Chapter 6
Thermomechanical Theory

Abstract The objective of this chapter is to present the balance laws for the thermomechanical theory. Specifically, the balances of entropy and energy are presented and different forms of second law of thermodynamics are discussed. Invariance under Superposed Rigid Body Motions (SRBM) is considered for the new thermal quantities and thermal constraints on material response are discussed. In addition, specific nonlinear constitutive equations are presented for a number of materials modeling: thermoelastic, thermoelastic–inelastic and porous responses. Also, constitutive equations for growth of thermoelastic–inelastic biological tissues are presented.

6.1 Thermomechanical Processes

A thermomechanical process is characterized by its velocity field **v** and its absolute temperature field θ

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)\,, \quad \theta = \theta(\mathbf{x}, t)\,, \tag{6.1.1}$$

the position of a material point **x** is determined by integrating the equation

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t)\,, \tag{6.1.2}$$

and the velocity gradient **L**, rate of deformation tensor **D** and temperature gradient **g** are defined by

$$\mathbf{L} = \partial\mathbf{v}/\partial\mathbf{x}\,, \quad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)\,, \quad \mathbf{g} = \frac{\partial\theta}{\partial\mathbf{x}}\,. \tag{6.1.3}$$

These quantities are defined at every material point in the material region P and on its closed boundary ∂P.

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Within the context of the thermomechanical theory proposed by Green and Naghdi [7, 8], in addition to the current mass density ρ, the specific (per unit mass) body force $\mathbf{b}$, the Cauchy stress $\mathbf{T}$, the unit outward normal $\mathbf{n}$ to ∂P and the traction vector $\mathbf{t} = \mathbf{Tn}$ per unit current area on ∂P, it is necessary to introduce the specific entropy η, the specific external rate of entropy supply s, the specific internal rate of entropy production ξ, the specific internal energy ε, the specific external rate of energy supply r on P and the entropy flux $\mathbf{p}$ and energy flux $\mathbf{q}$ vectors, both per unit present area on ∂P.

The external fields

$$\mathbf{b}\,,\; s\,, \tag{6.1.4}$$

need to be specified and constitutive equations must be provided for the response functions

$$\mathbf{T}\,,\; \eta\,,\; \xi\,,\; \mathbf{p}\,,\; \varepsilon\,, \tag{6.1.5}$$

with r and $\mathbf{q}$ determined by

$$r = \theta s\,, \quad \mathbf{q} = \theta \mathbf{p}\,. \tag{6.1.6}$$

6.2 Balance Laws for the Thermomechanical Theory

Within the context of the thermomechanical theory proposed by Green and Naghdi [7, 8] the current mass density ρ, the current position $\mathbf{x}$ of a material point and the absolute temperature θ are determined by the global forms of the conservation of mass and the balances of linear momentum and entropy

$$\begin{aligned}
&\frac{d}{dt}\int_P \rho dv = 0\\
&\frac{d}{dt}\int_P \rho \mathbf{v} dv = \int_P \rho \mathbf{b} dv + \int_{\partial P} \mathbf{t} da\,,\\
&\frac{d}{dt}\int_P \rho \eta dv = \int_P \rho (s+\xi) dv - \int_{\partial P} \mathbf{p}\cdot\mathbf{n} da\,.
\end{aligned} \tag{6.2.1}$$

The minus sign appears before the integral over the entropy flux because $\mathbf{p}\cdot\mathbf{n}$ is the rate of entropy expelled by the body through its surface. The global form of the balance of angular momentum is given by

$$\frac{d}{dt}\int_P (\mathbf{x}\times\rho\mathbf{v}) dv = \int_P (\mathbf{x}\times\rho\mathbf{b}) dv + \int_{\partial P} \mathbf{x}\times\mathbf{t} da\,, \tag{6.2.2}$$

and the balance of energy (i.e., the first law of thermodynamics) takes the form

$$\dot{\mathcal{E}} + \dot{\mathcal{K}} = \mathcal{W} + \mathcal{H}\,. \tag{6.2.3}$$

In words, the first law of thermodynamics states that the rate of change of the total internal energy $\mathcal{E}$ plus the rate of change of the total kinetic energy $\mathcal{K}$ is balanced by the total rate of work $\mathcal{W}$ done on the body and total rate of heat $\mathcal{H}$ supplied to the body, which indicates the equivalence of thermal and mechanical supplies of energy. Specifically, these quantities are defined by

$$\begin{aligned} &\mathcal{E} = \int_P \rho\varepsilon dv\,, && \mathcal{K} = \int_P \tfrac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}dv\,, \\ &\mathcal{W} = \int_P \rho\mathbf{b}\cdot\mathbf{v}dv + \int_{\partial P}\mathbf{t}\cdot\mathbf{v}da\,, && \mathcal{H} = \int_P \rho\theta s dv - \int_{\partial P}\theta\mathbf{p}\cdot\mathbf{n}da\,. \end{aligned} \tag{6.2.4}$$

Using standard continuity conditions, the local forms of the conservation of mass and balances of linear momentum and entropy are given by

$$\dot{\rho} + \rho\mathbf{D}\cdot\mathbf{I} = 0\,, \quad \rho\dot{\mathbf{v}} = \rho\mathbf{b} + \mathrm{div}\mathbf{T}\,, \quad \rho\dot{\eta} = \rho(s+\xi) - \mathrm{div}\mathbf{p}\,. \tag{6.2.5}$$

Also, using these balance laws, the reduced local form of the balance of angular momentum requires the Cauchy stress $\mathbf{T}$ to be symmetric

$$\mathbf{T}^T = \mathbf{T}\,, \tag{6.2.6}$$

and the reduced local form of the balance of energy requires

$$\rho\dot{\varepsilon} = \rho\theta s - \mathrm{div}(\theta\mathbf{p}) + \mathbf{T}\cdot\mathbf{D}\,. \tag{6.2.7}$$

Next, multiplying the balance of entropy in (6.2.5) by θ and using the expressions (6.1.6) it can be shown that

$$\rho\theta s - \mathrm{div}(\theta\mathbf{p}) = \rho\theta\dot{\eta} - \rho\theta\xi - \mathbf{p}\cdot\mathbf{g}\,. \tag{6.2.8}$$

Also, the internal rate of entropy production is separated into two parts [17]: a thermal part $-\mathbf{p}\cdot\mathbf{g}$ due to heat conduction and another part $\rho\theta\xi'$ due to the rate of material dissipation

$$\rho\theta\xi = -\mathbf{p}\cdot\mathbf{g} + \rho\theta\xi'\,, \tag{6.2.9}$$

so that the external rate of energy supply can be written in the form

$$\rho\theta s - \mathrm{div}(\theta\mathbf{p}) = -\rho\theta\xi' + \rho\theta\dot{\eta}\,. \tag{6.2.10}$$

In addition, the specific Helmholtz free energy ψ is defined by

$$\psi = \varepsilon - \theta\eta\,, \tag{6.2.11}$$

and the balance of energy (6.2.7) yields a constitutive equation for the rate of material dissipation $\rho\theta\xi'$

$$\rho\theta\xi' = \mathbf{T}\cdot\mathbf{D} - \rho(\dot{\psi} + \eta\dot{\theta})\,, \tag{6.2.12}$$

where use has been made of (6.2.10).

In this formulation of thermomechanics, ρ, $\mathbf{x}$ and θ are determined by the conservation of mass and the balances of linear momentum and entropy (6.2.5) and the balances of angular momentum and energy place restrictions on the constitutive equations which ensure that they are identically satisfied for all thermomechanical processes. Specifically, the reduced form of the balance of angular momentum (6.2.6) requires the Cauchy stress $\mathbf{T}$ to be symmetric and the reduced form of the balance of energy (6.2.12) determines a constitutive equation for the rate of material dissipation $\rho\theta\xi'$.

6.3 Second Laws of Thermomechanics

Observations indicate that thermomechanical processes progress is specific directions. For example, consider a body which is isolated with no rates of external work and heat supply

$$\begin{aligned} \mathbf{b}\cdot\mathbf{v} &= 0\,, \quad r = 0 \text{ on } P\,, \\ \mathbf{t}\cdot\mathbf{v} &= 0\,, \quad \mathbf{q}\cdot\mathbf{n} = 0 \text{ on } \partial P\,. \end{aligned} \tag{6.3.1}$$

Then, the global form (6.2.3) of the first law of thermodynamics indicates that an isolated body preserves total energy

$$\mathcal{E} + \mathcal{K} = \mathcal{E}(0) + \mathcal{K}(0) = \text{constant}\,. \tag{6.3.2}$$

Next, consider a body that is made from a homogeneous material which is in a zero-stress uniform material state at rest. In the absence of external forces and with no heat supply through its boundary, the body is heated by an external rate of energy supply to obtain an inhomogeneous temperature field in the body at rest. Then, in the absence of external forces and further external heat supply, the total energy would remain constant even if part of the body became hotter and another part of it became colder. However, observations indicate that this does not happen naturally. Instead, the body tends to reach a uniform temperature. Notions of entropy model the observed directions of thermomechanical processes.

Clausius–Duhem Inequality

In the classical approach to continuum thermomechanics proposed by Coleman and Noll [5], the conservation of mass and the balances of linear momentum, angular momentum and energy are supplimented by the Clausius–Duhem inequality

$$\frac{d}{dt}\int_P \rho\eta dv - \int_P \frac{\rho r}{\theta} dv + \int_{\partial P} \frac{\mathbf{q}\cdot\mathbf{n}}{\theta} da \geq 0\,, \tag{6.3.3}$$

which is a statement of the second law of thermodynamics that thermomechanical processes cause the internal rate of entropy production to have a tendency to increase.

Using standard continuity conditions, the local form of the conservation of mass and the definitions (6.1.3) and (6.1.6), the local form of (6.3.3) requires

$$\rho\theta\dot{\eta} - \rho\theta s + \theta \mathrm{div}\mathbf{p} \geq 0\,. \tag{6.3.4}$$

Moreover, with the help of the balance of energy (6.2.7) and the definition (6.2.11) of the Helmholtz free energy ψ, the Clausius–Duhem inequality requires

$$\mathbf{T}\cdot\mathbf{D} - \rho(\dot{\psi} + \eta\dot{\theta}) - \mathbf{p}\cdot\mathbf{g} \geq 0\,, \tag{6.3.5}$$

which places restrictions on constitutive equations.

Green–Naghdi Formulation
In the classical approach to thermomechanics, the Clausius–Duhem inequalty (6.3.5) is a single statement of the second law of thermodynamics that places restrictions on the constitutive equations. In contrast, the Green and Naghdi formulation places restrictions on the constitutive equations by requiring the reduced forms of the balance of angular momentum (6.2.6) and the balance of energy (6.2.7) to be satisfied identically, without any statement of the second law of thermodynamics.

To compare the two approaches to thermomechanics, use is made of the separation (6.2.9) and the constitutive Eq. (6.2.12) to rewrite the Clausius–Duhem inequality (6.3.4) in the form

$$\rho\theta\xi = -\mathbf{p}\cdot\mathbf{g} + \rho\theta\xi' > 0\,, \tag{6.3.6}$$

which requires the total internal rate of entropy production to be non-negative. However, the Green–Naghdi formulation allows for proposing different statements of the second law of thermodynmaics, as was discussed in [8].

Heat Flows From Hot to Cold Regions
One statement of the second law of thermodynamics is that heat flows from hot to cold regions

$$-\,\mathbf{p}\cdot\mathbf{g} > 0 \text{ for } \mathbf{g} \neq 0\,. \tag{6.3.7}$$

This indicates that the thermal part of the internal rate of entropy production in the separation (6.2.9) is non-negative.

Rate of Material Dissipation
To motivate a second statement of the second law of thermodynamics, it is noted from (6.1.6) and (6.2.10) that the rate of heat expelled by the body is given by

$$-\,(\rho r - \mathrm{div}\mathbf{q}) = -\big[\rho\theta s - \mathrm{div}(\theta\mathbf{p})\big] = \rho\theta\xi' - \rho\theta\dot{\eta}\,. \tag{6.3.8}$$

For general thermomechanical processes heat can be supplied or expelled. However, the notions of friction and viscous effects in fluids indicate that the rate of material dissipation causes a tendency for heat to be expelled by the body. Noting that positive values of $\rho\theta\xi'$ cause a tendency for heat to be expelled by the body, this second statement of the second law of thermodynamics requires the rate of material dissipation to be non-negative

$$\rho\theta\xi' = \mathbf{T}\cdot\mathbf{D} - \rho(\dot{\psi} + \eta\dot{\theta}) \geq 0\,. \tag{6.3.9}$$

Although the two statements (6.3.7) and (6.3.9) combined are consistent with the Clausius–Duhem inequality (6.3.6), this latter single statement of the second law of thermodynamics does not demand that (6.3.7) and (6.3.9) be satisfied individually, as in the Green–Naghdi formulation.

6.4 Invariance Under Superposed Rigid Body Motions (SRBM)

Although temperature θ is not a kinematic variable, it is an independent variable like the position vector $\mathbf{x}$ which needs to be determined by the balance laws, boundary and initial conditions. Consequently, in addition to the kinematic conditions (3.8.13) and (3.8.16)

$$\mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}\,, \quad \mathbf{Q}\mathbf{Q}^T = \mathbf{I}\,, \quad \det\mathbf{Q} = +1\,, \tag{6.4.1}$$

it is proposed that θ remains unaltered under SRBM

$$\theta^+ = \theta\,. \tag{6.4.2}$$

This means that the temperature gradient $\mathbf{g}$ transforms to $\mathbf{g}^+$, such that

$$\mathbf{g} = \frac{\partial\theta}{\partial\mathbf{x}}\,, \quad \mathbf{g}^+ = \frac{\partial\theta^+}{\partial\mathbf{x}^+} = \mathbf{g}\,(\partial\mathbf{x}/\partial\mathbf{x}^+) = \mathbf{g}\mathbf{Q}^T = \mathbf{Q}\mathbf{g}\,. \tag{6.4.3}$$

Section 4.7 introduced the notion of invariance under SRBM which is based on the two restrictions

(R-1): The balance laws must be form-invariant under SRBM. (6.4.4a)

(R-2): The constitutive response of the material relative to its orientation is the same for all SRBM. (6.4.4b)

The first restriction (R-1) in (6.4.4a) requires the global forms of the balance laws to be form-invariant in the superposed configuration P^+ with all independent and kinetic quantities taking their superposed values in P^+. Using the transformation relations (4.7.21)

$$\rho^+ = \rho\,, \quad \mathbf{T}^+ = \mathbf{Q}\mathbf{T}\mathbf{Q}^T\,, \quad \mathbf{b}^+ = \dot{\mathbf{v}}^+ + \mathbf{Q}(\mathbf{b} - \dot{\mathbf{v}})\,, \tag{6.4.5}$$

the conservation of mass, the balance of linear momentum (6.2.1) and the balance of angular momentum (6.2.2) are already form-invariant under SRBM. Consequently, within the context of the thermomechanical theory, the physical restriction (R-1) in (6.4.4a) requires the balance of entropy in (6.2.1) and the balance of energy (6.2.3) to remain form-invariant under SRBM

$$\frac{d}{dt}\int_{P^+} \rho^+\eta^+ dv^+ = \int_{P^+} \rho^+(s^+ + \xi^+)dv^+ - \int_{\partial P^+} \mathbf{p}^+ \cdot \mathbf{n}^+ da^+ \,, \tag{6.4.6a}$$

$$\dot{\mathcal{E}}^+ + \dot{\mathcal{K}}^+ = \mathcal{W}^+ + \mathcal{H}^+ \,, \tag{6.4.6b}$$

with the specifications

$$\begin{aligned}
&\mathcal{E}^+ = \int_{P^+} \rho^+\varepsilon^+ dv^+ \,, \qquad \mathcal{K}^+ = \int_{P^+} \frac{1}{2}\rho^+\mathbf{v}^+ \cdot \mathbf{v}^+ dv^+ \,, \\
&\mathcal{W}^+ = \int_{P^+} \rho^+\mathbf{b}^+ \cdot \mathbf{v}^+ dv^+ + \int_{\partial P^+} \mathbf{t}^+ \cdot \mathbf{v}^+ da^+ \,, \\
&\mathcal{H}^+ = \int_{P^+} \rho^+\theta^+ s^+ dv^+ - \int_{\partial P^+} \theta^+\mathbf{p}^+ \cdot \mathbf{n}^+ da^+ \,.
\end{aligned} \tag{6.4.7}$$

Using standard continuity arguments the local form the of the balance of entropy (6.4.6a) requires

$$(\dot{\rho}^+ + \rho^+\mathbf{D}^+ \cdot \mathbf{I})\,\eta^+ + \rho^+\dot{\eta}^+ = \rho^+(s^+ + \xi^+) - \mathrm{div}^+\mathbf{p}^+ \,, \tag{6.4.8}$$

and the local form of the balance of energy (6.4.6b) requires

$$\begin{aligned}
&(\dot{\rho}^+ + \rho^+\mathbf{D}^+ \cdot \mathbf{I})\left(\varepsilon^+ + \frac{1}{2}\rho^+\mathbf{v}^+ \cdot \mathbf{v}^+\right) + (\rho^+\dot{\mathbf{v}}^+ - \rho^+\mathbf{b}^+ - \mathrm{div}^+\mathbf{T}^+) \cdot \dot{\mathbf{v}}^+ \ + \rho^+\dot{\varepsilon}^+ \\
&\qquad = \mathbf{T}^+ \cdot \mathbf{L}^+ + \rho^+\theta^+ s^+ - \mathrm{div}^+(\theta^+\mathbf{p}^+) \,.
\end{aligned} \tag{6.4.9}$$

Then, using form-invariance of the local forms of the conservation of mass and the balances of linear and angular momentum, the local form of the balance of entropy requires

$$\rho^+\dot{\eta}^+ = \rho^+(s^+ + \xi^+) - \mathrm{div}^+\mathbf{p}^+ \,, \tag{6.4.10}$$

and the local form of the balance of energy requires

$$\rho^+\dot{\varepsilon}^+ = \mathbf{T}^+ \cdot \mathbf{D}^+ + \rho^+\theta^+ s^+ - \mathrm{div}^+(\theta^+\mathbf{p}^+) \,. \tag{6.4.11}$$

Now, with the help of (2.5.4) and (6.4.1) it can be shown that

$$\begin{aligned}
&\mathrm{div}^+\mathbf{p}^+ = (\partial\mathbf{p}^+/\partial\mathbf{x}^+) \cdot \mathbf{I} = (\partial\mathbf{p}^+/\partial\mathbf{x})(\partial\mathbf{x}/\partial\mathbf{x}^+) \cdot \mathbf{I} = (\partial\mathbf{p}^+/\partial\mathbf{x}) \cdot \mathbf{Q} \,, \\
&\mathrm{div}^+\mathbf{p}^+ = \mathrm{div}(\mathbf{Q}^T\mathbf{p}^+) \,.
\end{aligned} \tag{6.4.12}$$

Next, using the invariance of ρ in (6.4.5), the balance of entropy (6.4.10) can be solved for s^+ to obtain

$$s^+ = \dot{\eta}^+ - \xi^+ + \frac{1}{\rho}\text{div}(\mathbf{Q}^T \mathbf{p}^+)\,. \tag{6.4.13}$$

Moreover, using the local balance of entropy in (6.2.5) it can be shown that

$$s^+ = s + (\dot{\eta}^+ - \dot{\eta}) - (\xi^+ - \xi) + \frac{1}{\rho}\text{div}(\mathbf{Q}^T \mathbf{p}^+ - \mathbf{p})\,. \tag{6.4.14}$$

In this regard, it is noted that the restriction (R-1) tacitly assumes that the balance of entropy is valid for any specified external rate of entropy supply. Consequently, it is also valid for (6.4.14), which enforces SRBM.

To compete the restrictions for invariance under SRBM it is necessary to determine expressions for

$$\eta^+\,,\, \xi^+\,,\, \varepsilon^+\,,\, \mathbf{p}^+\,. \tag{6.4.15}$$

This requires use of the physical restriction (R-2) (6.4.4b) and recognition that in addition to the stress **T**, the quantities η, ξ, ε and **p** characterize the material response for thermomechanical processes. This means that η, ξ and ε, which do not depend on the orientation of the material, must be uninfluenced by SRBM

$$\eta^+ = \eta\,, \quad \xi^+ = \xi\,, \quad \varepsilon^+ = \varepsilon\,. \tag{6.4.16}$$

Moreover, the response due to the entropy flux vector relative to the orientation of the material will be the same if the restriction

$$\mathbf{p}^+ \cdot \mathbf{n}^+ = \mathbf{p} \cdot \mathbf{n} \tag{6.4.17}$$

is valid for all material points, all unit normals **n** and all SRBM. Now, using the kinematic result (3.8.20) that **n** rotates under SRBM

$$\mathbf{n}^+ = \mathbf{Q}\mathbf{n}\,, \tag{6.4.18}$$

the expression (6.4.17) for the entropy flux vector can be rewritten in the form

$$(\mathbf{p}^+ - \mathbf{Q}\mathbf{p}) \cdot \mathbf{n}^+ = 0\,. \tag{6.4.19}$$

Then, since the coefficient of $\mathbf{n}^+$ in this equation is independent of $\mathbf{n}^+$, and $\mathbf{n}^+$ is an arbitrary unit vector, the entropy flux vector **p** and heat flux vector **q** defined in (6.1.6) must satisfy the transformation relations

$$\mathbf{p}^+ = \mathbf{Q}\mathbf{p}\,, \quad \mathbf{q}^+ = \mathbf{Q}\mathbf{q}\,. \tag{6.4.20}$$

This means that with the help of (6.4.16), the restriction (6.4.14) requires the external rate of entropy supply s and the external rate of heat supply r defined in (6.1.6) to be unaffected by SRBM

$$s^+ = s\,, \quad r^+ = r\,. \tag{6.4.21}$$

In summary, under superposed rigid body motions SRBM the thermomechanical quantities θ, $\mathbf{g}$, η, ε, ψ, s, r, ξ, ξ', $\mathbf{p}$ and $\mathbf{q}$ transform to θ^+, $\mathbf{g}^+$, η^+, ε^+, ψ^+, s^+, r^+, ξ^+, ξ'^+, $\mathbf{p}^+$ and $\mathbf{q}^+$, such that

$$\begin{aligned} &\theta^+ = \theta\,, \quad \mathbf{g}^+ = \mathbf{Q}\mathbf{g}\,, \quad \eta^+ = \eta \quad \varepsilon^+ = \varepsilon\,, \quad \psi^+ = \psi\,, \\ &s^+ = s\,, \quad r^+ = r\,, \quad \xi^+ = \xi\,, \quad \xi'^+ = \xi'\,, \quad \mathbf{p}^+ = \mathbf{Q}\mathbf{p}\,, \quad \mathbf{q}^+ = \mathbf{Q}\mathbf{q}\,. \end{aligned} \tag{6.4.22}$$

6.5 Thermal Constraints

In general, it is possible to propose coupled thermomechanical constraints but such coupled constraints make it difficult to satisfy the forms (6.3.7) and (6.3.9) of the second law of thermodynamics individually. For this reason, this section considers thermal constraints which are independent of the kinematic constraints considered in Sect. 5.7. In this regard, it is noted that since the constraint response $\bar{\mathbf{T}}$ in (5.7.13) is workless (5.7.11)

$$\mathbf{T} \cdot \mathbf{D} = \hat{\mathbf{T}} \cdot \mathbf{D}\,, \tag{6.5.1}$$

so the constraint response makes no contribution to the rate of material dissipation in the second law of thermodynamics (6.3.9).

