



A Generalization of Bass Equation for Description of Diffusion of Cryptocurrencies and Other Payment Methods and Some Metrics for Cooperation on Market

Victor Dostov^{1,2}  and Pavel Shust^{1,2} 

¹ Saint-Petersburg State University, St Petersburg, Russia
{leonova, shoust}@npaed.ru

² Russian Electronic Money and Remittance Association, Moscow, Russia

Abstract. We use case of payment systems to discuss the typology of generalized Bass equation solution for audience growth in systems with cooperative and non-cooperative behavior of users. Based on the models for C2B and P2P payment system models analyzed in our previous papers, we propose an integrated approach. Different types of cooperation are discussed. The paper also proposes some criteria for estimating a degree of cooperation in given real-life system.

Keywords: Bass equation · Payment system · Cryptocurrency · Ricatti equation · Cooperative behavior coefficient

1 Introduction

Bass equation

$$\frac{dx}{dt} = (p + qx)(1 - x) \quad (1)$$

or

$$\frac{dx}{dt} = -qx^2 + (q - p)x + p \quad (2)$$

was proposed by Frank Bass [1] to describe diffusion on innovations, primarily for penetration of consumer durables such as refrigerators and home freezers [2]. It describes how innovations spread in audience with limited size where x is share of maximum audience that already accepted an innovation, p and q are empirical coefficients which, roughly, describe probability for someone to accept innovation as result of advertisement or under the influence of other acceptors, correspondingly. In Bass approach this is one-way process, a person who already uses innovative product or service is supposed to use them forever. This is good assumption for such durables as

home freezers but not accurate for wide spectrum of other technologies e.g. CD players or video players, which have relatively short lifecycle. However, Bass model became popular and was later applied to different kinds of innovative goods and services, e.g. in [3, 4].

Bass model is based on the assumption that there are two ways of how users can start using innovative product or service: first type of users are innovators who make independent decisions; second type are imitators, whose decisions depend of those who are already using the product or service. From other perspective, these two modalities are very similar to cooperative and non-cooperative games [5].

2 Generalization of Bass Model to Payment Systems

Initially, Bass model deals with “social” effects, such as advertisement and mutual influence between people. Social effect of mutual influence usually decreases with higher penetration of the technology. However, this is not always a case. In [6, 7] authors demonstrated that even in mature markets cooperation between clients may be important for technological reasons. For example, to make a first phone call, there should be two phone users, not one. This means that in certain technologies cooperation between parties plays an important role.

Interaction in a payment market demonstrates both cooperative and non-cooperative interactions. C2B (consumer to business) systems, like retail card payments are example of non-cooperative game because capacity of given POS terminal to accept payments is virtually infinite. The same is true for cash withdrawal from ATM: cardholder’s ability to use the card to withdraw cash does not depend on other cardholders. In other words, client’s behavior for such systems does not depend on other people’s behavior. This is not the case in P2P (peer to peer) systems, like international money transfers (remittance): if Alice transfers money to Bob, Bob needs to be part of the system too. If Alice wants to transfer her bitcoin from her wallet to Bob, Bob needs to have the wallet as well. All these examples mean that for payment systems (and other systems with “technical” cooperation between client) we can obtain generalization of Bass-like equation for mature market that reflects cooperation behavior driven both by technical specifics and human interaction. Some limited cases of pure C2B and P2P systems and Bass-like equation were considered in [7], in this paper we will draft a general approach from scratch.

Let’s consider general type of payment system with number of user x , maximum number of users (maximum audience capacity) N , and, consequently, number of non-users $N-x$. Then, in growth users rate dx/dt we will have non-cooperative contribution coefficient a_b , describing attractiveness of C2B payment functions and cooperative contribution coefficient a_p for attractiveness of P2P payment functions to describe a probability for non-involved customer to join the payment system at any point of time. For mature payment system or market we also need to introduce fatigue factor b , which describe how often users leave the system, disappointed in its functionality. Thus we obtain a following equation

$$\frac{dx}{dt} = a_p(N-x)x + a_b(N-x) - bx \quad (3)$$

or

$$\frac{dx}{dt} = -a_px^2 + (a_pN - a_b - b)x + a_bN \quad (4)$$

Then final equation in Ricatti equation [8] with constant coefficients

$$\frac{dx}{dt} = -a_px^2 + lx + a_bN \quad (5)$$

where

$$l = a_pN - a_b - b \quad (6)$$

Analytical solution of (5) yields

$$x = \frac{1}{2a_p} \left(-D \tanh \left(C - \frac{1}{2}tD \right) + l \right) \quad (7)$$

where D, given by

$$D^2 = l^2 + 4a_ba_pN \quad (8)$$

may be interpreted as reverse time of system evolution.

