1, V_2, \ldots, V_k\}$. Then consider all possible ways of selecting some nonempty collection of these k sets. For each nonempty collection of these k sets, form the union of all vertices contained in these sets, and then, in polynomial time, check to see if this union is a power dominating set. Among all of these different collections of k sets, output one that power dominates G and has the minimum total number of vertices. Note that in this algorithm, we can consider at most $2^k = 2^{\log n} = n$ different collections of the k sets $V_1, V_2, \ldots, V_k$. Clearly, this algorithm runs in polynomial time, since it can be tested in $O(|V| + |E|)$ time whether any set of vertices is a power dominating set. Let S^* be a minimum power dominating set, of cardinality $\gamma_P(G) = |S^*|$. It is easy to see that the union of the subsets in $V_1, V_2, \ldots, V_k$ that intersect S^* is a power dominating set of G and that the number of vertices in this union of sets is at most $\frac{n}{\log n}|S^*|$. This establishes the approximation guarantee of $O(\frac{n}{\log n})$.

In 2010 [2] Aazami introduces a variation of the power domination problem [also introduced by Liao, but as yet unpublished in his 2009 PhD Thesis [32]], which involves an integer, which we will denote r, for the number of *rounds* of propagation permitted before all vertices must be observed. Thus, for a graph of order n, $1 \leq r \leq n - 1$. The DOMINATING SET problem corresponds to $r = 1$, while the POWER DOMINATING SET (PDS) problem corresponds to $r = n - 1$. In PDS the goal is to find a minimum cardinality set of vertices S that power dominates all vertices $v \in V$, where a node v is power dominated if (1) $v \in S$ or it has a neighbor in S or (2) v has a neighbor u such that u and all of its neighbors except v are power dominated. Rule (1) is the DOMINATING SET problem, and Rule (2) is a propagation rule that applies iteratively. The *r-round PDS problem*, or *r*PDS, has the same set of rules as PDS, except that Rule (2) is applied in parallel to all vertices that are newly observed in the current *round*. The requirement is to find a minimum cardinality set S such that all vertices can be observed in at most r rounds. The *r-round power domination number* $\gamma_{rP}(G)$ equals the minimum cardinality of an r-round power dominating set in G.

The parameter $\gamma_{rP}(G)$ is a power dominating version of the *distance-r domination number*, denoted $\gamma_{\leq r}(G)$, which equals the minimum cardinality of a set $S \subseteq V$

such that for every vertex $w \in V - S$, $d(w, S) \leq r$, that is, every vertex $w \in V - S$ is at distance at most r from at least one vertex in S.

Aazami provides a proof that *rPDS* is NP-hard by means of a simple modification of the NP-hardness proof of PDS provided by both Guo et al. [23] and Kneis et al. [30], in which they attach a single leaf vertex to every vertex in a planar graph G.

Theorem 10 (Aazami) *For any $r \geq 1$, rPDS is NP-hard for planar graphs.*

Proof Sketch. Use a transformation from PLANAR DOMINATING SET. Given a planar graph $G = (V, E)$, construct a planar graph G' by attaching a path of length $r - 1$ to every vertex $v \in V(G)$. One can show that G has a dominating set of cardinality at most k if and only if G' has an r-round power dominating set of cardinality at most k.

Given a dominating set $S \subseteq V(G)$ of cardinality at most k, after the first round, S will dominate, or observe, all vertices in $V(G)$. Then, in the next $r - 1$ rounds, all vertices on all paths attached to the vertices $v \in V(G)$ will be observed. Thus, the same set S is an r-round power dominating set of G' of cardinality at most k.

Conversely, if S' is an r-round power dominating set of G', it is easy to see that there is another r-round power dominating set S'' of the same cardinality as S' which contains no vertices on any path attached to the vertices $v \in V(G)$. Assume that this set S'' is not a dominating set of G, and let $w \in V(G)$ be any vertex not dominated by S'' in the first round. This means that at best v is power dominated in the second round. But this implies that it will take at least $r + 1$ rounds to power dominate the last vertex on the path of length $r - 1$ attached to vertex w, a contradiction. $\square$

As shown previously with the Guo et al. [23] construction for power dominating set, the above construction holds for bipartite graphs, circular graphs, chordal graphs, and chordal bipartite graphs. Therefore we have the following corollary.

