Chapter 3
Kinematics

Abstract The objective of this chapter is to discuss nonlinear kinematics of deformable continua. Bodies, configurations and motion of continua are discussed along with a definition of the material time derivative, which is used to determine the velocity and acceleration of a material point. Deformation tensors and rate of deformation tensors are defined and analyzed. The notion of Superposed Rigid Body Motions (SRBM) is presented and the associated transformation relations of specific tensors are developed. In addition, an Eulerian formulation of evolution equations for elastic deformations is proposed and strongly objective, robust numerical integration algorithms for these evolution equations are developed.

3.1 Bodies, Configurations and Motion

Bodies

In an abstract sense a body $\mathcal{B}$ is a set of material particles which are denoted by Y (see Fig. 3.1). In mechanics a body is assumed to be smooth and can be put into correspondence with a domain of Euclidean 3-Space. Bodies are often mapped to their configurations, i.e., the regions of Euclidean 3-Space they occupy at each instant of time t $(-\infty < t < \infty)$. In the following, all position vectors are referred to a fixed point inertial in space.

Current Configuration and Motion

The current configuration of the body is the region of Euclidean 3-Space occupied by the body at the current time t. Let $\mathbf{x}$ be the position vector which identifies the place occupied by the particle Y at the time t. Since it is assumed that the body can be mapped smoothly into a domain of Euclidean 3-Space, a motion of the body can be represented as

$$\mathbf{x} = \bar{\mathbf{x}}(Y, t)\,. \tag{3.1.1}$$

In this expression, Y refers to the material particle, t refers to the current time, $\mathbf{x}$ refers to the value of the function and $\bar{\mathbf{x}}$ characterizes how each particle Y moves

M. B. Rubin, *Continuum Mechanics with Eulerian Formulations of Constitutive Equations*, Solid Mechanics and Its Applications 265,
https://doi.org/10.1007/978-3-030-57776-6_3

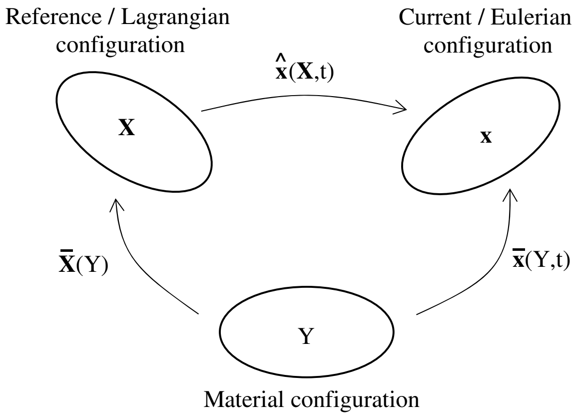

Fig. 3.1 Definition of the material Y, Referential/Lagrangian $\mathbf{X}$ and Current/Eulerian $\mathbf{x}$ configurations

through space as time progresses. It is assumed that this function is invertible so that

$$Y = \bar{\mathbf{x}}^{-1}(\mathbf{x}, t) = \tilde{Y}(\mathbf{x}, t)\,. \tag{3.1.2}$$

Reference Configuration
Sometimes it is convenient to select one particular configuration, called a reference configuration, and refer everything concerning the body and its motion to this configuration. The reference configuration need not necessarily be an actual configuration occupied by the body and in particular, the reference configuration need not be the *initial* configuration.

Let $\mathbf{X}$ be the position vector of the particle Y in the reference configuration. Then, the mapping from Y to the place $\mathbf{X}$ in the reference configuration can be written as

$$\mathbf{X} = \bar{\mathbf{X}}(Y)\,. \tag{3.1.3}$$

In this expression, $\mathbf{X}$ refers to the value of the function $\bar{\mathbf{X}}$ which characterizes the mapping. It is important to note that this mapping does not depend on time because the reference configuration is a single constant configuration. Moreover, this mapping is assumed to be invertible with its inverse given by

$$Y = \bar{\mathbf{X}}^{-1}(\mathbf{X}) = \hat{Y}(\mathbf{X})\,. \tag{3.1.4}$$

Motion
It follows that the mapping from the reference configuration to the current configuration can be obtained by substituting (3.1.4) into (3.1.1) to deduce that

$$\mathbf{x} = \bar{\mathbf{x}}(\hat{Y}(\mathbf{X}), t) = \hat{\mathbf{x}}(\mathbf{X}, t)\,, \tag{3.1.5}$$

which characterizes the motion all material points. From this expression, it is obvious that the functional form of the mapping $\hat{\mathbf{x}}$ depends on the specific choice of the reference configuration. Further in this regard, it is emphasized that the choice of the reference configuration is similar to the choice of coordinates in that it is arbitrary to the extent that a one-to-one correspondence exists between the material particles Y and their locations $\mathbf{X}$ in the reference configuration. Also, the inverse of this mapping can be written in the form

$$\mathbf{X} = \tilde{\mathbf{X}}(\mathbf{x}, t)\,. \tag{3.1.6}$$

In contrast to the material configuration, which is based on the abstract notion of a material point Y, the mapping (3.1.5) expresses $\mathbf{x}$ as a vector function of $\mathbf{X}$ and t and the inverse mapping (3.1.6) expresses $\mathbf{X}$ as a vector function of $\mathbf{x}$ and t. These vector functions are mathematical functions that are assumed to be smooth functions which can be differentiated with respect to either of their arguments as many times as necessary.

3.2 Representations

Material, Lagrangian and Eulerian Representations

There are several methods of describing properties of a body. The following considers three possible representations. To this end, let f be an arbitrary scalar or tensor function characterizing a property of the body which admits the following three representations

$$f = \bar{f}(Y, t) \qquad \text{Material representation}\,, \tag{3.2.1a}$$

$$f = \hat{f}(\mathbf{X}, t) \qquad \text{Lagrangian representation}\,, \tag{3.2.1b}$$

$$f = \tilde{f}(\mathbf{x}, t) \qquad \text{Eulerian representation}\,. \tag{3.2.1c}$$

For definiteness, a symbol is used to denote different functional forms from the value of a function. Whenever this is necessary, the functions that depend on Y are denoted with an overbar $\bar{(\ )}$, functions that depend on $\mathbf{X}$ are denoted with a hat $\hat{(\ )}$ and functions that depend on $\mathbf{x}$ are denoted with a tilde $\tilde{(\ )}$. Furthermore, the functional forms $\bar{f}$, $\hat{f}$, $\tilde{f}$ are related by the expressions

$$\hat{f}(\mathbf{X}, t) = \bar{f}(\hat{Y}(\mathbf{X}), t)\,, \quad \tilde{f}(\mathbf{x}, t) = \hat{f}(\tilde{\mathbf{X}}(\mathbf{x}, t), t)\,. \tag{3.2.2}$$

The representation (3.2.1a) is called *material* because the material point Y is used as an independent variable. The representation (3.2.1b) is called *referential* or *Lagrangian* because the position $\mathbf{X}$ of a material point in the reference configuration is an independent variable, and the representation (3.2.1c) is called *spatial* or *Eulerian* because the current position $\mathbf{x}$ in space is used as an independent variable. However, it is emphasized that in view of the invertibility of these functions, a one-to-one correspondence can be established between any two of these representations.

In this book, use is made of both the coordinate free forms of equations as well as their indicial counterparts. To this end, let $\mathbf{e}_A$ be a fixed right-handed orthonormal rectangular Cartesian basis associated with the reference configuration and let $\mathbf{e}_i$ be a fixed right-handed orthonormal rectangular Cartesian basis associated with the current configuration. Moreover, these base vectors are specified to coincide so that

$$\mathbf{e}_i \cdot \mathbf{e}_A = \delta_{iA} \,, \tag{3.2.3}$$

where δ_{iA} is the usual Kronecker delta. In the following, all tensor quantities are referred to either of these bases and for clarity use is made of upper case letters to indicate indices of quantities associated with the reference configuration and with lower case letters to indicate indices of quantities associated with the current configuration. For example,

$$\mathbf{X} = X_A \mathbf{e}_A \,, \quad \mathbf{x} = x_i \mathbf{e}_i \,, \tag{3.2.4}$$

where X_A are the rectangular Cartesian components of the position vector $\mathbf{X}$ and x_i are the rectangular Cartesian components of the position vector $\mathbf{x}$ and the usual summation convention over repeated indices is used. It follows that the motion (3.1.5) can be written in the form

$$x_i = \hat{x}_i(X_A, t) \,. \tag{3.2.5}$$

Velocity and Acceleration

The velocity $\mathbf{v}$ of a material point Y is defined as the rate of change with time t of position of the material point. Since the function $\bar{\mathbf{x}}(Y, t)$ characterizes the position of the material point Y at any time t, it follows that the velocity is defined conceptually by

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{\partial \bar{\mathbf{x}}(Y, t)}{\partial t} \,, \quad v_i = \dot{x}_i = \frac{\partial \bar{x}_i(Y, t)}{\partial t} \,, \tag{3.2.6}$$

where a superposed dot $\dot{(\;)}$ is used to denote partial differentiation with respect to time t holding the material particle Y fixed. Similarly, the acceleration $\mathbf{a}$ of a material point Y is defined by

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \bar{\mathbf{v}}(Y, t)}{\partial t} \,, \quad a_i = \dot{v}_i = \frac{\partial \bar{v}_i(Y, t)}{\partial t} \,. \tag{3.2.7}$$

Notice that in view of the mappings (3.1.4) and (3.1.6), the velocity and acceleration can be expressed as functions of either $(\mathbf{X}, t)$ or $(\mathbf{x}, t)$.

Material Derivative

The material derivative of an arbitrary function f is defined conceptually by

$$\dot{f} \equiv \left. \frac{\partial \bar{f}(Y, t)}{\partial t} \right|_Y \,. \tag{3.2.8}$$

It is important to emphasize that the material derivative, which is denoted by a superposed dot $\dot{()}$ is defined to be the rate of change with time t of the function holding the material particle Y fixed. In this sense the velocity $\mathbf{v}$ is the material derivative of the position $\mathbf{x}$ and the acceleration $\mathbf{a}$ is the material derivative of the velocity $\mathbf{v}$. Recalling that the function f can be expressed in terms of either the Material (3.2.1a), Lagrangian (3.2.1b) or Eulerian (3.2.1c) representations, it follows from the chain rule of differentiation that $\dot{f}$ admits the additional representations

$$\dot{f} = \frac{\partial \hat{f}(\mathbf{X},t)}{\partial t}\,\dot{t} + [\partial \hat{f}(\mathbf{X},t)/\partial \mathbf{X}]\,\dot{\mathbf{X}} = \frac{\partial \hat{f}(\mathbf{X},t)}{\partial t}\,,$$

$$\dot{f} = \frac{\partial \hat{f}(\mathbf{X},t)}{\partial t}\,\dot{t} + [\partial \hat{f}(\mathbf{X},t)/\partial X_A]\,\dot{X}_A = \frac{\partial \hat{f}(\mathbf{X},t)}{\partial t}\,, \tag{3.2.9a}$$

$$\dot{f} = \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial t}\,\dot{t} + [\partial \tilde{f}(\mathbf{x},t)/\partial \mathbf{x}]\,\dot{\mathbf{x}} = \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial t} + [\partial \tilde{f}(\mathbf{x},t)/\partial \mathbf{x}]\,\mathbf{v}\,,$$

$$\dot{f} = \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial t}\,\dot{t} + [\partial \tilde{f}(\mathbf{x},t)/\partial x_m]\,\dot{x}_m = \frac{\partial \tilde{f}(\mathbf{x},t)}{\partial t} + [\partial \tilde{f}(\mathbf{x},t)/\partial x_m]\,v_m\,, \tag{3.2.9b}$$

where in (3.2.9a) use has been made of the fact that the mapping (3.1.3) from the material point Y to its location $\mathbf{X}$ in the reference configuration is independent of time so that $\dot{\mathbf{X}}$ vanishes. It is important to emphasize that the physics of the material derivative defined by (3.2.8) remains unchanged even though its specific functional form for the different representations (3.2.9a) and (3.2.9b) changes.

3.3 Deformation Gradient and Deformation Measures

To describe the deformation of the body from the reference configuration to the current configuration, it is convenient to think of the body in its reference configuration as a finite collection of neighboring tetrahedrons. As the number of tetrahedrons increases it is possible to approximate a body having an arbitrary shape. If the deformation of each of these tetrahedrons from the reference configuration to the current configuration can be determined, then the shape (and volume) of the body in the current configuration can be determined by simply connecting the neighboring tetrahedrons. Since a tetrahedron is characterized by a triad of three vectors, the deformation of an arbitrary elemental tetrahedron (infinitesimally small) can be characterized by determining the deformation of an arbitrary material line element. This is because the material line element can be identified with each of the base vectors which represent the edges of the tetrahedron.

Deformation Gradient
For this reason it is sufficient to determine the deformation of a general material line element $d\mathbf{X}$ in the reference configuration to the material line element $d\mathbf{x}$ in the current configuration. Recalling that the mapping $\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X}, t)$ defines the position $\mathbf{x}$

in the current configuration of any material point **X** in the reference configuration at time t, it follows that

$$d\mathbf{x} = (\partial \hat{\mathbf{x}}/\partial \mathbf{X})d\mathbf{X} = \mathbf{F}d\mathbf{X}\,, \tag{3.3.1a}$$

$$dx_i = (\partial \hat{x}_i/\partial X_A)dX_A = x_{i,A}\,dX_A = F_{iA}dX_A\,, \tag{3.3.1b}$$

$$\mathbf{F} = (\partial \hat{\mathbf{x}}/\partial \mathbf{X}) = F_{iA}\mathbf{e}_i \otimes \mathbf{e}_A\,, \quad F_{iA} = x_{i,A}\,, \tag{3.3.1c}$$

where **F** is the deformation gradient with components F_{iA}. Unless otherwise stated, throughout the text a comma denotes partial differentiation with respect to X_A if the index is a capital letter and with respect to x_i if the index is a lower case letter. Since the mapping $\hat{\mathbf{x}}(\mathbf{X}, t)$ is invertible, **F** must satisfy the restriction

$$\det\mathbf{F} \neq 0\,, \quad \det(x_{i,A}) \neq 0\,, \tag{3.3.2}$$

for all time and all points in the spatial region occupied by the body. To ensure that the reference configuration has the possibility of coinciding with the current configuration at any time (i.e., $\mathbf{x} = \mathbf{X}$ and $\mathbf{F} = \mathbf{I}$), the deformation gradient must satisfy the restriction that

$$\det\mathbf{F} > 0\,, \quad \det(x_{i,A}) > 0\,. \tag{3.3.3}$$

Right and Left Cauchy–Green Deformation Tensors and the Cauchy Deformation Tensor

The magnitude ds of the material line element $d\mathbf{x}$ in the current configuration can be calculated using (3.3.1a), such that

$$(ds)^2 = d\mathbf{x}\cdot d\mathbf{x} = \mathbf{F}d\mathbf{X}\cdot\mathbf{F}d\mathbf{X} = d\mathbf{X}\cdot\mathbf{F}^T\mathbf{F}d\mathbf{X} = d\mathbf{X}\cdot\mathbf{C}d\mathbf{X}\,,$$

$$(ds)^2 = dx_i dx_i = F_{iA}dX_A F_{iB}dX_B = dX_A(x_{i,A}\,x_{i,B})dX_B = dX_A C_{AB} dX_B\,, \tag{3.3.4a}$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = C_{AB}\mathbf{e}_A \otimes \mathbf{e}_B\,, \quad C_{AB} = F_{iA}F_{iB} = x_{i,A}\,x_{i,B}\,, \tag{3.3.4b}$$

where **C** is called the *right Cauchy–Green deformation tensor*. Similarly, the magnitude dS of the material line element $d\mathbf{X}$ in the reference configuration can be calculated by inverting (3.3.1a) to obtain

$$d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}\,, \quad dX_A = X_{A,i}\,dx_i\,, \tag{3.3.5}$$

which yields

$$(dS)^2 = d\mathbf{X} \cdot d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x} \cdot \mathbf{F}^{-1}d\mathbf{x} = d\mathbf{x} \cdot \mathbf{F}^{-T}\mathbf{F}^{-1}d\mathbf{x} = d\mathbf{x} \cdot \mathbf{c}d\mathbf{x} \,,$$
$$(dS)^2 = dX_A dX_A = X_{A},_i \, dx_i X_{A},_j \, dx_j = dx_i (X_{A},_i \, X_{A},_j)dx_j = dx_i c_{ij} dx_j \,, \quad (3.3.6a)$$

$$\mathbf{c} = \mathbf{F}^{-T}\mathbf{F}^{-1} = c_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \,, \quad c_{ij} = X_{A},_i \, X_{A},_j \,, \quad (3.3.6b)$$

where $\mathbf{F}^{-T}$ is the transpose of $\mathbf{F}^{-1}$ and $\mathbf{c}$ is the *Cauchy deformation tensor*. It is also convenient to define the *left Cauchy–Green deformation tensor* $\mathbf{B}$ by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = B_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \,, \quad B_{ij} = F_{iA}F_{jA} = x_i,_A \, x_j,_A \,, \quad (3.3.7)$$

and it is noted that

$$\mathbf{c} = \mathbf{B}^{-1} \,. \quad (3.3.8)$$

Stretch and Extension

The stretch λ of a material line element is defined in terms of the ratio of the lengths ds and dS of the line element in the present and reference configurations, respectively, such that

$$\lambda = \frac{ds}{dS} \,. \quad (3.3.9)$$

Also, the extension ε of the same material line element is defined by

$$\varepsilon = \lambda - 1 = \frac{ds - dS}{dS} \,. \quad (3.3.10)$$

It follows from these definitions that the stretch is always positive. Also, the stretch is greater than one and the extension is greater than zero when the material line element is extended relative to its reference length.

For convenience let $\mathbf{S}$ be the unit vector defining the direction of the material line element $d\mathbf{X}$ in the reference configuration and let $\mathbf{s}$ be the unit vector defining the direction of the associated material line element $d\mathbf{x}$ in the current configuration, such that

$$d\mathbf{X} = \mathbf{S}dS \,, \quad dX_A = S_A dS \,, \quad \mathbf{S} \cdot \mathbf{S} = S_A S_A = 1 \,, \quad (3.3.11a)$$
$$d\mathbf{x} = \mathbf{s}ds \,, \quad dx_i = s_i ds \,, \quad \mathbf{s} \cdot \mathbf{s} = s_i s_i = 1 \,. \quad (3.3.11b)$$

Thus, using (3.3.4a) and (3.3.6a) it can be shown that

$$\lambda\mathbf{s} = \mathbf{F}\mathbf{S} \,, \quad \lambda s_i = x_i,_A \, S_A \,, \quad (3.3.12a)$$
$$\lambda^2 = \mathbf{C} \cdot (\mathbf{S} \otimes \mathbf{S}) \,, \quad \lambda^2 = C_{AB} S_A S_B \,, \quad (3.3.12b)$$
$$\frac{1}{\lambda^2} = \mathbf{c} \cdot (\mathbf{s} \otimes \mathbf{s}) \,, \quad \frac{1}{\lambda^2} = c_{ij} s_i s_j \,. \quad (3.3.12c)$$

Since the stretch is positive, it follows from (3.3.12b) and (3.3.12c) that **C** and **c** are positive-definite tensors. Similarly, it can be shown that **B** in (3.3.7) is also a positive-definite tensor. In addition, notice from (3.3.12b) that the stretch of a material line element depends not only on the value of **C** at the material point **X** and the time t, but it depends also on the orientation **S** of the material line element in the reference configuration.