We can also introduce stationary limit of (7)

$$x_\infty = \frac{1}{2a_p} (D + l) \quad (9)$$

If the initial condition is $x(0) = 0$, then arbitrary constant C is

$$C = \operatorname{arctanh}(l/D) \quad (10)$$

If we do not have information from beginning of system evolution and in starting point of observation $x(0) = x_0$, then

$$C = \operatorname{arctanh}((l - 2a_px_0)/D) \quad (11)$$

Actually, in practical analysis we have to use (11) with care as it is quite sensitive to errors in x_0 .

Here we actually demonstrated, that we can introduce fatigue factor to Bass equation without changing its general appearance, just setting up three independent coefficients instead of two. As seen from (5–6) these two Eqs. (5), one with arbitrary a_p , a_b , N and b and other with the coefficient replaced as following:

$$a_b + b \rightarrow a_b \tag{12}$$

$$N \rightarrow \frac{Na_b}{a_b - b} \tag{13}$$

$$a_p \rightarrow a_p \tag{14}$$

$$b \rightarrow 0 \tag{15}$$

coincide exactly. Such recalibration means that users fatigue $b \neq 0$ is qualitatively equivalent to increase of non-cooperative coefficient a_b and decrease of maximal audience N .

Examples of solutions (7) are given in Fig. 1.

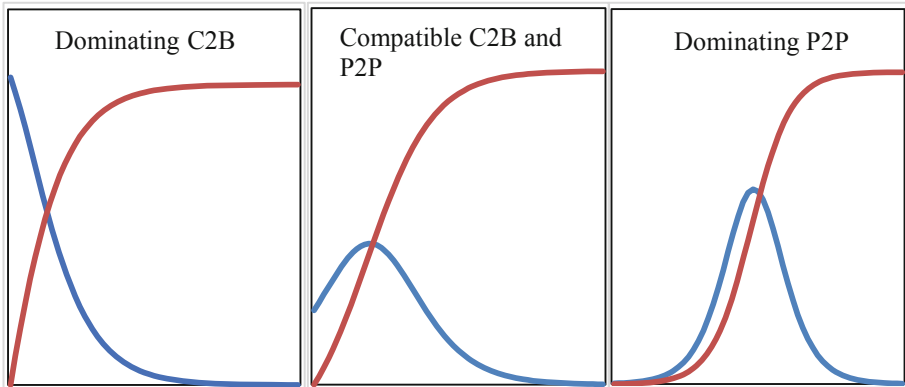


Fig. 1. Transition from C2B-dominating type, $a_b \gg a_p N$ to P2P-dominating type, $a_b \ll a_p N$ for number of users (red) and audience growth rate (blue) (Color figure online).

Also we can make formal but useful observation, that time t is presented in (7) only in combination τ

$$\tau = C - \frac{1}{2}tD \tag{16}$$

This means that, notwithstanding specific parameters, all systems described by (7) are moving along the same trajectory, giving us S-shaped curve like in last graph at Fig. 1, however on different parts of it. The observation period covers only part of this

curve, giving us different shapes shown at Fig. 1. This also means that if for current observation period we see C2B type shape with negative d^2x/dt^2 , this may mean both pure C2B system or P2P or mixed system in a later stage.

Also, if we start our observation at some point $t = 0$ which is not point of start of the system operation, we can make extrapolation to negative t , reconstructing previous system behavior. Formally we may find a starting point of system evolution t_0 where $x = 0$ as

$$t_0 = 2(C - \operatorname{arctanh}\left(\frac{l}{D}\right))/D \tag{17}$$

However, as (7) in part is very sensitive to variation of parameters, the extrapolation back in time leads to fast growing errors and may be irrelevant, so this estimation must be used with big caution.