As a physical example of a thermal constraint, consider a material that has fibers in one direction that allow for very rapid heat conduction relative to the surrounding matrix material. For this case, the temperature gradient $\mathbf{g}$ in the direction of the fibers will be very small relative to the temperature gradient in directions perpendicular to the fibers due to slow conduction through the matrix material only. Motivated by this simple example, consider a thermal constraint which constrains the temperature gradient in the direction $\boldsymbol{\gamma}$ of the form

$$\boldsymbol{\gamma} \cdot \mathbf{g} = 0\,, \tag{6.5.2}$$

where $\boldsymbol{\gamma}$ is a vector that is independent of $\mathbf{g}$ and which under SRBM satisfies the transformation relation

$$\boldsymbol{\gamma}^+ = \mathbf{Q}\boldsymbol{\gamma}\,. \tag{6.5.3}$$

Moreover, consider a general unconstrained material that is characterized by a constitutive equation $\hat{\mathbf{p}}$ for the entropy flux $\mathbf{p}$. Next, consider a model of a constrained material for which $\mathbf{p}$ is additively separated into the constitutive part $\hat{\mathbf{p}}$ and a part $\bar{\mathbf{p}}$, called the constraint response, which enforces the thermal constraint (6.5.2)

$$\mathbf{p} = \hat{\mathbf{p}} + \bar{\mathbf{p}} \, . \tag{6.5.4}$$

Although $\hat{\mathbf{p}}$ characterizes the response to general temperature gradients, its value in (6.5.4) is determined by evaluating $\hat{\mathbf{p}}$ only for temperature gradients that satisfy the imposed thermal constraint. Moreover, $\hat{\mathbf{p}}$ automatically transforms under SRBM, such that

$$\hat{\mathbf{p}}^{+} = \mathbf{Q}\hat{\mathbf{p}} \, . \tag{6.5.5}$$

Now, $\bar{\mathbf{p}}$ is assumed to satisfy the restriction

$$\bar{\mathbf{p}} \cdot \mathbf{g} = 0 \, , \tag{6.5.6}$$

and to be independent of the rate $\mathbf{g}$.

Next, multiplying (6.5.2) by an arbitrary scalar γ and subtracting the result from (6.5.6) yields

$$(\bar{\mathbf{p}} - \gamma \, \boldsymbol{\gamma}) \cdot \mathbf{g} = 0 \, . \tag{6.5.7}$$

Since $\boldsymbol{\gamma}$ is nonzero it is possible to specify γ by the equation

$$\gamma = \frac{\bar{\mathbf{p}} \cdot \boldsymbol{\gamma}}{|\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}|} \, . \tag{6.5.8}$$

Then, the only nonzero components in (6.5.7) are perpendicular to $\boldsymbol{\gamma}$. Since this equation must hold for arbitrary temperature gradients $\mathbf{g}$ that satisfy the constraint (6.5.2) and the coefficient of $\mathbf{g}$ is independent of $\mathbf{g}$, it follows that the constraint response $\bar{\mathbf{p}}$ must be given by

$$\bar{\mathbf{p}} = \gamma \, \boldsymbol{\gamma} \, , \tag{6.5.9}$$

with γ being an arbitrary function of $\mathbf{x}$ and t that is determined by the balance laws and boundary conditions. Since the restriction (R-2) in (4.7.3b), which defines how the constitutive response of the material relative to its orientation is the same for all SRBM, requires $\mathbf{p}$ to satisfy the transformation relation (6.4.20) and since $\hat{\mathbf{p}}$ satisfies the transformaton relation (6.5.5), it follows from (6.5.4) that the constraint response $\bar{\mathbf{p}}$ satisfies the transformation relation

$$\bar{\mathbf{p}}^{+} = \gamma^{+} \boldsymbol{\gamma}^{+} = \mathbf{Q}\bar{\mathbf{p}} = \gamma \, \mathbf{Q}\boldsymbol{\gamma} \tag{6.5.10}$$

for all SRBM. Then, with the help of (6.5.3) it can be shown that the arbitrary function γ must be unaffected by SRBM

$$\gamma^{+} = \gamma \, . \tag{6.5.11}$$

In addition, since the constraint response $\bar{\mathbf{p}}$ satisfies the restriction (6.5.6), it follows that

$$\mathbf{p} \cdot \mathbf{g} = \hat{\mathbf{p}} \cdot \mathbf{g} \, , \tag{6.5.12}$$

so the constraint response does not influence the restriction (6.3.7) of the second law of thermodynamics which requires heat to flow from hot to cold regions.

Furthermore, it is noted that up to two independent thermal constraints of the type (6.5.2) can be imposed simultaneously without causing **p** to be totally indeterminate.

6.6 Thermoelastic Materials

A thermoelastic solid is a special ideal material which is non-dissipative in the sense that the rate of material dissipation (6.2.12) vanishes

$$\rho\theta\xi' = \mathbf{T}\cdot\mathbf{D} - \rho(\dot{\psi} + \eta\dot{\theta}) = 0 \tag{6.6.1}$$

for all thermomechanical processes.

Within the context of the Eulerian formulation of constitutive equations, the microstructural vectors $\mathbf{m}_i$ and elastic metric m_{ij} described in Sect. 3.11 are determined by the evolution equations

$$\dot{\mathbf{m}}_i = \mathbf{L}\mathbf{m}_i\,, \quad m_{ij} = \mathbf{m}_i\cdot\mathbf{m}_j\,, \quad \dot{m}_{ij} = 2\mathbf{m}_i\otimes\mathbf{m}_j\cdot\mathbf{D}\,. \tag{6.6.2}$$

Moreover, for a thermoelastic material, the response functions ψ, η and $\mathbf{T}$ are specified in the forms

$$\psi = \hat{\psi}(m_{ij},\theta)\,, \quad \eta = \hat{\eta}(m_{ij},\theta)\,, \quad \mathbf{T} = \hat{\mathbf{T}}(\mathbf{m}_i,\theta)\,, \tag{6.6.3}$$

so the condition (6.6.1) requires

$$\left(\mathbf{T} - 2\rho\frac{\partial\hat{\psi}}{\partial m_{ij}}\mathbf{m}_i\otimes\mathbf{m}_j\right)\cdot\mathbf{D} - \rho(\frac{\partial\hat{\psi}}{\partial\theta} + \eta)\,\dot{\theta} = 0\,. \tag{6.6.4}$$

Since the coefficients of $\mathbf{D}$ and $\dot{\theta}$ are independent of these rates, and the coefficient of $\mathbf{D}$ is symmetric, $\mathbf{T}$ and η must be determined by the constitutive equations

$$\mathbf{T} = \hat{\mathbf{T}} = 2\rho\frac{\partial\hat{\psi}}{\partial m_{ij}}\mathbf{m}_i\otimes\mathbf{m}_j\,, \quad \eta = \hat{\eta} = -\frac{\partial\hat{\psi}}{\partial\theta}\,. \tag{6.6.5}$$

This expression for the Cauchy stress $\mathbf{T}$ automatically satisfies the restriction (6.2.6) due to angular momentum. Moreover, the entropy flux $\mathbf{p}$ takes the form

$$\mathbf{p} = p^i\mathbf{m}_i\,, \quad p^i = \hat{p}^i(m_{ij},\theta,g_i)\,, \quad g_i = \mathbf{g}\cdot\mathbf{m}_i\,, \tag{6.6.6}$$

which must satisfy the restriction (6.3.7) due to the second law of thermodynamics,

$$-\hat{\mathbf{p}}\cdot\mathbf{g} = -\hat{p}^i g_i > 0 \text{ for } \mathbf{g} \neq 0\,. \tag{6.6.7}$$

These functional forms are automatically properly invariant under SRBM. Also, using (6.2.11), it follows that the internal energy ε for a thermoelastic material is given by

$$\varepsilon = \hat{\varepsilon}(m_{ij},\theta) = \hat{\psi}(m_{ij},\theta) + \theta\hat{\eta}(m_{ij},\theta)\,. \tag{6.6.8}$$

Furthermore, these constitutive equations are restricted so that the material is in a zero-stress material state whenever the elastic deformation metric $m_{ij} = \delta_{ij}$ and the temperature equals the reference zero-stress temperature $\theta = \theta_z$

$$\mathbf{T} = 0 \text{ whenever } m_{ij} = \delta_{ij} \text{ and } \theta = \theta_z\,. \tag{6.6.9}$$

Rate-Dependent Response
Although the evolution Eq. (6.6.2) for $\mathbf{m}_i$ are homogeneous equations of order one in time and thus predict rate-independent response, and the response functions (6.6.3) and (6.6.6) are explicitly independent of the rates $\mathbf{D}$ and $\dot{\theta}$, the response of a thermoelastic material is rate-dependent. This is because the balance of entropy in $(6.2.5)_3$ predicts time-dependent response of the temperature for transient processes.

Path-Independent Response
Since $\mathbf{m}_i$ arematerial line elements, it follows that the values of the response functions ψ, η, $\mathbf{T}$ and $\mathbf{p}$ at a specified state characterized by $\mathbf{m}_i$ and θ are independent of the path of the thermomechanical process that attains this state. This also means that for any thermomechanical process that starts at the state

$$\mathbf{m}_i(\mathbf{x},t_1),\,\theta(\mathbf{x},t_1),\,\mathbf{v}(\mathbf{x},t_1)\,, \tag{6.6.10}$$

and ends at the state

$$\mathbf{m}_i(\mathbf{x},t_2),\,\theta(\mathbf{x},t_2),\,\mathbf{v}(\mathbf{x},t_2)\,, \tag{6.6.11}$$

the changes in internal and kinetic energies

$$\Delta\mathcal{E} = \mathcal{E}(t_2) - \mathcal{E}(t_1)\,, \quad \Delta\mathcal{K} = \mathcal{K}(t_2) - \mathcal{K}(t_1) \tag{6.6.12}$$

are independent of the path of the thermodynamic process. Moreover, with the help of the first law of thermodynamics (6.2.3), it follows that the total work done on the body plus the total heat supplied to the body during this process is also independent of the path of the process

$$\int_{t=t_1}^{t=t_2} (\mathcal{W} + \mathcal{H})dt = \Delta\mathcal{E} + \Delta\mathcal{K}\,. \tag{6.6.13}$$

In addition, the total work done on the body plus the total heat supplied to the body vanishes for any cyclic process which starts and ends at the same state $\mathbf{m}_i$, θ and $\mathbf{v}$.

An Irreversible Process

Although a thermoelastic material is an ideal material with no material dissipation, it can experience an irreversible process. For the present discussion it is assumed that

$$\hat{\varepsilon} \to \infty \text{ whenever } \hat{\eta} \to \infty \,. \tag{6.6.14}$$

Now consider a cantilever beam made from a homogeneous thermoelastic material. The external body force and external rate of heat supply both vanish $\mathbf{b} = 0$ and $r = 0$. Also, the velocity field on the clamped boundary vanishes, all other boundaries are traction free $\mathbf{t} = 0$ and all boundaries are insulated from heat flow $\mathbf{q} \cdot \mathbf{n} = 0$. For this problem the total rate of work $\mathcal{W}$ done on the body vanishes and the total rate of heat supply $\mathcal{H}$ vanishes. Also, consider the case when the body is initially in a zero-stress material state at constant density ρ_z and constant zero-stress reference temperature θ_z, but it has an initial velocity field with a non-uniform rate of deformation so that

$$\begin{aligned} &m_{ij}(\mathbf{x},0) = \delta_{ij}\,, \quad \theta(\mathbf{x},0) = \theta_z\,, \quad \partial\mathbf{D}(\mathbf{x},0)/\partial\mathbf{x} \neq 0\,, \\ &\mathbf{T}(\mathbf{x},0) = 0\,, \qquad \eta(\mathbf{x},0) = 0\,, \quad \varepsilon(\mathbf{x},0) = 0\,. \end{aligned} \tag{6.6.15}$$

Consequently, from (6.2.3) the sum of the total internal and kinetic energies remains constant

$$\mathcal{E} + \mathcal{K} = \mathcal{K}(0)\,, \tag{6.6.16}$$

where $\mathcal{K}(0)$ is the initial value of the total kinetic energy in the beam.

The global form of the balance of entropy in $(6.2.1)_3$ and the restrictions (6.3.1), (6.3.6), (6.3.7) and (6.6.1) require

$$\frac{d}{dt}\int_P \rho\eta dv = -\int_P (\frac{\mathbf{p}\cdot\mathbf{g}}{\theta}) dv \geq 0\,. \tag{6.6.17}$$

Moreover, the restriction (6.3.7) causes the total entropy to increase until the temperature becomes uniform with $\mathbf{g} = 0$. In particular, a non-uniform rate of deformation causes local temperature changes with a nonzero temperature gradient $\mathbf{g}$. However, due to assumption (6.6.14) and the result (6.6.16), the entropy cannot continue to increase without bound. This means that eventually the temperature must become uniform with $\mathbf{g} = 0$, the velocity field must go to zero and the elastic deformation metric m_{ij} must become independent of time, but it can be nonuniform due the clamped boundary. Since energy is preserved, the final values of $\mathcal{E}$ and $\mathcal{K}$ are constants given by

$$\mathcal{E}(\infty) = \mathcal{K}(0)\,, \quad \mathcal{K}(\infty) = 0\,. \tag{6.6.18}$$

This example shows the importance of the entropy in a thermomechanical process. In particular, since the entropy flux must satisfy the restriction (6.3.7), it follows that the process is thermodynamically irreversible even though the thermoelastic material is non-dissipative.

A Separation of Elastic Dilatation and Distortional Deformations

To introduce separate control over the response of the material to dilatation and distortional rates of deformation, it is convenient to use the work of Flory [6] and use the elastic dilatation J_e defined in (3.11.7), the elastic distortional deformation vectors $\mathbf{m}'_i$ defined in (3.11.14) and the elastic distortional deformation metric m'_{ij} defined in (3.11.16), which satisfy the evolution Eqs. (3.11.13), (3.11.15) and (3.11.17)

$$\begin{aligned}
&J_e = \mathbf{m}_1 \times \mathbf{m}_2 \cdot \mathbf{m}_3 > 0\,, && \dot{J}_e = J_e \mathbf{D} \cdot \mathbf{I}\,,\\
&\mathbf{m}'_i = J_e^{-1/3}\,\mathbf{m}_i\,, && \dot{\mathbf{m}}'_i = \mathbf{L}''\,\mathbf{m}'_i\,,\\
&m'_{ij} = \mathbf{m}'_i \cdot \mathbf{m}'_j = m'_{ji}\,, && \dot{m}'_{ij} = 2(\mathbf{m}'_i \otimes \mathbf{m}'_j - \tfrac{1}{3} m'_{ij}\mathbf{I}) \cdot \mathbf{D}\,,
\end{aligned} \tag{6.6.19}$$

where $\mathbf{L}''$ is the deviatoric part of $\mathbf{L}$. Moreover, since there is no inelastic volume change for a thermoelastic material, the elastic dilatation J_e can be expressed in the form (4.1.16)

$$J_e = \frac{\rho_z}{\rho}\,, \tag{6.6.20}$$

where ρ_z is the constant zero-stress mass density at zero-stress reference temperature θ_z.

Then, for a thermoelastic material, the response functions ψ, η and $\mathbf{T}$ are specified in the forms

$$\psi = \tilde{\psi}(J_e, m'_{ij}, \theta)\,, \quad \eta = \tilde{\eta}(J_e, m'_{ij}, \theta)\,, \quad \mathbf{T} = \tilde{\mathbf{T}}(J_e, \mathbf{m}'_i, \theta)\,, \tag{6.6.21}$$

so the condition (6.6.1) requires

$$\left[\mathbf{T} - \rho J_e \frac{\partial \tilde{\psi}}{\partial J_e}\mathbf{I} - 2\rho \frac{\partial \tilde{\psi}}{\partial m'_{ij}}(\mathbf{m}'_i \otimes \mathbf{m}'_j - \frac{1}{3} m'_{ij}\mathbf{I})\right] \cdot \mathbf{D} - \rho(\frac{\partial \tilde{\psi}}{\partial \theta} + \eta)\,\dot{\theta} = 0\,. \tag{6.6.22}$$

Since the coefficients of $\mathbf{D}$ and $\dot{\theta}$ are independent of these rates, and the coefficient of $\mathbf{D}$ is symmetric, $\mathbf{T}$ and η must be determined by the constitutive equations

$$\begin{aligned}
&\mathbf{T} = \tilde{\mathbf{T}} = -p\,\mathbf{I} + \mathbf{T}''\,, && p = \tilde{p} = -\rho_z \tfrac{\partial \tilde{\psi}}{\partial J_e}\,,\\
&\mathbf{T}'' = \tilde{\mathbf{T}}'' = 2J_e^{-1}\rho_z \tfrac{\partial \tilde{\psi}}{\partial m'_{ij}}(\mathbf{m}'_i \otimes \mathbf{m}'_j - \tfrac{1}{3} m'_{ij}\mathbf{I})\,, && \eta = \tilde{\eta} = -\tfrac{\partial \tilde{\psi}}{\partial \theta}\,,
\end{aligned} \tag{6.6.23}$$

where use has been made of (6.6.20). Moreover, the entropy flux $\mathbf{p}$ takes the form

$$\mathbf{p} = p^{i\prime}\mathbf{m}'_i\,, \quad p^{i\prime} = \tilde{p}^{i\prime}(J_e, m'_{ij}, \theta, g'_i)\,, \quad g'_i = \mathbf{g} \cdot \mathbf{m}'_i\,, \tag{6.6.24}$$

which must satisfy the restriction (6.3.7) due to the second law of thermodynamics,

$$-\,\mathbf{p} \cdot \mathbf{g} = -\tilde{p}^{i\prime} g'_i > 0 \text{ for } \mathbf{g} \neq 0\,. \tag{6.6.25}$$

An Elastically Isotropic Thermoelastic Material

If the material is elastically isotropic, then the elastic dilatation J_e and the symmetric, positive-definite, unimodular elastic distortional deformation tensor $\mathbf{B}'_e$ defined by (3.11.19) satisfy the evolution Eq. (3.11.30)

$$\dot{J}_e = J_e \mathbf{D} \cdot \mathbf{I} \,, \quad \dot{\mathbf{B}}'_e = \dot{\mathbf{B}}'_e = \mathbf{L}''\mathbf{B}'_e + \mathbf{B}'_e\mathbf{L}''^T \,, \tag{6.6.26}$$

where $\mathbf{L}''$ is the deviatoric part of the velocity gradient $\mathbf{L}$. Since $\mathbf{B}'_e$ is unimodular, it has only two independent non-trivial invariants α_1 and α_2 defined in (5.8.4)

$$\alpha_1 = \mathbf{B}'_e \cdot \mathbf{I} \,, \quad \alpha_2 = \mathbf{B}'_e \cdot \mathbf{B}'_e \,, \tag{6.6.27}$$

which satisfy the evolution Eq. (5.8.7)

$$\dot{\alpha}_1 = 2\mathbf{B}''_e \cdot \mathbf{D} \,, \quad \dot{\alpha}_2 = 4\left(\mathbf{B}'^2_e - \frac{1}{3}\alpha_2 \mathbf{I}\right) \cdot \mathbf{D} \,, \tag{6.6.28}$$

where $\mathbf{B}''_e$ is the deviatoric part of $\mathbf{B}'_e$. Moreover, the evolution equation for elastic dilatation can be integrated to obtain (6.6.20).

