Domination in Digraphs

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1 Introduction

Domination in digraphs is relatively unexplored if compared to its counterpart in graphs. In this chapter, we present selected results on domination in digraphs and give some background on the related topics of bases and kernels. The first two Ph.D. dissertations devoted to the study of domination in digraphs were written by Changwoo Lee [62] in 1994 and by Lisa Hansen [46] in 1997. A survey of results prior to 1998 on domination in directed graphs by Ghoshal, Laskar, and Pillone [43] is given in Chapter 15 of [54]. For completeness, many of these results are repeated

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here. We first present some terminology. For terminology and notation not found here, we refer the reader to the glossary in chapter "Glossary of Common Terms" of this volume.

1.1 Basic Terminology and Notation

Throughout this chapter, we let $D = (V, A)$ be a finite directed graph, or *digraph*, with a finite *vertex set* $V = V(D)$ and an *arc set* $A = A(D) \subseteq V \times V$, which is a subset of the Cartesian product $V \times V$, consisting of all ordered pairs of vertices in V, where neither loops (u, u) nor multiple arcs (u, v) and (u, v) are allowed, although pairs of opposite arcs, such as (u, v) and (v, u), are allowed. Also, $G = (V, E)$ stands for a simple, finite, *undirected* graph with vertex set $V(G)$ and *edge* set $E(G)$, which consists of a subset of the set of all unordered pairs $uv = vu$ of distinct vertices in V.

For two vertices $u, v \in V$ and an arc $(u, v) \in A$, we say that:

(i) (u, v) is an arc *from u to v*,
(ii) *u is adjacent to v*,
(iii) *v is adjacent from u*,
(iv) v is an *out-neighbor* of u,
(v) u is an *in-neighbor* of v,
(vi) v is a *successor* of u or the *terminal vertex* of the arc,
(vii) u is a *predecessor* of v or the *initial vertex* of the arc,
(viii) u and v are incident to arc (u, v), and
(ix) arc (v, u) is the *reverse* of arc (u, v).

We also denote an arc (u, v) by $u \to v$. If both arcs (u, v) and (v, u) are in A, we denote this by $u \leftrightarrow v$; and this is called a *bidirected* or *symmetric* arc. A digraph $D = (V, A)$ is called *oriented* or *anti-symmetric* if for every $(u, v) \in A$, we have $(v, u) \notin A$, that is, D has no symmetric arcs. Equivalently, an *oriented digraph* can be obtained from a graph G by assigning a direction, either $u \to v$ or $v \to u$, to each edge uv of G.

The *outset* or *out-neighborhood* of a vertex $u \in V$ is the set of vertices $N_D^+(u) = \{v \mid u \to v \in A\}$, while the *inset* or *in-neighborhood* of vertex u is the set $N_D^-(u) = \{v \mid u \leftarrow v \in A\}$. The *outdegree* of vertex u, denoted $\mathrm{od}_D(u)$ or $d_D^+(u)$ in the literature, equals $|N_D^+(u)|$, while the *indegree* of u, denoted $\mathrm{id}_D(u)$ or $d_D^-(u)$ in the literature, equals $|N_D^-(u)|$. The maximum indegree of a digraph D, denoted $\Delta^-(D)$, is the maximum indegree among the vertices in D. The maximum outdegree of D is defined as expected and is denoted $\Delta^+(D)$. Similarly, the minimum indegree and minimum outdegree of D are denoted $\delta^-(D)$ and $\delta^+(D)$, respectively. The *degree* of a vertex v in D is $d_D(v) = \mathrm{od}_D(v) + \mathrm{id}_D(v)$. We note that

$$\sum_{v \in V(D)} \mathrm{od}_D(v) = \sum_{v \in V(D)} \mathrm{id}_D(v).$$

A digraph is *r-regular* if $\text{od}_D(v) = \text{id}_D(v) = r$ for every vertex v of D. We also define the *closed out-neighborhood* of a vertex v to equal $N_D^+[v] = N_D^+(v) \cup \{v\}$ and similarly the *closed in*-neighborhood to equal $N_D^-[v] = N_D^-(v) \cup \{v\}$. The *out-neighborhood* of a set S of vertices is $N_D^+(S) = \cup_{v \in S} N_D^+(v)$, and the *closed out-neighborhood* of S is $N_D^+[S] = \cup_{v \in S} N_D^+[v]$. And finally, the *in-neighborhood* of S is $N_D^-(S) = \cup_{v \in S} N_D^-(v)$, and the *closed in-neighborhood* of S is $N_D^-[S] = \cup_{v \in S} N_D^-[v]$.

Let $S \subseteq V$ and $u \in S$. A vertex $v \in V \setminus S$ is called a *private out-neighbor* of u with respect to S if $N_D^-(v) \cap S = \{u\}$, that is, v is an out-neighbor of u, $u \to v$, but is not an out-neighbor of any other vertex in S. The set of all private out-neighbors of u with respect to S is denoted by $\text{pn}_D^+(u, S)$. Similarly, a vertex $v \in V \setminus S$ is called a *private in-neighbor* of u with respect to S if $N_D^+(v) \cap S = \{u\}$, that is, v is an in-neighbor of u, $u \leftarrow v$, but is not an in-neighbor of any other vertex in S. The set of all private in-neighbors of u with respect to S is denoted by $\text{pn}_D^-(u, S)$.

If the digraph D is clear from context, we omit the subscript D from the above notational definitions. For example, we simply write $\text{id}(u)$, $\text{od}(u)$, $N^-(u)$, $N^+(u)$, $\text{pn}^+(u, S)$, and $\text{pn}^-(u, S)$, rather than $\text{id}_D(u)$, $\text{od}_D(u)$, $N_D^-(u)$, $N_D^+(u)$, $\text{pn}_D^+(u, S)$, and $\text{pn}_D^-(u, S)$, respectively. A vertex u is called:

(i) an *isolated vertex* if $\text{od}(u) = \text{id}(u) = 0$,
(ii) a *source* or *transmitter* if $\text{id}(u) = 0$ and $\text{od}(u) > 0$, and
(iii) a *sink* or *receiver* if $\text{od}(u) = 0$ and $\text{id}(u) > 0$.

Given two sets $R, S \subseteq V$, we let (R, S) denote the set of all arcs in A from R to S, that is, $(R, S) = \{(u, v) \in A \mid u \in R, v \in S\}$.

For any integer $k \geq 1$, we use the standard notation $[k] = \{1, \ldots, k\}$ and $[k]_0 = [k] \cup \{0\} = \{0, 1, \ldots, k\}$. A *directed walk* in a digraph $D = (V, A)$ from a vertex u to a vertex w, called a (u, w)-*walk*, is a sequence of vertices of the form $u = v_0, v_1, \ldots, v_k = w$ such that for every $i \in [k]$, we have $(v_{i-1}, v_i) \in A$. Such a (u, w)-walk has *length* k. A directed walk having no repeated edges is called a *directed trail*. A directed walk having no repeated vertices is called a *directed path*. A directed walk in which $v_0 = v_k$ is called a *closed directed walk*, and a closed walk in which all vertices, except v_0 and v_k, are distinct is called a *directed cycle* or a *circuit*. Let $\vec{C}_n$ denote the directed cycle on n vertices.

The *distance* $d_D(u, v)$ from a vertex u to a vertex v in a digraph D is the minimum length of a directed (u, v)-path. If the digraph D is clear from the context, we write $d(u, v)$ rather than $d_D(u, v)$.

Given a digraph $D = (V, A)$, the *underlying graph of* D is the undirected graph $G(D) = (V, E)$, where $uv \in E$ if and only if $u \to v \in A$, $u \leftarrow v \in A$, or $u \leftrightarrow v \in A$. A digraph D is *connected* or *weakly connected* if its *underlying graph* $G(D)$ is connected.

A digraph D is said to be *strongly connected* if for every $u, w \in V$, there exist a directed (u, w)-path and a directed (w, u)-path. We note that one could consider the class of digraphs having the property that for every $u, w \in V$ either there is a directed walk from u to w or there is a directed walk from w to u.

A digraph $D = (V, A)$ is said to be *transitive* if $(u, v), (v, w) \in A$ implies that the arc $(u, w) \in A$. In other applications, a digraph D of order n is said to have a *transitive orientation* if there is an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that for every

$i \in [n-1]$, we have $(v_i, v_{i+1}) \in A$. A digraph is *complete* if for every $u, v \in V$, either (u, v), (v, u), or both arcs are in A. A *tournament* is an oriented complete graph.

We denote the *degree* of a vertex v in an undirected graph G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. The average degree in G is denoted by $d_{\mathrm{av}}(G)$. The minimum degree among the vertices of G is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$.

1.2 Domination and Independence

In this section we define independence and the types of domination in digraphs that will be discussed in this chapter. Let $D = (V, A)$ be a digraph with vertex set V and arc set A.

Definition 1 A set S of vertices in a digraph D is *independent* if no two vertices u, $v \in S$ are joined by an arc, that is, $(u, v) \notin A$ and $(v, u) \notin A$. The maximum cardinality of an independent set in a digraph D is called the *vertex independence number* of D and is denoted $\alpha(D)$, while the minimum cardinality of a maximal independent set of vertices in a digraph is the *lower vertex independence number*, denoted $\alpha_{\min}(D)$.

Definition 2 A set S of vertices in a digraph D is an *out-dominating set*, or just a *dominating set*, if for every vertex $v \in V \setminus S$, there exists a vertex $u \in S$ such that $u \to v \in A$, that is, every vertex in $V \setminus S$ is adjacent from a vertex in S. In other words, S is a dominating set of D if $V \setminus S \subseteq N^+[S]$. The minimum cardinality of dominating set in D is called the *out-domination number*, or simply the *domination number*, of D and is denoted $\gamma^+(D)$, or just $\gamma(D)$.

In general, we adopt the simplified terminology for out-dominating sets by omitting "out" and simply referring to dominating sets, domination number, and $\gamma(D)$.

Definition 3 A set S of vertices in a digraph D is an *in-dominating set* (also called a *converse dominating set* in the literature) if for every vertex $v \in V \setminus S$, there exists a vertex $u \in S$ such that $v \to u \in A$, that is, every vertex in $V \setminus S$ is adjacent to a vertex in S. In other words, S is an in-dominating set of D if $N^+(v) \cap S \neq \emptyset$. The minimum cardinality of an in-dominating set in a directed graph D is called the *in-domination number* of D and is denoted $\gamma^-(D)$.

Definition 4 A set S of vertices in a digraph D is a *twin dominating set* of D if it is both an in-dominating set and out-dominating set of D. The minimum cardinality of a twin dominating set is the *twin domination number* $\gamma^{\pm}(D)$ of D (also denoted $\gamma^*(D)$ in the literature).

To illustrate the above definitions, consider the digraph D shown in Figure 1. The darkened vertices in Figure 1(a) and 1(b) form a dominating set and an in-dominating set, respectively, of D, while the darkened vertices in Figure 1(c) form a twin dominating set of D.

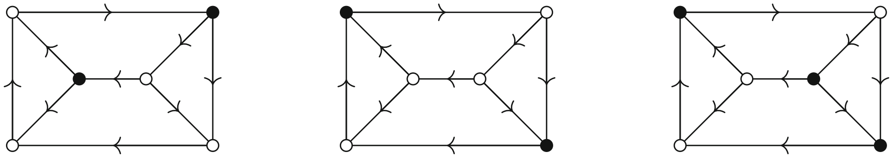


Fig. 1 A digraph D with $\gamma(D) = \gamma^-(D) = 2$ and $\gamma^\pm(D) = 3$

2 Background and History

In this section, we recognize and honor Dénes König for his pioneering work on domination in digraphs. His work on the *basis* of a digraph, which we shall see is an independent dominating set, comes some 30 years before any other mention of domination in the literature. Since König was the originator of domination in digraphs, we give several of his theorems along with their proofs. In the second part of this section, we present a brief overview of *kernels* in digraphs, which we shall see are independent in-dominating sets. We include some of Berge's early results on kernels with a sampling of proofs. We also give some results on the existence of kernels in digraphs. A survey of the expansive literature on kernels is beyond the scope of this chapter, so our brief overview is not meant to be complete. For more information we refer the reader to surveys by Boros and Gurvich [12] and Frankel [37], respectively.

2.1 *Basis of the Second Kind*

The concept of domination in digraphs was introduced as early as 1936 by König [61]. We present his original ideas in what follows, as they form a foundation on which many ideas for domination in digraphs can be built.

For any vertex $a \in V$ in a digraph $D = (V, A)$, let V_a equal the set consisting of a together with all vertices x for which there exists a directed path from a to x. If there is no vertex $b \in V$ such that $V_a \subset V_b$, then V_a is called a *basic set* with *source a*.

Theorem 1 ([61]) *Every vertex $a \in V$ of a finite directed graph $D = (V, A)$ is a member of some basic set of D.*

Proof Let $a \in V$. If V_a is a basic set, then clearly a is a member of a basic set. By definition, if V_a is not a basic set, then there exists a vertex $b \in V$ such that $V_a \subset V_b$, which implies that there must exist a directed path from b to a. Thus, if V_b is a basic set, then a is a member of the basic set V_b. Again, if V_b is not a basic set, then by definition, there exists a vertex $c \in V$ such that $V_a \subset V_b \subset V_c$. If V_c is a basic set, then a is a member of the basic set V_c. Since V is a finite set, this process must end with a vertex $x \in V$, such that V_x is a basic set containing a. □

König pointed out that this theorem does not hold for infinite directed graphs, using the example of an infinite directed path $v_1, v_2, v_3, \ldots$, in which every arc has the form (v_{i+1}, v_i). It is easy to see that this infinite directed path has no basic set.