This leads us to question of algorithm that could be used to obtain system type from practical data. A priori, if we write an Eq. (3) with given coefficients, we can introduce cooperation coefficient

$$K = \frac{a_p N}{a_b} \tag{18}$$

Evidently, $K \gg 1$ yields us cooperative behavior solution and vice versa. Combining K and D we may introduce four type classification for comparing of systems under consideration (Table 1).

Table 1. Typology of payment systems.

| | Small D | Big D |
|---------|---------------------------|---------------------------|
| Small K | Slow non-cooperative, C2B | Fast non-cooperative, C2B |
| Big K | Slow cooperative, P2P | Fast cooperative, P2P |

Unfortunately, this straightforward criterium (18) cannot be directly applied to practical analysis. In principle, applying LSD or LAD method [9] to real data set, can find best-fitting independent coefficients l , D , a_p for (8), and, correspondingly calculate C , then solve Eqs. (7) and (9) as a system finding N and a_b . From practical standpoint, looking at C is enough. If C is large negative, we get all-time non-cooperative behaviour, if C is a large positive, we get cooperative behaviour, for C around 0 we get mixed type. Therefore, depending on approach, we may use both C and K to estimate degree of cooperation in the system.

3 Practical Examples

To give practical examples, we use data on number payment cards in the European Union (in tens of millions) [10], Russia (in millions) [11, 12], South Korea debit cards (in millions) [13] and Spain bank cards (in millions) [14] at Fig. 2.

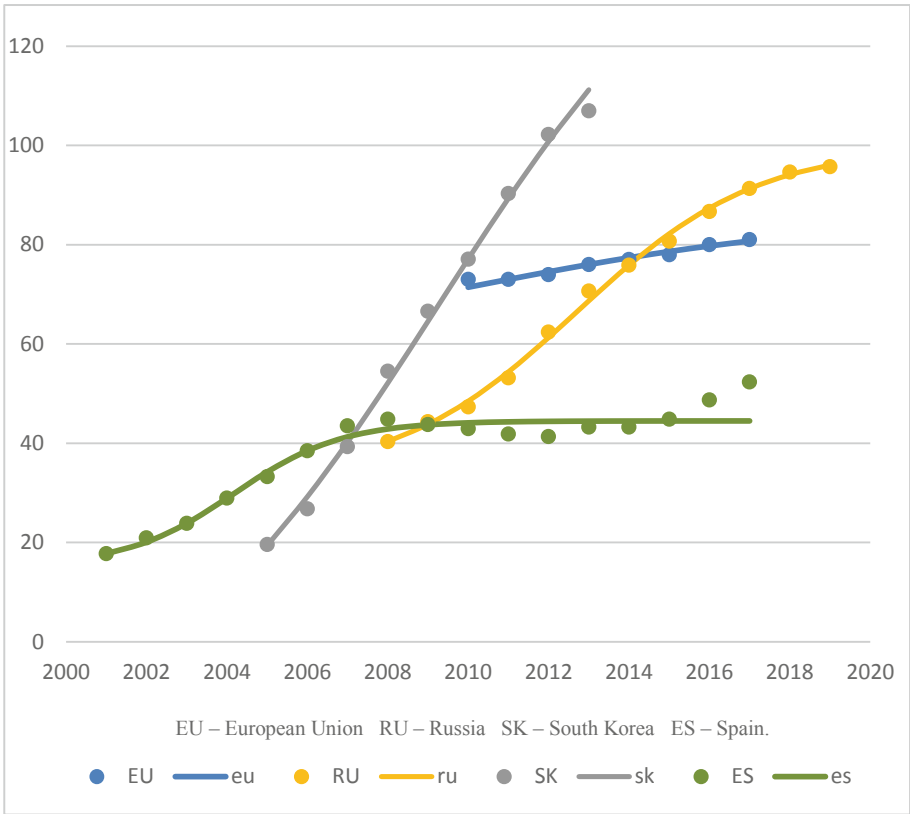


Fig. 2. Dynamics of card number in EU, Russia, South Korea and Spain interpolated by Eq. (8). Circles represent real data, curves represent interpolation by Eq. (7).

In the EU we see the best fit by LAD criterion at $C = 0.0$ with dominating C2B transactions and close to saturation market. In curve for Russia get $C = 1.0$, with both cooperative and non-cooperative contributions and also relatively close to saturation. We also give example of South Korea debit cards [13], for which model work quite well with $C = 0.57$ and far from saturation market and example of Spain bank cards [14], for which market behavior is more complicated. We can see quite a good fit in 2000–2014 with $C = 1,2$ but unpredicted growth later. We may attribute this growth to changes in regulation or market models which resulted in drastic change of model coefficients. We will discuss this situation below.