For an elastically isotropic thermoelastic material, the response functions ψ, η and $\mathbf{T}$ are specified in the forms

$$\psi = \hat{\psi}(J_e, \alpha_1, \alpha_2, \theta) \,, \quad \eta = \hat{\eta}(J_e, \alpha_1, \alpha_2, \theta) \,, \quad \mathbf{T} = \hat{\mathbf{T}}(\mathbf{B}'_e, \theta) \,, \tag{6.6.29}$$

so the condition (6.6.1) requires

$$\begin{aligned} &\left[\mathbf{T} - \rho_z \frac{\partial \hat{\psi}}{\partial J_e} \mathbf{I} - 2\rho_z J_e^{-1} \frac{\partial \hat{\psi}}{\partial \alpha_1} \mathbf{B}''_e - 4\rho_z J_e^{-1} \frac{\partial \hat{\psi}}{\partial \alpha_2} \left(\mathbf{B}'^2_e - \frac{1}{3}\alpha_2 \mathbf{I}\right)\right] \cdot \mathbf{D} \\ &\qquad - \rho \left(\frac{\partial \hat{\psi}}{\partial \theta} + \eta\right) \dot{\theta} = 0 \,, \end{aligned} \tag{6.6.30}$$

for all thermomechanical processes. Since the coefficients of $\mathbf{D}$ and $\dot{\theta}$ are independent of these rates, and the coefficient of $\mathbf{D}$ is symmetric, $\mathbf{T}$ and η must be determined by the constitutive equations

$$\begin{aligned} &\mathbf{T} = -p\,\mathbf{I} + \mathbf{T}'' \,, \quad p = \hat{p} = -\rho_z \frac{\partial \hat{\psi}}{\partial J_e} \,, \\ &\mathbf{T}'' = \hat{\mathbf{T}}'' = 2\rho_z J_e^{-1} \frac{\partial \hat{\psi}}{\partial \alpha_1} \mathbf{B}''_e + 4\rho_z J_e^{-1} \frac{\partial \hat{\psi}}{\partial \alpha_1} \left(\mathbf{B}'^2_e - \frac{1}{3}\alpha_2 \mathbf{I}\right) \,, \\ &\eta = \hat{\eta} = -\frac{\partial \hat{\psi}}{\partial \theta} \,. \end{aligned} \tag{6.6.31}$$

Also, for isotropic response the entropy flux vector can be specified by a generalized Fourier form

$$\mathbf{p} = -\frac{\bar{\kappa}}{\theta}\,\mathbf{g}\,, \quad \bar{\kappa} = \bar{\kappa}(J_e, \alpha_1, \alpha_2, \theta) \geq 0\,, \tag{6.6.32}$$

where the heat conduction coefficient $\bar{\kappa}$ should not be confused with the hardening variable κ defined for inelastic response.

6.7 Thermoelastic–Inelastic Materials

For elastically anisotropic thermoelastic–inelastic materials, the microstructural vectors $\mathbf{m}_i$ are determined by integrating the evolution Eq. (5.11.30)

$$\dot{\mathbf{m}}_i = (\mathbf{L} - \mathbf{L}_p)\,\mathbf{m}_i\,, \quad \mathbf{L}_p = \Gamma\bar{\mathbf{L}}_p\,, \quad \Gamma \geq 0\,, \tag{6.7.1}$$

where $\bar{\mathbf{L}}_p$ controls the direction of the inelastic rate $\mathbf{L}_p$ and Γ is a non-negative function that controls its magnitude. In general, $\bar{\mathbf{L}}_p$ has a symmetric part $\bar{\mathbf{D}}_p$ that controls the direction of inelastic deformation rate and a skew-symmetric part $\bar{\mathbf{W}}_p$ that controls the direction of inelastic spin defined by

$$\bar{\mathbf{L}}_p = \bar{\mathbf{D}}_p + \bar{\mathbf{W}}_p\,, \quad \bar{\mathbf{D}}_p = \frac{1}{2}(\bar{\mathbf{L}}_p + \bar{\mathbf{L}}_p^T)\,, \quad \bar{\mathbf{W}}_p = \frac{1}{2}(\bar{\mathbf{L}}_p - \bar{\mathbf{L}}_p^T)\,, \tag{6.7.2}$$

both of which require constitutive equations. Also, for isotropic hardening, the hardening κ is determined by the evolution Eq. (5.11.31)

$$\dot{\kappa} = \Gamma H\,, \tag{6.7.3}$$

where H is a function that controls the rate of hardening. More general directional hardening can be modeled by introducing directional hardening variables $\beta_{ij} = \beta_{ji}$ which satisfy the evolution Eq. (5.11.32)

$$\dot{\beta}_{ij} = \Gamma H_{ij}\,, \tag{6.7.4}$$

where $H_{ij} = H_{ji}$ are functions that control the relative magnitudes of β_{ij}. In addition, the elastic deformation metric m_{ij} defined in (5.11.34) satisfies the evolution equations

$$m_{ij} = \mathbf{m}_i \cdot \mathbf{m}_j\,, \quad \dot{m}_{ij} = 2(\mathbf{D} - \Gamma\bar{\mathbf{D}}_p) \cdot (\mathbf{m}_i \otimes \mathbf{m}_j)\,. \tag{6.7.5}$$

Now, for an anisotropic thermoelastic–inelastic material, the response functions ψ, η and $\mathbf{T}$ are specified in the forms

$$\psi = \hat{\psi}(m_{ij}, \theta, \kappa, \beta_{ij}) , \quad \eta = \hat{\eta}(m_{ij}, \theta, \kappa, \beta_{ij}) , \quad \mathbf{T} = \hat{\mathbf{T}}(\mathbf{m}_i, \theta, \kappa, \beta_{ij}) , \tag{6.7.6}$$

so the rate of material dissipation (6.3.9) requires

$$\begin{aligned} \rho\theta\xi' = & \left(\mathbf{T} - 2\rho \frac{\partial\hat{\psi}}{\partial m_{ij}} \mathbf{m}_i \otimes \mathbf{m}_j\right) \cdot \mathbf{D} - \rho \left(\frac{\partial\hat{\psi}}{\partial\theta} + \eta\right) \dot{\theta} , \\ & + \Gamma \left[2\rho \frac{\partial\hat{\psi}}{\partial m_{ij}} \mathbf{m}_i \otimes \mathbf{m}_j \cdot \bar{\mathbf{D}}_p - \rho \frac{\partial\hat{\psi}}{\partial\kappa} H - \rho \frac{\partial\hat{\psi}}{\partial\beta_{ij}} H_{ij}\right] \geq 0 . \end{aligned} \tag{6.7.7}$$

In general, without specifying details of the functional forms for Γ, $\bar{\mathbf{D}}_p$, H and H_{ij}, it is not possible to determine necessary restrictions on the constitutive equations for $\mathbf{T}$ and η. However, motivated by necessary restrictions for a rate-independent elastic–inelastic material with a yield function, the constitutive equations for $\mathbf{T}$ and η are specified by

$$\mathbf{T} = \hat{\mathbf{T}} = 2\rho \frac{\partial\hat{\psi}}{\partial m_{ij}} \mathbf{m}_i \otimes \mathbf{m}_j , \quad \eta = \hat{\eta} = -\frac{\partial\hat{\psi}}{\partial\theta} , \tag{6.7.8}$$

so the rate of material dissipation imposes the restriction

$$\rho\theta\xi' = \Gamma \left[\mathbf{T} \cdot \bar{\mathbf{D}}_p - \rho \frac{\partial\hat{\psi}}{\partial\kappa} H - \rho \frac{\partial\hat{\psi}}{\partial\beta_{ij}} H_{ij}\right] \geq 0 . \tag{6.7.9}$$

Also, the constitutive equation for stress is assumed to be restricted so that a zero-stress material state at zero-stress reference temperature θ_z [also called a Reference Lattice State (RLS)] is characterized by

$$\mathbf{T} = 0 , \quad \frac{\partial\hat{\psi}}{\partial m_{ij}} = 0 \quad \text{for} \quad m_{ij} = \delta_{ij} , \quad \theta = \theta_z , \tag{6.7.10}$$

where δ_{ij} is the Kronecker delta. This means that the triad $\mathbf{m}_i$ has been defined so that $\mathbf{m}_i$ are orthonormal vectors in a zero-stress material state at zero-stress reference temperature $\theta = \theta_z$. In addition, the entropy flux $\mathbf{p}$ can be specified in the form

$$\mathbf{p} = p^i \mathbf{m}_i , \quad p^i = \hat{p}^i(m_{ij}, \theta, \kappa, g_i) , \quad g_i = \mathbf{g} \cdot \mathbf{m}_i , \tag{6.7.11}$$

which must satisfy the restriction (6.3.7) due to the second law of thermodynamics,

$$-\hat{\mathbf{p}} \cdot \mathbf{g} = -\hat{p}^i g_i > 0 \text{ for } \mathbf{g} \neq 0 . \tag{6.7.12}$$

The evolution Eq. (6.7.1) for $\mathbf{m}_i$, (6.7.29) for κ and (6.7.4) for β_{ij} require initial conditions

$$\mathbf{m}_i(0), \kappa(0), \beta_{ij}(0) . \tag{6.7.13}$$

A Separation of Elastic Dilatation and Distortional Deformations
To introduce separate control over the response of the material to dilatation and distortional rates of deformation, it is convenient to use the elastic dilatation J_e, the distortional deformation vectors $\mathbf{m}'_i$ and the elastic distortional deformation metric m'_{ij}, which satisfy the Eq. (5.11.45)

$$\begin{aligned}
&J_e = \mathbf{m}_1 \times \mathbf{m}_2 \cdot \mathbf{m}_3 > 0\,, && \dot{J}_e = J_e(\mathbf{D} - \Gamma \bar{\mathbf{D}}_p) \cdot \mathbf{I}\,, \\
&\mathbf{m}'_i = J_e^{-1/3}\, \mathbf{m}_i\,, && \dot{\mathbf{m}}'_i = (\mathbf{L}'' - \Gamma \bar{\mathbf{L}}''_p)\, \mathbf{m}'_i\,, \\
&m'_{ij} = \mathbf{m}'_i \cdot \mathbf{m}'_j = m'_{ji}\,, && \dot{m}'_{ij} = 2\left(\mathbf{m}'_i \otimes \mathbf{m}'_j - \tfrac{1}{3} m'_{ij} \mathbf{I}\right) \cdot (\mathbf{D} - \Gamma \bar{\mathbf{D}}_p)\,,
\end{aligned} \tag{6.7.14}$$

where $\mathbf{L}''$ is the deviatoric part of the velocity gradient $\mathbf{L}$ and $\bar{\mathbf{L}}''_p$ is the deviatoric part of $\bar{\mathbf{L}}_p$. Then, for an anisotropic thermoelastic–inelastic material, the response functions ψ, η and $\mathbf{T}$ are specified in the forms

$$\begin{aligned}
&\psi = \tilde{\psi}(J_e, m'_{ij}, \theta, \kappa, \beta_{ij})\,, \quad \eta = \tilde{\eta}(J_e, m'_{ij}, \theta, \kappa, \beta_{ij})\,, \\
&\mathbf{T} = \tilde{\mathbf{T}}(J_e, \mathbf{m}'_i, \theta, \kappa, \beta_{ij})\,,
\end{aligned} \tag{6.7.15}$$

so the rate of material dissipation (6.3.9) requires

$$\begin{aligned}
\rho\theta\xi' = &\left[\mathbf{T} - \rho J_e \frac{\partial \tilde{\psi}}{\partial J_e} \mathbf{I} - 2\rho \frac{\partial \tilde{\psi}}{\partial m'_{ij}} \left(\mathbf{m}'_i \otimes \mathbf{m}'_j - \frac{1}{3} m'_{ij} \mathbf{I}\right)\right] \cdot \mathbf{D} - \rho \left(\frac{\partial \hat{\psi}}{\partial \theta} + \eta\right) \dot{\theta}\,, \\
&+ \Gamma \left[\rho J_e \frac{\partial \tilde{\psi}}{\partial J_e} \bar{\mathbf{D}}_p \cdot \mathbf{I} + 2\rho \frac{\partial \tilde{\psi}}{\partial m'_{ij}} \left(\mathbf{m}'_i \otimes \mathbf{m}'_j - \frac{1}{3} m'_{ij} \mathbf{I}\right) \cdot \bar{\mathbf{D}}_p \right. \\
&\left. - \rho \frac{\partial \hat{\psi}}{\partial \kappa} H - \rho \frac{\partial \hat{\psi}}{\partial \beta_{ij}} H_{ij}\right] \geq 0\,,
\end{aligned} \tag{6.7.16}$$

where use has been made of (6.6.20). Again, without specifying details of the rate of inelasticity and the hardening functions Γ, $\bar{\mathbf{D}}_p$, H and H_{ij}, it is not possible to obtain necessary restrictions on the constitutive equation for stress and entropy. However, motivated by the constitutive Eq. (6.6.23) for a thermoelastic–inelastic material, the constitutive equations for stress and entropy in a thermomelastic–inelastic material are specified by

$$\begin{aligned}
&\mathbf{T} = -p\,\mathbf{I} + \mathbf{T}'' = \tilde{\mathbf{T}}\,, \quad p = \tilde{p} = -\rho J_e \frac{\partial \tilde{\psi}}{\partial J_e}\,, \\
&\mathbf{T}'' = \tilde{\mathbf{T}}'' = 2\rho \frac{\partial \tilde{\psi}}{\partial m'_{ij}} \left(\mathbf{m}'_i \otimes \mathbf{m}'_j - \frac{1}{3} m'_{ij} \mathbf{I}\right)\,, \\
&\eta = \tilde{\eta} = -\frac{\partial \tilde{\psi}}{\partial \theta}\,,
\end{aligned} \tag{6.7.17}$$

so the rate of material dissipation imposes the restriction

$$\rho\theta\xi' = \Gamma\Big[-\tilde{p}\,(\mathbf{I}\cdot\bar{\mathbf{D}}_p) + \tilde{\mathbf{T}}''\cdot\bar{\mathbf{D}}_p - \rho\frac{\partial\tilde{\psi}}{\partial\kappa}H - \rho\frac{\partial\tilde{\psi}}{\partial\beta_{ij}}H_{ij}\Big] \geq 0\,. \tag{6.7.18}$$

Also, the constitutive equation for stress is assumed to be restricted so that a zero-stress material state at zero-stress reference temperature θ_z is characterized by

$$\begin{gathered}\mathbf{T}=0\,,\quad \frac{\partial\tilde{\psi}}{\partial J_e}=0\,,\quad \frac{\partial\tilde{\psi}}{\partial m'_{ij}} = \frac{1}{3}\frac{\partial\tilde{\psi}}{\partial m'_{nn}}\delta_{ij}\\ \text{for } J_e=1\,,\quad m'_{ij}=\delta_{ij}\,,\quad \theta=\theta_z\,,\end{gathered} \tag{6.7.19}$$

where δ_{ij} is the Kronecker delta. This means that the triad $\mathbf{m}'_i$ has been defined so that $\mathbf{m}'_i$ are orthonormal vectors in a zero-stress material state at zero-stress reference temperature $\theta=\theta_z$. In addition, the entropy flux $\mathbf{p}$ can be specified in the form (6.6.24)

$$\mathbf{p} = p^{i\prime}\mathbf{m}'_i\,,\quad p^{i\prime} = \tilde{p}^{i\prime}(J_e, m'_{ij}, \theta, g'_i)\,,\quad g'_i = \mathbf{g}\cdot\mathbf{m}'_i\,, \tag{6.7.20}$$

which must satisfy the restriction (6.3.7) due to the second law of thermodynamics,

$$-\mathbf{p}\cdot\mathbf{g} = -\tilde{p}^{i\prime}g'_i > 0 \text{ for } \mathbf{g}\neq 0\,. \tag{6.7.21}$$

The evolution Eq. (6.7.14) for J_e and $\mathbf{m}'_i$, (6.7.29) for κ and (6.7.4) for β_{ij} require initial conditions

$$J_e(0), \mathbf{m}'_i(0), \kappa(0), \beta_{ij}(0)\,. \tag{6.7.22}$$

Examples where this formulation has been used to model elastic and inelastic anisotropy in geological materials with joints can be found in [25, 35]. Also, notice that inelastic dilatation rate $\bar{\mathbf{D}}_p\cdot\mathbf{I}\neq 0$ in (6.7.14) prevents the elastic dilatation J_e from being written in a simple form like (6.6.20) since the zero-stress density of the material at zero-stress reference temperature need not be constant.

Elastically Isotropic Thermoelastic–Inelastic Response

For elastically isotropic thermoelastic–inelastic response, the Helmholtz free energy depends on the invariants of the metric m_{ij}. Then, following the definitions of pure dilatation and pure distortion proposed by Flory [6], the elastic dilatation J_e is defined by (3.11.7)

$$J_e = \mathbf{m}_1\times\mathbf{m}_2\cdot\mathbf{m}_3 > 0\,, \tag{6.7.23}$$

and the symmetric, positive-definite, unimodular elastic distortional deformation tensor $\mathbf{B}'_e$ is defined by (5.8.1)

$$\mathbf{B}'_e = \mathbf{m}'_i\otimes\mathbf{m}'_i\,. \tag{6.7.24}$$

Then, the evolution equation for J_e is given by (6.7.14)

$$\dot{J}_e = J_e(\mathbf{D} - \Gamma\bar{\mathbf{D}}_p)\cdot\mathbf{I}\,, \tag{6.7.25}$$

and $\mathbf{B}'_e$ satisfies the evolution Eq. (5.11.66) with the specification

$$\dot{\mathbf{B}}'_e = \mathbf{L}''\mathbf{B}'_e + \mathbf{B}'_e\mathbf{L}''^T - \Gamma\mathbf{A}_p\,, \quad \mathbf{A}_p = \mathbf{B}'_e - \left(\frac{3}{\mathbf{B}'^{-1}_e\cdot\mathbf{I}}\right)\mathbf{I}\,. \tag{6.7.26}$$

Also, the non-trivial invariants α_1 and α_2 of $\mathbf{B}'_e$ are given by (5.11.58)

$$\alpha_1 = \mathbf{B}'_e\cdot\mathbf{I}\,, \quad \alpha_2 = \mathbf{B}'_e\cdot\mathbf{B}'_e\,, \tag{6.7.27}$$

which satisfy the evolution Eq. (5.11.59)

$$\begin{aligned} \dot{\alpha}_1 &= 2\mathbf{B}''_e\cdot\mathbf{D} - \Gamma\mathbf{A}_p\cdot\mathbf{I}\,, \\ \dot{\alpha}_2 &= 4\left(\mathbf{B}'^2_e - \frac{1}{3}\alpha_2\mathbf{I}\right)\cdot\mathbf{D} - 2\Gamma\mathbf{A}_p\cdot\mathbf{B}'_e\,, \end{aligned} \tag{6.7.28}$$

where $\mathbf{B}''_e$ is the deviatoric part of $\mathbf{B}'_e$. In addition, attention is limited to isotropic hardening κ, which satisfies the evolution Eq. (6.7.29)

$$\dot{\kappa} = \Gamma H\,. \tag{6.7.29}$$

For an elastically isotropic thermoelastic–inelastic material, the response functions ψ, η and $\mathbf{T}$ are specified in the forms

$$\psi = \psi(J_e, \alpha_1, \alpha_2, \theta, \kappa)\,, \quad \eta = \eta(J_e, \alpha_1, \alpha_2, \theta, \kappa)\,, \quad \mathbf{T} = \mathbf{T}(J_e, \mathbf{B}'_e, \theta, \kappa)\,, \tag{6.7.30}$$

so the rate of material dissipation (6.3.9) requires

$$\begin{aligned} \rho\theta\xi' = &\left[\mathbf{T} - \rho J_e\frac{\partial\psi}{\partial J_e}\mathbf{I} - 2\rho\frac{\partial\psi}{\partial\alpha_1}\mathbf{B}''_e - 4\rho\frac{\partial\psi}{\partial\alpha_2}\left(\mathbf{B}'^2_e - \frac{1}{3}\alpha_2\mathbf{I}\right)\right]\cdot\mathbf{D} - \rho\left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta}\,, \\ &+ \Gamma\left[\rho J_e\frac{\partial\psi}{\partial J_e}\bar{\mathbf{D}}_p\cdot\mathbf{I} + 2\rho\frac{\partial\psi}{\partial\alpha_1}\mathbf{A}_p\cdot\mathbf{I} + 4\rho\frac{\partial\psi}{\partial\alpha_2}\mathbf{A}_p\cdot\mathbf{B}'_e - \rho\frac{\partial\psi}{\partial\kappa}H\right] \ge 0\,. \end{aligned} \tag{6.7.31}$$

In general, without specifying details of the functional forms for Γ, $\bar{\mathbf{D}}_p$ and H, it is not possible to determine necessary restrictions on the constitutive equations for $\mathbf{T}$ and η. However, motivated by necessary restrictions for a rate-independent elastic–inelastic material with a yield function, the constitutive equations for $\mathbf{T}$ and η are specified by

$$\begin{aligned}
&\mathbf{T} = -p\,\mathbf{I} + \mathbf{T}'' \,, \quad p = -\rho J_e \frac{\partial \psi}{\partial J_e} \,, \\
&\mathbf{T}'' = 2\rho \frac{\partial \psi}{\partial \alpha_1} \mathbf{B}_e'' + 4\rho \frac{\partial \psi}{\partial \alpha_2} \left(\mathbf{B}_e'^2 - \frac{1}{3}\alpha_2 \mathbf{I} \right) \,, \\
&\eta = -\frac{\partial \psi}{\partial \theta} \,,
\end{aligned} \tag{6.7.32}$$

so the rate of material dissipation imposes the restriction

$$\rho\theta\xi' = \Gamma \Big[-p\,(\bar{\mathbf{D}}_p \cdot \mathbf{I}) + 2\rho \frac{\partial \psi}{\partial \alpha_1} \mathbf{A}_p \cdot \mathbf{I} + 4\rho \frac{\partial \psi}{\partial \alpha_2} \mathbf{A}_p \cdot \mathbf{B}_e' - \rho \frac{\partial \psi}{\partial \kappa} H \Big] \geq 0 \,. \tag{6.7.33}$$

Also, the constitutive equation for stress is assumed to be restricted so that a zero-stress material state at zero-stress reference temperature θ_z is characterized by

$$\mathbf{T} = 0 \,, \quad \frac{\partial \psi}{\partial J_e} = 0 \,, \quad \text{for } J_e = 1 \,, \quad \mathbf{B}_e' = \mathbf{I} \,, \quad \theta = \theta_z \,. \tag{6.7.34}$$

In addition, for isotropic response the entropy flux can be specified by a generalized Fourier form (6.6.32)

$$\mathbf{p} = -\frac{\bar{\kappa}}{\theta} \mathbf{g} \,, \quad \bar{\kappa} = \bar{\kappa}(J_e, \alpha_1, \alpha_2, \theta, \kappa) \geq 0 \,, \tag{6.7.35}$$

where the heat conduction coefficient $\bar{\kappa}$ should not be confused with the hardening variable κ.

