Sensitive Instances of the Cutting Stock Problem

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Abstract. We consider the well-known cutting stock problem (CSP). The gap of a CSP instance is the difference between its optimal function value and optimal value of its continuous relaxation. For most instances of CSP the gap is less than 1 and the maximal known gap $6/5 = 1.2$ was found by Rietz and Dempe [11]. Their method is based on constructing instances with large gaps from so-called sensitive instances with some additional constraints, which are hard to fulfill. We adapt our method presented in [15] to search for sensitive instances with required properties and construct a CSP instance with gap $77/64 = 1.203125$. We also present several instances with large gaps much smaller than previously known.

Keywords: Cutting Stock Problem · Integer Round Up Property · Integrality gap · Sensitive instances

1 Introduction

In the classical formulation, the cutting stock problem (CSP) is stated as follows: there are infinite pieces of stock material of fixed length L. We have to produce $m \in \mathbb{N}$ groups of pieces of different lengths $l_1, \cdots, l_m$ and demanded quantities $b_1, \cdots, b_m$ by cutting initial pieces of stock material in such a way that the number of used initial pieces is minimized.

The cutting stock problem is one of the earliest problems that have been studied through methods of operational research [6]. This problem has many real-world applications, especially in industries where high-value material is being cut [3] (steel industry, paper industry). No exact algorithm is known that solves practical problem instances optimally, so there are lots of heuristic approaches. The number of publications about this problem increases each year, so we refer the reader to bibliography [18] and the most recent survey [2].

Throughout this paper we abbreviate an instance of CSP as $E := (L, l, b)$. The total number of pieces is $n = \sum_{i=1}^{m} b_i$. W.l.o.g., we assume that all numbers in the input data are positive integers and $L \geq l_1 > \cdots > l_m > 0$.

The classical approach for solving CSP is based on the formulation by Gilmore and Gomory [5]. Any subset of pieces (called a *pattern*) is formalized as

Supported by RFBR, project 19-07-00895.

Y. Kochetov et al. (Eds.): MOTOR 2020, CCIS 1275, pp. 80–87, 2020.
https://doi.org/10.1007/978-3-030-58657-7_9

a vector $a = (a_1, \cdots, a_m)^\top \in \mathbb{Z}_+^m$ where $a_i \in \mathbb{Z}_+$ denotes the number of pieces i in the pattern a. A pattern a of E is *feasible* if $a^\top l \le L$. So, we can define the set of all feasible patterns $P^f(L, l) = \{a \in \mathbb{Z}_+^m \mid a^\top l \le L\}$. For a given set of patterns $P = \{a^1, \cdots, a^r\}$, let $A(P)$ be the $(n \times r)$-matrix whose columns are given by the patterns a^i. Then the CSP can be formulated as follows:

$$z(E) := \sum_{i=1}^{r} x_i \to \min \text{ subject to } A(P^f(L, l))x = b, x \in \mathbb{Z}_+^r.$$

The common approximate solution approach involves considering *the continuous relaxation* of CSP

$$z_C(E) := \sum_{i=1}^{r} x_i^C \to \min \text{ subject to } A(P^f(L, l))x^C = b, x^C \in \mathbb{R}_+^r.$$

Here $z(E)$ and $z_C(E)$ are called *the optimal function values* for the instance E. The difference $\Delta(E) = z(E) - z_C(E)$ is called *the gap* of instance E. Practical experience and numerous computations have shown that for most instances the gap is very small. An instance E has *the integer round up property* (IRUP) if $\Delta(E) < 1$. Otherwise, E is called a non-IRUP instance. This notation was introduced by Baum and Trotter [1].

Subsequently, the largest known gap was increased. In 1986 Marcotte constructed the first known non-IRUP instance with the gap of exactly 1 [9]. Fieldhouse found an instance with gap $31/30 \approx 1.033333$ in 1990 [4]. In 1991 Schiethauer and Terno slightly improved this result to $137/132 \approx 1.037879$ [16]. Rietz, Scheithauer and Terno subsequently constructed non-IRUP instances with gaps $10/9 \approx 1.111111$ and $7/6 \approx 1.166666$ in 1998 and 2000 respectively [12,13] (both papers were published in 2002). Finally, Rietz constructed an instance with gap $6/5 = 1.2$ and published it in his PhD thesis in 2003 [10] and a slightly smaller instance with the same gap together with Dempe in 2008 [11].