Similar approach may be used to analyze behavior in cryptocurrency systems: such as bitcoin (abbreviation BTC marks real data) and Ethereum (ETH, correspondingly) using [15] as a source of data. Real-time curve contains a lot of investment-induced quasi-random noise, but we can still assume that average behavior is still described by aforementioned factors. In our previous work [7] we considered cryptocurrencies as pure cooperative systems. In this paper we use (7) and find coefficients on it using LAD criterion. Direct application of LAD to noisy curves does not work pretty well, therefore we first applied it to data averaged by years to get initial estimation: the results are shown on Fig. 3a. Then we applied LAD to full data set to improve this estimation with results shown on Fig. 3b. Evidently, the model is not able to describe hype behavior in 2017–2018 induced by dramatic change of bitcoin/USD exchange rates [16], but we see a good coincidence with average behavior. With $C = 1.6$ we see domination of cooperative behavior. We also see that cryptocurrency market is definitely reaching its saturation with no significant growth perspective. This fact was demonstrated earlier in [7] and recent data only prove it. The similar behavior is seen in Ethereum platform with $C = 3.0$. Cryptocurrencies are definitely fast cooperative P2P payment system type in terms of Table 1 and traditional systems are slower and less cooperative C2B system in comparison. However, as role of P2P payments in traditional card system is starting to grow recently [17], we expect a shift to larger values of C in their dynamics.

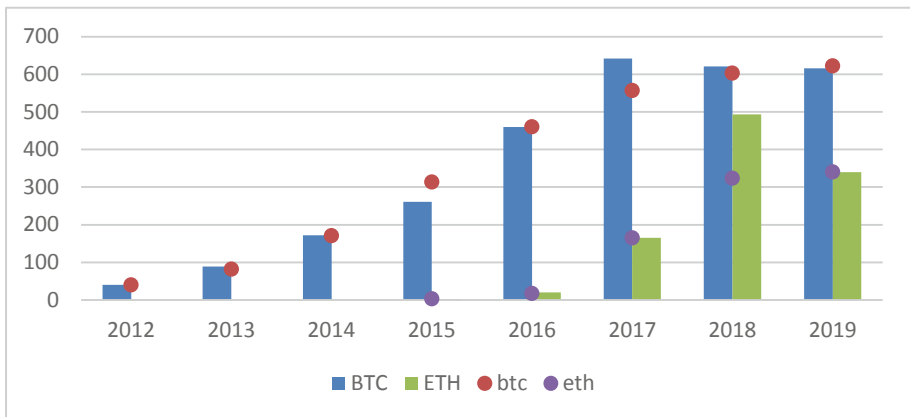


Fig. 3a. Dynamics of cryptocurrency wallets averaged by years for Bitcoin (blue) and Ethereum (green) and their approximation by Eq. (7) (Color figure online).