6.8 Orthotropic Thermoelastic–Inelastic Materials

A homogeneous, isotropic elastic material deformed from a uniform zero-stress material state to a Hydrostatic State of Stress (HSS)

$$\mathbf{T} = -p\,\mathbf{I} \,, \quad \mathbf{T}'' = 0 \tag{6.8.1}$$

is compressed but it experiences no distortional deformation. This means that a cube of a homogeneous, isotropic elastic material in a zero-stress material state will deform to a smaller cube when it is compressed by water causing a HSS. In contrast, a homogeneous, orthotropic elastic material deformed from a uniform zero-stress material state to a compressed HSS is distorted as well as being compressed. This means that a cube of a homogeneous, orthotropic elastic material in a zero-stress material state will deform to a rectangular parallelepiped when it is compressed by water causing a HSS. This also means that additional distortional deformation is required to produce deviatoric stress.

Rubin and Jabareen [20, 21] considered this problem for the purely mechanical theory of orthotropic elastic materials and developed physically based orthotropic invariants which characterize the added distortional deformations that cause deviatoric stress.

Recently, this approach was generalized to model an orthotropic thermoelastic–inelastic material [27] where additional details can be found. Motivated by the work in [14], the equations for this theory are slightly modified relative to those reported in [27]. For this theory use is made of the elastic dilatation J_e, the elastic distortional deformation vectors $\mathbf{m}'_i$ and the elastic distortional deformation metric m'_{ij} which satisfy the evolution Eq. (6.7.14)

$$\begin{aligned} &\dot{J}_e = J_e(\mathbf{D} - \Gamma\bar{\mathbf{D}}_p)\cdot\mathbf{I}\,, \quad \dot{\mathbf{m}}'_i = (\mathbf{L}'' - \Gamma\bar{\mathbf{L}}''_p)\,\mathbf{m}'_i\,, \\ &m'_{ij} = \mathbf{m}'_i\cdot\mathbf{m}'_j = m'_{ji}\,, \quad \dot{m}'_{ij} = 2(\mathbf{m}'_i\otimes\mathbf{m}'_j - \tfrac{1}{3}m'_{ij}\mathbf{I})\cdot(\mathbf{D} - \Gamma\bar{\mathbf{D}}_p)\,, \end{aligned} \tag{6.8.2}$$

where $\mathbf{L}''$ is the deviatoric part of the velocity gradient $\mathbf{L}$, $\bar{\mathbf{L}}''_p$ is the deviatoric part of the direction $\bar{\mathbf{L}}_p$ of inelastic rate and the non-negative function Γ controls the magnitude of the inelastic deformation rate. Using (6.7.2), the direction of inelastic deformation rate $\bar{\mathbf{D}}_p$ and the direction of inelastic spin $\bar{\mathbf{W}}_p$ are defined by

$$\bar{\mathbf{D}}_p = \frac{1}{2}\left(\bar{\mathbf{L}}_p + \bar{\mathbf{L}}_p^T\right)\,, \quad \bar{\mathbf{W}}_p = \frac{1}{2}\left(\bar{\mathbf{L}}_p - \bar{\mathbf{L}}_p^T\right)\,. \tag{6.8.3}$$

In addition, the isotropic hardening variable κ is determined by the evolution Eq. (6.7.29)

$$\dot{\kappa} = \Gamma H\,, \tag{6.8.4}$$

where H is a function that controls the rate of hardening and the directional hardening variables $\beta_{ij} = \beta_{ji}$ are determined by the evolution Eq. (6.7.4)

$$\dot{\beta}_{ij} = \Gamma H_{ij}\,, \tag{6.8.5}$$

where $H_{ij} = H_{ji}$ are functions that control the relative magnitudes of β_{ij}.

A thermoelastic orthotropic material in any HSS experiences mechanical distortion and can also experience distortion due to thermal effects. Specifically, in any HSS the elastic distortional vectors $\mathbf{m}'_i$, their reciprocal vectors $\mathbf{m}^{i\prime}$ and the metrics m'_{ij} and $m^{ij\prime}$ take the forms

$$\begin{aligned} &\tfrac{1}{\eta_1}\mathbf{m}'_1 = \eta_1\mathbf{m}^{1\prime}\,, && \tfrac{1}{\eta_2}\mathbf{m}'_2 = \eta_2\mathbf{m}^{2\prime}\,, && \tfrac{1}{\eta_3}\mathbf{m}'_3 = \eta_3\mathbf{m}^{3\prime}\,, \\ &m'_{11} = \eta_1^2\,, && m'_{22} = \eta_2^2\,, && m'_{33} = \eta_3^2\,, \\ &m'_{12} = 0\,, && m'_{13} = 0\,, && m'_{23} = 0\,, \\ &m^{11\prime} = \tfrac{1}{\eta_1^2}\,, && m^{22\prime} = \tfrac{1}{\eta_2^2}\,, && m^{33\prime} = \tfrac{1}{\eta_3^2}\,, \\ &m^{12\prime} = 0\,, && m^{13\prime} = 0\,, && m^{23\prime} = 0\,, \end{aligned} \tag{6.8.6}$$

which indicate that $\mathbf{m}'_i$ and $\mathbf{m}^{i\prime}$ are orthogonal triads of vectors parallel to the principal directions of orthotropy of the material. Moreover, η_i are dependent positive constitutive distortional deformation functions of the elastic dilatation J_e and temperature θ satisfying the restrictions

$$\eta_i = \eta_i(J_e, \theta) > 0\,, \quad \eta_1\eta_2\eta_3 = 1\,, \quad \eta_i(1, \theta_z) = 1\,, \tag{6.8.7}$$

with η_i being unity in any Reference Lattice State (RLS) at reference zero-stress temperature θ_z.

Since any HSS causes a cube of the orthotropic material to deform into a rectangular parallelepiped, in principle, it is possible to use a triaxial testing machine to measure the two independent functions η_1 and η_2 for different values of J_e and θ. Of course, experimental challenges related to friction on the sides of the specimen and difficulties with applying tension while letting the sides of the specimen slide freely, currently limit the range of J_e and θ that can be explored experimentally.

The Helmholtz free energy for an orthotropic thermoelastic–inelastic material can be expressed in the form

$$\psi = \psi(J_e, m'_{ij}, \theta)\,. \tag{6.8.8}$$

Specific functional forms for ψ must be proposed which match the distortions (6.8.6) in any HSS that are characterized by the measured values of the functions η_i as well as match additional experimental data for the added distortions that cause deviatoric stress.

The main idea in [27] is to rewrite the six generalized physically based orthotropic invariants β_i developed in [21] in terms of the elastic distortional deformation metric m'_{ij} and its inverse $m^{ij\prime}$, which are influenced by inelastic deformation rate, and to generalize the characterization of any HSS to include thermoelastic response. Then, the Helmholtz free energy ψ is specified in the form

$$\psi = \psi(J_e, \beta_i, \theta)\,. \tag{6.8.9}$$

Since the invariants β_i are based on the distortional deformation functions η_i, it follows that this representation of ψ automatically reproduces the distortions η_i and the results (6.8.6) in any HSS. This simplifies the constitutive modeling of an orthotropic thermoelastic–inelastic material because a specific form ψ in (6.8.9) need only consider the pressure in any HSS and experimental data for distortional deformations that cause deviatoric stress.

Specifically, using the functional forms for the distortions η_i, the physically based orthotropic invariants β_i are defined by

$$\begin{aligned}
&\beta_1 = \tfrac{m'_{11}}{\eta_1^2} + \eta_1^2\, m^{11\prime}\,, \quad \beta_2 = \tfrac{m'_{22}}{\eta_2^2} + \eta_2^2\, m^{22\prime}\,, \\
&\beta_3 = \tfrac{m'_{33}}{\eta_3^2} + \eta_3^2\, m^{33\prime}\,, \quad \beta_i \geq 2 \quad \text{for} \quad i = 1, 2, 3\,, \\
&\beta_4 = \tfrac{m'^2_{12}}{m'_{11}m'_{22}}\,, \qquad \beta_5 = \tfrac{m'^2_{13}}{m'_{11}m'_{33}}\,, \qquad \beta_6 = \tfrac{m'^2_{23}}{m'_{22}m'_{33}}\,,
\end{aligned} \tag{6.8.10}$$

which by definition attain the values

$$\beta_1 = \beta_2 = \beta_3 = 2\,, \quad \beta_4 = \beta_5 = \beta_6 = 0 \quad \text{for any } HSS\,. \tag{6.8.11}$$

Moreover, with the help of (6.8.7) it can be shown that

$$\begin{aligned} \dot{\eta}_i &= J_e \frac{\partial \eta_i}{\partial J_e} \mathbf{I} \cdot (\mathbf{D} - \Gamma \bar{\mathbf{D}}_p) + \frac{\partial \eta_i}{\partial \theta} \dot{\theta}\,, \\ \dot{\beta}_i &= 2(-N_i \mathbf{I} + \mathbf{B}_i'') \cdot (\mathbf{D} - \Gamma \bar{\mathbf{D}}_p) + 2 A_i \dot{\theta}\,, \end{aligned} \tag{6.8.12}$$

where the functions N_i and A_i are defined by

$$\begin{aligned} N_1 &= \tfrac{J_e}{\eta_1} \left(\tfrac{m_{11}'}{\eta_1^2} - \eta_1^2 m^{11\prime} \right) \tfrac{\partial \eta_1}{\partial J_e}\,, \quad & N_2 &= \tfrac{J_e}{\eta_2} \left(\tfrac{m_{22}'}{\eta_2^2} - \eta_2^2 m^{22\prime} \right) \tfrac{\partial \eta_2}{\partial J_e}\,, \\ N_3 &= \tfrac{J_e}{\eta_3} \left(\tfrac{m_{33}'}{\eta_3^2} - \eta_3^2 m^{33\prime} \right) \tfrac{\partial \eta_3}{\partial J_e}\,, \quad & N_4 &= N_5 = N_6 = 0\,, \\ A_1 &= -\tfrac{1}{\eta_1} \left(\tfrac{m_{11}'}{\eta_1^2} - \eta_1^2 m^{11\prime} \right) \tfrac{\partial \eta_1}{\partial \theta}\,, \quad & A_2 &= -\tfrac{1}{\eta_2} \left(\tfrac{m_{22}'}{\eta_2^2} - \eta_2^2 m^{22\prime} \right) \tfrac{\partial \eta_2}{\partial \theta}\,, \\ A_3 &= -\tfrac{1}{\eta_3} \left(\tfrac{m_{33}'}{\eta_3^2} - \eta_3^2 m^{33\prime} \right) \tfrac{\partial \eta_3}{\partial \theta}\,, \quad & A_4 &= A_5 = A_6 = 0\,. \end{aligned} \tag{6.8.13}$$

Also, the deviatoric tensors $\mathbf{B}_i''$ are defined by

$$\begin{aligned} \mathbf{B}_1'' &= \frac{1}{\eta_1^2} \mathbf{m}_1' \otimes \mathbf{m}_1' - \eta_1^2 \mathbf{m}^{1\prime} \otimes \mathbf{m}^{1\prime} - \frac{1}{3} \left(\frac{m_{11}'}{\eta_1^2} - \eta_1^2 m^{11\prime} \right) \mathbf{I}\,, \\ \mathbf{B}_2'' &= \frac{1}{\eta_2^2} \mathbf{m}_2' \otimes \mathbf{m}_2' - \eta_2^2 \mathbf{m}^{2\prime} \otimes \mathbf{m}^{2\prime} - \frac{1}{3} \left(\frac{m_{22}'}{\eta_2^2} - \eta_2^2 m^{22\prime} \right) \mathbf{I}\,, \\ \mathbf{B}_3'' &= \frac{1}{\eta_3^2} \mathbf{m}_3' \otimes \mathbf{m}_3' - \eta_3^2 \mathbf{m}^{3\prime} \otimes \mathbf{m}^{3\prime} - \frac{1}{3} \left(\frac{m_{33}'}{\eta_3^2} - \eta_3^2 m^{33\prime} \right) \mathbf{I}\,, \\ \mathbf{B}_4'' &= \frac{m_{12}'}{m_{11}' m_{22}'} \left[(\mathbf{m}_1' \otimes \mathbf{m}_2' + \mathbf{m}_2' \otimes \mathbf{m}_1') - \frac{m_{12}'}{m_{11}'} (\mathbf{m}_1' \otimes \mathbf{m}_1') - \frac{m_{12}'}{m_{22}'} (\mathbf{m}_2' \otimes \mathbf{m}_2') \right]\,, \\ \mathbf{B}_5'' &= \frac{m_{13}'}{m_{11}' m_{33}'} \left[(\mathbf{m}_1' \otimes \mathbf{m}_3' + \mathbf{m}_3' \otimes \mathbf{m}_1') - \frac{m_{13}'}{m_{11}'} (\mathbf{m}_1' \otimes \mathbf{m}_1') - \frac{m_{13}'}{m_{33}'} (\mathbf{m}_3' \otimes \mathbf{m}_3') \right]\,, \\ \mathbf{B}_6'' &= \frac{m_{23}'}{m_{22}' m_{33}'} \left[(\mathbf{m}_2' \otimes \mathbf{m}_3' + \mathbf{m}_3' \otimes \mathbf{m}_2') - \frac{m_{23}'}{m_{22}'} (\mathbf{m}_2' \otimes \mathbf{m}_2') - \frac{m_{23}'}{m_{33}'} (\mathbf{m}_3' \otimes \mathbf{m}_3') \right]\,. \end{aligned} \tag{6.8.14}$$

In particular, using (6.8.6), it follows that

$$N_i = 0\,, \quad A_i = 0\,, \quad \mathbf{B}_i'' = 0 \quad \text{for any } HSS\,. \tag{6.8.15}$$

Now, with the help of (6.8.9) and (6.8.12), the rate of material dissipation (6.3.9) requires

$$\rho\theta\xi' = \left[\mathbf{T}\cdot\mathbf{D} - \left\{\rho J_e \frac{\partial\psi}{\partial J_e} - 2\sum_{i=1}^{3}\rho\frac{\partial\psi}{\partial\beta_i}N_i\right\}\mathbf{I} - 2\sum_{i=1}^{6}\rho\frac{\partial\psi}{\partial\beta_i}\mathbf{B}_i''\right]\cdot(\mathbf{D}-\Gamma\bar{\mathbf{D}}_p)$$
$$-\rho\left[\frac{\partial\psi}{\partial\theta} + 2\sum_{i=1}^{6}\left(\frac{\partial\psi}{\partial\beta_i}A_i\right) + \eta\right]\dot{\theta} \geq 0\,. \tag{6.8.16}$$

Motivated by this expression, the constitutive equations for the Cauchy stress **T** and the entropy η are specified by

$$\begin{aligned} &\mathbf{T} = -p\,\mathbf{I} + \mathbf{T}''\,, && \mathbf{T}'' = 2\textstyle\sum_{i=1}^{6}\rho\frac{\partial\psi}{\partial\beta_i}\mathbf{B}_i''\,,\\ &p = -\rho J_e\frac{\partial\psi}{\partial J_e} + 2\textstyle\sum_{i=1}^{3}\rho\frac{\partial\psi}{\partial\beta_i}N_i\,, && \eta = -\frac{\partial\psi}{\partial\theta} - 2\textstyle\sum_{i=1}^{6}\frac{\partial\psi}{\partial\beta_i}A_i\,, \end{aligned} \tag{6.8.17}$$

where p is the pressure and $\mathbf{T}''$ is the deviatoric stress. It then follows that the rate of material dissipation requires

$$\rho\theta\xi' = -p\,(\Gamma\bar{\mathbf{D}}_p\cdot\mathbf{I}) + \mathbf{T}''\cdot\Gamma\bar{\mathbf{D}}_p \geq 0\,. \tag{6.8.18}$$

Also, the entropy flux vector can be specified by

$$\mathbf{p} = -\frac{\mathbf{K}}{\theta}\mathbf{g}\,,\quad \mathbf{g} = \frac{\partial\theta}{\partial\mathbf{x}}\,,\quad \mathbf{K} = \mathbf{K}^T = K^{ij}\mathbf{m}_i'\otimes\mathbf{m}_j'\,,\quad K^{ji} = K^{ij}\,, \tag{6.8.19}$$

where K^{ij} is a positive-definite symmetric matrix that characterizes anisotropic heat conduction coefficients. Since K^{ij} is a positive-definite, it follows that the restriction of the second law of thermodynamics which requires heat to flow from hot to cold (6.3.7)

$$-\,\mathbf{p}\cdot\mathbf{g} > 0 \text{ for } \mathbf{g}\neq 0\,, \tag{6.8.20}$$

is automatically satisfied.

Also, notice that inelastic dilatation rate $\bar{\mathbf{D}}_p\cdot\mathbf{I}\neq 0$ in (6.8.2) prevents the elastic dilatation J_e from being written in a simple form like (6.6.20) since the zero-stress density of the material at zero-stress reference temperature θ_z need not be constant.

Specific Constitutive Equations

To simplify the discussion, consider the case of metal plasticity for which inelastic deformation rate is isochoric so that

$$\bar{\mathbf{D}}_p\cdot\mathbf{I} = 0\,, \tag{6.8.21}$$

and the elastic dilatation J_e can be expressed in the form (4.1.16).

$$J_e = \frac{\rho_z}{\rho}\,, \tag{6.8.22}$$

where ρ_z is the constant zero-stress mass density at zero-stress reference temperature θ_z.

Next, motivated by the work in [16, 26] for a material with a constant specific heat C_v, the Helmholtz free energy is specified by

$$\begin{aligned}
&\psi = \psi_1(J_e, \theta) + \psi_2(J_e, \theta, \beta_i)\,, \\
&\rho_z\psi_1 = \rho_z C_v \left[\theta - \theta_z - \theta \ln\left(\frac{\theta}{\theta_z}\right)\right] - (\theta - \theta_z) f_1(J_e) + f_2(J_e)\,, \\
&\rho_z\psi_2 = \frac{1}{2}\sum_{i=1}^{3} K_i(\beta_i - 2) + \frac{1}{2}\sum_{i=4}^{6} K_i\beta_i\,, \quad K_i \geq 0\,,
\end{aligned} \tag{6.8.23}$$

where ψ_1 controls the isotropic thermomechanical response and ψ_2 controls the orthotropic response in any HSS as well as the influence of additional distortional deformation that causes deviatoric stress. In these expressions, f_1 controls strong thermomechanical coupling and f_2 controls the isotropic response to nonlinear compression that can be determined by plate impact experiments for shock waves as discussed in Sect. 6.9. In addition, K_i control the influences of the orthotropic invariants β_i. It then follows from (6.8.17) that the associated constitutive equations are given by

$$\begin{aligned}
&p = p_1(J_e, \theta) + p_2(J_e, \theta, \beta_i)\,, \qquad \mathbf{T}'' = J_e^{-1}\sum_{i=1}^{6} K_i \mathbf{B}_i''\,, \\
&p_1 = (\theta - \theta_z)\frac{df_1}{dJ_e} - \frac{df_2}{dJ_e}\,, \qquad p_2 = J_e^{-1}\sum_{i=1}^{3} K_i N_i\,, \\
&\eta = \hat{\eta}_1(J_e, \theta) + \hat{\eta}_2(J_e, \theta, \beta_i)\,, \\
&\rho_z\hat{\eta}_1 = \rho_z C_v \ln\left(\frac{\theta}{\theta_z}\right) + f_1(J_e)\,, \quad \rho_z\hat{\eta}_2 = -\sum_{i=1}^{3} K_i A_i\,.
\end{aligned} \tag{6.8.24}$$

In these expressions, p_1 and $\hat{\eta}_1$ control the isotropic thermomechanical response and p_2 and $\hat{\eta}_2$ control the orthotropic response in any HSS as well as the influence of additional distortional deformation that causes deviatoric stress. Although this form for ψ automatically predicts the correct distortions η_i in any HSS, the functions f_1 and f_2 need to be determined to match experimental data for the pressure in each HSS. Also, the functions $\hat{\eta}_1$ and $\hat{\eta}_2$ should not be confused with the function η_1 and η_2 in (6.8.6) which characterize distortional deformations in any HSS. Furthermore, using (6.2.11) the internal energy ε can be expressed in the form

$$\begin{aligned}
&\varepsilon = \psi + \theta\eta = \varepsilon_1(J_e, \theta) + \varepsilon_2(J_e, \theta, \beta_i)\,, \\
&\rho_z\varepsilon_1 = \rho_z C_v(\theta - \theta_z) + \theta_z f_1(J_e) + f_2(J_e)\,, \\
&\rho_z\varepsilon_2 = \frac{1}{2}\sum_{i=1}^{3} K_i(\beta_i - 2) + \frac{1}{2}\sum_{i=4}^{6} K_i\beta_i - \sum_{i=1}^{3} K_i\theta A_i\,.
\end{aligned} \tag{6.8.25}$$

Now, following the work in [27] the distortions η_i in (6.8.6) for any HSS are specified by the forms

$$\begin{aligned}
&\eta_i = J_e^{n_i/3}(\frac{\theta}{\theta_z})^{(\alpha_i\theta_z/3)}\,, \quad n_i = n_i(J_e, \theta)\,, \\
&\alpha_i = \alpha_i(J_e, \theta)\,,
\end{aligned} \tag{6.8.26}$$

where in view of (6.8.7) the functions n_i, α_i must satisfy the restrictions

$$n_1 + n_2 + n_3 = 0\,, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0\,. \tag{6.8.27}$$

Since the inelastic deformation rate is isochoric (6.8.21), the rate of material dissipation (6.8.18) requires

$$\rho\theta\xi' = \mathbf{T}'' \cdot \Gamma\bar{\mathbf{D}}_p \geq 0\,. \tag{6.8.28}$$

Then, following the work in [27] the direction of inelastic deformation rate $\bar{\mathbf{D}}_p$ is specified by

$$\begin{aligned}
&\bar{\mathbf{D}}_p = \sum_{i=1}^{6} d_i \mathrm{Sign}(\mathbf{T}'' \cdot \mathbf{B}_i'')\,, \quad d_i \geq 0\,, \\
&\mathrm{Sign}(x) = -1 \quad \text{for} \quad x < 0\,, \quad \mathrm{Sign}(x) = 1 \quad \text{for} \quad x \geq 0\,,
\end{aligned} \tag{6.8.29}$$

where d_i are non-negative constants so that the rate of material dissipation (6.8.28) is automatically satisfied. These restrictions on d_i are sufficient but not necessary conditions for (6.8.28) to be satisfied. For example, the work on metal forming with inelastic orthotropic deformation rate in [14] developed more relaxed restrictions on d_i which allow some of the d_i to be negative but which satisfy (6.8.28) for small elastic distortional deformations. Also, in [14] the direction of inelastic spin $\bar{\mathbf{W}}_p$ was specified by

$$\begin{aligned}
\bar{\mathbf{W}}_p = {}&\Omega_{12}\left[\bar{\mathbf{D}}_p \cdot (\mathbf{m}_1' \otimes \mathbf{m}_2')\right](\mathbf{m}^{1'} \otimes \mathbf{m}^{2'} - \mathbf{m}^{2'} \otimes \mathbf{m}^{1'}) \\
&+ \Omega_{13}\left[\bar{\mathbf{D}}_p \cdot (\mathbf{m}_1' \otimes \mathbf{m}_3')\right](\mathbf{m}^{3'} \otimes \mathbf{m}^{1'} - \mathbf{m}^{1'} \otimes \mathbf{m}^{3'}) \\
&+ \Omega_{23}\left[\bar{\mathbf{D}}_p \cdot (\mathbf{m}_2' \otimes \mathbf{m}_3')\right](\mathbf{m}^{2'} \otimes \mathbf{m}^{3'} - \mathbf{m}^{3'} \otimes \mathbf{m}^{2'})\,,
\end{aligned} \tag{6.8.30}$$

where Ω_{12}, Ω_{13} and Ω_{23} are constants that control the magnitude of the inelastic spin which influences the rate of rotation of the distortional microstructural vectors $\mathbf{m}_i'$.