Corollary 3 *For any $r \geq 1$, rPDS is NP-hard for bipartite, circular, chordal, and chordal bipartite graphs.*

Aazami shows that *rPDS*, or $\gamma_{rP}(G)$, cannot be approximated better than $2^{\log^{1-\epsilon} n}$ even for $r = 4$ in general graphs. He provides a dynamic programming algorithm to compute the value $\gamma_{rP}(G)$ in polynomial time for graphs of bounded treewidth. He also presents a PTAS (polynomial time approximation scheme) for *rPDS* on planar graphs for $r = O(\frac{\log n}{\log \log n})$. Finally, he gives an integer programming formulation of r-round PDS.

In 2011 [7] Barrera and Ferrero provide upper bounds for the power domination numbers of cylinders $P_m \square C_n$, and exact values of the power domination numbers of toroidal grid graphs $C_m \square C_n$ and some generalized Petersen graphs.

In 2016 [31] Liao continues the development of r-round power domination by presenting linear algorithms for computing $\gamma_{rP}(G)$ for trees and for block graphs.

In 2010 [38] Pai, Chang, and Wang consider a variation of power domination in which PMUs may only be placed within a restricted subset of the vertices V of a graph, called a *forbidden zone Z*. Thus, the parameter $\gamma_P(G, Z)$ equals the minimum cardinality of a power dominating set S such that $S \cap Z = \varnothing$. This leaves open the

possibility, of course, that for some restricted sets Z, no restricted power dominating set may exist. As an illustration, they present algorithms to solve this restricted type of power domination on grids, under the restriction that only certain consecutive rows or columns form a forbidden zone.

Pai et al. [38] also introduce another variation of power domination, as follows. Given a graph $G = (V, E)$ and an integer k, with $0 \leq k \leq |V|$, a set $S \subseteq V$ is called a *k-fault-tolerant power dominating set*, or *kFPDS*, of G if $S - F$ is still a PDS of G for any subset $F \subset S$ with $|F| \leq k$. The *k-fault-tolerant power domination number* of G, denoted $\gamma_{kFP}(G)$, equals the minimum cardinality of a *kFPDS* of G. Notice that from this definition, it follows that:

(i) for any $k \geq 0$, $\gamma_{kFP}(G) \leq \gamma_{k+1FP}(G)$,
(ii) $\gamma_{0FP}(G) = \gamma_P(G)$,
(iii) $\gamma_P(G) + k \leq \gamma_{kFP}(G)$, and
(iv) if G contains $k + 1$ mutually disjoint $\gamma_P(G)$-sets, then the union of these $k + 1$ sets can form a *kFPDS*, which implies that $\gamma_{kFP}(G) \leq (k + 1)\gamma_P(G)$.

Pai et al. establish the following results for $1 \times n$, $2 \times n$, and $3 \times n$ grid graphs.

Proposition 2 (Pai, Chang, Wang) *For $G_{1,n}$, $G_{2,n}$, $G_{3,n}$, $G_{4,n}$, and $G_{5,n}$,*

(1) $\gamma_{1FP}(G_{1,n}) = 2$*; let $S = \{(1, 1), (1, n)\}$,*
(2) $\gamma_{1FP}(G_{2,n}) = 2$*; let $S = \{(1, 1), (1, n)\}$,*
(3) $\gamma_{1FP}(G_{3,n}) = 2$*; let $S = \{(2, 1), (2, n)\}$,*
(4) $\gamma_{1FP}(G_{4,n}) = 3$*; let $S = \{(2, 1), (3, 3), (4, 1)\}$,*
(5) $\gamma_{1FP}(G_{5,})n = 3$*; let $S = \{(2, 1), (3, 3), (4, 1)\}$.*

Pai et al. conclude by presenting placement algorithms that do the following:

(i) approximate $\gamma_{1FP}(G_{m,n})$ within a factor of 1.60 for $6 \leq m \leq n$,
(ii) approximate $\gamma_{2FP}(G_{m,n})$ within a factor of 2.34 for $7 \leq mn$,
(iii) approximate $\gamma_{3FP}(G_{m,n})$ within a factor of 3.34 for $11 \leq m \leq n$.

