On Contractual Approach in Competitive Economies with Constrained Coalitional Structures

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Abstract. We establish a theorem that equilibria in an exchange economy can be described as allocations that are stable under the possibilities: (i) agents can partially and asymmetrically break current contracts, after that (ii) a new mutually beneficial contract can be concluded in a coalition of a size not more than 1 plus the maximum number of products that are not indifferent to the coalition members.

The presented result generalizes previous ones on a Pareto improvement in an exchange economy with l commodities that requires the active participation of no more than $l + 1$ traders. This concerned with Pareto optimal allocations, but we also describe equilibria. Thus according to the contractual approach to arrive at equilibrium only coalitions of constrained size can be applied that essentially raise the confidence of contractual modeling.

Keywords: Contractual economies · Coalitions of constrained size · Competitive equilibrium · Fuzzy contractual allocations

1 Introduction

I started to develop the theory of formal contractual economic interaction in the early 2000s and began to apply elaborated methods to the models of different types: Arrow–Debreu economies, incomplete markets, an economy with public goods, etc., see [1–4]. In the course of this activity, several specific characterizations of economic equilibria of different types were developed, but in all of them, the key feature was the admission of contract breakings—complete or partial. The idea of the barter exchange (contract) is by no means new in theoretical economics and seemingly goes back to classical Edgeworth results, but it usually appeared as an interpretation, in the form of net trades in a formal model. In the

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simplest version of the pure exchange economy, a barter contract is represented as a vector of acceptable exchanges of commodities among economic agents. A partial break involves the execution of the contract in an incomplete volume. Besides, in [1] there was proposed the notion of fuzzy contractual allocation and it became clear soon that this is the most meaningful concept among other methods of the contractual interaction. Fuzzy contractual interaction means that agents are able to break contracts partially and asymmetrically, i.e., it is admitted different agents can break contracts in a different amount. There was stated that under very weak assumptions in convex economy equilibria coincide with fuzzy contractual allocations. Nevertheless, the achieved results still are not satisfactory from the modeling point of view, because they assume the existence of agreements in many unrealistic coalitions between agents living at great distances, etc. This paper aims to fill this gap.

In this paper we consider a possibility to restrict the number of participants in the exchange transactions. We show that certain constrains of this type can be used without prejudice to its equilibrium properties of the final allocation. The idea goes back to [5–8], where it was found that Pareto optimal allocation can be achieved via mutually beneficial exchanges carried out in coalitions limited by the dimension of the commodity space, see also [9,10]. In these works, the contractual approach itself was not developed and the possibilities of individuals to break contracts were not considered. As a result, the obtained characterization does not appeal to Walrasian equilibria. Doing the admission of partially breaking of current contracts, we also take into account the fact that an agent may not be interested in absolutely all existing products. We show that the analysis can be reduced to an effective products' area of lower dimensionality—by eliminating products that are not of interest to the contracting parties. As a result, a coalition has a specific product space which dimension can be applied to restrict the size of coalitions. We will see that such restrictions on the size of coalitions do not prevent so-called fuzzy contractual allocations to be Walrasian equilibria.

The paper is organized as follows. In the second section, I present a contractual economic model and formulate some preliminary results that are the basis for the subsequent considerations. In the third one, I present the main result: new theorems on characterization of equilibria and other contractual allocations implemented via contracts of limited number of participants.

2 Contractual Exchange Economy

We consider a typical exchange economy in which $L = \mathbb{R}^l$ denotes the (finite dimensional) *space of commodities* (l is a number of commodities). Let $\mathcal{I} = \{1, \ldots, n\}$ be a set of agents (traders or consumers). A consumer $i \in \mathcal{I}$ is characterized by a consumption set $X_i \subset L$, an initial endowment $\mathbf{e}_i \in L$, and a preference relation described by a point-to-set mapping $\mathcal{P}_i : X_i \rightrightarrows X_i$ where $\mathcal{P}_i(x_i)$ denotes the set of all consumption bundles strictly preferred by the i-th agent to the bundle x_i. The notation $y_i \succ_i x_i$ is equivalent to $y_i \in \mathcal{P}_i(x_i)$.

So, the pure exchange model may be represented as a triplet

$$\mathcal{E} = \langle \mathcal{I}, L, (X_i, \mathcal{P}_i, \mathbf{e}_i)_{i \in \mathcal{I}} \rangle.$$

Let us denote by $\mathbf{e} = (\mathbf{e}_i)_{i \in \mathcal{I}}$ the vector of initial endowments of all traders of the economy. Denote $X = \prod_{i \in \mathcal{I}} X_i$ and let

$$\mathcal{A}(X) = \{ x \in X \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i \}$$

be the set of all *feasible allocations*. Everywhere below we assume that the model under study satisfies the following assumption.