Furthermore, in [14] an orthotropic yield function g for purely mechanical response was proposed in the form

$$g = g(T_{ij}, \kappa)\,, \tag{6.8.31}$$

where T_{ij} are the components of $\mathbf{T}$ relative to the elastic distortional deformation base vectors $\mathbf{m}'_i$ defined by

$$\begin{aligned} &T_{ij} = -p\, m'_{ij} + T''_{ij} = T_{ji}\,, && T''_{ij}\, m^{ij\prime} = 0\,, \\ &T''_{ij} = \mathbf{T}'' \cdot (\mathbf{m}'_i \otimes \mathbf{m}'_j)\,, && \mathbf{T}'' = T''_{ij}\, (\mathbf{m}^{i\prime} \otimes \mathbf{m}^{j\prime})\,. \end{aligned} \tag{6.8.32}$$

In this regard, it is important to emphasize that the components T_{ij} of $\mathbf{T}$ are uninfluenced by SRBM since their values T^+_{ij} in the superposed configuration are given by

$$T^+_{ij} = T_{ij}\,. \tag{6.8.33}$$

Next, the function Γ, which controls the magnitude of inelastic deformation rate, can be determined by the loading conditions (5.11.79) for rate-independent response or by (5.11.80) which models a smooth elastic–inelastic transition for rate-independent response and rate-dependent response.

Since the orthotropic invariants β_i are valid for large deformations, this formulation generalizes more standard formulations based on a quadratic strain energy function like (5.8.24) which are only accurate for moderate strains [see the end of Sect. 5.8]. Consequently, this formulation based on β_i can be used for soft materials which experience large deformations [27].

6.9 Thermoelastic–Inelastic Materials for Shock Waves

Plate impact experiments have been used for decades to study the dynamic response of materials, especially to strong shocks. Specifically, in the simplest form of a plate impact experiment, a circular cylindrical flyer plate of a known material is accelerated in a gas gun toward a circular cylindrical target plate of the material that is being studied, which has a circular cylindrical backup plate of a known transparent material. A laser is used to measure the axial velocity of a material point at the center of the back of the target plate. Since the wave propagation speed in each material is finite, there is a finite time window in which reflections from free surfaces do not reach material points on the centerline of the target plate. Moreover, since the material is shock loaded very rapidly, it is assumed that there is no time for heat conduction so the heat flux vector $\mathbf{q}$ is taken to be zero. Within this time window the deformation of the target is uniaxial strain and the equations of mass conservation and the balances of linear momentum and energy can be solved to obtain Rankine–Hugoniot jump conditions between the density, axial stress and internal energy as functions of the particle velocity and the shock velocity. The system of equations is closed by proposing a relationship between the shock velocity and the particle velocity, which are measured experimentally.

Since inelastic effects limit the magnitude of the deviatoric stress, the axial stress in strong shocks can be approximated by the pressure. The plate impact experimental data can be used to determine the Hugoniot curve for the target material which connects the pressure to the density for the equilibrium state after the shock has passed the material point of interest. For strong shocks, this creates strong thermomechanical coupling which must be modeled accurately to analyze the influence of the shock on material response.

The developments in this section are limited to an elastically isotropic thermoelastic–inelastic material. Use is made of the work in [3, 16] to propose a specific form for the Helmholtz free energy that is consistent with a Mie–Gruneisen equation of state (constitutive equation) for the pressure in shocked states. Also, for simplicity attention is limited to materials which have no inelastic dilatational deformation rate

$$\bar{\mathbf{D}}_p \cdot \mathbf{I} = 0\,, \tag{6.9.1}$$

so the elastic dilatation J_e is determined by the evolution equation

$$\frac{\dot{J}_e}{J_e} = \mathbf{D} \cdot \mathbf{I}\,, \tag{6.9.2}$$

which can be integrated and expressed in the form (4.1.16)

$$J_e = \frac{\rho_z}{\rho}\,, \tag{6.9.3}$$

where ρ_z is the constant zero-stress mass density at zero-stress reference temperature θ_z.

Also, the unimodular elastic distortional deformation tensor $\mathbf{B}'_e$ is determined by the evolution Eq. (5.11.55) with (5.11.65)

$$\dot{\mathbf{B}}'_e = \mathbf{L}''\mathbf{B}'_e + \mathbf{B}'_e\mathbf{L}''^T - \Gamma\mathbf{A}_p\,, \quad \mathbf{A}_p = \mathbf{B}'_e - \left(\frac{3}{\mathbf{B}'^{-1}_e \cdot \mathbf{I}}\right)\mathbf{I}\,, \tag{6.9.4}$$

with the elastic distortional deformation invariant α_1 satisfying the equations

$$\alpha_1 = \mathbf{B}'_e \cdot \mathbf{I}\,, \quad \dot{\alpha}_1 = 2\left(\mathbf{B}'_e - \frac{1}{3}\alpha_1\mathbf{I}\right) \cdot \mathbf{D} - \Gamma\mathbf{A}_p \cdot \mathbf{I}\,, \tag{6.9.5}$$

and the evolution equation for the isotropic hardening variable κ is given by (6.7.29)

$$\dot{\kappa} = \Gamma H\,. \tag{6.9.6}$$

Motivated by the work in [3, 16, 26] for a material with a constant specific heat C_v, the Helmholtz free energy ψ is specified by

$$\begin{aligned}
&\psi = \psi_1(J_e, \theta) + \psi_2(J_e, \alpha_1, \theta)\,, \\
&\psi_1 = C_v[\theta - \theta_z - \theta\, \ln\left(\frac{\theta}{\theta_z}\right)] + (\theta - \theta_z) f_1(J_e) + f_2(J_e)\,, \\
&\rho_z \psi_2 = \frac{1}{2}\mu(J_e, \theta)(\alpha_1 - 3)\,,
\end{aligned} \tag{6.9.7}$$

where ψ_1 controls the thermomechanical response to dilatation and μ is the shear modulus which is allowed to be a function of J_e and θ. The function f_1 controls strong thermomechanical coupling and f_2 controls nonlinear response to dilatation. These functions f_1 and f_2 need to be determined from plate impact experimental data.

Next, assuming that η and $\mathbf{T}$ are functions of the forms

$$\eta = \eta(J_e, \alpha_1, \theta)\,, \quad \mathbf{T} = \mathbf{T}(J_e, \mathbf{B}'_e, \theta)\,, \tag{6.9.8}$$

it follows from (6.7.32) that

$$\begin{aligned}
&\mathbf{T} = -p\,\mathbf{I} + \mathbf{T}''\,, && p = p_1(J_e, \theta) + p_2(J_e, \theta, \alpha_1)\,, \\
&p_1 = -\rho_z[(\theta - \theta_z)\tfrac{df_1}{dJ_e} + \tfrac{df_2}{dJe}]\,, && p_2 = -\tfrac{1}{2}\tfrac{\partial \mu}{\partial J_e}(\alpha_1 - 3)\,, \\
&\mathbf{T}'' = J_e^{-1}\mu\,\left(\mathbf{B}'_e - \tfrac{1}{3}\alpha_1 \mathbf{I}\right)\,, && \eta = \eta_1(J_e, \theta) + \eta_2(J_e, \theta, \alpha_1)\,, \\
&\eta_1 = C_v\, \ln\left(\tfrac{\theta}{\theta_z}\right) - f_1\,, && \rho_z \eta_2 = -\tfrac{1}{2}\tfrac{\partial \mu}{\partial \theta}(\alpha_1 - 3)\,,
\end{aligned} \tag{6.9.9}$$

and from (6.7.33) that the inelastic distortional deformation rate must satisfy the restriction

$$\rho_z \theta \xi' = \frac{1}{2}\Gamma \mu \mathbf{A}_p \cdot \mathbf{I} \geq 0\,. \tag{6.9.10}$$

However, since Γ is non-negative, the shear modulus μ is positive and in [18] it was shown that $\mathbf{A}_p \cdot \mathbf{I} \geq 0$ (5.11.67), it follows that nonzero inelastic deformation rate satisfies the condition (6.9.10) and is dissipative.

Then, with the help of (6.2.11) the specific internal energy ε takes the form

$$\begin{aligned}
&\varepsilon = \varepsilon_1(J_e, \theta) + \varepsilon_2(J_e, \alpha, \theta)\,, \\
&\varepsilon_1 = C_v(\theta - \theta_z) - \theta_z f_1 + f_2\,, \\
&\rho_z \varepsilon_2 = \frac{1}{2}\left(\mu - \theta\frac{\partial \mu}{\partial \theta}\right)(\alpha_1 - 3)\,.
\end{aligned} \tag{6.9.11}$$

In these expressions, the terms ψ_1, p_1, η_1 and ε_1 depend only on J_e and θ and represent the main thermomechanical response and the terms ψ_2, p_2, η_2 and ε_2 also depend on the elastic distortional deformation α_1.

For later reference it is noted that the expression ε_1 can be solved for θ and result can be substituted into the expression for p_1 to obtain

$$p_1 = -\rho_z \left[\left(\frac{\theta_z f_1 - f_2}{C_v} \right) \frac{df_1}{dJ_e} + \frac{df_2}{dJ_e} + \left(\frac{1}{C_v} \right) \frac{df_1}{dJ_e} \varepsilon_1 \right] , \tag{6.9.12}$$

which expresses p_1 as a linear function of the energy ε_1.

Next, neglecting body force, deviatoric stress, radiation and heat conduction and restricting attention to uniaxial strain in the $\mathbf{e}_1$ direction, the local equations of the conservation of mass, balance of linear momentum and balance of energy can be written in the forms

$$\rho J_e = \rho_z \, , \quad \rho \dot{u} = -\frac{\partial p}{\partial x} \, , \quad \rho \dot{\varepsilon} = -p \frac{\partial u}{\partial x} \, , \tag{6.9.13}$$

where x is the current position of a material point which was located at the position X in the reference configuration and u is the current velocity of the material point. Moreover, consider a steady-wave moving at the shock velocity U relative to the reference configuration into a zero-stress material state at zero-stress reference temperature and at rest with

$$J_e = 1 \, , \quad \theta = \theta_z \, , \quad \rho = \rho_z \, , \quad u = 0 \, , \quad p = 0 \, , \quad \varepsilon = 0 \, . \tag{6.9.14}$$

For this steady-wave it is convenient to introduce the auxiliary variable χ, such that

$$\chi = X - Ut \, . \tag{6.9.15}$$

Then, the position x and dilatation J_e are given by

$$x = X + \delta(\chi) \, , \quad J_e = \frac{\partial x}{\partial X} = 1 + \frac{d\delta}{dX} \, , \tag{6.9.16}$$

where $\delta(\chi)$ is the displacement of the material point relative to its reference position. Also, the particle velocity u, pressure p and internal energy ε are expressed in the forms

$$u = u(\chi) \, , \quad p = p_H(\chi) \, , \quad \varepsilon = \varepsilon_H(\chi) \, , \tag{6.9.17}$$

where p_H is the pressure and ε_H is the internal energy on the Hugoniot curve.

Differentiating the expression for x yields an expression for u given by

$$u = u(\chi) = \dot{x} = -U \frac{d\delta}{d\chi} = U(1 - J_e) \, . \tag{6.9.18}$$

Moreover, the balance of linear momentum and the balance of energy are expressed in the forms

$$-U\rho\frac{du}{d\chi}=-\frac{dp_H}{d\chi}J_e^{-1}\,,\quad -U\rho\frac{d\varepsilon_H}{d\chi}=-p_H\frac{du}{d\chi}J_e^{-1}\,,\tag{6.9.19}$$

which with the help of the conservation of mass (6.9.13) and the conditions (6.9.14) can be integrated to obtain

$$u=U(1-J_e)\,,\quad p_H=\rho_z U^2(1-J_e)\,,\quad \varepsilon_H=\frac{1}{2}U^2(1-J_e)^2\,,\tag{6.9.20}$$

which are the Rankine–Hugoniot jump conditions. These equations connect the states of the material on both sides of a uniaxial strain shock wave moving into a zero-stress material at rest. They are used to plot Hugoniot curves of the state variables as functions of the dilatation J_e or density $\rho=\rho_z J_e^{-1}$ for a specified shock velocity.

For a complete constitutive equation for pressure it is necessary to propose an expression for values of p off the Hugoniot curve $p=p_H$. In shock physics it is common to use a Mie–Gruneisen equation of state to determine the pressure for states off of the Hugoniot curve. This Mie–Gruneisen equation expresses the pressure as a function of dilatation and energy. Motivated by the work in [3, 16], this Mie–Gruneisen equation is written in terms of the main thermomechanical parts p_1, ε_1 of the pressure and energy in the form

$$p_1=p_H(J_e)+\rho\gamma(J_e)[\varepsilon_1-\varepsilon_H(J_e)]\,,\tag{6.9.21}$$

where the Gruneisen gamma γ is taken in the form

$$\rho\gamma=\rho_z\gamma_z\,,\tag{6.9.22}$$

with γ_z being the unshocked zero-stress value of γ.

Then, comparison of (6.9.12) with (6.9.21) shows that the constitutive Eq. (6.9.7) for the Helmholtz free energy will be consistent with the Mie–Gruneisen equation (6.9.21) provided that f_1 and f_2 satisfy the differential equations

$$\begin{aligned}&\frac{df_1}{dJ_e}=-\gamma_z C_v\,,\\&\frac{df_2}{dJ_e}+\gamma_z f_2=-\frac{p_H}{\rho_z}+\gamma_z\varepsilon_H+\gamma_z\theta_z f_1\,,\end{aligned}\tag{6.9.23}$$

which can be integrated subject to the conditions

$$f_1(1)=0\,,\quad f_2(1)=0\,,\tag{6.9.24}$$

to deduce that

$$\begin{aligned} f_1(J_e) &= \gamma_z C_v (1 - J_e)\,, \\ f_2(J_e) &= C_v \theta_z \left[1 + \gamma_z (1 - J_e) - \exp\{\gamma_z (1 - J_e)\}\right] + f_3\,, \\ f_3(J_e) &= \exp(-\gamma_z J_e) \int_{J_e}^{1} \left[\frac{p_H(x)}{\rho_z} - \gamma_z \varepsilon_H(x)\right] \exp(\gamma_z x)\, dx\,. \end{aligned} \tag{6.9.25}$$

Experimental data for shock waves in compression is used to determine the constant coefficients S_i in the approximation

$$U = U_z + S_1 u + S_2 (\frac{u}{U}) u + S_3 (\frac{u}{U})^2 u\,, \tag{6.9.26}$$

with U_z being the zero-stress shock velocity. Substituting (6.9.20) into this expression yields

$$U = \frac{U_z}{1 - S_1(1 - J_e) - S_2(1 - J_e)^2 - S_3(1 - J_e)^3} \text{ for } J_e \le 1\,, \tag{6.9.27}$$

for compression. This function is extended to the expansion regime using the form

$$U = \frac{U_z}{\sqrt{1 + \frac{\gamma_z}{2}(J_e - 1)}} \text{ for } J_e > 1\,. \tag{6.9.28}$$

Then, with the help of these expressions for the shock velocity U, it can be shown that the function f_3 in (6.9.25) can be determined in closed form for expansion with U given by (6.9.28), but it is necessary to integrate the function f_3 in (6.9.25) numerically with U given by (6.9.27) for compression. Recently, it was shown in [26] that by modifying the approximation (6.9.27) for compression to take the form

$$U = \frac{U_z \exp[\frac{1}{2}\gamma_z (1 - J_e)]}{1 - \tilde{S}_1(1 - J_e) - \tilde{S}_2(1 - J_e)^2} \text{ for } J_e \le 1\,, \tag{6.9.29}$$

it is possible to develop a closed form expression for f_3 for compression for general values of the constants $\tilde{S}_1$ and $\tilde{S}_2$. Numerous experiments have been conducted at great expense to obtain values of S_1, S_2, S_3 and γ_z for a large number of materials. Fortunately, it was shown in [26] that these values can be used without conducting additional experiments to determine values of $\tilde{S}_1$ and $\tilde{S}_2$ which yield excellent agreement with experimental data for most materials.

To complete the constitutive equations, it is necessary to specify functional forms for Γ in (6.9.4) and H in (6.9.6) for rate-independent or rate-dependent response as discussed in Sect. 5.8. Also, the entropy flux vector $\mathbf{p}$ must be specified, which could be taken in form (6.7.35).

6.10 Thermoelastic–Inelastic Porous Materials

Porosity appears naturally in a number of materials like rocks, soils, ceramics, metals and biological tissues. The $p - \alpha$ model developed in [11] and the modification in [4] have been used to model important dissipation due to porous compaction in shock wave problems. For these problems, the shock compaction occurs so quickly that even if the pores are partially or fully saturated with fluid, there is no time for the fluid to move, so fluid diffusion through the porous material can be neglected. Porous compaction, dilation and/or "bulking" (i.e., porous dilation under positive pressure) have been modeled in Nevada Tuff [19], granular media [22], sand [10] and ceramics [3]. Also, nucleation and growth of pores in metals have been modeled in [9].

Porosity in biological tissues is necessary for fluid flow that supplies essential nutrients for cell function. In contrast with shock loading, deformation of biological tissues is typically a slow process which can be significantly influenced by diffusion of fluid. An example of constitutive equations for slow deformation of biological tissues with fluid diffusion along with references to previous work can be found in [32]. In addition, prediction of the long-term quantity of production of an oil well that has been stimulated by hydrofracturing requires proper treatment of poroelasticity and inelasticity which characterizes porous compaction and the associated reduced permeability.

This section discusses the structure of constitutive equations for a porous material subjected to loading rapid enough to neglect fluid diffusion. Following the work in [10, 19, 22] an element of volume dv of the porous material is separated into an element of volume dv_s of the solid material and an element of volume dv_p of the pores

$$dv = dv_s + dv_p \,, \tag{6.10.1}$$

and the current porosity ϕ is defined by

$$\phi = \frac{dv_p}{dv} \,. \tag{6.10.2}$$

For simplicity, the pores are assumed so be evacuated so the element of mass dm of the porous material is due solely to the element of mass dm_s of the solid

$$dm = \rho dv = dm_s = \rho_s dv_s \,, \tag{6.10.3}$$

where ρ is the current mass density of the porous material and ρ_s is the current mass density of the solid matrix. Then, using (6.10.1)–(6.10.3) the mass density ρ can be expressed in the form

$$\rho = (1 - \phi)\rho_s \,. \tag{6.10.4}$$

Now, for an elastically isotropic, thermoelastic–inelastic porous material the elastic dilatation J_e satisfies the evolution equation

$$\frac{\dot{J}_e}{J_e} = \mathbf{D} \cdot \mathbf{I} - (\frac{\dot{\phi}}{1-\phi})\,, \tag{6.10.5}$$

where an evolution equation for ϕ must be specified. This equation can be rewritten in the form

$$\frac{d}{dt}\Big[\ln\Big\{\frac{J_e}{1-\phi}\Big\}\Big] = \mathbf{D} \cdot \mathbf{I}\,, \tag{6.10.6}$$

which with the help of the conservation of mass (6.2.5)

$$\dot{\rho} + \rho\,\mathbf{D} \cdot \mathbf{I} = 0\,, \tag{6.10.7}$$

and the expression (6.10.4) for ρ can be expressed in the form

$$\frac{d}{dt}\Big[\ln\Big\{\frac{\rho J_e}{1-\phi}\Big\}\Big] = \frac{d}{dt}\Big[\ln(\rho_s J_e)\Big] = 0\,. \tag{6.10.8}$$

This equation integrates to obtain

$$J_e = \frac{\rho_{sz}}{\rho_s}\,, \tag{6.10.9}$$

where ρ_{sz} is the zero-stress mass density of the solid material. This shows that J_e is the solid elastic dilatation.