Theorem 2 ([61]) *No proper subset of a basic set is a basic set.*

Proof Suppose, to the contrary, that a basic set V_b contains a basic set V_a as a proper subset. Since there is a directed path from b to a, and since V_a is a basic set, V_a cannot be properly contained in another basic set. Thus, it follows that there must be a directed path from a to b. From this it follows that V_b must be a subset of V_a and thus that $V_a = V_b$. But this means that V_a is not a proper subset of V_b, a contradiction. □

We can now define a basis of a directed graph.

Definition 5 A *basis* of a directed graph $D = (V, A)$ is a set $B \subset V$ having the following two properties:

(i) for every vertex $v \in V \setminus B$, there exist a vertex $u \in B$ and a directed path from u to v.
(ii) for every pair of vertices $u, v \in B$, there is no directed path from u to v.

Theorem 3 ([61]) *Every finite directed graph $D = (V, A)$ has a basis.*

Proof Let $\mathcal{V} = \{V_a, V_b, \ldots, V_k\}$ be the set of all basic sets of a finite directed graph $D = (V, A)$, and let $B = \{a, b, \ldots, k\}$ be sources for each of these basic sets. We claim that the set B is a basis of D.

Note that Theorem 1 says that every vertex $v \in V$ is a member of some basic set, say $v \in V' \in \mathcal{V}$. Assume that $v \in V \setminus B$. But $V' = V_w$ for some $V_w \in \mathcal{V}$ and $w \in B$, since $\mathcal{V}$ contains all basic sets. Thus, by definition there must be a directed path from w to v, and property (i) in Definition 5 is satisfied.

In order to show that B satisfies property (ii) in Definition 5, suppose, to the contrary, that for two sources a and b in B, where $V_a \neq V_b$, there is a directed path from a to b. But in this case, it follows that $V_b \subseteq V_a$. However, if $V_b \subset V_a$, then V_b cannot be a basic set, a contradiction. On the other hand, if $V_b = V_a$, then we contradict the supposition that $V_a \neq V_b$. □

Theorem 4 ([61]) *If a vertex $a \in V$ is contained in a basis B in a directed graph $D = (V, A)$, then V_a is a basic set.*

Proof Assume that a vertex $a \in V$ is contained in a basis. Suppose, to the contrary, that V_a is not a basic set. Then there must exist a vertex $b \in V$ not contained in V_a such that V_a is a proper subset of V_b. Therefore, there must be a directed path from b to a. But if this is the case, then b does not belong to the basis B, since by property (ii) there can be no directed path between two vertices in a basis. Therefore, there must be a directed path from a vertex c of B to b, where $c \neq a$, for otherwise b would belong to V_a. The directed paths from c to b and from b to a imply, by Theorem 1, that there exists a directed path from c to a, contradicting property (ii) in the definition of a basis. □

Theorem 5 ([61]) *Every basis B in a digraph* $D=(V, A)$ *consists of one source from each basic set.*

Proof By Theorem 4, every vertex of a basis B is a source of a basic set. In addition, two distinct vertices in B are never sources of the same basic set, since by property (ii) there can be no directed path between two vertices in B. It only remains to show that every basic set has a source in B. Suppose there exists a basic set V_a with source a such that $a \notin B$. By the definition of basis, there is a vertex $b \in B$ such that there is a directed path from b to a. But b is the source of a basic set V_b, and so the basic set V_a is a proper subset of the basic set V_b, contradicting Theorem 2. □

Corollary 6 ([61]) *Every basis of a digraph D has the same cardinality, which equals the number of source vertices in D.*

Proof By Theorem 5, since every basis has one source from each basic set, every basis has a cardinality equal to the number of basic sets in D. □

In his book, König pointed out that if every edge of a digraph D is symmetric, and the digraph D is basically an undirected graph, then the number of basic sets equals the number of components. König then defined a basis of the second kind as follows.

Definition 6 A *basis of the second kind* in a directed graph $D=(V, A)$ is a set $B \subset V$ satisfying the following two conditions:

(i) if v is a vertex in $V \setminus B$, then there is an arc (u, v) from a vertex $u \in B$ to v, and
(ii) there is no arc between two vertices in B.

Notice that by property (i) a basis of the second kind is a dominating set of D and by (ii) a basis of the second kind is an independent set of D. König noted that Corollary 6 is no longer true for bases of the second kind, i.e., for independent dominating sets.

In the case where a digraph D is symmetric, König's basis of the second kind appears to be the first time in the literature where an *independent dominating* set is defined in an undirected graph. It also, of course, defines an independent dominating set in a digraph for the first time. To illustrate a minimum independent dominating set in an undirected graph, König used as an example the classical problem of covering an 8×8 chessboard with the minimum number of queens. The *Queen's graph* consists of 64 vertices (one for each square on the chessboard), where two vertices/squares are adjacent if and only if a queen placed on one square can occupy the second square in 1 move. Thus, two vertices are adjacent if and only if they are in the same row, column, or diagonal. The minimum number of queens needed to cover the chessboard (the domination number of the Queen's graph) is 5. König's example of five queens, placed at the locations shown in Figure 2, covers the board with the added constraint that no two queens can attack each other, that is, this placement of these five queens represents a minimum independent dominating set of the Queen's graph.

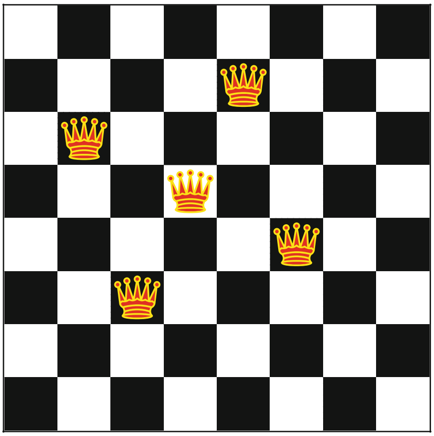


Fig. 2 Minimum independent dominating set of queens

An independent dominating set of a digraph is also called a *solution* in the literature. In the context of games, a solution is defined by Von Neumann and Morgenstern in their now classic book [92]. We formally state the definition of a solution in terms of digraphs and give notation for a minimum independent dominating set.

Definition 7 A *solution* in a digraph D is an independent dominating set of D. The *solution number* of D, denoted $i^+(D)$, equals the minimum cardinality of a solution in D, that is, $i^+(D) = \alpha_{\min}(D)$.

Richardson [79] showed that every digraph with no odd cycles has at least one solution.

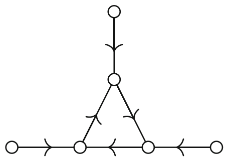
2.2 *Kernels in Digraphs*

In 1958, Berge [6] defined an in-dominating set, which he called an *absorbant set*. Although he called the in-domination number the absorption number and denoted it by $\beta(D)$, we shall continue with the terminology in-domination and denote the in-domination number as $\gamma^-(D)$, as defined in Section 1.2.

Definition 8 A *kernel* in a digraph D is an independent, in-dominating (absorbant) set of D. The *kernel number* of D equals the minimum cardinality of a kernel in D and is denoted $i^-(D)$.

The topic of kernels in digraphs has its roots in game theory and was introduced by Von Neumann and Morgenstern in 1944 [92]. Kernel applications have grown from n-person games and Nim-type games to more recent applications in artificial intelligence, combinatorics, and coding theory.

Fig. 3 A graph with a solution but no kernel



We note that not every digraph has a kernel; for example, a directed cycle $\vec{C}_5$ does not. Neither does $\vec{C}_5$ have a solution. The graph in Figure 3 has a solution, consisting of the three vertices of indegree zero, but it has no kernel.

For digraphs with kernels, Berge [6] proved the following.

Theorem 7 ([6]) *If S is a kernel, then S is both a maximal independent set and a minimal in-dominating set.*

Proof Let $S \subseteq V$ be a kernel in a digraph $D = (V, A)$. Since S is an in-dominating set, for each vertex $u \in V \setminus S$, there is an arc $(u, v) \in A$ where $v \in S$. Hence, $S \cup \{u\}$ is not an independent set, and so, S is a maximal independent set. Similarly, if $u \in S$, then $S \setminus \{u\}$ is not an in-dominating set since S is an independent set, and therefore there is no arc (u, v) for any $v \in S \setminus \{u\}$. Thus, S is a minimal in-dominating set. □

Since not all digraphs have kernels, a natural question to ask is: What structural properties of digraphs imply the existence of a kernel? The existence of a kernel in a given digraph has been studied in many papers, including [5, 25, 26, 41, 79]. Berge [7] gave a necessary and sufficient condition for a vertex set to be a kernel in terms of its characteristic function. Recall that the characteristic function $\phi_S : V \to \{0, 1\}$ of a set S is defined as: $\phi_S(x) = 1$ if $x \in S$ and $\phi_S(x) = 0$ if $x \notin S$. We will assume that if a vertex x has no out-neighbors, then $\max\{\phi_S(y) \mid y \in N^+(x)\} = 0$.

Theorem 8 ([7]) *A set $S \subseteq V$ is a kernel of a digraph $D = (V, A)$ if and only if for every $x \in V$, $\phi_S(x) = 1 - \max\{\phi_S(y) \mid y \in N^+(x)\}$.*

Proof Let S be a kernel in a digraph D, and assume that ϕ_S is the characteristic function defined on it. If $x \in S$, then $\phi_S(x) = 1$. Since S is an independent set, no out-neighbor of x is in S. Thus, $\max\{\phi_S(y) \mid y \in N^+(x)\} = 0$, and therefore, $\phi_S(x) = 1 = 1 - \max\{\phi_S(y) \mid y \in N^+(x)\}$.

If $x \notin S$, then $\phi_S(x) = 0$. Since S is an in-dominating set, it follows that there must be a vertex $v \in S$ and an arc $(x, v) \in A$. Thus, $\max\{\phi_S(y) \mid y \in N^+(x)\} = 1$, and therefore, $\phi_S(x) = 0 = 1 - \max\{\phi_S(y) \mid y \in N^+(x)\}$.

Conversely, let S be a set for which, for every $x \in V$, $\phi_S(x) = 1 - \max\{\phi_S(y) \mid y \in N^+(x)\}$. If $x \in S$, then $\phi_S(x) = 1$. Thus, since $\phi_S(x) = 1 - \max\{\phi_S(y) \mid y \in N^+(x)\}$, it must follow that $\max\{\phi_S(y) \mid y \in N^+(x)\} = 0$, but this means that no out-neighbor of x is in S. If an in-neighbor of x, say y, is in S, then x is an out-neighbor of y, and therefore, $\phi_S(y) = 1$. But $\max\{\phi_S(x) \mid x \in N^+(y)\} = 1$, and so, $1 - \max\{\phi_S(x) \mid x \in N^+(y)\} = 0$, a contradiction. Therefore, S is an independent set.

Similarly, if $x \notin S$, then $\phi_S(x) = 0$. But since, for every $x \in V$, $\phi_S(x) = 1 - \max\{\phi_S(y) \mid y \in N^+(x)\}$, this must mean that $\max\{\phi_S(y) \mid y \in N^+(x)\} = 1$. Hence, at least one neighbor of x, say y, is in S. Therefore, S is an in-dominating set. □

As early as 1936, König [61] proved the following result. A digraph $D = (V, A)$ is called *transitive* if whenever $(u, v) \in A$ and $(v, w) \in A$, then $(u, w) \in A$.

Theorem 9 ([61]) *If $D = (V, A)$ is a transitive digraph, then every minimal in-dominating set has the same cardinality. Furthermore, a set $S \subseteq V$ is a kernel if and only if S is a minimal in-dominating set.*

Corollary 10 *Every transitive digraph has a kernel, and all of its kernels have the same cardinality.*

In 1990 De la Vega [29] showed that although not all digraphs have kernels, probabilistically speaking, almost all digraphs do. Let $D(n, p) = (V, A)$ denote a *random digraph* of order n where for every $u, v \in V$, the arc (u, v) is chosen with probability p.

Theorem 11 ([29]) *For any probability p, where $0 \leq p \leq 1$, the probability that the random digraph $D(n, p)$ has a kernel goes to 1 as $n \to \infty$.*

Algorithms for determining all the kernels of a digraph D have been presented by Rudeanu [81] in 1966 and Roy [80] in 1970.

Many of the existence results for kernels are proved under an even stronger condition that the digraph is kernel-perfect. A digraph D is said to be *kernel-perfect* if D has a kernel and every induced subdigraph of D has a kernel. Meyniel conjectured that if every circuit of a digraph D has at least two chords, then D is kernel-perfect. Although Galeana-Sánchez [39] proved this conjecture to be false, the searching for a proof motivated results on sufficient conditions for the existence of a kernel in a digraph. The proof we present of the following result of Von Neumann and Morgenstern [92] is due to Berge [7].