(**A**) *For each $i \in \mathcal{I}$, X_i is a convex solid*[1] *closed set, $\mathbf{e}_i \in X_i$ and for every $x_i \in X_i$ there exists an open convex $G_i \subset L$ such that $\mathcal{P}_i(x_i) = G_i \cap X_i$ and if $\mathcal{P}_i(x_i) \neq \emptyset$ then $x_i \in \overline{\mathcal{P}}_i(x_i) \setminus \mathcal{P}_i(x_i)$.*[2]

Notice that due to (**A**) preferences may be satiated, i.e., $\mathcal{P}_i(x_i) = \emptyset$ is possible for some agent i and $x_i \in X_i$. However if $\mathcal{P}_i(x_i) \neq \emptyset$, then preference is *locally non-satiated* at the point x_i and this implies $\lambda(\mathcal{P}_i(x_i) - x_i) \subseteq \mathcal{P}_i(x_i) - x_i$ $\forall \lambda \in (0, 1]$. Next I recall some standard definitions and notions.

A pair (x, p) is said to be a *quasi-equilibrium* of $\mathcal{E}$ if $x \in \mathcal{A}(X)$ and there exists a linear functional $p \neq 0$ onto L such that

$$\langle p, \mathcal{P}_i(x_i) \rangle \geq px_i = p\mathbf{e}_i, \quad \forall\, i \in \mathcal{I}.$$

A quasi-equilibrium such that $x'_i \in \mathcal{P}_i(x_i)$ actually implies $px'_i > px_i$ is a *Walrasian or competitive equilibrium.*

An allocation $x \in \mathcal{A}(X)$ is said to be dominated (blocked) by a nonempty coalition $S \subseteq \mathcal{I}$ if there exists $y^S \in \prod_{i \in S} X_i$ such that $\sum_{i \in S} y_i^S = \sum_{i \in S} \mathbf{e}_i$ and $y_i^S \in \mathcal{P}_i(x_i)$ $\forall\, i \in S$.

The *core* of $\mathcal{E}$, denoted by $\mathcal{C}(\mathcal{E})$, is the set of all $x \in \mathcal{A}(X)$ that are blocked by no (nonempty) coalition.

Weak Pareto boundary for $\mathcal{E}$, denoted by $\mathcal{PB}^w(\mathcal{E})$, is the set of all $x \in \mathcal{A}(X)$ that cannot be dominated by the coalition $\mathcal{I}$ of all agents.

An allocation $x \in \mathcal{A}(X)$ is called *individual rational* if it cannot be dominated by singleton coalitions. $\mathcal{IR}(\mathcal{E})$ denotes the set of all these allocations.

Let $\mathfrak{L} = L^{\mathcal{I}}$ denote the space of all allocations of the economy $\mathcal{E}$. In the framework of model $\mathcal{E}$, we are going to introduce and study a formal mechanism of contractual interaction. This mechanism reflects the idea that any group of agents can find and realize some (permissible) within-the-group exchanges of commodities, referred to as contracts. The mechanism defines rules of contracting.

[1] Here "solid" is equivalent to "having nonempty interior."

[2] The symbol $\overline{A}$ denotes the closure of A and $\setminus$ is set for the set-theoretical difference.

By the formal definition, any reallocation of commodities $v = (v_i)_{i\in\mathcal{I}} \in \mathfrak{L}$, i.e., any vector $v \in \mathfrak{L}$ satisfying $\sum_{i\in\mathcal{I}} v_i = 0$, is called a *contract.*

Not every kind of possible reallocation may be realized in the economy; there are some institutional, physical, and behavioral restrictions in the economic models of different types. This is why we equip the abstract contractual economy model with a new element, the set of *permissible* contracts $\mathcal{W} \subset \mathfrak{L}$. Thus, the contractual (exchange) economy under study may be shortly represented by the 4-tuple

$$\mathcal{E}^c = \langle \mathcal{I}, L, \mathcal{W}, (X_i, \mathcal{P}_i, \mathbf{e}_i)_{i\in\mathcal{I}} \rangle.$$

For a contractual economy we study the sets of contracts which represent *feasible* allocations and introduce the operation of breaking a part of a given set of contracts. This motivates the following definition.