A Pure Measure of Dilatation (Volume Change)
To discuss the relative volume change of a material element, it is convenient to first prove that for any nonsingular second-order tensor **F** and any two vectors **a** and **b** that

$$\mathbf{Fa} \times \mathbf{Fb} = \det(\mathbf{F})\, \mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b})\,. \tag{3.3.13}$$

To prove this result, it is noted that the quantity $\mathbf{F}^{T}(\mathbf{a} \times \mathbf{b})$ is a vector that is orthogonal to the plane formed by the vectors **Fa** and **Fb** since

$$\begin{aligned} &\mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{Fa} = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{F}^{-T})^{T}\mathbf{Fa} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{F}^{-1}\mathbf{Fa} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0\,, \\ &\mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{Fb} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{F}^{-1}\mathbf{Fb} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0\,. \end{aligned} \tag{3.3.14}$$

This means that the quantity $\mathbf{Fa} \times \mathbf{Fb}$ must be a vector that is parallel to $\mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b})$ so that

$$\mathbf{Fa} \times \mathbf{Fb} = \alpha\, \mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b})\,. \tag{3.3.15}$$

Next, the value of the scalar α is determined by noting that both sides of Eq. (3.3.15) must be linear functions of **a** and **b**. This means that α is independent of the vectors **a** and **b**. Moreover, letting **c** be an arbitrary vector, it follows that

$$\mathbf{Fa} \times \mathbf{Fb} \cdot \mathbf{Fc} = \alpha\, \mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{Fc} = \alpha(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\,. \tag{3.3.16}$$

The proof is finished by recognizing that one definition of the determinant of **F** is

$$\alpha = \det \mathbf{F} = \frac{(\mathbf{Fa} \times \mathbf{Fb}) \cdot \mathbf{Fc}}{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}}\,, \tag{3.3.17}$$

for any set of linearly independent vectors **a**, **b** and **c**. Specifically, using the rectangular Cartesian base vectors $\mathbf{e}_i$ and taking $\mathbf{a} = \mathbf{e}_1$, $\mathbf{b} = \mathbf{e}_2$ and $\mathbf{c} = \mathbf{e}_3$, it follows that

$$\det \mathbf{F} = (\mathbf{Fe}_1 \times \mathbf{Fe}_2) \cdot \mathbf{Fe}_3\,, \tag{3.3.18}$$

which can be recognized as the scalar triple product of the columns of **F**.

Now, it will be shown that the determinant J of the deformation gradient **F**

$$J = \det \mathbf{F}\,, \tag{3.3.19}$$

is a pure measure of dilatation. To this end, consider an elemental material volume defined by the linearly independent material line elements $d\mathbf{X}^1$, $d\mathbf{X}^2$ and $d\mathbf{X}^3$ in the reference configuration and defined by the associated linearly independent material line elements $d\mathbf{x}^1, d\mathbf{x}^2$ and $d\mathbf{x}^3$ in the current configuration. Thus, the elemental volumes dV in the reference configuration and dv in the current configuration are given by

$$dV = d\mathbf{X}^1 \times d\mathbf{X}^2 \cdot d\mathbf{X}^3 \,, \tag{3.3.20a}$$

$$dv = d\mathbf{x}^1 \times d\mathbf{x}^2 \cdot d\mathbf{x}^3 \,. \tag{3.3.20b}$$

Since (3.3.1a) defines the mapping of each material line element from the reference configuration to the current configuration, it follows that

$$\begin{aligned}
&dv = \mathbf{F}d\mathbf{X}^1 \times \mathbf{F}d\mathbf{X}^2 \cdot \mathbf{F}d\mathbf{X}^3 = J\mathbf{F}^{-T}(d\mathbf{X}^1 \times d\mathbf{X}^2) \cdot \mathbf{F}d\mathbf{X}^3 \,, \\
&dv = J(d\mathbf{X}^1 \times d\mathbf{X}^2) \cdot \mathbf{F}^{-1}\mathbf{F}d\mathbf{X}^3 = J d\mathbf{X}^1 \times d\mathbf{X}^2 \cdot d\mathbf{X}^3 \,, \\
&dv = J dV \,.
\end{aligned} \tag{3.3.21}$$

This means that J is a pure measure of dilatation. It also follows from (3.3.4b) and (3.3.19) that

$$J^2 = \det \mathbf{C} \,. \tag{3.3.22}$$

Pure Measures of Distortion (Shape Change)

In general, the deformation gradient $\mathbf{F}$ characterizes the dilatation (volume change), distortion (shape change) and the orientation of a material region. Therefore, whenever $\mathbf{F}$ is a unimodular tensor (i.e., its determinant J equals unity), $\mathbf{F}$ is a measure of distortion and orientation. Using this idea, which originated with Flory [1], it is possible to separate $\mathbf{F}$ into its dilatational part $J^{1/3}\mathbf{I}$ and its distortional part $\mathbf{F}'$ such that

$$\mathbf{F} = (J^{1/3}\mathbf{I})\mathbf{F}' = J^{1/3}\mathbf{F}' \,, \quad \mathbf{F}' = J^{-1/3}\mathbf{F} \,, \quad \det \mathbf{F}' = 1 \,. \tag{3.3.23}$$

Note that since $\mathbf{F}'$ is unimodular (3.3.23), it is a pure measure of distortion and orientation. Similarly, the deformation tensor $\mathbf{C}$ can be separated into its dilatational part $J^{2/3}\mathbf{I}$ and its distortional part $\mathbf{C}'$ such that

$$\mathbf{C} = (J^{2/3}\mathbf{I})\mathbf{C}' = J^{2/3}\mathbf{C}' \,, \quad \mathbf{C}' = J^{-2/3}\mathbf{C} \,, \quad \det \mathbf{C}' = 1 \,, \tag{3.3.24}$$

in contrast to $\mathbf{F}'$, $\mathbf{C}'$ is a pure measure of distortional deformation only.

Strain Measures

Using (3.3.4a) and (3.3.6a), it follows that the change in length of a material line element can be expressed in the following forms

$$ds^2 - dS^2 = d\mathbf{X} \cdot (\mathbf{C} - \mathbf{I})d\mathbf{X} = d\mathbf{X} \cdot (2\mathbf{E})d\mathbf{X} = 2\mathbf{E} \cdot (d\mathbf{X} \otimes d\mathbf{X}) \,,$$
$$ds^2 - dS^2 = dX_A(C_{AB} - \delta_{AB})dX_B = dX_A(2E_{AB})dX_B \,, \tag{3.3.25a}$$
$$ds^2 - dS^2 = d\mathbf{x} \cdot (\mathbf{I} - \mathbf{c})d\mathbf{x} = d\mathbf{x} \cdot (2\mathbf{e})d\mathbf{x} = 2\mathbf{e} \cdot (d\mathbf{x} \otimes d\mathbf{x}) \,,$$
$$ds^2 - dS^2 = dx_i(\delta_{ij} - c_{ij})dx_j = dx_i(2e_{ij})dx_j \,, \tag{3.3.25b}$$

where the Lagrangian strain $\mathbf{E}$ and the Almansi strain $\mathbf{e}$ are defined by

$$2\mathbf{E} = \mathbf{C} - \mathbf{I} \,, \tag{3.3.26a}$$
$$2\mathbf{e} = \mathbf{I} - \mathbf{c} \,. \tag{3.3.26b}$$

Furthermore, in view of the separation (3.3.24) it is sometimes convenient to define a scalar measure of dilatational strain E_v and a tensorial measure of distortional strain $\mathbf{E}'$ by

$$2E_v = J^2 - 1 \,, \quad 2\mathbf{E}' = \mathbf{C}' - \mathbf{I} \,. \tag{3.3.27}$$

Eigenvalues of $\mathbf{C}$ *and* $\mathbf{B}$

The notions of eigenvalues, eigenvectors and the principal invariants of a tensor are briefly reviewed in Appendix A. Using the definitions (3.3.4b), (3.3.7) and (A.1.3) it is first shown that the principal invariants of $\mathbf{C}$ and $\mathbf{B}$ are equal. To this end, use is made of the properties of the dot product given by (2.6.19) to deduce that

$$\begin{aligned}
&\mathbf{C} \cdot \mathbf{I} = \mathbf{F}^T\mathbf{F} \cdot \mathbf{I} = \mathbf{F} \cdot \mathbf{F} = \mathbf{F}\mathbf{F}^T \cdot \mathbf{I} = \mathbf{B} \cdot \mathbf{I} \,, \\
&\mathbf{C} \cdot \mathbf{C} = \mathbf{F}^T\mathbf{F} \cdot \mathbf{F}^T\mathbf{F} = \mathbf{F} \cdot \mathbf{F}\mathbf{F}^T\mathbf{F} = \mathbf{F}\mathbf{F}^T \cdot \mathbf{F}\mathbf{F}^T = \mathbf{B} \cdot \mathbf{B} \,, \\
&\det \mathbf{C} = \det(\mathbf{F}^T\mathbf{F}) = \det \mathbf{F}^T \, \det \mathbf{F} = (\det \mathbf{F})^2 = \det(\mathbf{F}\mathbf{F}^T) = \det \mathbf{B} \,.
\end{aligned} \tag{3.3.28}$$

It follows from (A.1.3) that the principal invariants of $\mathbf{C}$ and $\mathbf{B}$ are equal

$$I_1(\mathbf{C}) = I_1(\mathbf{B}) \,, \quad I_2(\mathbf{C}) = I_2(\mathbf{B}) \,, \quad I_3(\mathbf{C}) = I_3(\mathbf{B}) \,. \tag{3.3.29}$$

Furthermore, using (3.3.12b) it can be seen that the eigenvalues of $\mathbf{C}$ are also the squares of the principal values of stretch λ, which are determined by the characteristic equation

$$\det(\mathbf{C} - \lambda^2 \mathbf{I}) = -\lambda^6 + \lambda^4 I_1(\mathbf{C}) - \lambda^2 I_2(\mathbf{C}) + I_3(\mathbf{C}) = \det(\mathbf{B} - \lambda^2 \mathbf{I}) = 0 \,. \tag{3.3.30}$$

Displacement Vector

The displacement vector $\mathbf{u}$ is the vector that connects the position $\mathbf{X}$ of a material point in the reference configuration to its position $\mathbf{x}$ in the current configuration so that

$$\begin{aligned}
&\mathbf{u} = \mathbf{x} - \mathbf{X} \,, \quad \mathbf{x} = \mathbf{X} + \mathbf{u} \,, \\
&\mathbf{X} = \mathbf{x} - \mathbf{u} \,, \quad \mathbf{u} = u_A\mathbf{e}_A = u_i\mathbf{e}_i \,.
\end{aligned} \tag{3.3.31}$$

It then follows from the definition (3.3.1c) of the deformation gradient $\mathbf{F}$ that

$$\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X} = \partial(\mathbf{X}+\mathbf{u})/\partial\mathbf{X} = \mathbf{I} + \partial\hat{\mathbf{u}}/\partial\mathbf{X}\,, \tag{3.3.32a}$$

$$\mathbf{F}^{-1} = \partial\mathbf{X}/\partial\mathbf{x} = \partial(\mathbf{x}-\mathbf{u})/\partial\mathbf{x} = \mathbf{I} - \partial\tilde{\mathbf{u}}/\partial\mathbf{x}\,, \tag{3.3.32b}$$

$$\begin{aligned} \mathbf{C} &= \mathbf{F}^T\mathbf{F} = (\mathbf{I} + \partial\hat{\mathbf{u}}/\partial\mathbf{X})^T(\mathbf{I} + \partial\hat{\mathbf{u}}/\partial\mathbf{X})\,,\\ &= \mathbf{I} + \partial\hat{\mathbf{u}}/\partial\mathbf{X} + (\partial\hat{\mathbf{u}}/\partial\mathbf{X})^T + (\partial\hat{\mathbf{u}}/\partial\mathbf{X})^T(\partial\hat{\mathbf{u}}/\partial\mathbf{X})\,,\\ C_{AB} &= \delta_{AB} + \hat{u}_{A,B} + \hat{u}_{B,A} + \hat{u}_{M,A}\,\hat{u}_{M,B}\;, \end{aligned} \tag{3.3.32c}$$

$$\begin{aligned} \mathbf{c} &= \mathbf{B}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1} = (\mathbf{I} - \partial\tilde{\mathbf{u}}/\partial\mathbf{x})^T(\mathbf{I} - \partial\tilde{\mathbf{u}}/\partial\mathbf{x})\,,\\ &= \mathbf{I} - \partial\tilde{\mathbf{u}}/\partial\mathbf{x} - (\partial\tilde{\mathbf{u}}/\partial\mathbf{x})^T + (\partial\tilde{\mathbf{u}}/\partial\mathbf{x})^T(\partial\tilde{\mathbf{u}}/\partial\mathbf{x})\,,\\ c_{ij} &= \delta_{ij} - \tilde{u}_{i,j} - \tilde{u}_{j,i} + \tilde{u}_{m,i}\,\tilde{u}_{m,j}\;. \end{aligned} \tag{3.3.32d}$$

Then, with the help of the definitions (3.3.26a) and (3.3.26b), the strains $\mathbf{E}$ and $\mathbf{e}$ can be expressed in terms of the displacement gradients $\partial\hat{\mathbf{u}}/\partial\mathbf{X}$ and $\partial\tilde{\mathbf{u}}/\partial\mathbf{x}$ by

$$\begin{aligned} \mathbf{E} &= \frac{1}{2}[\partial\hat{\mathbf{u}}/\partial\mathbf{X} + (\partial\hat{\mathbf{u}}/\partial\mathbf{X})^T + (\partial\hat{\mathbf{u}}/\partial\mathbf{X})^T(\partial\hat{\mathbf{u}}/\partial\mathbf{X})] = E_{AB}\mathbf{e}_A \otimes \mathbf{e}_B\,,\\ E_{AB} &= \frac{1}{2}(\hat{u}_{A,B} + \hat{u}_{B,A} + \hat{u}_{M,A}\,\hat{u}_{M,B})\,, \end{aligned} \tag{3.3.33a}$$

$$\begin{aligned} \mathbf{e} &= \frac{1}{2}[\partial\tilde{\mathbf{u}}/\partial\mathbf{x} + (\partial\tilde{\mathbf{u}}/\partial\mathbf{x})^T - (\partial\tilde{\mathbf{u}}/\partial\mathbf{x})^T(\partial\tilde{\mathbf{u}}/\partial\mathbf{x})] = e_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\,,\\ e_{ij} &= \frac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i} - \tilde{u}_{m,i}\,\tilde{u}_{m,j})\,. \end{aligned} \tag{3.3.33b}$$

Since these expressions have been obtained without any approximation they are exact and are sometimes referred to as finite strain measures. Notice the different signs in front of the quadratic terms in the displacement gradients appearing in the expressions (3.3.33a) and (3.3.33b).

Material Area Element

The material area element dA formed by the elemental parallelogram associated with the linearly independent material line elements $d\mathbf{X}^1$ and $d\mathbf{X}^2$ in the reference configuration, and the material area element da formed by the corresponding linearly independent material line elements $d\mathbf{x}^1$ and $d\mathbf{x}^2$ in the current configuration are given by

$$\mathbf{N}dA = d\mathbf{X}^1 \otimes d\mathbf{X}^2\,, \quad \mathbf{n}da = \mathbf{F}d\mathbf{X}^1 \otimes \mathbf{F}d\mathbf{X}^2\,, \tag{3.3.34}$$

where $\mathbf{N}$ and $\mathbf{n}$ are the unit vectors normal to the material surfaces defined by $d\mathbf{X}^1, d\mathbf{X}^2$ and $d\mathbf{x}^1, d\mathbf{x}^2$, respectively. It follows from (3.3.1a) and (3.3.13) that

$$\mathbf{n}da = \mathbf{F}d\mathbf{X}^1 \otimes \mathbf{F}d\mathbf{X}^2 = J\mathbf{F}^{-T}(d\mathbf{X}^1 \otimes d\mathbf{X}^2) = J\mathbf{F}^{-T}\mathbf{N}dA\,, \tag{3.3.35}$$

which is called Nanson's formula. It is important to emphasize that a material line element that was normal to the material surface in the reference configuration does not necessarily remain normal to the same material surface in the current configuration.

3.4 Polar Decomposition Theorem

The polar decomposition theorem states that any invertible second-order tensor $\mathbf{F}$ can be uniquely decomposed into its polar form

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}\,, \quad F_{iA} = R_{iM}U_{MA} = V_{im}R_{mA}\,, \tag{3.4.1}$$

where $\mathbf{R}$ is an orthogonal tensor

$$\begin{aligned} &\mathbf{R} = R_{iA}\mathbf{e}_i \otimes \mathbf{e}_A\,, \\ &\mathbf{R}^T\mathbf{R} = \mathbf{I}\,, \qquad R_{mA}R_{mB} = \delta_{AB}\,, \\ &\mathbf{R}\mathbf{R}^T = \mathbf{I}\,, \qquad R_{iA}R_{jA} = \delta_{ij}\,, \end{aligned} \tag{3.4.2}$$

$\mathbf{U}$ is the right stretch tensor and $\mathbf{V}$ is the left stretch tensor. These stretch tensors are symmetric, positive-definite tensors so that for an arbitrary vector $\mathbf{v}$, it follows that

$$\begin{aligned} &\mathbf{U}^T = \mathbf{U} = U_{AB}\mathbf{e}_A \otimes \mathbf{e}_B\,, \quad U_{BA} = U_{AB}\,, \\ &\mathbf{v}\cdot\mathbf{U}\mathbf{v} = \mathbf{U}\cdot\mathbf{v}\otimes\mathbf{v} > 0\,, \quad v_A U_{AB} v_B = U_{AB} v_A v_B > 0 \text{ for } \mathbf{v} \neq 0\,, \\ &\mathbf{V}^T = \mathbf{V} = V_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\,, \quad V_{ji} = V_{ij}\,, \\ &\mathbf{v}\cdot\mathbf{V}\mathbf{v} = \mathbf{V}\cdot\mathbf{v}\otimes\mathbf{v} > 0\,, \quad v_i V_{ij} v_j = V_{ij} v_1 v_j > 0 \text{ for } \mathbf{v} \neq 0\,. \end{aligned} \tag{3.4.3}$$

From these expressions and the definitions (3.3.4b) and (3.3.7) it can be deduced that

$$\mathbf{C} = \mathbf{U}^2\,, \quad \mathbf{B} = \mathbf{V}^2\,, \tag{3.4.4}$$

which explains why $\mathbf{C}$ is called the right Cauchy–Green deformation tensor and $\mathbf{B}$ is called the left Cauchy–Green deformation tensor.

To prove this theorem it is convenient to first consider the following Lemma.

Lemma *If* $\mathbf{S}$ *is an invertible second-order tensor then* $\mathbf{S}^T\mathbf{S}$ *and* $\mathbf{S}\mathbf{S}^T$ *are positive-definite symmetric tensors.*

Proof (i) Consider two vectors $\mathbf{v}$ and $\mathbf{w}$ defined by

$$\mathbf{w} = \mathbf{S}\mathbf{v}\,, \quad w_i = S_{ij}v_j\,. \tag{3.4.5}$$

Since $\mathbf{S}$ is invertible, it follows that

$$\begin{aligned} &\mathbf{w} = 0 \quad \text{if and only if} \quad \mathbf{v} = 0\,, \\ &\mathbf{w} \neq 0 \quad \text{if and only if} \quad \mathbf{v} \neq 0\,. \end{aligned} \tag{3.4.6}$$

Now, consider

$$\mathbf{w} \cdot \mathbf{w} = \mathbf{S}\mathbf{v} \cdot \mathbf{S}\mathbf{v} = \mathbf{v} \cdot \mathbf{S}^T\mathbf{S}\mathbf{v} = \mathbf{S}^T\mathbf{S} \cdot (\mathbf{v} \otimes \mathbf{v}) \,,$$

$$w_m w_m = S_{mi} v_i S_{mj} v_j = v_i (S^T_{im} S_{mj}) v_j \,. \tag{3.4.7}$$

Since $\mathbf{w} \cdot \mathbf{w} > 0$ whenever $\mathbf{v} \neq 0$, it follows that $\mathbf{S}^T\mathbf{S}$ is positive-definite.

