On the Semantic Concept of Logical Consequence

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Abstract. In this paper, we give a groundwork for the foundations of the semantic concept of logical consequence. We first give an opinionated survey of recent discussions on the model-theoretic concept, in particular Etchemendy's criticisms and responses, alluding to Kreisel's squeezing argument. We then present a view that in a sense the semantic concept of logical consequence irreducibly depends on the meaning of logical expressions but in another sense the extensional adequacy of the semantic account of first-order logical consequence is also of fundamental importance. We further point out a connection with proof-theoretic semantics.

Keywords: Logical consequence · Model-theoretic semantics · Natural language

1 Introduction

What is it to give a formal semantics of natural language? The traditional view in the semantics of natural language may be to give the truth condition of a sentence of natural language. This, in turn, is partly because by doing this we can give an account of a certain inferential relationship among sentences. E.g., one can see what conclusion can or cannot be drawn from certain sentences.

In such a semantic endeavor, it is a substantial problem which concept of logical consequence (we abbreviate this as "l.c.") we take to be the basis for our semantic studies of natural language. But our pre-theoretical concept of l.c. already appears to diverge; hence, we need to first discuss what data our account should be based on. Some take the pre-theoretical concept for our account of l.c. to be: it is impossible that the premises are true and the conclusion is false; however, others do: no argument with the same logical form has true premises and the false conclusion; yet others take the combination thereof (p. 366, [13]).[1]

To theorize these pre-theoretical concepts, the two major formal accounts of the semantic concept of l.c. have been proposed: 1. the **substitutional** account;

[1] In the following we often switch the terms i) "l.c." and ii) "logical truth or validity" as a special case. The difference never affects our philosophical points.

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2. the **model-theoretic** account. Quine, an advocate of 1, gives a formal substitutional account. For any sentence φ and a set of sentences Γ of a given language $\mathcal{L}$, we define an interpretation J (based on an appropriate substitution function) for $\mathcal{L}$ and define the notion of "truth in the interpretation J" (φ is true in J if the result of substitution φ^J is *truth simpliciter*). The substitutional consequence is defined: φ is a (substitutional) consequence of Γ if φ is true in each interpretation J in which all sentences in Γ are true.[2] On the other hand, Tarski, the founder of 2 states that X is a consequece of $\mathfrak{K}$ if "every model of the class $\mathfrak{K}$ [of sentences] is at the same time a model of the sentence X [27]."[3] The idea that the concept of l.c. can be identified with the model-theoretic concept of it is often considered to be a "thesis," i.e., **Tarski's thesis**: φ is an intuitive semantic (logical) consequence of Γ if and only if, for all model (structure) $\mathfrak{M}$, $\mathfrak{M} \models \Gamma$ lmplies $\mathfrak{M} \models \varphi$. The left side is an intuitive notion, so one cannot prove this as a theorem but needs to state it as a thesis like Church's thesis.

The difference between these accounts may be highly relevant to the semantics of natural language. When adopting the substitutional account, the central notion is *truth simpliciter*, and one can consider only the absolute truth condition of a sentence. In the model-theoretic one, the concept of consequence is based on *truth in a model*, and we need to take into consideration *truth conditions with respect to a model* (cf. [24]). Thus, in the two accounts, one considers different sorts of "truth conditions" to determine consequence relations.[4]

In this paper, we give an opinionated survey of the semantic concept of l.c. We take up two problems raised in [17] and criticisms of the accounts in [7,8]. Discussing these issues, we present a view that there is a sense in which the semantic concept of l.c. irreducibly depends on the meaning of logical expressions.

2 Criticisms of the Accounts

We discuss criticisms of the model-theoretic account of l.c., McGee's problems and Etchemendy's criticisms, one of which overlaps with one of the former. McGee's problems are: a) the reliability problem; b) the contingency problem. a) goes: "it is by no means obvious that being true in every model is any guarantee that a sentence is true [17]." The problem arises because "models" in Tarski's thesis are all sets but the extant entire universe of mathematics may not be

[2] This is obviously condensed. For details, see, e.g., [6].

[3] It may sound misleading to quote this sentence from [27], for the concept of "model" is in [27] is significantly different from the currently standard one since Tarski's does not seem to allow domain variations. We handle the issue later.

[4] Glanzberg [11] discusses the issue of what sort of "truth conditions" we consider in different schools of the semantics of natural language. According to him, Davidsonians consider the absolute notion of truth condition, Montagovians initially considered the one based on "truth in a model" but these days they also use the absolute notion.

identified with a set (presumably a proper class).[5] b) is "what sentences are valid ought not to be a matter of contingent fact, and Tarski's thesis would appear to make it so (p. 273, [17])." b) is essentially the same as one of Etchemendy's coming next.

Etchemendy's point can be summarized: "Tarski's analysis involves a simple, conceptual mistake: confusing the symptoms of logical consequence with their cause (p. 264, [8])." The extensional adequacy of the account is "at least as problematic as the conceptual adequacy of the analysis," although the critique "is not aimed at model-theoretic techniques, properly understood [8]."

Etchemendy begins his discussion by classifying semantics into the two kinds: representational and interpretational. The former fixes the meaning of expressions in a given sentence and considers possible worlds in which the sentence is true or not, but the latter modifies interpretations of expressions in the sentence (it becomes true or false depending on interpretations). E.g., concerning a situation in which a sentence "Snow is white" is false, representational semantics considers, say, a possible world in which snow is black, while interpretational semantics considers an interpretation where "white" means black.