Theorem 12 ([92]) *Every digraph D without directed cycles is kernel-perfect and has a unique kernel.*

Proof Given a digraph D having no directed cycles, define the set S_0 as the collection of sinks of D, and for each $k \geq 1$, define S_k as the set of all vertices u such that a longest (directed) path from u to a vertex in S_0 has length k. Thus,

$$S_0 = \{v \in V \mid N^+(v) = \emptyset\}.$$
$$S_1 = \{v \in V \mid N^+(v) \subseteq S_0\}.$$
$$S_2 = \{v \in V \mid N^+(v) \subseteq (S_0 \cup S_1)\}.$$

And in general, $S_k = \{v \in V \mid N^+(v) \subseteq (S_0 \cup S_1 \cup \ldots \cup S_{k-1})\}$.

Since D contains no directed cycles, the sets S_k form a partition of $V(D)$. One can then define a characteristic function $\phi_S(x) = 1 - \max\{\phi_S(y) \mid y \in N^+(x)\}$

iteratively, starting with the vertices $u \in S_0$ for each of which $\phi_S(u) = 1$, and then each vertex in S_1 receives the value 0. After this, a vertex x can be assigned a value $\phi_S(x)$ only after all of the vertices in $N^+(x)$ have been assigned a value, at which point the value of $\max\{\phi_S(y) \mid y \in N^+(x)\}$ can be determined. By Theorem 8, $S = \{v \in V \mid \phi_S(v) = 1\}$ is a kernel of D.

Since every subdigraph of D is acyclic, it follows that D is kernel-perfect. Moreover, the set S_0 of sinks is nonempty and unique, and so by definition, S_k is unique for each $k \geq 1$. The uniqueness of S follows from the fact that any kernel of D must contain S_0, and hence the vertices of $S_k \cap S$. □

We next mention classical results due to Richardson [79] and Duchet [25].

Theorem 13 ([79]) *Every digraph D without odd directed cycles is kernel-perfect.*

Theorem 14 ([25]) *If every circuit in a digraph D has at least one symmetric arc, then D is kernel-perfect.*

Recall that a kernel in a digraph D is an independent set S such that every vertex not in S dominates some vertex in S, where as usual by "dominates" we mean "out-dominates," that is, a vertex u dominates a vertex v if there is an arc (u, v) from u to v. We next define a semi-kernel in a digraph. Recall that the distance $d_D(u, v)$ from a vertex u to a vertex v in a digraph D is the shortest directed path from u to v. We note that $d_D(u, v)$ may be very different from $d_D(v, u)$.

Definition 9 A set S of vertices in a digraph D is a *semi-kernel* if S is an independent set and every vertex not in S either dominates some vertex in S or dominates a vertex which in turn dominates some vertex in S. Thus, S is a semi-kernel in D if S is an independent set and for every vertex $v \in V(D) \setminus S$, there is a vertex $u \in S$ such that $d_D(v, u) \leq 2$.

As observed earlier, not all digraphs have kernels. However, every digraph has a semi-kernel. This result is attributed to Chvátal and Lovász [24]. However, in this paper they proved Theorem 16, which we state shortly. It is not clear if Theorem 16 immediately implies Theorem 15. The proof of the following result is due to Bondy [11].

Theorem 15 ([11]) *Every digraph has a semi-kernel.*

Proof Let D be a digraph and let H be a maximal induced acyclic subdigraph of D. By Theorem 12, the acyclic digraph H has a (unique) kernel. Let S be the kernel of H. We claim that S is a semi-kernel of D. Since S is a kernel of H, every vertex of $H - S$ dominates some vertex of S. Let v be an arbitrary vertex outside H, and so $v \in V(D) \setminus V(H)$. By our choice of H, there is a directed cycle C in the subdigraph of D induced by $V(H) \cup \{v\}$. The vertex v therefore dominates its successor v^+ on C. Since $v^+ \in V(H)$, either $v^+ \in S$, in which case v dominates a vertex of S, or $v^+ \notin S$, in which v^+ dominates a vertex of S and therefore v dominates a vertex which in turn dominates some vertex of S. Thus, S is a semi-kernel of D. □

Definition 10 For any integer $k \geq 2$, a set S of vertices in a digraph D is a *k-dominating set* if S is an independent set and every vertex not in S can be reached from a vertex of S by a directed path of length at most k, that is, for every vertex $v \in V(D) \setminus S$, there is a vertex $u \in S$ such that $d(u, v) \leq k$.

We note that a 1-dominating set of a digraph D is an independent, (out-) dominating set of D. For $k \geq 1$, every k-dominating set is a $(k+1)$-dominating set. In particular, every 1-dominating set is a 2-dominating set. Not every digraph has a 1-dominating set; for example, $\vec{C}_5$ does not. In 1974 Chvátal and Lovász [24] proved that every digraph has a 2-dominating set.

Theorem 16 ([24]) *Every digraph has a* 2*-dominating set.*

Proof We proceed by induction on the order n of a digraph D. For $n = 1$ or $n = 2$, the result is immediate. Let $n \geq 3$ and assume that every digraph of order less than n has a 2-dominating set. Let w be an arbitrary vertex of D. If $V(D) = N_D^+(w)$, then the set $\{w\}$ is a 1-dominating set and therefore also a 2-dominating set. Hence, we may assume that $V(D) \neq N_D^+(w)$. Let D' be the subdigraph of D induced by the set of vertices at distance at least 2 from w in D. Thus, $V(D') = \{v \in V(D) \mid d_D(w, v) \geq 2\}$. Further, $(x, y) \in A(D')$ if and only if $x, y \in V(D')$ and $(x, y) \in A(D)$. Applying the inductive hypothesis, the digraph D' contains a 2-dominating set S'. Suppose firstly that there is an arc from u to w for some vertex $u \in S'$. Therefore, $d_D(u, w) = 1$, and every vertex in $N_D^+(w)$ is reachable from u by a directed path of length at most 2, that is, $d_D(u, x) \leq 2$ for every vertex $x \in N_D^+[w]$. In this case, let $S = S'$. Suppose secondly that there is no arc from a vertex in S' to the vertex w, and so $d_D(u, w) \geq 2$ for all vertices $u \in S'$. In this case, we let $S = S' \cup \{w\}$. In both cases, the set S is a 2-dominating set of D. □

As observed earlier, not every digraph has a 1-dominating set. In 1996 Jacob and Meyniel [59] proved that a digraph with no 1-dominating set contains at least three 2-dominating sets.

Theorem 17 ([59]) *Every digraph with no* 1*-dominating set contains at least three* 2*-dominating sets.*

Kernels have relations to Grundy functions in digraphs. We conclude this subsection with some results relating the two.

Definition 11 A non-negative function $g\colon V \to [n]_0$ from the vertex set V of a digraph D to the integers $[n]_0$ is called a *Grundy function* if for every vertex $u \in V$, $g(u)$ is the smallest non-negative integer not belonging to $\{g(v) \mid v \in N^+(u)\}$. It follows, therefore, that if g is a Grundy function, then the following hold.

(1) $g(u) = k$ implies that for each $0 \leq j < k$, there is a vertex $v \in N^+(u)$ with $g(v) = j$.
(2) $g(u) = k$ implies that for every $v \in N^+(u)$, $g(v) \neq g(u)$.

Proposition 18 ([7]) *If a digraph D has a Grundy function, then D has a kernel.*

Proof Let $g\colon V \to [n]_0$ be a Grundy function on a digraph $D = (V, A)$, and let $S = \{u \in V \mid g(u) = 0\}$. From condition (2) in Definition 11, we know that $g(u) = 0$ implies that for every $v \in N^+(u)$, $g(v) \neq g(u) = 0$, and therefore, S is an independent set.

If a vertex $v \notin S$, then $g(v) = k > 0$. From condition (1) in Definition 11, we know that $g(u) = k > 0$ implies that for each $j < k$, there is a vertex $u \in N^+(u)$ with $g(u) = j$, and in particular there is a vertex $w \in N^+(u)$ with $g(w) = 0$. Thus, S is an in-dominating set. Therefore, S is a kernel. □

While it can be verified that if a graph has a kernel, it need not have a Grundy function, the following interesting connection to kernel-perfect digraphs was shown by Berge [7].

Theorem 19 ([7]) *Every kernel-perfect digraph has a Grundy function.*

Proof Let $D = D_0$ be a kernel-perfect digraph, and let S_0 be a kernel of D_0. It follows from the definition of a kernel-perfect digraph that the digraph $D_1 = D_0 - S_0$ is a kernel-perfect digraph. Therefore, let S_1 be a kernel of D_1. Let $D_2 = D_1 - S_1$ and let S_2 be a kernel in D_2. In general for $k \geq 1$, let S_k be a kernel of the subdigraph D_k. The resulting sets $S_0, S_1, \ldots, S_k$ form a partition of $V(D)$. Define a function $g\colon V \to [k]_0$ by $g(u) = j$ if and only if $u \in S_j$. It follows that g is a Grundy function of D.

If $g(u) = j$, then vertex u is a vertex in every digraph $D_0, D_1, \ldots, D_{j-1}$. And $S_0, S_1, \ldots, S_{j-1}$ are in-dominating sets of these digraphs, respectively. Therefore, for each $i < j$, there is a vertex $w \in S_i$ where $w \in N^+(u)$. Thus, condition (1) of a Grundy function (see Definition 11) is satisfied. If $g(u) = j$, then $u \in S_j$, which is an in-dominating set of the digraph D_j. This means that the set S_j is an independent set. Therefore, if $g(u) = j$, then each $v \in N^+(u)$ satisfies $g(v) \neq j$. Therefore, every kernel-perfect digraph D has a Grundy function g. □

Fraenkel [36] has determined that deciding whether a finite digraph D has a kernel or a Grundy function is NP-complete, even when restricted to cyclic planar digraphs with $\mathrm{od}(x) \leq 2$, $\mathrm{id}(x) \leq 2$, and $\mathrm{od}(x) + \mathrm{id}(x) \leq 3$, and these bounds are best possible, since decreasing any of them results in a decision problem that can be solved in polynomial time. The proof of this theorem uses a simple transformation from *3*-Satisfiability.

3 Bounds on In, Out, and Twin Domination Numbers

In this section, we present bounds on the domination, in-domination, and twin domination numbers of digraphs.

3.1 (Out)-Domination

We begin with some well-known results of Ore [75] on dominating sets of graphs.

Theorem 20 ([75]) *If G is a graph having no isolated vertices, then the complement $V \setminus S$ of any minimal dominating set S is a dominating set of G.*

Corollary 21 *The vertices of any graph G having no isolated vertices can be partitioned into two dominating sets.*

Corollary 22 *For any graph G of order n having no isolated vertices, $\gamma(G) \leq \frac{1}{2}n$.*

Fu was interested in possible analogs of these results of Ore for digraphs. For example, can the vertices of a digraph D without isolated vertices be partitioned into two (directed) dominating sets? Fu [38] obtained the following results on dominating sets of directed graphs.

Theorem 23 ([38]) *A dominating set S in a digraph D is a minimal dominating set if for each $u \in S$, there is no arc (u, v) for any vertex $v \in S$.*

Proof Assume that S is a dominating set of a digraph D having the property that for no two vertices $u, v \in S$, $(u, v) \in A$, that is, S is an independent set. Then it follows that for every $u \in S$, $S \setminus \{u\}$ is a not a dominating set since there is no vertex in $S \setminus \{u\}$ that dominates vertex u. Thus, S is a minimal dominating set of D. □

As observed by Fu [38], in order that a digraph D has a dominating set S such that its complement $V \setminus S$ is also a dominating set, it is necessary and sufficient that each vertex $u \in S$ is dominated by a vertex in $V \setminus S$ and each vertex in $V \setminus S$ is dominated by a vertex in S. Moreover, in order that a digraph D has a dominating set S whose complement $V \setminus S$ is an in-dominating set, it is necessary and sufficient that each vertex in S dominates at least one vertex in $V \setminus S$.

Fu defined a digraph D to be *cyclic* or *strongly connected* if every pair of vertices are contained in a directed cycle.

Theorem 24 ([38]) *A strongly connected digraph D has a dominating set S whose complement $\overline{S} = V \setminus S$ is also a dominating set if and only if D contains a directed cycle of even length.*

Proof For the necessity part, assume that D has a dominating set S whose complement $\overline{S} = V \setminus S$ is also a dominating set. Assume that no vertices are colored. Select an arbitrary vertex $u \in S$. Color it blue. Since the complement $\overline{S}$ is a dominating set, there must be a vertex $v \in \overline{S}$ and an arc (v, u). Color vertex v red. Since S is a dominating set, there are a vertex $w \in S$ and an arc (w, v). If $w = u$, then we have found a directed cycle of length 2. If $w \neq u$, color vertex w blue. There must be a vertex $z \in \overline{S}$ which dominates w. If z has been previously colored, we have found a directed cycle beginning and ending in $\overline{S}$ and therefore having even length. If z has not been colored, color it red. Continuing in this way, all vertices encountered will either be in S and colored blue or in $\overline{S}$ and colored red.

Sooner or later we will have to encounter a previously colored vertex and hence have constructed a directed cycle of even length.