(ii) Alternatively, define the two vectors $\mathbf{v}$ and $\mathbf{w}$ by

$$\mathbf{w} = \mathbf{S}^T\mathbf{v} \,, \quad w_i = S^T_{ij} v_j = S_{ji} v_j \,. \tag{3.4.8}$$

Since $\mathbf{S}$ is invertible, it follows that

$$\mathbf{w} \cdot \mathbf{w} = \mathbf{S}^T\mathbf{v} \cdot \mathbf{S}^T\mathbf{v} = \mathbf{v} \cdot \mathbf{S}\mathbf{S}^T\mathbf{v} = \mathbf{S}\mathbf{S}^T \cdot (\mathbf{v} \otimes \mathbf{v}) \,,$$

$$w_m w_m = S_{im} v_i S_{jm} v_j = v_i (S_{im} S^T_{mj}) v_j \,. \tag{3.4.9}$$

Moreover, since $\mathbf{w} \cdot \mathbf{w} > 0$ whenever $\mathbf{v} \neq 0$ the tensor $\mathbf{S}^T\mathbf{S}$ is positive-definite.

To prove the polar decomposition theorem it is convenient to first prove existence of the forms $\mathbf{F} = \mathbf{R}\mathbf{U}$ and $\mathbf{F} = \mathbf{V}\mathbf{R}$ and then prove uniqueness of the quantities $\mathbf{R}$, $\mathbf{U}$ and $\mathbf{V}$.

Existence

(i) Since $\mathbf{F}$ is invertible the tensor $\mathbf{F}^T\mathbf{F}$ is symmetric and positive-definite so there exists a unique symmetric positive-definite square root $\mathbf{U}$ defined by

$$\mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2} \,, \quad \mathbf{U}^2 = \mathbf{F}^T\mathbf{F} \,, \quad U_{AM} U_{MB} = F_{mA} F_{mB} \,. \tag{3.4.10}$$

Now, let $\mathbf{R}_1$ be defined by

$$\mathbf{R}_1 = \mathbf{F}\mathbf{U}^{-1} \,, \quad \mathbf{F} = \mathbf{R}_1\mathbf{U} \,. \tag{3.4.11}$$

To prove that $\mathbf{R}_1$ is an orthogonal tensor consider

$$\mathbf{R}_1\mathbf{R}_1^T = \mathbf{F}\mathbf{U}^{-1}(\mathbf{F}\mathbf{U}^{-1})^T = \mathbf{F}\mathbf{U}^{-1}\mathbf{U}^{-T}\mathbf{F}^{-T} = \mathbf{F}(\mathbf{U}^2)^{-1}\mathbf{F}^T \,,$$

$$= \mathbf{F}(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T = \mathbf{F}(\mathbf{F}^{-1}\mathbf{F}^{-T})\mathbf{F}^T = \mathbf{I} \,, \tag{3.4.12a}$$

$$\mathbf{R}_1^T\mathbf{R}_1 = \mathbf{U}^{-T}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{I} \,. \tag{3.4.12b}$$

(ii) Similarly, since $\mathbf{F}$ is invertible the tensor $\mathbf{F}\mathbf{F}^T$ is symmetric and positive-definite there exists a unique symmetric, positive-definite square root $\mathbf{V}$

$$\mathbf{V} = (\mathbf{F}\mathbf{F}^T)^{1/2} \,, \quad \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T \,, \quad V_{im} V_{mj} = F_{iM} F_{jM} \,. \tag{3.4.13}$$

Now, let $\mathbf{R}_2$ be defined by

$$\mathbf{R}_2 = \mathbf{V}^{-1}\mathbf{F}\,, \quad \mathbf{F} = \mathbf{V}\mathbf{R}_2\,. \tag{3.4.14}$$

To prove that $\mathbf{R}_2$ is an orthogonal tensor consider

$$\mathbf{R}_2\mathbf{R}_2^T = \mathbf{V}^{-1}\mathbf{F}(\mathbf{V}^{-1}\mathbf{F})^T = \mathbf{V}^{-1}\mathbf{F}\mathbf{F}^T\mathbf{V}^{-1} = \mathbf{V}^{-1}\mathbf{V}^2\mathbf{V}^{-1} = \mathbf{I}\,,$$
$$\mathbf{R}_2^T\mathbf{R}_2 = \mathbf{F}^T\mathbf{V}^{-T}\mathbf{V}^{-1}\mathbf{F} = \mathbf{F}^T\mathbf{V}^{-2}\mathbf{F} = \mathbf{F}^T(\mathbf{F}\mathbf{F}^T)^{-1}\mathbf{F} = \mathbf{I}\,. \tag{3.4.15a}$$

Uniqueness

(i) Assume that $\mathbf{R}_1$ and $\mathbf{U}$ are not unique so that

$$\mathbf{F} = \mathbf{R}_1\mathbf{U} = \mathbf{R}_1^*\mathbf{U}^*\,. \tag{3.4.16}$$

Then consider

$$\mathbf{F}^T\mathbf{F} = \mathbf{U}^2 = (\mathbf{R}_1^*\mathbf{U}^*)^T(\mathbf{R}_1^*\mathbf{U}^*) = \mathbf{U}^{*T}\mathbf{R}_1^{*T}\mathbf{R}_1^*\mathbf{U}^* = \mathbf{U}^{*2}\,. \tag{3.4.17}$$

However, since $\mathbf{U}$ and $\mathbf{U}^*$ are both symmetric and positive-definite it can be deduced that $\mathbf{U}$ is unique

$$\mathbf{U}^* = \mathbf{U}\,. \tag{3.4.18}$$

Next, substituting (3.4.18) into (3.4.16) yields

$$\mathbf{R}_1\mathbf{U} = \mathbf{R}_1^*\mathbf{U}\,. \tag{3.4.19}$$

Then, multiplication of (3.4.19) on the right by $\mathbf{U}^{-1}$ proves that $\mathbf{R}_1$ is unique

$$\mathbf{R}_1 = \mathbf{R}_1^*\,. \tag{3.4.20}$$

(ii) Similarly, assume that $\mathbf{R}_2$ and $\mathbf{V}$ are not unique so that

$$\mathbf{F} = \mathbf{V}\mathbf{R}_2 = \mathbf{V}^*\mathbf{R}_2^*\,. \tag{3.4.21}$$

Then, consider

$$\mathbf{F}\mathbf{F}^T = \mathbf{V}^2 = (\mathbf{V}^*\mathbf{R}_2^*)(\mathbf{V}^*\mathbf{R}_2^*)^T = \mathbf{V}^*\mathbf{R}_2^*\mathbf{R}_2^{*T}\mathbf{V}^{*T} = \mathbf{V}^{*2}\,. \tag{3.4.22}$$

However, since $\mathbf{V}$ and $\mathbf{V}^*$ are both symmetric and positive-definite it can be deduced that $\mathbf{V}$ is unique

$$\mathbf{V}^* = \mathbf{V}\,. \tag{3.4.23}$$

Next, substituting (3.4.23) into (3.4.21) yields

$$\mathbf{V}\mathbf{R}_2 = \mathbf{V}\mathbf{R}_2^* \,. \tag{3.4.24}$$

Then, multiplication of (3.4.24) on the left by $\mathbf{V}^{-1}$ proves that $\mathbf{R}_2$ is unique

$$\mathbf{R}_2 = \mathbf{R}_2^* \,. \tag{3.4.25}$$

(iii) Finally, it is necessary to prove that $\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{R}$. To this end, define the auxiliary tensor $\mathbf{A}$ by

$$\mathbf{A} = \mathbf{R}_1\mathbf{U}\mathbf{R}_1^T = \mathbf{F}\mathbf{R}_1^T \,. \tag{3.4.26}$$

Clearly, $\mathbf{A}$ is symmetric so that

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A}^T = \mathbf{F}\mathbf{R}_1^T(\mathbf{F}\mathbf{R}_1^T)^T = \mathbf{F}\mathbf{R}_1^T\mathbf{R}_1\mathbf{F}^T = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 \,. \tag{3.4.27}$$

Since $\mathbf{A}$ and $\mathbf{V}$ are symmetric and nonsingular, it follows with the help of (3.4.14) and (3.4.26) that

$$\mathbf{V} = \mathbf{A} = \mathbf{F}\mathbf{R}_1^T = \mathbf{V}\mathbf{R}_2\mathbf{R}_1^T \,. \tag{3.4.28}$$

Now, multiplying (3.4.28) on the left by $\mathbf{V}^{-1}$ and on the right by $\mathbf{R}_1$, it follows that

$$\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{R} \,, \tag{3.4.29}$$

which completes the proof.

Example As an example, consider the simple deformation field for which $\mathbf{F}$ is given by

$$\mathbf{F} = F_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + F_{12}\mathbf{e}_1 \otimes \mathbf{e}_2 + F_{21}\mathbf{e}_2 \otimes \mathbf{e}_1 + F_{22}\mathbf{e}_2 \otimes \mathbf{e}_2 + F_{33}\mathbf{e}_3 \otimes \mathbf{e}_3 \,. \tag{3.4.30}$$

For this deformation field the rotation tensor $\mathbf{R}$ can be written in the form

$$\mathbf{R} = \cos\gamma\,(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \sin\gamma\,(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) + \mathbf{e}_3 \otimes \mathbf{e}_3 \,, \tag{3.4.31}$$

where the angle γ is determined by requiring $\mathbf{U} = \mathbf{R}^T\mathbf{F}$ to be a symmetric tensor

$$\gamma = \tan^{-1}\left(\frac{F_{12} - F_{21}}{F_{11} + F_{22}}\right) . \tag{3.4.32}$$

It then follows that $\mathbf{R}$, $\mathbf{U}$ and $\mathbf{V}$ for this deformation can be expressed in the forms

$$\mathbf{R} = \frac{1}{\sqrt{(F_{11}+F_{22})^2+(F_{12}-F_{21})^2}}\Big[(F_{11}+F_{22})(\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_2) + (F_{12}-F_{21})(\mathbf{e}_1\otimes\mathbf{e}_2-\mathbf{e}_2\otimes\mathbf{e}_1)\Big] + \mathbf{e}_3\otimes\mathbf{e}_3\,,$$

$$\mathbf{U} = \frac{1}{\sqrt{(F_{11}+F_{22})^2+(F_{12}-F_{21})^2}}\Big[\big\{F_{11}(F_{11}+F_{22})-F_{21}(F_{12}-F_{21})\big\}\mathbf{e}_1\otimes\mathbf{e}_1 + \big\{F_{22}(F_{11}+F_{22})+F_{12}(F_{12}-F_{21})\big\}\mathbf{e}_2\otimes\mathbf{e}_2 + (F_{11}F_{12}+F_{22}F_{21})(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1)\Big] + F_{33}\mathbf{e}_3\otimes\mathbf{e}_3\,,$$

$$\mathbf{V} = \frac{1}{\sqrt{(F_{11}+F_{22})^2+(F_{12}-F_{21})^2}}\Big[\big\{F_{11}(F_{11}+F_{22})+F_{12}(F_{12}-F_{21})\big\}\mathbf{e}_1\otimes\mathbf{e}_1 + \big\{F_{22}(F_{11}+F_{22})-F_{21}(F_{12}-F_{21})\big\}\mathbf{e}_2\otimes\mathbf{e}_2 + (F_{11}F_{21}+F_{22}F_{12})(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1)\Big] + F_{33}\mathbf{e}_3\otimes\mathbf{e}_3\,. \tag{3.4.33}$$

Physical Interpetation

To explain the physical interpretation of the polar decomposition theorem recall from (3.3.1a) that a material line element $d\mathbf{X}$ in the reference configuration is transformed by $\mathbf{F}$ into the material line element $d\mathbf{x}$ in the current configuration and define the elemental vectors $d\mathbf{X}'$ and $d\mathbf{x}'$ such that

$$d\mathbf{x} = \mathbf{RU}d\mathbf{X} \quad\Rightarrow\quad d\mathbf{X}' = \mathbf{U}d\mathbf{X}\,, \quad d\mathbf{x} = \mathbf{R}d\mathbf{X}'\,,$$
$$dx_i = R_{iA}U_{AB}dX_B \quad\Rightarrow\quad dX'_A = U_{AB}dX_B\,, \quad dx_i = R_{iA}dX'_A\,, \tag{3.4.34a}$$
$$d\mathbf{x} = \mathbf{VR}d\mathbf{X} \quad\Rightarrow\quad d\mathbf{x}' = \mathbf{R}d\mathbf{X}\,, \quad d\mathbf{x} = \mathbf{V}d\mathbf{x}'\,,$$
$$dx_i = V_{ij}R_{jB}dX_B \quad\Rightarrow\quad dx'_j = R_{jB}dX_B\,, \quad dx_i = V_{ij}dx'_j\,. \tag{3.4.34b}$$

In general, a material line element experiences both stretch and rotation as it deforms from $d\mathbf{X}$ to $d\mathbf{x}$. However, the polar decomposition theorem indicates that part of the deformation can be described as a pure rotation. To see this, use (3.3.4a) together with (3.4.34a) and (3.4.34b) and consider

$$ds^2 = d\mathbf{x}\cdot d\mathbf{x} = \mathbf{R}d\mathbf{X}'\cdot\mathbf{R}d\mathbf{X}' = d\mathbf{X}'\cdot\mathbf{R}^T\mathbf{R}d\mathbf{X}' = d\mathbf{X}'\cdot d\mathbf{X}'\,. \tag{3.4.35}$$

It follows that the magnitude of $d\mathbf{X}'$ is the same as that of $d\mathbf{x}$ so that all the stretching occurs during the transformation from $d\mathbf{X}$ to $d\mathbf{X}'$ and that the transformation from $d\mathbf{X}'$ to $d\mathbf{x}$ is a pure rotation. Similarly, with the help of (3.3.6a) and (3.4.34b) it can be shown that

$$d\mathbf{x}'\cdot d\mathbf{x}' = \mathbf{R}d\mathbf{X}\cdot\mathbf{R}d\mathbf{X} = d\mathbf{X}\cdot\mathbf{R}^T\mathbf{R}d\mathbf{X} = d\mathbf{X}\cdot d\mathbf{X} = dS^2\,. \tag{3.4.36}$$

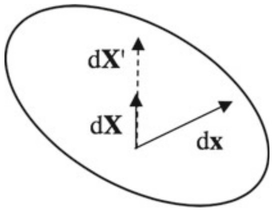

Fig. 3.2 Pure stretching followed by pure rotation: $\mathbf{F} = \mathbf{R}\mathbf{U}$; $d\mathbf{X}' = \mathbf{U}d\mathbf{X} = \lambda d\mathbf{X}$; $d\mathbf{x} = \mathbf{R}d\mathbf{X}'$

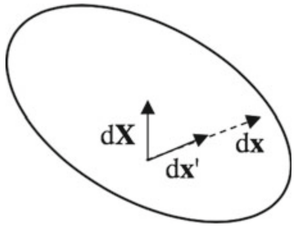

Fig. 3.3 Pure rotation followed by pure stretching: $\mathbf{F} = \mathbf{V}\mathbf{R}$; $d\mathbf{x}' = \mathbf{R}d\mathbf{X}$; $d\mathbf{x} = \mathbf{V}d\mathbf{x}' = \lambda d\mathbf{x}'$

Consequently, it follows that the magnitude of $d\mathbf{x}'$ is the same as that of $d\mathbf{X}$ so that all the stretching occurs during the transformation from $d\mathbf{x}'$ to $d\mathbf{x}$ and that the transformation from $d\mathbf{X}$ to $d\mathbf{x}'$ is a pure rotation.

Although the transformations from $d\mathbf{X}$ to $d\mathbf{X}'$ and from $d\mathbf{x}'$ to $d\mathbf{x}$ contain all of the stretching, they also tend to rotate a general line element. However, the special line element $d\mathbf{X}$ which is parallel to any of the three principal directions of $\mathbf{U}$ transforms $d\mathbf{X}$ to $d\mathbf{X}'$ as a pure stretch without rotation (see Fig. 3.2) because

$$d\mathbf{X}' = \mathbf{U}d\mathbf{X} = \lambda d\mathbf{X}\,, \tag{3.4.37}$$

where λ is the stretch defined by (3.3.9). It then follows that for this line element

$$\begin{aligned} d\mathbf{x} &= \mathbf{F}d\mathbf{X} = \mathbf{R}\mathbf{U}d\mathbf{X} = \mathbf{R}\lambda d\mathbf{X} = \lambda d\mathbf{x}'\,, \\ d\mathbf{x} &= \mathbf{F}d\mathbf{X} = \mathbf{V}\mathbf{R}d\mathbf{X} = \mathbf{V}d\mathbf{x}' = \lambda d\mathbf{x}'\,, \end{aligned} \tag{3.4.38}$$

so that $d\mathbf{x}'$ is also parallel to a principal direction of $\mathbf{V}$, which means that the transformation from $d\mathbf{x}'$ to $d\mathbf{x}$ is a pure stretch without rotation (see Fig. 3.3). This also means that the rotation tensor $\mathbf{R}$ describes the complete rotation of material line elements which are either parallel to principal directions of $\mathbf{U}$ in the reference configuration or parallel to principal directions of $\mathbf{V}$ in the current configuration.

3.5 Velocity Gradient and Rate of Deformation Tensors

The gradient of the velocity $\mathbf{v}$ with respect to the present position $\mathbf{x}$ is denoted by $\mathbf{L}$ and is defined by

$$\mathbf{L} = \partial\mathbf{v}/\partial\mathbf{x}\,, \quad L_{ij} = \frac{\partial v_i}{\partial x_j} = v_{i,j}\ . \tag{3.5.1}$$

The symmetric part **D** of **L** is called the rate of deformation tensor, while its skew-symmetric part **W** is called the spin tensor, which are defined by

$$\mathbf{L} = \mathbf{D} + \mathbf{W}\,, \qquad L_{ij} = v_{i,j} = D_{ij} + W_{ij}\,, \tag{3.5.2a}$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \mathbf{D}^T\,, \qquad D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) = D_{ji}\,, \tag{3.5.2b}$$

$$\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) - \mathbf{W}^T\,, \qquad W_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}) = -W_{ji}\,. \tag{3.5.2c}$$

Moreover, using the definition (2.5.4) of the divergence operator it can be shown that

$$\mathrm{div}\mathbf{v} = \mathbf{v},_m \cdot \mathbf{e}_m = (\partial \mathbf{v}/\partial \mathbf{x})\,\mathbf{e}_m \cdot \mathbf{e}_m = \mathbf{L}\mathbf{e}_m \cdot \mathbf{e}_m = \mathbf{L} \cdot (\mathbf{e}_m \otimes \mathbf{e}_m) = \mathbf{L} \cdot \mathbf{I} = \mathbf{D} \cdot \mathbf{I}\,. \tag{3.5.3}$$

Using the chain rule of differentiation, the continuity of the derivatives and the definition of the material derivative yields the expressions

$$\begin{aligned}
\dot{\mathbf{F}} &= \frac{\partial}{\partial t}(\partial \hat{\mathbf{x}}/\partial \mathbf{X}) = \partial^2 \hat{\mathbf{x}}/\partial \mathbf{X}\partial t = \partial(\partial \hat{\mathbf{x}}/\partial t)/\partial \mathbf{X} = \partial \hat{\mathbf{v}}/\partial \mathbf{X} = (\partial \tilde{\mathbf{v}}/\partial \mathbf{x})(\partial \hat{\mathbf{x}}/\partial \mathbf{X}) = \mathbf{L}\mathbf{F}\,, \\
\dot{\overline{x_{i,A}}} &= \frac{\partial}{\partial t}(\hat{x}_{i,A}) = \frac{\partial^2 \hat{x}_i}{\partial X_A \partial t} = \frac{\partial}{\partial X_A}(\frac{\partial \hat{x}_i}{\partial t}) = \hat{v}_{i,A} = \tilde{v}_{i,m}\,\hat{x}_{m,A}\;.
\end{aligned} \tag{3.5.4}$$

It then follows that the material derivative of **C** can be expressed in the form

$$\begin{aligned}
\dot{\mathbf{C}} &= \dot{\overline{\mathbf{F}^T\mathbf{F}}} = \dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}} = (\mathbf{L}\mathbf{F})^T\mathbf{F} + \mathbf{F}^T(\mathbf{L}\mathbf{F}) = \mathbf{F}^T(\mathbf{L}^T + \mathbf{L})\mathbf{F} = 2\mathbf{F}^T\mathbf{D}\mathbf{F}\,, \\
\dot{C}_{AB} &= \dot{\overline{x_{i,A}}}x_{i,B} + x_{i,A}\dot{\overline{x_{i,B}}} = v_{i,m}\,x_{m,A}\,x_{i,B} + x_{i,A}\,v_{i,m}\,x_{m,B}\;, \\
&= x_{m,A}\,(v_{i,m} + v_{m,i})x_{i,B} = 2x_{m,A}\,D_{mi}x_{i,B}\;.
\end{aligned} \tag{3.5.5}$$

Notice that the direct notation avoids the complications of changing repeated indices.