Based on this distinction, Etchemendy argues that the essence of Tarski's account of l.c. can be explained as follows. First, expressions in a given sentence are divided into the two sorts: one is "fixed terms" and the other is "variable terms" (not "variables"). The former are expressions which behave like logical constants and hence fixed. The latter are expressions whose interpretations can be varied. Then Tarski's original account of validity is not much different from: given a set of fix terms $\mathfrak{F}$ in a given sentence, say S, a sentence is logically true if it is true under all substitutions of (the variables replacing) the variable terms. Note that this is essentially the substitutional account. But, on this account, there is a possibility that, due to the poverty of the object language, a sentence clearly invalid may be artificially judged to be valid; hence, Tarski modifies the definition by using satisfaction. Despite its use of satisfaction, Etchemendy takes such a model-theoretic account to be "interpretational," which is essentially the same as the substitutional account to the extent that it explains l.c. by *the ordinary universal quantification* (over all satisfactions). He claims that this account cannot explain the notion of "necessity" involved in the concept of l.c.

This interpretational version of model-theoretic account (we call it "the I-model-theoretic account") that Etchemendy takes to be given in [27] is importantly different from the currently standard model-theoretic account. The latter not only considers all satisfactions but variations of domains (of quantifications) over all non-empty sets, whereas the I-model-theoretic account does not explicitly deal with domain variations and is apparently a fixed-domain account. Besides, Etchemendy even thinks that the I-model-theoretic account

[5] The problem has been aware of, e.g., "Mathematics as a whole – this is the lesson of the set theoretic antinomies – is not a structure itself, i.e., an object of mathematical investigation, nor is it isomorphic to one ([1], p. 7)."

is not congenial to the idea of domain variations.[6] But still he seems to take Tarski to reduce the representational aspects of semantics to the interpretational ones by the I-model-theoretic account. This introduces complications in handling Etchemendy's criticisms of the model-theoretic account because this is not based on the standard one. However, we keep discussing the I-model-theoretic account to examine Etchemendy's criticisms for a few reasons. It may be controversial both whether Etchemendy's interpretation of Tarski [27] is accurate or not[7] and whether Etchemendy's presentation accurately represents the standard model-theoretic account or not. But it does not matter much since there is a way of assimilating domain variations without explicitly talking about it, i.e., taking a relativization of quantifier by a monadic predicate in the language. Moreover, as we will see, his particular criticism forces us to consider domain variations by the construction in his case. Thus, if there is any substantial philosophical divergence between the I-model-theoretic account and the standard model-theoretic one, it cannot rest merely on the fact one takes the variable-domain account and the other does not, but on more substantial issues of how to understand l.c.

Etchemendy admits that Tarski's I-model-theoretic account provides a necessary condition for the concept of logical truth relativized w.r.t. $\mathfrak{F}$, but he claims that the account fails to give a sufficient condition for logical truth. He gives concrete arguments against the (I-) model-theoretic account to show this. We present two of them. In one, Etchemendy claims that it is ultimately difficult to make a distinction between the model-theoretic view and a view often criticized by the view. In the other, he argues that the account may fail to guarantee the extensional adequacy, in particular it overgenerates, i.e., generating more sentences as logical truths than our pre-theoretic concept allows.

1) Etchemendy argues that the foregoing account of l.c. is problematic when it comes to talking about quantifiers, since taking a quantifier to be either a fixed term or a variable term produces a problem. E.g., in the former case, if we take the equality symbol to be logical and hence in $\mathfrak{F}$, then $\exists x \exists y (x \neq y)$ would be a logical truth, which is absurd. But if we take "something" to be simply a variable term (to make the argument simpler, adopting Etchemendy's suggestion, i.e. taking "some" to be fixed and "thing" to be a variable term), then the inference (1) "Able Lincoln was president. Therefore something was president" may turn out to be invalid, provided we take "thing" to denote the class of dogs. This is invalid since the subcollection of the individuals over which the existential quantifier ranges is disjoint from that of humans. He then argues that, to avoid such cases, Tarski's view needs to adopt a maneuver called **cross-term restriction**. This is to put some constraints on our interpretations of two expressions often based on semantic categories. E.g., "Abe Lincoln" and "something" are so constrained that the interpretation of the former is in the latter. Any cross-term restriction has an effect of excluding some interpretations; hence, a cross-term restriction is similar

[6] Indeed, Etchemendy states, "it is hard to understand why in the semantics for first-order languages we vary the domain of quantification (p. 290, [8])."

[7] We do not go into historical issues here.

to meaning postulates. Meaning postulates are considered by model-theorists not to be determining logical truth, since they make the determination of logical truth circular. E.g., "If Abe Lincoln was president, then Abe Lincoln was an elected official" is not a logical truth, but if this case looks valid, this would be only because we exclude all the models invalidating this by appealing to the specific interpretations of pertinent expressions. But advocates of interpretational semantics criticizing meaning postulates while using cross-terms restrictions would be question begging since there is no in-principle distinction between the two.

This may be one of the strongest points in [7]. Indeed, in the I-model-theoretic account, there is no principled way of both excluding quantifiers from fixed terms *and* keeping valid the foregoing case of inference without appealing to the cross-term restriction; hence, using the restriction and criticizing meaning potulates is incoherent.

2) The second criticism is concerned with the size of the world. Let us consider

i) σ_2: $\exists x \exists y (x \neq y)$, σ_3: $\exists x \exists y \exists z (x \neq y \land y \neq z \land z \neq x), \cdots$
σ_n: $\exists x_1 \exists x_2 \exists x_3 \ldots \exists x_n (x_1 \neq x_2 \land x_2 \neq x_3 \land \ldots)$

If $\exists$, $\neq$ are in $\mathfrak{F}$, then they would be logically true, to which practically nobody would agree. First, one can easily see how to falsify these formulas. Second, the truths of these depend on the size of the world, although the logical truth should not depend on the size of the world (this ought to be carefully examined).