A finite collection V of permissible contracts is called *a web of contracts* iff

$$x_{\mathbf{e}}(U) = \mathbf{e} + \sum_{v\in U} v \in X, \quad \forall U \subseteq V.$$

So V being a web means that $\forall U \subseteq V$ its generated allocation $x_{\mathbf{e}}(U)$ is feasible one. Clearly, this notion can be considered with respect to any another allocation $y \in \mathcal{A}(X)$ chosen instead of $\mathbf{e}$. Note that $V = \emptyset$ is a web relative to every $y \in \mathcal{A}(X)$ (by convention $\sum_{v\in\emptyset} v = 0$).

Now we introduce the breaking operation of existing contracts and the signing of new ones. For any contract $v \in V$, let us set

$$S(v) = \mathrm{supp}\,(v) = \{i \in \mathcal{I} \mid v_i \neq 0\},$$

the support of the contract v. It is assumed that any contract $v \in V$ may be *broken* by any trader in $S(v)$, since he/she may not keep his/her contractual obligations. Also a non-empty group (coalition) of consumers can *sign* any number of new contracts. Being applied jointly, i.e., as a simultaneous procedure, these operations allow coalition $T \subseteq \mathcal{I}$ to yield new webs of contracts. The set of all such webs is denoted by $F(V, T)$.

Notice also that due to the definition of a web of contracts, a coalition can break any subset of contracts of a given web.[3]

Further, for the webs of contracts the notion of domination via a coalition is introduced that allows to consider different forms of web stabilities. This property, being written as $U \underset{T}{\succ} V$ (U dominates V via coalition T), means that

(i) $U \in F(V, T)$,
(ii) $x_i(U) \underset{i}{\succ} x_i(V) \quad$ for all $\quad i \in T$.

Definition 1. *A web of contracts V is called stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$.*

An allocation x is called ***contractual*** *if $x = x(V)$ for a stable web V.*

[3] Otherwise, it would occur that an allocation realized via breaking contracts is not feasible.

The property that a web of contracts is stable may be relaxed as well as strengthened. The most important possibilities are described below.

Definition 2. *A web of contracts V is called:*

(i) ***lower*** *stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$ and $U \subset V$;*
(ii) ***upper*** *stable if there is no web U and no coalition $T \subseteq \mathcal{I}$, $T \neq \emptyset$ such that $U \underset{T}{\succ} V$ and $V \subset U$.*
(iii) An upper and lower stable web of contracts V is called ***weakly*** *stable.*

An allocation x is called lower, upper, or weakly contractual if $x = x(V)$ for some lower, upper, or weakly stable web V, respectively.

The next possibility to strengthen contractual stability is to allow agents to break contracts partially. Partial breaking of the contract $v = (v_i)_{i\in\mathcal{I}}$ in the amount of $\alpha \in [0, 1]$ means that contract v is replaced by the contract $(1 - \alpha)v$. System (web) of contracts is called *proper* if no one is interested in the partial break off contracts: for each agent partial break (potentially different for different contracts) does not lead to the increase of utility. Only the proper web of contracts can be long-lived. Clearly, to admit agents apply partial breaking we have to assume the set $\mathcal{W}$ is a *star-shaped* at zero in $\mathfrak{L}$, i.e.,

$$v \in \mathcal{W} \;\Rightarrow\; \lambda v \in \mathcal{W}, \quad \forall\, 0 \leq \lambda \leq 1.$$

Allocation $x(V) = \mathbf{e} + \sum_{v\in V} v$, implemented by the web of contracts V is called *properly contractual* if the partial breaking of contracts is allowed to dominate and V is proper one.

One more notion is quite important in our analysis, it is the concept of *fuzzy contractual* allocation. To present it in a simplest way let us assume that the web consists on the only contract, i.e., $V = \{v\}$. So one has a feasible allocation to which the gross contract $x - \mathbf{e} = v = (v_i)_{i\in\mathcal{I}}$ (net trade) corresponds. It is assumed that the agents of the economy can (fuzzy and asymmetrically) break contract $v = (v_i)_{i\in\mathcal{I}}$, decreasing the individual consumption (fragment) from this contract in shares $(1 - t_i)_{i\in\mathcal{I}}$, $t_i \in [0, 1]$ forming a tuple[4]

$$v^t = (t_1 v_1, t_2 v_2, \ldots, t_n v_n)$$

of commodity bundles, which can be used in subsequent exchange transactions together with the initial endowments. After the conclusion of a new contract $w^S = (w_i)_{\mathcal{I}} \in L^{\mathcal{I}}$, $\sum_{\mathcal{I}} w_i = 0$ by a coalition $S \subseteq \mathcal{I}$ ($i \notin S \Rightarrow w_i = 0$) they yield (possibly unfeasible!) "allocation"

$$\xi(t, v, w) = w + v^t + \mathbf{e} = (w_1 + t_1 v_1^t + \mathbf{e}_1, \ldots, w_n + t_n v_n^t + \mathbf{e}_n).$$

[4] This is not a contract, because its key property $\sum_{\mathcal{I}} t_i v_i = 0$ is violated.