Furthermore, since the spin tensor **W** is skew-symmetric there exits a unique vector $\boldsymbol{\omega}$, called the axial vector of **W**, such that for any vector **a**

$$\mathbf{W}\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a}\,, \quad W_{ij}a_j = \varepsilon_{ikj}\omega_k a_j = -\varepsilon_{ijk}\omega_j a_k\,. \tag{3.5.6}$$

Since this equation must be true for any vector **a**, and **W** and $\boldsymbol{\omega}$ are independent of **a**, it follows that

$$\mathbf{W} = \boldsymbol{\varepsilon}^T\boldsymbol{\omega} = -\boldsymbol{\varepsilon}\boldsymbol{\omega}\,, \quad W_{ij} = \varepsilon_{ikj}\omega_k = -\varepsilon_{ijk}\omega_k\,. \tag{3.5.7}$$

Multiplying (3.5.7) by ε_{ijm} and using the identity

$$\varepsilon_{ijk}\varepsilon_{ijm} = 2\delta_{km}\,, \tag{3.5.8}$$

it is possible to solve for ω_m in terms of W_{ij} to obtain

$$\boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbf{W}\,, \quad \omega_m = -\frac{1}{2}\varepsilon_{ijm} W_{ij} = -\frac{1}{2}\varepsilon_{mij} W_{ij}\,. \tag{3.5.9}$$

Next, substituting (3.5.2c) into this equation and using (2.5.6) yields

$$\omega_m = -\frac{1}{4}\varepsilon_{mij}(v_{i,j} - v_{j,i}) = -\frac{1}{4}(-\varepsilon_{mij}v_{j,i} - \varepsilon_{mij}v_{j,i}) = \frac{1}{2}\varepsilon_{mij}v_{j,i} = \frac{1}{2}\varepsilon_{mji}v_{i,j}\,,$$
$$\boldsymbol{\omega} = \frac{1}{2}\mathrm{curl}\mathbf{v} = \frac{1}{2}\,\nabla \times \mathbf{v}\,, \tag{3.5.10}$$

where the symbol ∇ denotes the gradient operator

$$\nabla\,\phi = \partial\phi/\partial\mathbf{x} = \phi_{,i}\,\mathbf{e}_i\,. \tag{3.5.11}$$

For later reference, use is made of (3.3.13), (3.3.18) and (3.3.19) to deduce that

$$\dot{J} = \mathbf{F}\mathbf{e}_2 \times \mathbf{F}\mathbf{e}_3 \cdot \dot{\mathbf{F}}\mathbf{e}_1 + \mathbf{F}\mathbf{e}_3 \times \mathbf{F}\mathbf{e}_1 \cdot \dot{\mathbf{F}}\mathbf{e}_2 + \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2 \cdot \dot{\mathbf{F}}\mathbf{e}_3 = J\mathbf{F}^{-T} \cdot \dot{\mathbf{F}}\,. \tag{3.5.12}$$

Next, thinking of J as a function of $\mathbf{F}$ and using the chain rule of differentiation it can be shown that

$$\dot{J} = \frac{\partial J}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}}\,, \tag{3.5.13}$$

so that

$$\left(\frac{\partial J}{\partial \mathbf{F}} - J\mathbf{F}^{-T}\right) \cdot \dot{\mathbf{F}} = 0\,. \tag{3.5.14}$$

Since this equation must be valid for all values of $\mathbf{F}$ and $\dot{\mathbf{F}}$, and the coefficient of $\dot{\mathbf{F}}$ is independent of the rate $\dot{\mathbf{F}}$, it follows that

$$\frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{-T}\,. \tag{3.5.15}$$

This procedure of using the material derivative of a scalar function to determine its derivative respect to its tensorial argument is often easier than differentiating the scalar function directly with respect to its argument. Now, with the help of (3.5.4) it can be shown that

$$\dot{J} = J\,\mathbf{D} \cdot \mathbf{I}\,. \tag{3.5.16}$$

Derivative of a Unimodular Tensor

With the help of (3.3.23) and (3.5.16), it follows that the unimodular tensor $\mathbf{F}'$ satisfies the evolution equation

$$\dot{\mathbf{F}}' = \mathbf{L}''\,\mathbf{F}'\,, \quad \mathbf{L}'' = \mathbf{L} - \frac{1}{3}(\mathbf{L} \cdot \mathbf{I})\,\mathbf{I}\,, \tag{3.5.17}$$

so that $\dot{\mathbf{F}}'$ is orthogonal to $\mathbf{F}'^{-T}$

$$\dot{\mathbf{F}}' \cdot \mathbf{F}'^{-T} = 0\,. \tag{3.5.18}$$

Rates of Stretch and Rotation of a Material Line Element
Using the expression (3.3.1a) and the result (3.5.4) it can be shown that the material derivative of a material line element $d\mathbf{x}$ is given by

$$\dot{\overline{d\mathbf{x}}} = \mathbf{L}\,d\mathbf{x}\,. \tag{3.5.19}$$

Next, consider a material line elment which in the current configuration has stretch λ and unit direction $\mathbf{s}$. Taking the material derivative of (3.3.12a) and using (3.5.4), it follows that

$$\dot{\lambda}\mathbf{s} + \lambda\dot{\mathbf{s}} = \lambda\mathbf{L}\mathbf{s}\,. \tag{3.5.20}$$

Now, taking the dot product of this equation with $\mathbf{s}$ and using the fact that $\mathbf{s}$ is a unit vector so that $\dot{\mathbf{s}}$ is orthogonal to $\mathbf{s}$ yields an expression for the rate of stretch

$$\frac{\dot{\lambda}}{\lambda} = \mathbf{D} \cdot \mathbf{s} \otimes \mathbf{s}\,. \tag{3.5.21}$$

Then, substituting this result into (3.5.20) yields an equation for the rate of rotation of a material line element

$$\dot{\mathbf{s}} = [\mathbf{L} - (\mathbf{D} \cdot \mathbf{s} \otimes \mathbf{s})\mathbf{I}]\,\mathbf{s}\,. \tag{3.5.22}$$

Rates of Material Area Stretch and Rotation of the Normal to a Material Surface
Consider a material surface with unit normal $\mathbf{n}$ and element of area da in the present configuration. Taking the material derivative of Nanson's formula (3.3.35) and using (3.5.4) and (3.5.16) and the result

$$\dot{\overline{\mathbf{F}^{-1}}} = -\mathbf{F}^{-1}\mathbf{L}\,, \tag{3.5.23}$$

it follows that

$$\dot{\mathbf{n}}da + \mathbf{n}\dot{\overline{da}} = -\mathbf{L}^T\mathbf{n}da + (\mathbf{D} \cdot \mathbf{I})\,\mathbf{n}da\,. \tag{3.5.24}$$

Next, taking the dot product of this equation with $\mathbf{n}$ and using the fact that $\mathbf{n}$ is a unit vector so that $\dot{\mathbf{n}}$ is orthogonal to $\mathbf{n}$ yields an expression for the rate of material area stretch

$$\frac{\dot{\overline{da}}}{da} = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{D}\,. \tag{3.5.25}$$

Then, substituting this result into (3.5.24) yields an equation for the rate of rotation of the normal $\mathbf{n}$ to the material surface

$$\dot{\mathbf{n}} = -[\mathbf{L}^T - (\mathbf{D} \cdot \mathbf{n} \otimes \mathbf{n})\,\mathbf{I}]\,\mathbf{n}\,. \tag{3.5.26}$$

3.6 Deformation: Interpretations and Examples

To interpret the various deformation measures, it is recalled from (3.3.9), (3.3.11a) and (3.3.11b) that

$$\begin{aligned} &\lambda \mathbf{s} = \mathbf{F}\,, \qquad \lambda s_i = x_{i,A}\, S_A\,, \qquad \lambda = \frac{ds}{dS}\,, \\ &\mathbf{s} = \frac{d\mathbf{x}}{ds}\,, \qquad \mathbf{s}\cdot\mathbf{s} = 1\,, \qquad \mathbf{S} = \frac{d\mathbf{X}}{dS}\,, \qquad \mathbf{S}\cdot\mathbf{S} = 1\,, \end{aligned} \tag{3.6.1}$$

where $\mathbf{S}$ is the unit vector in the direction of the material line element $d\mathbf{X}$ of length dS in the reference configuration, $\mathbf{s}$ is the unit vector in the direction of the same material line element $d\mathbf{x}$ of length ds in the current configuration, and λ is the stretch of the material line element. Now, from (3.3.12b) and the definition (3.3.26a) of Lagrangian strain $\mathbf{E}$, it follows that

$$\lambda^2 = \mathbf{C}\cdot(\mathbf{S}\otimes\mathbf{S}) = 1 + 2\mathbf{E}\cdot(\mathbf{S}\otimes\mathbf{S}) = 1 + 2E_{AB}S_A S_B\,. \tag{3.6.2}$$

Also, the extension ε defined in (3.3.10) becomes

$$\varepsilon = \frac{ds - dS}{dS} = \lambda - 1 = \sqrt{1 + 2\mathbf{E}\cdot\mathbf{S}\otimes\mathbf{S}} - 1 = \sqrt{1 + 2E_{AB}S_A S_B} - 1\,. \tag{3.6.3}$$

For the purpose of interpreting the diagonal components of the strain tensor E_{AB}, it is convenient to calculate the extensions ε_1, ε_2 and ε_3 of the material line elements which were parallel to the coordinate axes with base vectors $\mathbf{e}_A$ in the reference configuration

$$\begin{aligned} \varepsilon = \varepsilon_1 &= \sqrt{1 + 2E_{11}} - 1 \quad \text{for} \quad \mathbf{S} = \mathbf{e}_1\,, \\ \varepsilon = \varepsilon_2 &= \sqrt{1 + 2E_{22}} - 1 \quad \text{for} \quad \mathbf{S} = \mathbf{e}_2\,, \\ \varepsilon = \varepsilon_3 &= \sqrt{1 + 2E_{33}} - 1 \quad \text{for} \quad \mathbf{S} = \mathbf{e}_3\,. \end{aligned} \tag{3.6.4}$$

This clearly shows that the diagonal components of the strain tensor are measures of the extensions of material line elements which were parallel to the coordinate directions in the reference configuration.

To interpret the off-diagonal components of the strain tensor E_{AB} as measures of shear, consider two material line elements $d\mathbf{X}$ and $d\bar{\mathbf{X}}$ in the reference configuration which are deformed, respectively, into $d\mathbf{x}$ and $d\bar{\mathbf{x}}$ in the present configuration. Letting $\bar{\mathbf{S}}$ and $d\bar{S}$ and $\bar{\mathbf{s}}$ and $d\bar{s}$ be the directions and lengths of the material line elements $d\bar{\mathbf{X}}$ and $d\bar{\mathbf{x}}$, respectively, it follows from (3.6.1) that

$$\bar{\lambda}\,\bar{\mathbf{s}} = \mathbf{F}\bar{\mathbf{S}}\,, \quad \bar{\lambda} = \frac{d\bar{s}}{d\bar{S}}\,. \tag{3.6.5}$$

Notice that there is no over bar on $\mathbf{F}$ in this equation because (3.6.1) is valid for any material line element, including the particular material line element $d\bar{\mathbf{X}}$. Moreover, it follows that the angle Θ between the undeformed material line elements $d\mathbf{X}$ and $d\bar{\mathbf{X}}$ and the angle θ between the deformed material line elements $d\mathbf{x}$ and $d\bar{\mathbf{x}}$ can be calculated by (see Fig. 3.4)

$$\cos\Theta = \frac{d\mathbf{X}}{dS} \cdot \frac{d\bar{\mathbf{X}}}{d\bar{S}} = \mathbf{S} \cdot \bar{\mathbf{S}}\,, \quad \cos\theta = \frac{d\mathbf{x}}{ds} \cdot \frac{d\bar{\mathbf{x}}}{d\bar{s}} = \mathbf{s} \cdot \bar{\mathbf{s}}\,. \tag{3.6.6}$$

Furthermore, using (3.6.1), it follows that

$$\cos\theta = \frac{\mathbf{C} \cdot \mathbf{S} \otimes \bar{\mathbf{S}}}{\sqrt{\mathbf{C} \cdot \mathbf{S} \otimes \mathbf{S}}\sqrt{\mathbf{C} \cdot \bar{\mathbf{S}} \otimes \bar{\mathbf{S}}}} = \frac{\cos\Theta + 2\mathbf{E} \cdot \mathbf{S} \otimes \bar{\mathbf{S}}}{\sqrt{1 + 2\mathbf{E} \cdot \mathbf{S} \otimes \mathbf{S}}\sqrt{1 + 2\mathbf{E} \cdot \bar{\mathbf{S}} \otimes \bar{\mathbf{S}}}}\,. \tag{3.6.7}$$

Defining the reduction angle ψ between the two material line elements, this equation can be rewritten in the form

$$\begin{aligned} \theta &= \Theta - \psi\,, \\ \cos\Theta\cos\psi + \sin\Theta\sin\psi &= \frac{\cos\Theta + 2E_{AB}S_A\bar{S}_B}{\sqrt{1 + 2E_{MN}S_MS_N}\sqrt{1 + 2E_{RS}\bar{S}_R\bar{S}_S}}\,. \end{aligned} \tag{3.6.8}$$

Notice that, in general, the reduction angle ψ depends on the reference angle Θ and on all of the components of strain.

As a specific example, consider two material line elements which in the reference configuration were orthogonal and aligned along the coordinate axes such that (see Fig. 3.4)

$$\mathbf{S} = \mathbf{e}_1\,, \quad \bar{\mathbf{S}} = \mathbf{e}_2\,, \quad \Theta = \frac{\pi}{2}\,. \tag{3.6.9}$$

Then, (3.6.8) yields

$$\sin\psi = \frac{2E_{12}}{\sqrt{1 + 2E_{11}}\sqrt{1 + 2E_{22}}}\,. \tag{3.6.10}$$

This shows that the shear depends on the off-diagonal components of strain as well as on the normal components of strain. However, if the strain is small (i.e., $E_{AB} << 1$) then (3.6.10) can be approximated by

$$\psi \approx 2E_{12}\,, \tag{3.6.11}$$

which shows that the off-diagonal components of strain are related to shear deformations.

Using the work in [1], it follows that in the absence of distortional deformation the unimodular part $\mathbf{C}'$ of the deformation tensor $\mathbf{C}$ is the identity

$$\mathbf{C}' = J^{-2/3}\mathbf{C} = \mathbf{I}\,, \quad \mathbf{C} = J^{2/3}\mathbf{I}\,, \tag{3.6.12}$$

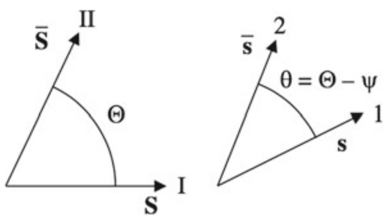

Fig. 3.4 Shear angle: Points I, II in the reference configuration move to points 1, 2 in the current configuration. Notice that the plane of $\mathbf{s}$ and $\bar{\mathbf{s}}$ is not necessarily parallel to the plane of $\mathbf{S}$ and $\bar{\mathbf{S}}$

so the associated deformation gradient $\mathbf{F}$ is determined by the total dilatation J and an arbitrary proper orthogonal rotation tensor $\mathbf{R}$, such that

$$\mathbf{F} = J^{1/3}\mathbf{R}\,, \quad \mathbf{R}^T\mathbf{R} = \mathbf{I}\,, \quad \det \mathbf{R} = +1\,. \tag{3.6.13}$$

Using this expression for $\mathbf{C}$ in (3.6.7) yields

$$\cos\theta = \cos\Theta\,, \tag{3.6.14}$$

which means that the angle between any two material line elements remains the same unless there is some distortional deformation ($\mathbf{C}' \neq \mathbf{I}$).

3.7 Rate of Deformation: Interpretations and Examples

Recall the expressions (3.5.21) for the rate of stretch $\dot{\lambda}$ and (3.5.22) for the rate of rotation $\dot{\mathbf{s}}$ of a material line element

$$\frac{\dot{\lambda}}{\lambda} = \mathbf{D}\cdot\mathbf{s}\otimes\mathbf{s}\,, \tag{3.7.1a}$$

$$\dot{\mathbf{s}} = [\mathbf{L} - (\mathbf{D}\cdot\mathbf{s}\otimes\mathbf{s})\mathbf{I}]\,\mathbf{s}\,. \tag{3.7.1b}$$

It follows from (3.7.1a) that the logarithmic derivative of the stretch is determined by the rate of deformation tensor $\mathbf{D}$ for the material line element that is currently in the $\mathbf{s}$ direction. Moreover, substituting (3.5.2a) into (3.7.1b) yields

$$\dot{\mathbf{s}} = \mathbf{W}\mathbf{s} + [\mathbf{D} - (\mathbf{D}\cdot\mathbf{s}\otimes\mathbf{s})\mathbf{I}]\,\mathbf{s}\,, \tag{3.7.2}$$

which shows that, in general, the rate of rotation of the material line element which is currently in the direction $\mathbf{s}$ is dependent on both the rate of deformation tensor $\mathbf{D}$ and the spin tensor $\mathbf{W}$. However, if $\mathbf{s}$ is parallel to a principal direction of $\mathbf{D}$ then

$$\mathbf{D}\mathbf{s} = (\mathbf{D}\cdot\mathbf{s}\otimes\mathbf{s})\mathbf{s}\,, \quad \dot{\mathbf{s}} = \mathbf{W}\mathbf{s}\,. \tag{3.7.3}$$

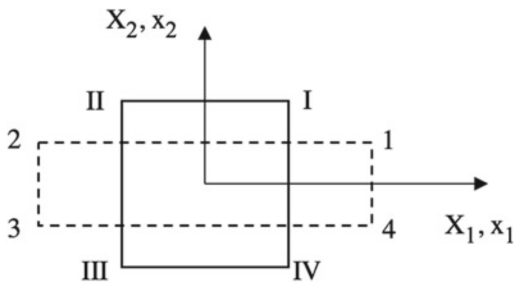

Fig. 3.5 Extension and contraction: Points I, II, II, IV in the reference configuration move to points 1, 2, 3, 4 in the current configuration

This shows that the spin tensor **W** controls the rate of rotation of the material line element $d\mathbf{x}$ which in the current configuration is parallel to a principal direction of **D**. Furthermore, using (3.5.6) it can be seen that for this case the axial vector $\boldsymbol{\omega}$ of the **W** determines the rate of rotation of **s**

$$\dot{\mathbf{s}} = \boldsymbol{\omega} \times \mathbf{s} \quad \text{for} \quad \mathbf{D}\mathbf{s} = (\mathbf{D} \cdot \mathbf{s} \otimes \mathbf{s})\mathbf{s} \,. \tag{3.7.4}$$

Example: Extension and Contraction (Fig. 3.5)
By way of example, let X_A be the Cartesian components of **X**, x_i be the Cartesian components of **x** and the Cartesian base vectors $\mathbf{e}_A$ and $\mathbf{e}_i$ coincide ($\mathbf{e}_i = \delta_{iA}\mathbf{e}_A$). Also, consider the motion defined by

$$x_1 = e^{at} X_1 \,, \quad x_2 = e^{-bt} X_2 \,, \quad x_3 = X_3 \,, \tag{3.7.5}$$

where a, b are positive numbers. The inverse mapping is given by

$$X_1 = e^{-at} x_1 \,, \quad X_2 = e^{bt} x_2 \,, \quad X_3 = x_3 \,. \tag{3.7.6}$$