Figure 7 presents two examples of one-fault power dominating sets, one in a $15 \times n$ grid graph and the other in a $17 \times n$ grid graph.

In 2012 [15] Chang, Dorbec, Montassier, and Raspaud introduce the concept of *k*-power domination, a direct generalization of power domination, by changing in a
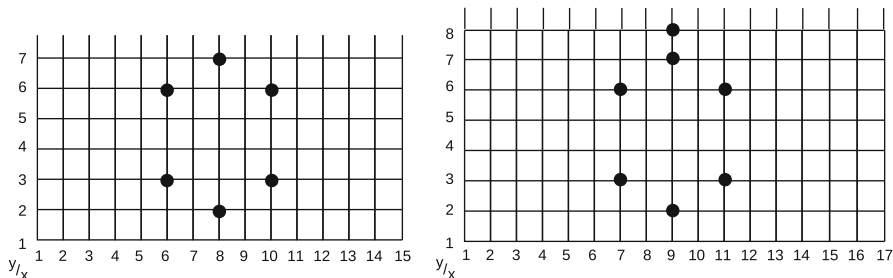

Fig. 7 One-fault power dominating sets (figure reproduced from [38])

natural way Rule 2 (Kirchhoff's Rule), sometimes called the *propagation rule*, as follows, for some fixed nonnegative integer k:

Rule B1. If a vertex $v \in S$, then v and all vertices in $N(v)$ are observed.

Rule B2. (Kirchhoff's Rule) If a vertex v is observed and there is a vertex $u \in N(v)$ that is the only unobserved vertex in $N(v)$, then vertex u is observed.

Rule B3. (Chang, Dorbec, Montassier, Raspaud) If a vertex v is observed and there are at most k vertices in $N(v)$ that are unobserved, then all vertices in $N(v)$ are observed.

This can also be stated as follows:

For a connected graph G and a vertex set $S \subseteq V$, the set $M_k(S)$ of vertices k-observed, or k-monitored, by S is defined recursively as follows:

(1) $M_k(S) \leftarrow S \cup N(S)$ (v and all of its neighbors in $N(v)$ are k-monitored),
(2) While there is a vertex $v \in M_k(S)$ having at most k unmonitored neighbors, that is, $|N(v) \cap (V(G) - M(S))| \leq k$, set $M_k(S) \leftarrow M_k(S) \cup N(v)$.

A set S is called a *k-power dominating set* of G if $M_k(S) = V(G)$. The *k-power domination number* $\gamma_{kP}(G)$ is the minimum cardinality of a k-power dominating set of G.

One can observe that when $k = 0$, $\gamma_{0P}(G) = \gamma(G)$, and when $k = 1$, $\gamma_{1P}(G) = \gamma_P(G)$.

The authors quickly establish an upper bound for $\gamma_{kP}(G)$ which generalizes known results for $k = 0$ and $k = 1$.

Theorem 11 (Chang, Dorbec, Montassier, Raspaud) *For any connected graph G of order $n \geq k + 2$,*

$$\gamma_{kP}(G) \leq n/(k+2)$$

and this bound is best possible.

Chang et al. provide one complexity result and one algorithm.

KPDS
Instance: Graph $G = (V, E)$, positive integer t.
Question: Does G have a k-power dominating set of cardinality at most t?

Theorem 12 (Chang, Dorbec, Montassier, Raspaud) *KPDS is NP-complete for chordal graphs and bipartite graphs.*

Proof Sketch. For any graph G and any nonnegative integer k, let G_k be the graph obtained from G by attaching k leaves to every vertex $v \in V(G)$. It is easy to show that $\gamma_{kP}(G_k) = \gamma(G)$. Furthermore, if G is chordal or bipartite, so is G_k.   □

Theorem 13 (Chang, Dorbec, Montassier, Raspaud) *There is a linear algorithm for computing the value $\gamma_{kP}(T)$ for any tree T.*