To prove the sufficiency, we assume that there is a directed cycle of even length, and we need only show that there is a way to assign the vertices of D either to S or $\overline{S}$, in such a way that both sets are dominating sets. We begin with any directed cycle C_0 of even length and alternately assign its vertices to S and $\overline{S}$. Thus, all of the vertices on C_0 are assigned to a dominating set of C_0. If this includes all vertices of D, then the theorem is proved. Thus, we may assume that there is an unassigned vertex, say w. Since D is strongly connected, w and u are on a directed cycle for any vertex u on C_0. We may then find a directed path from u to w and continue until a vertex is encountered which has already been assigned. The vertices on this directed path can be alternately assigned to either S or $\overline{S}$. This directed path may end with two consecutive vertices assigned to the same set, but each vertex thus encountered is always dominated by the vertex which precedes it on the directed path. Since w is an arbitrary unassigned vertex, every vertex of D can be assigned to one of S and $\overline{S}$. □

Corollary 25 ([38]) *A strongly connected digraph D has a dominating set S whose complement $\overline{S}$ is also a dominating set, and furthermore both S and $\overline{S}$ are in-dominating sets if and only if every vertex of V is in some directed cycle of even length.*

Corollary 26 ([38]) *In order that a strongly connected digraph D has a dominating set S whose complement $\overline{S}$ is an in-dominating set, it is sufficient that D contains a directed cycle of even length.*

Corollary 27 ([38]) *If D is a strongly connected digraph of order n having a cycle of even length, then $\gamma(D) \leq \frac{1}{2}n$.*

We observe that if D is a Hamiltonian digraph of order n, then $\gamma(D) \leq \lceil \frac{n}{2} \rceil$. In 1998 Lee [63] improved the result of Corollary 27 as follows.

Theorem 28 ([63]) *If D is a strongly connected digraph of order n, then $1 \leq \gamma(D) \leq \lceil \frac{n}{2} \rceil$.*

In order to prove Theorem 28, Lee [63] proved that if D is a directed tree of order n that contains a vertex u such that every vertex in D is reachable from u, that is, for every v in D different from u, there is a directed path from u to v, then $1 \leq \gamma(D) \leq \lceil \frac{n}{2} \rceil$. The proof of this result given in [63] is algorithmic in nature and finds a dominating set S in such a directed tree D satisfying $1 \leq |S| \leq \lceil \frac{n}{2} \rceil$. From this result, we can readily deduce Theorem 28, noting that a strongly connected digraph has as a subdigraph a directed spanning tree with the desired property.

Lee [62] proved the following upper bound on the domination number of a digraph D in terms of its order and the minimum indegree $\delta^-(D)$.

Theorem 29 ([62]) *If D is a digraph of order n with $\delta^-(D) = \delta^- \geq 1$, then*

$$\gamma(D) \leq \left(\frac{\delta^{-}+1}{2\delta^{-}+1}\right)n.$$

As a consequence of Theorem 29, we have the following upper bound on the domination number of a digraph in which every vertex has indegree at least 1.

Corollary 30 ([62]) *If D is a digraph of order n with $\delta^{-}(D) \geq 1$, then $\gamma(D) \leq \frac{2}{3}n$.*

Using standard probabilistic arguments, Lee [62] established the following upper bound on the domination of a digraph.

Theorem 31 ([62]) *If D is a digraph of order n with $\delta^{-}(D) = \delta^{-} \geq 1$, then*

$$\gamma(D) \leq \left(1 - \left(\frac{1}{1+\delta^{-}}\right)^{\frac{1}{\delta^{-}}} + \left(\frac{1}{1+\delta^{-}}\right)^{\frac{1+\delta^{-}}{\delta^{-}}}\right)n.$$

We remark that when the minimum indegree $\delta^{-}(D)$ is small, namely, $\delta^{-}(D) \in \{1, 2\}$, then the upper bound given by Theorem 29 is better than that given by Theorem 31.

As before, let $D(n, p) = (V, A)$ denote a *random digraph* of order n where for every $u, v \in V$, the arc (u, v) is chosen with probability p. Let Q be a property of digraphs. If $\mathcal{A}$ is the set of digraphs of order n with property Q and the probability $\Pr(\mathcal{A})$ of $\mathcal{A}$ has limit 1 as $n \to \infty$, then we say *almost all digraphs have property Q* or a *random digraph has property Q almost surely*. Lee [62] established the following result for random digraphs.

Theorem 32 ([62]) *For a fixed p with $0 < p < 1$, a random digraph $D \in D(n, p)$ satisfies*

$$\gamma(D) = \lfloor k^{*} \rfloor + 1 \quad \text{or} \quad \gamma(D) = \lfloor k^{*} \rfloor + 2$$

almost surely, where $k^{*} = \log n - 2\log\log n + \log\log e$ and where $\log$ denotes the logarithm with base $1/(1-p)$.

Ghoshal, Laskar, and Pillone [43] determined lower and upper bounds on the domination number of a digraph in terms of its order and maximum outdegree.

Theorem 33 ([43]) *If D is a digraph of order n, then*

$$\frac{n}{1+\Delta^{+}(D)} \leq \gamma(D) \leq n - \Delta^{+}(D).$$

Proof Let $x \in V$ be any vertex having maximum outdegree in D, that is, $\text{od}(x) = \Delta^{+}(D)$. Let $S = V \setminus N^{+}(x)$. It follows that S is an out-dominating set. Thus, $\gamma(D) \leq |S| = n - \Delta^{+}(D)$. This establishes the upper bound. To prove the lower bound, let $S \subseteq V$ be a minimum dominating set of D, that is, $\gamma^{+}(D) = |S|$.

Every vertex in S dominates at most $\Delta^+(D)$ vertices outside S, implying that $n - |S| = |V \setminus S| \leq |S| \cdot \Delta^+(D)$, and so $\gamma^+(D) = |S| \geq 1/(1 + \Delta^+(D))$. □

Hao and Qian [52] strengthened the lower bound of Theorem 33. The *Slater number* sl(D) of a digraph D is the smallest integer t such that adding t to the sum of the first t terms of the non-increasing outdegree sequence of D is at least as large as the order of D.

Theorem 34 ([52]) *If D is a digraph of order n, then*

$$\frac{n}{1 + \Delta^+(D)} \leq \mathrm{sl}(D) \leq \gamma(D).$$

Moreover, the authors [52] showed that the difference between sl(D) and $\left\lceil \frac{n}{1+\Delta^+(D)} \right\rceil$ can be arbitrarily large.

3.2 In-Domination

We turn our attention to bounds on the in-domination number of a digraph and give the following classical 1973 results due to Berge [7].

Proposition 35 ([7]) *If D is a digraph of order n and size m, then* $\gamma^-(D) \geq n - m$.

Proof Let $S \subseteq V$ be a minimum in-dominating set, that is, $\gamma^-(D) = |S|$. Since for every vertex $w \in V \setminus S$, there exist a vertex $v \in S$ and an arc (w, v), it follows that $n - |S| = |V \setminus S| \leq m$, and so $\gamma^-(D) = |S| \geq n - m$. □

Proposition 36 ([7]) *For any digraph D of order n having maximum indegree* $\Delta^-(D)$,

$$\left\lceil \frac{n}{1 + \Delta^-(D)} \right\rceil \leq \gamma^-(D) \leq n - \Delta^-(D).$$

Proof Let $x \in V$ be any vertex having maximum indegree in D, that is, id$(x) = \Delta^-(D)$. Let $S = V \setminus N^-(x)$. It follows that S is an in-dominating set. Thus, $\gamma^-(D) \leq |S| = n - \Delta^-(D)$. This establishes the upper bound. To prove the lower bound, let $S \subseteq V$ be a minimum in-dominating set of D, that is, $\gamma^-(D) = |S|$. Every vertex in S is dominated by at most $\Delta^-(D)$ vertices outside S, implying that $n - |S| = |V \setminus S| \leq |S| \cdot \Delta^-(D)$, and so $\gamma^-(D) = |S| \geq 1/(1 + \Delta^-(D))$. □

We note that both bounds of Proposition 36 are sharp for a digraph of order n having $\Delta^-(D) = n - 1$.

3.3 Domination and In-Domination

In 1999 Chartrand, Harary, and Yue [19] proved the following upper bound on the sum of the domination number and the in-domination number of a digraph. Recall that $\vec{C}_3$ denotes the directed cycle on three vertices and an endvertex is a vertex of degree 1.

Theorem 37 ([19]) *If D is a digraph of order n with* $\delta^-(D) \geq 1$ *and* $\delta^+(D) \geq 1$*, then*

$$\gamma(D) + \gamma^-(D) \leq \frac{4}{3}n.$$

Further, equality holds if and only if $D = \vec{C}_3$*, or if every vertex of D is an endvertex or is adjacent to exactly one endvertex and adjacent from exactly one endvertex.*

In 2015 Hao and Qian [51] improved the upper bound of Theorem 37 as follows.

Theorem 38 ([51]) *Let D be a digraph of order n with* $\delta^-(D) \geq 1$ *and* $\delta^+(D) \geq 1$*. If* $2k+1$ *is the length of a shortest odd circuit of D, then*

$$\gamma(D) + \gamma^-(D) \leq \left(\frac{2k+2}{2k-1}\right) n.$$

As a consequence of Theorem 38, we have the following result.

Corollary 39 ([51]) *If D is a digraph of order n with* $\delta^-(D) \geq 1$ *and* $\delta^+(D) \geq 1$ *with no odd directed cycle, then* $\gamma(D) + \gamma^-(D) \leq n$*.*

3.4 Twin Domination

In this section, we present results on the twin domination number of a digraph. We first present the following key lemma. Recall that for $r \geq 1$ an integer, a graph G is *r-degenerate* if every induced subgraph of G has minimum degree at most r. When we say that digraph D is *minimal* with respect to some property $\mathcal{P}$, we mean arc-minimal, that is, removing any arc from D destroys property $\mathcal{P}$.

Lemma 40 *If a digraph D is minimal with respect to the property of every vertex of D having indegree and outdegree at least k, then the underlying graph is* $2k$*-degenerate.*

Proof Let D be a digraph that is minimal with respect to the property $\mathcal{P}$ that every vertex of D has indegree and outdegree at least k. Let G be the underlying (undirected) graph of D. We show that G is $2k$-degenerate. Suppose, to the contrary, that there is a set V' of vertices such that the subgraph, say G', of G induced by the set V' has minimum degree at least $2k+1$. Let D' be the subdigraph of D

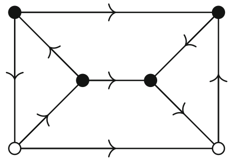


Fig. 4 A digraph D with $\gamma^{\pm}(D) = \frac{2}{3}n$

induced by the set V', and so G' is the underlying graph of the digraph D'. Each vertex $v \in V'$ has an excess of in- or out-arcs in D', noting that $d_{G'}(v) \geq 2k+1$. Suppose there is an arc a_v whose removal from D' destroys the property of v having indegree and outdegree at least k. If $\text{od}_{D'}(v) \geq k+1$, then a_v is an arc into v and in this case $\text{id}_{D'}(v) = k$. If $\text{id}_{D'}(v) \geq k+1$, then a_v is an arc out of v and in this case $\text{od}_{D'}(v) = k$. Thus, the number of arcs incident to v whose removal from D' destroys property $\mathcal{P}$ is either zero or k. Hence, there are at most $k|V'|$ arcs in D whose removal destroys property $\mathcal{P}$. But every arc removal from D' destroys property $\mathcal{P}$ for some vertex of D', implying that there are at most $k|V'|$ arcs in D'. This in turn implies that every vertex has indegree and outdegree exactly k in D', and therefore G' is a $(2k)$-regular graph, contradicting the supposition that $\delta(G') \geq 2k+1$. □

In 2003 Chartrand, Dankelmann, Schultz, and Swart [20] established the following upper bound on the twin domination number of a digraph. We present here a simple proof of this result, using the key Lemma 40. Our proof is based on the fact that a k-degenerate graph has chromatic number at most $k+1$, as shown by Szekeres and Wilf [88] in 1968. Recall that a vertex and an edge *cover* each other in a graph G if they are incident in G. A *vertex cover* in G is a set of vertices that covers all the edges of G. The *vertex cover number* $\beta(G)$ (also denoted by $\tau(G)$ or $\text{vc}(G)$ in the literature) is the minimum cardinality of a vertex cover in G.

Theorem 41 ([20]) *If D is a digraph of order n with $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$, then $\gamma^{\pm}(D) \leq \frac{2}{3}n$.*

Proof We may assume the digraph D is minimal with respect to this property of $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$, since adding arcs cannot increase the twin domination number. With this assumption, the underlying graph G of D is 2-degenerate by Lemma 40 and hence 3-colorable. Thus, the independence number of G is at least $n/3$, which means that the vertex cover number of G is at most $2n/3$. But a vertex cover of G is a twin dominating set in D since $\delta^-(D) \geq 1$ and $\delta^+(D) \geq 1$. Thus, $\gamma^{\pm}(D) \leq \frac{2}{3}n$. □

The simplest example of a digraph achieving equality in the upper bound of Theorem 41 is $\vec{C}_3$. As a further small example, the digraph D shown in Figure 4 has order $n=6$ and satisfies $\gamma^{\pm}(D) = 4 = \frac{2}{3}n$, where the darkened vertices form a twin dominating set of D of cardinality 4.

In 2013 Arumugam, Ebadi, and Sathikala [4] gave the following upper bound on the twin domination number.

Theorem 42 ([4]) *If D is a digraph of order n and $\ell(D)$ is the length of a longest directed path in D, then* $\gamma^{\pm}(D) \le n - \left\lfloor \frac{\ell(D)}{2} \right\rfloor$.