It then follows that

$$F_{iA} = \begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{-bt} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad C_{AB} = \begin{pmatrix} e^{2at} & 0 & 0 \\ 0 & e^{-2bt} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{3.7.7}$$

$$E_{AB} = \frac{1}{2} \begin{pmatrix} e^{2at} - 1 & 0 & 0 \\ 0 & e^{-2bt} - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.7.8}$$

To better understand this deformation, it is convenient to calculate the stretch λ and the extension ε of line elements which were parallel to the coordinate directions in the reference configuration

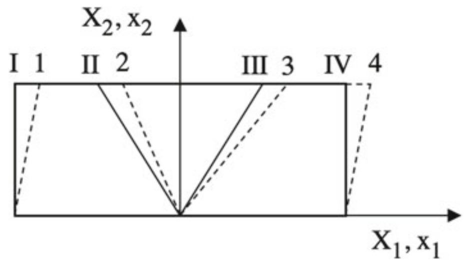

Fig. 3.6 Simple shear: Points I, II, II, IV in the reference configuration move to points 1, 2, 3, 4 in the current configuration

$$\begin{array}{llll} \text{For} \quad \mathbf{S} = \mathbf{e}_1\,, & \lambda = e^{at} \geq 1\,, & \varepsilon = e^{at} - 1 \geq 0\,, & \text{(extension)}\,, \\ \text{For} \quad \mathbf{S} = \mathbf{e}_2\,, & \lambda = e^{-bt} \leq 1\,, & \varepsilon = e^{-bt} - 1 \leq 0\,, & \text{(contraction)}\,, \\ \text{For} \quad \mathbf{S} = \mathbf{e}_3\,, & \lambda = 1\,, & \varepsilon = 0\,, & \text{(no deformation)}\,. \end{array} \tag{3.7.9}$$

Next, consider the rate of deformation to deduce that

$$\begin{gathered} v_1 = a x_1\,, \qquad v_2 = -b x_2\,, \quad v_3 = 0\,, \\ L_{ij} = D_{ij} = \begin{pmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad \mathbf{W} = 0\,, \qquad \boldsymbol{\omega} = 0\,. \end{gathered} \tag{3.7.10}$$

The principal directions of $\mathbf{D}$ are $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ so since $\mathbf{W} = 0$, it follows that the material line elements that are parallel to these principal directions in the current configuration experience pure stretching without rotation

$$\begin{array}{llll} \text{For} \quad \mathbf{s} = \mathbf{e}_1\,, & \frac{\dot{\lambda}}{\lambda} = a > 0\,, & \dot{\mathbf{s}} = 0\,, & \text{(rate of extension)}\,, \\ \text{For} \quad \mathbf{s} = \mathbf{e}_2\,, & \frac{\dot{\lambda}}{\lambda} = -b > 0\,, & \dot{\mathbf{s}} = 0\,, & \text{(rate of contraction)}\,, \\ \text{For} \quad \mathbf{s} = \mathbf{e}_3\,, & \frac{\dot{\lambda}}{\lambda} = 0\,, & \dot{\mathbf{s}} = 0\,, & \text{(no deformation)}\,. \end{array} \tag{3.7.11}$$

It is emphasized that although $\mathbf{W}$ vanishes, other material line elements can rotate during this motion.

Example: Simple Shear (Fig. 3.6)
To clarify the meaning of the spin tensor $\mathbf{W}$ consider a simple shearing deformation which is defined by

$$x_1 = X_1 + \kappa(t) X_2\,, \quad x_2 = X_2\,, \quad x_3 = X_3\,, \tag{3.7.12}$$

where $\kappa(t)$ is a monotonically increasing nonnegative function of time

$$\kappa \geq 0\,, \quad \dot{\kappa} > 0\,. \tag{3.7.13}$$

The inverse mapping is given by

$$X_1 = x_1 - \kappa x_2\,, \quad X_2 = x_2\,, \quad X_3 = x_3\,, \tag{3.7.14}$$

and it follows that

$$F_{iA} = \begin{pmatrix} 1 & \kappa & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_{AB} = \begin{pmatrix} 1 & \kappa & 0 \\ \kappa & 1+\kappa^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{AB} = \frac{1}{2}\begin{pmatrix} 0 & \kappa & 0 \\ \kappa & \kappa^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.7.15}$$

To better understand this deformation, it is convenient to calculate the stretch λ and the extension ε of material line elements which were parallel to the coordinate directions in the reference configuration

$$\begin{array}{llll} \text{For} \quad \mathbf{S} = \mathbf{e}_1\,, & \lambda = 1\,, & \varepsilon = 0\,, & \text{(no deformation)}\,, \\ \text{For} \quad \mathbf{S} = \mathbf{e}_2\,, & \lambda = \sqrt{1+\lambda^2}\,, & \varepsilon = \sqrt{1+\lambda^2} - 1 \geq 0\,, & \text{(extension)}\,, \\ \text{For} \quad \mathbf{S} = \mathbf{e}_3\,, & \lambda = 1\,, & \varepsilon = 0\,, & \text{(no deformation)}\,. \end{array} \tag{3.7.16}$$

Notice that the result for $\mathbf{S} = \mathbf{e}_2$ could be obtained by direct calculation using elementary geometry. Next, consider the rate of deformation to deduce that

$$\begin{aligned} &v_1 = \dot{\kappa} x_2\,, \qquad v_2 = 0\,, \qquad v_3 = 0\,, \\ &L_{ij} = \begin{pmatrix} 0 & \dot{\kappa} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D_{ij} = \frac{1}{2}\begin{pmatrix} 0 & \dot{\kappa} & 0 \\ \dot{\kappa} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &W_{ij} = \frac{1}{2}\begin{pmatrix} 0 & \dot{\kappa} & 0 \\ -\dot{\kappa} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega} = -\frac{1}{2}\dot{\kappa}\mathbf{e}_3\,. \end{aligned} \tag{3.7.17}$$

Since the principal directions of $\mathbf{D}$ are $\frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$, $\frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2)$ and $\mathbf{e}_3$, with the help of (3.7.1a), it follows that

$$\begin{array}{lll} \text{For} & \mathbf{s} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)\,, & \frac{\dot{\lambda}}{\lambda} = \frac{1}{2}\dot{\kappa} > 0\,, \\ & \dot{\mathbf{s}} = (\frac{1}{2}\dot{\kappa})\frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2)\,, & \text{(rate of extension)}\,, \\ \text{For} & \mathbf{s} = \frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2)\,, & \frac{\dot{\lambda}}{\lambda} = -\frac{1}{2}\dot{\kappa} < 0\,, \\ & \dot{\mathbf{s}} = (\frac{1}{2}\dot{\kappa})\frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)\,, & \text{(rate of contraction)}\,, \\ \text{For} & \mathbf{s} = \mathbf{e}_3\,, & \frac{\dot{\lambda}}{\lambda} = 0\,, \\ & \dot{\mathbf{s}} = 0\,, & \text{(no deformation)}\,. \end{array} \tag{3.7.18}$$

Thus, from (3.7.17), it follows that these special material line elements in (3.7.18) are rotating in the clockwise direction about the $\mathbf{e}_3$ axis with angular speed $\frac{1}{2}\dot{\kappa}$. In addition, it noted that this motion is isochoric (3.5.16) (no change in volume) with

$$J = \det \mathbf{F} = 1\,, \quad \mathbf{D} \cdot \mathbf{I} = 0\,. \tag{3.7.19}$$

3.8 Superposed Rigid Body Motions (SRBM)

This section develops the kinematics of Superposed Rigid Body Motions (SRBM) which will be used later to place restrictions on constitutive equations for material response. Consider a group of motions associated with configurations P^+ which differ from an arbitrary prescribed motion such as (3.1.5)

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X}, t)\,, \tag{3.8.1}$$

by SRBM of the entire body, (i.e., motions which in addition to the prescribed motion include purely rigid motions of the body).

To this end, consider a material point $\mathbf{X}$ of the body, which in $\mathcal{P}$ at time t occupies the location $\mathbf{x}$ as specified by (3.8.1). Suppose that under a SRBM the material point, which occupies the location $\mathbf{x}$ at time t in the configuration $\mathcal{P}$, moves to the location $\mathbf{x}^+$ at time t^+

$$t^+ = t + c\,, \tag{3.8.2}$$

in the superposed configuration $\mathcal{P}^+$, where c is a constant time shift. Throughout the text, quantities associated with the superposed configuration $\mathcal{P}^+$ are denoted using the same symbol as associated with the configuration P but with a superscript $(\,)^+$. In particular, the position $\mathbf{x}^+$ of the same material point in the superposed configuration is written in the form

$$\mathbf{x}^+ = \hat{\hat{\mathbf{x}}}^+(\mathbf{X}, t^+) = \hat{\mathbf{x}}^+(\mathbf{X}, t)\,, \tag{3.8.3}$$

where the notation $\hat{\hat{\mathbf{x}}}^+$ and $\hat{\mathbf{x}}^+$ has been used to distinguish between the function $\hat{\hat{\mathbf{x}}}^+$, which depends on t^+, and the function $\hat{\mathbf{x}}^+$, which depends on t and includes the influence of c .

Similarly, consider another material point $\mathbf{Y}$ of the body, which in the current configuration P at time t occupies the location $\mathbf{y}$ specified by

$$\mathbf{y} = \hat{\mathbf{x}}(\mathbf{Y}, t)\,. \tag{3.8.4}$$

It is important to emphasize that the function $\hat{\mathbf{x}}$ in (3.8.4) is the same function as that in (3.8.1). Furthermore, suppose that under the same SRBM the material point which occupies the location $\mathbf{y}$ at time t in the configuration $\mathcal{P}$ moves to the location $\mathbf{y}^+$ at time t^+. Then, with the help of (3.8.3), it follows that

$$\mathbf{y}^+ = \hat{\hat{\mathbf{x}}}^+(\mathbf{Y}, t^+) = \hat{\mathbf{x}}^+(\mathbf{Y}, t)\,. \tag{3.8.5}$$

Recalling the inverse relationships

$$\mathbf{X} = \tilde{\mathbf{X}}(\mathbf{x}, t)\,, \quad \mathbf{Y} = \tilde{\mathbf{X}}(\mathbf{y}, t)\,, \tag{3.8.6}$$

the function $\hat{\mathbf{x}}^+$ on the right-hand sides of (3.8.3) and (3.8.6) can be expressed as different functions of $\mathbf{x}$ and t and $\mathbf{y}$ and t, respectively, such that

$$\mathbf{x}^+ = \hat{\mathbf{x}}^+(\tilde{\mathbf{X}}(\mathbf{x}, t), t) = \tilde{\mathbf{x}}^+(\mathbf{x}, t)\,, \quad \mathbf{y}^+ = \hat{\mathbf{x}}^+(\tilde{\mathbf{X}}(\mathbf{y}, t), t) = \tilde{\mathbf{x}}^+(\mathbf{y}, t)\,. \tag{3.8.7}$$

Since the superposed motion of the body is restricted to be rigid, the magnitude of the relative displacement $\mathbf{y}^+ - \mathbf{x}^+$ must remain equal to the magnitude of the relative displacement $\mathbf{y} - \mathbf{x}$ for all pairs of material points $\mathbf{X}$ and $\mathbf{Y}$, and for all time. Thus,

$$[\tilde{\mathbf{x}}^+(\mathbf{y}, t) - \tilde{\mathbf{x}}^+(\mathbf{x}, t)] \cdot [\tilde{\mathbf{x}}^+(\mathbf{y}, t) - \tilde{\mathbf{x}}^+(\mathbf{x}, t)] = (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})\,. \tag{3.8.8}$$

Recognizing that $\mathbf{x}$ and $\mathbf{y}$ are independent, (3.8.8) can be differentiated first with respect to $\mathbf{x}$ and then with respect to $\mathbf{y}$ to obtain

$$\begin{aligned} -2[\partial\tilde{\mathbf{x}}^+(\mathbf{x}, t)/\partial\mathbf{x}]^T[\tilde{\mathbf{x}}^+(\mathbf{y}, t) - \tilde{\mathbf{x}}^+(\mathbf{x}, t)] &= -2(\mathbf{y} - \mathbf{x})\,, \\ [\partial\tilde{\mathbf{x}}^+(\mathbf{x}, t)/\partial\mathbf{x}]^T[\partial\tilde{\mathbf{x}}^+(\mathbf{y}, t)/\partial\mathbf{y}] &= \mathbf{I}\,. \end{aligned} \tag{3.8.9}$$

In this equation the transpose has been used to retain the inner product of $\tilde{\mathbf{x}}^+(\mathbf{x}, t)$ with $\tilde{\mathbf{x}}^+(\mathbf{y}, t)$. Moreover, it follows that the determinant of the tensor $\partial\tilde{\mathbf{x}}^+(\mathbf{x}, t)/\partial\mathbf{x}$ does not vanish so that this tensor is invertible and (3.8.9) can be rewritten in the alternative form

$$[\partial\tilde{\mathbf{x}}^+(\mathbf{x}, t)/\partial\mathbf{x}]^T = [\partial\tilde{\mathbf{x}}^+(\mathbf{y}, t)/\partial\mathbf{y}]^{-1}\,, \tag{3.8.10}$$

for all $\mathbf{x}$ and $\mathbf{y}$ in the region and all t. Thus, each side of this equation must be a tensor function of time only, say $\mathbf{Q}^T(t)$, so that

$$\partial\tilde{\mathbf{x}}^+(\mathbf{x}, t)/\partial\mathbf{x} = \mathbf{Q}(t)\,, \tag{3.8.11}$$

for all $\mathbf{x}$ in the region and all time t. Using the fact that $\mathbf{Q}$ in (3.8.11) is independent of $\mathbf{x}$, it also follows that

$$\partial\tilde{\mathbf{x}}^+(\mathbf{y}, t)/\partial\mathbf{y} = \mathbf{Q}(t)\,, \tag{3.8.12}$$

so that (3.8.9) restricts $\mathbf{Q}$ to be an orthogonal tensor

$$\mathbf{Q}^T(t)\mathbf{Q}(t) = \mathbf{I}\,, \quad \det\mathbf{Q} = \pm 1\,. \tag{3.8.13}$$

Since (3.8.7) represents a SRBM it must include the trivial motion

$$\tilde{\mathbf{x}}^+(\mathbf{x}, t) = \mathbf{x}\,, \quad \mathbf{Q} = \mathbf{I}\,, \quad \det\mathbf{Q} = +1\,. \tag{3.8.14}$$

Furthermore, since the motions are assumed to be continuous and $\det\mathbf{Q}$ cannot vanish, $\mathbf{Q}$ must remain a proper orthogonal tensor function of time only

$$\mathbf{Q}^T(t)\mathbf{Q}(t) = \mathbf{Q}(t)\mathbf{Q}^T(t) = \mathbf{I}\,, \quad \det\mathbf{Q} = +1\,. \tag{3.8.15}$$

Next, integrating (3.8.11) yields the general solution for SRBM

$$\mathbf{x}^+ = \tilde{\mathbf{x}}^+(\mathbf{x}, t) = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}\,, \tag{3.8.16}$$

where $\mathbf{c}(t)$ is an arbitrary vector function of time only representing an arbitrary translation of the body and $\mathbf{Q}(t)$ represents an arbitrary rotation of the body.

By definition, the superposed part of the motion defined by (3.8.16) is a rigid body motion. This means that the lengths of line elements are preserved

$$\begin{aligned}|\mathbf{x}^+ - \mathbf{y}^+|^2 &= (\mathbf{x}^+ - \mathbf{y}^+)\cdot(\mathbf{x}^+ - \mathbf{y}^+) = \mathbf{Q}(\mathbf{x}-\mathbf{y})\cdot\mathbf{Q}(\mathbf{x}-\mathbf{y}) = (\mathbf{x}-\mathbf{y})\cdot\mathbf{Q}^T\mathbf{Q}(\mathbf{x}-\mathbf{y})\,,\\ &= (\mathbf{x}-\mathbf{y})\cdot\mathbf{I}(\mathbf{x}-\mathbf{y}) = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^2\,,\end{aligned} \tag{3.8.17}$$

and the angles between two material line elements are also preserved so that

$$\begin{aligned}\cos\theta^+ &= \frac{(\mathbf{x}^+ - \mathbf{y}^+)}{|\mathbf{x}^+ - \mathbf{y}^+|}\cdot\frac{(\mathbf{x}^+ - \mathbf{z}^+)}{|\mathbf{x}^+ - \mathbf{z}^+|} = \frac{\mathbf{Q}(\mathbf{x}-\mathbf{y})}{|(\mathbf{x}-\mathbf{y})|}\cdot\frac{\mathbf{Q}(\mathbf{x}-\mathbf{z})}{|(\mathbf{x}-\mathbf{z})|}\,,\\ &= \frac{(\mathbf{x}-\mathbf{y})}{|(\mathbf{x}-\mathbf{y})|}\cdot\frac{\mathbf{Q}^T\mathbf{Q}(\mathbf{x}-\mathbf{z})}{|(\mathbf{x}-\mathbf{z})|} = \frac{(\mathbf{x}-\mathbf{y})}{|(\mathbf{x}-\mathbf{y})|}\cdot\frac{(\mathbf{x}-\mathbf{z})}{|(\mathbf{x}-\mathbf{z})|} = \cos\theta\,,\end{aligned} \tag{3.8.18}$$

where $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are material points in the body which move to $\mathbf{x}^+$, $\mathbf{y}^+$ and $\mathbf{z}^+$ under SRBM. Furthermore, this means that material areas, and volumes are preserved under SRBM. To show this use is made of (3.8.16) with $\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X}, t)$ to calculate the deformation gradient $\mathbf{F}^+$ from the reference configuration to the superposed configuration

$$\mathbf{F}^+ = \partial\hat{\mathbf{x}}^+(\mathbf{X}, t)/\partial\mathbf{X} = \mathbf{Q}(\partial\mathbf{x}/\partial\mathbf{X}) = \mathbf{Q}\mathbf{F}\,, \tag{3.8.19}$$

so that from (3.3.21), (3.3.35) and (3.8.19), it follows that

$$\begin{aligned}&J^+ = \frac{dv^+}{dV} = \det\mathbf{F}^+ = \det(\mathbf{Q}\mathbf{F}) = \det\mathbf{Q}\det\mathbf{F} = J\,,\\ &\mathbf{n}^+da^+ = d\mathbf{x}^{1+}\times d\mathbf{x}^{2+} = J^+(\mathbf{F}^+)^{-T}\mathbf{N}dA = J\mathbf{Q}\mathbf{F}^{-T}\mathbf{N}dA = \mathbf{Q}\mathbf{n}da\,,\\ &(da^+)^2 = \mathbf{n}^+da^+\cdot\mathbf{n}^+da^+ = \mathbf{Q}\mathbf{n}da\cdot\mathbf{Q}\mathbf{n}da = \mathbf{n}\cdot\mathbf{Q}^T\mathbf{Q}(da)^2 = (da)^2\,,\\ &\mathbf{n}^+ = \mathbf{Q}\mathbf{n}\,.\end{aligned} \tag{3.8.20}$$

For later convenience it is desirable to calculate expressions for the velocity and rate of deformation tensors associated with the superposed configuration. To this end, take the material derivative of (3.8.13) to deduce that

$$\dot{\mathbf{Q}}^T\mathbf{Q} + \mathbf{Q}^T\dot{\mathbf{Q}} = 0 \quad\Rightarrow\quad \dot{\mathbf{Q}} = \mathbf{\Omega}\mathbf{Q} \quad\Rightarrow\quad \mathbf{\Omega}^T = -\mathbf{\Omega}\,, \tag{3.8.21}$$

where $\mathbf{\Omega}(t)$ is a skew-symmetric tensor function of time only. Letting $\boldsymbol{\omega}$ be the axial vector of $\mathbf{\Omega}$ it is recalled from (3.5.6) that for an arbitrary vector $\mathbf{a}$

$$\mathbf{\Omega}\mathbf{a} = \boldsymbol{\omega}\times\mathbf{a}\,. \tag{3.8.22}$$