Proof Sketch. The algorithm is based on the method used to compute the value of $\gamma(T)$ for any tree T given by Mitchell, Cockayne, and Hedetniemi in [35] and Cockayne, Goodman, and Hedetniemi in [16]. Assign two values $L(v) = (a_v, b_v)$ to each vertex v in a rooted tree T, where $a_v \in \{B, F, R\}$ and $b_v \in \{0, 1, \ldots, k\}$. The value $a_v = R$ (required) means that the vertex will be a vertex in the minimum k-power dominating set; the value $a_v = F$ (free) means that the vertex has been monitored/observed; and the value $a_v = B$ (bound) means that the vertex has not been monitored. The second label b_v is only applied to vertices labeled F or B and records the number of neighbors of v, which is at most k, that may yet be monitored once v is monitored. The algorithm works on a rooted tree from the leaves up to the root and is based on the following five rules for a leaf vertex x and its parent vertex y, where $T' = T - x$ is the tree remaining after leaf x has been deleted and where initially all vertices v are assigned the label $L(v) = (B, k)$.

(1) if $a_x = R$, then $\gamma_{kP}(T) = 1 + \gamma_{kP}(T')$ and change $a_y = F$ if $a_y = B$.
(2) if $(a_y = R)$ or $(a_x = F$ and $b_x = 0)$, then $\gamma_{kP}(T) = \gamma_{kP}(T')$.
(3) if $a_x = B$ and $b_y > 0$, then $\gamma_{kP}(T) = \gamma_{kP}(T')$, where $b_y = b_y - 1$ in T'.
(4) if $a_x = B$ and $b_y = 0$, then $\gamma_{kP}(T) = \gamma_{kP}(T')$, where $a_y = R$ in T'.
(5) otherwise, if $a_x = F$, $b_x > 0$ and $a_y \neq R$, then $\gamma_{kP}(T) = \gamma_{kP}(T')$, where $a_y = F$ in T'.

Chang et al. illustrate their algorithm with the example in Figure 8.

In 2013 [18] Dorbec, Henning, Löwenstein, Montassier, and Raspaud continue the development of k-power domination by applying it to the consideration of
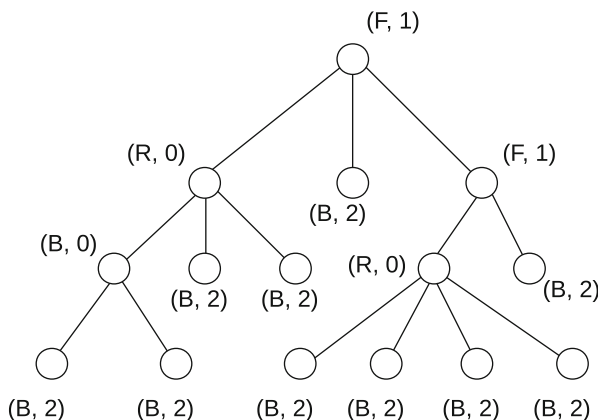
Fig. 8 2-power domination in trees (figure reproduced from [15])

regular graphs. The main result of their paper is a 12-page proof of the following theorem.

Theorem 14 (Dorbec, Henning, Löwenstein, Montassier, Raspaud) *For $k \geq 1$ and G a connected $(k+2)$-regular graph of order n, if $G \neq K_{k+2,k+2}$, then $\gamma_{kP}(G) \leq \frac{n}{(k+3)}$, and this bound is tight.*

They conclude with the following conjecture.

Conjecture 1 (Dorbec, Henning, Löwenstein, Montassier and Raspaud) *For $k \geq 1$ and $r \geq 3$, if $G \neq K_{r,r}$ is a connected r-regular graph of order n, then $\gamma_{kP}(G) \leq \frac{n}{(r+1)}$.*

In 2016 [41] Wang, Chen, and Lu generalize the linear, k-power domination algorithm by Chang, Dorbec, Montassier, and Raspaud [15] for computing $\gamma_{kP}(G)$ for trees, to an $O(|V| + |E|)$ algorithm for computing $\gamma_{kP}(G)$ for block graphs.

In 2013 [34] Liao and Lee give an 11-page presentation of a linear algorithm for computing $\gamma_P(G)$ for interval graphs, if the interval ordering of the graph is provided as input. In addition, they show that if the interval ordering is not given, their algorithm runs in $\theta(n \log n)$ time, where n is the number of intervals, and is asymptotically optimal.