Also, the elastic distortional deformation tensor is characterized by the symmetric, positive-definite, unimodular tensor $\mathbf{B}'_e$ which satisfies the evolution Eq. (5.11.66)

$$\dot{\mathbf{B}}'_e = \mathbf{L}''\mathbf{B}'_e + \mathbf{B}'_e\mathbf{L}''^T - \Gamma\mathbf{A}_p\,, \quad \mathbf{A}_p = \mathbf{B}'_e - \left(\frac{3}{\mathbf{B}'^{-1}_e \cdot \mathbf{I}}\right)\mathbf{I}\,, \tag{6.10.10}$$

where Γ is a non-negative function that controls the rate of distortional inelasticity. Also, for simplicity, attention is limited to the first invariant α_1 of elastic distortional deformation defined by (6.7.27) and the evolution Eq. (6.7.28)

$$\alpha_1 = \mathbf{B}'_e \cdot \mathbf{I}\,, \quad \dot{\alpha}_1 = 2\mathbf{B}''_e \cdot \mathbf{D} - \Gamma\mathbf{A}_p \cdot \mathbf{I}\,, \tag{6.10.11}$$

where $\mathbf{B}''_e$ is the deviatoric part of $\mathbf{B}'_e$. In addition, attention is limited to isotropic hardening κ, which satisfies the evolution Eq. (6.7.29)

$$\dot{\kappa} = \Gamma H\,, \tag{6.10.12}$$

where H controls the hardening rate.

For an elastically isotropic thermoelastic–inelastic porous material, the response functions ψ, η and $\mathbf{T}$ are specified in the forms

$$\psi = \psi_s(J_e, \alpha_1, \theta)\,, \quad \eta = \eta_s(J_e, \alpha_1, \theta)\,, \quad \mathbf{T} = \mathbf{T}(J_e, \mathbf{B}'_e, \theta, \phi)\,, \tag{6.10.13}$$

where ψ_s and η_s are the Helmholtz free energy and entropy of the solid material, both per unit mass. Then, with the help of (6.10.5)–(6.10.13), the rate of material dissipation (6.3.9) requires

$$\begin{aligned} \rho\theta\xi' &= \left[\mathbf{T} - \rho J_e \frac{\partial\psi_s}{\partial J_e}\mathbf{I} - 2\rho\frac{\partial\psi_s}{\partial\alpha_1}\mathbf{B}_e''\right]\cdot\mathbf{D} - \rho\left(\frac{\partial\psi_s}{\partial\theta} + \eta_s\right)\dot{\theta}\,, \\ &+ \left[\rho J_e \frac{\partial\psi_s}{\partial J_e}\left(\frac{\dot{\phi}}{1-\phi}\right) + \Gamma\rho\frac{\partial\psi_s}{\partial\alpha_1}\mathbf{A}_p\cdot\mathbf{I}\right] \geq 0\,. \end{aligned} \tag{6.10.14}$$

In general, without specifying details of the functional forms for Γ, $\dot{\phi}$ and H it is not possible to determine necessary restrictions on the constitutive equations for $\mathbf{T}$ and η_s. Specifically, in [19] the added compressibility of porosity was modeled with both elastic and inelastic rates of porosity. Here, for simplicity, changes in ϕ are assumed to be inelastic only. Moreover, motivated by necessary restrictions for a rate-independent elastic–inelastic material with a yield function, the constitutive equations for $\mathbf{T}$ and η are specified by

$$\begin{aligned} &\mathbf{T} = -p\,\mathbf{I} + \mathbf{T}''\,, \quad p = (1-\phi)p_s\,, \qquad p_s = -\rho_{sz}\tfrac{\partial\psi_s}{\partial J_e}\,, \\ &\mathbf{T}'' = (1-\phi)\mathbf{T}_s''\,, \ \ \mathbf{T}_s'' = 2J_e^{-1}\rho_{sz}\tfrac{\partial\psi_s}{\partial\alpha_1}\mathbf{B}_e''\,, \\ &\eta = \eta_s = -\tfrac{\partial\psi_s}{\partial\theta}\,, \end{aligned} \tag{6.10.15}$$

where p_s and $\mathbf{T}_s''$ are the pressure and deviatoric stress in the solid. It then follows that the rate of material dissipation imposes the restriction

$$\rho\theta\xi' = \rho\theta\xi_\phi' + \rho\theta\xi_d' \geq 0\,, \quad \rho\theta\xi_\phi' = -p\left(\frac{\dot{\phi}}{1-\phi}\right)\,, \quad \rho\theta\xi_d' = \Gamma\rho\frac{\partial\psi}{\partial\alpha_1}\mathbf{A}_p\cdot\mathbf{I}\,, \tag{6.10.16}$$

where $\rho\theta\xi_\phi'$ is the material dissipation rate due to porosity changes and $\rho\theta\xi_d'$ is the material dissipation rate due to inelastic distortional deformations. Assuming that the effective shear modulus is positive

$$\frac{\partial\psi}{\partial\alpha_1} > 0\,, \tag{6.10.17}$$

and using the fact that $\mathbf{A}_p\cdot\mathbf{I} \geq 0$, it follows that the inelastic distortional deformation is dissipative

$$\rho\theta\xi_d' \geq 0\,. \tag{6.10.18}$$

Also, the constitutive equation for stress is assumed to be restricted so that a zero-stress material state at zero-stress reference temperature θ_z is characterized by

$$\mathbf{T} = 0\,, \quad \frac{\partial\psi}{\partial J_e} = 0\,, \quad \text{for } J_e = 1\,, \quad \mathbf{B}_e' = \mathbf{I}\,, \quad \theta = \theta_z\,. \tag{6.10.19}$$

In addition, for isotropic response the entropy flux vector is specified by a generalized Fourier form (6.6.32)

$$\mathbf{p} = -\frac{\bar{\kappa}}{\theta}\,\mathbf{g}\,, \quad \mathbf{g} = \frac{\partial \theta}{\partial \mathbf{x}}\,, \quad \bar{\kappa} = \bar{\kappa}(J_e, \alpha_1, \theta, \kappa) \geq 0\,, \tag{6.10.20}$$

where the heat conduction coefficient $\bar{\kappa}$ should not be confused with the isotropic hardening variable κ.

From the definition of $\rho\theta\xi'_\phi$ in (6.10.16), it follows that porous compaction ($\dot{\phi} < 0$) at positive pressure and porous dilation ($\dot{\phi} > 0$) at negative pressure are both dissipative processes, but “bulking” (i.e., porous dilation at positive pressure) is a non-dissipative process. These three processes are modeled in the nonlinear breakage model developed in [22]. Also, the work in [10] discusses functional forms that include these effects in a thermomechanical theory.

As a special case, the Helmholtz free energy ψ_s of the solid material is specified in the form

$$\begin{aligned}
\rho_{sz}\psi_s &= \rho_{sz}C_v\left[\theta - \theta_z - \theta\ln\left(\frac{\theta}{\theta_z}\right)\right] + f_s(J_e,\theta) + \frac{1}{2}\mu_z(\alpha_1 - 3)\,,\\
f_s &= \rho_{sz}C_z^2\left[\frac{1}{S^2}\ln\left\{\frac{1}{1 - S(1 - Je)}\right\} - \left(\frac{1 - Je}{S}\right) + \alpha(1 - J_e)(\theta - \theta_z)\right] \quad \text{for} \quad S > 0\,,\\
f_s &= \rho_{sz}C_z^2\left[\frac{1}{2}(1 - J_e)^2 + \alpha(1 - J_e)(\theta - \theta_z)\right] \quad \text{for} \quad S = 0\,,
\end{aligned} \tag{6.10.21}$$

where C_v is the constant specific heat, C_z is the zero-stress shock wave speed, S is a positive constant controlling the slope of the shock velocity versus particle velocity curve, α is the constant coefficient of linear expansion and μ_z is the zero-stress shear modulus. It then follows from (6.10.15) that

$$\begin{aligned}
p_s &= \rho_{sz}C_z^2\left[\frac{(1 - J_e)}{1 - S(1 - J_e)} + \alpha(\theta - \theta_z)\right]\,, \quad \mathbf{T}''_s = J_e^{-1}\mu_z\mathbf{B}''_e\,,\\
\eta_s &= C_v\ln\left(\frac{\theta}{\theta_z}\right) - C_z^2\alpha(1 - J_e)\,.
\end{aligned} \tag{6.10.22}$$

Next, for simplicity, attention is further limited to isothermal response with $\theta = \theta_z$ for which

$$p = (1 - \phi)p_s\,, \quad p_s = p_s(J_e) = \rho_{sz}C_z^2\Big[\frac{(1 - J_e)}{1 - S(1 - J_e)}\Big]\,. \tag{6.10.23}$$

To motivate a form for the evolution equation for porosity consider the expression

$$\dot{p} = (1 - \phi)J_e\frac{dp_s}{dJ_e}\Big[\mathbf{D}\cdot\mathbf{I} - \frac{1}{\Gamma_\phi}\,\frac{\dot{\phi}}{1 - \phi}\Big]\,, \quad \Gamma_\phi = \frac{J_e\frac{dp_s}{dJ_e}}{J_e\frac{dp_s}{dJ_e} + p_s}\,, \tag{6.10.24}$$

which with the help of (6.10.23) yields

$$\begin{aligned} &J_e \frac{dp_s}{dJ_e} = -\rho_{sz} C_z^2 \left[\frac{J_e}{\{1 - S(1 - J_e)\}^2} \right] < 0 \,, \\ &\Gamma_\phi = \frac{J_e}{2J_e - 1 + S(1 - J_e)^2} > 0 \quad \text{for } J_e > \max\left(0, 1 - \frac{1}{S}\right) \,, \end{aligned} \tag{6.10.25}$$

where J_e is restricted so that the denominators in these expressions do not vanish. Also, for later convenience it can be shown that the constitutive equation (6.10.23) for the pressure p can be solved for J_e to deduce that

$$J_e(P, \phi) = \frac{1 - \phi + (S - 1)\,P}{1 - \phi + S\,P} \quad \text{for } S \geq 0 \,, \quad P = \frac{p}{\rho_{sz} C_z^2} \,, \tag{6.10.26}$$

where P is the normalized pressure.

Next, it is convenient to introduce the constant pressures $p_T < 0$, $p_c > 0$ and the maximum pressure p_{max} and the minimum porosity ϕ_{min} attained. Then, motivated by the work in [32] the evolution equation for ϕ is specified by

$$\frac{\dot{\phi}}{1 - \phi} = \Gamma_T \, \mathbf{D} \cdot \mathbf{I} \qquad \text{for} \quad p = p_T \text{ and } \mathbf{D} \cdot \mathbf{I} \geq 0 \,, \tag{6.10.27a}$$

$$\dot{\phi} = 0 \qquad \text{for} \quad p_T < p < p_c \,, \tag{6.10.27b}$$

$$\frac{\dot{\phi}}{1 - \phi} = \Gamma_c \, \mathbf{D} \cdot \mathbf{I} \qquad \text{for} \quad p = p_c \,, \quad \mathbf{D} \cdot \mathbf{I} < 0 \text{ and } \phi > \phi_{min} \,, \tag{6.10.27c}$$

$$\dot{\phi} = 0 \qquad \text{for} \quad p_c < p < p_{max} \,, \tag{6.10.27d}$$

$$\frac{\dot{\phi}}{1 - \phi} = \Gamma_\phi \left(\frac{\phi}{m + \phi} \right) \mathbf{D} \cdot \mathbf{I} + \frac{\beta_d (\rho \theta_z \xi_d)}{1 + \beta_d \, p} \quad \text{for} \quad p = p_{max} \text{ and } \mathbf{D} \cdot \mathbf{I} \leq 0 \,, \tag{6.10.27e}$$

where m and β_d are non-negative constants. These equations define five regions of response: porous dilatation ($\dot{\phi} \geq 0$) with $p = p_T < 0$ for (6.10.27a); elastic response for (6.10.27b); porous compaction ($\dot{\phi} \leq 0$) with $p = p_c > 0$ for (6.10.27c); elastic response with $p_c < p < p_{max}$ for (6.10.27d) and porous compaction ($\dot{\phi} \leq 0$) with $p = p_{max}$ for (6.10.27e). The non-negative functions Γ_T, Γ_c and Γ_ϕ are determined by the conditions $p = p_T$, $p = p_c$ and (6.10.24), respectively. Also, it follows from (6.10.27e) that for compaction at positive pressure, the rate of compaction, which is controlled by the constant m, competes with the rate of dilation due to bulking, which is controlled by the constant β_d. Moreover, from this evolution equation it can be seen that ϕ is automatically limited to its physical range

$$0 < \phi < 1 \,. \tag{6.10.28}$$

Furthermore, for compaction at maximum pressure (6.10.27e), it can be shown that (6.10.24) reduces to

$$\dot{p} = (1-\phi) J_e \frac{dp_s}{dJ_e} \left[\left(\frac{m}{m+\phi} \right) \mathbf{D} \cdot \mathbf{I} - \frac{\beta_d (\rho \theta_z \xi_d)}{\Gamma_\phi (1+\beta_d\, p)} \right] . \tag{6.10.29}$$

In view of the restrictions (6.10.25), it follows that small values of m cause a tendency for a slow increase in pressure for $\phi >> m$, and nonzero values of β_d, with nonzero rate of dissipation due to inelastic distortional deformation $\rho\theta_z\xi_d > 0$ causing additional increase in pressure due to bulking. Also, when $\phi \to 0$ and the pressure is large, the response asymptotically approaches that of the nonporous solid matrix.

Moreover, for these evolution equations, it follows that porosity changes are dissipative

$$\rho\theta_z \xi'_\phi = -p \frac{\dot{\phi}}{1-\phi} \geq 0 , \tag{6.10.30}$$

for the response regions without bulking (6.10.27a), (6.10.27b), (6.10.27c) and (6.10.27d). Also, for compaction with bulking (6.10.27e), it can be shown with the help of (6.10.18) that the rate of material dissipation (6.10.16) for $\theta = \theta_z$ requires

$$\rho\theta_z \xi' = -p \left(\frac{\phi}{m+\phi} \right) \Gamma_\phi \, \mathbf{D} \cdot \mathbf{I} + \frac{\rho\theta_z \xi'_d}{1+\beta_d p} \geq 0 . \tag{6.10.31}$$

Consequently, the rate of material dissipation (6.10.16) is satisfied for all processes

$$\rho\theta_z \xi' \geq 0 . \tag{6.10.32}$$

Numerical Integration Algorithm

Consider a time step which begins at $t = t_n$, ends at t_{n+1} with time increment $\Delta t = t_{n+1} - t_n$. A strongly objective numerical integration algorithm (5.11.89a) for $\mathbf{B}'_e$ was discussed in Sect. 5.11. Here, a strongly objective numerical integration algorithm is developed for the evolution equation for J_e. To this end, it is recalled that the relative dilatation J_r satisfies the Eq. (5.11.84)

$$\dot{J}_r = J_r \, \mathbf{D} \cdot \mathbf{I} , \quad J_r(t_n) = 1 . \tag{6.10.33}$$

Then, the evolution Eq. (6.10.6) for J_e can be expressed in the form

$$\frac{d}{dt} \left[\ln \left\{ \frac{J_e}{(1-\phi) J_r} \right\} \right] = 0 , \tag{6.10.34}$$

which can be integrated to deduce that

$$J_e(t_{n+1}) = \left[\frac{1-\phi(t_{n+1})}{1-\phi(t_n)} \right] J_r(t_{n+1}) \, J_e(t_n) . \tag{6.10.35}$$

Equating $J_e(t_{n+1})$ in (6.10.35) with J_e in (6.10.26) yields the result

$$\begin{aligned}
\phi(t_{n+1}, P) &= \frac{A(P) - \sqrt{A^2(P) - B(P)}}{2J_e(t_n)J_r(t_{n+1})} \,, \\
A(P) &= (2 + SP)J_e(t_n)J_r(t_{n+1}) + \phi(t_n) - 1 \,, \\
B(P) &= 4J_e(t_n)J_r(t_{n+1})\Big[\big\{(S-1)\phi(t_n) + SJ_e(t_n)J_r(t_{n+1}) - S + 1\big\}P \\
&\qquad + J_e(t_n)J_r(t_{n+1}) + \phi(t_n) - 1\Big] \,,
\end{aligned} \tag{6.10.36}$$

which can be used to determine

$$\begin{aligned}
\phi(t_{n+1}) &= \phi(t_{n+1}, P_T) \quad \text{for } P_T = \frac{p_T}{\rho_{sz}C_z^2} \,, \\
\phi(t_{n+1}) &= \phi(t_{n+1}, P_c) \quad \text{for } P_c = \frac{p_c}{\rho_{sz}C_z^2} \,.
\end{aligned} \tag{6.10.37}$$

This means that the values of $\phi(t_{n+1})$ and $J_e(t_{n+1})$ for the solutions of (6.10.27a), (6.10.27c) and (6.10.27d) at the end of the time step are determined by (6.10.35) and (6.10.37).

To obtain a solution for (6.10.27e), this equation is rewritten in the form

$$\frac{d}{dt}\left[\ln\left\{\frac{(1-\phi)^{1+m}}{\phi^m}\right\}\right] = -\Gamma_\phi \, \mathbf{D}\cdot\mathbf{I} - \left(\frac{m+\phi}{\phi}\right)\left[\frac{\beta_d(\rho\theta_z\xi_d)}{1+\beta_d\, p}\right] \quad \text{for } p = p_{max} \text{ and } \mathbf{D}\cdot\mathbf{I} \le 0 \,. \tag{6.10.38}$$

Assuming that Γ_ϕ can be approximated as constant over the time step, this equation is rewritten in the form

$$\frac{d}{dt}\left[\ln\left\{\frac{(1-\phi)^{1+m}J_r^{\Gamma_\phi(t_{n+1})}}{\phi^m}\right\}\right] \approx -\left(\frac{m+\phi}{\phi}\right)\left[\frac{\beta_d(\rho\theta_z\xi_d)}{1+\beta_d\, p}\right]. \tag{6.10.39}$$

The solution of this equation is further approximated by the solution of the implicit expression

$$\frac{[1-\phi(t_{n+1})]^{1+m}}{\phi(t_{n+1})^m} = \frac{[1-\phi(t_n)]^{1+m}}{\phi(t_n)^m J_r(t_{n+1})^{\Gamma_\phi(t_{n+1})}} \exp\left[-\left\{\frac{m+\phi(t_{n+1})}{\phi(t_{n+1})}\right\}\frac{\Delta t\beta_d(\rho\theta_z\xi_d)(t_{n+1})}{1+\beta_d\, p(t_{n+1})}\right] \quad \text{for } J_r(t_{n+1}) \le 1 \,, \tag{6.10.40}$$

where $(\rho\theta\xi_d)(t_{n+1})$ is an estimate of $\rho\theta\xi_d$ at the end of the time step, and $p(t_{n+1})$ and $\Gamma_\phi(t_{n+1})$ are determined by replacing J_e in (6.10.23) and (6.10.25), respectively, with (6.10.35) to obtain a nonlinear equation for $\phi(t_{n+1})$ to be solved numerically. Once the values $\phi(t_{n+1})$ and $J_e(t_{n+1})$ have been determined, the pressure p at the end of the time step is determined by (6.10.23).

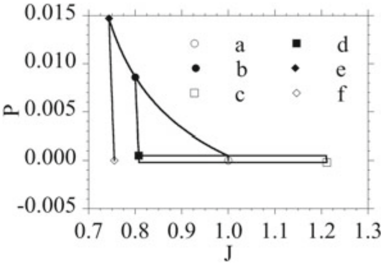

Fig. 6.1 Pure dilatation cyclic loading. Compression $a-b$; expansion $b-c$; compression $c-d$; compression $d-e$ and expansion $e-f$ for $S = 1.3$, $P_c = 0.005$, $P_T = -0.002$ and $m = 0.01$

To display the compaction response, it is convenient to use the total dilatation J from the initial state determined by the evolution equation and initial condition

$$\dot{J} = J\,\mathbf{D} \cdot \mathbf{I}\,, \quad J(0) = 1\,. \tag{6.10.41}$$

The exact integration of this equation over a time step is given by the expression

$$J(t_{n+1}) = J_r(t_{n+1})J(t_n)\,. \tag{6.10.42}$$

The following examples consider the case of no bulking with $\beta_d = 0$.

Cyclic Dilatational Loading

Next, to understand the influence of the parameter m in the evolution Eq. (6.10.27e) for ϕ, consider the case when the material is initially at zero stress with

$$J_e(0) = 1\,, \quad \mathbf{B}'_e(0) = \mathbf{I}\,, \tag{6.10.43}$$

and confine attention to pure dilatational loading for which

$$\mathbf{D} = \frac{1}{3}\,(\mathbf{D} \cdot \mathbf{I})\,\mathbf{I}\,. \tag{6.10.44}$$

For these conditions, the deviatoric stress remains zero $\mathbf{T}'' = 0$ and the pressure is determined by (6.10.23). Moreover, the initial value of porosity ϕ and the material constants S, P_T and P_c for this example are specified by

$$\phi(0) = 0.3\,, \quad S = 1.3\,, \quad P_T = -0.002\,, \quad P_c = 0.005\,. \tag{6.10.45}$$

Figure 6.1 shows the response to cyclic loading with compression $a-b$; expansion $b-c$; compression $c-d$; compression $d-e$ and expansion $e-f$ for $m = 0.01$. During the compression cycle $a-b$ the response is elastic with constant porosity ϕ until $P = P_c$ and then compaction occurs with decrease in porosity. The expansion cycle $b-c$ is elastic with constant porosity until $P = P_T$ and then dilation occurs at constant pressure $P = P_T$ with increase in porosity. The recompression cycle $c-d$ is elastic with constant porosity until $P = P_c$ and then compaction occurs at constant

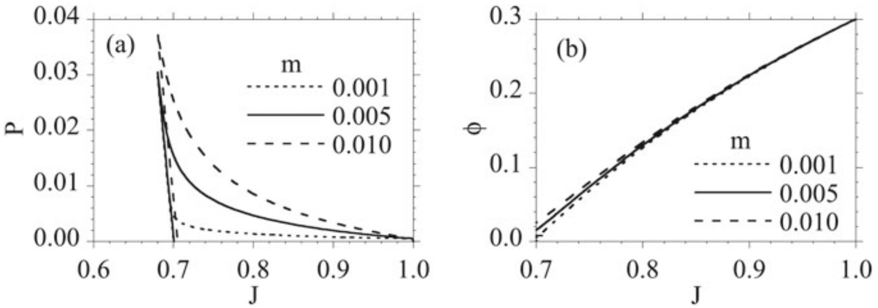

Fig. 6.2 Influence of m on pure dilatational compression followed by expansion for $S = 1.3$ and $P_c = 0.005$

pressure $P = P_c$ with decrease in porosity until $\phi = \phi_{min}$. The compression cycle $d{-}e$ is elastic with constant porosity until $P = P_{max} = p_{max}/(\rho_{sz} C_z^2)$ and continued compaction occurs with decrease in porosity. The expansion $e{-}f$ is elastic with constant porosity.