The MIRUP (modified IRUP) conjecture [17] states that $\Delta(E) < 2$ for all CSP instances E, but it is still open. More investigations about non-IRUP instances can be found in [7,8,14].

The main idea of our paper is to connect our algorithm for enumeration of instances published in [15] together with ideas of Rietz and Dempe [11] in aim to construct CSP instances with the gap larger than currently known.

The paper has the following structure. In Sect. 2, we describe the construction of Rietz and Dempe, in Sect. 3, we describe our enumeration algorithm. In Sect. 4, we present the computational results and, finally, we draw a conclusion in Sect. 5.

2 Preliminaries

The construction principles of Rietz and Dempe are based on the instance

$$E_0(p, q) = (33 + 3p + q, (21 + p + q, 19 + p + q, 15 + p + q, 10 + p, 9 + p, 7 + p, 6 + p, 4 + p)^\top, b_0),$$

where p and q are positive integers, $b_0 = (1,1,1,1,1,2,1,1)^\top$, and the following theorem:

Theorem 1 (Rietz and Dempe). *Consider an instance $E = (L, l, b)$ of CSP with the following properties: $l_1 > l_2 > \ldots > l_{m-1} > 2l_m$ and $l_m \leq L/4$. Moreover, assume that this instance is sensitive, i.e. its optimal function value increases if b_m is increased by 1. Then, there are integers p and q such that instance $E' = E \oplus E_0(p, q)$ has gap $\Delta(E') = 1 + \Delta(E)$.*

Here $\oplus$ means a composition of instances. Let $E_1 = (L_1, l_1, b_1)$ and $E_2 = (L_2, l_2, b_2)$ denote two instances of CSP having n_1 and n_2 pieces respectively and with $L_1 = L_2$. The composed instance $E := E_1 \oplus E_2$ of CSP consists of the task of cutting all the $n_1 + n_2$ pieces of lengths from the both vectors l_1 and l_2 and with demands according to both vectors b_1 and b_2. In case when L_1 and L_2 are different, they can be multiplied by one common multiplier (together with piece lengths) to adjust the stock material lengths of both instances. For example, the instances $(2, (1)^\top, (1)^\top)$ and $(5, (2)^\top, (2)^\top)$ can be composed into the new instance $(2, (1)^\top, (1)^\top) \oplus (5, (2)^\top, (2)^\top) = (10, (5, 4)^\top, (1, 2)^\top)$.

Note that $b_m = 0$ is possible in Theorem 1, this means that the maximal possible trimloss in a cutting pattern used in an optimal solution is smaller than half of the length of the shortest piece.

Searching for sensitive instances with properties described in Theorem 1 is a very difficult task. An example of a suitable instance mentioned by Rietz and Dempe in their paper is the following:

$$E_{ST'} = (132, (44, 33, 12)^\top, (2, 3, 5)^\top).$$

Indeed, this instance is sensitive, because its optimal function value $z(E_{ST'}) = 2$ increases to 3 when we insert an additional piece of length 12. Also, $l_1 > l_2 > 2l_3$ and $l_3 < L/4$. $\Delta(E_{ST'}) = 17/132$, so by Theorem 1 there are integers p and q such that $\Delta(E_0(p, q) \oplus E_{ST'}) = 149/132 \approx 1.128787$. Namely, the instance $E_1 = E_0(p, q) \oplus E_{ST'}$ for $p = 74$ and $q = 669$ is the following:

$$E_1 = (924, (764, 762, 758, 308, 231, 84, 83, 81, 80, 78)^\top, (1, 1, 1, 2, 3, 6, 1, 2, 1, 1)^\top).$$

3 Enumeration Algorithm

Consider an instance $E = (L, l, b)$. If L and l are fixed, then the matrix of patterns $A(P^f(L, l))$ is fixed too. We will consider vector b as a vector of variables. Setting $l = (L - l_m, L - l_m - 1, \ldots, 2l_m + 2, 2l_m + 1, l_m)$, where $l_m \leq L/4$, we ensure that the most of required properties of Theorem 1 are satisfied, and now we have to ensure that E is sensitive.