Liao and Lee also give a 7-page presentation in which they extend their methods to the class of circular arc graphs. They construct a linear algorithm for computing $\gamma_P(G)$, first for proper circular arc graphs, in which no arc is properly contained within another arc, and then for general circular arc graphs, provided the circular arc endpoints are sorted.

In 2015 [40] Stephen, Rajan, Ryan, Grigorious, and William apply power domination to the graphs of the chemical compounds of polyphenylene, dendrimers (cf. Figure 9), rhenium trioxide (cf. Figure 10), and silicate networks (cf. Figure 11). The rhenium trioxide (ReO_3) graphs, cf. Figure 8, are essentially subdivision graphs of three-dimensional grid graphs, that is, the graphs $S(P_p \square P_q \square P_r)$, but denoted by the authors as $RO(p, q, r)$, where the subdivision vertices, the vertices of degree 2, represent oxygen atoms and the vertices of degree 3 represent rhenium atoms.

Although the authors do not solve the problem of computing the power domination number of rhenium trioxide lattices, they do provide several interesting observations.

Theorem 15 (Stephen, Rajan, Ryan, Gregorious, William) *For any rhenium trioxide graph $RO(p, q, r)$, the following must hold:*

(i) *Each unit cell in $RO(p, q, r)$ must contain at least three vertices of any power dominating set.*

(ii) *The three vertices in every unit cell of $RO(p, q, r)$ belonging to a power dominating set cannot all lie on the same face.*

(iii) *Every face in each unit cell must have at least one vertex in any power dominating set.*

(iv) *In any power dominating set of $RO(p, q, r)$, every unit cell must contain at least two rhenium atoms (vertices of degree 3).*

**Fig. 9** Polyphenylene dendrimers (figure reproduced from Stephen et al. [40])
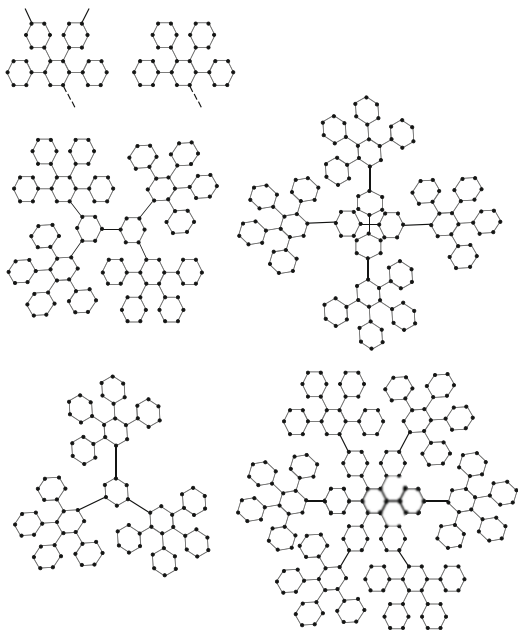
Fig. 10 Rhenium trioxide (figure reproduced from Stephen at el. [40])
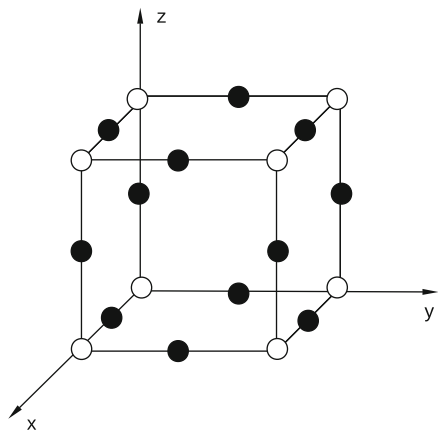
The graphs of *silicate networks SL(n)* are obtained from the graphs of *honeycomb* networks *HC(n)* as follows. The graphs *HC(n)* are defined recursively: (i) $HC(1) = C_6$, (ii) *HC(2)* is obtained from *HC(1)* by attaching a layer of six hexagons to the outer edges of *HC(1)*, and (iii) *HC(n)* is obtained from *HC(n − 1)* by attaching hexagons to the outer edges of *HC(n − 1)*.