Thus, by taking the material derivative of (3.8.16) the velocity $\mathbf{v}^+$ of the material point in the superposed configuration can be expressed in the forms

$$\begin{aligned}\mathbf{v}^+ &= \dot{\mathbf{x}}^+ = \dot{\mathbf{c}} + \dot{\mathbf{Q}}\mathbf{x} + \mathbf{Q}\dot{\mathbf{x}} = \dot{\mathbf{c}} + \boldsymbol{\Omega}\mathbf{Q}\mathbf{x} + \mathbf{Q}\mathbf{v}\,,\\ \mathbf{v}^+ &= \dot{\mathbf{c}} + \boldsymbol{\Omega}(\mathbf{x}^+ - \mathbf{c}) + \mathbf{Q}\mathbf{v} = \dot{\mathbf{c}} + \boldsymbol{\omega}\times(\mathbf{x}^+ - \mathbf{c}) + \mathbf{Q}\mathbf{v}\,.\end{aligned} \tag{3.8.23}$$

It follows that the velocity gradient $\mathbf{L}^+$ and rate of deformation $\mathbf{D}^+$ and spin $\mathbf{W}^+$ tensors associated with the superposed configuration are given by

$$\begin{aligned}&\mathbf{L}^+ = \partial\mathbf{v}^+/\partial\mathbf{x}^+ = \mathbf{Q}(\partial\mathbf{v}/\partial\mathbf{x})(\partial\mathbf{x}/\partial\mathbf{x}^+) + \boldsymbol{\Omega} = \mathbf{Q}\mathbf{L}\mathbf{Q}^T + \boldsymbol{\Omega}\,,\\ &\mathbf{D}^+ = \mathbf{Q}\mathbf{D}\mathbf{Q}^T\,, \quad \mathbf{W}^+ = \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \boldsymbol{\Omega}\,,\end{aligned} \tag{3.8.24}$$

where use has been made of the condition (3.8.21) and (3.8.16) has been differentiated to obtain

$$\partial\mathbf{x}^+/\partial\mathbf{x} = \mathbf{Q}\,, \quad \partial\mathbf{x}/\partial\mathbf{x}^+ = \mathbf{Q}^T\,. \tag{3.8.25}$$

In general, SRBM are in addition to the general motion $\mathbf{x}(\mathbf{X}, t)$ of a deformable body. However, the kinematics of rigid body motions can be obtained as a special case by identifying $\mathbf{x}$ with its value $\mathbf{X}$ in the fixed reference configuration so that distortion and dilatation of the body are eliminated and (3.8.23) yields

$$\mathbf{x} = \mathbf{X} \quad \Rightarrow \quad \dot{\mathbf{x}}^+ = \dot{\mathbf{c}} + \boldsymbol{\omega}\times(\mathbf{x}^+ - \mathbf{c})\,. \tag{3.8.26}$$

In this form, it is easy to recognize that $\mathbf{c}(t)$ represents the translation of a point moving with the rigid body and $\boldsymbol{\omega}$ is the absolute angular velocity of the rigid body.

In summary, the most general SRBM is characterized by Eqs. (3.8.16), (3.8.13) and (3.8.21)

$$\mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{x}\,, \quad \mathbf{Q}^T\mathbf{Q} = \mathbf{I}\,, \quad \det\mathbf{Q} = +1\,, \quad \dot{\mathbf{Q}} = \boldsymbol{\Omega}\mathbf{Q}\,, \quad \boldsymbol{\Omega}^T = -\boldsymbol{\Omega}\,. \tag{3.8.27}$$

3.9 Material Line, Material Surface and Material Volume

Recall that a material point Y is mapped into its location $\mathbf{X}$ in the reference configuration and that this mapping is independent of time. Consequently, lines, surfaces and volumes, which remain constant in the reference configuration, always contain the same material points and therefore are called material.

Material Line
A material line is a fixed curve in the reference configuration that can be parameterized by its archlength S, which is independent of time. It follows that the Lagrangian representation of a material line becomes

$$\mathbf{X} = \mathbf{X}(S)\,. \tag{3.9.1}$$

Alternatively, using the mapping (3.1.5) the current positions of material points on the same material line are determined by

$$\mathbf{x} = \mathbf{x}(S, t) = \hat{\mathbf{x}}(\mathbf{X}(S), t)\,. \tag{3.9.2}$$

Material Surface
A material surface is a fixed surface in the reference configuration that can be parameterized by two coordinates S_1 and S_2 that are independent of time. It follows that the Lagrangian representation of a material surface becomes

$$\mathbf{X} = \mathbf{X}(S_1, S_2) \quad \text{or} \quad \hat{f}(\mathbf{X}) = 0\,, \tag{3.9.3}$$

where $\hat{f}(\mathbf{X}) = 0$ is a constraint on the three components of $\mathbf{X}$ which ensures that $\mathbf{X}$ identifies points in the space of the reference configuration on the material surface. Alternatively, using the mapping (3.1.5) and its inverse (3.1.6), the current positions of material points on this surface and the Eulerian representation of the same material surface can be characterized by the expressions

$$\mathbf{x} = \mathbf{x}(S_1, S_2, t) = \hat{\mathbf{x}}(\mathbf{X}(S_1, S_2), t) \quad \text{or} \quad \tilde{f}(\mathbf{x}, t) = \hat{f}(\tilde{\mathbf{X}}(\mathbf{x}, t)) = 0\,, \tag{3.9.4}$$

where $\tilde{f}(\mathbf{x}, t) = 0$ is a constraint on the three components of $\mathbf{x}$ which ensures that $\mathbf{x}$ identifies points in the space of the current configuration on the material surface.

Lagrange's Criterion for a Material Surface
The surface defined by the constraint $\tilde{f}(\mathbf{x}, t) = 0$ is material if and only if

$$\dot{\tilde{f}} = \frac{\partial \tilde{f}}{\partial t} + \partial \tilde{f}/\partial \mathbf{x} \cdot \mathbf{v} = 0\,. \tag{3.9.5}$$

Proof In general, the mapping (3.1.5) can be used to deduce that

$$\hat{f}(\mathbf{X}, t) = \tilde{f}(\hat{\mathbf{x}}(\mathbf{X}, t), t)\,, \tag{3.9.6}$$

which can be used to rewrite (3.9.5) in the form

$$\dot{\hat{f}}(\mathbf{X}, t) = \frac{\partial \hat{f}}{\partial t} = \dot{\tilde{f}} = 0\,, \tag{3.9.7}$$

so that $\hat{f}$ is independent of time and the surface $\hat{f} = 0$ is fixed in the reference configuration, which means that $\hat{f} = \tilde{f} = 0$ characterizes a material surface. Alternatively, if $\hat{f}$ is independent of time, then $\dot{\hat{f}} = 0$ and $\dot{\tilde{f}} = 0$.

Material Region

A material region is a region of space bounded by a closed material surface. For example, if ∂P_0 is a closed material surface in the reference configuration then the region of space P_0 enclosed by ∂P_0 is a material region that contains the same material points for all time if P_0 and ∂P_0 are fixed in the reference configuration. Alternatively, using the mapping (3.1.5) each point of the material surface ∂P_0 maps into a point on the closed material surface ∂P in the current configuration so the region P enclosed by ∂P is the associated material region in the current configuration.

3.10 Reynolds Transport Theorem

Reynolds transport theorem is used to calculate the time derivative of an integral over a material region P in the current configuration whose closed boundary ∂P is changing with time.

Leibniz's Rule
By way of introduction, consider the simpler one-dimensional case of Leibniz's rule and recall that

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} \phi(x,t)dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial \phi(x,t)}{\partial t}dx + \phi(\beta(t),t)\dot{\beta} - \phi(\alpha(t),t)\dot{\alpha}\,, \quad (3.10.1)$$

where $\phi(x,t)$ is an arbitrary function of position x and time t, and $\alpha(t)$ and $\beta(t)$ define the changing boundaries of integration. It is important to notice that the rates of change of the boundaries enter the expression in (3.10.1).
Reynolds Transport Theorem for a Material Region
To develop the generalization of (3.10.1) for a three-dimensional material region, it is convenient to consider an arbitrary scalar or tensor valued function $\boldsymbol{\phi}$ which admits the representations

$$\boldsymbol{\phi} = \tilde{\boldsymbol{\phi}}(\mathbf{x},t) = \hat{\boldsymbol{\phi}}(\mathbf{X},t)\,. \quad (3.10.2)$$

By mapping the material region P from the current configuration back to the reference configuration P_0, it is possible to calculate the derivative of the integral of $\boldsymbol{\phi}$ over the changing region P as follows

$$\begin{aligned} \frac{d}{dt}\int_P \tilde{\boldsymbol{\phi}}(\mathbf{x},t)dv &= \frac{d}{dt}\int_{P_0} \hat{\boldsymbol{\phi}}(\mathbf{X},t)JdV\,, \\ &= \int_{P_0} \frac{\partial\{\hat{\boldsymbol{\phi}}(\mathbf{X},t)J\}}{\partial t}\Big|_{\mathbf{X}}dV = \int_{P_0}(\dot{\hat{\boldsymbol{\phi}}} + \hat{\boldsymbol{\phi}}\mathrm{div}\mathbf{v})JdV\,, \end{aligned} \quad (3.10.3)$$

which can be transformed back to an integral over the present region P to obtain

$$\frac{d}{dt}\int_P \boldsymbol{\phi}(\mathbf{x},t)dv = \int_P (\dot{\boldsymbol{\phi}} + \boldsymbol{\phi}\,\mathrm{div}\mathbf{v})dv\,, \quad (3.10.4)$$

where $\dot{\boldsymbol{\phi}}$ is the usual material derivative of $\boldsymbol{\phi}$

$$\dot{\boldsymbol{\phi}} = \frac{\partial \hat{\boldsymbol{\phi}}(\mathbf{X}, t)}{\partial t} = \frac{\partial \tilde{\boldsymbol{\phi}}(\mathbf{x}, t)}{\partial t} + (\partial \tilde{\boldsymbol{\phi}}(\mathbf{x}, t)/\partial \mathbf{x}) \cdot \mathbf{v}\,. \tag{3.10.5}$$

Next, substituting (3.10.5) into (3.10.4) yields

$$\begin{aligned}\frac{d}{dt}\int_P \boldsymbol{\phi}(\mathbf{x}, t)dv &= \int_P [\frac{\partial \tilde{\boldsymbol{\phi}}(\mathbf{x}, t)}{\partial t} + (\partial \tilde{\boldsymbol{\phi}}(\mathbf{x}, t)/\partial \mathbf{x}) \cdot \mathbf{v} + \boldsymbol{\phi}\, \mathrm{div}\mathbf{v}]dv \\ &= \int_P [\frac{\partial \tilde{\boldsymbol{\phi}}(\mathbf{x}, t)}{\partial t} + \mathrm{div}(\tilde{\boldsymbol{\phi}} \otimes \mathbf{v})]dv\,,\end{aligned} \tag{3.10.6}$$

which with the help of the divergence theorem (2.5.10) can be written in the form

$$\frac{d}{dt}\int_P \boldsymbol{\phi}(\mathbf{x}, t)dv = \int_P \frac{\partial \tilde{\boldsymbol{\phi}}(\mathbf{x}, t)}{\partial t}dv + \int_{\partial P} \tilde{\boldsymbol{\phi}}\, (\mathbf{v} \cdot \mathbf{n})da\,, \tag{3.10.7}$$

where $\mathbf{n}$ is the unit outward normal to the material surface ∂P. It should be emphasized that the time differentiation and the integration over space operations commute in (3.10.3) because the region P_0 is independent of time. In contrast, the time differentiation and the integration over space operations in (3.10.7) do not commute because the region P depends on time. However, sometimes in fluid mechanics the region P in space at time t is considered to be a control volume and is identified as a fixed region $\bar{P}$ with fixed boundary $\partial \bar{P}$ which instantaneously coincide with the material region $\mathcal{P}$ and the material boundary $\partial\mathcal{P}$. Then, the time differentiation is interchanged with the integration over space operation to obtain

$$\frac{d}{dt}\int_P \boldsymbol{\phi}(\mathbf{x}, t)dv = \frac{\partial}{\partial t}\int_{\bar{P}} \tilde{\boldsymbol{\phi}}(\mathbf{x}, t)dv + \int_{\partial \bar{P}} \tilde{\boldsymbol{\phi}}\, (\mathbf{v} \cdot \mathbf{n})da\,, \tag{3.10.8}$$

where P on the left-hand side of this equation represents a material region that changes with time. In this regard, it is *essential* to interpret the partial differentiation operation in (3.10.8) as differentiation with respect to time holding $\mathbf{x}$ fixed. To avoid possible confusion, it is preferable to use the form (3.10.7) instead of (3.10.8).

Transport Theorem for a Non-material Region
To develop a generalized version of Leibnitz's rule (3.10.1) consider a general non-material region $\mathcal{V}(t)$ with general non-material closed boundary $\partial\mathcal{V}(t)$ for which

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \tilde{\boldsymbol{\phi}}(\mathbf{x}, t)dv = \int_{\mathcal{V}(t)} \frac{\partial \tilde{\boldsymbol{\phi}}}{\partial t}dv + \int_{\partial\mathcal{V}(t)} \tilde{\boldsymbol{\phi}}\, (\mathbf{w} \cdot \mathbf{n})da\,, \tag{3.10.9}$$

where $\tilde{\boldsymbol{\phi}}(\mathbf{x}, t)$ is a general tensor field and $\mathbf{w}$ is the velocity of points on the moving boundary $\partial\mathcal{V}(t)$. Next, using the divergence theorem (2.5.10), it follows that

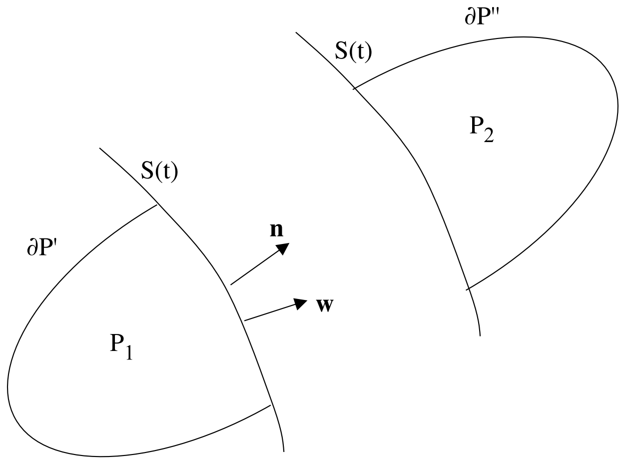

Fig. 3.7 A material region with a singular moving surface $S(t)$

$$\int_{\mathcal{V}(t)} \text{div}(\boldsymbol{\phi} \otimes \mathbf{v}) dv = \int_{\partial\mathcal{V}(t)} \tilde{\boldsymbol{\phi}}\ (\mathbf{v} \cdot \mathbf{n}) da\,, \tag{3.10.10}$$

where $\mathbf{v}$ is the velocity of material points $\mathbf{x}$ in the region $\mathcal{V}(t)$ or the velocity of material points which instantaneously lie on the moving surface $\mathcal{V}(t)$. Thus, (3.10.9) can be rewritten in the form

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \tilde{\boldsymbol{\phi}}(\mathbf{x}, t) dv = \int_{\mathcal{V}(t)} \left[\frac{\partial \tilde{\boldsymbol{\phi}}}{\partial t} + \text{div}(\boldsymbol{\phi} \otimes \mathbf{v})\right] dv + \int_{\partial\mathcal{V}(t)} \tilde{\boldsymbol{\phi}}\ [(\mathbf{w} - \mathbf{v}) \cdot \mathbf{n}] da\,. \tag{3.10.11}$$

Moreover, using (2.5.4) and (3.10.5) it can be shown that

$$\text{div}(\tilde{\boldsymbol{\phi}} \otimes \mathbf{v}) = (\partial \tilde{\boldsymbol{\phi}}/\partial \mathbf{x}) \cdot \mathbf{v} + \tilde{\boldsymbol{\phi}}\, \text{div}\mathbf{v} = \dot{\boldsymbol{\phi}} + \tilde{\boldsymbol{\phi}}\, \text{div}\mathbf{v} - \frac{\partial \tilde{\boldsymbol{\phi}}}{\partial t}\,. \tag{3.10.12}$$

Then, using this expression the generalized transport theorem for a non-material region becomes

$$\frac{d}{dt}\int_{\mathcal{V}(t)} \boldsymbol{\phi}(\mathbf{x}, t) dv = \int_{\mathcal{V}(t)} (\dot{\boldsymbol{\phi}} + \boldsymbol{\phi}\, \text{div}\mathbf{v}) dv + \int_{\partial\mathcal{V}(t)} \boldsymbol{\phi}\ [(\mathbf{w} - \mathbf{v}) \cdot \mathbf{n}] da\,, \tag{3.10.13}$$

where the last term in this equation represents the flux of $\boldsymbol{\phi}$ entering $\mathcal{V}(t)$ through the moving boundary $\partial\mathcal{V}(t)$. When $\mathcal{V}$ is a material region P and $\partial\mathcal{V}$ is a material boundary ∂P then $(\mathbf{w} - \mathbf{v}) \cdot \mathbf{n} = 0$ and (3.10.13) reduces to the simple form (3.10.4).

Transport Theorem for a Material Region with a Singular Moving Surface
Impulsive loading of materials cause shock waves that travel through the material region. At the front of a shock wave the state of the material can change rapidly.

Mathematically, it is convenient to approximate the front of the shock wave as a singular surface $S(t)$ moving through the material at which quantities other than the positions of material particles can be discontinuous across the surface $S(t)$. Figure 3.7 shows a material region P with closed material boundary ∂P that is divided by a singular moving surface $S(t)$ into two parts P_1 and P_2 with closed boundaries ∂P_1 and ∂P_2, respectively. Furthermore, let the intersection of ∂P_1 with ∂P be denoted by $\partial P'$ and the intersection of ∂P_2 with ∂P be denoted by $\partial P''$. Mathematically, this separation is summarized by

$$\begin{aligned} &P = P_1 \cup P_2 \,, && \partial P' = \partial P_1 \cap \partial P \,, && \partial P'' = \partial P_2 \cap \partial P \,, \\ &\partial P = \partial P' \cup P'' \,, && \partial P_1 = \partial P' \cup S \,, && \partial P_2 = \partial P'' \cup S \,. \end{aligned} \tag{3.10.14}$$

Points on this singular surface move with velocity $\mathbf{w}$ and the unit normal to $S(t)$ outward from the part P_1 is denoted by $\mathbf{n}$.