We will enumerate all sensitive instances with a fixed objective function value. Namely, let $S_k(L, l)$ be the set of all patterns b such that $z((L, l, b)) = k$ and b corresponds to a sensitive instance (L, l, b).

Consider the set of *inextensible* feasible patterns $P^f_*(E) = \{a \in \mathbb{Z}^m_+ \mid a^\top l \leq L \ \wedge \ a^\top l + l_1 > L\}$. Obviously, $S_0(L, l) = \{\mathbf{0}\}$, and $S_1(L, l) = P^f_*(L, l)$. Now we will build the set $S_{i+1}(L, l)$ from $S_i(L, l)$ by adding vectors from $P^f_*(E)$ and considering only those patterns which lead to sensitive instances.

To transform the set $S_i(L, l)$ into the set $S_{i+1}(L, l)$ we need a data structure called a "map", which contains a set of pairs <key, value> (all keys are pairwise distinct) and allows us to make the following operations: insert a pair, find a value by a key (or determine that there is no pair with this key), modify a value by a key and return the list of all pairs. The algorithm is the following:

create an empty map A
for all $s \in S_i(L, l)$
    for all $a \in P^f_*(L, l)$
        $x \leftarrow (s_1 + a_1, \ldots, s_{m-1} + a_{m-1})$
        $y \leftarrow s_m + a_m$
        if A has no key x, then
            insert into A the pair (x, y)
        else $A[x] \leftarrow \max(A[x], y)$
$S_{i+1}(L, l) = \{(x_1, \ldots, x_{m-1}, y) \mid (x, y) \in A\}$

To find a sensitive instance with maximum gap with fixed L, l and k we generate $S_k(L, l)$ and then simply calculate $\Delta(E)$ over all $E = (L, l, s)$, $s \in S_k(L, l)$.

4 Results

We implemented our algorithm as a C++ program using CPLEX 12.7. The program was run on an Intel Core i7-5820K 4.2 GHz machine with 6 cores and 32 Gb RAM.

Results for the runs where $l = (L - l_m, L - l_m - 1, \ldots, 2l_m + 1, l_m)$ are presented in Table 1 and Table 2. Maximum gaps greater than 0.1 are marked in bold, and the maximal gap in every column is underlined.

Several sensitive instances with large gaps found during the search are presented in Table 3. Here E_1, E_2 and E_3 correspond to some maximum gaps presented in Table 1 and Table 2. For instance E_4 we continued the search up to $L = 250$ setting $l = (\lfloor L/2 \rfloor, \lfloor L/2 \rfloor - 1, \ldots, 2l_m + 1, l_m)$. The gap 0.1875 is the maximal over all considered instances with $k \leq 4$.

The instance E_5 is built from E_4 and a non-IRUP instance

$$E_T(t) = (3t, (t + 4, t + 3, t, t - 2, t - 6)^\top, (1, 1, 2, 1, 1)^\top)$$

for some integer t. E_6 is a combination of E_4 and some pieces from two copies of $E_T(t)$ with different values of t.

Using Theorem 1, we constructed a series of non-IRUP instances $E'_1, \ldots, E'_6$ from the sensitive instances $E_1, \ldots, E_6$. They are presented in Table 4. In Table 5 we compare our instances with the previously known ones considering the number of piece types m.