We should interject here that the power domination number of these honeycomb networks *HC(n)* were determined by Ferrero and Varghese in 2011 [22] to be: $\gamma_P(HC(n)) = \lceil \frac{2n}{3} \rceil$.
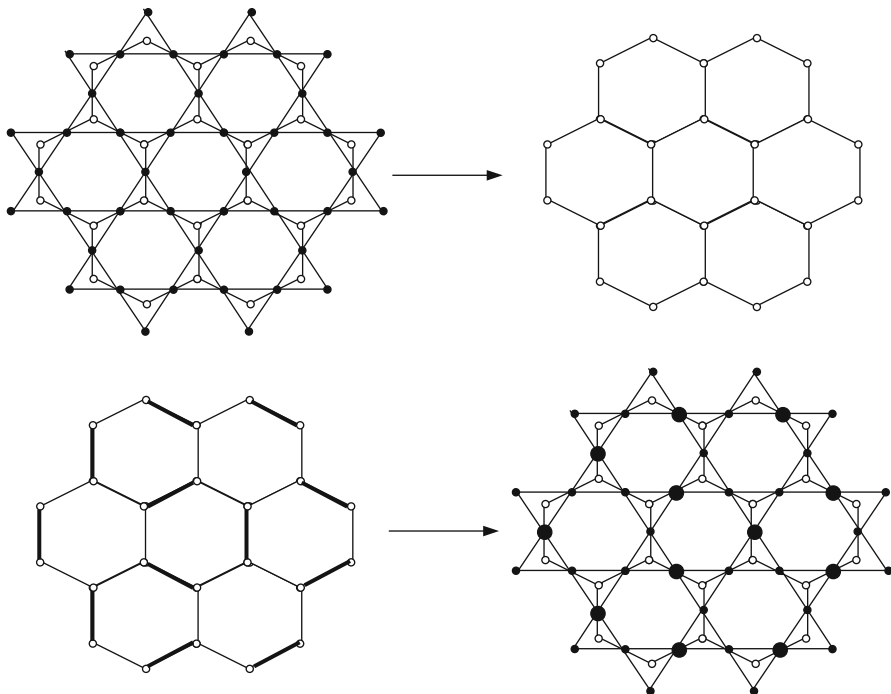
Fig. 11 Silicate networks (figure reproduced from Stephen et al. [40])

In order to construct a silicate network $SL(n)$, to each vertex in $HC(n)$, assign a silicon ion. Then subdivide every edge of $HC(n)$, and assign to each of these subdivision vertices an oxygen ion. To each of the $6n$ degree-2 vertices on the outer face of $HC(n)$, attach a leaf and assign an oxygen atom to these $6n$ leaves. Finally, add edges to form a triangle of oxygen ions surrounding each silicon ion, thereby creating a tetrahedron, the center of which is a silicate ion, cf. Figure 11. The resulting graph is the silicate network $SL(n)$, which has $15n^2 + 3n$ vertices, $36n^2$ edges, and diameter $4n$.

The algorithm for constructing a minimum power dominating set of a silicate network $SL(n)$ becomes very simple, as follows:

Step 1. In each bounding cycle of $HC(i)$, for $1 \le i \le n$, choose alternate edges. [In $HC(1)$ choose 3 alternating edges; in $HC(2)$ choose 12 alternating edges; etc.]

Step 2. For every $1 \le i \le n$, choose the oxygen ion vertices in $SL(i)$ that subdivide the edges chosen in the bounding cycle of $HC(i)$.

The authors then show that the $3n^2$ vertices so chosen form a minimum cardinality power dominating set of $SL(n)$.

Theorem 16 (Stephen et al.) *For every silicate network SL(n),* $\gamma_P(SL(n)) = 3n^2$.

In 2017 [28] Kang and Wormald present two heuristics for finding a small power dominating set in a random cubic graph. They analyze the performance of these

heuristics on random cubic graphs using differential equations. In this way, they prove that the proportion of vertices in a minimum power dominating set of a random cubic graph is asymptotically almost surely at most 0.067801. They also provide a corresponding lower bound of 1/29.7, which is approximately 0.03367, using known results on bisection width.

In this paper, lower and upper bounds for $\gamma_P(G)$ are given for a random cubic graph G.

For the lower bound, Kang and Wormald first prove that $\gamma_P(G) \geq bw(G) - 13$ for any cubic graph G, where $bw(G)$ denotes the bisection width of G, which is defined as $min\{|\partial S| : S \subset V(G)\}$, $|S| = |V(G)|/2\}$. Coupled with a result of Kostochka and Melnikov [29], this gives $\gamma_p(G) > 0.03367n$ asymptotically almost surely for an n-vertex random cubic graph G, as $n \to \infty$.