The bound of Theorem 42 is attained, for example, by directed paths and also by any digraph D obtained from a directed path P_k: $u_1, u_2, \ldots, u_k$ by adding a new vertex u_i' and arc (u_i', u_i) for each u_i for $i \in [k]$.

3.5 Reverse Domination

The digraph obtained from a digraph D by reversing all the arcs of D is called the *reverse digraph* (also called the *converse* in the literature) of D, denoted D^-. We note that $\gamma(D) = \gamma^-(D^-)$ for every digraph D. Thus by Theorem 37, if D is a digraph of order n with $\delta^-(D) \ge 1$ and $\delta^+(D) \ge 1$, then $\gamma(D) + \gamma(D^-) \le \frac{4}{3}n$.

For $r \ge 1$, let $\mathcal{D}_r$ be the class of r-regular strongly connected digraphs. We note that the only 1-regular strongly connected digraphs are the directed cycles, and so $\mathcal{D}_1 = \{\vec{C}_n \mid n \ge 3\}$. Since a directed cycle is isomorphic to its reverse, if $D \in \mathcal{D}_1$, then $\gamma(D^-) - \gamma(D) = 0$. For $r \ge 2$, the difference $\gamma(D^-) - \gamma(D)$ can be arbitrarily large in the class $\mathcal{D}_r$, as shown by Gyürki [45] in the case when $r = 2$ and by Niepel and Knor [73] for all $r \ge 3$. However, for a fixed $r \ge 2$, it remains an open problem to determine the greatest ratio $\gamma(D^-)/\gamma(D)$ of an r-regular strongly connected digraph. The best known results to date are the following.

Theorem 43 ([45]) *For digraph* $D \in \mathcal{D}_2$, $\sup_{D \in \mathcal{D}_2} \dfrac{\gamma(D^-)}{\gamma(D)} \ge \dfrac{4}{3}$.

Theorem 44 ([45, 73]) *For* $r \ge 3$, *we have* $\sup_{D \in \mathcal{D}_r} \dfrac{\gamma(D^-)}{\gamma(D)} \ge \dfrac{7}{6}$.

4 Domination in Digraph Products

Vizing's conjecture [90] asserts that the domination number of the Cartesian product of two graphs is at least as large as the Cartesian product of their domination numbers. This conjecture was first stated in 1963 as a problem in [89] and later in 1968 formally posed as a conjecture in [90]. It is considered by many to be the main open problem in the area of domination in graphs. It is natural then that the study of domination in digraphs considers results for Cartesian products of digraphs.

The *Cartesian product* of two digraphs $G = (V(G), A(G))$ and $H = (V(H), A(H))$, denoted by $G \Box H$, is the digraph with vertex set $V(G) \times V(H)$, and there exists an arc $((u_1, v_1), (u_2, v_2)) \in A(G \Box H)$ if and only if either $(u_1, u_2) \in A(G)$ and $v_1 = v_2$

or $(v_1, v_2) \in A(H)$ and $u_1 = u_2$. Much of the work on Cartesian products in digraphs considers directed cycles.

In 2009 Shaheen [84] and in 2010 Liu, Zhang, Chen, and Meng [64, 93] independently determined the domination number of $\vec{C}_m \,\square\, \vec{C}_n$ for $m \le 6$ and arbitrary $n \ge 2$.

Theorem 45 ([64, 84, 93]) *For $n \ge 2$, the following hold.*

(a) $\gamma(\vec{C}_2 \,\square\, \vec{C}_n) = n$.
(b) $\gamma(\vec{C}_3 \,\square\, \vec{C}_n) = n$ *if* $n \equiv 0 \pmod 3$*; otherwise,* $\gamma(\vec{C}_3 \,\square\, \vec{C}_n) = n + 1$.
(c) $\gamma(\vec{C}_4 \,\square\, \vec{C}_n) = \frac{3}{2}n$ *if* $n \equiv 0 \pmod 8$*; otherwise,* $\gamma(\vec{C}_4 \,\square\, \vec{C}_n) = n + \left\lceil \frac{n+1}{2} \right\rceil$.
(d) $\gamma(\vec{C}_5 \,\square\, \vec{C}_n) = 2n$.
(e) $\gamma(\vec{C}_6 \,\square\, \vec{C}_n) = 2n + 2$.

Zhang et al. [93] also determined $\gamma(\vec{C}_m \,\square\, \vec{C}_n)$ when both m and n are divisible by 3.

Theorem 46 ([93]) *If* $m \equiv 0 \pmod 3$ *and* $n \equiv 0 \pmod 3$*, then* $\gamma(\vec{C}_m \,\square\, \vec{C}_n) = \frac{1}{3}mn$.

In 2013, Mollard [71] determined the exact values of $\gamma(C_m \square C_n)$ for m congruent to 2 modulo 3, with the exception of one subcase.

Theorem 47 ([71]) *If* $m, n \ge 2$, $m \equiv 2 \pmod 3$, $k = \lfloor m/3 \rfloor$*, and* $\ell = \lfloor n/3 \rfloor$*, then*

$$\gamma(\vec{C}_m \,\square\, \vec{C}_n) = \begin{cases} n(k+1) & \text{if } n = 3\ell \\ n(k+1) & \text{if } n = 3\ell + 1 \text{ and } 2\ell \ge k \\ n(k+1) & \text{if } n = 3\ell + 2 \text{ and } n \ge m \\ m(\ell+1) & \text{if } n = 3\ell + 2 \text{ and } n \le m. \end{cases}$$

Furthermore, $\gamma(\vec{C}_m \,\square\, \vec{C}_n)$ *if* $n = 3\ell + 1$ *and* $2\ell < k$.

Zhang et al. [93] conjectured that if $k \ge 2$ where $k = \lfloor \frac{m}{3} \rfloor$, then $\gamma(\vec{C}_m \,\square\, \vec{C}_n) = k(n+1)$ for $n \not\equiv 0 \pmod 3$, but Mollard [71] disproved this conjecture by showing that it doesn't always hold when $n \equiv 1 \pmod 3$. For example, they noted that $\gamma(\vec{C}_{3k} \,\square\, \vec{C}_4) = \gamma(\vec{C}_4 \,\square\, \vec{C}_{3k}) = 3k + \left\lceil \frac{3k+1}{2} \right\rceil$ when $k \not\equiv 0 \pmod 8$, while the conjecture claims that $\gamma(\vec{C}_4 \,\square\, \vec{C}_{3k}) = 5k$. These values are different for $k \ge 3$.

Mollard [71] also established the following bounds.

Theorem 48 ([71]) *If* $m, n \ge 2$ *and* $k = \lfloor \frac{m}{3} \rfloor$*, then*

$$\gamma(\vec{C}_m \,\square\, \vec{C}_n) \ge \begin{cases} nk & \text{if } m \equiv 0 \pmod 3 \\ nk + \frac{n}{2} & \text{if } m \equiv 1 \pmod 3 \\ nk + n & \text{if } m \equiv 2 \pmod 3). \end{cases}$$

In 2013 Shao, Zhu, and Lang [86] determined upper and lower bounds on $\gamma(\vec{C}_m \Box \vec{C}_n)$ for the case when m is congruent to 1 modulo 3.

Theorem 49 ([86]) *If* $k \geq 1$ *and* $n \geq 3$ *are integers, then*

$$\left\lceil \frac{(2k+1)n}{2} \right\rceil \leq \gamma(\vec{C}_{3k+1} \Box \vec{C}_n) \leq \left\lceil \frac{(2k+1)n}{2} \right\rceil + k.$$

Based on the bounds of Theorem 49, Shao et al. [86] determined the exact values of $\gamma(\vec{C}_m \Box \vec{C}_n)$ for $m \in \{7, 10\}$.

We conclude this section by noting that Liu, Zhang, and Meng [65] investigated domination numbers of Cartesian products of directed paths in 2011, and Ma and Liu [67] studied the twin domination number of the Cartesian products of directed cycles in 2016. For domination and twin domination in other types of digraph products, see [15, 66, 68, 69, 74].

5 Domination in Oriented Graphs

Recall that an *oriented graph* D is a digraph that can be obtained from a graph G by assigning a direction to (i.e., orienting) each edge of G. The resulting digraph D is called an *orientation* of G. Thus, if D is an oriented graph, then for every pair u and v of distinct vertices of D, at most one of (u, v) and (v, u) is an arc of D. For example, a *tournament* is an oriented complete graph. Recall also that the independence number of a directed graph D is denoted by $\alpha(D)$. As before, unless otherwise stated, we refer to an out-dominating set in a digraph simply as a dominating set.

5.1 Oriented Graphs

In 1996 Chartrand, Vanderjagt, and Yue [18] studied domination in oriented graphs. They defined the *lower orientable domination number* of a graph G, which they denoted as dom(G) (denoted by $\gamma_d(G)$ in [17]), to equal the minimum domination number over all orientations of G. Further, they defined the *upper orientable domination number*, or simply the *orientable domination number*, of a graph G, which they denoted as DOM(G) (denoted by $\Gamma_d(G)$ in [17]), as the maximum domination number over all orientations of G. Thus,

$$\begin{aligned} \text{dom}(G) &= \min\{\gamma(D) \mid \text{ over all orientations } D \text{ of } G\} \\ \text{DOM}(G) &= \max\{\gamma(D) \mid \text{ over all orientations } D \text{ of } G\}. \end{aligned}$$

The orientable domination number of a complete graph was first studied by Erdős in 1963 [28], albeit in disguised form. In 1962, Schütte [28] raised the question of given any positive integer $k > 0$, does there exist a tournament $T_{n(k)}$ on $n(k)$ vertices in which for any set S of k vertices, there is a vertex u that dominates all vertices in S. Erdős [28] showed, by probabilistic arguments, that such a tournament $T_{n(k)}$ does exist, for every positive integer k. The proof of the following bounds on the orientable domination number of a complete graph is along identical lines to that presented by Erdős [28]. This result can also be found in [78]. Here, log is to the base 2.

Theorem 50 ([28]) *For* $n \geq 2$, $\log n - 2\log(\log n) \leq \mathrm{DOM}(K_n) \leq \log(n+1)$.

This notion of orientable domination in a complete graph was subsequently extended to orientable domination of all graphs by Chartrand et al. [18]. They proved the following result.

Theorem 51 ([18]) *For every graph G,* $\mathrm{dom}(G) = \gamma(G)$.

In view of Theorem 51, it is not interesting to ask about the lower orientable domination number, $\mathrm{dom}(G)$, of a graph G since this is precisely its domination number, which is very well studied. We therefore focus our attention on the (upper) orientable domination number of a graph. Chartrand et al. [18] determined $DOM(G)$ for special classes of graphs, including paths, cycles, complete bipartite graphs, and regular complete tripartite graphs. They also proved the following result.

Theorem 52 ([18]) *For every graph G and for every integer c with* $\mathrm{dom}(G) \leq c \leq DOM(G)$*, there exists an orientation D of G such that* $\gamma(D) = c$.

In 2010 Blidia and Ould-Rabah [8] continued the study of domination in oriented graphs. For an oriented graph D, let $\alpha'(D)$ denote the matching number of D and let $s(D)$ denote the number of support vertices in the underlying graph of D. The authors in [8] proved the following result. In fact, they proved a slightly stronger result involving the irredundance number of an oriented graph (which we do not define here).

Theorem 53 ([8]) *If D is an oriented graph of order n, then* $s(D) \leq \gamma(D) \leq n - \alpha'(D)$.

Blidia and Ould-Rabah [8] characterized the oriented trees T satisfying $\gamma(T) - \alpha'(T)$ and the oriented graphs D satisfying $\gamma(D) = s(D)$ and $s(D) = n - \alpha'(D)$.

In 2011 Caro and Henning [16] also studied domination in oriented graphs. In this paper, they proved a Greedy Partition Lemma, which they used to present an upper bound on the orientable domination number of a graph in terms of its independence number. To state their result, let $\alpha \geq 1$ be an integer and let $\mathcal{G}_\alpha$ be the class of all graphs G with $\alpha \geq \alpha(G)$.

Theorem 54 ([16]) *For* $\alpha \geq 1$ *an integer, if* $G \in \mathcal{G}_\alpha$ *has order* $n \geq \alpha$*, then*

$$\mathrm{DOM}(G) \leq \alpha\left(1 + 2\ln\left(\frac{n}{\alpha}\right)\right).$$

The next result follows as a consequence of Theorem 54, where $\chi(G)$ denotes the chromatic number of G and $d_{\text{av}}(G)$ denotes the average degree in G.

Corollary 55 ([16]) *If G is a graph of order n, then the following hold.*

(a) $\text{DOM}(G) \le \alpha(G)\,(1 + 2\ln(\chi(G)))$.
(b) $\text{DOM}(G) \le \alpha(G)\,(1 + 2\ln(d_{\text{av}}(G) + 1))$.

For any integer $d \ge 1$, let $\mathcal{F}_d$ be the class of all graphs G whose complement is a d-degenerate graph. The property of being d-degenerate is a hereditary property that is closed under induced subgraphs, as is the property of the complement of a graph being d-degenerate. Applying their Greedy Partition Lemma for domination in oriented graphs, the authors in [16] proved the following result.