The implausibility of the claim of the logical truth of i) is due to taking both $\exists$ and the equality symbol to be fixed terms. Indeed, both may well be variable terms. But we present Etchemendy's argument which fixes only equality.[8]

To fix i), Etchemendy first takes the negation of each of the σ_n sentences.

ii) $\neg\sigma_2$: $\neg\exists x \exists y (x \neq y)$, $\neg\sigma_3$: $\neg\exists x \exists y \exists z (x \neq y \land y \neq z \land z \neq x)$, etc.

He treats $\exists$ as a variable term and introduces an *existential quantifier variable* E, whose satisfaction domain consists of various subcollection of the universe.[9]

iii) $\forall E[\neg\sigma_2(\exists/E)]$: $\forall E \neg ExEy(x \neq y)$
$\forall E[\neg\sigma_3(\exists/E)]$: $\forall E \neg ExEyEz(x \neq y \land y \neq z \land z \neq x)$, etc.

For each n, iv) $\forall E[\neg\sigma_n(\exists/E)]$ claims that every subcollection of the universe contains fewer than n objects. On the I-model-theoretic account: if the universe is finite, then the account tells us that $\neg\sigma_2$, $\neg\sigma_3$, $\ldots$ are logically true; if the universe is infinite, then the account tells us that none of these is logically true.

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[8] The second weakness can be fixed by avoiding equality and by using any relation symbol s.t. (*) that R is transitive and irreflexive implies that there exists x for all y s.t. $\neg R(x, y)$. The negation of (*) has only infinite models. But we keep using equality.

[9] This is a kind of relativization, but we universally quantify over relativizations.

Then Etchemendy argues as follows. $\neg\sigma_n$ is indeed not (usually counted as) logically true. But this is only because iv) is false, i.e., there are more than n objects in the universe. Nothing in the (standard or I-) model-theoretic account can assure that this is not logically true by any purely logical ground, but the logical status of iv) depends on a non-logical feature (the size of the universe), i.e., the axiom of infinity. Hence, the definition does not capture "the ordinary concept of logical truth" and gives no internal guarantee of the extensional adequacy.

Note that the example is so chosen as to show that his point is also valid for the variable-domain account. In the example, the idea of variable-domain is incorporated in terms of E. So what is at issue is the size of the universe out of which each domain is taken. Hence, the variable-domain account and the fixed-domain one make no substantial difference. Anyways, on the model-theoretic account the logical status of a sentence depends on extralogical facts. The upshot is that there is no guarantee that the account does not overgenerate. Etchemendy claims an overgeneration can happen if there is a substantive generalization – generalization making a substantive claim of the world.[10]

After these critiques are given, Etchemendy's alternative view is stated as: "a sentence is *logically* true if it is true *solely by virtue of the meaning of the logical vocabulary it contains* (p. 103, [7])." Endorsing this point enables him to take the representational view and to explain how the notion of necessity is involved in l.c. He claims that the representational view "makes perfectly good sense of model-theoretic practice – much better sense, in fact, than the Tarskian view (p. 286, [8])." The idea is: "[t]he set-theoretic structures that we construct in giving a model-theoretic semantics are meant to be mathematical models of logically possible ways the world ... might be or might have been [8]."[11] This suggests that he criticizes the *traditional* model-theoretic (both I-model-theoretic and standard) account to the extent that the one is essentially reducible to the substitutional one, also called quantificational.

Instead of criticizing the techniques in model theory, he gives an *alternative* view of model-theoretic account. Etchemendy endorses a view that the logical truth of a sentence in an object language ultimately has its source in the metatheory by appealing to an observation (due to Carnap): "if the truth of a sentence follows logically from the recursive definition of truth for the language in which it occurs, then that sentence must be logically true." Consider: (1) Lincoln was president or Lincoln was not a president. To establish its logical truth, one can

[10] He adds that the issue of the choice of logical constants is red herring, for the dependence on extralogical facts can arise even when all expressions are "logical constants," e.g., $\forall x \forall y \forall P(Px \rightarrow Py)$. This is true in a world with essentially one object.

[11] In [8], Etchemendy emphasizes that Kripke semantics is a good case of representational semantics. However, he elsewhere suggests that there is a severe limitation in representational semantics. "$2+2 \neq 4$ (p. 62, [7])" is easy to make sense in interpretational semantics but makes no sense in representational one (since mathematical truth is a necessary truth). This can be a reason why interpretational semantics is also needed in addition to the representational view. See Sect. 3, Sect. 4.

start from an elementary logical truth of the metatheory: (M) For any f, either f satisfies 'xP' or f does not satisfy 'xP.' From this, by the recursive clause for 'not' and 'or' in the definition of satisfaction, it follows: (2) For any f, f satisfies 'xP or not xP.' By the closure principle: "[i]f a universally quantified sentence is logically true, then all of its instances are logically true as well," this is sufficient to show that (1) is logically true. The moral is that "the fact that the above demonstration requires no appeal external to the semantics of the language and the logic of the metatheory provides us with a genuine assurance, quite independent of Tarski's account, of the logical truth of the associated universal closure (2) (p. 140, [7])."[12] The source of logical truth is claimed to be the recursive clauses in the logic of the metatheory, not ordinary universal quantifications.