Application of the generalized transport theorem (3.10.13) to each of the parts P_1 and P_2 yields

$$\begin{aligned} \frac{d}{dt}\int_{P_1} \boldsymbol{\phi}(\mathbf{x},t)dv &= \int_{P_1} (\dot{\boldsymbol{\phi}} + \boldsymbol{\phi}\,\mathrm{div}\mathbf{v})dv + \int_{S(t)} \boldsymbol{\phi}_1\ \{(\mathbf{w}-\mathbf{v}_1)\cdot\mathbf{n}\}da \,, \\ \frac{d}{dt}\int_{P_2} \boldsymbol{\phi}(\mathbf{x},t)dv &= \int_{P_2} (\dot{\boldsymbol{\phi}} + \boldsymbol{\phi}\,\mathrm{div}\mathbf{v})dv - \int_{S(t)} \boldsymbol{\phi}_2\ \{(\mathbf{w}-\mathbf{v}_2)\cdot\mathbf{n}\}da \,, \end{aligned} \tag{3.10.15}$$

where $\boldsymbol{\phi}_1$ and $\mathbf{v}_1$ are the values of $\boldsymbol{\phi}$ and $\mathbf{v}$ in part P_1 and $\boldsymbol{\phi}_2$ and $\mathbf{v}_2$ are the values of $\boldsymbol{\phi}$ and $\mathbf{v}$ in part P_2, all on the singular surface $S(t)$. Next, adding these expressions yields

$$\begin{aligned} \frac{d}{dt}\int_{P} \boldsymbol{\phi}(\mathbf{x},t)dv = \int_{P_1} (\dot{\boldsymbol{\phi}} + \boldsymbol{\phi}\,\mathrm{div}\mathbf{v})dv + \int_{P_2} (\dot{\boldsymbol{\phi}} + \boldsymbol{\phi}\,\mathrm{div}\mathbf{v})dv \\ - \int_{S(t)} [[\boldsymbol{\phi}\,\{(\mathbf{w}-\mathbf{v})\cdot\mathbf{n}\}]]da \,, \end{aligned} \tag{3.10.16}$$

where the jump operator $[[\boldsymbol{\phi}]]$ is defined by

$$[[\boldsymbol{\phi}]] = \boldsymbol{\phi}_2 - \boldsymbol{\phi}_1 \,. \tag{3.10.17}$$

In addition, $\mathbf{w}$ and $\mathbf{n}$ are the same on both sides of $S(t)$

$$\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{w} \,, \quad \mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n} \,. \tag{3.10.18}$$

3.11 An Eulerian Formulation of Evolution Equations for Elastic Deformations

Recall from (3.5.4) that the deformation gradient **F** from the reference configuration satisfies the evolution equation

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}\,, \tag{3.11.1}$$

where **L** is the velocity gradient. Also, recall that the total dilatation J and the unimodular part $\mathbf{F}'$ of **F**, both from the reference configuration, satisfy the evolution equations (3.5.16) and (3.5.17), respectively

$$\dot{J} = J\,\mathbf{D}\cdot\mathbf{I}\,, \quad \dot{\mathbf{F}}' = \mathbf{L}''\mathbf{F}'\,, \tag{3.11.2}$$

where $\mathbf{L}''$ is the deviatoric part of **L**. To integrate these equations from an arbitrary time $t = t_1$ it is necessary to know the initial values

$$\mathbf{F}(t_1)\,, \quad J(t_1) = \det\mathbf{F}(t_1) > 0\,, \quad \mathbf{F}'(t_1)\,, \tag{3.11.3}$$

where the dependence on space has been suppressed for notational convenience. These initial values depend on an arbitrary choice of the reference configuration, with $\mathbf{F}(t_1)$ depending explicitly on the choice of the orientation of the body in the reference configuration.

Onat [4] discussed physical restrictions on internal state variables. This discussion proposed that internal state variables, which are determined by integrating time evolution equations, are specified to measure properties of the material response that define the current state of the material. Moreover, since these evolution equations need initial conditions, it is necessary that the values of the internal state variables be, in principle, measurable directly or indirectly by experiments on multiple identical samples of the material in its current state. Thus, all variables that define the current material state must be characterized by internal state variables whose values in the current state are measurable.

In this regard, it is noted that the reference configuration can be chosen to be an arbitrary configuration which admits a one-to-one mapping between material points in the reference configuration and the same material points in the current configuration. This requires **F** to be nonsingular with $\det\mathbf{F} > 0$. For example, let **A** be an arbitrary second-order tensor function of **X** only with positive determinant $\det\mathbf{A} > 0$. It then follows that **FA** satisfies the evolution equation (3.11.1)

$$\dot{\overline{\mathbf{F}\mathbf{A}}} = \mathbf{L}(\mathbf{F}\mathbf{A})\,. \tag{3.11.4}$$

However, since the choice of the reference configuration is arbitrary, it is not possible to determine the value of **FA** in the current state from experiments on identical samples of the material in its current state. This is true even if it is known that the the material in the reference configuration is in a uniform stress-free material state, since **FA** in the reference state could have an arbitrary orientation described by three arbitrary orientation angles of a proper orthogonal rotation tensor. This means that **F** is not an internal state variable in the sense of Onat [4] and therefore should not be used in constitutive equations, even for an elastic material. Similarly, the total dilatation $J = \det \mathbf{F}$ and the unimodular tensor $\mathbf{F}'$ from the reference configuration are also not internal state variables. However, **F**, J and $\mathbf{F}'$ can be used to *parameterize* the solution of a particular problem for which the initial value of **F** is specified.

The Eulerian formulation for the purely mechanical theory of a compressible elastic material proposes an evolution equation for the elastic dilatation J_e in the form

$$\dot{J}_e = J_e \mathbf{D} \cdot \mathbf{I} \,. \tag{3.11.5}$$

Since the constitutive equation for stress is restricted to be invertible (1.2.12), it follows from (1.2.9) that J_e is an internal state variable in the sense of Onat [4] since it can be measured by experiments on identical samples of the material in the current configuration. Moreover, the evolution equation (3.11.5) is considered to be an Eulerian formulation of an evolution equation for the elastic dilatation J_e since it depends only on the current state of the material characterized by the values of J_e and **D**, which are measurable in the current state.

Anisotropic Elastic Solids

Following the work in [6] for elastically anisotropic materials, consider a triad of linearly independent microstructural vectors $\mathbf{m}_i$ $(i = 1, 2, 3)$ defined by the evolution equations

$$\dot{\mathbf{m}}_i = \mathbf{L}\,\mathbf{m}_i \,. \tag{3.11.6}$$

From (3.5.19) it is clear that $\mathbf{m}_i$ deform like material line elements. Moreover, since $\mathbf{m}_i$ are linearly independent they can be defined so that they form a right-handed triad with the elastic dilatation defined by (1.2.9)

$$J_e = \mathbf{m}_1 \times \mathbf{m}_2 \cdot \mathbf{m}_3 > 0 \,. \tag{3.11.7}$$

These vectors characterize both elastic deformations and rotations of material line elements. In particular, the elastic deformations can be defined by the elastic metric

$$m_{ij} = \mathbf{m}_i \cdot \mathbf{m}_j = m_{ji} \,, \tag{3.11.8}$$

and the vectors $\mathbf{m}_i$ can be specified so that they form an orthonormal triad in any zero-stress material state with

$$m_{ij} = \delta_{ij} \quad \text{for any zero-stress material state} \,. \tag{3.11.9}$$

Moreover, using (3.11.6) it can be shown that the elastic metric satisfies the evolution equation

$$\dot{m}_{ij} = 2(\mathbf{m}_i \otimes \mathbf{m}_j) \cdot \mathbf{D} \,. \tag{3.11.10}$$

In contrast to the total deformation gradient **F**, which is not an internal state variable, the microstructural vectors $\mathbf{m}_i$ are internal state variables in the sense of Onat [4]. Specifically, since the constitutive equation for stress is restricted to be invertible (1.2.12), it follows that the values of $\mathbf{m}_i$ are measurable by experiments on identical samples of a material in its current state (see the more detailed discussion in Sect. 5.3).

As the material deforms, $\mathbf{m}_i$ do not remain orthonormal. However, since $\mathbf{m}_i$ are linearly independent, their reciprocal vectors $\mathbf{m}^i$ can be defined by

$$\mathbf{m}^1 = J_e^{-1}\mathbf{m}_2 \times \mathbf{m}_3\,, \qquad \mathbf{m}^2 = J_e^{-1}\mathbf{m}_3 \times \mathbf{m}_1\,, \qquad \mathbf{m}^3 = J_e^{-1}\mathbf{m}_1 \times \mathbf{m}_2\,,$$
$$J_e^{-1} = \mathbf{m}^1 \times \mathbf{m}^2 \cdot \mathbf{m}^3\,, \tag{3.11.11}$$

which have the properties that

$$\mathbf{m}_i \otimes \mathbf{m}^i = \mathbf{I}\,. \tag{3.11.12}$$

Then, taking the material derivative of J_e in (3.11.7) and using the evolution equation (3.11.6), the definitions (3.11.11) and the result (3.11.12), it follows that

$$\begin{aligned} \dot{J}_e &= \dot{\mathbf{m}}_1 \cdot \mathbf{m}_2 \times \mathbf{m}_3 + \dot{\mathbf{m}}_2 \cdot \mathbf{m}_3 \times \mathbf{m}_1 + \dot{\mathbf{m}}_3 \cdot \mathbf{m}_1 \times \mathbf{m}_2\,, \\ &= J_e\dot{\mathbf{m}}_i \cdot \mathbf{m}^i = J_e\mathbf{L} \cdot \mathbf{m}^i \otimes \mathbf{m}_i = J_e\,\mathbf{L} \cdot \mathbf{I} = J_e\,\mathbf{D} \cdot \mathbf{I}\,, \end{aligned} \tag{3.11.13}$$

which proves that the specification (3.11.7) satisfies the evolution equation (3.11.5) for elastic dilatation and the condition that $J_e = 1$ in any zero-stress material state.

Next, using the work of Flory [1] it is possible to develop pure measures of elastic distortional deformation. Specifically, the elastic distortional vectors $\mathbf{m}'_i$ are defined by

$$\mathbf{m}'_i = J_e^{-1/3}\,\mathbf{m}_i\,, \quad \mathbf{m}'_1 \times \mathbf{m}'_2 \cdot \mathbf{m}'_3 = 1\,, \tag{3.11.14}$$

which satisfy the evolution equations

$$\dot{\mathbf{m}}'_i = \mathbf{L}''\,\mathbf{m}'_i\,, \tag{3.11.15}$$

where $\mathbf{L}''$ is the deviatoric part of **L**. Also, the elastic distortional deformation metric m'_{ij} is defined by

$$m'_{ij} = \mathbf{m}'_i \cdot \mathbf{m}'_j = m'_{ji}\,. \tag{3.11.16}$$

This metric satisfies the evolution equation

$$\dot{m}'_{ij} = 2\mathbf{m}'_i \otimes \mathbf{m}'_j \cdot \mathbf{D}'' = 2(\mathbf{m}'_i \otimes \mathbf{m}'_j - \frac{1}{3}m'_{ij}\mathbf{I}) \cdot \mathbf{D}\,, \tag{3.11.17}$$

where $\mathbf{D}''$ is the deviatoric part of **D**. In addition, the associated reciprocal vectors $\mathbf{m}^{i\prime}$ satisfy equations

$$\mathbf{m}^{i\prime} = J_e^{1/3}\mathbf{m}^i\,, \quad \mathbf{m}^{1\prime} \times \mathbf{m}^{2\prime} \cdot \mathbf{m}^{3\prime} = 1\,. \tag{3.11.18}$$

Isotropic Elastic Solids
For isotropic elastic solids, it is not possible to distinguish between the three microstructural vectors $\mathbf{m}_i$ and it is convenient to introduce a symmetric, positive deformation elastic deformation tensor $\mathbf{B}_e$ defined by

$$\mathbf{B}_e = \mathbf{m}_i \otimes \mathbf{m}_i\,, \tag{3.11.19}$$

which with the help of (3.11.6) can be shown to satisfy the evolution equation

$$\dot{\mathbf{B}}_e = \mathbf{L}\mathbf{B}_e + \mathbf{B}_e\mathbf{L}^T\,. \tag{3.11.20}$$

Since the constitutive equation for stress is restricted to be invertible (1.2.4), it follows that the value of $\mathbf{B}_e$ is measurable by experiments on identical samples of a material in its current state (see the more detailed discussion in Sect. 5.8). Consequently, $\mathbf{B}_e$ is an internal state variable in the sense of Onat [4].

Next, using the fact that (3.5.12) is valid for any nonsingular tensor, it follows that

$$\dot{\overline{\det\mathbf{B}_e}} = (\det\mathbf{B}_e)\,\mathbf{B}_e^{-1} \cdot \dot{\mathbf{B}}_e = 2(\det\mathbf{B}_e)(\mathbf{D} \cdot \mathbf{I})\,, \tag{3.11.21}$$

so that J_e in (3.11.5) can be identified as

$$J_e = (\det\mathbf{B}_e)^{1/2}\,. \tag{3.11.22}$$

In addition, using the work of Flory [1] it is convenient to define the symmetric, positive-definite, elastic distortional deformation tensor $\mathbf{B}'_e$ by

$$\mathbf{B}'_e = J_e^{-2/3}\mathbf{B}_e = \mathbf{m}'_i \otimes \mathbf{m}'_i\,, \tag{3.11.23}$$

which can be seen to be a unimodular tensor

$$\begin{aligned} \det\mathbf{B}'_e &= \det(J_e^{-2/3}\mathbf{B}_e) = (J_e^{-2/3})^3\det\mathbf{B}_e = 1\,, \\ \det\mathbf{B}'_e &= \frac{\mathbf{B}'_e\mathbf{m}^{1\prime} \times \mathbf{B}'_e\mathbf{m}^{2\prime} \cdot \mathbf{B}'_e\mathbf{m}^{3\prime}}{\mathbf{m}^{1\prime} \times \mathbf{m}^{2\prime} \cdot \mathbf{m}^{3\prime}} = \mathbf{m}'_1 \times \mathbf{m}'_2 \cdot \mathbf{m}'_3 = 1\,. \end{aligned} \tag{3.11.24}$$

Moreover, using the evolution equations (3.11.15) it can be shown that $\mathbf{B}'_e$ satisfies the evolution equation

$$\dot{\mathbf{B}}'_e = \mathbf{L}''\mathbf{B}'_e + \mathbf{B}'_e\mathbf{L}''^T\,, \tag{3.11.25}$$

which with the help of (3.11.21) ensures that $\mathbf{B}'_e$ remains unimodular since

$$\dot{\mathbf{B}}'_e \cdot \mathbf{B}_e'^{-1} = 0\,. \tag{3.11.26}$$

Summary
For anisotropic response, the elastic deformations can be characterized by a right-handed triad of linearly independent microstructural vectors $\mathbf{m}_i$, which satisfy the evolution equations

$$\dot{\mathbf{m}}_i = \mathbf{L}\,\mathbf{m}_i\ . \tag{3.11.27}$$

Alternatively, the elastic deformations can be characterized by the elastic dilatation J_e and the elastic distortional deformation vectors $\mathbf{m}'_i$, which satisfy the evolution equations

$$\dot{J}_e = J_e\,\mathbf{D}\cdot\mathbf{I}\,, \quad \dot{\mathbf{m}}'_i = \mathbf{L}''\,\mathbf{m}'_i\,, \tag{3.11.28}$$

where $\mathbf{L}''$ is the deviatoric part of $\mathbf{L}$.

For isotropic response the elastic deformations can be characterized by the elastic deformation tensor $\mathbf{B}_e$, which satisfies the evolution equation

$$\dot{\mathbf{B}}_e = \mathbf{L}\mathbf{B}_e + \mathbf{B}_e\mathbf{L}^T\,, \tag{3.11.29}$$

or, alternatively, by the elastic dilatation J_e and the elastic distortional deformation $\mathbf{B}'_e$, which satisfy the evolution equations

$$\dot{J}_e = J_e\,\mathbf{D}\cdot\mathbf{I}\,, \quad \dot{\mathbf{B}}'_e = \mathbf{L}''\mathbf{B}'_e + \mathbf{B}'_e\mathbf{L}''^T\ . \tag{3.11.30}$$

Equations (3.11.27)–(3.11.30) represent Eulerian formulations of evolution equations because they depend only on quantities that can be determined in the current state of the material.

Transformations Under SRBM
Under Superposed Rigid Body Motions SRBM the quantities $\mathbf{m}_i$, m_{ij}, $\mathbf{m}^i$, J_e, $\mathbf{m}'_i$, m'_{ij}, $\mathbf{m}^{i\prime}$, $\mathbf{B}_e$ and $\mathbf{B}'_e$ transform to $\mathbf{m}^+_i$, m^+_{ij}, $\mathbf{m}^{i+}$, J^+_e, $\mathbf{m}'^+_i$, m'^+_{ij}, $\mathbf{m}^{i\prime+}$, $\mathbf{B}^+_e$, and $\mathbf{B}'^+_e$, such that

$$\begin{array}{lll} \mathbf{m}^+_i = \mathbf{Q}\mathbf{m}_i\,, & m^+_{ij} = m_{ij}\,, & \mathbf{m}^{i+} = \mathbf{Q}\mathbf{m}^i\,, \\ J^+_e = J_e\,, & \mathbf{m}'^+_i = \mathbf{Q}\mathbf{m}'_i\,, & m'^+_{ij} = m'_{ij}\,, \\ \mathbf{m}^{i\prime+} = \mathbf{Q}\mathbf{m}^{i\prime}\,, & \mathbf{B}^+_e = \mathbf{Q}\mathbf{B}_e\mathbf{Q}^T\,, & \mathbf{B}'^+_e = \mathbf{Q}\mathbf{B}'_e\mathbf{Q}^T\ . \end{array} \tag{3.11.31}$$

These transformation relations make the evolution equations form-invariant under SRBM, so for examples (3.11.6) and (3.11.20) are consistent with the evolution equations

$$\dot{\mathbf{m}}^+_i = \mathbf{L}^+\,\mathbf{m}^+_i\,, \quad \dot{\mathbf{B}}^+_e = \mathbf{L}^+\mathbf{B}^+_e + \mathbf{B}^+_e\mathbf{L}^{+T}\,, \tag{3.11.32}$$

where under SRBM $\mathbf{L}$ transforms to $\mathbf{L}^+$.

Additional Eulerian Strain Measures
Using the condition (3.11.9) it is convenient to introduce elastic strains e_{ij} measured relative to zero-stress material states by

$$e_{ij} = \frac{1}{2}(m_{ij} - \delta_{ij}) \,, \tag{3.11.33}$$

which in view of (3.11.10) satisfy the evolution equations

$$\dot{e}_{ij} = (\mathbf{m}_i \otimes \mathbf{m}_j) \cdot \mathbf{D} \,. \tag{3.11.34}$$

Similarly, using the elastic distortional deformation (3.11.16), the elastic distortional strains e'_{ij} relative to zero-stress material states are defined by

$$e'_{ij} = \frac{1}{2}(m'_{ij} - \delta_{ij}) \,, \tag{3.11.35}$$

which in view of (3.11.17) satisfy the evolution equations

$$\dot{e}'_{ij} = (\mathbf{m}'_i \otimes \mathbf{m}'_j - \frac{1}{3} m'_{ij} \mathbf{I}) \cdot \mathbf{D} \,. \tag{3.11.36}$$

In addition, for elastically isotropic response, the elastic distortional strain $\mathbf{g}'_e$ and its deviatoric part $\mathbf{g}''_e$ can be defined by

$$\mathbf{g}'_e = \frac{1}{2}(\mathbf{B}'_e - \mathbf{I}) \,, \quad \mathbf{g}''_e = \frac{1}{2}\mathbf{B}''_e \,, \tag{3.11.37}$$

where $\mathbf{B}''_e$ is the deviatoric part of $\mathbf{B}'_e$

$$\mathbf{B}''_e = \mathbf{B}'_e - \frac{1}{3}(\mathbf{B}'_e \cdot \mathbf{I})\,\mathbf{I} \,. \tag{3.11.38}$$

3.12 Compatibility

Since the velocity gradient $\mathbf{L}$ is defined by the gradient of a velocity field $\mathbf{v}$, it follows that if $\mathbf{v}$ is continuously differentiable with respect to $\mathbf{x}$ then the total deformations are compatible in the sense that a motion $\hat{\mathbf{x}}(\mathbf{X}, t)$ exists and the deformation gradient $\mathbf{F}$ defined in (3.3.1c) is consistent with the value of $\mathbf{F}$ obtained by integrating the evolution equation (3.5.4).