Definition 3. *An allocation $x \in \mathcal{A}(X)$ is called* ***fuzzy contractual*** *if for every $t = (t_i)_{i\in\mathcal{I}}$, $0 \le t_i \le 1$, $\forall i \in \mathcal{I}$ and for $x - \mathbf{e} = v$ there is no barter contract $w = (w_1, \dots, w_n) \in L^{\mathcal{I}}$, $\sum_{\mathcal{I}} w_i = 0$, such that*

$$\xi_i = \xi_i(t, v, w) = w_i + t_i v_i + \mathbf{e}_i, \quad i \in \mathcal{I} \tag{1}$$

$$\xi_i \succ_i x_i \quad \forall i: \ \xi_i \neq x_i. \tag{2}$$

Note that by virtue of (2) $w = 0$ is permissible, i.e. only partial breaking of contracts is possible. Denying the possibility of such domination means that the web of contracts is *proper* and the allocation is *stable* with respect to asymmetric *partial break* of contracts.

Depending on the structure of permissible contracts, specified as a new element $\mathcal{W} \subset L^{\mathcal{I}}$ of the model, one can describe well known economic theoretical notions in terms of a stable web of contracts. In a standard exchange model (every contract is permissible) they are the core (contractual allocations, only full break off contracts), competitive equilibria (admission of partial break), the Pareto frontier (upper contractual allocations), etc. The most interesting among others is the presentation of competitive equilibrium as a fuzzy contractual allocation, described in the following technical lemma and proposition.

The following characteristic lemma can be directly produced from Definition 3.

Lemma 1. *Suppose $\mathcal{W} = L^{\mathcal{I}}$. Then an allocation $x \in \mathcal{A}(X)$ is fuzzy contractual if and only if*[5]

$$\mathcal{P}_i(x_i) \cap [x_i, \mathbf{e}_i] = \emptyset \quad \forall i \in \mathcal{I} \tag{3}$$

and

$$\prod_{\mathcal{I}} [(\mathcal{P}_i(x_i) + [0, \mathbf{e}_i - x_i]) \cup \{\mathbf{e}_i\}] \bigcap \{(z_i)_{\mathcal{I}} \in L^{\mathcal{I}} \mid \sum_{i\in\mathcal{I}} z_i = \sum_{i\in\mathcal{I}} \mathbf{e}_i\} = \{\mathbf{e}\}. \tag{4}$$

Here condition (3) indicates that a partial break off contracts without signing of a new one cannot be beneficial. The requirement (4) denies the existence of a dominating coalition after the partial asymmetric break of the contract $v = (x - \mathbf{e})$. Now applying separation theorem one can easily state (see [1] for details) the following

Proposition 1. *Every equilibrium is a fuzzy contractual allocation and vice versa: any non-satiated fuzzy contractual allocation is a nontrivial quasi-equilibrium.*

So, if the model is such that every nontrivial quasi-equilibrium is an equilibrium[6] then the notion of competitive equilibrium and fuzzy contractual allocation is equivalent. This and similar statements from [1–4] allow us to state that our

[5] *A linear* segment with ends $a, b \in L$ is the set $[a, b] = \mathrm{conv}\{a, b\} = \{\lambda a + (1 - \lambda)b \mid 0 \le \lambda \le 1\}$.

[6] Conditions, providing this fact are well known in the literature, e.g. it can be *irreducibility*.

contractual approach presents a model of perfect competition (simplest among others).

The sketch of the proof of Proposition 1. Separating sets in (4) by a (non-zero) linear functional $\pi = (p_1, \dots, p_n) \in L^{\mathcal{I}}$ one can conclude:

(i) $p_i = p_j = p \neq 0$ for each $i, j \in \mathcal{I}$; this is so because π is bounded on

$$\mathcal{A}(L^{\mathcal{I}}) = \{(z_1, \dots, z_n) \in L^{\mathcal{I}} \mid \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} \mathbf{e}_i\}.$$

So, one can take p as a price vector.

