On the Proximity of the Optimal Values of the Multi-dimensional Knapsack Problem with and Without the Cardinality Constraint

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Abstract. We study the proximity of the optimal value of the m-dimensional knapsack problem to the optimal value of that problem with the additional restriction that only one type of items is allowed to include in the solution. We derive exact and asymptotic formulas for the precision of such approximation, i.e. for the infinum of the ratio of the optimal value for the objective functions of the problem with the cardinality constraint and without it. In particular, we prove that the precision tends to $0.59136\ldots/m$ if $n \to \infty$ and m is fixed. Also, we give the class of the worst multi-dimensional knapsack problems for which the bound is attained. Previously, similar results were known only for the case $m = 1$.

Keywords: Multi-dimensional knapsack problem · Approximate solution · Cardinality constraints

1 Introduction

In [1, 2, 4, 5] the proximity of the optimal value of the (one-dimensional) knapsack problem to the optimal value of the problem with the cardinality constraints was studied. The cardinality constraint is the additional restriction that only k type of items is allowed to include in the solution (i.e. that only k coordinates of the optimal solution vector can be non-zero). Different upper and lower bounds for the guaranteed precision, i.e. for the infinum of the ratio of the optimal value for the objective functions of the problem with the cardinality constraints and without them, were obtained. Also, in some cases the classes of worst problems were constructed.

   The importance of such kind of research is due to the fact that some algorithms for solving the knapsack problems require to find an optimal solution to that problem with the cardinality constraints; see, for example [4, 5], where this

approach is used for constructing greedy heuristics for the integer knapsack problem. Moreover, the results of research can be potentially useful for constructing new fully polynomial approximation schemes.

Here, from this point of view, we consider the m-dimensional knapsack problem. The solution to that problem with the additional constraint that only 1 coordinate can be non-zero is called the approximate solution. We derive exact and asymptotic formulas for the precision of such approximation. In particular, we prove that the precision tends to $0.59136\ldots/m$ if $n \to \infty$ and m is fixed. Also, we give a class of worst multi-dimensional knapsack problems for which the bound is attained.

2   Definitions

Denote by $\mathbb{Z}_+$, $\mathbb{R}_+$ the sets of all non-negative integer and real numbers respectively. Let

$$L(A, b) = \left\{ x \in \mathbb{Z}_+^n : \ Ax \le b \right\}, \qquad A = (a_{ij}) \in \mathbb{R}_+^{m \times n}, \qquad b = (b_i) \in \mathbb{R}_+^m.$$

The *integer m-dimensional knapsack problem* is to find x such that

$$cx \to \max \qquad \text{s.t. } x \in L(A, b), \tag{1}$$

where $c = (c_j) \in \mathbb{R}_+^n$ [3,6].

Denote by $v^{(j)}$ $(j = 1, 2, \ldots, n)$ a point in $L(A, b)$, all of whose coordinates $v_i^{(j)}$ are 0, except for of $v_j^{(j)}$, which is

$$v_j^{(j)} = \min_{i:\ a_{ij} > 0} \lfloor b_i/a_{ij} \rfloor.$$

It is not hard to see that $v^{(j)} \in L(A, b)$ and $cv^{(j)} = c_j v_j^{(j)}$. Denote $V(A, b) = \left\{ v^{(1)}, \ldots, v^{(n)} \right\}$. A point $v^{(j)}$, on which the maximum

$$\max_j cv^{(j)}$$

attained is called an *approximate solution* to the problem (1). The *precision of the approximate solution* is

$$\alpha(A, b, c) = \frac{\max\limits_{x \in V(A,b)} cx}{\max\limits_{x \in L(A,b)} cx}.$$

In this paper we study the value

$$\alpha_{mn} = \inf_{\substack{A \in \mathbb{R}_+^{m \times n} \\ b \in \mathbb{R}_+^m,\ c \in \mathbb{R}_+^n}} \alpha(A, b, c).$$

Table 1. Values of δ_n, ε_n and α_{1n} for small n

n	$\delta_n = \delta_{n-1}(\delta_{n-1}+1)$	$\varepsilon_n = 1 + \varepsilon_{n-1}(\delta_{n-1}+1)$	$\alpha_{1n} = \delta_n/\varepsilon_n$
1	1	1	1.000000000000000
2	2	3	0.666666666666667
3	6	10	0.600000000000000
4	42	71	0.591549295774648
5	1806	3054	0.591355599214145
6	3263442	5518579	0.591355492056923
7	10650056950806	18009568007498	0.591355492056890
8	113423713055421844361000442	191802924939285448393150887	0.591355492056890