3 Critical Assessments of the Criticisms

Here we give critical assessments of the criticisms. We first give a survey of the extant discussions on l.c., most of which directly handles Etchemendy's second criticism, and then move on to presenting our own view. Before going into concrete cases, let us give some caveats on general points. One is about one's goal. Etchemendy uses the phrase "conceptual analysis" but is not clear about the adequacy conditions for the concept of l.c.,[13] while some critiques aims for only "the extensional adequacy." Such a difference of pursued goals may raise complications in the assessments of the arguments. Another is the scope of one's argument. Depending on which logic one has in mind, e.g., first-order, higher-order, etc., one may have different conclusions, and this affects an evaluation of a view. Most critiques confine their discussions to classical first-order logic, but Etchemendy puts no limitation in his general conceptual discussions, in which first-order logic is merely a special case satisfying some desirable properties.

3.1 Prawitz's Anticipation of Etchemendy's Critiques

We first discuss Prawitz's [19] neglected criticism of the model-theoretic account, which anticipates Etchemendy's. Prawitz reconstructs Tarski's account as follows. Let $A(c_1, \ldots, c_n)$ and $B(c_1, \ldots, c_n)$ be sentences where $c_1, \ldots, c_n$ stand for the nonlogical constants in A and B, and let $A(c_1, \ldots, c_n)$ and let $A(v_1, \ldots, v_n)$ and $B(v_1, \ldots, v_n)$ be the open formulas obtained by replacing the constants c_i by variables of v_i. Then $B(c_1, \ldots, c_n)$ is a logical consequence of $A(c_1, \ldots, c_n)$ iff

(1) every assignment or model satisfying $A(v_1, \ldots, v_n)$ satisfies $B(v_1, \ldots, v_n)$, or
(2) $(\forall v_1 \in D_1) \ldots (\forall v_n \in D_n)(A(v_1, \ldots, v_n) \rightarrow B(v_1, \ldots, v_n))$ is true regardless of how *independent* (not determined by fixing the range of quantifers) domains are chosen.

[12] Schurz [23] also claims that the intensions of logical terms are determined by the recursive truth definition.

[13] He does not explain the phrase "the ordinary concept of logical truth," either.

(1) is closer to Tarski's original (Etchemendy's I-model-theoretic) version, and (2) relativizes quantifiers, which can be an alternative to domain variations.

Prawtiz observes that "I think that there is no doubt that … the material equivalence asserted between logical consequence and (1) [is] correct …" but "the analysis does not go very far" and argues as follows. i) Suppose A and B are sentences (e.g., in predicative second order logic) without descriptive [non-logical] constants. Then (2) says only that B is a l.c. of A if and only if $A \to B$ is true. Then, there is no way of considering "a variation of descriptive constants" in Tarski's account (cf. footnote 10). ii) " [A]nalysis makes no distinction between logical sentences (containing only logical constants) and factual sentences (containing also descriptive constants)." Hence, a logical sentence is logically true just in case it is true in the same sense as factual sentences are true. So "no analysis is made of the necessity involved in logical truth" nor "the *ground* for a universal truth like (2)." Prawitz argue for bringing in the notions of proof to fix this. It is notable that his points substantially overlap with Etchemendy's.

3.2 The Extensional and Intensional (Conceptual) Adequacy

Etchemendy appears to claim that the dependence on the size of the universe shows that the model-theoretic account is wrong since i) whether or not a sentence follows from premises should not depend on extralogical facts; ii) due to the dependence of nonlogical facts we have no guarantee that the account does not overgenerate. One can find the following objections in the literature: a) showing that there is a fallacy in Etchemendy's argument; b) showing that dependence on extralogical facts is not enough to show that the conceptual analysis is defective; c) showing that the dependence on extralogical facts does not occur; d) giving a justification for the claim that the account does not overgenerate. We will discuss not a)[14] but b), c), d). In discussing d), we focus on a method of *proving* that a logic with a complete formal proof system is extensionally correct: **Kreisel's squeezing argument**.

3.2.1 With Kreisel's Squeezing Argument

Kreisel [14] gives the squeezing argument in order to illustrate the role of intuitive concepts in foundational studies under the methodological concept "informal rigour." First, for any first-order formula φ, let us write $D(\varphi)$ for "φ is formally derivable (in a given system of formal rules)," $V(\varphi)$ for "φ is valid in all set-theoretic structures" (model-theoretically valid) and $Val(\varphi)$ for "φ is intuitively valid," which means that φ is true in arbitrary (not necessarily set-theoretic) "structures."[15] Then we postulate the two principles: (1): $D(\varphi) \to Val(\varphi)$; (2): $Val(\varphi) \to V(\varphi)$. (1) expresses intuitive soundness. (2) states that intuitive validity implies set-theoretic validity because a formula valid in all structures is valid in all set-theoretic ones. Then we have $(*)$ $V(\varphi) \to D(\varphi)$, which is simply

[14] [12,26], etc. analyze fallacies in his argument. We omit these partly because we have reason to reject his claim even if we find no alleged fallacy in the argument.

[15] We omit the details of the complicated background of this notion of Val.

Gödel's completeness theorem. By the principle (1), (2) and (∗), we can establish: $Val(\varphi) \leftrightarrow D(\varphi) \leftrightarrow V(\varphi)$.