Theorem 56 ([16]) *For any integer $d \ge 1$, if $G \in \mathcal{F}_d$ has order n, then*

$$\text{DOM}(G) \le 2d + 1 + 2\ln\left(\frac{n - 2d + 1}{2}\right).$$

The following upper bound on the orientable domination number of a $K_{1,m}$-free graph is established in [16], where a graph is F-free if it does not contain F as an induced subgraph.

Theorem 57 ([16]) *For $m \ge 3$, if G is a $K_{1,m}$-free graph of order n with $\delta(G) = \delta$, then*

$$\text{DOM}(G) < 2(m-1)n\ln\left(\frac{\delta + m - 1}{\delta + m - 1}\right).$$

Let $\mathcal{G}_n$ denote the family of all graphs of order n. We define

$$\begin{aligned} \text{NG}_{\min}(n) &= \min\{\text{DOM}(G) + \text{DOM}(\overline{G})\} \\ \text{NG}_{\max}(n) &= \max\{\text{DOM}(G) + \text{DOM}(\overline{G})\} \end{aligned}$$

where the minimum and maximum are taken over all graphs $G \in \mathcal{G}_n$. The following Nordhaus-Gaddum-type bounds for the orientable domination of a graph were established in [16].

Theorem 58 ([16]) *The following hold.*

(a) $c_1 \log n \le \text{NG}_{\min}(n) \le c_2(\log n)^2$ *for some constants c_1 and c_2.*
(b) $n + \log n - 2\log(\log n) \le \text{NG}_{\max}(n) \le n + \lceil \frac{n}{2} \rceil$.

Caro and Henning continued their study of the orientable domination number in [17]. They defined the *maximum average degree* in a graph G, denoted by $\text{mad}(G)$, as the maximum of the average degrees taken over all subgraphs H of G, that is,

$$\text{mad}(G) = \max_{H \subseteq G}\left\{\frac{2|E(H)|}{|V(H)|}\right\}.$$

Theorem 59 ([17]) *If G is a graph of order n, then the following hold.*

(a) $\text{DOM}(G) \geq \alpha(G) \geq \gamma(G)$.
(b) $\text{DOM}(G) \geq n/\chi(G)$.
(c) $\text{DOM}(G) \geq \lceil(\text{diam}(G)+1)/2)\rceil$.
(d) $\text{DOM}(G) \geq n/(\lceil\text{mad}(G)/2\rceil+1)$.

Proof We present here only a proof of part (a). Let I be a maximum independent set in G, and let D be the digraph obtained from G by orienting all arcs from I to $V \setminus I$ and orienting all arcs in $G[V \setminus I]$, if any, arbitrarily. Every dominating set of D contains the set I, and so $\gamma(D) \geq |I|$. However, the set I itself is a dominating set of D, and so $\gamma(D) \leq |I|$. Consequently, $DOM(G) \geq \gamma(D) = |I| = \alpha(G) \geq \gamma(G)$. □

As remarked in [17], since $\text{mad}(G) \leq \Delta(G)$ for every graph G, as an immediate consequence of Theorem 59(d), we have that $\text{DOM}(G) \geq n/(\lceil\Delta(G)/2\rceil+1)$. The following lemma is useful when establishing upper bounds on the orientable domination number of a graph.

Lemma 60 ([17]) *Let $G = (V, E)$ be a graph and let $V_1, V_2, \ldots, V_k$ be subsets of V, not necessarily disjoint, such that $\cup_{i=1}^{k} V_i = V$. If $G_i = G[V_i]$ for $i \in [k]$, then*

$$\text{DOM}(G) \leq \sum_{i=1}^{k} \text{DOM}(G_i).$$

Proof Consider an arbitrary orientation D of G. Let D_i be the orientation of the edges of G_i induced by D, and let S_i be a γ-set of D_i for each $i \in [k]$. By Theorem 59(a), $\text{DOM}(G_i) \geq \gamma(D_i) = |S_i|$ for each $i \in [k]$. Since the set $S = \cup_{i=1}^{k} S_i$ is a dominating set of D, we have that

$$\gamma(D) \leq |S| \leq \sum_{i=1}^{k} |S_i| \leq \sum_{i=1}^{k} \text{DOM}(G_i).$$

Since this is true for every orientation D of G, the desired upper bound of $\text{DOM}(G)$ follows. □

As a consequence of Lemma 60, the authors in [17] proved the following upper bounds on the orientable domination number of a graph.

Theorem 61 ([17]) *If G is a graph of order n, then the following hold.*

(a) $\text{DOM}(G) \leq n - \alpha'(G)$.
(b) *If G has a perfect matching, then* $\text{DOM}(G) \leq n/2$.
(c) $\text{DOM}(G) \leq n$ *with equality if and only if* $G = \overline{K}_n$.
(d) *If G has minimum degree δ and $n \geq 2\delta$, then* $\text{DOM}(G) \leq n - \delta$.
(e) $\text{DOM}(G) = n - 1$ *if and only if every component of G is a K_1-component, except for one component which is either a star or a complete graph K_3.*

Proof We present here only a proof of part (a). Let $M = \{u_1v_1, u_2v_2, \ldots, u_tv_t\}$ be a maximum matching in G, and so $t = \alpha'(G)$. Let $V_i = \{u_i, v_i\}$ for $i \in [t]$. If $n > 2t$, let $(V_{t+1}, \ldots, V_{n-2t})$ be a partition of the remaining vertices of G into $n - 2t$ subsets each consisting of a single vertex. By Lemma 60,

$$\mathrm{DOM}(G) \le \sum_{i=1}^{n-t} \mathrm{DOM}(G_i) = t + (n - 2t) = n - t = n - \alpha'(G). \qquad \square$$

Applying results on the size of a maximum matching in a regular graph established in [57], we have the following consequence of Theorem 61(a).

Theorem 62 ([17]) *For $r \ge 2$, if G is a connected r-regular graph of order n, then*

$$\mathrm{DOM}(G) \le \begin{cases} \max\left\{\left(\dfrac{r^2+2r}{r^2+r+2}\right) \times \dfrac{n}{2}, \dfrac{n+1}{2}\right\} & \text{if } r \text{ is even} \\[2ex] \dfrac{(r^3+r^2-6r+2)\,n+2r-2}{2(r^3-3r)} & \text{if } r \text{ is odd.} \end{cases}$$

The orientable domination number of a bipartite graph is precisely its independence number. Recall that König [60] and Egerváry [27] showed that if G is a bipartite graph, then $\alpha'(G) = \beta(G)$. Hence by Gallai's Theorem [42], if G is a bipartite graph of order n, then $\alpha(G) + \alpha'(G) = n$.

Theorem 63 ([17]) *If G is a bipartite graph, then* $\mathrm{DOM}(G) = \alpha(G)$.

Proof Since G is a bipartite graph, we have that $n - \alpha'(G) = \alpha(G)$. Thus, by Theorem 59(a) and Theorem 61(a), we have that $\alpha(G) \le \mathrm{DOM}(G) \le n - \alpha'(G) = \alpha(G)$. Consequently, we must have equality throughout this inequality chain. In particular, $\mathrm{DOM}(G) = \alpha(G)$. $\square$

In 2018 Harutyunyan, Le, Newman, and Thomassé [53] observed that in general there is no upper bound on the orientable domination number of a graph solely in terms of its independence number. Nevertheless, they showed that these two quantities can be related.

Theorem 64 ([53]) *If G is a graph of order n, then* $\mathrm{DOM}(G) \le \alpha(G) \cdot \log n$.

Theorem 64 implies that when the independence number of an oriented graph is sufficiently large, it is possible to bound the orientable domination number of the graph purely in terms of its independence number.

Theorem 65 ([53]) *If D is a graph of order n and $\alpha(G) \ge \log n$, then* $\mathrm{DOM}(D) \le (\alpha(D))^2$.

Harutyunyan et al. [53] concluded their paper with the following conjecture.

Conjecture 1 *There exists an integer k such that for any $\vec{C}_3$-free oriented graph D with $\alpha(D) = \alpha$, we have $\gamma(D) \le \alpha^k$.*

The following result establishes an upper bound on the orientable domination number of a graph in terms of its independence number and chromatic number.

Theorem 66 ([17]) *If G is a graph of order n, then the following hold.*

(a) $\mathrm{DOM}(G) \le \alpha(G) \cdot \lceil \chi(G)/2 \rceil$.
(b) $\mathrm{DOM}(G) \le n - \lfloor \chi(G)/2 \rfloor$.
(c) $\mathrm{DOM}(G) \le (n + \alpha(G))/2$.

The following result establishes an upper bound on the orientable domination of a graph in terms of the chromatic number of its complement.

Theorem 67 ([17]) *If G is a graph of order n, then*

$$\mathrm{DOM}(G) \le \chi(\overline{G}) \cdot \log\left(\left\lceil \frac{n}{\chi(\overline{G})} \right\rceil + 1\right).$$

As a consequence of Theorem 67, we have the following result on the orientable domination number of a graph with sufficiently large minimum degree.

Theorem 68 ([17]) *If G is a graph of order n with minimum degree $\delta(G) \ge (k-1)n/k$ where k divides n, then* $\mathrm{DOM}(G) \le \frac{n}{k}\log(k+1)$.

Let $\mathrm{Mop}(n) = \max\{\mathrm{DOM}(G)\}$, where the maximum is taken over all maximal outerplanar graphs of order n.

Theorem 69 ([17]) *For maximal outerplanar graphs of order n,* $\mathrm{Mop}(n) = \lceil \frac{n}{2} \rceil$.

5.2 Tournaments

Since a tournament is an oriented complete graph, many applications interpret a tournament as a competition graph. That is, a tournament on n vertices represents a competition between n teams (each represented by a vertex) in which the teams play each other once. No ties are allowed, and there is an arc from a vertex u to a vertex v if and only if u defeats v. The score of a vertex v is its outdegree (the number of teams it defeats). Hence, a dominating set S of a tournament represents a collection of teams such that every team not in S is defeated by at least one team in S. Tournaments are popular, in part, because of this pairwise comparison and ranking of competitors.

The following result is attributed by Moon to Erdős (cf. Moon [72] p. 28). As before, unless otherwise stated, log is to the base 2.

Theorem 70 (Erdős) *If T is a tournament with $n \ge 2$ vertices, then $\gamma(T) \le \lceil \log n \rceil$.*

Proof The sum of the outdegrees of the vertices in a tournament $T=(V,A)$ of order n is the number of arcs in T, that is,

$$\sum_{u\in V} \mathrm{od}_T(u) = \frac{1}{2}n(n-1).$$

Thus, there must be a vertex $x\in V$ with $\mathrm{od}_T(x) \geq \lceil \frac{1}{2}(n-1)\rceil$. We remove this vertex x and all out-neighbors of x, thereby removing at least half the vertices. We now repeat this process on the remaining tournament, which has at most $\lceil \frac{1}{2}(n-1)\rceil$ vertices, by again selecting a vertex which dominates at least half of the remaining vertices and then deleting this second vertex and all of its out-neighbors. Repeating this process will produce a dominating set with no more than $\lceil \log n\rceil$ vertices. □

A *random tournament* is obtained by orienting the edges of a complete graph randomly, independently, with equal probabilities. Let T_n be the probability space consisting of the random tournaments on n vertices. In 1997 Bollobás and Szabó [9] showed that the domination number of a random tournament is one of two values, where log is to the base 2. We remark that this result was obtained by Lee [62] in 1994.

Theorem 71 ([9, 62]) *A random tournament* $T\in T_n$ *has domination number* $\lfloor k\rfloor+1$ *or* $\lfloor k\rfloor+2$, *where* $k=\log(n)-2\log(\log(n))+\log(\log(e))$.

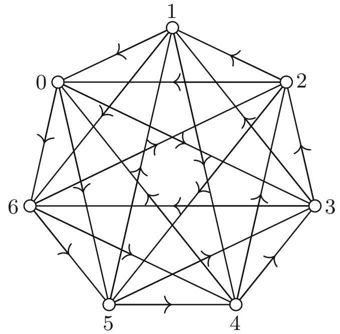
By Theorem 71, there are tournaments having arbitrarily large domination numbers. This leads to the question: Which tournaments have bounded domination number (not dependent on the order n of the tournament)? To partially answer this question, we first define a k-majority tournament.

Definition 12 As usual, by a linear order in a tournament, we mean with respect to the transitive orientation of the tournament. A tournament T is a *k-majority tournament* if there are $2k-1$ linear orders of $V(T)$ such that for all distinct vertices u and v in T, if u is adjacent to v, then u is before v in at least k of the $2k-1$ orders. Let $F(k)$ be the supremum of the size of a minimum dominating set in a k-majority tournament, where the supremum is taken over all k-majority tournaments, with no restriction on their size.

Trivially, $F(1)=1$. In 2006 Alon, Brightwell, Kierstead, Kostochka, and Winkler [2] proved that $F(2)=3$. To do this, they first showed that every 2-majority tournament has a dominating set of size at most 3, that is, $F(2)\leq 3$. We omit their proof.