(ii) Due to construction and in view of preferences are locally non-satiated at the point $x \in \mathcal{A}(X)$ the points x_i and $\mathbf{e}_i$ belong to the closure of

$$\mathcal{P}_i(x) + \operatorname{conv}\{0, \mathbf{e}_i - x_i\}.$$

Therefore via separating property we have

$$\sum_{j \neq i} p\mathbf{e}_j + px_i \geq \sum_{\mathcal{I}} p\mathbf{e}_j \quad \Rightarrow \quad px_i \geq p\mathbf{e}_i \quad \forall i \in \mathcal{I},$$

that is possible only if $px_i = p\mathbf{e}_i$ $\forall i \in \mathcal{I}$. So, we obtain budget constrains for consumption bundles.

(iii) By separation property for each i we also have

$$\langle p, \mathcal{P}_i(x) + \operatorname{conv}\{0, \mathbf{e}_i - x_i\}\rangle \geq p\mathbf{e}_i,$$

that by (ii) implies $\langle p, \mathcal{P}_i(x)\rangle \geq px_i = p\mathbf{e}_i$. So we proved that p is quasi-equilibrium prices for allocation $x = (x_i)_{i \in \mathcal{I}}$.

As a result one can see that if an economic model is such that every quasi-equilibrium is equilibrium, then fuzzy contractual allocation is an equilibrium one. Conditions delivering this fact are well known in literature; for example, it is the case when an economy is irreducible. ■

3 Result

In a real economy, consumers may not be interested in all existing products, i.e., individuals may be indifferent to some products[7]. Excluding them from consideration, one can reduce the dimension of the actual product space for each agent. The exact definition is given below.

Definition 4. *A commodity j is* ***indifferent*** *for $i \in \mathcal{I}$ if $\forall x \in \mathcal{A}(X)$*[8]

$$\forall y_i = ((y_i)_{-j}, y_i^j) \in \mathcal{P}_i(x_i) \iff ((y_i)_{-j}, \mathbf{e}_i^j) \in \mathcal{P}_i((x_i)_{-j}, \mathbf{e}_i^j) = \mathcal{P}_i(x_i).$$

[7] For example, an ordinary consumer on the market is not interested in all kinds of spare parts, parts and structural elements (bolts, nuts, gears, transistors ...).

[8] Here we indirectly assume that all bundles we need belong to consumption set, i.e., $((y_i)_{-j}, \mathbf{e}_i^j)$, $((x_i)_{-j}, \mathbf{e}_i^j) \in X_i$; it is a specific constraint for X_i, $i \in \mathcal{I}$.

Here $y_i = (y_i^j)_{j=1,...,l} \in \mathbb{R}^l$ and $(y_i)_{-j} = (y_i^k)_{k \neq j, k=1,...,l}$ is a vector consisting of all components of y_i excluding y_i^j.

Two properties are postulated in this definition: a product j is indifferent to a given individual i, if in any consumption bundle $y_i = ((y_i)_{-j}, y_i^j) \in X_i$ his/her consumption can be replaced by the initial one (to nullify?), i.e., one goes to a bundle $((y_i)_{-j}, \mathbf{e}_i^j)$ such that $((y_i)_{-j}, \mathbf{e}_i^j) \in X_i$ and this does not lead to the change of consumption properties of $y_i \in L$. Clearly, for preferences specified via utility functions for indifferent commodity j we have $\forall y_i \in X_i \; u_i((y_i)_{-j}, y_i^j) = u_i((y_i)_{-j}, \mathbf{e}_i^j)$, $i \in \mathcal{I}$.

Let $L_i \subseteq L$ be the space of *non-indifferent* commodities (interesting) for individual i and let $L_S \subseteq L$ be a subspace of commodities that are interesting for the members of coalition $S \subseteq \mathcal{I}$:

$$L_S = \sum_{i \in S} L_i.$$

In this section, the notation z^S means the projection of the vector $z \in L$ onto the subspace $L_S \subseteq L$. Recall that for contracts $v \in \mathcal{W}$ there is defined $S(v) = \operatorname{supp}(v)$, this is the support of the contract. Given the possible indifference to some products, as a product space for a coalition $S(v)$, one can specify

$$L_{S(v)} = \sum_{i \in S(v)} L_i.$$

Now let us consider the following restriction for the set of all permissible contracts.

$$v = (v_i)_{i \in \mathcal{I}} \in \mathcal{W} \iff v_i \in L_{S(v)}, \; i \in S(v), \quad |S(v)| \leq \dim(L_{S(v)}) + 1. \quad (5)$$

This specification restricts the size of permissible contracting coalitions.