This shows that, under some natural assumptions, the completeness theorem is sufficient to establish the co-extensiveness of the model-theoretic validity and the intuitive validity. Kreisel's main motivation to give the "proof" is to argue that from the viewpoint of "informal rigour" even our intuitive concept may have a precise characterization. But Kreisel's squeezing argument has been adapted by philosophers of logic, who have their own purposes, as follows.

i) Assuring that validity implies truth. Recall that the reliability problem has been raised because there is no reason why the actual world itself is a structure or a set, and if we define the concept of validity as "truth in all structures," then it would be unclear whether even a valid sentence is true (simpliciter). However, by Kreisel's squeezing argument, as far as first-order logic is concerned, the established equivalence between Val and V can assure that if a sentence is invalidated by something too large to be a set, then there is a set-size structure invalidating it. By taking the contrapositive, if the actual mathematical world can be identified with one of "class-size structures," then the sentence being valid, i.e., having no set-theoretic countermodel of it, suffices to show that the sentence is true simpliciter. Hence, Kreisel's squeezing argument can give *a* solution to the reliability problem.[16]

ii) Preventing overgeneration. Etchemendy himself modified the squeezing argument to show that, although there is no conceptual guarantee that the traditional model-theoretic account does not overgenerate (in fact, he argues that the second-order logic overgenerates by using the case of the continuum hypothesis (CH)), first-order logic does not overgenerate. Etchemendy modifies Kreisel's argument as follows. Kreisel identifies the intuitive validity Val with truth in all "structures." However, once the identification is lifted, the latter makes (2) correct, but the former makes (2) "dubious." Thus, Etchemendy introduces a new predicate LTr, meaning "intuitive notion of logical truth" with the principle: (1′) $D(\varphi) \rightarrow LTr(\varphi)$ (intuitive soundness). By (1′), (2) and $V(\varphi) \rightarrow D(\varphi)$ (completeness), we can prove $V(\varphi) \rightarrow LTr(\varphi)$ (p. 149, [7]). Hence, even the traditional model-theoretic account does not overgenerate for first-order l.c.

iii) Accommodating modality. Both Shapiro and Hanson appear to adapt Kreisel's squeezing argument to accommodate "modality" involved in the pre-theoretical characterization of l.c. Thus we explain their cases in a uniform scheme. First, they both take the issue of modality (necessity) involved in the concept of l.c. very seriously. Accordingly, they adopt the combined pre-theoretic concept mentioned in Sect. 1: it is impossible that the premises are true and the conclusion is false by the form of the argument. Then Shapiro [24] adopts a *blended* view, combining Etchemendy's representational and interpretational view, which means taking the combination of the views, i.e., i) fixing the language and considering possible worlds and ii) considering reinterpretation of all non-logical expressions. Hanson [13] takes three factors: necessity; generality; a priority to be properly treated in any satisfactory account of l.c. Second, at

[16] There are some other solutions, e.g. Boolos' in [3] and McGee's [17].

least in [13, 24, 25], they focus on first-order logic as a paradigmatic case. Third, they define the standard, precise model-theoretic l.c. $\Gamma \models \varphi$ with no explicit mention of modality; hence, they "accommodate" modality, i.e., the modality is represented by structures and "reduced" to them. Fourth, adapting Kreisel's squeezing argument, both Shapiro and Hanson aim for obtaining only extensional adequacy.[17]

iv) Characterizing the primitive concept of consequence. Field thinks that the genuine l.c. is neither (standard) model-theoretic nor proof-theoretic and should be treated as a primitive concept. The traditional soundness and completeness theorems "merely connect two different notions (p. 62, [9])," but we need to show that the model theory (also the proof theory) is sound and complete with respect to the primitive concept. Kreisel's squeezing argument tells us how to use completeness to characterize the primitive concept.

Let $\Gamma \Rightarrow B$ mean that the argument from Γ to B is logically valid in the "primitive" sense. Let Γ be a set of sentences and B be a sentence of a particular fixed language. Then we can state the following properties of $\Rightarrow$.

(P-Sound) [Genuine Soundness of the proof theory]: if $\Gamma \vdash_S B$ then $\Gamma \Rightarrow B$.
(P-Comp) [Genuine Completeness of the proof theory]: if $\Gamma \Rightarrow B$ then $\Gamma \vdash_S B$.
(M-Sound) [Genuine Soundness of the model theory]: if $\Gamma \models_M B$ then $\Gamma \Rightarrow B$.
(M-Comp) [Genuine Completeness of the model theory]: if $\Gamma \Rightarrow B$ then $\Gamma \models_M B$.

Formal soundness and completeness theorems can be formulated as follows: (FST) if $\Gamma \vdash_S B$ then $\Gamma \models_M B$; (FCT) if $\Gamma \models_M B$ then $\Gamma \vdash_S B$. In this setting, one can reconstruct Kreisel's squeezing argument and more.

a) P-sound, M-comp, FST: $\Gamma \vdash_S B \Leftrightarrow \Gamma \models_M B \Leftrightarrow \Gamma \Rightarrow B$.
b) M-sound, P-comp, FCT: $\Gamma \vdash_S B \Leftrightarrow \Gamma \models_M B \Gamma \models_M B \Gamma \Rightarrow B$.

a) is essentially a reconstruction of Kreisel's squeezing argument. By this equivalence, Field's primitive consequence can be proven to at least extensionally coincide with the two traditional concepts of l.c., whereas b) is pointless since M-soundness is not obvious for the same reason why $V(\varphi) \rightarrow Val(\varphi)$ was not.

3.2.2 Without Kreisel's Squeezing Argument

Critiques argue that Etchemendy's claim that his argument shows that we can't ensure to avoid overgeneration is mistaken. The first two disagree with Etchemendy's understanding of the traditional model-theoretic account.

i) Conceptual considerations. **MacFarlane** [16] considers a modified version of Etchemendy's argument, using a case of modal finitist, who believes (n-fin): "there could not be an infinite number of objects." She can consistently assert (n-fin) and (*) in footnote 8 is not logically true. But she cannot consistently assert (n-fin) and (*) is not true in all models. Therefore, logical truth and

[17] Shapiro takes modality involved in l.c. to be a "logical modality" given with respect to the isomorphism property (a necessary condition for a logical term). But Hanson's modality is not particularly "logical."