For upper bounds, Kang and Wormald present two greedy algorithms that find power dominating sets and analyze them, again using the differential equation method. The analysis of the second algorithm gives the main upper bound: asymptotically almost surely, $\gamma_p(G) \leq 0.067801n$ for a random cubic graph G of order n.

In 2018 [8] Benson, Ferrero, Flagg, Furst, Hogben, Vasilevska, and Wissman discuss the close connection between the power domination number, what is called the *zero forcing number* $Z(G)$ of a graph G, and the *maximum nullity* of G. We will need a few definitions.

The concept of *zero forcing* can be explained in terms of a process of coloring the vertices of a graph G. Initially some subset $S \subseteq V$ of vertices are colored, say blue, and all vertices in $V - S$ are colored white. Then, just as in the power domination propagation rule, if u is a blue vertex and exactly one neighbor $v \in N(u)$ is white, then the color of v changes to blue. In this way vertex u *forces* vertex v to change color. This can be denoted by $u \to v$ and is called a *zero forcing rule*. A set S is called a *zero forcing set* of G if after coloring all vertices in S blue, repeated applications of the zero forcing rule result in all vertices being colored blue. The *zero forcing number*, $Z(G)$, equals the cardinality of a zero forcing set in G.

It is easy to see that the power domination process in a graph G can be described as choosing a set $S \subseteq V$ and applying the zero forcing process to the closed neighborhood $N[S]$ of S. Thus, as first observed by Aazami [1], a set S is a power dominating set of a graph G if and only if $N[S]$ is a zero forcing set of G.

Notice also that for any graph G with minimum degree $\delta(G)$, $\delta(G) \leq Z(G)$.

Next, let $S_n(\mathcal{R})$ denote the set of all $n \times n$ real symmetric matrices. For $A = [a_{ij}] \in S_n(\mathcal{R})$, the graph of A, denoted by $G(A)$, is the graph with vertices $V = \{v_1, v_2, \ldots, v_n\}$ and edges $\{v_i v_j : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Note that the diagonal of A is ignored in defining $G(A)$.

The set of symmetric matrices described by a graph G of order n is defined as $S(G) = \{A \in S_n(\mathcal{R}) : G(A) = G\}$. The *maximum nullity* of G is $M(G) = max\{nullA : A \in S(G)\}$, where *nullA* is the dimension of the null space of A, and the *minimum rank* of G is $mr(G) = min\{rankA : A \in S(G)\}$, where *rankA* is the dimension of the column space of A. By definition, $M(G) + mr(G) = |V(G)|$.

The term *zero forcing* comes from the process of forcing zeros in a null vector of a matrix $A \in S(G)$. Thus, we have the basic inequality, as observed in [4]:

Proposition 3 (AIM) *For any graph G, $M(G) \leq Z(G)$.*

Starting from this basic inequality, Benson et al. prove the following.

Theorem 17 (Benson et al.) *For any non-empty graph G, $\frac{Z(G)}{\Delta(G)} \leq \gamma_P(G)$.*

Having established a connection between the power domination number and the zero forcing number, the authors mention an interesting variation of the power domination propagation rule, or the zero forcing rule, called a *skew zero forcing rule*, first introduced in [27] by the IMA-ISU research group on minimum rank.

In the skew zero forcing rule, a vertex u can force a neighbor $v \in N(u)$ to change color from white to blue if v is the only neighbor of u colored white. But it is permitted that the color of blue itself can be white, whereas in the zero forcing rule, the color of vertex u must be blue. This modified forcing rule then gives rise to the *skew zero forcing number*, $Z^-(G)$, which equals the minimum cardinality of a skew zero forcing set, that is, a set S of vertices that can force all vertices $v \in V$ to be colored blue using the skew zero forcing rule. This variation, essentially in power domination, seems worth studying.