Remark 1. In the process of manufacturing high-tech products, a huge number of elements are used, the range of which can be counted in millions—for example, in modern aircraft construction. However, the final user needs the resulting product (the plane!), and not some of its components, bolts, nuts, ailerons, and other structural elements, the existence of which he may not know at all. However, this is important for service companies, etc. Production unions enter into contracts for the supply of the element base of the final product can be very large, but consumer unions can be much smaller—this fact can be concluded from Theorem 1 and Corollary 1 below. Formal examples also can be easily constructed. Indeed one can consider several exchange economies $\mathcal{E}_1$, ..., $\mathcal{E}_k$ having product ranges $S_1, \dots, S_k \subseteq \{1, 2, \dots, l\}$. Assume that utilities of individuals from $\mathcal{E}_\xi$ may depend of only commodities from S_ξ. One can consider extended commodity space $L = \mathbb{R}^l$ and formally extend these utilities to this space, supposing that they do not depend of new variables. Now we consider the united economy $\mathcal{E} = \bigcup_{\xi=1}^{k} \mathcal{E}_\xi$. The first result below describes Pareto frontier and says us that if coalition contains only individuals of one economy $\mathcal{E}_\xi$, then number of coalition members may be restricted by $card(S_\xi)$; for coalition of two individual kinds from $\mathcal{E}_{\xi_1}, \mathcal{E}_{\xi_2}$ the number of its members may be not more of $card(S_{\xi_1} \cup S_{\xi_2})$ and so on. Similar conclusion is done for equilibria. ■

Further we first discuss the concept of upper stable web and upper contractual allocation, see Definition 2. Now let us consider a slightly modified classical concept of Pareto optimality[9]. We call an allocation $x \in \mathcal{A}(X)$ *strictly* Pareto optimal iff

$$\nexists S \subseteq \mathcal{I} \ \&\ y^S \in \prod_{i\in S} X_i \ \textit{ such that } \ \sum_{i\in S} y_i^S = \sum_{i\in S} x_i \ \ \&\ \ y_i^S \in \mathcal{P}_i(x_i)\ \forall i \in S.$$

It is easy to see, that according to the definitions if there are no permissibility constrains for contracts, the notions of upper contractual and *strictly* Pareto optimal allocation are equivalent.

It is said that a vector (consumption bundle) $\kappa \in L$ is *extremely desirable* if for each $x_i \in X_i$ one has

$$x_i + \kappa \succ_i x_i, \quad i \in \mathcal{I}.$$

In the literature, it is standardly assumed that cumulative initial endowments $\sum_{i\in\mathcal{I}} \mathbf{e}_i = \bar{\mathbf{e}}$ presents an extremely desirable bundle.

Recall that binary relation $\succ$ is transitive iff

$$\forall x, y, z \in \mathrm{Dom}(\succ) \ \ x \succ y \succ z \ \Rightarrow \ x \succ z.$$

Theorem 1. *If $\mathcal{W}$ obeys (5) then every upper contractual allocation is strictly Pareto optimal. Moreover, if preferences of $\mathcal{E}$ are transitive and there is an extremely desirable bundle $\kappa \in L$, then (5) can be weakened and one can require*

$$v \in \mathcal{W} \quad \Longleftrightarrow \quad v_i \in L_{S(v)},\ i \in S(v) \ \ \&\ \ |S(v)| \le \dim(L_{S(v)}).$$

So, the Theorem states that the economic system can arrive at Pareto optimal commodity allocation via a contractual process with coalitions size constrained by (5). In further analysis, we apply the following

Theorem 2 (Carathéodory, 1907). *Let $A \subset L$ be a subset of a vector space L. If $\dim \mathrm{aff}(A) = d < \infty$, then any element $x \in \mathrm{conv}A$ can be presented as a convex hull of not more than $d+1$ elements of A.*

Proof of Theorem 1. Suppose that an upper contractual allocation $x \in \mathcal{A}(X)$ is not strictly Pareto optimal. Therefore, there exists a coalition $S \subset \mathcal{I}$ and contract $v = (v_i)_{i\in\mathcal{I}} \in \mathfrak{L} = L^{\mathcal{I}}$, $\mathrm{supp}\,(v) = S$ such that

$$\forall i \in S \ \ x_i + v_i \in \mathcal{P}_i(x). \tag{6}$$

[9] Under classical assumptions they are equivalent, but it is not so in general case.