Table 1. Maximum gaps for sensitive instances with fixed L, l_m and $k \leq 4$

$L \backslash l_m$	2	3	4	5	6	7
8	0.000000					
9	0.000000					
10	0.100000					
11	0.000000					
12	0.083333	0.000000				
13	0.000000	0.000000				
14	0.071429	0.000000				
15	0.083333	0.100000				
16	0.062500	0.100000	0.000000			
17	0.058824	0.083333	0.000000			
18	0.100000	0.083333	0.000000			
19	0.075000	0.083333	0.000000			
20	0.068182	0.071429	0.100000	0.000000		
21	0.066667	**0.119048**	0.100000	0.000000		
22	0.078947	0.100000	0.100000	0.000000		
23	0.066667	0.093750	0.083333	0.000000		
24	0.083333	**0.129630**	0.083333	0.000000	0.000000	
25	0.060606	0.100000	0.083333	0.100000	0.000000	
26	0.078125	0.083333	0.083333	0.100000	0.000000	
27	0.069444	**0.111111**	**0.119048**	0.100000	0.000000	
28	0.071429	0.100000	**0.119048**	0.100000	0.000000	0.000000
29	0.064815	0.087500	**0.113636**	0.083333	0.000000	0.000000
30	0.076389	**0.125000**	**0.145833**	0.083333	0.100000	0.000000
31		0.097222	**0.129630**	0.083333	0.100000	0.000000
32		0.100000	**0.127907**	0.083333	0.100000	0.000000
33		**0.102564**	**0.106061**	**0.119048**	0.100000	0.000000
34		0.096154	**0.129630**	**0.119048**	0.100000	0.000000
35		0.092857	**0.111111**	**0.125000**	0.083333	0.100000
36		**0.106061**	**0.133333**	**0.138889**	0.083333	0.100000
37			**0.105263**	**0.145833**	0.083333	0.100000
38			**0.125000**	**0.131579**	0.083333	0.100000
39			**0.128788**	**0.153333**	**0.119048**	0.100000
40			**0.130435**	**0.138889**	**0.119048**	0.100000
41			**0.105263**	**0.136364**	**0.125000**	0.083333
42			**0.125000**	**0.136364**	**0.142857**	0.083333
43				**0.133333**	**0.138889**	0.083333
44				**0.136364**	**0.156250**	0.083333
45				**0.130952**	**0.161458**	**0.119048**
46				**0.133333**	**0.149123**	**0.119048**
47				**0.136364**	**0.144068**	**0.125000**
48				**0.136364**	**0.156863**	**0.142857**
49					**0.136364**	**0.142857**
50					**0.148148**	**0.140000**
51					**0.141026**	**0.166667**

Table 2. Maximum gaps for sensitive instances with fixed L, l_m and $k \leq 4$

L	$l_m = 7$	L	$l_m = 8$	L	$l_m = 9$	L	$l_m = 10$
45	**0.119048**	51	**0.119048**	57	**0.119048**	63	**0.119048**
46	**0.119048**	52	**0.119048**	58	**0.119048**	64	**0.119048**
47	**0.125000**	53	**0.125000**	59	**0.125000**	65	**0.125000**
48	**0.142857**	54	**0.142857**	60	**0.142857**	66	**0.142857**
49	**0.142857**	55	**0.142857**	61	**0.142857**	67	**0.142857**
50	**0.140000**	56	**0.142857**	62	**0.142857**	68	**0.142857**
51	**0.166667**	57	**0.149123**	63	**0.150794**	69	**0.150794**
52	**0.150000**	58	**0.171875**	64	**0.149123**	70	**0.150794**
53	**0.160000**	59	**0.167969**	65	**0.175000**	71	**0.149123**
54	**0.154762**	60	**0.166667**	66	**0.166667**	72	**0.177083**
55	**0.151515**	61	**0.153333**	67	**0.171875**	73	**0.171875**
56	**0.145833**	62	**0.161765**	68	**0.160000**	74	**0.175000**
57	**0.166667**	63	**0.166667**	69	**0.172043**	75	**0.166667**
58	**0.156863**	64	**0.161765**	70	**0.166667**	76	**0.171875**