3   Previous Work

The precision of the approximate solution to the 1-dimensional $(m = 1)$ knapsack problem was studied in [1,2,4,5]. In particular, in [2,4] it was proven that

$$\delta_n = \delta_{n-1}(\delta_{n-1}+1), \qquad \varepsilon_n = 1 + \varepsilon_{n-1}(\delta_{n-1}+1), \qquad \delta_1 = \varepsilon_1 = 1.$$

The sequence $\{\delta_n\}$ is the A007018 sequence in On-Line Encyclopedia of Integer Sequences (OEIS) [7]. The sequence $\{\epsilon_n\}$ is currently absent in OEIS.

The sequence $\alpha_{1n} = \delta_n/\varepsilon_n$ decreases monotonously and tends to the value $\alpha_{1\infty} = 0.591355492056890\ldots$ The values for δ_n, ε_n and α_{1n} for small n are presented in Table 1.

In [4,5] these results are used in constructing the approximate scheme for the integer knapsack problem. Note that α_{1n} is even higher than the guaranteed precision 0.5 of the greedy algorithm [6].

The infinum for α_{1n} is achieved on the problem (the worst case)

$$\sum_{j=1}^{n} \frac{x_j}{\delta_j} \to \max$$

s.t.

$$\sum_{j=1}^{n} \frac{x_j}{\delta_j + \mu_n} \le 1,$$

where $0 \le \mu_n < 1$ and $\sum_{j=1}^{n} \frac{1}{\delta_j + \mu_n} = 1$. In particular,

$$\mu_1 = 1, \quad \mu_2 = \frac{\sqrt{5}-1}{2} = 0.61803\ldots, \quad \mu_3 = 0.93923\ldots, \quad \mu_4 = 0.99855\ldots$$

The optimal solution vector to this problem is $(1, 1, \ldots, 1)$ and the optimal solution value is ε_n/δ_n, whereas the approximate solution vectors are

$$(1, 0, 0 \ldots, 0), \quad (0, \delta_2, 0, \ldots, 0), \quad (0, 0, \delta_3, \ldots, 0), \quad \ldots, \quad (0, 0, 0, \ldots, \delta_n)$$

and the corresponding value of the objective function is 1.

Lower and upper bounds for the guaranteed precision for $k \geq 2$ are obtained in [2].

In this paper we obtain formulas for α_{mn} for $m \geq 1$. In particular, we prove that $\alpha_{mn} \to \dfrac{\alpha_{1\infty}}{m}$ if $n \to \infty$ and m is fixed.

4   Preliminaries

Lemma 1. *For any fixed m the sequence $\{\alpha_{mn}\}$ decreases monotonously.*

Proof. Let $A \in \mathbb{R}_+^{m \times n}$, $h, b \in \mathbb{R}_+^m$, $c \in \mathbb{R}_+^n$ and $h > b$. Consider a matrix $A' = (A \mid h) \in \mathbb{R}_+^{m \times (n+1)}$ and a vector $c' = (c, 0) \in \mathbb{R}_+^{n+1}$. It is not hard to see that all points in $L(A', b)$ are obtained from the points in $L(A, b)$ by writing the zero component to the end. Hence $\alpha(A, b, c) = \alpha(A', b, c') \geq \alpha_{m,n+1}$. Due to the arbitrariness of A, b, c, we get $\alpha_{mn} \geq \alpha_{m,n+1}$.

Lemma 2. $\alpha(A, b, c) = \alpha(A', b, c)$ *for some $A' \leq A$, where each column of A' contains at least one non-zero element.*

Proof. Let for some s, t we have $a_{st} > 0$ and for all $i \neq s$

$$\left\lfloor \frac{b_s}{a_{st}} \right\rfloor \leq \left\lfloor \frac{b_i}{a_{it}} \right\rfloor$$

(if there are no such s, t, then put $A' = A$ and A' has the required form). From the matrix A we construct a matrix A' by setting $a'_{it} = 0$ for all $i \neq s$ and $a'_{ij} = a_{ij}$ otherwise.