In 2019 [11] Bozeman, Brimkov, Erickson, Ferrero, Flagg, and Hogben consider the problem of determining the minimum number of additional PMUs needed to observe a power network when the network is expanded, but the existing devices S remain in place. They also study the related problem of finding the smallest zero forcing set that must contain a given set of vertices S. The sizes of such sets in a graph G are, respectively, called the *restricted power domination number* and the *restricted zero forcing number* of G subject to S, which can be denoted $\gamma_P(G, S)$ and $Z(G, S)$.

Bozeman et al. present a linear algorithm for computing $\gamma_P(G, S)$ on graphs with bounded treewidth.

Theorem 18 *For any graph $G = (V, E)$ of order n and bounded treewidth, and any set $S \subseteq V$, a minimum power dominating set of G containing S can be computed in $O(n)$ time.*

In 2019 [12] Brimkov, Mikesell, and Smith consider the problem of finding a minimum power dominating set in which the subgraph $G[S]$ induced by the initial set S of vertices is connected. The minimum cardinality of such a power dominating set is called the *connected power domination number*, denoted $\gamma_{cP}(G)$. They show that the connected power domination problem (CPDS) is NP-hard for arbitrary graphs, but can be computed in linear time for trees, cactus graphs, and block graphs.

Tables 1 and 2 and Figure 12 summarize the complexity results for power domination and the several variations of power domination discussed in this chapter.

Table 1 Complexity results for power dominating set

NP-complete	Polynomial
Bipartite graphs	Trees
Haynes et al. 2002 [25], Guo et al. 2005, 2008 [23, 24]	Haynes et al. 2002 [25], Guo et al. 2005 [23]
Chordal graphs	Graphs with bounded treewidth
Haynes et al. 2002 [25]	Guo et al. 2008 [24], Kneis et al. 2006 [30]
Split graphs	Interval graphs
Guo et al. 2005, 2008 [23, 24], Liao and Lee 2005 [33]	Liao and Lee 2005, 2013 [33, 34]
Planar graphs	Proper circular arc graphs
Guo et al. 2005, 2008 [23, 24]	Liao and Lee 2005, 2013 [33, 34]
Planar bipartite graphs	Circular arc graphs
Brueni and Heath 2005 [14]	Liao and Lee 2013 [34]
Circle graphs	Block graphs
Guo et al. 2005, 2008 [23, 24]	Xu et al. 2006 [42]
Planar cubic graphs	Block-cactus graphs
Raible and Fernau 2008, 2012 [9, 39]	Hon et al. 2007 [26]
	Grid graphs
	Dorfling and Henning 2006 [19]
	Silicate networks
	Stephen et al. 2015 [40]

Table 2 Complexity results for variations of power domination

|	NP-complete	Polynomial
r-PDS	Planar, bipartite, chordal, circle graphs	Bounded treewidth, trees, block graphs
	Aazami 2010 [2]	Liao 2016 [31]
k-PDS	Chordal, bipartite, circle, planar, chordal bipartite graphs	Trees
	Chang et al. 2012 [15]	Chang et al. 2012 [15]
		Block graphs
		Wang et al. 2016 [41]
PDS with forbidden zone		Bounded treewidth
		Bozeman et al. 2019 [11]
c-PDS	Arbitrary graphs	Trees
	Brimkov et al. 2019 [12]	Bozeman et al. 2019 [11]
		Cactus, block graphs
		Brimkov et al. 2019 [12]
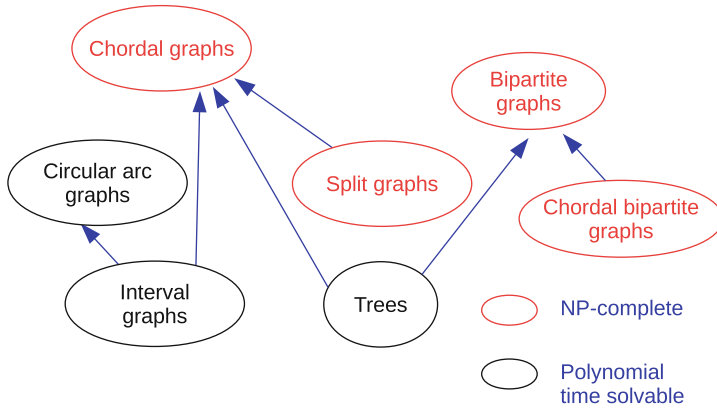
Fig. 12 Summary of power domination results

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