Table 3. Sensitive instances with required properties and large gaps

E_i	$z(E_i)$	$\Delta(E_i)$	
$E_1 = (30, (14, 13, 10, 4)^\top, (1, 1, 2, 2)^\top)$	2	7/48	0.145833
$E_2 = (51, (23, 22, 19, 17, 16, 7)^\top, (2, 1, 1, 1, 1, 3)^\top)$	3	1/6	0.166667
$E_3 = (72, (32, 31, 28, 25, 24, 22, 10)^\top, (2, 1, 1, 1, 2, 2, 3)^\top)$	4	17/96	0.177083
$E_4 = (183, (81, 79, 65, 64, 61, 59, 55, 25)^\top, (1, 1, 2, 1, 2, 1, 1, 4)^\top)$	4	3/16	0.187500
$E_5 = (1281, (567, 553, 455, 448, 430, 427, 425, 413, 385, 175)^\top, (1, 1, 2, 1, 2, 1, 1, 2, 1, 4)^\top)$	5	19/96	0.197917
$E_6 = (1281, (567, 553, 455, 448, 431, 430, 427, 425, 421, 413, 385, 175)^\top, (1, 1, 2, 1, 2, 1, 2, 1, 1, 2, 1, 4)^\top)$	6	13/64	0.203125

Table 4. Non-IRUP instances with large gaps

$E'_i = E_0(p, q) \oplus E_i$	$z(E'_i)$	$\Delta(E'_i)$	
$E'_1 = (300, (228, 226, 222, 140, 130, 100, 40, 39, 37, 36, 34)^\top, (1, 1, 1, 1, 1, 2, 3, 1, 2, 1, 1)^\top)$	6	55/48	1.145833
$E'_2 = (510, (378, 376, 372, 230, 220, 190, 170, 160, 70, 69, 67, 66, 64)^\top, (1, 1, 1, 2, 1, 1, 1, 1, 4, 1, 2, 1, 1)^\top)$	7	7/6	1.166667
$E'_3 = (720, (528, 526, 522, 320, 310, 280, 250, 240, 220, 100, 99, 97, 96, 94)^\top, (1, 1, 1, 2, 1, 1, 1, 2, 2, 4, 1, 2, 1, 1)^\top)$	8	113/96	1.177083
$E'_4 = (1830, (1338, 1336, 1332, 810, 790, 650, 640, 610, 590, 550, 250, 249, 247, 246, 244)^\top, (1, 1, 1, 1, 1, 2, 1, 2, 1, 1, 5, 1, 2, 1, 1)^\top)$	8	19/16	1.187500
$E'_5 = (12810, (9318, 9316, 9312, 5670, 5530, 4550, 4480, 4300, 4270, 4250, 4130, 3850, 1750, 1749, 1747, 1746, 1744)^\top, (1, 1, 1, 1, 1, 2, 1, 2, 1, 1, 2, 1, 5, 1, 2, 1, 1)^\top)$	9	115/96	1.197917
$E'_6 = (12810, (9318, 9316, 9312, 5670, 5530, 4550, 4480, 4310, 4300, 4270, 4250, 4210, 4130, 3850, 1750, 1749, 1747, 1746, 1744)^\top, (1, 1, 1, 1, 1, 2, 1, 2, 1, 2, 1, 1, 2, 1, 5, 1, 2, 1, 1)^\top)$	10	77/64	1.203125

Table 5. The number of piece types in old and new non-IRUP instances

m	Old		New	
3	137/132	1.0378787		
4				
5	16/15	1.0666667		
6	38/35	1.0857143		
7	11/10	1.1000000		
8	10/9	1.1111111		
9				
10	149/132	1.1287879		
11			55/48	1.1458333
12				
13			7/6	1.1666667
14	51/44	1.1590909	113/96	1.1770833
15			19/16	1.1875000
16	7/6	1.1666667		
17			115/96	1.1979167
18	13/11	1.1818182		
19			77/64	1.2031250
⋮				
28	6/5	1.2000000		

5 Conclusion

We have combined the construction of Rietz and Dempe and our enumeration algorithm for searching for sensitive instances. We have found a lot of sensitive instances with large gaps. This allowed us to construct a lot of non-IRUP instances with gap, say, greater than 1.17. We also constructed a non-IRUP instance with gap 1.203125 which is greater than the previously known world record 1.2. Also the non-IRUP instances with large gaps that we found are smaller than the previously known ones.

Producing instances with large gaps using our search method requires a lot of computational resources, so we do not expect that it will handle the MIRUP conjecture directly. But the instances we found may provide the hints about improved constructions. In the future research we are going to improve our technique of combining instances (using which we produced E_5 and E_6) and construct new instances with much larger gaps.

Acknowledgements. The authors would like to thank the anonymous referees for their valuable remarks.

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