Within the context of the Eulerian formulation for anisotropic elastic solids, the microstructural vectors $\mathbf{m}_i$ obtained by integrating the evolution equations (3.11.6) will also be compatible in the sense that a motion can be characterized by the invertible mapping

$$\mathbf{x} = \mathbf{x}(\theta^i, t) \,, \tag{3.12.1}$$

where θ^i are convected coordinates. Moreover, since for the elastic case $\mathbf{m}_i$ can be identified as material line elements these convected coordinates can be defined so that

$$\mathbf{m}_i = \frac{\partial \mathbf{x}}{\partial \theta^i} \,. \tag{3.12.2}$$

For a continuously differentiable motion

$$\frac{\partial^2 \mathbf{x}}{\partial \theta^i \partial \theta^j} = \frac{\partial^2 \mathbf{x}}{\partial \theta^j \partial \theta^i} \,, \tag{3.12.3}$$

with $\mathbf{m}_i$ satisfying the integrability conditions

$$\frac{\partial \mathbf{m}_i}{\partial \theta^j} = \frac{\partial \mathbf{m}_j}{\partial \theta^i} \,. \tag{3.12.4}$$

Now, expressing $\mathbf{m}_i$ as functions of $\mathbf{x}$, t and using the fact that

$$\frac{\partial \mathbf{m}_i}{\partial \theta^j} = (\partial \mathbf{m}_i / \partial \mathbf{x}) \, \mathbf{m}_j \,, \tag{3.12.5}$$

it is convenient to define the three vectors $\mathbf{c}_k$ by

$$\mathbf{c}_k = \varepsilon_{kij} (\partial \mathbf{m}_i / \partial \mathbf{x}) \, \mathbf{m}_j \,. \tag{3.12.6}$$

Then, for elastic response the integrability conditions (3.12.4) require

$$\mathbf{c}_k = 0 \,. \tag{3.12.7}$$

For inelastic material response that will be discussed later, the vectors $\mathbf{m}_i$ will be obtained by integrating evolution equations which include an inelastic deformation rate, with $\mathbf{m}_i$ still characterizing elastic deformations. For the general case when $\mathbf{L}$ depends on $\mathbf{x}$ the total deformations and the inelastic deformation rate will be inhomogeneous so the elastic deformations need not be compatible in the sense that $\mathbf{c}_k$ in (3.12.6) need not satisfy the compatibility conditions (3.12.7).

3.13 Strongly Objective, Robust Numerical Integration Algorithms

Since the general equations of continuum mechanics are nonlinear, it is necessary to use numerical methods to obtain solutions to challenging problems. Computational mechanics is a field of mechanics that develops computational methods and applies them to analyze fundamental and practical problems in continuum mechanics.

To this end, a numerical algorithm must be proposed to integrate the Eulerian formulations of the evolution equations for the internal state variables, discussed in the previous sections, over a typical time step that begins at time $t = t_n$ and ends at time $t = t_{n+1}$, with time increment $\Delta t = t_{n+1} - t_n$. Specifically, given the values

$$\mathbf{m}_i(t_n)\,, \quad J_e(t_n)\,, \quad \mathbf{m}'_i(t_n)\,, \quad \mathbf{B}_e(t_n)\,, \quad \mathbf{B}'_e(t_n) \tag{3.13.1}$$

of these internal state variables at the beginning of the time step, it is necessary to develop a numerical algorithm to determine their values

$$\mathbf{m}_i(t_{n+1})\,, \quad J_e(t_{n+1})\,, \quad \mathbf{m}'_i(t_{n+1})\,, \quad \mathbf{B}_e(t_{n+1})\,, \quad \mathbf{B}'_e(t_{n+1}) \tag{3.13.2}$$

at the end of the time step.

Following the work of Simo [12], it is convenient to introduce the relative deformation gradient $\mathbf{F}_r(t)$ from the beginning of a time step, which satisfies the evolution equation and initial condition

$$\dot{\mathbf{F}}_r = \mathbf{L}\mathbf{F}_r\,, \quad \mathbf{F}_r(t_n) = \mathbf{I}\,. \tag{3.13.3}$$

The associated relative dilatation $J_r(t)$ from the beginning of the time step is defined by

$$J_r = \det\mathbf{F}_r\,, \tag{3.13.4}$$

which with the help of (3.5.12) can be seen to satisfy the evolution equation and initial condition

$$\dot{J}_r = J_r\mathbf{F}_r^{-T} \cdot \dot{\mathbf{F}}_r = J_r\,\mathbf{D} \cdot \mathbf{I}\,, \quad J_r(t_n) = 1\,. \tag{3.13.5}$$

Also, the unimodular part $\mathbf{F}'_r$ of $\mathbf{F}_r$ is defined by

$$\mathbf{F}'_r = J_r^{-1/3}\,\mathbf{F}_r\,, \quad \det\mathbf{F}'_r = 1\,, \tag{3.13.6}$$

which satisfies the evolution equation and initial condition

$$\dot{\mathbf{F}}'_r = \mathbf{L}''\,\mathbf{F}'_r\,, \quad \mathbf{F}'_r(t_n) = \mathbf{I}\,, \tag{3.13.7}$$

where $\mathbf{L}''$ is the deviatoric part of $\mathbf{L}$.

These relative deformation quantities $J_r, \mathbf{F}_r$ and $\mathbf{F}'_r$ are independent of arbitrary choices of a reference configuration and therefore can used to integrate Eulerian forms of evolution equations for internal state variables that are themselves independent of arbitrariness of the reference configuration. Also, under SRBM $J_r, \mathbf{F}_r$ and $\mathbf{F}'_r$ transform to $J_r^+, \mathbf{F}_r^+$ and $\mathbf{F}_r'^+$ according to the transformation relations

$$J_r^+ = J_r\,, \quad \mathbf{F}_r^+ = \mathbf{Q}\mathbf{F}_r \quad \mathbf{F}_r'^+ = \mathbf{Q}\mathbf{F}'_r\,. \tag{3.13.8}$$

Specifically, the elastic trial quantities

$$\begin{aligned}&\mathbf{m}_i^*(t) = \mathbf{F}_r(t)\mathbf{m}_i(t_n)\,, \qquad J_e^*(t) = J_r(t)J_e(t_n)\,,\\&\mathbf{m}_i'^*(t) = \mathbf{F}_r'(t)\mathbf{m}_i'(t_n)\,, \qquad \mathbf{B}_e^*(t) = \mathbf{F}_r(t)\mathbf{B}_e(t_n)\mathbf{F}_r^T(t)\,,\\&\mathbf{B}_e'^*(t) = \mathbf{F}_r'(t)\mathbf{B}_e'(t_n)\mathbf{F}_r'^T(t)\end{aligned} \tag{3.13.9}$$

satisfy the evolution equations and initial conditions

$$\begin{aligned}&\dot{\mathbf{m}}_i^* = \mathbf{L}\mathbf{m}_i^*\,, && \mathbf{m}_i^*(t_n) = \mathbf{m}_i(t_n)\,,\\&\dot{J}_e^* = J_e^*\,\mathbf{D}\cdot\mathbf{I}\,, && J_e^*(t_n) = J_e(t_n)\,,\\&\dot{\mathbf{m}}_i'^* = \mathbf{L}''\,\mathbf{m}_i'^*\,, && \mathbf{m}_i'^*(t_n) = \mathbf{m}_i'(t_n)\,.\\&\dot{\mathbf{B}}_e^* = \mathbf{L}\mathbf{B}_e^* + \mathbf{B}_e^*\mathbf{L}^T\,, && \mathbf{B}_e^*(t_n) = \mathbf{B}_e(t_n)\,,\\&\dot{\mathbf{B}}_e'^* = \mathbf{L}''\mathbf{B}_e'^* + \mathbf{B}_e'^*\mathbf{L}''^T\,, && \mathbf{B}_e'^*(t_n) = \mathbf{B}_e'(t_n)\,.\end{aligned} \tag{3.13.10}$$

Also, for later reference it is noted that the deviatoric part $\mathbf{B}_e''^*$

$$\mathbf{B}_e''^* = \mathbf{B}_e'^* - \frac{1}{3}(\mathbf{B}_e'^*\cdot\mathbf{I})\,\mathbf{I}\,, \tag{3.13.11}$$

of the elastic trial $\mathbf{B}_e'^*$ satisfies the evolution equation and initial condition

$$\dot{\mathbf{B}}_e''^* = \mathbf{L}''\mathbf{B}_e'^* + \mathbf{B}_e'^*\mathbf{L}''^T - \frac{2}{3}(\mathbf{B}_e'\cdot\mathbf{D}'')\,\mathbf{I}\,, \quad \mathbf{B}_e''^*(t_n) = \mathbf{B}_e''(t_n)\,. \tag{3.13.12}$$

Consequently, the elastic trial values (3.13.9) and (3.13.11) are exact solutions of the evolution equations (3.11.6), (3.11.5), (3.11.15), (3.11.20), (3.11.25) and (3.13.12), respectively. A fundamental feature of these elastic trial values is that they satisfy the same transformation relations under SRBM as the exact values

$$\begin{aligned}&\mathbf{m}_i^{*+} = \mathbf{Q}\mathbf{m}_i^*\,, && J_e^{*+} = J_e^*\,, && \mathbf{m}_i'^{*+} = \mathbf{Q}\mathbf{m}_i'^*\,,\\&\mathbf{B}_e^{*+}(t) = \mathbf{Q}\mathbf{B}_e^*\mathbf{Q}^T\,, && \mathbf{B}_e'^{*+} = \mathbf{Q}\mathbf{B}_e'^*\mathbf{Q}^T\,, && \mathbf{B}_e''^{*+} = \mathbf{Q}\mathbf{B}_e''^*\mathbf{Q}^T\,.\end{aligned} \tag{3.13.13}$$

In particular, robust, strongly objective numerical algorithms can be developed using these elastic trial values (e.g., [2, 3, 5, 7–10]).

Average Total Deformation Rate
Following the work in [11] the average deformation rate $\tilde{\mathbf{D}}$ in a time step $t_n \le t \le t_{n+1}$ is expressed in the form

$$\tilde{\mathbf{D}} = \frac{1}{3}(\tilde{\mathbf{D}}\cdot\mathbf{I})\,\mathbf{I} + \tilde{\mathbf{D}}''\,, \tag{3.13.14}$$

where $\tilde{\mathbf{D}}''$ is the deviatoric part of $\tilde{\mathbf{D}}$. Integration of the evolution equation (3.13.5) for J_r yields an expression for the average total dilatational rate

$$\tilde{\mathbf{D}}\cdot\mathbf{I} = \frac{1}{\Delta t}\,\ln[J_r(t_{n+1})]\,. \tag{3.13.15}$$

To develop an expression for the average total distortional deformation rate tensor $\tilde{\mathbf{D}}''$ it is convenient to define the unimodular relative deformation tensors

$$\mathbf{C}'_r = \mathbf{F}'^T_r \mathbf{F}'_r \,, \quad \mathbf{B}'_r = \mathbf{F}'_r \mathbf{F}'^T_r \,. \tag{3.13.16}$$

Then, with the help of (3.13.7) it can be shown that

$$\dot{\mathbf{C}}'_r = 2\mathbf{F}'^T_r \mathbf{D}'' \mathbf{F}'_r \,, \quad \mathbf{D}'' = \frac{1}{2}\mathbf{F}'^{-T}_r \dot{\mathbf{C}}'_r \mathbf{F}'^{-1}_r \,, \tag{3.13.17}$$

where $\mathbf{D}''$ is the deviatoric part of $\mathbf{D}$. Moreover, since $\mathbf{C}'_r$ is unimodular, it follows that

$$\dot{\mathbf{C}}'_r \cdot \mathbf{C}'^{-1}_r = 0 \,. \tag{3.13.18}$$

This property is satisfied when the derivative $\dot{\mathbf{C}}'_r$ is approximated by

$$\dot{\mathbf{C}}'_r \approx \frac{1}{\Delta t}[\mathbf{C}'_r(t_{n+1}) - \{\frac{3}{\mathbf{C}'^{-1}_r(t_{n+1}) \cdot \mathbf{I}}\} \mathbf{I}] \,. \tag{3.13.19}$$

Then, using the fact that $\mathbf{C}'^{-1}_r \cdot \mathbf{I} = \mathbf{B}'^{-1}_r \cdot \mathbf{I}$, the average total distortional deformation rate $\tilde{\mathbf{D}}''$ during the time step can be approximated by

$$\tilde{\mathbf{D}}'' = \frac{1}{2\Delta t}\left[\mathbf{I} - \{\frac{3}{\mathbf{B}'^{-1}_r(t_{n+1}) \cdot \mathbf{I}}\} \mathbf{B}'^{-1}_r(t_{n+1})\right] , \tag{3.13.20}$$

with $\tilde{\mathbf{D}}$ given by

$$\tilde{\mathbf{D}} = \frac{1}{3}\frac{1}{\Delta t} \ln[J_r(t_{n+1})] \mathbf{I} + \frac{1}{2\Delta t}\left[\mathbf{I} - \{\frac{3}{\mathbf{B}'^{-1}_r(t_{n+1}) \cdot \mathbf{I}}\} \mathbf{B}'^{-1}_r(t_{n+1})\right] . \tag{3.13.21}$$

3.14 The Total Deformation Gradient Used to Parameterize Specific Solutions

The objective of this section is to discuss differences between an elastic deformation variable that characterizes material response and a total deformation measure which is used to parameterize the solution of a particular problem for an elastic material.

Recall from Sect. 3.13 that the deformation gradient $\mathbf{F}_r(t)$ relative to the initial configuration at $t = 0$ satisfies the evolution equation and initial condition (3.13.3)

$$\dot{\mathbf{F}}_r = \mathbf{L}\mathbf{F}_r \,, \quad \mathbf{F}_r(0) = \mathbf{I} \,, \tag{3.14.1}$$

where $\mathbf{L}$ is the velocity gradient. Also, the relative dilatation J_r and the unimodular part $\mathbf{F}'_r$ of $\mathbf{F}_r$, defined by (3.13.4) and (3.13.6)

$$J_r = \det\mathbf{F}_r \,, \quad \mathbf{F}'_r = J_r^{-1/3}\,\mathbf{F}_r \,, \tag{3.14.2}$$

satisfy the evolution equations and initial conditions (3.13.5) and (3.13.7)

$$\begin{aligned} \dot{J}_r &= J_r\,\mathbf{D}\cdot\mathbf{I}\,, \quad J_r(0) = 1\,, \\ \dot{\mathbf{F}}'_r &= \mathbf{L}''\,\mathbf{F}'_r\,, \quad \mathbf{F}'_r(0) = \mathbf{I}\,, \end{aligned} \tag{3.14.3}$$

where $\mathbf{L}''$ is the deviatoic part of $\mathbf{L}$.

It has been shown in Sect. 3.11 that the microstructural vectors $\mathbf{m}_i$ in the Eulerian formulation are internal state variables in the sense of Onat [4], since their values are measurable by experiments on identical samples of the material in its current state. These vectors characterize elastic deformations and orientations of anisotropy and satisfy the evolution equations (3.11.27)

$$\dot{\mathbf{m}}_i = \mathbf{L}\mathbf{m}_i\,. \tag{3.14.4}$$

It then follows that the elastic dilatation J_e and the elastic distortional deformation vectors $\mathbf{m}'_i$, which satisfy the evolution equations (3.11.28)

$$\dot{J}_e = J_e\,\mathbf{D}\cdot\mathbf{I}\,, \quad \dot{\mathbf{m}}'_i = \mathbf{L}''\,\mathbf{m}'_i\,, \tag{3.14.5}$$

are also internal state variables. Next, let $\mathbf{m}_i(0)$, $J_e(0)$ and $\mathbf{m}'_i(0)$ be the measured values of $\mathbf{m}_i$, J_e and $\mathbf{m}'_i$, respectively, in the initial configuration. Then, using $\mathbf{F}_r$ in (3.14.1) and J_r and $\mathbf{F}'_r$ in (3.14.3), the evolution equations (3.14.4) and (3.14.5) can be integrated to obtain

$$\mathbf{m}_i(t) = \mathbf{F}_r(t)\mathbf{m}_i(0)\,, \quad J_e(t) = J_r(t)J_e(0)\,, \quad \mathbf{m}'_i(t) = \mathbf{F}'_r(t)\mathbf{m}'_i(0)\,. \tag{3.14.6}$$

Next, recall from (3.5.4) that the total deformation gradient $\mathbf{F}$ from the reference configuration satisfies the evolution equation

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}\,. \tag{3.14.7}$$

Consequently, with the help of $\mathbf{F}_r$ in (3.14.1) the total deformation gradient $\mathbf{F}$ is given by

$$\mathbf{F}(t) = \mathbf{F}_r(t)\mathbf{F}(0)\,, \tag{3.14.8}$$

where $\mathbf{F}(0)$ is the initial value of $\mathbf{F}$. Furthermore, $\mathbf{F}_r$ can be written in the form

$$\mathbf{F}_r(t) = \mathbf{F}(t)\mathbf{F}^{-1}(0) = [\mathbf{F}(t)\mathbf{A}][\mathbf{F}(0)\mathbf{A}]^{-1}\,, \quad \det\mathbf{A} > 0\,, \tag{3.14.9}$$

where $\mathbf{A}$ is an arbitrary, time independent, second-order tensor with positive determinant which can be a function of the position of material points. This expression shows that the relative deformation tensor $\mathbf{F}_r$ is insensitive to arbitrariness of the choice of the reference configuration for defining the total deformation gradient $\mathbf{F}$.

For the solution of a particular problem, the initial configuration can be specified to be the reference configuration with

$$\mathbf{F}(t) = \mathbf{F}_r(t) \quad \text{for} \quad \mathbf{F}(0) = \mathbf{I}\,, \tag{3.14.10}$$

so that $\mathbf{F}$ or $\mathbf{F}_r$ can be used to parameterize the solution of a particular problem. However, neither of these tensors can be used to determine the microstructural vectors $\mathbf{m}_i(t)$, which determine elastic deformations and the orientation of directions of anisotropy, without measuring $\mathbf{m}_i(0)$ in the initial configuration.

In summary, although $\mathbf{F}$ or $\mathbf{F}_r$ can be used to parameterize the solution of a particular problem, they are not internal state variables and cannot be used by themselves to characterize the material response of an elastic material.

References

1. Flory PJ (1961) Thermodynamic relations for high elastic materials. Trans Faraday Soc 57:829–838
2. Hollenstein M, Jabareen M, Rubin MB (2013) Modeling a smooth elastic-inelastic transition with a strongly objective numerical integrator needing no iteration. Comput Mech 52:649–667
3. Hollenstein M, Jabareen M, Rubin MB (2015) Erratum to: Modeling a smooth elastic-inelastic transition with a strongly objective numerical integrator needing no iteration. Comput Mech 55:649–667
4. Onat ET (1968) The notion of state and its implications in thermodynamics of inelastic solids. In: Irreversible aspects of continuum mechanics and transfer of physical characteristics in moving fluids, pp 292–314
5. Papes O (2013) Nonlinear continuum mechanics in modern engineering applications. Ph.D. dissertation DISS ETH NO 19956
6. Rubin MB (1994) Plasticity theory formulated in terms of physically based microstructural variables - Part I. Theory. Int J Solids Struct 31:2615–2634
7. Rubin MB, Papes O (2011) Advantages of formulating evolution equations for elastic-viscoplastic materials in terms of the velocity gradient instead of the spin tensor. J Mech Mater Struct 6:529–543
8. Rubin MB (2012) Removal of unphysical arbitrariness in constitutive equations for elastically anisotropic nonlinear elastic-viscoplastic solids. Int J Eng Sci 53:38–45
9. Rubin MB (2019) An Eulerian formulation of Inelasticity - From metal plasticity to growth of biological tissues. Trans R Soc A 377:20180071
10. Rubin MB, Cardiff P (2017) Advantages of formulating an evolution equation directly for elastic distortional deformation in finite deformation plasticity. Comput Mech 60:703–707
11. Rubin MB (2020) A strongly objective expression for the average deformation rate with application to numerical integration algorithms. Finite Elem Anal Des 175:103409
12. Simo JC (1988) A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition. Part II: Computational aspects. Comput Methods Appl Mech Eng 68:1–31