To show that $F(2)\geq 3$, Alon et al. [2] provided the following example. Recall that if there is an integer x with $0<x<p$ such that $x^2\equiv q \pmod p$, then q is a *quadratic residue modulo p*. In practice, it suffices to restrict the range of x to $0<x\leq\lfloor p/2\rfloor$ because of the symmetry $(p-x)^2\equiv x^2 \pmod p$. For example, the quadratic residues modulo 7 are given by 1, 2, 4 since $1^1\equiv 1 \pmod 7$, $2^2\equiv 4 \pmod 7$, and $3^2\equiv 2$ (mod 7). Let T be the quadratic residue tournament whose vertices are the elements of the finite field $GF(7)$ in which $i\to j$ if and only if $i-j$ is a quadratic residue modulo 7,

Fig. 5 A 2-majority tournament T' with $\gamma(T') = 3$



i.e., $(i-j) \bmod 7 \in \{1, 2, 4\}$. Since the edges of T are preserved under translation, it suffices for us to consider the subtournament T' of T with vertex set $\{0, 1, \ldots, 6\}$ as illustrated in Figure 5.

No two vertices dominate T', while the set $\{0, 1, 2\}$, for example, is a dominating set of T', and so $\gamma(T') = 3$. Further, T' is a 2-majority tournament realized by the orders P_1, P_2, and P_3, where

$$P_1 : 0 < 1 < 2 < 3 < 4 < 5 < 6,$$
$$P_2 : 4 < 6 < 1 < 3 < 5 < 0 < 2,$$
$$P_3 : 5 < 2 < 6 < 3 < 0 < 4 < 1.$$

Thus, T' is a 2-majority tournament satisfying $\gamma(T') = 3$. As observed earlier, the edges of T are preserved under translation, implying that T is a 2-majority tournament satisfying $\gamma(T) = 3$. This example shows that $F(2) \geq 3$. As observed earlier, $F(2) \leq 3$. Consequently, $F(2) = 3$. We state this result formally as follows.

Theorem 72 ([2]) *For 2-majority tournaments, $F(2) = 3$.*

The value of $F(k)$ has yet to be determined for any value of $k \geq 3$. The following nontrivial result shows that $F(3) \geq 4$.

Theorem 73 ([2]) *There exists a 3-majority tournament T with $\gamma(T) = 4$, that is, $F(3) \geq 4$.*

As observed earlier, there are tournaments having arbitrarily large domination numbers. Kierstead and Trotter (see [2] for a discussion) conjectured that this is not the case for k-majority tournaments for some fixed k. Alon et al. [2] proved this conjecture and showed that $F(k)$ is finite for each fixed k.

Theorem 74 ([2]) *For an arbitrary fixed integer $k \geq 1$, if T is a k-majority tournament, then*

$$\gamma(T) \leq 20(2 + o(1))k \log(k(2\log 2)) \leq (80 + o(1))k \log(k).$$

We remark that their paper was the first to introduce the idea of using the VC dimension to study domination in tournaments, where the VC dimension (Vapnik-Chervonenkis dimension) of a hypergraph H is the largest cardinality of a vertex subset X *shattered by* H, that is, for any $Y \subseteq X$, the hypergraph H has an edge A such that $A \cap X = Y$. The upper bound in the following theorem follows as a consequence of Theorem 74.

Theorem 75 ([2]) *For an arbitrary fixed integer* $k \geq 1$,

$$\left(\frac{1}{5} + \mathrm{o}(1)\right)\frac{k}{\log k} \leq F(k) \leq (80 + \mathrm{o}(1))k\log(k).$$

A tournament is *k-transitive* if its edge set can be partitioned into k sets each of which is transitively oriented. András Gyárfás made the conjecture that k-transitive tournaments have bounded domination number, and this was explored in 2014 by Pálvölgyi and Gyárfás [76].

Conjecture 2 (Gyárfás) *For each positive integer k, there exists a (least)* $p(k)$ *such that every k-transitive tournament has a dominating set of at most* $p(k)$ *vertices.*

We proceed further with the following definitions.

Definition 13 A class $\mathcal{C}$ of tournaments has *bounded domination* if there exists a constant c such that every tournament in $\mathcal{C}$ has domination number at most c. If S and T are tournaments, then T is called *S-free* if no subtournament of T is isomorphic to S. A tournament S is a *rebel* if the class of all S-free tournaments has bounded domination.

In 2018 Chudnovsky, Ringi, Chun-Hung, Seymour, and Thomassé [23] investigated the following conjecture posed by HeHui Wu.

Conjecture 3 (HeHui Wu) *Every tournament is a rebel.*

Chudnovsky et al. [23] disproved Conjecture 3. For this purpose, they defined the notion of a poset tournament.

Definition 14 A tournament T is a *poset tournament* if its vertex set can be ordered $\{v_1, \ldots, v_n\}$ such that for all $1 \leq i < j < k \leq n$, if v_j is adjacent from v_i and adjacent to v_k, then v_i is adjacent to v_k; that is, the "forward" edges under this linear order form the comparability graph of a partial order.

Chudnovsky et al. [23] observed that not every tournament is a poset tournament. Thereafter, they proved the following result, hence disproving Conjecture 3.

Theorem 76 ([23]) *Every rebel is a poset tournament.*

However, it remains an open problem to determine if every poset tournament is a rebel. Since Wu's Conjecture, that every tournament is a rebel, is false, it naturally raises the question: Which tournaments are rebels? Theorem 76 provides a partial

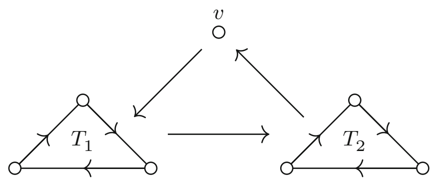


Fig. 6 The non-2-colorable tournament T^*

answer to this question. To further answer this question, we need the definition of a coloring of a tournament.

Definition 15 A k-coloring of a tournament T is a partition of $V(T)$ into k transitive sets, or, equivalently, into k acyclic sets. A tournament T with a k-coloring is called k-colorable.

Chudnovsky et al. [23] proved that Conjecture 3 is true for 2-colorable tournaments. Their proof followed from a direct application of VC dimension.

Theorem 77 ([23]) *All* 2*-colorable tournaments are rebels.*

A breakthrough in their paper [23] is that Chudnovsky et al. overcame the unboundedness of the VC dimension by showing that large shattered sets in a hypergraph are sparse, which turns out to be enough to carry over the proof of Theorem 76. This enabled them to give a non-2-colorable tournament T^* on seven vertices that satisfies Conjecture 3. Such a tournament T^* is constructed from a cyclic triangle by substituting a copy of a cyclic triangle for two of the three vertices of an original cyclic triangle. A sketch of the tournament T^* is given in Figure 6, where the arrow from v to the cyclic triangle T_1 indicates that all three arcs from v to T_1 are arcs out of v while the arrow from the cyclic triangle T_2 to v indicates that all three arcs from T_2 to v are arcs into v. Further, the arc from T_1 to T_2 indicates that every vertex in T_1 is adjacent to every vertex in T_2.

Theorem 78 ([23]) *The non-*2*-colorable tournament* T^* *is a rebel.*

Thus, Theorem 78 gives a counterexample to the converse of Theorem 77, that all rebels are 2-colorable. As a consequence of Theorem 78, the following result is proven, where the *odd girth* of a tournament T is the smallest k for which there exists a subtournament of T with k vertices that is not 2-colorable (and is undefined if T is 2-colorable).

Theorem 79 ([23]) *For* $k \geq 8$*, the class of tournaments with odd girth at least k has bounded domination.*

We close this section on domination in tournaments, with a brief discussion on what we define next as a domination graph of a digraph.

Definition 16 Two vertices x and y dominate an oriented graph $D = (V, A)$ if the set $\{x, y\}$ is a dominating set of D, that is, every vertex z different from x and y is adjacent from at least one of x and y, and so $(x, z) \in A$ or $(y, z) \in A$. The *domination graph* of an oriented graph D is the graph G with $V(G) = V(D)$ and with an edge between two vertices x and y if x and y dominate T, that is, if every other vertex loses to at least one of x and y.

Domination graphs were introduced and studied by Fisher et al. [30–35] and [21, 22], who largely considered the domination graphs of tournaments. In particular, Fisher et al. showed that the domination graph of a tournament is either an odd cycle with or without isolated and/or pendant vertices or a forest of caterpillars. They also showed that any graph consisting of an odd cycle with or without isolated and/or pendant vertices is the domination graph of some tournament.

6 Total Domination in Digraphs

There are several possibilities for defining the counterpart of a total dominating set in a digraph D. We consider four such versions in the following subsections.

6.1 Total Domination: Version 1

In this version of total domination, we define a set S in a digraph D to be a *total in-dominating set* if S is an in-dominating set in D with the added property that the subdigraph induced by S has no isolated vertices. Here we define the *total in-domination number* $\gamma_{ti}^-(D)$ of a digraph D to equal the minimum cardinality of such a set S according to Version 1. We note that if the underlying graph of D has no isolated vertices, then $V(D)$ is vacuously a total in-domination set of D, and so $\gamma_{ti}^-(D)$ is well-defined and $\gamma_{ti}^-(D) \leq |V(D)|$.

6.2 Total Domination: Version 2

In this version of total domination, a set S in a digraph D is a *total dominating set* if S is a dominating set in D with the added property that the subdigraph induced by S has no isolated vertices. This is a version defined by Arumugam, Jacob, and Volkmann [3] in 2007 and Hao [49] in 2017. We define the *total domination number* $\gamma_t(D)$ of a digraph D with no isolated vertices to equal the minimum cardinality of such a set S according to Version 2. As with version 1 above, we note that $\gamma_t(D)$ is well-defined and $\gamma_t(D) \leq |V(D)|$. Arumugam et al. [3] established the following lower bound on the total domination number of a digraph.

Theorem 80 ([3]) *If D is a digraph of order n, with maximum outdegree Δ^+ and without isolated vertices, then*

$$\gamma_t(D) \geq \left\lceil \frac{2n}{2\Delta^+ + 1} \right\rceil.$$

Hao and Chen [50] improved the lower bound in Theorem 80. For this purpose, they define the *out-Slater number* of a digraph D of order n as

$$\mathrm{sl}^+(D) = \min\{k : \lfloor k/2 \rfloor + (d_1^+ + d_2^+ + \cdots + d_k^+) \geq n\},$$

where $d_1^+, d_2^+, \ldots, d_k^+$ are the first k largest outdegrees of D.

Theorem 81 ([50]) *If D is a digraph of order n, with maximum outdegree Δ^+ and without isolated vertices, then*

$$\gamma_t(D) \geq \mathrm{sl}^+(D) \geq \left\lceil \frac{2n}{2\Delta^+ + 1} \right\rceil.$$

Further, the gap between the rightmost two numbers can be arbitrarily large.

The authors in [50] also determined the following lower bound on the total domination number of an oriented tree in terms of its order and number of vertices of outdegree 0.

Theorem 82 ([50]) *If T is an oriented tree of order $n \geq 2$, with n_0 vertices of outdegree 0 and with non-increasing outdegree sequence $d_1^+, d_2^+, \ldots, d_n^+$, then*

$$\gamma_t(T) \geq \mathrm{sl}^+(D) \geq \frac{2}{3}(n - n_0 + 1),$$

with equality if and only if $n - n_0 \equiv 2 \pmod 3$ and $d_{k+1}^+ \leq 1$, where $k = \frac{2}{3}(n - n_0 + 1)$.

6.3 Total Domination: Version 3

In this version of total domination, a set S in a digraph $D = (V, E)$ is a *total in-dominating set* if every vertex in V is adjacent *to* a vertex in S, that is, $N^-(S) = V$. This is equivalent to saying that S is an in-dominating set and the subdigraph induced by S has no isolated vertices and no sources. The minimum cardinality of such a set could be called the *total absorption number*, denoted $\gamma_t^-(D)$. We note that every digraph D with $\delta^-(D) \geq 1$ has a total dominating set according to this definition since $V(D)$ is such a set. For example, the digraph D shown in Figure 7 satisfies $\gamma_t^-(D) = 3$, where the darkened vertices form a total dominating set of D of cardinality 3.

For a digraph $D = (V, E)$ and for a real-valued function $f : V \to \mathbb{R}$, the *weight* of f is $\mathrm{w}(f) = \sum_{v \in V} f(v)$. Further, for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$; in particular,

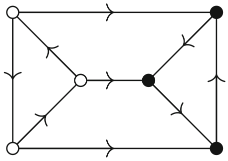


Fig. 7 A digraph D with $\gamma_t^-(D) = 3$

this means that w(f) = $f(V)$. Let $f : V \to \{0, 1\}$ be a function which assigns to each vertex of a graph an element of the set $\{0, 1\}$. We say f is a *total dominating function* if for every $v \in V$, the sum of the function values under f in every out-neighborhood of a vertex is at least 1, that is, for every vertex $v \in V$, we have

$$\sum_{u \in N^+(v)} f(u) \geq 1.$$

The total absorption number of D can be defined as

$$\gamma_t^-(D) = \min\{\mathrm{w}(f) \mid f \text{ is a total dominating function on } D\}.$$

In order to present a lower bound on the total absorption number of a digraph, St-Louis, Gendron, and Hertz [87] in 2012 considered the fractional version of a total dominating set where vertices have fractional weights in the range [0, 1]. A real-valued function $f : V \to [0, 1]$ is called a *fractional total dominating function* of a digraph D if $\sum_{u \in N^+(v)} f(u) \geq 1$ for each $v \in V$. The minimum weight of a fractional total dominating function of D is the *fractional total domination number*, which we denote here by $\gamma_{\mathrm{tf}}^-(D)$. Thus,

$$\gamma_{\mathrm{tf}}^-(D) = \min\{\mathrm{w}(f) \mid f \text{ is a fractional total dominating function for } D\}.$$

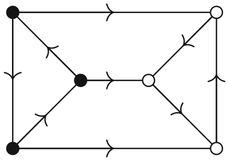
We remark that the fractional total domination number is readily viewed as a linear program. Thus we can talk of minimum, rather than infimum in the above definition. By definition, $\gamma_t^-(D) \geq \gamma_{\mathrm{tf}}^-(D)$, and so the fractional version provides a lower bound on the total absorption number of D. The *girth* $g(D)$ of a digraph D is the number of vertices of the smallest directed cycle in D. St-Louis et al. [87] posed two conjectures, one of which is the following.