Figure 6.2 shows the influence of m on the compaction curve. Specifically, Fig. 6.2a shows the pressure response and Fig. 6.2b shows the porosity response. From these figures, it can be seen that m has a strong effect on the pressure during compaction with only small differences in the functional form of ϕ required for the range of values of m. It can also be seen that as the porosity approaches zero the response asymptotically approaches that of the nonporous solid material.

As mentioned before, more complicated evolution equations for ϕ which deal thermomechanical unloading can be found in [10, 22]. Also, to complete the constitutive equations, it is necessary to specify a functional form for Γ in (6.10.10) and H in (6.10.12) for rate-independent or rate-dependent response as discussed in Sect. 5.8.

6.11 Thermoelastic–Inelastic Theory for Growth of Biological Tissues

Biological tissues are complicated materials which are mixtures of many components that can flow relative to each other and interact mechanically, chemically and electrically (e.g., [1, 2, 12, 31]). A simplified constrained theory of mixtures with only one velocity field was developed by Humphrey and Rajagopal [12]. Also, review articles on growth and remodeling of tissues can be found in (e.g., [1, 13, 33]).

When the tissue is considered to be a homogenized solid, the standard approach to modeling growth for finite deformations is the Lagrangian formulation of growth proposed by Rodriguez et al. [15]. This formulation is based on the multiplicative form (5.11.11) by replacing the plastic deformation tensor $\mathbf{F}_p$ with a growth tensor

$\mathbf{F}_g$, such that

$$\mathbf{F}_e = \mathbf{F}\mathbf{F}_g^{-1} , \quad \dot{\mathbf{F}}_g = \boldsymbol{\Lambda}_g \mathbf{F}_g , \tag{6.11.1}$$

with the rate of growth $\boldsymbol{\Lambda}_g$ specified by a constitutive equation. However, this multiplicative formulation has the same arbitrariness as discussed previously for the Lagrangian formulation of plasticity, which can be removed by the Eulerian formulation discussed below.

Rubin et al. [23] developed an Eulerian unified theoretical structure for modeling interstitial growth and muscle activation in soft tissues. This Eulerian formulation of growth was used: in [30] to study significant differences in the mechanical modeling of confined growth predicted by the Lagrangian and Eulerian formulations; in [28] to analyze stresses in arteries and in [29] to model early cardiac morphogenesis during c-looping. This section reviews this Eulerian thermomechanical formulation of growth.

Growth requires an influx of nutrients to the tissue. Consequently, the theory developed in [23] treats the tissue as an open system with an external rate of mass supply. Specifically, the current mass density ρ of the tissue is determined by the balance of mass

$$\dot{\rho} + \rho(\mathbf{D} \cdot \mathbf{I}) = r_m \rho , \tag{6.11.2}$$

where r_m is the external rate of mass supply per unit mass. This simplifies the formulation by neglecting diffusion of fluid and allowing for a single velocity field to describe deformation of the tissue.

The balances of linear momentum and entropy in this theory are given by (6.2.5)

$$\rho \dot{\mathbf{v}} = \rho \mathbf{b} + \mathrm{div}\mathbf{T} , \quad \rho \dot{\eta} = \rho(s + \xi) - \mathrm{div}\mathbf{p} . \tag{6.11.3}$$

The balance of angular momentum (6.2.6) again requires the Cauchy stress to be symmetric

$$\mathbf{T}^T = \mathbf{T} , \tag{6.11.4}$$

but the balance of energy (6.2.7) is modified to include an external rate of energy supply b per unit mass due to mechanobiological processes

$$\rho \dot{\varepsilon} = \rho r - \mathrm{div}\mathbf{q} + \mathbf{T} \cdot \mathbf{D} + \rho b . \tag{6.11.5}$$

The Helmholtz free energy ψ is defined by (6.2.11)

$$\psi = \varepsilon - \theta\eta , \tag{6.11.6}$$

and the internal rate of entropy production $\rho\xi$ in [17] is separated into a thermal part $(-\mathbf{p} \cdot \mathbf{g})$ and a rate of material dissipation $\rho\theta\xi'$ (6.2.9)

$$\rho\theta\xi = -\mathbf{p} \cdot \mathbf{g} + \rho\theta\xi' , \tag{6.11.7}$$

with $\mathbf{g} = \partial\theta/\partial\mathbf{x}$ being the temperature gradient and

$$\rho\theta\xi' = \mathbf{T}\cdot\mathbf{D} - \rho(\dot{\psi} + \eta\dot{\theta}) + \rho b\,. \tag{6.11.8}$$

In these equations, the superposed $\dot{(\,)}$ denotes the standard material derivative with respect to the single velocity field.

The second law of thermodynamics for heat conduction requires the entropy flux $\mathbf{p}$ to satisfy the restriction (6.3.7) that heat flows from hot to cold regions

$$-\mathbf{p}\cdot\mathbf{g} > 0 \text{ for } \mathbf{g} \neq 0\,, \tag{6.11.9}$$

and the second law for the rate of material dissipation requires (6.3.9)

$$\rho\theta\xi' \geq 0\,. \tag{6.11.10}$$

This model can be used for growth of biological tissues as well as for muscle activation, both of which require an external supply of energy, which is characterized by term b in the balance of energy (6.11.5). Here, the mechanobiological processes which control growth and muscle activation are not modeled explicitly and it is assumed that b is large enough to ensure that (6.11.10) is satisfied for all thermomechanical processes with growth.

The elastic dilatation J_e for the growing tissue is determined by integrating the evolution equation

$$\frac{\dot{J}_e}{J_e} = \mathbf{D}\cdot\mathbf{I} - r_m\,, \tag{6.11.11}$$

which includes the external rate of mass supply r_m. Following the work in [23] and using the modification in [24], r_m is specified by

$$r_m = \Gamma_m \ln\left(\frac{J_e}{J_h}\right)\,, \tag{6.11.12}$$

so the evolution equation for the elastic dilatation J_e becomes

$$\frac{\dot{J}_e}{J_e} = \mathbf{D}\cdot\mathbf{I} - \Gamma_m \ln\left(\frac{J_e}{J_h}\right)\,, \quad \Gamma_m \geq 0\,, \quad J_h > 0\,. \tag{6.11.13}$$

Also, the symmetric, positive-definite, unimodular tensor $\mathbf{B}'_e$ that characterizes elastic distortional deformations is determined by the evolution equation

$$\begin{aligned} &\dot{\mathbf{B}}'_e = \mathbf{L}''\mathbf{B}'_e + \mathbf{B}'_e\mathbf{L}''^T - \Gamma\mathbf{A}_g\,, \\ &\mathbf{A}_g = \mathbf{B}'_e - \left(\frac{3}{\mathbf{B}'^{-1}_e\cdot\mathbf{H}}\right)\mathbf{H}\,, \quad \Gamma \geq 0\,, \quad \mathbf{H}' = \det(\mathbf{H})^{-1/3}\mathbf{H}\,, \end{aligned} \tag{6.11.14}$$

where $\mathbf{H}$ is a positive-definite, symmetric tensor and $\mathbf{H}'$ is its unimodular part. These evolution equations model homeostasis, which is the process that causes a tendency for J_e to approach its homeostatic value J_h and for $\mathbf{B}'_e$ to approach its homeostatic value $\mathbf{H}'$. In particular, it can be seen that in the absence of loading ($\mathbf{L} = 0$) the rates at which J_e and $\mathbf{B}'_e$ approach their homeostatic values J_h and $\mathbf{H}'$ are controlled by the functions Γ_m and Γ, respectively. Moreover, it is noted that the modified form (6.11.12) simplifies the numerical algorithm for solving the evolution Eq. (6.11.13). Also, the two nontrivial invariants of $\mathbf{B}'_e$ satisfy the equations

$$\begin{aligned} \alpha_1 &= \mathbf{B}'_e \cdot \mathbf{I}\,, \quad \dot{\alpha}_1 = 2\mathbf{B}''_e \cdot \mathbf{D} - \Gamma \mathbf{A}_g \cdot \mathbf{I}\,, \\ \alpha_2 &= \mathbf{B}'_e \cdot \mathbf{B}'_e\,, \quad \dot{\alpha}_2 = 4({\mathbf{B}'_e}^2 - \tfrac{1}{3}\alpha_2 \mathbf{I}) \cdot \mathbf{D} - 2\Gamma \mathbf{A}_g \cdot \mathbf{B}'_e\,, \end{aligned} \tag{6.11.15}$$

where $\mathbf{B}''_e$ is the deviatoric part of $\mathbf{B}'_e$.

In this model, the Cauchy stress $\mathbf{T}$ is a function of the elastic deformations J_e, $\mathbf{B}'_e$ and the temperature θ

$$\mathbf{T} = \mathbf{T}(J_e, \mathbf{B}'_e, \theta)\,. \tag{6.11.16}$$

This constitutive equation is restricted so that zero-stress material states occur whenever the elastic deformations are given by $J_e = 1$ and $\mathbf{B}'_e = \mathbf{I}$ and the temperature equals the zero-stress reference temperature $\theta = \theta_z$,

$$\mathbf{T} = 0 \quad \text{whenever} \quad J_e = 1\,, \quad \mathbf{B}'_e = \mathbf{I}\,, \quad \theta = \theta_z\,. \tag{6.11.17}$$

It is well known that the homeostatic state of the skin on the human body is not in a zero-stress material state. For this reason, surgeons cut the skin parallel to tension lines to minimize scarring. Within the context of this model, the stress in the homeostatic state of the tissue can be nonzero

$$\mathbf{T}(J_h, \mathbf{H}', \theta_z) \neq 0\,. \tag{6.11.18}$$

In particular, constitutive equations can be proposed for the homeostatic values J_h and $\mathbf{H}'$ of J_e and $\mathbf{B}'_e$, respectively, to ensure that the stress field in the homeostatic state of the tissue matches measured nonzero values. Additional constitutive equations need to be proposed for the homeostasis rates Γ_m and Γ.

Zero-Stress Growth:
To understand the influence of the homeostatic values J_h of J_e and $\mathbf{H}'$ of $\mathbf{B}'_e$, it is convenient to consider zero-stress growth (6.11.17) for which the evolution Eqs. (6.11.13) and (6.11.14) require

$$\begin{aligned} &\mathbf{D}_z \cdot \mathbf{I} + \Gamma_m \ln(J_h) = 0\,, \\ &2\mathbf{D}''_z - \Gamma \left[\mathbf{I} - \left(\frac{3}{\mathbf{I} \cdot \mathbf{H}} \right) \mathbf{H} \right] = 0\,, \end{aligned} \tag{6.11.19}$$

where $\mathbf{D}_z''$ denotes the deviatoric part of the zero-stress rate of deformation value $\mathbf{D}_z$ of $\mathbf{D}$ for zero-stress growth. As a special case, specify $\mathbf{H}$ in the form

$$\mathbf{H} = \mathbf{I} + \mathbf{H}'' \,, \quad \mathbf{H}'' \cdot \mathbf{I} = 0 \,, \quad \mathbf{H}'' \cdot \mathbf{H}'' < \frac{2}{3} \,, \tag{6.11.20}$$

where $\mathbf{H}''$ is the deviatoric part of $\mathbf{H}$ and its magnitude is bounded to ensure that $\mathbf{H}$ remains positive-definite. It then follows that (6.11.19) can be solved for the zero-stress growth rate $\mathbf{D}_z$ to obtain

$$\mathbf{D}_z = -\frac{1}{3}\Gamma_m \ln(J_h)\,\mathbf{I} + \mathbf{D}_z'' \,, \quad \mathbf{D}_z'' = -\frac{1}{2}\Gamma \mathbf{H}'' \,. \tag{6.11.21}$$

Since Γ_m is non-negative, it follows that $J_h > 1$ causes a volumetric rate of contraction and $J_h < 1$ causes a volumetric rate of expansion. Moreover, the deviatoric part $\mathbf{D}_z''$ of $\mathbf{D}_z$ is in the opposite direction to $\mathbf{H}''$. Furthermore, the limited magnitude of $\mathbf{H}''$ does not limit the magnitude of $\mathbf{D}_z''$, which is determined by the value of Γ.

In a review of growth in living systems, Kuhl [13] presented evolution equations which model volumetric, area and fiber growth. Elastic deformation measures J_e, λ_n and λ_s associated with these growth processes and used in elastically anisotropic constitutive equations were developed in [23]. In addition, expressions for the homeostatic values J_h and $\mathbf{H}'$ and the homeostasis rates Γ_m and Γ associated with these growth processes were discussed in [23].

Elastic Volumetric Growth:
The elastic dilatation J_e associated with this growth process satisfies the evolution Eq. (6.11.13).

Elastic Area Growth:
The elastic area stretch λ_n associated with growth of an area element on a material surface with unit normal vector $\mathbf{n}$ in the current configuration is motivated by expressions for the material area element da and the unit normal $\mathbf{n}$ to a material surface. To develop these expressions, it is convenient to define the second-order tensor $\mathbf{N}$ by

$$\mathbf{N} = \mathbf{n} \otimes \mathbf{n} \,, \tag{6.11.22}$$

which should not be confused with the unit normal vector $\mathbf{N}$ in Nanson's formula (3.3.35). Moreover, using the result (3.5.26)

$$\dot{\mathbf{n}} = -[\mathbf{L}^T - (\mathbf{D} \cdot \mathbf{n} \otimes \mathbf{n})\,\mathbf{I}]\,\mathbf{n} \,, \tag{6.11.23}$$

it can be shown that $\mathbf{N}$ satisfies the evolution equation

$$\dot{\mathbf{N}} = 2(\mathbf{D} \cdot \mathbf{N})\,\mathbf{N} - \mathbf{L}^T\mathbf{N} - \mathbf{N}\mathbf{L} \,. \tag{6.11.24}$$

Also, using (3.3.35) is can be shown that

$$\frac{da}{dA} = (J^{-4/3}\mathbf{B}' \cdot \mathbf{N})^{-1/2}\,, \quad \mathbf{B}' = J^{-2/3}\mathbf{F}\mathbf{F}^T\,. \tag{6.11.25}$$

Motivated by this expression, the elastic area stretch λ_n is defined by

$$\lambda_n = (J_e^{-4/3}\mathbf{B}'_e \cdot \mathbf{N})^{-1/2}\,. \tag{6.11.26}$$

Then, with the help of the evolution Eqs. (6.11.13), (6.11.14) and (6.11.24), the elastic area stretch λ_n satisfies the evolution equation

$$\frac{\dot{\lambda}_n}{\lambda_n} = (\mathbf{I} - \mathbf{N}) \cdot \mathbf{D} - \frac{2}{3}\Gamma_m \ln\left(\frac{J_e}{J_h}\right) + \frac{1}{2}\Gamma(\mathbf{B}'_e \cdot \mathbf{N})^{-1}(\mathbf{A}_g \cdot \mathbf{N})\,. \tag{6.11.27}$$

Elastic Fiber Growth:
The elastic fiber stretch λ_s associated with growth of a fiber in the direction of the unit vector **s** in the current configuration is motivated by expressions for the stretch λ and unit direction **s** of a material fiber. To develop these expressions, it is convenient to define the second-order tensor **S** by

$$\mathbf{S} = \mathbf{s} \otimes \mathbf{s}\,, \tag{6.11.28}$$

which should not be confused with the unit vector **S** in (3.3.12a) or the symmetric Piola-Kirchhoff stress in (4.6.14). Moreover, using the result (3.5.22)

$$\dot{\mathbf{s}} = [\mathbf{L} - (\mathbf{D} \cdot \mathbf{s} \otimes \mathbf{s})\,\mathbf{I}]\,\mathbf{s}\,, \tag{6.11.29}$$

it can be shown that **S** satisfies the evolution equation

$$\dot{\mathbf{S}} = \mathbf{L}\mathbf{S} + \mathbf{S}\mathbf{L}^T - 2(\mathbf{D} \cdot \mathbf{S})\,\mathbf{S}\,. \tag{6.11.30}$$

Also, using (3.3.8) and (3.3.12c) it can be shown that

$$\lambda = (J^{-2/3}\mathbf{B}'^{-1} \cdot \mathbf{S})^{-1/2}\,. \tag{6.11.31}$$

Motivated by this expression, the elastic fiber stretch λ_s is defined by

$$\lambda_s = (J_e^{-2/3}\mathbf{B}'^{-1}_e \cdot \mathbf{S})^{-1/2}\,. \tag{6.11.32}$$

Then, with the help of the evolution Eqs. (6.11.13), (6.11.14) and (6.11.30), the elastic fiber stretch λ_s satisfies the evolution equation

$$\frac{\dot{\lambda}_s}{\lambda_s} = \mathbf{S} \cdot \mathbf{D} - \frac{1}{3}\Gamma_m \ln\left(\frac{J_e}{J_h}\right) - \frac{1}{2}\Gamma(\mathbf{B}'^{-1}_e \cdot \mathbf{S})^{-1}(\mathbf{B}'^{-1}_e\mathbf{A}_g\mathbf{B}'^{-1}_e \cdot \mathbf{S})\,. \tag{6.11.33}$$

Constitutive Equations:
Following the work in [23], the constitutive equations for an elastically anisotropic thermoelastic material with growth are proposed in the forms

$$\begin{aligned}&\psi = \psi(J_e, \theta, \mathcal{V})\,, \quad \eta = \eta(J_e, \theta, \mathcal{V})\,, \qquad \mathbf{T} = \mathbf{T}(J_e, \theta, \mathbf{B}'_e, \mathcal{V})\,,\\ &\mathbf{p} = -\tfrac{\bar{\kappa}(J_e,\theta,\mathcal{V})}{\theta}\,\mathbf{g}\,, \quad \mathcal{V} = (\alpha_1, \alpha_2, \lambda_n, \lambda_s)\,, \quad \mathbf{g} = \tfrac{\partial\theta}{\partial\mathbf{x}}\,,\end{aligned} \tag{6.11.34}$$

where the non-negative function $\bar{\kappa}$ that represents the heat conduction coefficient should not be confused with an isotropic hardening variable κ. Then, using the evolution Eqs. (6.11.13), (6.11.15), (6.11.27) and (6.11.33), the stress $\mathbf{T}$ and entropy η are specified by

$$\begin{aligned}\mathbf{T} = \rho\Big[&J_e\frac{\partial\psi}{\partial J_e}\mathbf{I} + 2\frac{\partial\psi}{\partial\alpha_1}\mathbf{B}''_e + 4\frac{\partial\psi}{\partial\alpha_2}\left(\mathbf{B}'^{\,2}_e - \frac{1}{3}\alpha_2\mathbf{I}\right)\\ &+ \frac{\partial\psi}{\partial\lambda_n}\,\lambda_n(\mathbf{I}-\mathbf{N}) + \frac{\partial\psi}{\partial\lambda_s}\,\lambda_s\mathbf{S}\Big],\\ \eta = -&\frac{\partial\psi}{\partial\theta}\,.\end{aligned} \tag{6.11.35}$$

Notice that the component of stress due to the elastic area stretch λ_n is isotropic in the plane normal to $\mathbf{n}$ and the stress due to the elastic fiber stretch λ_s is in the $\mathbf{S}$ direction. Also, the stress $\mathbf{T}$ can be written in terms of the pressure p and its deviatoric part $\mathbf{T}''$, such that

$$\begin{aligned}&\mathbf{T} = -p\,\mathbf{I} + \mathbf{T}''\,, \quad p = -\rho\left(J_e\,\frac{\partial\psi}{\partial J_e} + \frac{2}{3}\frac{\partial\psi}{\partial\lambda_n}\lambda_n + \frac{1}{3}\frac{\partial\psi}{\partial\lambda_s}\lambda_s\right),\\ &\mathbf{T}'' = \rho\Big[2\frac{\partial\psi}{\partial\alpha_1}\mathbf{B}''_e + 4\frac{\partial\psi}{\partial\alpha_2}\left(\mathbf{B}'^{\,2}_e - \frac{1}{3}\alpha_2\mathbf{I}\right) + \frac{1}{3}\frac{\partial\psi}{\partial\lambda_n}\lambda_n(\mathbf{I}-3\mathbf{N})\\ &\qquad + \frac{1}{3}\frac{\partial\psi}{\partial\lambda_s}\lambda_s(3\mathbf{S}-\mathbf{I})\Big].\end{aligned} \tag{6.11.36}$$

Modeling Area Growth:
To model area growth, it is assumed that the growth is isotropic in a material surface that has unit normal $\mathbf{n}$ in the current configuration and $\mathbf{H}'$ is specified by

$$\mathbf{H}' = \frac{1}{h}(\mathbf{I}-\mathbf{N}) + h^2\mathbf{N}\,, \quad h > 0\,, \tag{6.11.37}$$

where $\mathbf{N}$ is defined by (6.11.22), $\mathbf{n}$ is defined by the evolution Eq. (6.11.23) and h is a positive scalar that controls the rate of area growth which needs to be specified by an evolution equation for $\dot{h}$.