Here by Definition 4 for each member of the coalition S, the components of v_i corresponding to indifferent products can be considered as zero, i.e., $v_i \in L_{S(v)}$ $\forall i \in \mathcal{I}$. Since x is upper contractual, then $v \notin \mathcal{W}$ (here $v_i = 0$, $i \in \mathcal{I} \setminus S$) and, therefore, $|S(v)| > \dim(L_{S(v)}) + 1$. Now we can assume that S is a coalition of minimal size among those having this property. We have $\frac{1}{|S|}\sum_{i\in S} v_i = 0$, $S = S(v)$. Using the Caratheodory theorem, one can find a coalition $T \subset S$ such that

$$\forall i \in T \ \ \exists \alpha_i \in (0,1]: \quad \sum_{i\in T} \alpha_i = 1, \quad \sum_{i\in T} \alpha_i v_i = 0 \ \ \& \ \ |T| \leq \dim(L_S) + 1.$$

Define $w_i = \alpha_i v_i \neq 0$, and think without loss of generality that $w_i \in L_T$, $i \in T$ (if necessary, one replaces some components with zeros). Now due to the main assumption (**A**) one has $\lambda(\mathcal{P}_i(x_i) - x_i) \subseteq \mathcal{P}_i(x_i) - x_i$ $\forall \lambda \in (0,1]$, that implies $x_i + w_i \in \mathcal{P}_i(x)$, $i \in T$. Since $\sum_{i\in T} w_i = 0$ and $|T| < |S|$, we come to a contradiction with the choice of S as a coalition of minimal size. Therefore, there are no such coalitions at all and x is a strictly Pareto optimal allocation.

In the second part of the statement of the Theorem, we again argue from the contrary and find a coalition $S \subset \mathcal{I}$ of minimal size and a contract $v \in \mathfrak{L}$, $\mathrm{supp}\,(v) = S$, $v_i \in L_S$, $i \in \mathcal{I}$ satisfying (6) and such that $|S| > \dim L_S$. Let us specify

$$\Gamma = \mathrm{conv}\{v_i \in L_S \mid i \in S\}.$$

By construction one has $\frac{1}{|S|}\sum_{i\in S} v_i = 0 \in \Gamma$. Next, we take an extremely desirable $\kappa \in L$, consider its projection κ^S onto L_S and find a real $\lambda \geq 0$ such that $-\lambda\kappa^S$ belongs to the face of (bounded) polyhedron Γ. This can be done from the condition

$$\lambda = \max\{\lambda' \mid -\lambda'\kappa^S \in \Gamma\}.$$

Since the dimension of any proper face is at most $\dim L_S - 1$, there is a coalition $T \subset S$ such that $|T| \leq \dim L_S$ and

$$\forall i \in T \ \ \exists \alpha_i \in (0,1]: \quad \sum_{i\in T} \alpha_i = 1, \quad \sum_{i\in T} \alpha_i v_i = -\lambda\kappa^S.$$

Next one defines $w_i = \alpha_i(v_i + \lambda\kappa^S)$, $i \in T$ and $w_i = 0$, $i \in \mathcal{I} \setminus T$. As a result one has:

$$x_i \prec_i x_i + \alpha_i v_i \prec_i x_i + \alpha_i v_i + \alpha_i \lambda \kappa^S = x_i + w_i, \quad i \in T,$$
$$\sum_{i\in T} w_i^S = \sum_{i\in T} \alpha_i v_i^S + (\sum_{i\in T} \alpha_i)\lambda\kappa^S = 0.$$

These relations indicate that $w = (w_i)_{i\in\mathcal{I}}$ is a mutually beneficial contract, the support of which is the coalition T, no larger than $\dim(L_S)$. Thus, we again have found the coalition that dominates the current allocation, and its size is strictly less than $|S|$, which is impossible. ∎

Let us turn now to the characterization of fuzzy contractual allocation, which represents the main result of the section.

Lemma 2. *Let x be a fuzzy contractual allocation and $\mathcal{W}$ obey (5). Then (4) is true:*

$$\prod_{i\in\mathcal{I}}[(\mathcal{P}_i(x_i)+\mathrm{co}\{0,\mathbf{e}_i-x_i\})\cup\{\mathbf{e}_i\}]\bigcap\mathcal{A}(L^{\mathcal{I}})=\{\mathbf{e}\}.$$

Now by virtue of the characterization presented in Proposition 1 we directly conclude

Corollary 1. *Let $\mathcal{W}$ obey (5). Then every non-satiated fuzzy contractual allocation is a quasi-equilibrium one.*

So, these Lemma and Corollary state that applying partial break and contracts specified in (5), a contractual process can arrive the economy to Walrasian equilibrium.