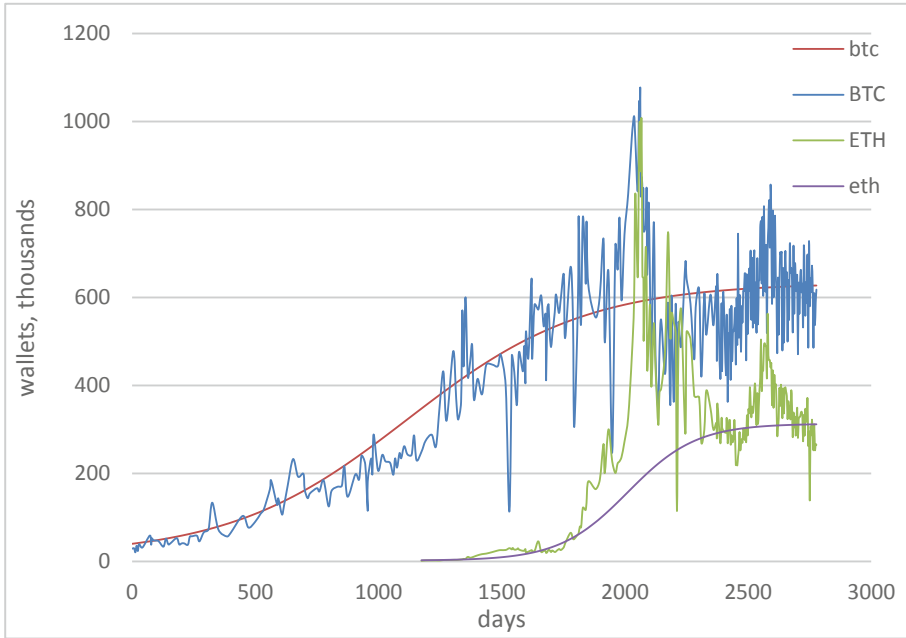


Fig. 3b. Dynamics of cryptocurrency wallets for Bitcoin (red) and Ethereum (green) and their approximation by Eq. (7) (Color figure online).

Table 2. Parameters C, D, x_∞ for different payment instruments.

| | EU | SK | RU | ES | BTC | ETH |
|------------------|------|------|------|------|------|------|
| C | 0,0 | 0,6 | 1,0 | 1,2 | 1.6 | 3.0 |
| $D, 1/year$ | 0,21 | 0,26 | 0,44 | 0,74 | 1,0 | 3,0 |
| $x_\infty, mlns$ | 850 | 160 | 99 | 44 | 0,63 | 0,34 |

We summarize different cases in Table 2, which allows to compare essential parameters for different payment instruments.

We suppose that future studies may provide us with even less-cooperative cases with negative C .

Figure 4 represents graphically Table 2 for card segment with circle size representing audience size. We see clear connection between C, D and x_∞ which will be subject of further studies.

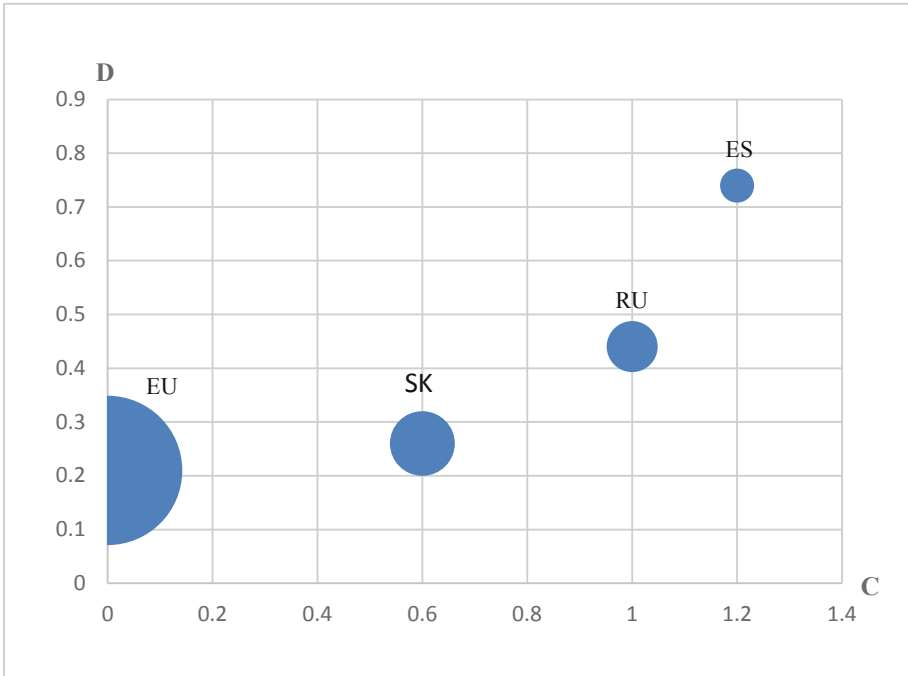


Fig. 4. Representation of significant parameters for card dynamics in different countries.

We also have to underline that that effectiveness of using this approach for longer-term analysis shall not be overestimated. Evidently, coefficients in (3) change slowly but steadily. While typical time of such change is much bigger than $1/D$, we can neglect these changes. But sooner or later these changes may become important. E.g. in Figs. 3a and 3b we describe quasi-steady states successfully reached by cryptocurrencies. However, if, e.g. regulatory situation will change and most of countries adopt cryptocurrency regulation, a_p and a_b coefficients will change dramatically and, after transition period, they will switch to new trajectory of development with different coefficients in (7). Evidently, this option is relevant to other types of payment systems as well, including card systems considered above. Very probably, aforementioned card statistics for Spain demonstrates us such situation. However, in short- and middle-time perspective assumption of almost constant coefficients in (3) works well for empirical purposes.