Conjecture 4 ([87]) *If D is a digraph with $\delta^+(D) \geq 1$, then $\gamma_{\mathrm{tf}}^-(D) > g(D) - 1$.*

St-Louis et al. [87] proved that Conjecture 4 is equivalent to the 1978 Caccetta-Häggkvist Conjecture which we state below.

Conjecture 5 ([14]) *If D is a digraph of order n with $\delta^+(D) \geq r \geq 1$, then $g(D) \leq \lceil \frac{n}{r} \rceil$.*

Fig. 8 A digraph D with $\gamma_{to}^{+}(D) = 3$



6.4 Total Domination: Version 4

In this version of total domination, a set S in a digraph $D=(V, E)$ is a *total dominating set* if every vertex in V is adjacent *from* a vertex in S, that is, $N^{+}(S)=V$. This is equivalent to saying that S is a dominating set and the subdigraph induced by S has no isolated vertices and no sinks. This is a version defined by Hansen, Lai, and Yue [47] in 1999 and by Schaudt [83] in 2012. We shall call this type of total dominating set a *total open dominating set* and let $\gamma_{to}^{+}(D)$ equal the minimum cardinality of a total open dominating set in a digraph D. For example, the digraph D shown in Figure 8 satisfies $\gamma_{to}^{+}(D) = 3$, where the darkened vertices form a total open dominating set of D of cardinality 3.

In 1999 Hansen et al. [47] defined the *lower orientable open domination number* $\mathrm{dom}_1(G)$ of a graph G as the minimum total open domination number among all orientations of G. The *upper orientable total open domination number* $\mathrm{DOM}_1(G)$ equals the maximum such total open domination number.

Theorem 83 ([47]) *For a connected graph G,* $\mathrm{dom}_1(G)$ *and* $\mathrm{DOM}_1(G)$ *exist if and only if G is not a tree.*

Hansen et al. [47] also investigated the function $\mathrm{DOM}_1(K_n)$. They showed this to be a non-decreasing function and unbounded and determined specific values. Analogous to Theorem 52, they proved the following result.

Theorem 84 ([47]) *For every integer c with* $\mathrm{dom}_1(K_n) \le c \le \mathrm{DOM}_1(K_n)$*, there exists an orientation D of K_n such that $\gamma_{to}^{+}(D) = c$.*

In 2012 Schaudt [83] studied efficient total domination in digraphs, where an *efficient total dominating set* of a digraph D is a total open dominating set S with the property that for each vertex v of D, there is a unique vertex $u \in S$ that is adjacent to v. Graphs that permit an orientation having such a set were studied in [83]. Further, complexity results and characterizations were given.

6.5 Fractional Domination in Digraphs

In Section 6.3, we considered the fractional version of total domination in digraphs. In this section, we present results on the fractional version of domination in digraphs. Adopting our earlier notation, a real-valued function $f : V \to \mathbb{R}$ in a digraph D is a *dominating function* if for every $v \in V$, the sum of the function values under f in every closed out-neighborhood of a vertex is at least 1, that is, for every vertex $v \in V$, we have

$$\sum_{u \in N^+[v]} f(u) \geq 1.$$

The domination number of D can be defined as

$$\gamma(D) = \min\{\mathrm{w}(f) \mid f \text{ is a dominating function on } D\}.$$

A real-valued function $f : V \to [0, 1]$ is called a *fractional dominating function* of a digraph D if $\sum_{u \in N^+[v]} f(u) \geq 1$ for each $v \in V$. The minimum weight of a fractional dominating function of D is the *fractional domination number*, which we denote here by $\gamma_f(D)$. Thus,

$$\gamma_f(D) = \min\{\mathrm{w}(f) \mid f \text{ is a fractional dominating function for } D\}.$$

In 1982 Sands, Sauer, and Woodrow [82] (also due to Erdős) posed the following conjecture.

Conjecture 6 ([82]) *For each n, there is a (least) positive integer $f(n)$ so that every finite tournament whose edges are colored with n colors contains a set S of $f(n)$ vertices with the property that for every vertex u not in S, there is a monochromatic directed path from u to a vertex of S.*

A *complete multidigraph* is a directed graph in which multiple arcs and circuits of length 2 are allowed and such that there always exists an arc between two distinct vertices. A tournament, for example, is a complete multidigraph in the special case when the directed graph is simple (and contains no multiple arcs or circuits of length 2). As remarked in [13], the transitive closure of each color class is a quasi-order (i.e., a transitive digraph); hence, the Erdős-Sands-Sauer-Woodrow conjecture can be restated as follows.

Conjecture 7 ([82]) *For every k, there exists an integer $f(k)$ such that if T is a complete multidigraph whose arcs are the union of k quasi-orders, then $\gamma(T) \leq f(k)$.*

In 2019 Bousquet, Lochet, and Thomassé [13] succeeded in proving this long-standing 1982 Erdős-Sands-Sauer-Woodrow conjecture. The main ingredient in their proof is that the fractional domination number of complete multidigraphs (and therefore of tournaments) is bounded.

Theorem 85 ([13]) *For every k, if T is a complete multidigraph whose arcs are the union of k quasi-orders, then*

$$\gamma(T) = \mathrm{O}(\ln(2k) \cdot k^{k+2}).$$

Harutyunyan, Le, Newman, and Thomassé [53] continued the study of fractional domination in digraphs. Recall that in general there is no upper bound on the domination number of an oriented graph solely in terms of its independence number. However, by Theorem 64, if G is a graph of order n, then $\mathrm{DOM}(G) \le \alpha(G) \cdot \log n$. In contrast to this result, Harutyunyan et al. [53] showed that for any digraph, its fractional domination number is at most twice its independence number.

Theorem 86 ([53]) *For every digraph D, we have $\gamma_f(D) \le 2\alpha(D)$, and this bound is sharp.*

The authors in [53] presented two proofs of Theorem 86. The first proof uses the duality of linear programming, while the second proof is by induction. To show sharpness of the bound, given an arbitrary small real number $\epsilon > 0$, for any integer $k \ge 1$, they constructed a digraph D such that $\alpha(G) = k$ and $\gamma_f(D) > 2k - \epsilon$. Further, they showed that almost surely a random tournament has fractional domination number close to the upper bound of 2.

7 The Oriented Version of the Domination Game

In 2002 Alon, Balogh, Bollobás, and Szabó [1] introduced and first studied the oriented domination game, which belongs to the growing family of competitive optimization graph games. The oriented domination game describes a process in which two players with conflicting goals alternately orient an edge of a graph G until all of the edges are oriented. One player's goal is to minimize the domination number of the resulting oriented graph, while the other player wants to maximize it.

Formally, the oriented domination game on a graph G consists of two players, *Minimizer* and *Maximizer* (called *Dominator* and *Avoider* in [1]), who take turns orienting an unoriented edge of a graph G, until all edges are oriented. The goal of Minimizer is to minimize the domination number of the resulting digraph, while the goal of Maximizer is to maximize the domination number. The Minimizer-start oriented domination game is the oriented domination game when Minimizer plays first. The *oriented game domination number* $\gamma_{\rm og}(G)$ of G is the minimum possible domination number of the resulting digraph when both players play according to the rule that on each move a player may only orient an unoriented edge. To illustrate the game, Alon et al. [1] determined the oriented game domination number of a complete graph.

Proposition 87 ([1]) *For a complete graph K_n of order $n \ge 4$, we have $\gamma_{\rm og}(K_n) = 2$.*

Proof Minimizer's strategy is to pick two arbitrary vertices, say u and v. On each of his turns, Minimizer orients an edge from u or v to a vertex w different from u and v. His strategy is to orient these edges in such a way that at least one of u and v is oriented towards w. He can always achieve his goal as follows. Whenever Maximizer orients the edge uw from w to u, then Minimizer immediately replies by orienting the edge vw from v to w, if it is not already oriented. Analogously, whenever Maximizer orients the edge vw from w to v, then Minimizer immediately replies by orienting the edge uw from u to w, if it is not already oriented. In this way, he ensures that the set $\{u, v\}$ is a dominating set in the resulting oriented graph. Thus, $\gamma_{\rm og}(K_n) \le 2$.

To show that $\gamma_{\rm og}(K_n) \ge 2$, Maximizer adopts the following strategy. Maximizer can clearly prevent a source in the oriented graph resulting when $n = 4$. In the case when $n \ge 5$, there exists a collection of n edge-disjoint paths of length 2, one centered at each of the n vertices of K_n (see [10]). Maximizer's strategy is whenever Minimizer orients one of these edges from a central vertex on one of these paths, Maximizer responds by orienting the other edge of the corresponding path towards the central vertex. In this way, Maximizer guarantees that the indegree of each vertex in the resulting oriented graph becomes at least 1, implying that $\gamma_{\rm og}(K_n) \ge 2$. Consequently, $\gamma_{\rm og}(K_n) \ge 2$. □

In [1], the authors obtained a sharp lower bound for the oriented game domination number of trees.

Theorem 88 ([1]) *If G is a tree of order n, then $\frac{1}{2}n \le \gamma_{\rm og}(G) \le \frac{2}{3}n$.*

The proof of Theorem 88 implies that the upper bound holds for any connected graph G, as Minimizer can concentrate his attention on a spanning tree T of G and play according to his strategy in the tree T. Whenever Maximizer orients an edge not in T, Minimizer continues to orient edges according to his strategy in the tree. As shown in [1], both bounds in Theorem 88 are sharp. For graphs with minimum degree at least 2, the following improved upper bound was given in [1].

Theorem 89 ([1]) *If G is a graph of order n with $\delta(G) \ge 2$, then $\gamma_{\rm og}(G) \le \frac{1}{2}n$.*

If G is a graph of order n with maximum degree Δ, then a trivial lower bound on the domination number is $\gamma(G) \ge n/\Delta$. In the oriented domination game, Maximizer orients half of the edges. As observed by Alon et al. [1], Maximizer might succeed in decreasing the outdegree of each vertex to about $\Delta/2$, in which case the resulting domination number is at least $2n/\Delta$. This prompted them to pose the following conjecture.

Conjecture 8 ([1]) *If G is a graph of order n with maximum degree Δ, then*

$$\gamma_{\rm og}(G) \ge \left(\frac{2}{(1 + {\rm o}(1))\Delta}\right) n.$$

Conjecture 8 has yet to be settled. The best general lower bound to date on the oriented game domination number in terms of the maximum degree and order of the graph is the following result in [1].

Theorem 90 ([1]) *If G is a graph of order n with maximum degree Δ, then*

$$\gamma_{\rm og}(G) \ge \left(\frac{4}{3\Delta + 7}\right) n.$$

Nordhaus-Gaddum-type inequalities for the oriented domination game are given in [1]. Here, $\overline{G}$ denotes the complement of a graph G.

Theorem 91 ([1]) *If G is a graph of order n, then $\gamma_{\rm og}(G) + \gamma_{\rm og}(\overline{G}) \le n + 2$, and this bound is sharp.*

We note that if G is the complete graph K_n where $n \ge 4$, then $\gamma_{\rm og}(\overline{G}) = n$ and, by Proposition 87, $\gamma_{\rm og}(G) = 2$. Thus, if $G = K_n$, then $\gamma_{\rm og}(G) + \gamma_{\rm og}(\overline{G}) = n+2$, showing sharpness of the bound in Theorem 91. We close this section with the following conjecture posed in [1], that the inequality in Theorem 91 can be strengthened for connected graphs.

Conjecture 9 ([1]) *If both G and its complement $\overline{G}$ are connected graphs of order n, then*

$$\gamma_{\rm og}(G) + \gamma_{\rm og}(\overline{G}) \le \frac{2}{3}n + 3.$$

8 Concluding Comments

In this chapter, we have surveyed selected results on domination in digraphs. Many results have been omitted to prevent the chapter from growing too large. For example, topics such as signed domination in digraphs, efficient domination in digraphs, packing in digraphs, reinforcement numbers of digraphs, rainbow domination in digraphs, and Roman domination in digraphs, to name a few, are omitted. Additional references on domination in digraphs can be found in [40, 44, 48, 55, 56, 58, 70, 77, 85, 91]. Due to space limitations, we have also omitted proofs of many important results on domination in digraphs presented in this chapter, including the proofs of results due to Alon, Brightwell, Kierstead, Kostochka, and Winkler [2]; Chudnovsky, Ringi, Chun-Hung, Seymour, and Thomassé [23]; Harutyunyan, Le, Newman, and Thomassé [53]; and Bousquet, Lochet, and Thomassé [13] which have significantly impacted the latest developments in the field of domination in digraphs and tournaments. We apologize for these omissions.

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