For all $x \in \mathbb{R}_+^n$ we have $A'x \leq Ax$. Hence $L(A, b) \subseteq L(A', b)$. Hence

$$\max_{x \in L(A,b)} cx \leq \max_{x \in L'(A,b)} cx.$$

But

$$\min_{k:\ a_{kj} > 0} \left\lfloor \frac{b_k}{a_{kj}} \right\rfloor = \min_{k:\ a'_{kj} > 0} \left\lfloor \frac{b_k}{a'_{kj}} \right\rfloor \qquad (j = 1, 2, \ldots, n),$$

hence $V(A, b) = V(A', b)$. Now we have

$$\alpha(A, b, c) = \frac{\max\limits_{x \in V(A,b)} cx}{\max\limits_{x \in L(A,b)} cx} \geq \frac{\max\limits_{x \in V(A',b)} cx}{\max\limits_{x \in L(A',b)} cx} = \alpha(A', b, c).$$

To complete the proof we note that the procedure described above can be performed until the matrix A' acquires the required form.

From Lemma 2 it follows that to study α_{mn} it is enough to consider only multi-dimensional knapsack problems with constraints

$$\begin{cases} a_{11}x_1 + \ldots + a_{1l_1}x_{l_1} & \leq b_1, \\ \qquad\qquad a_{2,l_1+1}x_{l_1+1} + \ldots + a_{2,l_2}x_{l_2} & \leq b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \qquad\qquad\qquad\qquad\qquad a_{m,l_{m-1}+1}x_1 + \ldots + a_{mn}x_n \leq b_m, \end{cases}$$

that can be called a *direct product of m knapsack problems*. All inequalities $0 \leq b_i$ have to be deleted due to Lemma 1. Denote $n_i = l_i - l_{k-1}$, where $l_0 = 0$, $l_m = n$ $(i = 1, 2, \ldots, m)$. Thus, we have proved the following.

Lemma 3. *For each m, n the infimum α_{mn} is attained on the direct product of knapsack problems.*

5   The Main Result

The main result of the paper is formulated in the following theorem.

Theorem 1. *For each m, n*

$$\alpha_{mn} = \frac{\alpha_{1q}}{m + r\left(\dfrac{\alpha_{1q}}{\alpha_{1,q+1}} - 1\right)}, \tag{2}$$

where $n = qm + r$, $q = \lfloor n/m \rfloor$.

The theorem follows from two lemmas below.

Lemma 4. *For each m, n*

$$\alpha_{mn} \geq \frac{\alpha_{1q}}{m + r\left(\dfrac{\alpha_{1q}}{\alpha_{1,q+1}} - 1\right)}.$$

Proof. Thanks to Lemma 3, it is enough to consider only direct products of m knapsack problems. Let $\tau_i = \gamma_i/\beta_i$ be the precision of approximate solution to the i-th knapsack problem $(i = 1, 2, \ldots, m)$, where γ_i is the approximate solution value, β_i is the optimal solution value. For their product we have

$$\alpha(A, b, c) = \frac{\max\limits_{i=1,\ldots,m} \gamma_i}{\sum\limits_{i=1}^{m} \beta_i} = \frac{\gamma_s}{\sum\limits_{i=1}^{m} \beta_i} = \frac{1}{\sum\limits_{i=1}^{m} \dfrac{\beta_i}{\gamma_s}} = \frac{1}{\sum\limits_{i=1}^{m} \dfrac{\gamma_i}{\gamma_s \tau_i}} \geq \frac{1}{\sum\limits_{i=1}^{m} \dfrac{1}{\tau_i}}.$$

The inequality turns into equality if and only if $\gamma_1 = \gamma_2 = \cdots = \gamma_m$. Since $\tau_s \geq \alpha_{1n_1}$ then

$$\alpha(A, b, c) \geq \frac{1}{\sum\limits_{i=1}^{m} \dfrac{1}{\alpha_{1n_i}}}.$$

Thus, we obtain the problem to find $n_1, n_2, \ldots, n_m$ such that

$$\frac{1}{\sum\limits_{i=1}^{m} \dfrac{1}{\alpha_{1n_i}}} \to \min \qquad \text{s.t.} \sum\limits_{i=1}^{m} n_i = n. \tag{3}$$