Proof of Lemma 2. Let x be a fuzzy contractual allocation, $\mathcal{W}$ obey (5) and conclusion of the Lemma be false. Let us consider the left part of intersection (4). Now we first show that there is no $y=(y_i)_{\mathcal{I}}\neq\mathbf{e}$ such that the coalition

$$T(y)=\{i\in\mathcal{I}\mid y_i\neq\mathbf{e}_i\}\neq\emptyset \tag{7}$$

satisfies $|T(y)|\leq\dim(L_T)+1$. Indeed, otherwise according to the construction one can find $z_i\in\mathcal{P}_i(x_i)$, $\alpha_i\in[0,1]$, $i\in T$ such that

$$y_i=z_i+\alpha_i(\mathbf{e}_i-x_i)\neq\mathbf{e}_i,\quad i\in T,\quad \sum_T y_i=\sum_T\mathbf{e}_i.$$

Applying now Definition 4, we may think that $z_i,x_i\in X_i\cap(L_T+\mathbf{e}_i)$ (for i and the bundle x_i one has to change indifferent components with his/her initial endowments and do not change all other). Now, specifying $v_i=(y_i-\mathbf{e}_i)\in L_T$, $i\in\mathcal{I}$, via construction and Definition 4 we obtain

$$z_i=v_i+\alpha_i(x_i-\mathbf{e}_i)+\mathbf{e}_i\succ_i x_i,\quad i\in T,\quad \sum_{i\in\mathcal{I}}v_i=0\ \ \&\ \ \mathrm{supp}\,(v)=T,$$

that contradicts Definition 3 and condition (5).

Thus, if the conclusion of the Lemma is false, then $|T(y)|>\dim(L_T)+1$ for each coalition specified by (7). But in the (finite) set of all such coalitions there is a coalition of minimal size, which we denote $S\subset\mathcal{I}$. Again, one can think $x_i\in X_i\cap(L_S+\mathbf{e}_i)$, $i\in S$. By construction there are $z_i\in\mathcal{P}_i(x_i)\cap(L_S+\mathbf{e}_i)$, $\alpha_i\in[0,1]$, $i\in S$ such that

$$y_i=z_i+\alpha_i(\mathbf{e}_i-x_i)\neq\mathbf{e}_i,\quad i\in S,\quad \sum_S y_i=\sum_S\mathbf{e}_i.$$

We have $\frac{1}{|S|}\sum_S(y_i-\mathbf{e}_i)=0$. Since by assumption $|S(y)|>\dim(L_S)+1$, then using Caratheodory theorem one concludes there exists $R\subset S$ and $\beta_i\in(0,1]$, $i\in R$ such that $|R|<|S|$ and

$$\sum_{i\in R}\beta_i(y_i-\mathbf{e}_i)=0,\quad \sum_{i\in R}\beta_i=1\quad\Rightarrow\quad \sum_{i\in R}(\beta_i z_i+\beta_i\alpha_i(\mathbf{e}_i-x_i)-\beta_i\mathbf{e}_i)=0.$$

Since $\lambda(\mathcal{P}_i(x_i) - x_i) \subseteq \mathcal{P}_i(x_i) - x_i \ \forall \lambda \in (0, 1]$, the terms on the left-hand side of the latter equality can be rewritten in the form

$$\beta_i(z_i - x_i) + \beta_i\alpha_i(\mathbf{e}_i - x_i) - \beta_i(\mathbf{e}_i - x_i) = \xi_i - x_i - \beta_i(1 - \alpha_i)(\mathbf{e}_i - x_i) = v_i$$

for some $\xi_i \succ_i x_i$, $i \in R$. By construction $\sum_{i\in R} v_i = 0$ and defining $y'_i = v_i + \mathbf{e}_i$, $i \in R$ and $y'_i = \mathbf{e}_i$ for $i \in \mathcal{I} \setminus R$ one obtains $\sum_{i\in\mathcal{I}} y'_i = \sum_{i\in\mathcal{I}} \mathbf{e}_i$ and

$$y'_i = \xi_i + (1 - \beta_i(1 - \alpha_i))(\mathbf{e}_i - x_i) \in \mathcal{P}_i(x_i) + \mathrm{co}\{0, \mathbf{e}_i - x_i\}, \quad i \in R.$$

Thus, we found y' such that under condition (7) we have $T(y') = R$, where $|R| < |S|$, which contradicts the minimality of $S \subset \mathcal{I}$. This contradiction completes the proof. ■

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