truth in all models are not identical. MacFarlane argues against this by claiming that this is based on the hidden assumption: "[t]he semantic value of an expression depends only on its meaning and the state of the world (p. 9, [16])," and this does not support the conclusion. Indeed, the semantic value of a term is determined by the two items "only against the background of a specification of the term's semantic category (p. 10, [16])." To specify this means to specify "the range of possible semantic values" by our sortal concepts. "Provided that our sortal concepts themselves do not rule out an infinite number of instances, there is a sense in which there can be an infinite number of possible semantic values for singular terms (p. 11, [16])." Hence, even the modal finitist cannot assert both (n-fin) and "(*) is logically true" solely on a logical ground.

ii) Informal proofs. According to **Garcia-Carpintero**, the model-theoretic account of l.c., as he understands it, is different from the quantificational account, since it involves a *semantic* theory for the logical particles (cf. [5] for a similar point). The *partial semantic theory* gives us a syntactic formation of sentences and semantically "determine[s] the truth conditions of complex sentences" (p. 115, [10])." "*Relative to that partial semantic theory* [10]," one can say that logical truth (truth in virtue of the meanings of expressions) is a truth in all *preformal models.* Here a *preformal model* means: "a possible set of logical values such that expressions belonging to the same logical category as the nonlogical expressions in the sentence or argument could have those values [10]," where logical values are, roughly, the semantic properties contributing the determination of truth condition of a sentence. Then Garcia-Carpintero informally proves: a sentence being true in virtue of the meaning of logical constants is equivalent to being true in all preformal models. From his viewpoint, set theory is not a core part of the partial semantic theory that describes preformal models but "only a tool to give us a more precisely defined sense of 'model' (p. 121, [10])." The semantics already "involves the existence of an infinite preformal model [10]."

iii) The entanglement of logic and mathematics. Purporting to show that the ground for a sentence being logically true may not be purely logical, **Parsons** argues that Etchemendy demands too much, when he does: "if a sentence is not logically true, this has to be by virtue of statements that are logical truths (p. 158 [18])." Parsons says that he doesn't see how this demand can be satisfied, no matter what one's criterion of logical truth is. The reason is quite general. "Given a sentence A, the statements 'A is logically true' and 'A is not logically true' are neither of them logical truths on the usual criteria, for a rather trivial reason: they depend on the existence of the sentence A and of the elements making up its structures, as well as its truth-conditions [18]" These are matters of logic, broadly speaking, but, since Etchemendy endorses "the ontological minimalism of logical truth," Parsons doesn't "see how one could show a sentence not logically true without appeal to extra-logical facts [18]" and concludes that Etchemendy's case is not sufficient to show that we have no guarantee that there is no overgeneration. (cf. p. 151, [24], for a similar point.)

4 Some Reflections on Logical Consequence

In the final section, we both take stock of our discussions and present our own view of the concept of l.c.[18] First of all, we are pessimistic about obtaining a uniform view of the concept of l.c. throughout various contexts. Hence, the discussions on the semantic concept of l.c. should be given by distinguishing the contexts in a two-dimensional manner: I) Adequacy; II) Scope. The distinction concerning I) is made between 1) the intensional and 2) the extensional adequacy. The distinctions concerning II) are made in the hierarchy of logics.

Concerning the intensional (conceptual) adequacy, we require that the semantic concept of l.c. satisfy more conditions than mere extensional adequacy. E.g., we take the modal intuition of it to be heeded. Also, the semantic concept of l.c. should take into account the necessity and the formality of the consequence due to the meaning of logical expressions. To be intensionally adequate, our account should take the pre-theoretic concept to be the combined version in Sect. 1: it is impossible that premises are true but the conclusion is false due to the form of the argument. According to this view, the substitutional account, which can at most give an extensionally adequate account due to its use of truth simpliciter, does not respect this pre-theoretic concept. Consider $\exists x \exists y (x \neq y)$ again. If we take $\neg$, $=$, $\exists$ to be logical expressions, this would be logically true, although it is obviously not.[19] Indeed, it is easily conceivable that there is a "possibility" that there is only one object in the domain of quantification. Considering variable domains is close to considering "possible" situations in which there are things of different cardinalities. Hence, concerning the fundamental viewpoint, we are inclined to share with Prawitz and Etchemendy the view that the quantificational account of l.c. does not explain the modal feature of l.c. (see Sect. 2, Sect. 3). Logical necessity (or justification) should be caused via the "guarantee" of the truth of the conclusion raised by the connection between the premises and the conclusion by virtue of the meaning of the (logical) expressions.[20]

Let us additionally note that we are in partial agreement with some views presented in Sect. 3. First, when we consider l.c., it is necessary to adopt a conceptual framework in which two independent dimensions, explained by Etchemendy as representational/interpretational, can be taken into account (cf. [24,28], see footnote 11).[21] Secondly, we take the pre-theoretic concept to be neither proof-theoretic nor (standard) model-theoretic, the latter of which must be not a pre-theoretically given datum, but a result of theorization, although they are

[18] Due to the limited space, we often state our points without detailed arguments.

[19] One might immediately object that = is not a logical symbol. However, such a change's raising significant difference would already make dubious the robustness of the view. Also, from an intensional viewpoint, it is out of the question whether we can extensionally accommodate this pre-theoretic concept by a subsitutional account.

[20] L.c. can be a special case of "analytic" consequence. A system of transformation rules which transforms an atomic formula to another can be taken to give an analytic consequence. Formal systems of logic are often conservative extensions thereof (cf. [21]).

[21] We refrain from entirely agreeing with Etchemendy and Shapiro about the details.

often conflated in logic. Both model-theoretic and proof-theoretic views may work together to characterize both of the concepts.