The sequence

$$\frac{1}{\alpha_{1,n+1}} - \frac{1}{\alpha_{1n}} = \frac{\varepsilon_{n+1}}{\delta_{n+1}} - \frac{\varepsilon_n}{\delta_n} = \frac{1 + \varepsilon_n(\delta_n + 1)}{\delta_{n+1}} - \frac{\varepsilon_n(\delta_n + 1)}{\delta_{n+1}} = \frac{1}{\delta_{n+1}}$$

decreases monotonously as $n \to \infty$, hence

$$\frac{1}{\alpha_{1,n+2}} + \frac{1}{\alpha_{1n}} \leq \frac{2}{\alpha_{1,n+1}}.$$

We conclude that the minimum for (3) is reached if $n_1 = \cdots = n_r = q + 1$, $n_{r+1} = \cdots = n_m = q$. Thus,

$$\alpha(A, b, c) \geq \frac{1}{\sum_{i=1}^{m} \frac{1}{\alpha_{1n_i}}} = \frac{1}{\frac{r}{\alpha_{1,q+1}} + \frac{m-r}{\alpha_{1q}}} = \frac{\alpha_{1q}}{m + r\left(\frac{\alpha_{1q}}{\alpha_{1,q+1}} - 1\right)}.$$

In the following lemma we construct a class of (worst) multi-dimensional knapsack problems on which the bound (2) is attained.

Lemma 5. *For each m and n*

$$\alpha_{mn} \leq \frac{\alpha_{1q}}{m + r\left(\frac{\alpha_{1q}}{\alpha_{1,q+1}} - 1\right)},$$

where $n = qm + r$, $q = \lfloor n/m \rfloor$.

Proof. Consider the direct product of r knapsack problems of the form

$$\max \sum_{j=1}^{q+1} \frac{x_j}{\delta_j} \to \max \qquad \text{s.t.} \sum_{j=1}^{q+1} \frac{x_j}{\delta_j + \mu_{q+1}} \leq 1$$

and $m - r$ knapsack problems of the form

$$\max \sum_{j=1}^{q} \frac{x_j}{\delta_j} \to \max \qquad \text{s.t.} \sum_{j=1}^{q} \frac{x_j}{\delta_j + \mu_q} \leq 1.$$

The precision of the approximate solutions to these problems is α_{1q} and $\alpha_{1,q+1}$ respectively (see Sect. 3). For the product of these problems the optimal solution value is

$$r\frac{\varepsilon_{q+1}}{\delta_{q+1}} + (m - r)\frac{\varepsilon_q}{\delta_q} = \frac{r}{\alpha_{1,q+1}} + \frac{m-r}{\alpha_{1q}}$$

and the approximate solution value is 1, hence the precision of the approximate solution is

$$\alpha(A, b, c) = \frac{1}{\frac{r}{\alpha_{1,q+1}} + \frac{m-r}{\alpha_{1q}}} = \frac{\alpha_{1q}}{m + r\left(\frac{\alpha_{1q}}{\alpha_{1,q+1}} - 1\right)}.$$

Corollary 1.

$$\frac{\alpha_{1,\lceil n/m \rceil}}{m} \leq \alpha_{mn} \leq \frac{\alpha_{1,\lfloor n/m \rfloor}}{m}.$$

Proof. The first inequality obviously follows from (2). Let us prove the second one. If $r = 0$ then

$$\alpha_{mn} = \frac{\alpha_{1q}}{m} = \frac{\alpha_{1,\lceil n/m \rceil}}{m}.$$

If $0 < r < m$ then

$$\alpha_{mn} = \frac{\alpha_{1q}}{m + r\left(\frac{\alpha_{1q}}{\alpha_{1,q+1}} - 1\right)} > \frac{\alpha_{1q}}{m + m\left(\frac{\alpha_{1q}}{\alpha_{1,q+1}} - 1\right)} = \frac{\alpha_{1,q+1}}{m} = \frac{\alpha_{1,\lceil n/m \rceil}}{m}.$$

From Corollary 1 we obtain the following.

Corollary 2. *If $n \to \infty$, $m = o(n)$ then $\alpha_{mn} \sim \frac{\alpha_{1,\lfloor n/m \rfloor}}{m}$.*

Corollary 3. *If $n \to \infty$ and m is fixed then $\alpha_{mn} \to \frac{\alpha_{1\infty}}{m}$.*

6   Conclusion

In this paper we derived exact and asymptotic formulas for the precision of approximate solutions to the m-dimensional knapsack problem. In particular, we proved that the precision tends to $0.59136\ldots/m$ if $n \to \infty$ and m is fixed. The proof of the attainability of the obtained bounds for the precision is constructive.

In the future, our results can be base for new fully polynomial time approximation schemes.

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