4 Conclusion

We used payment systems to propose the generalization of Bass approach to the case of technology-driven cooperative behavior and possible loss of existing users. In addition to terms of equation describing increase in audience we also introduce a coefficient, responsible for customer outflow and show that this churn shifts effectively system

audience to more non-cooperative behavior. We provide solution of Riccati equation, compare them to the original Bass solution and propose different numerical criteria to describe users' behavior. These results can be applied both to cryptocurrencies audience and "classic" payment system dynamics. We believe that proposed solutions might be used not only for payment systems analysis, but also to describe dynamics of innovations in other fields.

References

1. Bass, F.M.: A dynamic model of market share and sales behavior. In: Proceedings, Winter Conference American Marketing Association, Chicago, IL, USA (1963)
2. Bass, F.M.: A new product growth for model consumer durables. *Manage. Sci.* **15**(5), 215–227 (1969)
3. Ganjeizadeh, F., Lei, H., Goraya, P., Olivar, E.: Applying looks-like analysis and bass diffusion model techniques to forecast a neurostimulator device with no historical data. In: 27th International Conference on Flexible Automation and Intelligent Manufacturing (FAIM), pp. 1916–1924. *Procedia Manufacturing*, Modena (2017)
4. Soffer, T., Nachmias, R., Ram, J.: Diffusion of web supported instruction in higher education: The case of tel-aviv university. *Educ. Technol. Soc.* **13**(3), 212–223 (2010)
5. Takako, F.-G.: *Non-Cooperative Game Theory*. Springer, Japan (2015)
6. Dostov, V., Shust, P., Popova, E.: Using mathematical models for analysis and prediction of payment systems behavior. In: 34th International Business Information Management Association Conference (IBIMA). IBIMA Publishing, King of Prussia, Madrid (2019)
7. Dostov, V., Shoust, P., Popova, E.: Using mathematical models to describe the dynamics of the spread of traditional and cryptocurrency payment systems. In: 19th International Conference Computational Science and Its Applications – ICCSA, pp. 457–471. Springer, Saint Petersburg (2019)
8. Simmons, G., Robertson, J.: *Differential Equations with Applications and Historical Notes*, 2nd Edn. International Series in Pure and Applied Mathematics, 640 p. McGraw-Hill, New York (1991)
9. Dodge, Y., Jureckova, J.: *Adaptive Regression*. Springer, New York (2000)
10. Cherowbrier, J.: Total number of payment cards in the European Union from 2010 to 2018. <https://www.statista.com/statistics/444328/number-of-payment-cards-european-union>. Accessed 23 Mar 2020
11. Central Bank of Russia Statistics: The number of payment cards issued by credit institutions and the Bank of Russia. <http://www.cbr.ru/eng/statistics/nps/>, <http://www.cbr.ru/Content/Document/File/69593/t13.xlsx>. Accessed 23 Mar 2020
12. Odud, A., Platonova, Y.: Analysis of The Bank Cards Market in Russia. *Economic vector* 11 (2019). http://www.vectoreconomy.ru/images/publications/2019/11/financeandcredit/Odud_Platonova.pdf. Accessed 23 Mar 2020
13. The Korea Bizwire Business Statistics. http://koreabizwire.com/category/feature/kobiz_stats/business-statistics. Accessed 25 Mar 2020
14. CEIC Spain Payment and Cards Statistics. <https://www.ceicdata.com/en/spain/payment-and-cards-statistics>. Accessed 25 Mar 2020
15. BitInfoChart Bitcoin Active Addresses historical chart. <https://bitinfocharts.com/comparison/bitcoin-activeaddresses.html>. Accessed 01 Apr 2020

16. Bitnews Today, The annual cryptocurrency market review: what's left after the hype, 2.01.2029. <https://bitnewstoday.com/news/the-annual-cryptocurrency-market-review-what-s-left-after-the-hype>. Accessed 03 Apr 2020
17. Eremina, A., Yumabaev, D.: Russians make more C2C transactions that card payments in stores (Vedomosti 16.03.2018)