To understand the implications of the constitutive form (6.11.37), consider the special case when the velocity gradient $\mathbf{L}$ is specified by

$$\mathbf{L} = \mathbf{D} = \frac{1}{2}\left(\frac{\dot{a}}{a}\right)(\mathbf{I} - \mathbf{N}) + \frac{\dot{\lambda}}{\lambda}\mathbf{N}\,, \quad \mathbf{D}\cdot\mathbf{I} = \frac{\dot{a}}{a} + \frac{\dot{\lambda}}{\lambda}\,, \tag{6.11.38}$$

with a and λ being arbitrary functions of time. For this velocity field, it follows from (6.11.23) and (6.11.24) that **n** and **N** remain constant. Using (6.11.27) and [(6.11.33) with **S** replaced by **N**], it can be shown that a represents the area stretch of the material surface that is normal to **n** and λ represents the stretch of a material fiber that is normal to this material surface.

Next, $\mathbf{B}_e'$ and its inverse are specified by

$$\mathbf{B}_e' = \frac{1}{b_e}(\mathbf{I} - \mathbf{N}) + b_e^2\mathbf{N}\,, \quad \mathbf{B}_e'^{-1} = b_e(\mathbf{I} - \mathbf{N}) + \frac{1}{b_e^2}\mathbf{N}\,, \quad b_e > 0\,, \tag{6.11.39}$$

where b_e is a positive scalar to be determined. Using this expression, it follows that h is the homeostatic value of the elastic stretch b_e of the fiber normal to the material surface. Moreover, the distortional invariant α_1 in (6.11.15), the elastic area stretch λ_n in (6.11.26) and the elastic fiber stretch λ_s in (6.11.32) with **S** replaced by **N**, become

$$\alpha_1 = \frac{2 + b_e^3}{b_e}\,, \quad \lambda_n = \frac{J_e^{2/3}}{b_e}\,, \quad \lambda_s = J_e^{1/3} b_e\,. \tag{6.11.40}$$

In addition, the evolution Eqs. (6.11.13) and (6.11.14) yield two scalar equations to determine J_e and b_e of the forms

$$\frac{\dot{J}_e}{J_e} = \frac{\dot{a}}{a} + \frac{\dot{\lambda}}{\lambda} - \Gamma_m \ln\left(\frac{J_e}{J_h}\right)\,, \quad \frac{\dot{b}_e}{b_e} = -\frac{1}{3}\left(\frac{\dot{a}}{a}\right) + \frac{2}{3}\left(\frac{\dot{\lambda}}{\lambda}\right) - \Gamma\left[\frac{1 - \left(\frac{h}{b_e}\right)^3}{2 + \left(\frac{h}{b_e}\right)^3}\right]. \tag{6.11.41}$$

Therefore, steady-state solutions of these equations exist with

$$\{\, J_e\,, J_h\,, \Gamma_m\,, \frac{\dot{a}}{a}\,, \frac{\dot{\lambda}}{\lambda}\,, b_e\,, \mathbf{B}_e'\,, h\,, \Gamma\,\}\,, \tag{6.11.42}$$

being constants, such that

$$\frac{\dot{a}}{a} = \frac{2}{3}\Gamma_m \ln\left(\frac{J_e}{J_h}\right) - \Gamma\left[\frac{1 - \left(\frac{h}{b_e}\right)^3}{2 + \left(\frac{h}{b_e}\right)^3}\right], \quad \frac{\dot{\lambda}}{\lambda} = \frac{1}{3}\Gamma_m \ln\left(\frac{J_e}{J_h}\right) + \Gamma\left[\frac{1 - \left(\frac{h}{b_e}\right)^3}{2 + \left(\frac{h}{b_e}\right)^3}\right]. \tag{6.11.43}$$

In particular, area growth can occur without extension in the **n** direction with

$$\frac{\dot{a}}{a} = \Gamma_m \ln\left(\frac{J_e}{J_h}\right)\,, \quad \frac{\dot{\lambda}}{\lambda} = 0\,, \quad \mathbf{D}\cdot\mathbf{I} = \frac{\dot{a}}{a} \quad \text{for} \quad \frac{1}{3}\Gamma_m \ln\left(\frac{J_e}{J_h}\right) = -\Gamma\left[\frac{1 - \left(\frac{h}{b_e}\right)^3}{2 + \left(\frac{h}{b_e}\right)^3}\right], \tag{6.11.44}$$

and area growth can occur without volume change with

$$\frac{\dot{a}}{a} = -\Gamma \left[\frac{1 - \left(\frac{h}{b_e}\right)^3}{2 + \left(\frac{h}{b_e}\right)^3} \right], \quad \frac{\dot{\lambda}}{\lambda} = -\frac{\dot{a}}{a}, \quad \mathbf{D} \cdot \mathbf{I} = 0 \quad \text{for} \quad \Gamma_m \ln\left(\frac{J_e}{J_h}\right) = 0 \,. \tag{6.11.45}$$

Modeling Fiber Growth:

To model fiber growth, it is assumed that the growth is isotropic in a surface normal to the unit direction $\mathbf{s}$ of the fiber in the current configuration and $\mathbf{H}'$ is specified by

$$\mathbf{H}' = \frac{1}{h}(\mathbf{I} - \mathbf{S}) + h^2 \mathbf{S} \,, \quad h > 0 \,, \tag{6.11.46}$$

where $\mathbf{S}$ is defined by (6.11.28), $\mathbf{s}$ is defined by the evolution Eq. (6.11.29) and h is a positive scalar that controls the rate of fiber growth which needs to be specified by an evolution equation for $\dot{h}$.

To understand the implications of the constitutive form (6.11.46), consider the special case when the velocity gradient $\mathbf{L}$ is specified by

$$\mathbf{L} = \mathbf{D} = \frac{1}{2}\left(\frac{\dot{a}}{a}\right)(\mathbf{I} - \mathbf{S}) + \frac{\dot{\lambda}}{\lambda}\mathbf{S} \,, \quad \mathbf{D} \cdot \mathbf{I} = \frac{\dot{a}}{a} + \frac{\dot{\lambda}}{\lambda} \,, \tag{6.11.47}$$

with a and λ being arbitrary functions of time. For this velocity field, it follows from (6.11.29) and (6.11.30) that $\mathbf{s}$ and $\mathbf{S}$ remain constant. Using [(6.11.27) with $\mathbf{N}$ replaced by $\mathbf{S}$] and (6.11.33), it can be shown that a represents the area stretch of the material surface that is normal to $\mathbf{s}$ and λ represents the stretch of the material fiber that is in the direction $\mathbf{s}$.

Next, $\mathbf{B}'_e$ and its inverse are specified in the forms (6.11.39) with $\mathbf{N}$ replaced by $\mathbf{S}$

$$\mathbf{B}'_e = \frac{1}{b_e}(\mathbf{I} - \mathbf{S}) + b_e^2 \mathbf{S} \,, \quad \mathbf{B}'^{-1}_e = b_e(\mathbf{I} - \mathbf{S}) + \frac{1}{b_e^2}\mathbf{S} \,, \quad b_e > 0 \,, \tag{6.11.48}$$

where b_e is a positive scalar to be determined. Using this expression, it follows that h is the homeostatic value of the elastic stretch b_e of the fiber in the direction $\mathbf{s}$ normal to the material surface. Moreover, the distortional invariant α_1 in (6.11.15), the elastic area stretch λ_n in [(6.11.26) with $\mathbf{N}$ replaced by $\mathbf{S}$] and the elastic fiber stretch λ_s in (6.11.32) are given by (6.11.40). In addition, the evolution Eqs. (6.11.13) and (6.11.14) yield two scalar Eq. (6.11.41) to determine J_e and b_e. Therefore, steady-state solutions of these equations exist with

$$J_e \,, J_h \,, \Gamma_m \,, \frac{\dot{a}}{a} \,, \frac{\dot{\lambda}}{\lambda} \,, b_e \,, \mathbf{B}'_e \,, h \,, \Gamma \,, \tag{6.11.49}$$

being constants, such that

$$\frac{\dot{a}}{a} = \frac{2}{3}\Gamma_m \ln\left(\frac{J_e}{J_h}\right) - \Gamma\left[\frac{1-\left(\frac{h}{b_e}\right)^3}{2+\left(\frac{h}{b_e}\right)^3}\right], \quad \frac{\dot{\lambda}}{\lambda} = \frac{1}{3}\Gamma_m \ln\left(\frac{J_e}{J_h}\right) + \Gamma\left[\frac{1-\left(\frac{h}{b_e}\right)^3}{2+\left(\frac{h}{b_e}\right)^3}\right]. \tag{6.11.50}$$

In particular, fiber growth can occur without area change normal to the fiber with

$$\frac{\dot{a}}{a} = 0, \quad \frac{\dot{\lambda}}{\lambda} = \Gamma_m \ln\left(\frac{J_e}{J_h}\right), \quad \mathbf{D}\cdot\mathbf{I} = \frac{\dot{\lambda}}{\lambda}, \quad \frac{2}{3}\Gamma_m \ln\left(\frac{J_e}{J_h}\right) = \Gamma\left[\frac{1-\left(\frac{h}{b_e}\right)^3}{2+\left(\frac{h}{b_e}\right)^3}\right], \tag{6.11.51}$$

and fiber growth can occur without volume change with

$$\frac{\dot{a}}{a} = -\frac{\dot{\lambda}}{\lambda}, \quad \frac{\dot{\lambda}}{\lambda} = \Gamma\left[\frac{1-\left(\frac{h}{b_e}\right)^3}{2+\left(\frac{h}{b_e}\right)^3}\right], \quad \mathbf{D}\cdot\mathbf{I} = 0, \quad \Gamma_m \ln\left(\frac{J_e}{J_h}\right) = 0, \tag{6.11.52}$$

which is the same as the solution (6.11.45).

Modeling Muscle Activation:
These equations have also been used to model muscle activation and details can be found in [23].

6.12 Jump Conditions for the Thermomechanical Balance Laws

The purpose of this section is to develop jump conditions for the global thermomechanical balance laws. Specifically, it is recalled from Sect. 6.2 that within the context of the thermomechanical theory proposed by Green and Naghdi [7, 8], the current mass density ρ, the current position $\mathbf{x}$ of a material point and the absolute temperature θ are determined by the global forms of the conservation of mass and the balances of linear momentum and entropy (6.2.1)

$$\frac{d}{dt}\int_P \rho dv = 0, \tag{6.12.1a}$$

$$\frac{d}{dt}\int_P \rho\mathbf{v} dv = \int_P \rho\mathbf{b} dv + \int_{\partial P} \mathbf{t} da, \tag{6.12.1b}$$

$$\frac{d}{dt}\int_P \rho\eta dv = \int_P \rho(s+\xi) dv - \int_{\partial P} \mathbf{p}\cdot\mathbf{n} da. \tag{6.12.1c}$$

Moreover, the balance of angular momentum (6.2.2) and the balance of energy (6.2.3)

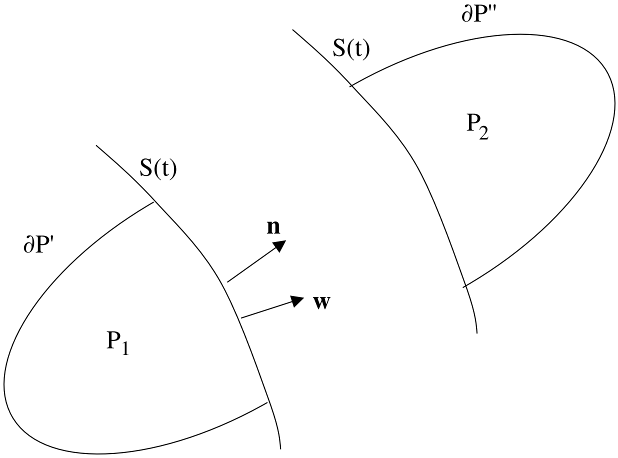

Fig. 6.3 A material region with a singular moving surface $S(t)$

$$\frac{d}{dt}\int_P (\mathbf{x}\times\rho\mathbf{v})dv = \int_P (\mathbf{x}\times\rho\mathbf{b})dv + \int_{\partial P} \mathbf{x}\times\mathbf{t}da\,, \tag{6.12.2a}$$

$$\begin{aligned}\frac{d}{dt}\int_P (\rho\varepsilon+\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v})dv &= \int_P (\rho\mathbf{b}\cdot\mathbf{v})dv + \int_{\partial P}\mathbf{t}\cdot\mathbf{v}da\\ &+\int_P \rho\theta s dv - \int_{\partial P}\theta\mathbf{p}\cdot\mathbf{n}da\end{aligned} \tag{6.12.2b}$$

are identically satisfied for all thermomechanical processes.

The discussion in Sect. 3.10 considered the material region P with closed material boundary ∂P to be divided into two parts P_1 and P_2 by the singular $S(t)$ that moves through the material (see Fig. 6.3). Furthermore, the intersection of ∂P_1 with ∂P was denoted by $\partial P'$ and the intersection of ∂P_2 with ∂P was denoted by $\partial P''$. Mathematically, this separation is summarized by (3.10.14)

$$\begin{aligned} &P = P_1\cup P_2\,, &&\partial P' = \partial P_1\cap\partial P\,, &&\partial P'' = \partial P_2\cap\partial P\,,\\ &\partial P = \partial P'\cup P''\,, &&\partial P_1 = \partial P'\cup S\,, &&\partial P_2 = \partial P''\cup S\,.\end{aligned} \tag{6.12.3}$$

A discussion of the motion of singular surfaces in fluid mechanics can be found in [34].

Next, the generalized transport theorem (3.10.16) is given by

$$\begin{aligned}\frac{d}{dt}\int_P \boldsymbol{\phi}(\mathbf{x},t)dv &= \int_{P_1}(\dot{\boldsymbol{\phi}}+\boldsymbol{\phi}\,\mathrm{div}\mathbf{v})dv + \int_{P_2}(\dot{\boldsymbol{\phi}}+\boldsymbol{\phi}\,\mathrm{div}\mathbf{v})dv\\ &-\int_{S(t)}[[\boldsymbol{\phi}\,\{(\mathbf{w}-\mathbf{v})\cdot\mathbf{n}\}]]da\,,\end{aligned} \tag{6.12.4}$$

where points on this singular surface move with velocity $\mathbf{w}$ and the unit normal to $S(t)$ outward from the part P_1 is denoted by $\mathbf{n}$. Also, the jump operator $[[\boldsymbol{\phi}]]$ is defined by (3.10.17)

$$[[\boldsymbol{\phi}\,\{(\mathbf{w}-\mathbf{v})\cdot\mathbf{n}\}]] = \boldsymbol{\phi}_2\{(\mathbf{w}-\mathbf{v}_2)\cdot\mathbf{n}\} - \boldsymbol{\phi}_1\{(\mathbf{w}-\mathbf{v}_1)\cdot\mathbf{n}\}\,, \tag{6.12.5}$$

where $\boldsymbol{\phi}_1$ and $\mathbf{v}_1$ are the values of $\boldsymbol{\phi}$ and $\mathbf{v}$ in part P_1 and $\boldsymbol{\phi}_2$ and $\mathbf{v}_2$ are the values of $\boldsymbol{\phi}$ and $\mathbf{v}$ in part P_2, all evaluated on the singular surface $S(t)$. In addition, $\mathbf{w}$ and $\mathbf{n}$ are the same on both sides of $S(t)$ (3.10.18)

$$\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{w}\,, \quad \mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n}\,. \tag{6.12.6}$$

Now, it is assumed that the local forms of the balance laws (6.12.1)

$$\dot{\rho} + \rho\,\mathrm{div}\mathbf{v} = 0\,, \tag{6.12.7a}$$
$$\rho\dot{\mathbf{v}} = \rho\mathbf{b} + \mathrm{div}\mathbf{T}\,, \tag{6.12.7b}$$
$$\rho\dot{\eta} = \rho(s+\xi) - \mathrm{div}\mathbf{p}\,, \tag{6.12.7c}$$

and the local forms of the balance laws (6.12.2)

$$\mathbf{x}\times\rho\dot{\mathbf{v}} = \mathbf{x}\times\rho\mathbf{b} + \mathrm{div}(\mathbf{x}\times\mathbf{T})\,, \tag{6.12.8a}$$
$$\rho\dot{\varepsilon} + \rho\dot{\mathbf{v}}\cdot\mathbf{v} = \rho\mathbf{b}\cdot\mathbf{v} + \rho\theta s + \mathrm{div}(\mathbf{v}\cdot\mathbf{T} - \theta\mathbf{p}) \tag{6.12.8b}$$

are valid in each part P_1 and P_2 where use has been made of the expression for (4.3.24) for the traction vector $\mathbf{t}$

$$\mathbf{t} = \mathbf{T}\mathbf{n}\,. \tag{6.12.9}$$

Applying the generalized tranport theorem (6.12.4) to the global form (6.12.1a) of the conservation of mass and using the local Eq. (6.12.7a) in each of the parts P_1 and P_2 yields

$$\int_{S(t)} [[\rho\,\{(\mathbf{w}-\mathbf{v})\cdot\mathbf{n}\}]]da = 0\,. \tag{6.12.10}$$

Assuming continuity of the integrand along $S(t)$ requires the jump condition on mass

$$[[m]] = 0\,, \quad m = \rho\,[(\mathbf{w}-\mathbf{v})\cdot\mathbf{n}]\,, \tag{6.12.11}$$

to be valid for all points on $S(t)$.

Since the internal rate of entropy production ξ in the balance of entropy can be singular at the singular surface, this balance law needs special attention so it will be used as an example for the other balance laws. Specifically, due to this singularity, it follows that

$$\int_P \rho\xi dv = \int_{P_1} \rho\xi dv + \int_{P_2} \rho\xi dv + \int_{S(t)} m\,\Xi da\,, \tag{6.12.12}$$

where it is assumed that the singularity in ξ is integrable across $S(t)$ to yield the finite value Ξ. In contrast, the external rate of entropy supply s is assumed to be bounded across $S(t)$ so that

$$\int_P \rho s dv = \int_{P_1} \rho s dv + \int_{P_2} \rho s dv\,. \tag{6.12.13}$$

Now, applying the generalized transport theorem (6.12.4) to the rate of change of entropy and using (6.12.11) and the local Eqs. (6.12.7a) and (6.12.7c) yields

$$\begin{aligned}\frac{d}{dt}\int_P \rho\eta dv = &\int_{P_1} [\rho(s+\xi) - \mathrm{div}\mathbf{p}]dv + \int_{P_2} [\rho(s+\xi) - \mathrm{div}\mathbf{p}]dv \\ &- \int_{S(t)} [[m\,\eta]]da\,.\end{aligned} \tag{6.12.14}$$

However, application of the divergence theorem yields

$$\begin{aligned}\int_{P_1} \mathrm{div}\mathbf{p}dv &= \int_{\partial P'} \mathbf{p}\cdot\mathbf{n}da + \int_{S(t)} \mathbf{p}_1\cdot\mathbf{n}da \\ \int_{P_2} \mathrm{div}\mathbf{p}dv &= \int_{\partial P''} \mathbf{p}\cdot\mathbf{n}da - \int_{S(t)} \mathbf{p}_2\cdot\mathbf{n}da\,,\end{aligned} \tag{6.12.15}$$

so that

$$\int_{P_1} \mathrm{div}\mathbf{p}dv + \int_{P_2} \mathrm{div}\mathbf{p}dv = \int_{\partial P} \mathbf{p}\cdot\mathbf{n}da - \int_{S(t)} [[\mathbf{p}\cdot\mathbf{n}]]da\,. \tag{6.12.16}$$

Thus, with the help of (6.12.12) and (6.12.13), (6.12.14) can be rewritten in the form

$$\frac{d}{dt}\int_P \rho\eta dv = \int_P \rho(s+\xi)dv - \int_{\partial P} \mathbf{p}\cdot\mathbf{n}da - \int_{S(t)} \Big(m\,\Xi + [[m\,\eta - \mathbf{p}\cdot\mathbf{n}]]\Big)da\,. \tag{6.12.17}$$

Now, using the global balance laws (6.12.1c) and assuming continuity of the integrand over $S(t)$ requires the jump condition on entropy

$$m\,\Xi + [[m\,\eta - \mathbf{p}\cdot\mathbf{n}]] = 0\,, \tag{6.12.18}$$

to be valid for all points on $S(t)$.

Following this same procedure for the other balance laws and assuming that $\rho\mathbf{b}$ and θ are bounded across $S(t)$, the jump conditions for balance laws (6.12.1) can be summarized as

$$[[m]] = 0\,, \tag{6.12.19a}$$

$$[[m\mathbf{v} + \mathbf{Tn}]] = 0\,, \tag{6.12.19b}$$

$$m\,\Xi + [[m\,\eta - \mathbf{p}\cdot\mathbf{n}]] = 0\,, \tag{6.12.19c}$$

and the jump conditions for the balance laws (6.12.2) can be summarized as

$$\mathbf{x} \times [[m\mathbf{v} + \mathbf{Tn}]] = 0\,, \tag{6.12.20a}$$

$$[[m(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{Tn} - \theta\mathbf{p}\cdot\mathbf{n}]] = 0\,, \tag{6.12.20b}$$

where m is defined by (6.12.11) and $\mathbf{x}$ is continuous across $S(t)$.

Notice that the jump in angular momentum (6.12.20a) is automatically satisfied when the jump in linear momentum (6.12.19b) is satisfied. In contrast with the local Eq. (6.12.7), the jump condition (6.12.19c) is used to determine the internal rate of entropy production Ξ due to the jump in entropy across $S(t)$ and the jump condition (6.12.20b) on energy is used to determine the jump in temperature θ.

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