Besides, Etchemendy's point on the cross-term restriction (among his two arguments) is well-taken. To this extent, we do not consider that the I-model-theoretic (quantificational) account is completely intact. However, these do not imply that his overgeneration argument is convincing. The argument needs to be examined carefully. Etchemendy's view is that since the conceptual analysis given in the account is wrong, ultimately the account may get the extension of l.c. wrong, unless we use a kind of squeezing argument to save the account. We argue both that Etchemendy's argument to show the lack of extensional adequacy, based on conceptual considerations, is dubious and that, concerning the extensional adequacy w.r.t. first-order logic, there is reason to be content with an argument given independently of Etchemendy's. In order to argue this way, we need to make clear in which context we give our argument, i.e., which logic in the hierarchy of logics is at issue. Hence, we now handle the distinctions of scope. We have hardly any problem about propositional logic.[22] But Etchemendy's argument related to the cross-term restriction suggests that first-order quantifiers be treated carefully, since the interpretation is varied over every non-empty domain, despite their being "(logical) constants."[23] In addition to this, second-order logic has yet other numerous meta-logical differences from first-order logic. Thus, putting aside propositional logic, we make only a distinction between first-order and second-order logic. Then, combined with the distinction concerning the adequacy, we have four combinations: i) first-order/extensional; ii) first-order/intensional; iii) beyond first-order/extensional; iv) beyond first-order/intensional. In the literature, nobody takes iii), the others take i) or ii), and Etchemendy seems to be the only one taking iv). He criticizes the model-theoretic account in an unlimited scope, i.e., first- and second-order logic in terms of intensional adequacy, even claiming that in both cases it may get their extensions wrong, but we mainly focus on first-order logic.

We recapitulate the structure of Etchemendy's second argument. 1. Suppose a situation in which the universe of sets is finite. 2. People in the finite universe ("finitists") may take a non-logical truth in our sense to be true in all models. 3. But the case of non-logical truth is not logically true (in our sense), since it is false in some infinite model. 4. Hence, logical truth and truth in all models cannot be conceptually identified (intensional inadequacy). 5. In fact, our identification of logical truth with truth in all models is made possible only by the axiom of infinity, but it is a non-logical fact that the axiom holds. 6. No extensional correctness is guaranteed due to an extra-logical fact (extensional inadequacy?). 7. But the extensional adequacy of the account of logical truth is guaranteed by the help of a (sound) proof system and completeness for first-order logic. The traditional model-theoretic account of logical truth for second-order logic, where

[22] Even this may not be entirely unproblematic. There is an issue called "non-categoricity" in propositional logic first pointed out by Carnap [4].

[23] This point seems to be underrated (see [23]), although this is not unnoticed, e.g., Enderton's textbook treats quantifiers as "parameters [24]."

completeness fails, is indeed extensionally inadequate (overgeneration), since CH or its negation is a logical truth in second-order logic (cf. [14]).

Let us give a tentative examination of the argument only for the case i), i.e., without going into the case of CH. First, in general, it is unclear whether we should appeal to such a "counterfactual" situation to discuss the intensional adequacy. Secondly, the steps from 2 to 4 depends on a tacit assumption that our concept of "logic" is at least common both to ourselves and the "finitists." But this needs to be independently argued. Third, Etchemendy claims that the step 5, i.e., our model-theoretic account of logical truth depends on an extra-logical fact (the axiom of infinity), suggests that the conceptual analysis is defective, but he does not explicitly give a condition for the intensional adequacy of an account to succeed. Fourth, note that the completeness theorem itself actually depends on the infinity of the universe of sets in the meta-theory, so if only the finite universe is available to the "finitist," then it would be unclear whether one can coherently appeal to Etchemendy's modified squeezing argument to guarantee the extensional adequacy of the I-model-theoretic account of first-order l.c.[24]

We are now discussing what we can be sure of. Unlike Etchemendy, most authors take (i) and think that we should be content with an extensionally adequate account of classical first-order logic, and there seems to be an almost uniform agreement that the model-theoretic (however conceived) account of l.c. of it is extensionally adequate. Kreisel's squeezing argument plays a major role in this agreement.

We will give quick comments on Kreisel's squeezing argument in this context. Let us first remind the reader of why we care about the semantic concept of l.c. at all. At least in the traditional viewpoint, proof theoretic systems are for provability; the model-theoretic concept of l.c. is primarily for proving unprovability or independence.[25] This role of the model-theoretic concept is still technically fundamental, and the extensional adequacy is sufficient for this. This has an important effect on the debate between the substitutional and the model-theoretic view. If one confines her attention only to the extensional adequacy of the model-theoretic account of l.c. of first-order logic, then Kreisel's squeezing argument can go further. Combined with the extensional equivalence between Val and V, Kreisel's equivalence in [15]: $V(\varphi) \leftrightarrow V^{\omega}(\varphi) \leftrightarrow V^{a}$, where V^{ω} stands for "valid in all countable models" and V^{a} stands for "valid in all arithmetic interpretations," establishes the co-extensionality of Val and the other validities. The former equivalence is based on Löwnheim-Skolem theorem and the latter on Hilbert-Bernays arithmetized completeness theorem. In particular, the latter uses the notion of "substitution" of formulas in the language of arithmetic. Thus, as far as first-order logical truth and the extensional adequacy are at issue, there is no substantial difference between the substitutional and

[24] Etchemendy addresses the issue (p. 275, footnote 6, [8]), saying that the finitist overgenerates. But there may be a further problem: the use of the axiom of infinity in proving completeness makes Etchemendy's squeezing argument circular.

[25] This roughly means that logical truth corresponds to a lack of counterexample.

the model-theoretic view (however conceived).[26] In general, showing the extensional adequacy of an account of first-order l.c. by a convincing argument is a significant contribution. Kreisel's ingenious squeezing argument gives a kind of "proof" of Tarski's thesis for first-order logic and deserves credit for that. More specifically, to a) The reliability problem; b) The contingency (overgeneration) problem, Kreisel's squeezing argument can give solutions, concerning the extensional adequacy of first order l.c., no matter what one's background motivation is (see Sect. 3). Even Etchemendy uses a variant of Kreisel's argument to prevent overgeneration for first-order logic. We take these to be evidence for the claim that the issue of the concept of l.c. is pretty much settled for the extensional adequacy w.r.t. first-order logic and to this extent we should be content with the result given by Kreisel's argument. This does not mean that the point of Kreisel's squeezing argument consists in to secure the "faulty" analysis of quantificational account. It shows that it is possible to establish the extensional adequacy by using an extensive variety of semantic methods. Hence, it is not necessarily fruitful nor feasible to criticize the extensional inadequacy of an account among them by arguing that it is inadequate.

Still, some critiques are engaged in the debate on the intensional adequacy of the model-theoretic account of first-order l.c., i.e., Sect. 3.2.2 (ii). They try to give informal proofs or conceptual observations that the model-theoretic account of first-order l.c. is even intensionally adequate. MacFarlane's and Garcia-Carpintero (Sect. 3) contend that we can "require" the (possible) existence of infinitely many objects to falsify a sentence on a purely logical ground. Their strategy is to undermine the step 5 of the foregoing summary of Etchemendy's argument, i.e., to show that there is no conceptual gap between logical truth and truth in all models. However, Parsons has a different view of this. Grounds for a sentence being logically true may not be purely logical but this is not sufficient to show that the model-theoretic account of first-order l.c. overgenerates because it is unclear that the dependence on extralogical (mathematical) facts is a good reason for overgeneration.

The arguments given by MacFarlane and Garcia-Carpintero presented in Sect. 3 show that there is something more in the model-theoretic concept than those reducible to the substitutional (or quantificational) account. Although Etchemendy appears to think otherwise, the concept of truth by virtue of the meaning of logical expressions seems not to be incompatible with fulfilling a sort of ontological requirement (especially when one shows that a sentences is invalid). Their arguments look convincing to this extent. But it is another issue whether or not their views *completely* capture intensional aspects of the model-theoretic concept. It is yet another issue whether their view is correct or Parsons' view is correct as a reason why Etchemendy's argument against the extensional

[26] Quine's substitutional view in [20] is based on the Hilbert-Bernays arithemtiized completeness theorem. There is a subtlety on the issue of compactness. See [2,6].

adequacy of the model-theoretic account is wanting,[27] although either of these seems to give a sufficient ground for the claim that Etchemendy's contention of overgeneration needs to be reconsidered. However, we leave these issues open.

Whereas Etchemendy criticizes the traditional model-theoretic account as an quantificational one and proposes an alternative view and MacFarlane and Garcia-Carpintero take "the model-theoretic" concept to have something more than that, their ultimate views are not very different (see [8] footnote 20) though certainly not identical. Be that as it may, in our current understanding, the intensional aspects of the concept of l.c. is far from being settled. Indeed, we can describe our current situation as: we are in agreement with Prawitz and Etchemendy on the fundamental view, but we consider Etchemendy's particular argument to be problematic; hence, although the extensional adequacy of a variety of accounts of first-order l.c. is settled, anything beyond that is still widely open, e.g., the intensional adequacy of a semantic account of first-order l.c. We consider investigations of it to be carried out by focusing on the meaning of expressions and along the line suggested by Etchemendy on the logic of the metatheory,[28] i.e., the logical power of a logical expression is conferred to it via the recursive clause for the expression in Tarskian inductive characterization of satisfaction.

We then point out a connection between this idea and proof-theoretic semantics. Interestingly, one can find a similar idea in proof-theoretic semantics. Sambin et al. [22] suggest a general scheme of introducing the logical constants called **the principle of reflection**. E.g.,"The common explanation of the truth of a compound proposition like $A\&B$ is that $A\&B$ is true if and only if A is true and B is true." More schematically, they claim that for any connective $\circ$, "[i]n our terms, a connective $\circ$ between propositions, like & above, reflects at the level of object language a *link* between assertions in the meta-language. (*link* is an expression for a meta-linguistic device corresponding to $\circ$.) The equivalence "A $\circ$ B true if and only if A true *link* B true," which we call *definitional equation* for $\circ$, "gives all we need to know about it. A $\circ$ B is semantically defined as that proposition which, when asserted true, behaves exactly as the compound assertion A true *link* B true. The inference rules for $\circ$ are derived in a system for $\circ$ by using reflexivity $\varphi \vdash \varphi$ and transitivity (cut) for $\vdash$ (we eliminate cut afterwards). We say that $\circ$ is introduced by the principle of reflection. E.g., for $\otimes$, from the definitional equation $\Gamma, A \otimes B \vdash \Delta$ iff $\Gamma, A, B \vdash \Delta$, we derive

$$\mathrm{L}\otimes : \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \qquad \mathrm{R}\otimes : \frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}.$$

[27] This issue is not simple, since those who argue that the infinity of a domain can be equipped on purely logical ground consider only first-order logic. To invalidate a formula in first-order language, we only need a countable model (see V^ω). But things are more complicated in second-order logic, since falsifying a sentence in the language of second-order logic may require staggering ontology (p. 151, [24]). For second-order logic, Parsons' entanglement view is more reasonable.

[28] The concept is so fundamental that it may be difficult to reduce it to something more fundamental.

The similarity of these ideas may suggest that the idea of reflecting the metatheory by the theory in the object language is worth further investigating. Perhaps, there is a convergence between these ideas.

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